HITTING PROBABILITIES OF CONSTRAINED SIMPLE RANDOM WALKS IN THREE DIMENSIONS

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ABSTRACT

HITTING PROBABILITIES OF CONSTRAINED SIMPLE RANDOM WALKS IN THREE DIMENSIONS

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We study the constrained simple random walk in three dimensions modeling the state of a queueing system with three nodes working in parallel. The process is assumed to be stable, i.e., the service rate at each node is greater than the arrival rate. The stability assumption implies that the process follows a repeating cycle, starting anew each time the process hits the origin. Consider the probability p_n that the sum of the components of the process equals n before the process hits the origin, which can be thought of as the probability of a buffer overflow in a cycle. The stability assumption implies that p_n decays exponentially in n. The goal of the present thesis is to develop approximation formulas for p_n. In the literature, this problem is treated for two dimensional simple walks using an affine transformation of the problem. We extend this analysis to three dimensions. As in two dimensions, the affine transformation yields a limit process and a limit hitting probability. We show, for the case of the three dimensional stable constrained simple random walk, the limit probability approximates p_n with an exponentially diminishing relative error, assuming that the first component of the initial point of the process is nonzero. We further approximate the limit probability by harmonic functions of the limit process constructed from solutions of harmonic systems associated with the problem. We provide a numerical example and discuss a possible application to finance.

Keywords: constrained simple random walks, rare events, queueing systems, financial modelling, harmonic systems

ÖZ

ÜÇ BOYUTTA SINIRLI BASİT RASTGELE YÜRÜYÜŞLERİN ÇARPMA OLASILIKLARI

Aktepe İlter, Cansu
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Çalışmada, üç ağın paralel olarak çalıştığı bir kuyruk sisteminin durumunu modelleyen üç boyutlu sınırlı basit rastgele yürüyüş incelenmektedir. Sürecin dengeli olduğu varsayılmaktadır, diğer bir deyişle, her ağdaki servis oranı varış oranından daha büyüktür. Dengelilik varsayımı, sürecin orijine her ulaştığında yeniden başlayarak tekrarlayan bir döngüyü takip ettiği anlamına gelmektedir. Sürecin başlangıç noktasına ulaşmadan önce bileşenlerinin toplamının n'ye eşit olma olasılığı p_n olsun. Bu olasılık, bir döngüde bir arabellek aşım olasılığı olarak düşünülebilir. Sürecin dengeli olması varsayımı, p_n'nin n arttıkça üssel olarak azaldığını ima etmektedir. Bu tezin amacı, p_n için yaklaşık hesaplama formülleri geliştirmektir. Literatürde bu problem, problemin afin dönüşümü kullanılarak iki boyutlu basit rastgele yürüyüşler için ele alınmaktadır. Bu analiz, mevcut çalışmada üç boyuta genişletilmektedir. İki boyutta olduğu gibi, afin dönüşüm sonrasında bir limit süreci ve bir limit çarpma olasılığı elde edilmektedir. Üç boyutlu dengeli kısıtlı basit rastgele yürüyüş için, elde edilen limit olasılığının, sürecin başlangıç noktasının ilk bileşeninin sıfır olmadığı varsayılarak, üstel olarak azalan bir göreli hata ile p_n'ye yaklaştığı gösterilmektedir. Ayrıca, problemle ilişkili olan harmonik sistemin çözümlerinden elde edilerek oluşturulan harmonik fonksiyonlar ile limit olasılığı yaklaşık olarak hesaplanmaktadır. Sayısal bir örnek sağlanmış ve finans sisteminde olası bir uygulamadan bahsedilmiştir.

Anahtar Kelimeler: sınırlı basit rastgele yürüyüşler, nadir olaylar, kuyruk sistemleri, finansal modelleme, harmonik sistemler

To My Family

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LIST OF ABBREVIATIONS

$\mathbb{R}$	Real Numbers
$\mathbb{C}$	Complex Numbers
$\mathbb{Z}$	Integers
$\mathbb{Z}_+$	Positive Integers
IS	Importance Sampling
LDA	Large Deviations Analysis
iid	Independently and Identically Distributed
NPL	Non-Performing Loan
HJB	Hamilton Jacobi Bellmann

CHAPTER 1

INTRODUCTION

Constrained random walks are commonly used in queuing theory for modeling of processes that have natural constraints forcing them to stay in a given specific set. Such systems could be computer networks, sources of a company, or a business servicing customers. In the simplest term, one dimensional constrained random walk can be used to model a single queue in which customers/objects arrive to the system and get service with a specific service rate. Systems that have multiple components can be modeled with multidimensional constrained random walks. In this thesis we will focus on three dimensional constrained simple random walk model corresponding to three parallel queues. An illustration of such a system is provided in Figure 1.1.

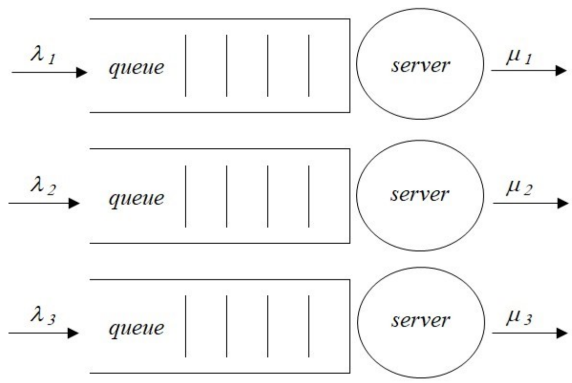


Figure 1.1: Three parallel queues

In this system, objects arrive to the system to get service with arrival rates Poisson

λ_i and serviced with service rates Exponential μ_i for $i = 1, 2, 3$. We assume without loss of generality that the service and arrival rates sum to 1. The embedded random walk X of this system is obtained by observing the system at its jump times (service completion and customer arrival times). The embedded random walk is a simple constrained random walk on the positive orthant $\mathbb{Z}_+^3$. Here each axis represents the number of objects in the corresponding queue. A jump forward in the first [second, third] axis occurring with probability λ_1 [λ_2, λ_3] represents an arrival in the first [second, third] queue and a jump backwards in the first [second, third] axis occurring with probability μ_1 [μ_2, μ_3] represents a completion of service in the first [second, third] queue. The dynamics of three dimensional constrained simple random walk is illustrated in Figure 1.2.

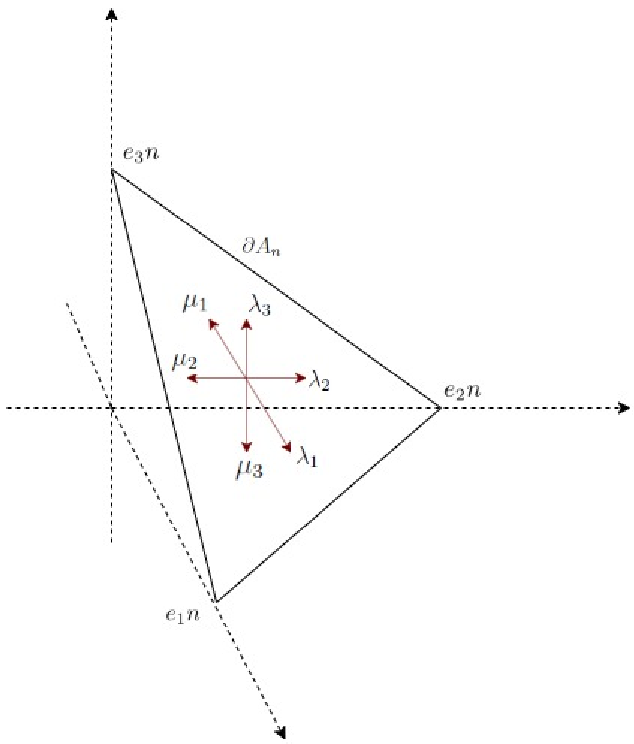


Figure 1.2: Dynamics of 3-dimensional constrained simple random walk

We assume that X is stable, i.e., $\lambda_i < \mu_i$ so that the serving performance of the system is faster on average than the arrivals. We have this stability assumption since it implies a functioning/reliable system.

Let us define the following hitting times:

$$\tau_n \doteq \inf\{k > 0 : X_k(1) + X_k(2) + X_k(3) = n\};$$

τ_n is the first time that the sum of the components of X equals n. An important performance measure for the queuing system modeled by the random walk X is the following probability:

$$p_n(x) \doteq P_x(\tau_n < \tau_0) \tag{1.1}$$

If we further make the following definition of the region A_n:

$$A_n \doteq \{x \in \mathbb{Z}_+^3 : x(1) + x(2) + x(3) \leq n\}$$

and its boundary

$$\partial A_n \doteq \{x \in \mathbb{Z}_+^3 : x(1) + x(2) + x(3) = n\}$$

Then p_n is the probability that X hits the boundary of A_n before hitting 0. One interpretation of this problem is to calculate the probability that the number of objects in the system hits a certain amount, i.e., "n" before the entire system empties.

Calculation of the probability given in 1.1 has been treated with different dimensions and different constraints. For one dimensional case, calculation of p_n is straightforward, as explicit formulas can be derived by solving the corresponding one dimensional recursive equation. The problem for two or more dimensional cases turn out to be nontrivial. A natural approach to the computation of p_n is through simulation; see, for example, [20]. See [16] for the construction of asymptotically optimal importance algorithms for p_n. The current thesis is related to the recent studies [38], [43], [42], [6] and [39] which develop approximation formulas for p_n (with relative error decaying exponentially in n) for a range of constrained random walks based on an affine transformation and limit analysis of the problem. The works [43] and [42], in particular, treat the two dimensional simple constrained random walk.

The goal of this thesis is to extend the results in [43] for the two dimensional simple walk case to the three dimensions and prove that the relative error similarly approaches to 0. The main approach of this thesis is parallel with the studies aforementioned. Compared to [43], introduction of a third dimension to the problem significantly complicates the construction of the revelant upper and lower bounds. These differences are discussed throughout the thesis.

1.1 Definitions

We model three queues that work in parallel with a random walk X constrained to the positive orthant $\mathbb{Z}^3_+$. Let us define the constraining boundaries for X as follows:

$$\partial_a \doteq \{x \in \mathbb{Z}^3_+ : x(i) = 0 \text{ if } i \in a\}$$

for the set $a \subset \{1, 2, 3\}$ and $a \neq \emptyset$. $x(i) = 0$ means that there are no objects in the i^{th} queue. We will also need what we call as strict boundaries:

$$\Pi_a \doteq \{x \in \mathbb{Z}^3_+ : x(i) = 0 \iff i \in a\}.$$

Note that

$$\partial_a = \bigcup_{a \supset b} \Pi_b, \Pi_a = \partial_a \bigcap (\partial_{a^c})^c.$$

As an example, for the point $x = (0, 1, 0)$ we have $x \in \partial_1$, $x \in \partial_{1,3}$ and $x \in \Pi_{1,3}$ but $x \notin \Pi_1$.

We define the constraining map limiting X to remain on the positive orthant with the function π:

$$\pi(x, w) \doteq \begin{cases} w, & x + w \in \mathbb{Z}^3_+ \\ 0, & otherwise. \end{cases}$$

X has increments e_i and $-e_i$ for $i = 1, 2, 3$ where $\{e_1, e_2, e_3\}$ are the unit vectors. The probabilities for e_i and $-e_i$ are λ_i and μ_i, respectively. Let I_k be an iid sequence with values drawn from the set $\{e_1, e_2, e_3, -e_1, -e_2, -e_3\}$. Then we can write X precisely as follows:

$$X_0 = x \in \mathbb{Z}^3, X_{k+1} = X_k + \pi(X_k, I_k), k = 1, 2, ..,$$

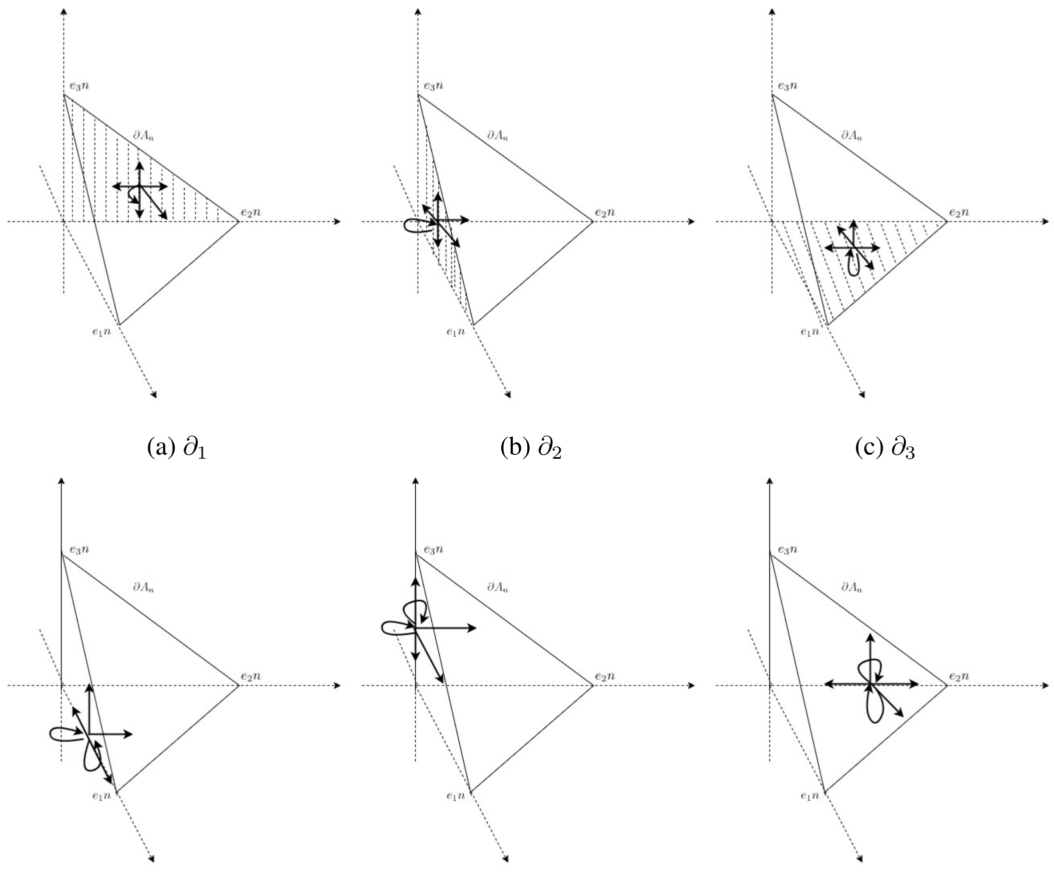
The dynamics of X at the boundaries is illustrated in Figure 1.3.

We define ρ_i as:

$$\rho_i = \frac{\lambda_i}{\mu_i}, i = 1, 2, 3.$$

Since we assume X to be stable, we have that $\rho_i = \dfrac{\lambda_i}{\mu_i} < 1$ for all $i = 1, 2, 3$. We define the parameters r_a for $a \subset \{1, 2, 3\}$ as the following:

$$r_a \doteq \frac{\sum_{i \in a} \lambda_i}{\sum_{i \in a} \mu_i}. \tag{1.2}$$



(a) ∂_1 (b) ∂_2 (c) ∂_3

(d) $\partial_{2,3}$ (e) $\partial_{1,2}$ (f) $\partial_{1,3}$

Figure 1.3: Dynamics of X at the boundaries

Hence we have $r_1 = \rho_1$, $r_2 = \rho_2$ and $r_3 = \rho_3$.

Since p_n does not depend on the order of the nodes, we can assume without loss of generality that

$$\rho_3 < \rho_2 < \rho_1.$$

Further assumptions needed for the construction of harmonic functions of Y and error analysis are:

$$r_{1,2,3}^2/\rho_3, r_{1,2,3}^2/\rho_2, r_{1,2}^2/\rho_2, r_{1,3}^2/\rho_3 < 1. \tag{1.3}$$

In order to approximate p_n, we will make use of an affine transformation of X as done in [37, 38, 43]. Let

$$\mathcal{I} \doteq \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and describe the affine transformation functions with $T_n = ne_1 + \mathcal{I}$. With the transformation T_n, we observe the random walk X from the hitting boundary. The natural point of view on the hitting boundary is proposed as $(n, 0, 0)$ [38]. Hence, we obtain the following process:

$$Y^n \doteq T_n(X), T_n(x) \doteq y$$

Y^n is a process on $(n - \mathbb{Z}_+) \times \mathbb{Z}_+^2$. The components of Y^n are defined as follows:

$$y(j) \doteq \begin{cases} n - x(i), & \text{if } j = i \\ x(j), & \text{otherwise.} \end{cases} \quad j = 1, 2, 3$$

Y^n is the same process as X but viewed from the corner ne_1 so that the probabilities of the increments e_1 and $-e_1$ are interchanged. By applying T_n, the origin of the coordinate system is shifted to ne_1. Defining $B_n = T_n(A_n)$, ∂A_n is mapped to:

$$\partial B_n = \{y \in (n - \mathbb{Z}_+) \times \mathbb{Z}_+^2, y(1) = y(2) + y(3)\}$$

Moreover, the set $\{x \in \mathbb{Z}_+^3 : x(1) = 0\}$ is mapped to:

$$\{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(i) = n\}$$

Taking the limits as $n \to \infty$, Y^n converges to the limit process Y on the domain $D_Y = \mathbb{Z} \times \mathbb{Z}_+^2$. Y is a process on D_Y, i.e. it is not constrained on $\partial_1 : \{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(1) = 0\}$. The set B_n is mapped onto

$$B \doteq \{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(1) \geq y(2) + y(3)\}$$

The boundary of B becomes the following:

$$\partial B \doteq \{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(1) = y(2) + y(3)\}$$

The constraining map π_1 on Y is defined as:

$$\pi_1(x, v) \doteq \begin{cases} v & x + v \in \mathbb{Z} \times \mathbb{Z}_+^2 \\ 0 & otherwise \end{cases}$$

Let us denote the increments of the process Y with J_k where $J_k \doteq \mathcal{I}I_k$ and

$$Y_{k+1} = Y_k + \pi_1(Y_k, J_k), k = 1, 2, ..,$$

Let τ denote the first time Y hits the boundary, ∂B:

$$\tau \doteq \inf\{k : Y_k \in \partial B\}$$

The hitting boundary for the limit problem becomes ∂B and the limit stopping time turns into τ. Applying T_n and taking limits as $n \to \infty$ leads to the limit problem of computing $P_y(\tau < \infty)$, i.e. if Y ever hits the boundary of B. Note that removing the first boundary constraint ∂_1 and the assumption of X being stable leads to the unstable process Y.

1.2 Summary of the results

This thesis aims to approximate the probability $P_x(\tau_n < \tau_0)$ for a three dimensional constrained simple random walk X, the dynamics of which are defined as in the previous subsection. In order to approximate such a probability, we implement the arguments provided in previous works [37, 38, 43, 39]. We make an affine transformation T_n of X and observe the system from one of the exit points, e.g. $(n, 0, 0)$. The new process $Y^n = T_n(X)$ is basically the same process as X except for the first coordinates are reversed $(y(1) = n - x(1), y(2) = x(2), y(3) = x(3))$. We would like to approximate $P_x(\tau_n < \tau_0)$ with the probability $P_{T_n(x)}(\tau < \infty)$. Applying the affine transformation and taking limit as n goes to ∞ allows us to remove some of the constraints and we are left with the limit problem of computing the probability $P_y(\tau < \infty)$.

For the case corresponding to the initial point of the process is fixed in y and if we set $x_n = T_n(y)$, [38] states that $P_{x_n}(\tau_n < \tau_0)$ converges to $P_y(\tau < \infty)$ as n goes to ∞ for any dimension d. For the case where the initial position of the process is specified in scaled x coordinates, a convergence analysis studying the following relative error is provided for different cases.

$$\frac{|P_{x_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)|}{P_{x_n}(\tau_n < \tau_0)} \tag{1.4}$$

With the process is fixed in scaled x, [38] treats the case for two dimensional tandem walk, [43] treats the case for two dimensional constrained simple random walk and [6] deals with the case where $d = 2$, and the dynamics of the constrained process is Markov modulated. All of the studies mentioned prove that the relative error given in Equation 1.4 decays exponentially to 0 with an amount in terms of x. [39] studies the relative error given in Equation 1.4 for a d dimensional tandem system for an initial point of unscaled x. In these studies, the probability $P_y(\tau < \infty)$ is approximated (or even explicitly formulated in some cases) by using the harmonic functions/systems of the process Y and conjugate points on the characteristic surfaces.
In this thesis we extend the results for a two dimensional constrained simple random walk given in [43] to three dimensional constrained simple random walk. We show that $P_{x_n}(\tau_n < \tau_0)$ can be approximated with $P_{T_n(x_n)}(\tau < \infty)$ with exponentially diminishing relative error. The main result of the thesis is the following:

Theorem 1.1. *For any $x \in \mathbb{R}^3_+$, $x(1) + x(2) + x(3) < 1$, $x(1) > 0$, $N > 0$ such that*

$$\frac{|P_{x_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)|}{P_{x_n}(\tau_n < \tau_0)} = \frac{|P_{x_n}(\tau_n < \tau_0) - P_{x_n}(\bar{\tau}_n < \infty)|}{P_{x_n}(\tau_n < \tau_0)} \tag{1.5}$$

decays exponentially in n for $n > N$, where $x_n = \lfloor xn \rfloor$.

where we define

$$\bar{\tau}_n \doteq \inf\{k > 0 : \sum_{j=1}^{3} \bar{X}_k(j) = n\}$$

for $\bar{X}_k \doteq T_n(Y_k)$ and $\bar{X}_{k+1} = \bar{X}_k + \pi_1(\bar{X}_k, I_k)$. Note that $\bar{X}_k$ and X_k share similar dynamics except for the boundary ∂_1. Since the hitting time of Y on ∂B exactly matches with the hitting time of $\bar{X}$ on $\{x \in \mathbb{Z} \times \mathbb{Z}^2_+ : x(1) + x(2) + x(3) = n\}$ we have $\bar{\tau}_n = \tau$. Therefore, we can write:

$$P_{x_n}(\bar{\tau}_n < \infty) = P_{T_n(x_n)}(\tau < \infty). \tag{1.6}$$

The proof of this theorem is provided in Subsection 3.4. We further construct a class of harmonic functions of Y in order to approximate $P_y(\tau < \infty)$.

1.3 Organization of the thesis

This thesis is organized as six chapters. In Chapter 1 an introduction to the problem is provided. The subject of the thesis is defined and the construction of the problem is explained. A brief background on the subject is also mentioned in this chapter. Definitions which are used in the rest of the thesis are provided. The method implemented in addressing the problem is further clarified.

Chapter 2 covers the construction of the Y-harmonic functions from harmonic systems which are later on linearly combined in order to approximate the probability $P_y(\tau < \infty)$. Y-harmonic functions are built up using pair of nodes and using all four nodes on the graph of the harmonic system; and a point in the intersection of characteristic surfaces defined from the characteristic polynomials of Y.

Chapter 3 deals with the error analysis. In this chapter, we provide a convergence analysis between the probabilities $P_{x_n}(\tau_n < \tau_0)$ and $P_{T_n(x)}(\tau < \infty)$. Our main result is provided in Theorem 3.1. For the proof of this theorem, upper bounds and lower bounds on the probabilities are constructed.

Constrained random walk models, in addition to many other research areas related to the queuing theory, arises also in finance applications. In Chapter 4 possible application to the banking sector is provided in order to provide an example of such application. Moreover, a numerical example is also given in order to demonstrate the numerical performance of the approximation algorithm.

A study of the literature on the thesis subject is presented in Chapter 5. Background on the subject and recent studies with which the thesis is related are further explained.

Lastly, conclusion of the thesis and a comparison to previous studies is given in Chapter 6. Possible future work on the subject is also discussed in this chapter.

CHAPTER 2

CONSTRUCTION OF Y-HARMONIC FUNCTIONS

This chapter is devoted to the construction of Y-harmonic functions that will be used in the approximation of the probability $P_y(\tau < \infty)$. We first define a Y-harmonic function and how to construct such functions from characteristic surfaces of Y. We introduce our four node harmonic system, points on which corresponds to the roots on the characteristic surfaces. These roots are used in the construction of Y-harmonic functions using all four nodes on the graph of the harmonic system and using pair of nodes. Another harmonic function comes from the intersection of the characteristic surfaces. Finally, a suitable linear combination of such functions will be the main function in the approximation of $P_y(\tau < \infty)$. In this chapter, for the ease of notation we will call $r_{1,2,3} \doteq r$.

2.1 Y-harmonic functions from harmonic systems

To approximate $P_y(\tau < \infty)$, we construct a class of harmonic functions of Y and apply superposition principle. These functions will be the solutions of a harmonic system, which will be defined by a graph whose vertices indicate points on the characteristic surface of Y. The approach is similar to the studies [38], [43] and [39].

Definition 2.1. *A function h is Y-harmonic if it satisfies the following equation*

$$h(y) = h(y-e_1)\lambda_1+h(y+e_1)\mu_1+\sum_{i=2}^{3}(h(y+e_i)\lambda_i+h(y+\pi_1(y,-e_i))\mu_i) = \mathbb{E}_y[h(Y_1)] \tag{2.1}$$

Let us define the region D_Y^o as follows:

$$D_Y^o \doteq \{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(2) > 0, y(3) > 0\}$$

Define the characteristic polynomial of Y on D_Y^o as follows:

$$p(\beta, \alpha_2, \alpha_3) \doteq \frac{\lambda_1}{\beta} + \beta\mu_1 + \frac{\alpha_2\lambda_2}{\beta} + \frac{\beta\mu_2}{\alpha_2} + \frac{\alpha_3\lambda_3}{\beta} + \frac{\beta\mu_3}{\alpha_3} \tag{2.2}$$

We also define characteristic polynomials of Y on the constrained coordinates ∂_2, ∂_3, $\partial_{2,3}$ as the following:

$$p_2(\beta, \alpha_2, \alpha_3) \doteq \frac{\lambda_1}{\beta} + \beta\mu_1 + \frac{\alpha_2\lambda_2}{\beta} + \frac{\alpha_3\lambda_3}{\beta} + \frac{\beta\mu_3}{\alpha_3} + \mu_2$$

$$p_3(\beta, \alpha_2, \alpha_3) \doteq \frac{\lambda_1}{\beta} + \beta\mu_1 + \frac{\alpha_2\lambda_2}{\beta} + \frac{\alpha_3\lambda_3}{\beta} + \frac{\beta\mu_2}{\alpha_2} + \mu_3$$

$$p_{2,3}(\beta, \alpha_2, \alpha_3) \doteq \frac{\lambda_1}{\beta} + \beta\mu_1 + \frac{\alpha_2\lambda_2}{\beta} + \frac{\alpha_3\lambda_3}{\beta} + \mu_2 + \mu_3$$

Y harmonic functions will be constructed from the equations $p = 1$ and $p_a = 1$ for $a \subset \{2, 3\}$, solutions of which define the following the characteristic surfaces $\mathcal{H}$ and $\mathcal{H}_a$:

$$\mathcal{H} \doteq \{(\beta, \alpha_2, \alpha_3) \in \mathbb{C}^3 : p(\beta, \alpha_2, \alpha_3) = 1\}$$

$$\mathcal{H}_a \doteq \{(\beta, \alpha_2, \alpha_3) \in \mathbb{C}^3 : p_a(\beta, \alpha_2, \alpha_3) = 1\}$$

We multiply both sides of Equation 2.2 by α_2 and obtain:

$$\alpha_2^2\frac{\lambda_2}{\beta} + \alpha_2\left[\frac{\lambda_1}{\beta} + \beta\mu_1 + \lambda_3\frac{\alpha_3}{\beta} + \mu_3\frac{\beta}{\alpha_3} - 1\right] + \mu_2\beta = 0 \tag{2.3}$$

If we solve Equation 2.3 for β and α_3 fixed, and $\alpha_{2,1}$, $\alpha_{2,2}$ are distinct roots of Equation 2.3, then they will satisfy the following:

$$\alpha_{2,1} = \frac{1}{\alpha_{2,2}}\frac{\beta^2}{\rho_2}$$

Likewise, we multiply both sides of Equation 2.2 by α_3 and obtain:

$$\alpha_3^2\frac{\lambda_3}{\beta} + \alpha_3\left[\frac{\lambda_1}{\beta} + \beta\mu_1 + \lambda_2\frac{\alpha_2}{\beta} + \mu_2\frac{\beta}{\alpha_2} - 1\right] + \mu_3\beta = 0 \tag{2.4}$$

If we solve Equation 2.3 for β and α_2 fixed, and $\alpha_{3,1}$, $\alpha_{3,2}$ are distinct roots of Equation 2.4, then they will satisfy the following:

$$\alpha_{3,1} = \frac{1}{\alpha_{3,2}} \frac{\beta^2}{\rho_3}$$

Definition 2.2. *The function*

$$\boldsymbol{\alpha}(i, (\beta, \alpha_{i,1})) = \alpha_{i,2} \tag{2.5}$$

for $i = 2, 3$ *is called the conjugator function.*

Distinct roots defined above will satify Equation 2.5. Such roots on $\mathcal{H}$ will be called i-conjugate.

Note that $(r, 1, 1) \in \mathcal{H}$ as it satisfies the equation $p = 1$.
Now, let $\beta = r$ and $\alpha_3 = 1$ be fixed. If we multiply both sides of Equation 2.2 with α_2, we get a second order equation:

$$\alpha_2^2 \frac{\lambda_2}{r} + \alpha_2 \left(-\frac{\lambda_2}{r} - r\mu_2 \right) + r\mu_2 = 0$$

$$\Delta = \left(\frac{\lambda_2}{r} + r\mu_2 \right)^2 - 4\lambda_2\mu_2 = \frac{\lambda_2^2}{r^2} + 2\lambda_2\mu_2 + r^2\mu_2^2 - 4\lambda_2\mu_2 = \left(\frac{\lambda_2}{r} - r\mu_2 \right)^2$$

Roots of this polynomial will be

$$\alpha_2 = \frac{\frac{\lambda_2}{r} + r\mu_2 + \sqrt{\left(\frac{\lambda_2}{r} - r\mu_2 \right)^2}}{\frac{2\lambda_2}{r}} = 1 \implies (r, 1, 1)$$

$$\alpha_2 = \frac{\frac{\lambda_2}{r} + r\mu_2 - \sqrt{\left(\frac{\lambda_2}{r} - r\mu_2 \right)^2}}{\frac{2\lambda_2}{r}} = \frac{r^2}{\rho_2} \implies (r, \frac{r^2}{\rho_2}, 1)$$

Therefore, for $\beta = r$ and $\alpha_3 = 1$ fixed, we obtain $(r, 1, 1)$ and $(r, \frac{r^2}{\rho_2}, 1)$ as 2-conjugate points.

Similarly, for fixed $\beta = r$ and $\alpha_2 = 1$, we can obtain 3-conjugate points $(r, 1, 1)$ and $(r, 1, \frac{r^2}{\rho_3})$. For fixed $\beta = r$ and $\alpha_3 = \frac{r^2}{\rho_3}$, we can derive 2-conjugate points $(r, 1, \frac{r^2}{\rho_3})$ and $(r, \frac{r^2}{\rho_2}, \frac{r^2}{\rho_3})$.

These four roots are related to each other as shown in Figure 2.1. This is in fact a graph of a harmonic system (see Definition 2.4) with four nodes. Points we found from characteristic surfaces of Y are represented as nodes on the graph and edges between the points denotes the conjugacy relation between them. The harmonic

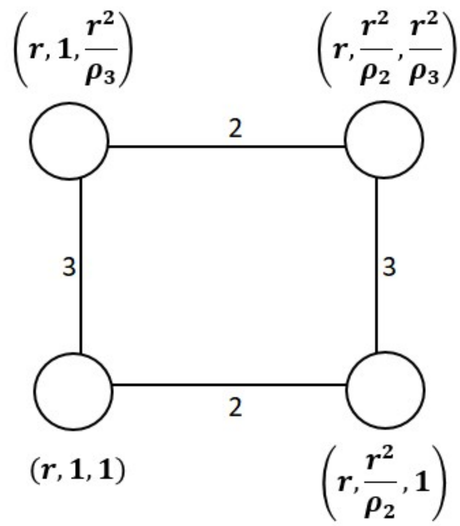


Figure 2.1: The graph of a harmonic system for the three dimensional constrained simple random walk

system and its solution given in Figure 2.1 are one of the important novelties of the analysis of the constrained simple random walk in three dimensions. The analysis in two dimensions depend only on systems with two nodes; the existence of the four dimensional harmonic system given in Figure 2.1 and its solution do not directly follow from the two node systems used in two dimensions.

For a point $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}$ we introduce the following function:

$$y \mapsto [(\beta, \alpha_2, \alpha_3), y] \doteq \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)}$$

The following proposition is a special case of [39, Lemma 7]; we provide a full proof for our special case for the reader's convenience.

Proposition 2.1. $[(\beta, \alpha_2, \alpha_3), y]$ *is* Y*-harmonic for* $y \in D_Y^o$ *when* $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}$.

Proof. Let us substitute $[(\beta, \alpha_2, \alpha_3), y]$ into Equation 2.1:

$$\begin{aligned}
\mathbb{E}_y[h_\beta(Y_1)] - h_\beta(y) &= \lambda_1 \beta^{y(1)-1-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \mu_1 \beta^{y(1)+1-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \lambda_2 \beta^{y(1)-(y(2)+1+y(3))} \alpha_2^{y(2)+1} \alpha_3^{y(3)} \\
&+ \mu_2 \beta^{y(1)-(y(2)-1+y(3))} \alpha_2^{y(2)-1} \alpha_3^{y(3)} \\
&+ \lambda_3 \beta^{y(1)-(y(2)+y(3)+1)} \alpha_2^{y(2)} \alpha_3^{y(3)+1} \\
&+ \mu_3 \beta^{y(1)-(y(2)+y(3)-1)} \alpha_2^{y(2)} \alpha_3^{y(3)-1} - \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&= \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} [\beta^{-1}\lambda_1 + \beta\mu_1 + \beta^{-1}\alpha_2\lambda_2 \\
&+ \beta\alpha_2^{-1}\mu_2 + \beta^{-1}\alpha_3\lambda_3 + \beta\alpha_3^{-1}\mu_3] - \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&= 0
\end{aligned}$$

as inside of brackets is 1 since $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}$. Hence $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic for $y \in D_Y^o$. □

Now, the functions $[(\beta, \alpha_2, \alpha_3), y]$ can be used to further introduce the following class of Y-harmonic functions. The following proposition is a special case of [39, Lemma 10]. For reader's convenience we provide a full proof for the three dimensional simple random walk treated in the present work:

Proposition 2.2. *Let* $(\beta, \alpha_2, \alpha_3) \in \mathcal{H} \cap \mathcal{H}_a$ *for* $a \subset \{2, 3\}$. *Then* $[(\beta, \alpha_2, \alpha_3), y]$ *is* Y*-harmonic.*

Proof. We already have from Proposition 2.1 that for $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}$, $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic on D_Y^o. Now we need to show that $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic on the boundaries ∂_a for $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}_a$.

On boundary ∂_2 :

$$\begin{aligned}
\mathbb{E}_y[h_\beta(Y_1)] - h_\beta(y) &= \lambda_1 \beta^{y(1)-1-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \mu_1 \beta^{y(1)+1-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \lambda_2 \beta^{y(1)-(y(2)+1+y(3))} \alpha_2^{y(2)+1} \alpha_3^{y(3)} \\
&+ \mu_2 \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \lambda_3 \beta^{y(1)-(y(2)+y(3)+1)} \alpha_2^{y(2)} \alpha_3^{y(3)+1} \\
&+ \mu_3 \beta^{y(1)-(y(2)+y(3)-1)} \alpha_2^{y(2)} \alpha_3^{y(3)-1} - \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&= \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} [\beta^{-1}\lambda_1 + \beta\mu_1 + \beta^{-1}\alpha_2\lambda_2 \\
&+ \mu_2 + \beta^{-1}\alpha_3\lambda_3 + \beta\alpha_3^{-1}\mu_3] - \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&= 0
\end{aligned}$$

since $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}_2$. Hence $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic.

On boundary ∂_3 :

$$\begin{aligned}
\mathbb{E}_y[h_\beta(Y_1)] - h_\beta(y) &= \lambda_1 \beta^{y(1)-1-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \mu_1 \beta^{y(1)+1-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&+ \lambda_2 \beta^{y(1)-(y(2)+1+y(3))} \alpha_2^{y(2)+1} \alpha_3^{y(3)} \\
&+ \mu_2 \beta^{y(1)-(y(2)-1+y(3))} \alpha_2^{y(2)-1} \alpha_3^{y(3)} \\
&+ \lambda_3 \beta^{y(1)-(y(2)+y(3)+1)} \alpha_2^{y(2)} \alpha_3^{y(3)+1} \\
&+ \mu_3 \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} - \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&= \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} [\beta^{-1}\lambda_1 + \beta\mu_1 + \beta^{-1}\alpha_2\lambda_2 \\
&+ \beta\mu_2\alpha_2^{-1} + \beta^{-1}\alpha_3\lambda_3 + \mu_3] - \beta^{y(1)-(y(2)+y(3))} \alpha_2^{y(2)} \alpha_3^{y(3)} \\
&= 0
\end{aligned}$$

since $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}_3$. Hence $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic.

On boundary $\partial_{2,3}$:

$$\begin{aligned}\mathbb{E}_y[h_\beta(Y_1)] - h_\beta(y) &= \lambda_1 \beta^{y(1)-1-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)} \\ &+ \mu_1 \beta^{y(1)+1-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)} \\ &+ \lambda_2 \beta^{y(1)-(y(2)+1+y(3))}\alpha_2^{y(2)+1}\alpha_3^{y(3)} \\ &+ \mu_2 \beta^{y(1)-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)} \\ &+ \lambda_3 \beta^{y(1)-(y(2)+y(3)+1)}\alpha_2^{y(2)}\alpha_3^{y(3)+1} \\ &+ \mu_3 \beta^{y(1)-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)} - \beta^{y(1)-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)} \\ &= \beta^{y(1)-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)}[\beta^{-1}\lambda_1 + \beta\mu_1 + \beta^{-1}\alpha_2\lambda_2 \\ &+ \mu_2 + \beta^{-1}\alpha_3\lambda_3 + \mu_3] - \beta^{y(1)-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)} \\ &= 0\end{aligned}$$

since $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}_{2,3}$. Hence $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic. Therefore, for $(\beta, \alpha_2, \alpha_3) \in \mathcal{H} \cap \mathcal{H}_a$ for $a \subset \{2, 3\}$, $[(\beta, \alpha_2, \alpha_3), y]$ is Y-harmonic. □

Parallel to the case treated in [43], note that the point (ρ_1, ρ_1, ρ_1) lies on the intersection of the characteristic surfaces $\mathcal{H}$ and $\mathcal{H}_a$ for $a \subset \{2, 3\}$. By Proposition 2.2 we attain our first nontrivial Y-harmonic function:

$$h_{\rho_1}(y) \doteq [(\rho_1, \rho_1, \rho_1), y] = \rho_1^{y(1)-(y(2)+y(3))}\rho_1^{y(2)}\rho_1^{y(3)} = \rho_1^{y(1)}. \tag{2.6}$$

Let us define the following functions which can be used in the construction of further harmonic functions of Y:

$$C(2, \beta, \alpha_{2,i}, \alpha_3) \doteq 1 - \frac{\beta}{\alpha_{2,i}} \tag{2.7}$$

and

$$C(3, \beta, \alpha_2, \alpha_{3,i}) \doteq 1 - \frac{\beta}{\alpha_{3,i}} \tag{2.8}$$

for $i = 1, 2$. The following proposition is a special case of [39, Proposition 3.1]; we provide a proof for the reader's convenience.

Proposition 2.3. *Suppose that $(\beta, \alpha_{2,1}, \alpha_3)$ and $(\beta, \alpha_{2,2}, \alpha_3)$ are distinct 2-conjugate points on the characteristic surface $\mathcal{H}$. Then the function*

$$h_\beta \doteq C(2, \beta, \alpha_{2,2}, \alpha_3)[(\beta, \alpha_{2,1}, \alpha_3), .] - C(2, \beta, \alpha_{2,1}, \alpha_3)[(\beta, \alpha_{2,2}, \alpha_3), .]$$

is Y-harmonic on ∂_2.

Proof. For $y \in \partial_2$ let us look at

$$\begin{aligned}\mathbb{E}_y[h_\beta(Y_1)] - h_\beta(y) &= C(2,\beta,\alpha_{2,2},\alpha_3)\mathbb{E}_y[(\beta,\alpha_{2,1},\alpha_3), y+\pi_1] \\ &- C(2,\beta,\alpha_{2,1},\alpha_3)\mathbb{E}_y[(\beta,\alpha_{2,2},\alpha_3), y+\pi_1] \\ &- C(2,\beta,\alpha_{2,2},\alpha_3)[(\beta,\alpha_{2,1},\alpha_3), y] \\ &+ C(2,\beta,\alpha_{2,1},\alpha_3)[(\beta,\alpha_{2,2},\alpha_3), y] \\ &= C(2,\beta,\alpha_{2,2},\alpha_3)\Big[\mathbb{E}_y[(\beta,\alpha_{2,1},\alpha_3), y+\pi_1] - [(\beta,\alpha_{2,1},\alpha_3), y]\Big] \\ &- C(2,\beta,\alpha_{2,1},\alpha_3)\Big[\mathbb{E}_y[(\beta,\alpha_{2,2},\alpha_3), y+\pi_1] - [(\beta,\alpha_{2,2},\alpha_3), y]\Big]\end{aligned}$$

We first look at $C(2,\beta,\alpha_{2,2},\alpha_3)\Big[\mathbb{E}_y[(\beta,\alpha_{2,1},\alpha_3), y+\pi_1] - [(\beta,\alpha_{2,1},\alpha_3), y]\Big]$.

$$\begin{aligned}&\mathbb{E}_y[(\beta,\alpha_{2,1},\alpha_3), y+\pi_1] - [(\beta,\alpha_{2,1},\alpha_3), y] \\ &\qquad = \lambda_1\beta^{y(1)-1-y(3)}\alpha_3^{y(3)} + \mu_1\beta^{y(1)+1-y(3)}\alpha_3^{y(3)} \\ &\qquad +\lambda_2\beta^{y(1)-1-y(3)}\alpha_{2,1}\alpha_3^{y(3)} + \mu_2\beta^{y(1)-y(3)}\alpha_3^{y(3)} \\ &\qquad +\lambda_3\beta^{y(1)-y(3)-1}\alpha_3^{y(3)+1} - \beta^{y(1)-y(3)}\alpha_3^{y(3)} \\ &\qquad +\mu_3\beta^{y(1)-y(3)+1}\alpha_3^{y(3)-1} \\ &\qquad = \beta^{y(1)-y(3)}\alpha_3^{y(3)}\Big[\frac{\lambda_1}{\beta} + \mu_1\beta + \frac{\lambda_2}{\beta}\alpha_{2,1} + \mu_2 + \frac{\lambda_3}{\beta}\alpha_3 \\ &\qquad\qquad + \frac{\beta}{\alpha_3}\mu_3 - 1\Big]\end{aligned}$$

Adding and subtracting the term $\frac{\beta}{\alpha_{2,1}}\mu_2$ inside brackets, we obtain the following equation:

$$\begin{aligned}&\mathbb{E}_y[(\beta,\alpha_{2,1},\alpha_3), y+\pi_1] - [(\beta,\alpha_{2,1},\alpha_3), y] \\ &\qquad = \beta^{y(1)-y(3)}\alpha_3^{y(3)}\Big[p(\beta,\alpha_{2,1},\alpha_3) - \frac{\beta}{\alpha_{2,1}}\mu_2 + \mu_2 - 1\Big] \\ &\qquad = \beta^{y(1)-y(3)}\alpha_3^{y(3)}\Big[\mu_2(1-\frac{\beta}{\alpha_{2,1}})\Big] \\ &\qquad = \beta^{y(1)-y(3)}\alpha_3^{y(3)}\mu_2 C(2,\beta,\alpha_{2,1},\alpha_3)\end{aligned}$$

Since $p(\beta,\alpha_{2,1},\alpha_3) = 1$ for $(\beta,\alpha_{2,1},\alpha_3) \in \mathcal{H}$.

Similar calculations can be done to obtain the equation:

$$\begin{aligned}&\mathbb{E}_y[(\beta,\alpha_{2,2},\alpha_3), y+\pi_1] - [(\beta,\alpha_{2,2},\alpha_3), y] \\ &\qquad = \beta^{y(1)-y(3)}\alpha_3^{y(3)}\mu_2 C(2,\beta,\alpha_{2,2},\alpha_3)\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}_y[h_\beta(Y_1)] - h_\beta(y) &= C(2,\beta,\alpha_{2,2},\alpha_3)\beta^{y(1)-y(3)}\alpha_3^{y(3)}\mu_2 C(2,\beta,\alpha_{2,1},\alpha_3) \\ &\quad - C(2,\beta,\alpha_{2,1},\alpha_3)\beta^{y(1)-y(3)}\alpha_3^{y(3)}\mu_2 C(2,\beta,\alpha_{2,2},\alpha_3) \\ &= 0.\end{aligned}$$

Hence, harmonicity condition is satisfied for $y \in \partial_2$. □

Note that similar proof can be done to show that

$$h_\beta \doteq C(3,\beta,\alpha_2,\alpha_{3,2})[(\beta,\alpha_2,\alpha_{3,1}),.] - C(3,\beta,\alpha_2,\alpha_{3,1})[(\beta,\alpha_2,\alpha_{3,2}),.]$$

is Y-harmonic on ∂_3 for $(\beta,\alpha_2,\alpha_{3,1})$ and $(\beta,\alpha_2,\alpha_{3,2})$ being distinct 3-conjugate points on the characteristice surface $\mathcal{H}$.

2.1.1 Y-harmonic functions using four nodes

In order to construct a harmonic function using all of the four nodes appearing in Figure 2.1, we will refer to the definition of a harmonic system proposed in [37]. Let G be an adjacency matrix associated with any graph; V_G denote the set of nodes of G; L be a finite set representing the set of labels of edges in G. If distinct vertices $i, j \in V_G$ are not connected, then $G(i,j) = 0$, and if they are connected by an edge labeled $l \in L$ (which will be called l-edge), then $G(i,j) = l$. For a node $j \in V_G$, $G(j,j)$ denotes the set of the labels of the loops on node j. The following two definitions are from [37].

Definition 2.3. *Let G and L be defined as above. G is called edge-complete with respect to L if each node $j \in V_G$ has a unique l-edge for all $l \in L$.*

Definition 2.4. *A Y-harmonic system consists of an edge-complete graph G with respect to $\mathcal{N} = \{2, ..., d\}$, the variables $(\beta, \alpha_j) \in \mathbb{C}^d, \boldsymbol{c}_j \in \mathbb{C}, j \in V_G$, and these constraints:*

1. $(\beta,\alpha_j) \in \mathcal{H}, \boldsymbol{c}_j \in \mathbb{C} - \{0\}, j \in V_G,$
2. $\alpha_i \neq \alpha_j$, *if* $i \neq j, i, j \in V_G,$

3. *α_i, α_j are $G(i,j)$-conjugate if $G(i,j) \neq 0, i \neq j, i, j \in V_G$,*

4. *If $G(i,j) \neq 0$,*
$$\frac{\boldsymbol{c}_i}{\boldsymbol{c}_j} = -\frac{C(G(i,j), \beta, \alpha_j)}{C(G(i,j), \beta, \alpha_i)},$$

5. *$(\beta, \alpha_j) \in \mathcal{H}_l$ for all $l \in G(j,j), j \in V_G$.*

The following proposition provides a harmonic function constructed from all four nodes given in Figure 2.1.

Proposition 2.4. *The coefficients*
$$1, -\frac{1-r}{1-\rho_3/r}, -\frac{1-r}{1-\rho_2/r}, \frac{(1-r)^2}{(1-\rho_2/r)(1-\rho_3/r)}$$
for the points $(r,1,1), (r,1,r^2/\rho_3), (r, r^2/\rho_2, 1), (r, r^2/\rho_2, r^2/\rho_3)$ respectively, solve the harmonic system defined by the graph given in Figure 2.1.

Proof. It is enough to show that given coefficients satisfy the constraints given in Definition 2.4. Now we know that all the points:
$$(r,1,1), (r, r^2/\rho_2, 1), (r, 1, r^2/\rho_3), (r, r^2/\rho_2, r^2/\rho_3)$$
are already in $\mathcal{H}$. Also, we know that;
$$\begin{aligned}
(r,1,1)\&(r,1,r^2/\rho_3) &\implies 3 - conjugate \\
(r,1,1)\&(r,r^2/\rho_2,1) &\implies 2 - conjugate \\
(r,r^2/\rho_2,1)\&(r,r^2/\rho_2,r^2/\rho_3) &\implies 3 - conjugate \\
(r,1,r^2/\rho_3)\&(r,r^2/\rho_2,r^2/\rho_3) &\implies 2 - conjugate
\end{aligned}$$
from the Figure 2.1. It is only left to show that these coefficients satisfy the 4^{th} condition of the Definition 2.4.

- Let $(r,1,1)$ be the 1^{st} point and $(r,1,r^2/\rho_3)$ be the 2^{nd} point:

$$\frac{\boldsymbol{c}_1}{\boldsymbol{c}_2} = -\frac{C(3,\beta,\alpha_2)}{C(3,\beta,\alpha_1)} = -\frac{C(3,r,1,r^2/\rho_3)}{C(3,r,1,1)} = -\frac{1-\dfrac{r}{r^2/\rho_3}}{1-r} = -\frac{r-\rho_3}{r(1-r)}$$

where we have used the definition provided in 2.8. The last value in the above equation can be rewritten as:

$$-\frac{r-\rho_3}{r(1-r)} = -\frac{1-\rho_3/r}{(1-r)} = \frac{1}{-\dfrac{1-r}{1-\rho_3/r}}$$

The numerator and the denominator are actually the coefficients we defined for the points $(r, 1, 1)$ and $(r, 1, r^2/\rho_3)$ in the Proposition 2.4. Hence, we have shown that they satisfy the 4^{th} condition of Definition 2.4.

- Let $(r, 1, r^2/\rho_3)$ be the 2^{nd} point and $(r, r^2/\rho_2, r^2/\rho_3)$ be the 3^{rd} point:

$$\frac{\boldsymbol{c}_2}{\boldsymbol{c}_3} = -\frac{C(2,\beta,\alpha_3)}{C(2,\beta,\alpha_2)} = -\frac{C(2,r,r^2/\rho_2,r^2/\rho_3)}{C(2,r,1,r^2/\rho_3)} = -\frac{1-\dfrac{r}{r^2/\rho_2}}{1-r} = -\frac{r-\rho_2}{r(1-r)}$$

where we have used the definition provided in 2.7. The last value in the above equation can be rewritten as:

$$-\frac{r-\rho_2}{r(1-r)} = -\frac{1-\rho_2/r}{(1-r)} = \frac{-\dfrac{1-r}{1-\rho_3/r}}{\dfrac{(1-r)^2}{(1-\rho_2/r)(1-\rho_3/r)}}$$

The numerator and the denominator are actually the coefficients we defined for the points $(r, 1, r^2/\rho_3)$ and $(r, r^2/\rho_2, r^2/\rho_3)$ in this proposition. Hence, we have shown that they satisfy the 4^{th} condition of Definition 2.4.

- Let $(r, r^2/\rho_2, r^2/\rho_3)$ be the 3^{rd} point and $(r, r^2/\rho_2, 1)$ be the 4^{th} point:

$$\frac{\boldsymbol{c}_3}{\boldsymbol{c}_4} = -\frac{C(3,\beta,\alpha_4)}{C(3,\beta,\alpha_3)} = -\frac{C(3,r,r^2/\rho_2,1)}{C(3,r,r^2/\rho_2,r^2/\rho_3)} = -\frac{1-r}{1-\dfrac{r}{r^2/\rho_3}} = -\frac{r(1-r)}{r-\rho_3}$$

where we have used the definition provided in 2.8. The last value in the above equation can be rewritten as:

$$-\frac{r(1-r)}{r-\rho_3} = -\frac{(1-r)^2(1-\rho_2/r)}{(1-r)(1-\rho_2/r)(1-\rho_3/r)} = \frac{\dfrac{(1-r)^2}{(1-\rho_2/r)(1-\rho_3/r)}}{-\dfrac{1-r}{1-\rho_2/r}}$$

The numerator and the denominator are actually the coefficients we defined for the points $(r, r^2/\rho_2, r^2/\rho_3)$ and $(r, r^2/\rho_2, 1)$ in this proposition. Hence, we have shown that they satisfy the 4^{th} condition of the Definition 2.4.

- Let $(r, 1, 1)$ be the 1^{st} point and $(r, r^2/\rho_2, 1)$ be the 4^{th} point:

$$\frac{\boldsymbol{c}_1}{\boldsymbol{c}_4} = -\frac{C(2, \beta, \alpha_4)}{C(2, \beta, \alpha_1)} = -\frac{C(2, r, r^2/\rho_2, 1)}{C(2, r, 1, 1)} = -\frac{1 - \dfrac{r}{r^2/\rho_2}}{1 - r} = -\frac{r - \rho_2}{r(1 - r)}$$

where we have used the definition provided in 2.7. The last value in the above equation can be rewritten as:

$$-\frac{r - \rho_2}{r(1 - r)} = \frac{1 - \rho_2/r}{1 - r} = \frac{1}{-\dfrac{1 - r}{1 - \rho_2/r}}$$

The numerator and the denominator are actually the coefficients we defined for the points $(r, 1, 1)$ and $(r, r^2/\rho_2, 1)$ in the Proposition. Hence, we have shown that they satisfy the 4^{th} condition of Definition 2.4. □

Using [[37], Proposition 5.2] and Proposition 2.4, the following function becomes a Y-harmonic function constructed from all four nodes given in Figure 2.1:

$$\begin{aligned} h_r \doteq & [(r, 1, 1), .] - \frac{1 - r}{1 - \rho_2/r}[(r, r^2/\rho_2, 1), .] \\ & - \frac{1 - r}{1 - \rho_3/r}[(r, 1, r^2/\rho_3), .] + \frac{(1 - r)^2}{(1 - \rho_2/r)(1 - \rho_3/r)}[(r, r^2/\rho_2, r^2/\rho_3), .] \end{aligned} \tag{2.9}$$

or

$$\begin{aligned} h_r(y) = r^{y(1)-y(2)-y(3)}\Big[& 1 - \frac{1 - r}{1 - \rho_2/r}\Big(\frac{r^2}{\rho_2}\Big)^{y(2)} - \frac{1 - r}{1 - \rho_3/r}\Big(\frac{r^2}{\rho_3}\Big)^{y(3)} \\ & + \frac{(1 - r)^2}{(1 - \rho_2/r)(1 - \rho_3/r)}\Big(\frac{r^2}{\rho_2}\Big)^{y(2)}\Big(\frac{r^2}{\rho_3}\Big)^{y(3)}\Big] \end{aligned}$$

We look for harmonic functions which are positive and if possible take value 1 on ∂B. As an example, if we choose $\lambda_1 = 0.15, \lambda_2 = 0.10, \lambda_3 = 0.10$ and $\mu_1 = 0.20, \mu_2 = 0.20, \mu_3 = 0.25$, h_r becomes positive for suitable choice of $y(2)$ and $y(3)$ on

$$\partial B \doteq \{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(1) = y(2) + y(3)\}$$

2.1.2 Y-harmonic functions using pair of nodes

For the construction of Y-harmonic functions using the pair of nodes, we implement the argument proposed in [37]. According to this argument, for a given harmonic system and the solutions with respect to this system, it is possible to construct solutions of harmonic systems for higher dimensional walks. And they will be extensions of the given system. For a given harmonic system and edge-complete graph G_0 with respect to L_0, its edge-complete extension G_1 with respect to L_1 is defined to be adding to each node of G_0 an l-loop where l belongs to the finite set $L_1 - L_0$. [[37], Proposition 5.4] explains a way to build harmonic systems (and solutions) for the extension of a given harmonic system. The proposition states that if the harmonic system with the corresponding edge-complete graph G_0 can be solved with $(\beta, \alpha_j), \boldsymbol{c}_j, j \in V_{G_0}$, then the harmonic system with the corresponding edge-complete graph G_1 can be solved with $(\beta, \alpha_j^1), \boldsymbol{c}_j, j \in V_{G_1} = V_{G_0}$ where

$$\alpha_j^1|_{\mathcal{N}} \doteq \alpha_j,$$
$$\alpha_j^1|_{\mathcal{N}^1 - \mathcal{N}} \doteq \beta$$

and $\mathcal{N} \subset \mathcal{N}^1$, $\mathcal{N}^1$ being the set of constrained coordinates of the extended process.
In our case, we have a 3 dimensional constrained random walk. We first induce the problem to 2 dimensional case by considering the boundaries 1 and 2, 1 and 3 together, and then implement the above given argument to construct harmonic functions for the 3 dimensional case.
First, we consider the boundaries 1 and 2 together and ignore the third coordinate. Previous work [42] shows that for the 2 dimensional case of the same problem, there exists a Y-harmonic function constructed from the conjugate points $(r_{1,2}, 1)$ and $(r_{1,2}, \frac{r_{1,2}^2}{\rho_2})$, where $r_{1,2} = \frac{\lambda_1 + \lambda_2}{\mu_1 + \mu_2}, \rho_2 = \frac{\lambda_2}{\mu_2}$. Y-harmonic function constructed from these conjugate points is:

$$h_{r_{1,2}} = [(r_{1,2}, 1), .] - \frac{1 - r_{1,2}}{1 - \rho_2/r_{1,2}}[(r_{1,2}, r_{1,2}^2/\rho_2), .] \tag{2.10}$$

An edge complete graph of this system consists of 2 conjugate points (one at each node) connected with label 2. Now by adding each node a 3-loop, we obtain its edge-complete extension. According to [[37], Proposition 5.4], we assign the value $r_{1,2}$ to the new component arising from the new dimension so that $(r_{1,2}, 1, \boldsymbol{r_{1,2}})$,

$(r_{1,2}, r_{1,2}^2/\rho_2, \boldsymbol{r_{1,2}})$ and the same coefficients 1 and $-\dfrac{1-r_{1,2}}{1-\rho_2/r_{1,2}}$ solve the harmonic system associated with the edge-complete extended graph. Hence we obtain a Y-harmonic function constructed from the points $(r_{1,2}, 1, \boldsymbol{r_{1,2}})$, $(r_{1,2}, r_{1,2}^2/\rho_2, \boldsymbol{r_{1,2}})$:

$$h_{r_{1,2}} = [(r_{1,2}, 1, r_{1,2}), .] - \frac{1-r_{1,2}}{1-\rho_2/r_{1,2}}[(r_{1,2}, r_{1,2}^2/\rho_2, r_{1,2}), .] \tag{2.11}$$

$$\begin{aligned} h_{r_{1,2}}(y) &= [(r_{1,2}, 1, r_{1,2}), y] - \frac{1-r_{1,2}}{1-\rho_2/r_{1,2}}[(r_{1,2}, r_{1,2}^2/\rho_2, r_{1,2}), y] \\ &= r_{1,2}^{y(1)-y(2)}\Big(1 - \frac{1-r_{1,2}}{1-\rho_2/r_{1,2}}\Big(\frac{r_{1,2}^2}{\rho_2}\Big)^{y(2)}\Big) \end{aligned}$$

Another harmonic function can be constructed by considering the boundaries 1 and 3 together. Similar to the above argument we consider the 2 dimensional walk and define $r_{1,3} = \dfrac{\lambda_1+\lambda_3}{\mu_1+\mu_3}, \rho_3 = \dfrac{\lambda_3}{\mu_3}$. Y-harmonic function constructed from the conjugate points $(r_{1,3}, 1)$ and $(r_{1,3}, \dfrac{r_{1,3}^2}{\rho_3})$ is:

$$h_{r_{1,3}} = [(r_{1,3}, 1), .] - \frac{1-r_{1,3}}{1-\rho_3/r_{1,3}}[(r_{1,3}, r_{1,3}^2/\rho_3), .]$$

Following [[37], Proposition 5.4], we assign the value $r_{1,3}$ to the new component coming from the new dimension so that $(r_{1,3}, \boldsymbol{r_{1,3}}, 1), (r_{1,3}, \boldsymbol{r_{1,3}}, r_{1,3}^2/\rho_3)$ and the same coefficients 1 and $-\dfrac{1-r_{1,3}}{1-\rho_3/r_{1,3}}$ solve the harmonic system associated with the edge-complete extended graph. Y-harmonic function which is built from these two points is:

$$h_{r_{1,3}} = [(r_{1,3}, r_{1,3}, 1), .] - \frac{1-r_{1,3}}{1-\rho_3/r_{1,3}}[(r_{1,3}, r_{1,3}, r_{1,3}^2/\rho_3), .] \tag{2.12}$$

$$\begin{aligned} h_{r_{1,3}}(y) &= [(r_{1,3}, r_{1,3}, 1), y] - \frac{1-r_{1,3}}{1-\rho_3/r_{1,3}}[(r_{1,3}, r_{1,3}, r_{1,3}^2/\rho_3), y] \\ &= r_{1,3}^{y(1)-y(3)}\Big(1 - \frac{1-r_{1,3}}{1-\rho_3/r_{1,3}}\Big(\frac{r_{1,3}^2}{\rho_3}\Big)^{y(3)}\Big) \end{aligned}$$

Therefore, using the pairs of nodes, we obtained the Y harmonic functions $h_{r_{1,2}}$ and $h_{r_{1,3}}$. These functions are used in the calculation of our final Y harmonic function. The following subsection is devoted to this end.

2.1.3 The linear combination of Y-harmonic functions

Together with Equations 2.6, 2.9, 2.11 and 2.12 we have four Y-harmonic functions. Efforts until here are made in order to obtain the following harmonic function, which is a linear combination of harmonic functions given in Equations 2.6, 2.9, 2.11 and 2.12.

Proposition 2.5. *There exists constants c_1, c_2 and c_3 so that*

$$\boldsymbol{h_r} \doteq h_r(y) + c_1 h_{\rho_1}(y) + c_2 h_{r_{1,2}}(y) + c_3 h_{r_{1,3}}(y) > 1/2 \tag{2.13}$$

for $y \in \partial B$.

Proof. Note that

$$\begin{aligned} h_r(y) = r^{y(1)-(y(2)+y(3))} &- \frac{1-r}{1-\rho_2/r} r^{y(1)-(y(2)+y(3))} \left(\frac{r^2}{\rho_2}\right)^{y(2)} \\ &- \frac{1-r}{1-\rho_3/r} r^{y(1)-(y(2)+y(3))} \left(\frac{r^2}{\rho_3}\right)^{y(3)} \\ &+ \frac{(1-r)^2}{(1-\rho_2/r)(1-\rho_3/r)} r^{y(1)-(y(2)+y(3))} \left(\frac{r^2}{\rho_2}\right)^{y(2)} \left(\frac{r^2}{\rho_3}\right)^{y(3)} \end{aligned}$$

This implies

$$\begin{aligned} h_r(y) = 1 &- \frac{1-r}{1-\rho_2/r} \left(\frac{r^2}{\rho_2}\right)^{y(2)} - \frac{1-r}{1-\rho_3/r} \left(\frac{r^2}{\rho_3}\right)^{y(3)} \\ &+ \frac{(1-r)^2}{(1-\rho_2/r)(1-\rho_3/r)} \left(\frac{r^2}{\rho_2}\right)^{y(2)} \left(\frac{r^2}{\rho_3}\right)^{y(3)} \end{aligned} \tag{2.14}$$

for $y \in \partial B$.

First choose $K_2 > 0$ and $K_3 > 0$ so that

$$\left| \frac{1-r}{1-\rho_2/r} \right| \left(r^2/\rho_2\right)^{y(2)} < 1/10$$

for $y(2) > K_2$ and

$$\left| \frac{1-r}{1-\rho_3/r} \right| \left(r^2/\rho_3\right)^{y(3)} < 1/10$$

for $y(3) > K_3$. Next we note that

$$\begin{aligned} h_{r_{1,2}}(y) &= {r_{1,2}}^{y(3)} - \frac{1-r_{1,2}}{1-\rho_2/r_{1,2}}\Big(\frac{{r_{1,2}}^2}{\rho_2}\Big)^{y(2)}{r_{1,2}}^{y(3)} \\ h_{r_{1,3}}(y) &= {r_{1,3}}^{y(2)} - \frac{1-r_{1,3}}{1-\rho_3/r_{1,3}}\Big(\frac{{r_{1,3}}^2}{\rho_3}\Big)^{y(3)}{r_{1,3}}^{y(2)} \\ h_{\rho_1}(y) &= {\rho_1}^{y(2)+y(3)} \end{aligned} \tag{2.15}$$

for $y \in \partial B$. Now choose c_2 and c_3 so that

$$c_2{r_{1,2}}^{y(3)} - \frac{1-r}{1-\rho_3/r}\left(r^2/\rho_3\right)^{y(3)} > 0$$

for $y(3) \leq K_3$ and

$$c_3{r_{1,3}}^{y(2)} - \frac{1-r}{1-\rho_2/r}\left(r^2/\rho_2\right)^{y(2)} > 0$$

for $y(2) \leq K_2$. With these choices of c_2, c_3 we have

$$\begin{aligned} f_1(y) \doteq 1 &- \frac{1-r}{1-\rho_2/r}\left(r^2/\rho_2\right)^{y(2)} - \frac{1-r}{1-\rho_3/r}\left(r^2/\rho_3\right)^{y(3)} \\ &+ c_2{r_{1,2}}^{y(3)} + c_3{r_{1,3}}^{y(2)} > 8/10 \end{aligned} \tag{2.16}$$

for $y \in \partial B$. Define further

$$\begin{aligned} f_2(y) \doteq &-c_2\frac{1-r_{1,2}}{1-\rho_2/r_{1,2}}\left(r_{1,2}^2/\rho_2\right)^{y(2)} r_{1,2}^{y(3)} - c_3\frac{1-r_{1,3}}{1-\rho_3/r_{1,3}} r_{1,3}^{y(2)}\left(r_{1,3}^2/\rho_3\right)^{y(3)} \\ &+ \frac{(1-r)^2}{(1-\rho_2/r)(1-\rho_3/r)}\left(r^2/\rho_2\right)^{y(2)}\left(r^2/\rho_3\right)^{y(3)} \end{aligned} \tag{2.17}$$

It follows from the above definitions and Equations 2.14, 2.15 that we can write

$$\boldsymbol{h_r} = f_1(y) + f_2(y) + c_1\rho_1^{y(2)+y(3)}, y \in \partial B, \tag{2.18}$$

where c_1 is still to be determined. The fact that r^2/ρ_2, r^2/ρ_3, $r_{1,2}^2/\rho_2$, $r_{1,2}$, $r_{1,3}$, $r_{1,3}^2/\rho_3 < 1$ and the definition of f_2 imply that there exists a constant $K > 0$ such that

$$|f_2(y)| < 1/10, y \in \{y \in \partial B, y(2) > K \text{ or } y(3) > K\}.$$

Since $R_K = \{y \in \partial B, y(1) \leq K \text{ and } y(2) \leq K\}$ is finite, one can choose c_1 large enough so that

$$f_2(y) + c_1\rho_1^{y(2)+y(3)} > 0$$

holds for $y \in R_K$. Combining the last two displays give

$$f_2(y) + c_1\rho_1^{y(2)+y(3)} > -1/10, y \in \partial B.$$

This, 2.16 and 2.18 imply 2.13. □

$\boldsymbol{h}_r$ will be used as an approximating function for the probability $P_y(\tau < \infty)$, see subsection 4.1 of Chapter 4. A perturbation of $\boldsymbol{h}_r$ will also be used in the error analysis, see Proposition 3.2.

CHAPTER 3

ERROR ANALYSIS

The goal of this chapter is to prove the following theorem:

Theorem 3.1. *For any* $x \in \mathbb{R}^3_+$, $x(1) + x(2) + x(3) < 1$, $x(1) > 0$, $N > 0$ *such that*

$$\frac{|P_{x_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)|}{P_{x_n}(\tau_n < \tau_0)} = \frac{|P_{x_n}(\tau_n < \tau_0) - P_{x_n}(\bar{\tau}_n < \infty)|}{P_{x_n}(\tau_n < \tau_0)} \tag{3.1}$$

decays exponentially in n *for* $n > N$, *where* $x_n = \lfloor xn \rfloor$.

This generalizes [43, Theorem 6.1], which treats the case $d = 2$, to $d = 3$. The main argument of the proof remains the same: since X and $\bar{X}$ have the same dynamics up to time σ_1 the portion of the events $\{\tau_n < \tau_0\}$ and $\{\tau < \infty\}$ that happen before time σ_1 have the same probability. It turns out that the remaining parts, i.e., those sample paths for which these events happen after time σ_1 have very small probability compared to the probability of interest $\mathbb{P}_x(\tau_n < \tau_0)$, if the initial position x is away from the boundary ∂_1. The implementation of this argument consists of the following steps:

1. Derive an upper bound on $\mathbb{P}_{x_n}(\sigma_1 < \tau_n < \tau_0)$ (Subsection 3.1).
2. Derive an upper bound on $\mathbb{P}_{x_n}(\sigma_1 < \tau < \infty)$ (Subsection 3.2).
3. Derive a lower bound on $\mathbb{P}_{x_n}(\tau_n < \tau_0)$ (Subsection 3.3).

These steps are put together to produce a proof of Theorem 3.1 in Subsection 3.4. The mathematical novelties compared to $d = 2$ treated in [43] are in the implementation of the above steps, especially in the first and the third steps.

As it is discussed in the previous works [38, 43, 6, 39], the proof is based on the idea that the events $\{\tau_n < \tau_0\}$ and $\{\tau < \infty\}$ predominantly coincides since the two processes X and Y differentiates only at the first constraining boundary, ∂_1.

Let us define the following stopping time:

$$\sigma_1 \doteq inf\{k > 0 : X_k(1) = 0\} \tag{3.2}$$

Let $\bar{X}_k \doteq T_n(Y_k)$ and $\bar{X}_{k+1} = \bar{X}_k + \pi_1(\bar{X}_k, I_k)$. We assume the starting point of the processes is $\bar{X}_0 = X_0$. Note that $\bar{X}_k$ and X_k share similar dynamics except for the boundary ∂_1 and the two processes move identically until the stopping time σ_1. Let us further define $\bar{\tau}_n$ as:

$$\bar{\tau}_n \doteq inf\{k > 0 : \sum_{j=1}^{3} \bar{X}_k(j) = n\}$$

Since the hitting time of Y on $\partial B = \{y \in \mathbb{Z} \times \mathbb{Z}_+^2 : y(1) = y(2) + y(3)\}$ exactly matches with the hitting time of $\bar{X}$ on $\{x \in \mathbb{Z} \times \mathbb{Z}_+^2 : x(1) + x(2) + x(3) = n\}$ we have $\bar{\tau}_n = \tau$. Therefore, we can write:

$$P_{x_n}(\bar{\tau}_n < \infty) = P_{T_n(x_n)}(\tau < \infty). \tag{3.3}$$

Hence, if the starting point of the process X has a distance from the constraining boundary ∂_1, the difference between the events $\{\tau_n < \tau_0\}$ and $\{\tau < \infty\}$ lies in the union of the events $\{\sigma_1 < \tau_n < \tau_0\}$ and $\{\sigma_1 < \tau = \bar{\tau}_n < \infty\}$. For the convergence analysis, we find upper bounds on the probabilities $P(\sigma_1 < \tau_n < \tau_0)$ and $P(\sigma_1 < \tau < \infty)$ by constructing a corresponding supermartingale; the supermartingale is constructed by applying X-superharmonic functions to the process X.

For the error analysis, we first introduce some elementary facts. Lemma 1, Lemma 2, Lemma 3 and Lemma 4 are used repeatedly in the convergence analysis proofs.

Lemma 1. *Suppose $a, b, c, d > 0$ satisfy*

$$a/b \leq c/d. \tag{3.4}$$

Then

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}. \tag{3.5}$$

Furthermore, the equalities hold if and only if $a/b = c/d$.

Proof. Let us define

$$d' = c\frac{b}{a},$$
$$d'' = d - b + \frac{ad}{c},$$
$$f : x \mapsto \frac{a+c}{b+x}$$

where f is a strictly decreasing function for $x > 0$. From the definition of d' we have $ad' = bc$. Now if we have the equality $a/b = c/d$ then $d = d'$. So, the equalities $a/b = c/d' = (a+c)/(b+d')$ directly follow. (3.4) and the definition of d' implies $c/d \geq c/d'$ and $d' \geq d$. On the other hand, (3.4) and the definition of d'' implies $d'' \leq d$. Then overall we have $d' \geq d \geq d''$. Applying the function f we have

$$f(d') = \frac{a}{b} \leq f(d) = \frac{a+c}{b+d} \leq f(d'') = \frac{c}{d}.$$

The statement on the equalities follows from the strict monotonicity of the function f. □

Recall that as in two dimensions we assume

$$\rho_1 > \rho_2 > \rho_3. \tag{3.6}$$

Moreover we define r_a as in 1.2. In two dimensions the definition of r_a reduces to $r_1 = \rho_1$, $r_2 = \rho_2$ and $r_{1,2}$. And $r_1 > r_{1,2} > r_2$ follows from the assumption $\rho_1 > \rho_2$.

In three dimensions we also need to compare r_a and r_b for $a, b \subset \{1, 2, 3\}$.

The necessary comparisons are listed in the lemmas below.

Lemma 2. *Under assumption 3.6*

$$\rho_1 > r_{1,2} > \rho_2, \;\; \rho_2 > r_{2,3} > \rho_3, \;\; \rho_1 > r_{1,3} > \rho_3, \;\; \rho_1 > r_{1,2} > r_{1,2,3} > r_{2,3} > \rho_3 \tag{3.7}$$

always hold.

Proof. $\rho_1 > \rho_2$ implies $\lambda_1/\mu_1 > \lambda_2/\mu_2$. By Lemma 1 we have $\frac{\lambda_1}{\mu_1} > \frac{\lambda_1 + \lambda_2}{\mu_1 + \mu_2} > \frac{\lambda_1}{\mu_1}$. Hence, $\rho_1 > r_{1,2} > \rho_2$; A similar argument applied to the pairs $\rho_2 > \rho_3$ and $\rho_1 > \rho_3$ gives $\rho_2 > r_{2,3} > \rho_3$ and $\rho_1 > r_{1,3} > \rho_3$.

We saw above that $r_{1,2} > \rho_2$; by assumption we have $\rho_2 > \rho_3$, therefore $r_{1,2} > \rho_3$. Lemma 1 applied with $c = \lambda_1 + \lambda_2$, $d = \mu_1 + \mu_2$, $c/d = r_{1,2}$ and $a = \lambda_3$ $b = \mu_3$, $a/b = \rho_3$) imply $r_{1,2} > r_{1,2,3} > \rho_3$.

The inequalities $\rho_1 > \rho_2$ and $\rho_2 > r_{2,3} > \rho_3$; imply $\rho_1 > r_{2,3}$; once again an application of Lemma 1 (this time with $a = \lambda_2 + \lambda_3$, $b = \mu_2 + \mu_3$, $a/b = r_{2,3}$ and $c = \lambda_1$, $b = \mu_1$, $c/d = \rho_1$) implies $\rho_1 > r_{1,2,3} > r_{2,3}$. Therefore we have $\rho_1 > r_{1,2} > r_{1,2,3} > r_{2,3} > \rho_3$. $\square$

By the previous lemma, Assumption 3.6 allows us to compare most of the r_a with each other. However, the above lemma doesn't resolve how $r_{1,2,3}$, $r_{1,3}$ and ρ_2 compare with each other. As already noted this situation does not arise in 2 dimensions, since the only comparisons that arise in that case are included in $\rho_1 > r_{1,2} > \rho_2$, which always holds under the assumption $\rho_1 > \rho_2$. The arguments below depend on the order of $r_{1,2,3}$ and $r_{1,3}$ (see subsections 3.1.1 and 3.1.2 below).

As in previous works we find an upper bound on a probability by constructing a corresponding supermartingale; the supermartingale is constructed by applying X-superharmonic functions to the process X. Which functions are applied depends on the probability being bounded.

For $a \subset \{1,2,3\}$ let a^c denote $\{1,2,3\} - a$. All of our functions will be constructed by taking linear combinations of functions of the following form:

$$h(r,b,\cdot) : x \mapsto \doteq r^{n-\sum_{j\notin b} x(j)}, b \subset \{1,2,3\}, x \in \mathbb{Z}_+^3, r \in (0,1). \tag{3.8}$$

If $r = r_a$ for some $a \subset \{1,2,3\}$ we will write $h(a,b,\cdot)$ instead of $h(r_a,b,\cdot)$. The harmonicity properties of these functions are established using the following two lemmas:

Lemma 3. *For $b \subset \{1,2,3\}$, $b \neq \emptyset$ the function $f : x \mapsto \sum_{i\in b}\left(\lambda_i \frac{1}{x} + \mu_i x\right)$ $x > 0$ is strictly convex and satisfies:*

1. $f(r_b) = f(1) = \sum_{i\in b}(\lambda_i + \mu_i)$,
2. $f(x) < \sum_{i\in b}(\lambda_i + \mu_i)$ *for* $x \in (r_b, 1)$,
3. $f(x) > \sum_{i\in b}(\lambda_i + \mu_i)$ *for* $x \in (0, r_b) \cup (1, \infty)$.

The proof follows from the definition of f.

Lemma 4. *For* $a \subset \{1,2,3\}$*, the function* $= h(r,a,\cdot) : x \mapsto r^{n-\sum_{j\notin a} x(j)}$ *satisfies*

$$\mathbb{E}_x[h(X_1)] = h(x)\left(1 + \sum_{j\notin a}\left(\lambda_j \frac{1}{r} + \mu_j r - (\lambda_j + \mu_j)\right) + \sum_{i\notin a} \mathbf{1}_{\partial_i}(x)\mu_i(1-r)\right). \tag{3.9}$$

For $r \in [r_{a^c}, 1]$*,* $h(r,a,\cdot)$ *is* X*-superharmonic on* $(\cup_{i\notin a}\partial_i)^c$*.*

Proof. The case $a = \{1,2,3\}$ is trivial. For $a \neq \{1,2,3\}$, the proof of (3.9) follows from the dynamics of X and the definition of h. For $x \in (\cup_{i\notin a}\partial_i)^c$ we have $x \notin \partial_i$ for $i \notin a$. Therefore for such x (3.9) reduces to

$$\mathbb{E}_x[h(X_1)] = h(x)\left(1 + \sum_{i\notin a}\left(\lambda_i \frac{1}{r} + \mu_i r - (\lambda_i + \mu_i)\right)\right).$$

For $r \in [r_{a^c}, 1]$, Lemma 3 (with $b = a^c$) implies that the last sum is 0 or negative; this implies that h is X-superharmonic on $(\cup_{i\notin a}\partial_i)^c$. □

The functions we identify below to construct supermartingales slightly deviate from being X-superharmonic (see, for example, (3.14)); this deviation causes errors to accumulate with each step of the processes X and Y. Therefore, to get meaningful bounds, the number of steps X takes in the convergence argument must be bounded, i.e., we need to truncate time. A similar truncation of time argument is used in all of the previous works [16], [34], [37] [43], [6]. To truncate time for the X process we use a general result from [42]. We need a new result for the truncation of time for the Y process. For this we will generalize the argument given in [42] to three dimensions: this consists of finding an upper bound on the moment generating function $\mathbb{E}[\beta^\tau \mathbf{1}_{\{\tau<\infty\}}]$ for some $\beta > 1$. In Subsection 3.1 we find an upper bound on the probability $\mathbb{P}_x(\sigma_1 < \tau_n < \tau_0)$.

3.1 Upper bound on the probability $P_x(\sigma_1 < \tau_n < \tau_0)$

Let us now identify the "almost" X-superharmonic functions that will be used to find an upper bound on the probability of the the event $\{\sigma_1 < \tau_n < \tau_0\}$. This event happens in two stages: first the process hits ∂_1 without hitting 0 or ∂A_n and then hits

∂A_n before hitting 0. We will use a different function for each stage. The functions depend on whether $r_{1,2,3} > r_{1,3}$ or otherwise. The functions for the case $r_{1,2,3} \geq r_{1,3}$ are presented in Theorems 3.2 and 3.3 in Subsection 3.1.1; the other case in Theorem 3.4 in Subsection 3.1.2 These theorems are verification arguments: they prove that some proposed functions have the right properties. In the intervening paragraphs we explain the process through which we identify the proposed functions.

The second stage of the event consists of X hitting ∂A_n before 0. This suggests that we mimic the function $x \mapsto \mathbb{P}_x(\tau_n < \tau_0)$ in trying to construct the X-superharmonic function for this stage. Note that $\mathbb{P}_x(\tau_n < \tau_0)$ is the unique X-harmonic function taking the value 1 on ∂A_n and 0 at $(0, 0, 0)$. The simplest function of the form (3.8) that equals 1 on ∂A_n is $x \mapsto h(r, a, x) = r^{n - \sum_{i=1}^{3} x(i)}$ and for $r = r_{1,2,3}$ this function is X-harmonic on $\mathbb{Z}_+^{3,o}$ by Lemma 4.

The function $h(\{1, 2, 3\}, \emptyset, \cdot)$, however, can't serve by itself as our function for the second stage because it is not X-superhamonic on the constraining boundaries. To get a function that is also X-superharmonic on these boundaries we can try to linearly combine $h(\{1, 2, 3\}, \emptyset, \cdot)$ with functions that are superharmonic on $\mathbb{Z}_+^{3,o}$ as well as on the constraining boundaries. Note that

$$h(\{1, 2, 3\}, \emptyset, x) = h(\{1, 2, 3\}, \{i\}, x) \text{ for } x \in \Pi_i. \tag{3.10}$$

Recall that by Lemma 2 $r_{1,2,3} > r_{2,3}$. Then by Lemma 4 (with $a = \{1\}$ and $r = r_{1,2,3} > r_{2,3}$) $h(\{1, 2, 3\}, \{1\}, \cdot)$ is X-superharmonic on $\mathbb{Z}_+^3 \cup \Pi_1 = (\partial_2 \cup \partial_3)^c$. Then nearly combining $h(\{1, 2, 3\}, \emptyset, \cdot)$ with may give a function that is X-superharmonic on $\mathbb{Z}_+^{3,o} \cup \Pi_1$. The usefulness of $h(\{1, 2, 3\}, \{2\}, \cdot)$ to treat the boundary Π_2 in this way depends on the order $r_{1,2,3}$ and $r_{1,3}$. The function $h(\{1, 2, 3\}, \{3\}, \cdot)$, on the other hand, is X-subharmonic (this follows from (3.9) with $a = \{3\}$, $r_{1,2,3} < r_{1,2}$ (Lemma 2)) and is therefore not useful to treat the boundary Π_3. The right function for this boundary turns out to be $h(\bar{r}_{1,2}, \{3\}, \cdot)$ where $\bar{r}_{1,2} \in (r_{1,2}, 1)$ is a variable whose value will be fixed later. Since $\bar{r}_{1,2} > r_{1,2,3}$, we don't have $h(\{1, 2, 3\}, \emptyset, x) = h(\bar{r}_{1,2}, \{i\}, x)$ for $x \in \Pi_3$. But $\bar{r}_{1,2} > r_{1,2,3}$ implies $h(\{1, 2, 3\}, \emptyset, x) \leq h(\bar{r}_{1,2}, \{i\}, x)$ $x \in \Pi_3$, which suffices for our purposes (see (3.23) below).

Remark 1. *Let us comment on the choice of the function $h(\bar{r}_{1,2}, \{3\}, \cdot)$ for the boundary Π_3. The goal is to find a function that is X-superharmonic on $\mathbb{Z}_+^3 \cup \Pi_3$ so* *that when it is linearly combined with $h(\{1,2,3\},\emptyset,\cdot)$ we end up with a function that is also X-superharmonic on Π_3. We may want to proceed as we did with Π_1 and try $h(\{1,2,3\},\{3\},x)$ but as noted above this function is X-subharmonic. The second natural choice is $h(\{1,2\},\{3\},\cdot)$, which is the two dimensional version of $h(\{1,2,3\},\emptyset,\cdot)$ for the first two dimensions; but this function is X-harmonic on Π_3 and cannot be used to cancel out the X-subharmonicity of $h(\{1,2,3\},\emptyset,\cdot)$ on Π_3. So we perturb $r_{1,2}$ slightly upward to obtain a strictly X-superharmonic function.*

We have thus far have identified two additional functions: $h(\{1,2,3\},\{1\},\cdot)$ for the boundary Π_1 and $h(\bar{r}_{1,2},\{3\},\cdot)$ for Π_3. These functions however themselves fail to be X-superharmonic on the lower dimensional boundaries $\Pi_{3,1}=\Pi_{1,3}$ and $\Pi_{3,2}$ and $\Pi_{1,2}$. To handle these we use a reasoning similar to above to identify further functions of the form (3.8) with the right X-superharmonicity properties, which lead to: $h(\bar{r}_{1,2},\{1,3\},\cdot)$ for $\Pi_{1,3}$, $h(\bar{\rho}_1,\{2,3\},\cdot)$ for $\Pi_{2,3}$ and $h(r_{1,2,3},\{1,2\},\cdot)$ for $\Pi_{1,2}$.

These functions will suffice for the treatment of Π_1 and Π_3 and the lower dimensional boundaries $\Pi_{1,2}$, $\Pi_{1,3}$ and $\Pi_{3,2}$. The function for Π_2 depends on the order of $r_{1,3}$ and $r_{1,2,3}$. Let us continue our discussion with the case $r_{1,3}\leq r_{1,2,3}$.

3.1.1 The case $r_{1,2,3}\geq r_{1,3}$

In this subsection we assume

$$r_{1,2,3}\geq r_{1,3}. \tag{3.11}$$

Note that (3.11) implies $\rho_2\geq r_{1,2,3}\geq r_{1,3}$ by Lemma 1.
Our main function, $h(\{1,2,3\},\emptyset,\cdot)$ satisfies

$$h(\{1,2,3\},\emptyset,x)=h(\{1,2,3\},\{2\},x)$$

for $x\in\Pi_2$. For $r_{1,2,3}\geq r_{1,3}$, $h(\{1,2,3\},\{2\},\cdot)$ is X-superharmonic on $\mathbb{Z}_+^3\cup\Pi_2$ by Lemma 4 (with $a=\{2\}$ and $r=r_{1,2,3}\geq r_{1,3}$). and we will use this function as the X-superharmonic function to deal with Π_2 in our linear combination.

With this we have all of the functions we need to identify our X-superharmonic function corresponding to the second stage of the event $\{\sigma_1<\tau_n<\tau_0\}$ for the case $r_{1,3}\leq r_{1,2,3}\leq\rho_2$:

1. For the interior $\mathbb{Z}_+^{3,o}$: $h(\{1,2,3\},\emptyset,\cdot)$,
2. For boundaries Π_1, Π_2: $h(\{1,2,3\},\{i\},\cdot)$, $i = 1,2$,
3. For boundary Π_3: $h(\bar{r}_{1,2},\{3\},\cdot)$,
4. For boundaries $\Pi_{2,3}$, $\Pi_{1,3}$ and $\Pi_{1,2}$: $h(\bar{\rho}_1,\{2,3\},\cdot)$, $h(\bar{r}_{1,2},\{1,3\},\cdot)$ and $h(\{1,2,3\},\{1,2\},\cdot)$.

For ease of notation set:

$$\begin{aligned}
h_0(x) &= h(\{1,2,3\},\emptyset,x) = r_{1,2,3}^{n-\sum_{j=1}^3 x(j)} \\
h_1(x) &= h(\{1,2,3\},\{1\},x) = r_{1,2,3}^{n-(x(2)+x(3))} \\
h_2(x) &= h(\{1,2,3\},\{2\},x) = r_{1,2,3}^{n-(x(1)+x(3))} \\
h_3(x) &= h(\bar{r}_{1,2},\{3\},x) = \bar{r}_{1,2}^{n-(x(1)+x(2))} \\
h_4(x) &= h(\bar{\rho}_1,\{2,3\},x) = \bar{\rho}_1^{n-x(1)} \\
h_5(x) &= h(\bar{r}_{1,2},\{1,3\},x) = \bar{r}_{1,2}^{n-x(2)} \\
h_6(x) &= h(\{1,2,3\},\{1,2\},x) = r_{1,2,3}^{n-x(3)}.
\end{aligned} \tag{3.12}$$

We want to linearly combine these functions to get a function that is "almost" X superharmonic in all $\mathbb{Z}_+^3$. The linear combination is

$$\boldsymbol{h}_1 \doteq h_0 + \sum_{j=1}^{6} c_j h_j \tag{3.13}$$

where $c_i > 0$, $i = 1,2,3...,6$ are constants to be determined.

Recall that $\bar{r}_{1,2} \in (r_{1,2},1)$ and $\bar{\rho}_1 \in (\rho_1,1)$ are variables; their values will be fixed later.

Theorem 3.2. *The constants c_i, $i = 1,2,3...,6$ can be chosen so that $\boldsymbol{h}_1$ satisfies the following*

$$\mathbb{E}_x[h_1(X_1)] - h_1(x) \le c_{10}\rho_1^n \tag{3.14}$$

for all $x \in \mathbb{Z}_+^3$ where $c_{10} > 0$ is a constant.

Proof. We begin by recalling that all of h_i, $i \in \{0,1,2,3,4,5,6\}$ are functions of the form $h(r,\{a\},\cdot)$ with $r \in [r_{a^c},1]$:

1. h_0: $a = \emptyset$, $r = r_{1,2,3}$,

2. h_1: $a = \{1\}$, $r = r_{1,2,3}$, where we use $r_{1,2,3} > r_{2,3}$ (Lemma 2),

3. h_2: $a = \{2\}$, $r = r_{1,2,3}$, where we use $r_{1,2,3} \geq r_{1,3}$ (Assumption 3.11),

4. h_3: $a = \{3\}$, $r = \bar{r}_{1,2}$, where we use $\bar{r}_{1,2} > r_{1,2}$, by choice,

5. h_4: $a = \{2, 3\}$, $r = \bar{\rho}_1$, where we use $\bar{\rho}_1 > \rho_1$, by choice,

6. h_5: $a = \{1, 3\}$, $r = \bar{r}_{1,2}$, where we use $\bar{r}_{1,2} > r_{1,2} > \rho_2$ (the second inequality is by Lemma 2),

7. h_6: $a = \{1, 2\}$, $r = r_{1,2,3}$ where we use $r_{1,2,3} > \rho_3$ (Lemma 2).

Therefore, by Lemma 4, all of the functions in the linear combination (3.13) are X-superharmonic on $\mathbb{Z}_+^{3,o}$. This and $c_i > 0$ imply that h_1 is X-superharmonic on $\mathbb{Z}_+^{3,o}$. Therefore, (3.14) holds with $c_{10} = 0$ for $x \in \mathbb{Z}_+^{3,o}$. It remains to treat the constraining boundaries which are Π_i, $i = 1, 2, 3$, $\Pi_{1,2}$, $\Pi_{1,3}$, $\Pi_{2,3}$ and $\Pi_{1,2,3} = \{0\}$.

$x \in \Pi_1$: by Lemma 4:

$$
\begin{aligned}
&\mathbb{E}_x[h_0(X_1)] - h_0(x) = r_{1,2,3}^{n-(x(2)+x(3))}(\mu_1(1 - r_{1,2,3})) \\
&\mathbb{E}_x[h_1(X_1)] - h_1(x) = -r_{1,2,3}^{n-(x(2)+x(3))}\left(\frac{\lambda_1}{r_{1,2,3}} + \mu_1 r_{1,2,3} - (\lambda_1 + \mu_1)\right) < 0 \\
&\mathbb{E}_x[h_2(X_1)] - h_2(x) \leq r_{1,2,3}^{n-x(3)}\mu_1(1 - r_{1,2,3}) \qquad (3.15)\\
&\mathbb{E}_x[h_3(X_1)] - h_3(x) \leq \bar{r}_{1,2}^{n-x(2)}\mu_1(1 - \bar{r}_{1,2}) \\
&\mathbb{E}_x[h_4(X_1)] - h_4(x) \leq \bar{\rho}_1^n\mu_1(1 - \bar{\rho}_1) \\
&\mathbb{E}_x[h_5(X_1)] - h_5(x) = -\bar{r}_{1,2}^{n-x(2)}\left(\lambda_2 + \mu_2 - \frac{\lambda_2}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}\right) < 0 \\
&\mathbb{E}_x[h_6(X_1)] - h_6(x) = -r_{1,2,3}^{n-x(3)}\left(\lambda_3 + \mu_3 - \frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3}\right).
\end{aligned}
$$

Note that the difference associated with h_4 already satisfies (3.14) when c_{10} is chosen large enough. Then it suffices to choose c_1, c_5 and c_6 so that the negative terms associated with h_1, h_5 and h_6 balance the positive terms associated with h_0, h_2 and

h_3:

$$c_1 \geq \frac{\mu_1(1-r_{1,2,3})}{\frac{\lambda_1}{r_{1,2,3}} + \mu_1 r_{1,2,3} - (\lambda_1+\mu_1)} > 0,$$
$$c_5 \geq c_3 \frac{\mu_1(1-\bar{r}_{1,2})}{\lambda_2+\mu_2-\frac{\lambda_2}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}} > 0, \tag{3.16}$$
$$c_6 \geq c_2 \frac{\mu_1(1-r_{1,2,3})}{\lambda_3+\mu_3-\frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3}} > 0.$$

These choices of c_1, c_5 and c_6 and (3.15) imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_4\bar{\rho}_1^n\mu_1(1-\bar{\rho}_1). \tag{3.17}$$

$x \in \Pi_2$: by Lemma 4:

$$\mathbb{E}_x[h_0(X_1)] - h_0(x) = r_{1,2,3}^{n-(x(1)+x(3))}(\mu_2(1-r_{1,2,3}))$$
$$\mathbb{E}_x[h_1(X_1)] - h_1(x) \leq r_{1,2,3}^{n-x(3)}\mu_2(1-r_{1,2,3})$$
$$\mathbb{E}_x[h_2(X_1)] - h_2(x) = -r_{1,2,3}^{n-(x(1)+x(3))}\left(\frac{\lambda_2}{r_{1,2,3}} + \mu_2 r_{1,2,3} - (\lambda_2+\mu_2)\right) < 0 \tag{3.18}$$
$$\mathbb{E}_x[h_3(X_1)] - h_3(x) \leq \bar{r}_{1,2}^{n-x(1)}\mu_2(1-\bar{r}_{1,2})$$
$$\mathbb{E}_x[h_4(X_1)] - h_4(x) = -\bar{\rho}_1^{n-x(1)}\left(\lambda_1+\mu_1-\frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right) < 0$$
$$\mathbb{E}_x[h_5(X_1)] - h_5(x) \leq \bar{r}_{1,2}^n\mu_2(1-\bar{r}_{1,2})$$
$$\mathbb{E}_x[h_6(X_1)] - h_6(x) = -r_{1,2,3}^{n-x(3)}\left(\lambda_3+\mu_3-\frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3}\right) < 0.$$

To balance the positive terms associated with h_0, h_1, h_3 with the negative terms associated with h_2, h_4 and h_6 it suffices to choose c_2, c_4 and c_6 as follows:

$$c_2 \geq \frac{\mu_2(1-r_{1,2,3})}{\frac{\lambda_2}{r_{1,2,3}} + \mu_2 r_{1,2,3} - (\lambda_2+\mu_2)} > 0,$$
$$c_4 \geq c_3 \frac{\mu_2(1-\bar{r}_{1,2})}{\lambda_1+\mu_1-\frac{\lambda_1}{\bar{\rho}_1} + \mu_1\bar{\rho}_1} > 0 \tag{3.19}$$
$$c_6 \geq c_1 \frac{\mu_2(1-r_{1,2,3})}{\lambda_3+\mu_3-\frac{\lambda_2}{r_{1,2,3}} + \mu_2 r_{1,2,3}} > 0$$

These choices of c_2, c_4 and c_6 and (3.18) imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_5\bar{r}_{1,2}^n\mu_2(1-\bar{\rho}_1) \tag{3.20}$$

for $x \in \Pi_2$.

$x \in \Pi_3$: by Lemma 4:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-(x(1)+x(2))}(\mu_3(1-r_{1,2,3})) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n-x(2)}\mu_3(1-r_{1,2,3}) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq r_{1,2,3}^{n-x(1)}\mu_3(1-r_{1,2,3}) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &= -\bar{r}_{1,2}^{n-(x(1)+x(2))}\left(\sum_{i=1}^{2}\left(\lambda_i + \mu_i - \lambda_i \frac{1}{\bar{r}_{1,2}} - \mu_i \bar{r}_{1,2})\right)\right) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &= -\bar{\rho}_1^{n-x(1)}\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &= -\bar{r}_{1,2}^{n-x(2)}\left(\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2})\right) \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\leq r_{1,2,3}^{n}\mu_3(1-r_{1,2,3}).
\end{aligned} \tag{3.21}$$

We now choose c_3 so that the first three positive terms are balanced by the negative term arising from h_3:

$$c_3 \geq \frac{(1+c_1+c_2)\mu_3(1-r_{1,2,3})}{\sum_{i=1}^{2}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}}_{1,2} - \mu_i\bar{r}_{1,2}\right)} > 0. \tag{3.22}$$

This choice of c_3, (3.21), $\bar{r}_{1,2} > r_{1,2,3}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_6 r_{1,2,3}^{n}\mu_3(1-r_{1,2,3}) \tag{3.23}$$

for $x \in \Pi_3$.

Let us now consider the lower dimensional boundaries; in all of the calculations below we use Lemma 4:

For $x \in \Pi_{1,2}$:

$$
\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-x(3)} \left(\sum_{i=1}^{2} \mu_i (1 - r_{1,2,3}) \right) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n-x(3)} \mu_2 (1 - r_{1,2,3}) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq r_{1,2,3}^{n-x(3)} \mu_1 (1 - r_{1,2,3}) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^{n} \sum_{i=1}^{2} \mu_i (1 - \bar{r}_{1,2}) \qquad (3.24) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &\leq \bar{\rho}_1^n \mu_1 (1 - \bar{\rho}_1) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &\leq \bar{r}_{1,2}^{n} \mu_2 (1 - \bar{r}_{1,2}) \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &= -r_{1,2,3}^{n-x(3)} \left(\lambda_3 + \mu_3 - \frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3} \right) < 0.
\end{aligned}
$$

We choose c_6 so that the last term balances the first three terms:

$$
c_6 \geq \frac{(1 + c_2)\mu_1(1 - r_{1,2,3}) + (1 + c_1)\mu_2(1 - r_{1,2,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3}}. \qquad (3.25)
$$

This choice, (3.24) and $\bar{\rho}_1 > \bar{r}_{1,2}$ imply

$$
\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_7 \bar{\rho}_1^n \qquad (3.26)
$$

for $x \in \Pi_{1,2}$ where

$$
c_7 \doteq c_3 \sum_{i=1}^{2} \mu_i (1 - \bar{r}_{1,2}) + c_4 \mu_1 (1 - \bar{\rho}_1) + c_5 \mu_2 (1 - \bar{r}_{1,2}).
$$

For $x \in \Pi_{1,3}$ we have:

$$
\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-x(2)} \left(\sum_{i \in \{1,3\}} \mu_i (1 - r_{1,2,3}) \right) \qquad (3.27) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n-x(2)} \mu_3 (1 - r_{1,2,3}) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq r_{1,2,3}^{n} \left(\sum_{i \in \{1,3\}} \mu_i (1 - r_{1,2,3}) \right) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^{n-x(2)} \left(\mu_1 (1 - \bar{r}_{1,2}) \right) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &\leq \bar{\rho}_1^n \mu_1 (1 - \bar{\rho}_1) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &= -\bar{r}_{1,2}^{n-x(2)} \left(\lambda_2 + \mu_2 - \lambda_2 \frac{1}{\bar{r}_{1,2}} - \mu_2 \bar{r}_{1,2} \right) < 0 \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\leq r_{1,2,3}^{n} \mu_3 (1 - r_{1,2,3}).
\end{aligned}
$$

Now choose c_5 so that the negative term associated with h_5 balances the positive terms associated with h_0, h_1 and h_3:

$$c_5 \geq \frac{\sum_{i\in\{1,3\}} \mu_i(1-r_{1,2,3}) + c_1\mu_3(1-r_{1,2,3}) + c_3\mu_1(1-\bar{r}_{1,2})}{\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}} > 0. \tag{3.28}$$

This choice, (3.27), $\bar{r}_{1,2} \geq r_{1,2,3}$ and $\bar{\rho}_1 > r_{1,2,3}, \bar{r}_{1,2}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_8\bar{\rho}_1^n \tag{3.29}$$

for $x \in \Pi_{1,3}$ where

$$c_8 \doteq c_2\left(\sum_{i\in\{1,3\}} \mu_i(1-r_{1,2,3})\right) + c_4\mu_1(1-\bar{\rho}_1) + c_6\mu_3(1-r_{1,2,3}).$$

For $x \in \Pi_{2,3}$:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-x(1)}\left(\sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3})\right)\\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n}\left(\sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3})\right)\\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq r_{1,2,3}^{n-x(1)}\left(\mu_3(1-r_{1,2,3})\right)\\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^{n-x(1)}\left(\mu_2(1-\bar{r}_{1,2})\right)\\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &= -\bar{\rho}_1^{n-x(1)}\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right)\\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &\leq \bar{r}_{1,2}^{n}\mu_2(1-\bar{r}_{1,2})\\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\leq r_{1,2,3}^{n}\mu_3(1-r_{1,2,3}).
\end{aligned} \tag{3.30}$$

We choose c_4 so that the negative term arising from h_4 balances the positives terms arising from h_0, h_2 and h_3:

$$c_4 \geq \frac{(\mu_2(1-r_{1,2,3}) + (1+c_2)\mu_3(1-r_{1,2,3})) + c_3\left(\mu_2(1-\bar{r}_{1,2})\right)}{\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right)}. \tag{3.31}$$

This choice, (3.30), $\bar{\rho}_1 > \bar{r}_{1,2}, r_{1,2,3}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_9\bar{\rho}_1^n \tag{3.32}$$

for $x \in \Pi_{2,3}$ where

$$c_9 \doteq c_6\mu_3(1 - r_{1,2,3}) + c_5\mu_2(1 - \bar{r}_{1,2}) + c_1 \left(\sum_{i \in \{2,3\}} \mu_i(1 - r_{1,2,3}) \right).$$

Finally we consider $x \in \Pi_{1,2,3}$, i.e., $x = (0, 0, 0)$:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^n \left(\sum_{i \in \{1,2,3\}} \mu_i(1 - r_{1,2,3}) \right) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^n \left(\sum_{i \in \{2,3\}} \mu_i(1 - r_{1,2,3}) \right) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq r_{1,2,3}^n \left(\sum_{i \in \{1,3\}} \mu_i(1 - r_{1,2,3}) \right) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^n \sum_{i=1}^{2} (\mu_i(1 - \bar{r}_{1,2})) \qquad (3.33) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &\leq \bar{\rho}_1^n \mu_1(1 - \bar{\rho}_1) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &\leq \bar{r}_{1,2}^n \mu_2(1 - \bar{r}_{1,2}) \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\leq r_{1,2,3}^n \mu_3(1 - r_{1,2,3}).
\end{aligned}$$

Then $\bar{\rho}_1 > r_{1,2,3}, \bar{r}_{1,2}$ implies

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_{10}\bar{\rho}_1^n \qquad (3.34)$$

for $x = (0, 0, 0)$ where

$$\begin{aligned}
c_{10} \doteq& \sum_{i \in \{1,2,3\}} \mu_i(1 - r_{1,2,3}) + c_1 \sum_{i \in \{2,3\}} \mu_i(1 - r_{1,2,3}) + c_2 \sum_{i \in \{1,3\}} \mu_i(1 - r_{1,2,3}) \\
&+ c_3 \sum_{i=1}^{2} \mu_i(1 - \bar{r}_{1,2}) + c_4\mu_1(1 - \bar{\rho}_1) + c_5\mu_2(1 - \bar{r}_{1,2}) + c_6\mu_3(1 - r_{1,2,3}).
\end{aligned}$$

Let us now combine the results above. First note that

$$c_{10} \geq c_4\mu_1(1 - \bar{\rho}_1) \vee c_5\mu_2(1 - \bar{r}_{1,2}) \vee c_6\mu_3(1 - r_{1,2,3}) \vee_{i=7}^{9} c_i \qquad (3.35)$$

(the terms on the right are the constants appearing on the right of (3.17) (3.20) (3.23) (3.26) (3.29) and (3.32)). Note further that:

1) (3.25) implies the constraints on c_6 given in (3.16) and (3.19),

2) (3.28) implies the constraint on c_5 given in (3.16),

3) (3.31) implies the constraint on c_4 given in (3.19).

Then if we choose c_i, $i = 1, 2, ..., 6$ as below, all of (3.16), (3.19), (3.22), (3.25), (3.28) and (3.31) are satisfied:

$$
\begin{aligned}
c_1 &\geq \frac{\mu_1(1 - r_{1,2,3})}{\frac{\lambda_1}{r_{1,2,3}} + \mu_1 r_{1,2,3} - (\lambda_1 + \mu_1)} > 0, \\
c_2 &\geq \frac{\mu_2(1 - r_{1,2,3})}{\frac{\lambda_2}{r_{1,2,3}} + \mu_2 r_{1,2,3} - (\lambda_2 + \mu_2)} > 0, \\
c_3 &\geq \frac{(1 + c_1 + c_2)\mu_3(1 - r_{1,2,3})}{\sum_{i=1}^{2}\left(\lambda_i + \mu_i - \lambda_i \frac{1}{\bar{r}_{1,2}} - \mu_i \bar{r}_{1,2}\right)} > 0. \\
c_4 &\geq \frac{(\mu_2(1 - r_{1,2,3}) + (1 + c_2)\mu_3(1 - r_{1,2,3})) + c_3\left(\mu_2(1 - \bar{r}_{1,2})\right)}{\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1 \bar{\rho}_1\right)} > 0 \\
c_5 &\geq \frac{\sum_{i\in\{1,3\}} \mu_i(1 - r_{1,2,3}) + c_1\mu_3(1 - r_{1,2,3}) + c_3\mu_1(1 - \bar{r}_{1,2})}{\lambda_2 + \mu_2 - \lambda_2 \frac{1}{\bar{r}_{1,2}} - \mu_2 \bar{r}_{1,2}} > 0. \\
c_6 &\geq \frac{(1 + c_2)\mu_1(1 - r_{1,2,3}) + (1 + c_1)\mu_2(1 - r_{1,2,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3}} > 0.
\end{aligned}
$$

For these choices of c_i, $i = 1, 2, 3, 4, 5, 6$, (3.17), (3.20), (3.23), (3.26), (3.29), and (3.32) all hold and imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_{10}\bar{\rho}_1^n,$$

for all $x \in \mathbb{Z}_+^3$ where we also used $\bar{\rho}_1 \geq r_{1,2,3}, \bar{r}_{1,2}$, (3.34) and (3.35). □

We next identify a function $\boldsymbol{h}_0$ for the first stage of the event $\{\sigma_1 < \tau_n < \tau_0\}$. The first stage consists of the process hitting ∂_1 at time σ_1 before hitting ∂A_n and $(0, 0, 0)$. Upon hitting ∂_1 the process enters the second stage for which we have constructed the function $\boldsymbol{h}_1$ above. Recall that we will apply these functions to X to get a supermartingale. For this construction to work we need $\boldsymbol{h}_0(x) \geq \boldsymbol{h}_1(x)$ for $x \in \partial_1$. The function $\boldsymbol{h}_1$ restricted to ∂_1 equals:

$$\boldsymbol{h}_1(x) = h_0(x) + \sum_{j=1}^{6} c_j h_j(x) = (1+c_1)h_1(x) + (c_2+c_6)h_6(x) + (c_3+c_5)h_5(x) + c_4\bar{\rho}_1^n \tag{3.36}$$

where we used $h_0(x) = h_1(x)$, $h_2(x) = h_6(x)$, $h_3(x) = h_5(x)$ for $x \in \partial_1$.

Our starting point for choosing a function for the first stage is the right side of (3.36). We will follow a process similar to above to modify the coefficients in (3.36) so that the resulting function is "almost" X-superharmonic. Define

$$\boldsymbol{h}_0 \doteq (1+c_1)h_1(x) + c_5' h_5(x) + c_6' h_6(x) + c_4\bar{\rho}_1^n \tag{3.37}$$

Theorem 3.3. *The constants $c_5' > 0$, $c_6' > 0$ can be chosen so that $\boldsymbol{h}_0(x) \geq \boldsymbol{h}_1(x)$, $x \in \partial_1$ and $\boldsymbol{h}_0$ satisfies*

$$\mathbb{E}_x[h_0(X_1)] - h_0(x) \leq c_{10}' \rho_1^n, x \in \mathbb{Z}_+^3, \tag{3.38}$$

for some constant $c_{10}' > 0$

Proof. The functions h_1, h_5 and h_6 are all X-superharmonic on $\mathbb{Z}_+^{3,o}$ so (3.38) holds for $x \in \mathbb{Z}_+^{3,o}$ with $c_{10}' = 0$. The treatment of the constraining boundaries proceed similar to the proof of Theorem 3.2. The function h_1, h_5 and h_6 depend only on $x(2)$ and $x(3)$; therefore it suffices to consider only the boundaries Π_2, Π_3 and $\Pi_{2,3}$. For $x \in \Pi_2$: (3.18) implies the following choice for c_6':

$$c_6' \geq (1+c_1)\frac{\mu_2(1-r_{1,2,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{r_{1,2,3}} - \mu_3 r_{1,2,3}} > 0. \tag{3.39}$$

This choice of c_6' and (3.18) imply

$$\mathbb{E}_x[h_0(X_1)] - \boldsymbol{h}_0(x) \leq c_5'\mu_2(1-\bar{r}_{1,2})\bar{r}_{1,2}^n, x \in \Pi_2. \tag{3.40}$$

For $x \in \Pi_3$: (3.21) implies the following choice for c_5':

$$c_5' \geq (1+c_1)\frac{\mu_3(1-r_{1,2,3})}{\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}}. \tag{3.41}$$

This choice of c_5', $\bar{r}_{1,2} > r_{1,2,3}$ and (3.21) imply

$$\mathbb{E}_x[h_0(X_1)] - \boldsymbol{h}_0(x) \leq c_6'\mu_3(1-r_{1,2,3})r_{1,2,3}^n, x \in \Pi_3. \tag{3.42}$$

For $x \in \Pi_{2,3}$ (3.30) and $\bar{\rho}_1 > \bar{r}_{1,2}, r_{1,2,3}$ imply

$$\mathbb{E}_x[h_0(X_1)] - \boldsymbol{h}_0(x) \leq c_{10'}\bar{\rho}_1^n \tag{3.43}$$

where

$$c_{10}' \doteq (1+c_1)\sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3}) + c_5'\mu_2(1-\bar{r}_{1,2}) + c_6'\mu_3(1-r_{1,2,3}). \tag{3.44}$$

Putting all of the above together we have: choose c'_5, c'_6 and c'_{10} as in (3.41), (3.39) and (3.44); (3.40), (3.42), (3.43), $c'_{10} > c'_5\mu_2(1-\bar{r}_{1,2})$, $c'_6\mu_3(1-r_{1,2,3})$ and $\bar{\rho}_1 > \bar{r}_{1,2}, r_{1,2,3}$ imply (3.38).

Finally, increase c'_5 and c'_6 if necessary so that $c'_5 > c_2+c_6$ and $c'_6 > c_3+c_5$; this choice of the constants c'_5 and c'_6, (3.36) and the definition (3.37) of $\boldsymbol{h}_0$ imply $\boldsymbol{h}_0(x) \geq \boldsymbol{h}_1(x)$ for $x \in \partial_1$. $\square$

3.1.2 The case $r_{1,2,3} < r_{1,3}$

Note that $r_{1,2,3} < r_{1,3}$ implies $\rho_2 < r_{1,2,3} < r_{1,3}$ by Lemma 1.
Recall from Subsection 3.1.1 that in the case $r_{1,2,3} \geq r_{1,3}$ we used the function $h(\{1,2,3\},\{2\},\cdot) : x \mapsto r_{1,2,3}^{n-(x(1)+x(3)}$ to deal with the constraining boundary Π_2. The Equation (3.9) (with $a = \{2\}$ and $r = r_{1,2,3}$), the assumption $r_{1,2,3} < r_{1,3}$ and Lemma 3 (with $b = a^c = \{1,3\}$) imply that $h(\{1,2,3\},\{2\},\cdot) : x \mapsto r_{1,2,3}^{n-(x(1)+x(3)}$ is strictly X-subharmonic even on $\mathbb{Z}_+^{3,o}$. So this function is not useful to treat Π_2 when $r_{1,2,3} < r_{1,3}$ and we need to identify another function to deal with this constraining boundary. In the construction of $\boldsymbol{h}_1$ of (3.13) we used the function $h(\bar{r}_{1,2},\{3\},\cdot)$ to deal with the boundary Π_3. The corresponding function for Π_2 is $h(\bar{r}_{1,3},\{2\},\cdot)$ where $\bar{r}_{1,3}$ is a variable to be chosen in the interval $(r_{1,3},\rho_1)$. By Lemma 4 (with $a = \{2\}$ and $r = \bar{r}_{1,3} > r_{1,3}$) this function is X-superharmonic on $\mathbb{Z}_+^{3,o} \cup \Pi_2$. Recall that $h(\{1,2,3\},\{2\},x) = h(\{1,2,3\},\emptyset,x)$ for $x \in \Pi_2$; $h(\bar{r}_{1,3},\{2\},\cdot)$ doesn't have this property but the assumption $r_{1,3} > r_{1,2,3}$ implies $\bar{r}_{1,3} > r_{1,2,3}$ and hence $h(\bar{r}_{1,3},\{2\},x) \geq h(\{1,2,3\},\emptyset,x)$ for $x \in \Pi_2$, which is sufficient for our purposes.

We need one further modification: the function $h(\bar{r}_{1,3},\{2\},\cdot)$ is X-subharmonic on Π_1 and Π_3 and this needs to be balanced by some X-superharmonic functions on these boundaries. The function $h_4 = h(\bar{\rho}_1,\{2,3\},\cdot)$ can serve this purpose since $\bar{\rho}_1 > \rho_1 > \bar{r}_{1,3}$. Using a similar reasoning we replace $h(\{1,2,3\},\{1,2\},\cdot)$ with $h(\bar{r}_{1,3},\{1,2\},\cdot)$ Since $\bar{r}_{1,3} > r_{1,3} > \rho_3$ (Lemma 2) this function is X-superharmonic on $(\cup_{i\in 1,2}\partial_i)^c$.

With these modifications, the construction of the functions $\boldsymbol{h}_1$ and $\boldsymbol{h}_0$ proceed as in the previous subsection. We modify the following functions in (3.12):

$$h_2 = h(\bar{r}_{1,3}, \{2\}, \cdot) : x \mapsto \bar{r}_{1,3}^{n-(x(1)+x(3))} \tag{3.45}$$
$$h_6 = h(\bar{r}_{1,3}, \{1,2\}, \cdot) : x \mapsto \bar{r}_{1,3}^{n-x(3)}.$$

After this change the definitions of $\boldsymbol{h}_1$ and $\boldsymbol{h}_0$ ((3.13) and (3.37)) remain the same for the current case and we have the following theorem:

Theorem 3.4. *Suppose $r_{1,3} > r_{1,2,3}$. Let $\boldsymbol{h}_1$ and $\boldsymbol{h}_0$ be defined as in* (3.13) *and* (3.37) *with h_2 and h_6 as in* (3.45). *Then Theorems 3.2 and 3.3 continue to hold.*

Proof. The proof proceeds exactly as the proofs of Theorems 3.2 and 3.3 where we only modify calculations pertaining to h_2 and h_6.

Proof of Theorem 3.2 for the case $r_{1,3} > r_{1,2,3}$:
Let us rewrite h_i, $i \in \{0,1,2,3,4,5,6\}$ where $h(r, \{a\}, \cdot)$ with $a \in [r_{a^c}, 1]$ as in the proof of Theorem 3.2:

1. h_0: $a = \emptyset$, $r = r_{1,2,3}$, $h_0(x) = r_{1,2,3}^{n-\sum_{j=1}^{3} x(j)}$
2. h_1: $a = \{1\}$, $r = r_{1,2,3}$, $h_1(x) = r_{1,2,3}^{n-(x(2)+x(3))}$
3. h_2: $a = \{2\}$, $r = \bar{r}_{1,3}$, $h_2(x) = \bar{r}_{1,3}^{n-(x(1)+x(3))}$
4. h_3: $a = \{3\}$, $r = \bar{r}_{1,2}$, $h_3(x) = \bar{r}_{1,2}^{n-(x(1)+x(2))}$
5. h_4: $a = \{2,3\}$, $r = \bar{\rho}_1$, $h_4(x) = \bar{\rho}_1^{n-x(1)}$
6. h_5: $a = \{1,3\}$, $r = \bar{r}_{1,2}$, $h_5(x) = \bar{r}_{1,2}^{n-x(2)}$
7. h_6: $a = \{1,2\}$, $r = \bar{r}_{1,3}$, $h_6(x) = \bar{r}_{1,3}^{n-x(3)}$

Once again, by Lemma 4, all of the functions in the linear combination (3.13) are X-superharmonic on $\mathbb{Z}_+^{3,o}$. This and $c_i > 0$ imply that h_1 is X-superharmonic on $\mathbb{Z}_+^{3,o}$. We only to treat the constraining boundaries Π_i, $i = 1,2,3$, $\Pi_{1,2}$, $\Pi_{1,3}$, $\Pi_{2,3}$ and $\Pi_{1,2,3} = \{0\}$.

$x \in \Pi_1$: Using Lemma 4:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-(x(2)+x(3))}(\mu_1(1 - r_{1,2,3})) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &= -r_{1,2,3}^{n-(x(2)+x(3))}\left(\frac{\lambda_1}{r_{1,2,3}} + \mu_1 r_{1,2,3} - (\lambda_1 + \mu_1)\right) < 0 \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\le \bar{r}_{1,3}^{n-x(3)}\mu_1(1 - \bar{r}_{1,3}) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\le \bar{r}_{1,2}^{n-x(2)}\mu_1(1 - \bar{r}_{1,2}) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &\le \bar{\rho}_1^n\mu_1(1 - \bar{\rho}_1) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &= -\bar{r}_{1,2}^{n-x(2)}\left(\lambda_2 + \mu_2 - \frac{\lambda_2}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}\right) < 0 \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &= -\bar{r}_{1,3}^{n-x(3)}\left(\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3\bar{r}_{1,3}\right).
\end{aligned} \tag{3.46}$$

Choose c_1, c_5 and c_6 so that the negative terms coming from h_1, h_5 and h_6 balance the positive terms coming from h_0, h_2 and h_3:

$$\begin{aligned}
c_1 &\ge \frac{\mu_1(1 - \bar{r}_{1,3})}{\frac{\lambda_1}{r_{1,2,3}} + \mu_1 r_{1,2,3} - (\lambda_1 + \mu_1)} > 0, \\
c_5 &\ge c_3\frac{\mu_1(1 - \bar{r}_{1,2})}{\lambda_2 + \mu_2 - \frac{\lambda_2}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}} > 0, \\
c_6 &\ge c_2\frac{\mu_1(1 - \bar{r}_{1,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3\bar{r}_{1,3}} > 0.
\end{aligned} \tag{3.47}$$

These choices of c_1, c_5 and c_6 and (3.46) imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \le c_4\bar{\rho}_1^n\mu_1(1 - \bar{\rho}_1). \tag{3.48}$$

for $x \in \Pi_1$.

$x \in \Pi_2$: Using Lemma 4:

$$
\begin{aligned}
&\mathbb{E}_x[h_0(X_1)] - h_0(x) = r_{1,2,3}^{n-(x(1)+x(3))}(\mu_2(1 - r_{1,2,3}))\\
&\mathbb{E}_x[h_1(X_1)] - h_1(x) \leq r_{1,2,3}^{n-x(3)}\mu_2(1 - r_{1,2,3})\\
&\mathbb{E}_x[h_2(X_1)] - h_2(x) = -\bar{r}_{1,3}^{n-(x(1)+x(3))}\left(\sum_{i\in\{1,3\}}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}_{1,3}} - \mu_i\bar{r}_{1,3}\right)\right) < 0 \qquad (3.49)\\
&\mathbb{E}_x[h_3(X_1)] - h_3(x) \leq \bar{r}_{1,2}^{n-x(1)}\mu_2(1 - \bar{r}_{1,2})\\
&\mathbb{E}_x[h_4(X_1)] - h_4(x) = -\bar{\rho}_1^{n-x(1)}\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right) < 0\\
&\mathbb{E}_x[h_5(X_1)] - h_5(x) \leq \bar{r}_{1,2}^{n}\mu_2(1 - \bar{r}_{1,2})\\
&\mathbb{E}_x[h_6(X_1)] - h_6(x) = -\bar{r}_{1,3}^{n-x(3)}\left(\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3\bar{r}_{1,3}\right) < 0.
\end{aligned}
$$

Choose c_2, c_4 and c_6 so that the negative terms coming from h_2, h_4 and h_6 are balanced by the positive terms associated with h_0, h_1, h_3:

$$
\begin{aligned}
c_2 &\geq \frac{\mu_2(1 - \bar{r}_{1,3})}{\sum_{i\in\{1,3\}}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}_{1,3}} - \mu_i\bar{r}_{1,3}\right)} > 0,\\
c_4 &\geq c_3\frac{\mu_2(1 - \bar{r}_{1,2})}{\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1} > 0 \qquad (3.50)\\
c_6 &\geq c_1\frac{\mu_2(1 - \bar{r}_{1,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3\bar{r}_{1,3}} > 0
\end{aligned}
$$

These choices of c_2, c_4, c_6 and (3.49) together with $\bar{r}_{1,3} > r_{1,2,3}$, $\bar{\rho}_1 > \bar{r}_{1,2}$ imply

$$
\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_5\bar{r}_{1,2}^{n}\mu_2(1 - \bar{\rho}_1) \qquad (3.51)
$$

for $x \in \Pi_2$.

$x \in \Pi_3$: Using Lemma 4:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-(x(1)+x(2))}(\mu_3(1 - r_{1,2,3})) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\le r_{1,2,3}^{n-x(2)}\mu_3(1 - r_{1,2,3}) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\le \bar{r}_{1,3}^{n-x(1)}\mu_3(1 - \bar{r}_{1,3}) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &= -\bar{r}_{1,2}^{n-(x(1)+x(2))}\left(\sum_{i=1}^{2}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}_{1,2}} - \mu_i\bar{r}_{1,2}\right)\right) \qquad (3.52) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &= -\bar{\rho}_1^{n-x(1)}\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &= -\bar{r}_{1,2}^{n-x(2)}\left(\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2})\right) \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\le \bar{r}_{1,3}^{n}\mu_3(1 - \bar{r}_{1,3}).
\end{aligned}$$

Choose c_3, c_4, c_5 so that the positive terms are balanced by the negative terms arising from h_3, h_4 and h_5:

$$\begin{aligned}
c_3 &\ge \frac{(1 + c_1 + c_2)\mu_3(1 - \bar{r}_{1,3})}{\sum_{i=1}^{2}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}_{1,2}} - \mu_i\bar{r}_{1,2}\right)} > 0, \\
c_4 &\ge c_2\frac{\mu_3(1 - \bar{r}_{1,3})}{\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1} > 0 \qquad (3.53) \\
c_5 &\ge c_1\frac{\mu_3(1 - r_{1,2,3})}{\lambda_2 + \mu_2 - \frac{\lambda_2}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}} > 0
\end{aligned}$$

This choice of c_3, (3.52), $\bar{r}_{1,3} > r_{1,2,3}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \le c_6\bar{r}_{1,3}^{n}\mu_3(1 - \bar{r}_{1,3}) \qquad (3.54)$$

for $x \in \Pi_3$.

For $x \in \Pi_{1,2}$:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-x(3)} \left(\sum_{i=1}^{2} \mu_i (1 - r_{1,2,3}) \right) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n-x(3)} \mu_2 (1 - r_{1,2,3}) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq \bar{r}_{1,3}^{n-x(3)} \mu_1 (1 - \bar{r}_{1,3}) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^{n} \sum_{i=1}^{2} \mu_i (1 - \bar{r}_{1,2}) \qquad (3.55) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &\leq \bar{\rho}_1^{n} \mu_1 (1 - \bar{\rho}_1) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &\leq \bar{r}_{1,2}^{n} \mu_2 (1 - \bar{r}_{1,2}) \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &= -\bar{r}_{1,3}^{n-x(3)} \left(\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3 \bar{r}_{1,3} \right) < 0.
\end{aligned}$$

We choose c_6 so that the last term balances the first three terms:

$$c_6 \geq \frac{(1 + c_2)\mu_1(1 - \bar{r}_{1,3}) + (1 + c_1)\mu_2(1 - \bar{r}_{1,3}))}{\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3 \bar{r}_{1,3}}. \qquad (3.56)$$

This choice, (3.55), $\bar{\rho_1} > \bar{r}_{1,2}$ and $\bar{r}_{1,3} > r_{1,3} > r_{1,2,3}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_7 \bar{\rho}_1^n \qquad (3.57)$$

for $x \in \Pi_{1,2}$ where

$$c_7 \doteq c_3 \sum_{i=1}^{2} \mu_i (1 - \bar{r}_{1,2}) + c_4 \mu_1 (1 - \bar{\rho}_1) + c_5 \mu_2 (1 - \bar{r}_{1,2}).$$

For $x \in \Pi_{1,3}$ we have:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-x(2)} \left(\sum_{i \in \{1,3\}} \mu_i (1 - r_{1,2,3}) \right) \qquad (3.58) \\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n-x(2)} \mu_3 (1 - r_{1,2,3}) \\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq \bar{r}_{1,3}^{n} \left(\sum_{i \in \{1,3\}} \mu_i (1 - \bar{r}_{1,3}) \right) \\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^{n-x(2)} \left(\mu_1 (1 - \bar{r}_{1,2}) \right) \\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &\leq \bar{\rho}_1^{n} \mu_1 (1 - \bar{\rho}_1) \\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &= -\bar{r}_{1,2}^{n-x(2)} \left(\lambda_2 + \mu_2 - \lambda_2 \frac{1}{\bar{r}_{1,2}} - \mu_2 \bar{r}_{1,2} \right) < 0 \\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\leq \bar{r}_{1,3}^{n} \mu_3 (1 - \bar{r}_{1,3}).
\end{aligned}$$

Choose c_5 so that the negative term coming from h_5 balances the positive terms coming from h_0, h_1 and h_3:

$$c_5 \geq \frac{\sum_{i\in\{1,3\}} \mu_i(1-r_{1,2,3}) + c_1\mu_3(1-r_{1,2,3}) + c_3\mu_1(1-\bar{r}_{1,2})}{\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}} > 0. \tag{3.59}$$

This choice, (3.58), $\bar{r}_{1,2} \geq r_{1,2,3}$ and $\bar{\rho}_1 > r_{1,3}, \bar{r}_{1,2}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_8\bar{\rho}_1^n \tag{3.60}$$

for $x \in \Pi_{1,3}$ where

$$c_8 \doteq c_2\left(\sum_{i\in\{1,3\}} \mu_i(1-\bar{r}_{1,3})\right) + c_4\mu_1(1-\bar{\rho}_1) + c_6\mu_3(1-\bar{r}_{1,3}).$$

For $x \in \Pi_{2,3}$:

$$\begin{aligned}
\mathbb{E}_x[h_0(X_1)] - h_0(x) &= r_{1,2,3}^{n-x(1)}\left(\sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3})\right)\\
\mathbb{E}_x[h_1(X_1)] - h_1(x) &\leq r_{1,2,3}^{n}\left(\sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3})\right)\\
\mathbb{E}_x[h_2(X_1)] - h_2(x) &\leq \bar{r}_{1,3}^{n-x(1)}\left(\mu_3(1-\bar{r}_{1,3})\right)\\
\mathbb{E}_x[h_3(X_1)] - h_3(x) &\leq \bar{r}_{1,2}^{n-x(1)}\left(\mu_2(1-\bar{r}_{1,2})\right)\\
\mathbb{E}_x[h_4(X_1)] - h_4(x) &= -\bar{\rho}_1^{n-x(1)}\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right)\\
\mathbb{E}_x[h_5(X_1)] - h_5(x) &\leq \bar{r}_{1,2}^{n}\mu_2(1-\bar{r}_{1,2})\\
\mathbb{E}_x[h_6(X_1)] - h_6(x) &\leq \bar{r}_{1,3}^{n}\mu_3(1-\bar{r}_{1,3}).
\end{aligned} \tag{3.61}$$

Choose c_4 so that the negative term stemming from h_4 balances the positive terms arising from h_0, h_2 and h_3:

$$c_4 \geq \frac{(\mu_2(1-\bar{r}_{1,3}) + (1+c_2)\mu_3(1-\bar{r}_{1,3})) + c_3\left(\mu_2(1-\bar{r}_{1,3})\right)}{\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right)}. \tag{3.62}$$

This choice of c_4, (3.61), $\bar{r}_{1,3} > \bar{r}_{1,3} > r_{1,2,3}$, $\bar{\rho}_1 > \bar{r}_{1,2}, r_{1,3}$ imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \leq c_9\bar{\rho}_1^n \tag{3.63}$$

for $x \in \Pi_{2,3}$ where

$$c_9 \doteq c_6\mu_3(1-\bar{r}_{1,3}) + c_5\mu_2(1-\bar{r}_{1,2}) + c_1\left(\sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3})\right).$$

For $x \in \Pi_{1,2,3}$, i.e., $x = (0,0,0)$ applying Lemma 4 we obtain:

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \le c_{10}\bar{\rho}_1^n \tag{3.64}$$

for $x = (0,0,0)$ where

$$\begin{aligned} c_{10} \doteq & \sum_{i\in\{1,2,3\}} \mu_i(1-r_{1,2,3}) + c_1 \sum_{i\in\{2,3\}} \mu_i(1-r_{1,2,3}) + c_2 \sum_{i\in\{1,3\}} \mu_i(1-\bar{r}_{1,3}) \\ & + c_3 \sum_{i=1}^{2} \mu_i(1-\bar{r}_{1,2}) + c_4\mu_1(1-\bar{\rho}_1) + c_5\mu_2(1-\bar{r}_{1,2}) + c_6\mu_3(1-\bar{r}_{1,3}). \end{aligned}$$

Combining the results above implies:

$$c_{10} \ge c_4\mu_1(1-\bar{\rho}_1) \vee c_5\mu_2(1-\bar{r}_{1,2}) \vee c_6\mu_3(1-\bar{r}_{1,3}) \vee c_7 \vee c_8 \vee c_9 \tag{3.65}$$

Choose c_i, $i = 1,2,...,6$ as below so that all the conditions provided in (3.47), (3.50), (3.53), (3.56), (3.59) and (3.62) are satisfied:

$$\begin{aligned} c_1 &\ge \frac{\mu_1(1-\bar{r}_{1,3})}{\frac{\lambda_1}{r_{1,2,3}} + \mu_1 r_{1,2,3} - (\lambda_1+\mu_1)} > 0, \\ c_2 &\ge \frac{\mu_2(1-\bar{r}_{1,3})}{\sum_{i\in\{1,3\}}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}_{1,3}} - \mu_i\bar{r}_{1,3}\right)} > 0, \\ c_3 &\ge \frac{(1+c_1+c_2)\mu_3(1-\bar{r}_{1,3})}{\sum_{i=1}^{2}\left(\lambda_i + \mu_i - \lambda_i\frac{1}{\bar{r}_{1,2}} - \mu_i\bar{r}_{1,2}\right)} > 0. \\ c_4 &\ge \frac{(\mu_2(1-\bar{r}_{1,3}) + (1+c_2)\mu_3(1-\bar{r}_{1,3})) + c_3\left(\mu_2(1-\bar{r}_{1,2})\right)}{\left(\lambda_1 + \mu_1 - \frac{\lambda_1}{\bar{\rho}_1} - \mu_1\bar{\rho}_1\right)} > 0 \\ c_5 &\ge \frac{\sum_{i\in\{1,3\}}\mu_i(1-r_{1,2,3}) + c_1\mu_3(1-r_{1,2,3}) + c_3\mu_1(1-\bar{r}_{1,2})}{\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}} > 0. \\ c_6 &\ge \frac{(1+c_2)\mu_1(1-\bar{r}_{1,3}) + (1+c_1)\mu_2(1-\bar{r}_{1,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3\bar{r}_{1,3}} > 0. \end{aligned}$$

For these choices of c_i, $i = 1,2,3,4,5,6$, (3.48), (3.51), (3.54), (3.57), (3.60), and (3.63) all hold and imply

$$\mathbb{E}_x[\boldsymbol{h}_1(X_1)] - \boldsymbol{h}_1(x) \le c_{10}\bar{\rho}_1^n,$$

for all $x \in \mathbb{Z}_+^3$ where we also used, (3.64) and (3.65)

Proof of Theorem 3.3 for the case $r_{1,3} > r_{1,2,3}$.:
Similar to the arguments provided in Subsection 3.1.1, for the first stage of the event

$\{\sigma_1 < \tau_n < \tau_0\}$, we can identify $\boldsymbol{h}_0$ as in 3.37 where the functions h_1, h_5 and h_6 are provided in Subsection 3.1.2. The functions h_1, h_5 and h_6 are in terms of $x(2)$ and $x(3)$; hence, it is enough to check only the boundaries Π_2, Π_3 and $\Pi_{2,3}$.

For $x \in \Pi_2$: (3.49) indicates the choice of c_6' as follows:

$$c_6' \geq (1 + c_1)\frac{\mu_2(1 - r_{1,2,3})}{\lambda_3 + \mu_3 - \frac{\lambda_3}{\bar{r}_{1,3}} - \mu_3\bar{r}_{1,3}} > 0. \tag{3.66}$$

This choice of c_6' and (3.49) imply

$$\mathbb{E}_x[h_0(X_1)] - \boldsymbol{h}_0(x) \leq c_5'\mu_2(1 - \bar{r}_{1,2})\bar{r}_{1,2}^n, x \in \Pi_2. \tag{3.67}$$

For $x \in \Pi_3$: (3.52) indicates the choice for c_5' as follows:

$$c_5' \geq (1 + c_1)\frac{\mu_3(1 - r_{1,2,3})}{\lambda_2 + \mu_2 - \lambda_2\frac{1}{\bar{r}_{1,2}} - \mu_2\bar{r}_{1,2}}. \tag{3.68}$$

This choice of c_5', $\bar{r}_{1,2} > r_{1,2,3}$ and (3.52) imply

$$\mathbb{E}_x[h_0(X_1)] - \boldsymbol{h}_0(x) \leq c_6'\mu_3(1 - \bar{r}_{1,3})\bar{r}_{1,3}^n, x \in \Pi_3. \tag{3.69}$$

For $x \in \Pi_{2,3}$ (3.61) and $\bar{\rho}_1 > \bar{r}_{1,3} > r_{1,2,3}$, $\bar{\rho}_1 > \bar{r}_{1,2}$ imply

$$\mathbb{E}_x[h_0(X_1)] - \boldsymbol{h}_0(x) \leq c_{10}'\bar{\rho}_1^n \tag{3.70}$$

where

$$c_{10}' \doteq (1 + c_1)\sum_{i \in \{2,3\}} \mu_i(1 - r_{1,2,3}) + c_5'\mu_2(1 - \bar{r}_{1,2}) + c_6'\mu_3(1 - \bar{r}_{1,3}). \tag{3.71}$$

Now, taking into account all of the calculations above, we have: choose c_5', c_6' and c_{10}' as in (3.68), (3.66) and (3.71); (3.67), (3.69), (3.70), $c_{10}' > c_5'\mu_2(1 - \bar{r}_{1,2})$, $c_6'\mu_3(1 - \bar{r}_{1,3})$ and $\bar{\rho}_1 > \bar{r}_{1,3}, r_{1,2,3}$ imply (3.38).

Finally, we can increase c_5' and c_6' so that $c_5' > c_2 + c_6$ and $c_6' > c_3 + c_5$ holds. this choice of the constants c_5' and c_6', and the definition of $\boldsymbol{h}_0$ and $\boldsymbol{h}_1$ imply $\boldsymbol{h}_0(x) \geq \boldsymbol{h}_1(x)$ for $x \in \partial_1$.

□

3.1.3 Statement of the upperbound

Using the functions constructed in the previous sections we can now derive our upper bound on the probability $\mathbb{P}_x(\sigma_1 < \tau_n < \tau_0)$:

Proposition 3.1. *There exists a constant* $c_{11} > 0$ *and* $N_0 > 0$ *such that*

$$\mathbb{P}_{x_n}(\sigma_1 < \tau_n < \tau_0) \leq \boldsymbol{h}_0(x_n) + c_{11} n \rho_1^n. \tag{3.72}$$

for any $x \in A_n$ *and* $n > N_0$

Proof. Define

$$M_n = \boldsymbol{h}_0(X_n)\mathbf{1}_{\{n \leq \sigma_1\}} + \boldsymbol{h}_1(X_n)\mathbf{1}_{\{n > \sigma_1\}} - c_{12} n \rho_1^n.$$

By Theorems 3.2, 3.3 and 3.4, M_n is a supermartingale if we choose $c_{12} > \max(c_{10}, c'_{10})$. To deal with the last term in the definition of M, we need to argue that X cannot spend too much time without hitting the origin or ∂A_n. This is accomplished by using the following result ([32, Theorem A.1.13]): there exists $c_{13} > 0$ and N_0 such that for any $x \in A_n$

$$\mathbb{P}_x(\tau_n \wedge \tau_0 > c_{13} n) \leq \rho_1^{2n} \tag{3.73}$$

for $n > N_0$. This implies

$$\mathbb{P}_x(\sigma_1 < \tau_n < \tau_0) \leq \rho_1^{2n} + \mathbb{P}_x(\sigma_1 < \tau_n < \tau_0 \leq c_{13} n) \tag{3.74}$$

for $n > N_0$. To bound the last probability we apply the optional sampling theorem to the supermartingale M at the bounded stopping time $\tau_n \wedge \tau_0 \wedge c_{13} n$:

$$\begin{aligned}\boldsymbol{h}_0(x) = M_0 &\geq \mathbb{E}_x[M_{\tau_n \wedge \tau_0 \wedge c_{13} n}] \\ &\geq \mathbb{E}_x[\boldsymbol{h}_1(X_{\tau_n})\mathbf{1}_{\{\sigma_1 < \tau_n < \tau_0 \wedge c_{13} n\}}] - c_{12} c_{13} n \rho_1^n.\end{aligned}$$

By its Definition (3.13), $\boldsymbol{h}_1(x) \geq h_0(x)$ and for $x \in \partial A_n$ we have $\sum_{j=1}^3 x(j) = n$ and $h_0(x) = 1$ (see (3.12)). Then if we replace $\boldsymbol{h}_1(X_{\tau_n})$ in the above display with 1 we get something smaller:

$$\begin{aligned}&\geq \mathbb{E}_x[\mathbf{1}_{\{\sigma_1 < \tau_n < \tau_0 \wedge c_{13} n\}}] - c_{12} c_{13} n \rho_1^n \\ &= \mathbb{P}_x(\sigma_1 < \tau_n < \tau_0 \leq c_{13} n) - c_{12} c_{13} n \rho_1^n.\end{aligned}$$

Now choose $c_{11} > c_{12} c_{13}$; the last display, $\rho_1^n > \rho_1^{2n}$ and (3.74) imply (3.72) for $n > N_0$. □

3.2 Upper bound on the probability $P_x(\sigma_1 < \tau < \infty)$

The main difference between the events $\{\sigma_1 < \tau < \infty\}$ and $\{\sigma_1 < \tau_n < \tau_0\}$ is that X is constrained on ∂_1 whereas $\bar{X}$ is not. This implies that the supermartingale constructed in the previous section for the event $\{\sigma_1 < \tau_n < \tau_0\}$ can also be used to bound the event $\{\sigma_1 < \tau < \infty\}$; in fact, since $\bar{X}$ is not constrained on ∂_1, the terms introduced into the functions h_i to deal with ∂_1 can be omitted which leads to a tighter upperbound for the probability $\mathbb{P}_x(\{\sigma_1 < \tau < \infty\})$. The only nontrivial change that is needed to adapt the supermartingale of the previous section to the event $\{\sigma_1 < \tau < \infty\}$ is in the truncation of time argument. Recall that we used the bound (3.73), [32, Theorem A.1.13] for this purpose. This result covers only the fully contrained process X and not $\bar{X}$ so it can't be used for truncation of time in deriving an upperbound for $\mathbb{P}_x(\{\sigma_1 < \tau < \infty\})$. The work [43] obtains an an upperbound of the form (3.73) by finding an upper bound on the probability generating function $y \mapsto \mathbb{E}_y[z^\tau \mathbf{1}_{\{\tau<\infty\}}]$ for some $z > 1$; to find the upper bound on the probability generating function, it introduces and uses the concept of Y-z-harmonic functions. The following subsection gives a definition of these functions in the current context and constructs Y-z-harmonic functions to find an upperbound on the probability generating function.

3.2.1 Y-z-harmonic functions and their construction

A function h is said to be Y-z-harmonic if it satisfies

$$z\mathbb{E}_y[h(Y_1)] = h(y).$$

Recall that we constructed Y-harmonic functions from points on the characteristic surface $\mathcal{H}$. In an entirely similar way one can define a characteristic surface $\mathcal{H}^{(z)}$ and construct Y-z-harmonic functions from points on this surface. A two dimensional version of this construction was done in [43]. Let us now do it for our case.

For $z \in \mathbb{C}$, the characteristic surface $\mathcal{H}^{(z)}$ is defined as

$$\mathcal{H}^{(z)} = \{(\beta, \alpha_2, \alpha_3) \in \mathbb{C}^3 : p(\beta, \alpha_2, \alpha_3) = 1/z\}, \tag{3.75}$$

where

$$p(\beta, \alpha_2, \alpha_3) = \frac{\lambda_1}{\beta} + \mu_1\beta + \lambda_2\frac{\alpha_2}{\beta} + \mu_2\frac{\beta}{\alpha_2} + \lambda_3\frac{\alpha_3}{\beta} + \mu_3\frac{\beta}{\alpha_3}.$$

Note that $\mathcal{H} = \mathcal{H}^{(1)}$. Recall that the function h_r was constructed from points $(r, 1, 1)$, $(r, r^2/\rho_2, 1)$, $(r, 1, r^2/\rho_3)$ and $(r, r^2/\rho_2, r^2/\rho_3)$ on the surface $\mathcal{H}$. This construction continues to work with minor modifications when we introduce the $1/z$ term above. We identified the value r by solving $p(\beta, 1, 1) = 1$ which is a quadratic equation in β. To find the corresponding value for z, we solve $p(\beta, 1, 1) = 1/z$ which is again quadratic. Denote the solution by $\boldsymbol{\beta}(z)$. Being the solution of a quadratic equation $\boldsymbol{\beta}(z)$ is continuous in β and we already know from Chapter 2 that $\boldsymbol{\beta}(1) = r < 1$. Then for z near 1 we have $\boldsymbol{\beta}(z) < 1$. Now one solves $p(\boldsymbol{\beta}(z), \alpha_2, 1) = 1/z$ for α_2 to find the value corresponding to r^2/ρ_2. Multiplying $p - 1/z = 0$ by α_2 gives the following quadratic equation in α_2:

$$\alpha_2^2\frac{\lambda_2}{\beta} + \alpha_2\left[\frac{\lambda_1}{\beta} + \mu_1\beta + \lambda_3\frac{1}{\beta} + \mu_3\beta - 1/z\right] + \mu_2\beta = 0$$

where we take $\beta = \boldsymbol{\beta}(z)$. Note that $\alpha_2 = 1$ is already a solution to this equation (since $p(\boldsymbol{\beta}(z), 1, 1) = 1/z$). Then the other root must be

$$\frac{\boldsymbol{\beta}(z)^2}{\rho_2}.$$

The point corresponding to r^2/ρ_3 is identified similarly to be $\frac{\boldsymbol{\beta}(z)^2}{\rho_3}$. These calculations yield us the following points on $\mathcal{H}^{(z)}$:

$$(\boldsymbol{\beta}(z), 1, 1), (\boldsymbol{\beta}(z), \boldsymbol{\beta}(z)^2/\rho_2, 1), (\boldsymbol{\beta}(z), 1, \boldsymbol{\beta}(z)^2/\rho_3), (\boldsymbol{\beta}(z), \boldsymbol{\beta}(z)^2/\rho_2, \boldsymbol{\beta}(z)^2/\rho_3).$$

Each of these points define a corresponding Y-z-harmonic function on D_Y^o. We now want to linearly combine them to obtain a Y-z-harmonic function on D_Y. To identify the coefficients to be used in this linear combination we need to generalize the functions C given in Equations (2.7) and (2.8) to the current case. Following the definition for the case $d = 2$ given in [43], the correct generalization turns out to be:

$$C_z(i, \beta, \alpha_2, \alpha_3) = z\left(1 - \beta/\alpha_i\right).$$

Once the C_z and the above characteristic points are available the coefficients directly generalize from those identified in Proposition 2.4: 1, $-\frac{1-\boldsymbol{\beta}(z)}{(1-\rho_2/\boldsymbol{\beta}(z))}$, $-\frac{1-\boldsymbol{\beta}(z)}{(1-\rho_3/\boldsymbol{\beta}(z))}$, $\frac{1-\boldsymbol{\beta}(z)}{(1-\rho_2/\boldsymbol{\beta}(z))}\frac{1-\boldsymbol{\beta}(z)}{(1-\rho_3/\boldsymbol{\beta}(z))}$.

All of these considerations give the following Y-z-harmonic version of h_r:

Proposition 3.2.

$$
\begin{aligned}
h_{r,z} = [(\boldsymbol{\beta}(z), 1, 1), \cdot] &- \frac{1-\boldsymbol{\beta}(z)}{(1-\rho_2/\boldsymbol{\beta}(z))}[(\boldsymbol{\beta}(z), \boldsymbol{\beta}(z)^2/\rho_2, 1), \cdot] \\
&- \frac{1-\boldsymbol{\beta}(z)}{(1-\rho_3/\boldsymbol{\beta}(z))}[(\boldsymbol{\beta}(z), 1, \boldsymbol{\beta}(z)^2/\rho_3), \cdot] \\
&+ \frac{1-\boldsymbol{\beta}(z)}{(1-\rho_2/\boldsymbol{\beta}(z))}\frac{1-\boldsymbol{\beta}(z)}{(1-\rho_3/\boldsymbol{\beta}(z))}[(\boldsymbol{\beta}(z), \boldsymbol{\beta}(z)^2/\rho_2, \boldsymbol{\beta}(z)^2/\rho_3), \cdot]
\end{aligned}
$$

defines a Y-z-harmonic function on D_Y.

Proof. The proof proceeds exactly as the proof of Proposition 2.4 and uses the calculations given above. □

To find a bound on the probability generating function we need a Y-z-harmonic function that is strictly positive on ∂B; the function $h_{r,z}$ identified is not even positive on ∂B. So we need further Y-z-harmonic functions. We can get these by generalizing $h_{r_{1,2}}$, $h_{r_{1,3}}$ and h_{ρ_1} identified in Chapter 2. Since all of these functions are constructed from points on the characteristic surface $\mathcal{H}$ their generalization to the current case proceeds exactly as the generalization of h_r to $h_{r,z}$ given above. For this we need the following functions. Let $\boldsymbol{\beta}_2(z)$ denote the solution of $p(\beta, 1, \beta) = 1/z$ such that $\boldsymbol{\beta}_2(1) = r_{1,2}$; $\boldsymbol{\beta}_2$ is continuous since it is defined as the root of a quadratic equation. In particular $\boldsymbol{\beta}_2(z) < 1$ for z near 1. Similarly, let $\boldsymbol{\beta}_3(z)$ denote the solution of $p(\beta, \beta, 1) = 1/z$ such that $\boldsymbol{\beta}_3(1) = r_{1,3}$. Finally, we define $\boldsymbol{\beta}_4(z)$ to be the solution of $p(\beta, \beta, \beta) = 1/z$ with $\boldsymbol{\beta}_4(1) = \rho_1$.

Proposition 3.3.

$$
\begin{aligned}
h_{2,z} &\doteq [(\boldsymbol{\beta}_2(z), 1, \boldsymbol{\beta}_2(z)), \cdot] - \frac{1-\boldsymbol{\beta}_2(z)}{1-\frac{\rho_2}{\boldsymbol{\beta}_2(z)}}[(\boldsymbol{\beta}_2(z), \boldsymbol{\beta}_2(z)^2/\rho_2, \boldsymbol{\beta}_2(z)), \cdot] \\
h_{3,z} &\doteq [(\boldsymbol{\beta}_3(z), \boldsymbol{\beta}_3(z), 1), \cdot] - \frac{1-\boldsymbol{\beta}_3(z)}{1-\frac{\rho_3}{\boldsymbol{\beta}_3(z)}}[(\boldsymbol{\beta}_3(z), \boldsymbol{\beta}_3(z), \boldsymbol{\beta}_3(z)^2/\rho_3), \cdot] \\
h_{4,z} &\doteq [(\boldsymbol{\beta}_4(z), \boldsymbol{\beta}_4(z), \boldsymbol{\beta}_4(z)), \cdot]
\end{aligned}
$$

define Y-z-harmonic functions.

Proof. The proof is based on the preceding calculations and proceeds as the derivations of $h_{r_{1,2}}$, $h_{r_{1,3}}$ and h_{ρ_1} given in Equations 2.11, 2.12, and 2.6. □

Proposition 3.4. *There exist $z_0 > 1$, c_{2,z_0}, c_{3,z_0} and c_{4,z_0} such that*

$$\boldsymbol{h}_{r,z_0} \doteq h_{r,z_0} + c_{2,z_0} h_{2,z_0} + c_{3,z_0} h_{3,z_0} + c_{4,z_0} h_{4,z_0} \tag{3.76}$$

satisfies $\boldsymbol{h}_{r,z_0} > 1/2$ on ∂B and

$$\begin{aligned} &\boldsymbol{\beta}(z_0), \boldsymbol{\beta}_2(z_0), \boldsymbol{\beta}_3(z_0), \boldsymbol{\beta}_4(z_0) < 1, \\ &\boldsymbol{\beta}_2(z_0)^2/\rho_2, \boldsymbol{\beta}_3(z_0)^2/\rho_3 < 1 \end{aligned} \tag{3.77}$$

Proof. First recall that $\boldsymbol{\beta}(1) = r < 1$, $\boldsymbol{\beta}_2(1) = r_{1,2} < 1$ and $\boldsymbol{\beta}_3(1) = r_{1,3} < 1$ $\boldsymbol{\beta}_4(1) = \rho_1 < 1$ by the stability assumption and $\boldsymbol{\beta}_2(1)^2/\rho_2 < 1, \boldsymbol{\beta}_3(1)^2/\rho_3 < 1$ by assumption 1.3. Since all of these functions are continuous in z, if we choose $z_0 > 1$ close to 1 all of these inequalities will continue to hold. Once $z_0 > 1$ is chosen in this way the choice of the constants c_{i,z_0}, $i = 1, 2, 3, 4$ proceeds exactly as in Proposition 2.5. □

Using the function constructed in the previous proposition we can establish our upperbound on the probability generating function:

Proposition 3.5. *Let z_0 be as in the previous proposition. Then there exists $c_{14} > 0$ such that*

$$\mathbb{E}_y[z_0^\tau \mathbf{1}_{\{\tau<\infty\}}] < c_{14} \tag{3.78}$$

for $y \in B$.

The difficult part of this result was the construction of the $\boldsymbol{h}_{r,z_0}$ which was given above. Once this function is available the proof of Proposition 3.5 is essentially unchanged from the case $d = 2$ presented in [43]. For completeness we provide a proof below.

Proof. The inequality (3.77) implies that $\boldsymbol{h}_{r,z_0}$ is bounded on ∂B. The same inequality also implies that h_{r,z_0}, h_{2,z_0}, h_{3,z_0} and h_{4,z_0} are exponentially decreasing in $y(1)$. These imply that $\boldsymbol{h}_{r,z_0}(y)$ is bounded by some constant $c_{14}/2 > 0$ for $y \in B$. That $\boldsymbol{h}_{r,z_0}$ is Y-z-harmonic implies that $z^n \boldsymbol{h}_{r,z_0}(Y_n)$ is a martingale. An application of the optional sampling theorem to this martingale at the time τ gives (3.78). □

3.2.2 The bound on the probability $P_y(c_{15}n < \tau < \infty)$

Using the bound (3.78) we can now derive a bound on $P_y(c_{15}n < \tau < \infty)$, which will be used in the truncation of time when bounding $P_y(\sigma_1 < \tau < \infty)$.

Proposition 3.6. *There exists $c_{15} > 0$ such that*

$$P_y(c_{15}n < \tau < \infty) < \rho_1^{2n} \tag{3.79}$$

for $y \in B$.

Proof. As in [43] the proof consists of an application of Markov's inequality to the random variable z_0^τ:

$$\begin{aligned} z_0^{c_{15}n}\mathbb{P}(c_{15}n < \tau < \infty) &\leq \mathbb{E}_y[z_0^\tau \mathbf{1}_{\tau<\infty}] \\ \mathbb{P}(c_{15}n < \tau < \infty) &\leq \left(z_0^{-c_{15}}\right)^n c_{14}. \end{aligned}$$

Now choose c_{15} large enough so that $z_0^{-c_{15}}c_{14} < \rho_1^2$, which is possible since $z_0 > 1$. □

3.2.3 The bound on the probability $P_y(\sigma_1 < \tau < \infty)$

With the truncation bound provided by Proposition 3.6 we are able to derive the upperbound we seek on $P_y(\sigma_1 < \tau < \infty)$:

Proposition 3.7. *Let c_{11} and N_0 be as in Proposition 3.1. Then*

$$\mathbb{P}_x(\sigma_1 < \tau < \infty) \leq \boldsymbol{h}_0(x) + c_{11}n\rho_1^n. \tag{3.80}$$

for any $x \in A_n$ and $n > N_0$.

Proof. The proof proceeds exactly as the proof of Proposition 3.1 except that instead of truncating time using (3.73) we use the bound (3.79). □

3.3 Lower bound on the probability $P_x(\tau_n < \tau_0)$

Recall that for the 3-dimensional case we defined the parameters r_a for $a \subset \{1,2,3\}$ and ρ_i for $i = 1,2,3$ as the following

$$r_a \doteq \frac{\sum_{i\in a} \lambda_i}{\sum_{i\in a} \mu_i} \tag{3.81}$$

Furthermore, we assumed that $\rho_3 < \rho_2 < \rho_1$.

Proposition 3.8. *Let*

$$f_n(x) \doteq r^{n-[x(1)+x(2)+x(3)]} \vee r_{1,2}^{n-[x(1)+x(2)]} \vee \rho_1^{n-x(1)} \tag{3.82}$$

Then $f_n(x)$ is a subharmonic function of X on $A_n - \partial A_n$.

Proof. Firstly, write

$$\begin{aligned} {r_{1,2}}^{n-[x(1)+x(2)]} &= {r_{1,2}}^{n-[x(1)+x(2)+x(3)]}{r_{1,2}}^{x(3)} \\ &= {r_{1,2}}^{n-[x(1)+x(2)+x(3)]}1^{x(2)}{r_{1,2}}^{x(3)} \\ &= [(r_{1,2}, 1, r_{1,2}), T_n(x)] \end{aligned}$$

Since for $(\beta, \alpha_2, \alpha_3) \in \mathcal{H}$, $x \to [(\beta, \alpha_2, \alpha_3), T_n(x)]$ is X-harmonic on $\mathbb{Z}_+^3 - \bigcup_{a\subset\{1,2,3\}} \partial_a$, and we know that $(r_{1,2}, 1, r_{1,2}) \in \mathcal{H}$, we have ${r_{1,2}}^{n-[x(1)+x(2)]}$ is X-harmonic for $x \in \mathbb{Z}_+^3 - \bigcup_{a\subset\{1,2,3\}} \partial_a$. Similarly, write

$$\begin{aligned} r^{n-[x(1)+x(2)+x(2)]} &= r^{n-[x(1)+x(2)+x(3)]}1^{x(2)}1^{x(3)} \\ &= [(r, 1, 1), T_n(x)] \end{aligned}$$

we know that $(r, 1, 1) \in \mathcal{H}$. So $r^{n-[x(1)+x(2)+x(3)]}$ is X-harmonic for $x \in \mathbb{Z}_+^3 - \bigcup_{a\subset\{1,2,3\}} \partial_a$. Also write

$$\begin{aligned} \rho_1^{n-x(1)} &= \rho_1^{n-[x(1)+x(2)+x(3)]}\rho_1^{x(2)}\rho_1^{x(3)} \\ &= [(\rho_1, \rho_1, \rho_1), T_n(x)] \end{aligned}$$

we know that $(\rho_1, \rho_1, \rho_1) \in \mathcal{H}$. So $\rho_1^{n-x(1)}$ is X-harmonic for $x \in \mathbb{Z}_+^3 - \bigcup_{a\subset\{1,2,3\}} \partial_a$. Hence, their maximum $f_n(x)$ is subharmonic on $\mathbb{Z}_+^3 - \bigcup_{a\subset\{1,2,3\}} \partial_a$. Now we need to check if $f_n(x)$ is subharmonic on the boundaries.

For $\mathbf{x} \in \partial_{\mathbf{1}}$, let $x(1) = 0, x(2), x(3) > 0$.

- ${r_{1,2}}^{n-[x(1)+x(2)]}$

$$\begin{aligned}
\mathbb{E}_x[h(X)] &= \lambda_1 {r_{1,2}}^{n-[1+x(2)]} + \mu_1 {r_{1,2}}^{n-x(2)} + \lambda_2 {r_{1,2}}^{n-[x(2)+1]} + \mu_2 {r_{1,2}}^{n-[x(2)-1]} \\
&+ \lambda_3 {r_{1,2}}^{n-x(2)} + \mu_3 {r_{1,2}}^{n-x(2)} \\
&= {r_{1,2}}^{n-x(2)} [\lambda_1 {r_{1,2}}^{-1} + \mu_1 + \lambda_2 {r_{1,2}}^{-1} + \mu_2 r_{1,2} + \lambda_3 + \mu_3] \\
&= {r_{1,2}}^{n-x(2)} [\lambda_1 {r_{1,2}}^{-1} + \mu_1 + \lambda_2 {r_{1,2}}^{-1} + \mu_2 r_{1,2} + \lambda_3 + \mu_3 \\
&+ \mu_{\mathbf{1}} \mathbf{r_{1,2}} - \mu_{\mathbf{1}} \mathbf{r_{1,2}}] \\
&= {r_{1,2}}^{n-x(2)} [1 + \mu_1 (1 - r_{1,2})] \\
&> {r_{1,2}}^{n-x(2)}
\end{aligned}$$

So, ${r_{1,2}}^{n-[x(1)+x(2)]}$ is subharmonic on ∂_1.

- $\rho_1^{n-x(1)}$

$$\begin{aligned}
\mathbb{E}_x[h(X)] &= \lambda_1 \rho_1^{n-1} + \mu_1 \rho_1^n + \lambda_2 \rho_1^n + \mu_2 \rho_1^n + \lambda_3 \rho_1^n + \mu_3 \rho_1^n \\
&= \rho_1^n [\lambda_1 \rho_1^{-1} + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3] \\
&= \rho_1^n [\mu_1 + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3 + \lambda_{\mathbf{1}} - \lambda_{\mathbf{1}}] \\
&= \rho_1^n [1 + (\mu_1 - \lambda_1)] \\
&> \rho_1^{n-x(1)}
\end{aligned}$$

where we have used the stability assumption. So, $\rho_1^{n-x(1)}$ is subharmonic on ∂_1.

- $r^{n-[x(1)+x(2)+x(3)]}$

$$\begin{aligned}
\mathbb{E}_x[h(X)] &= \lambda_1 r^{n-[1+x(2)+x(3)]} + \mu_1 r^{n-[x(2)+x(3)]} + \lambda_2 r^{n-[x(2)+1+x(3)]} + \\
&\mu_2 r^{n-[x(2)-1+x(3)]} + \lambda_3 r^{n-[x(2)+x(3)+1]} + \mu_3 r^{n-[x(2)+x(3)-1]} \\
&= r^{n-[x(2)+x(3)]} [\lambda_1 r^{-1} + \mu_1 + \lambda_2 r^{-1} + \mu_2 r + \lambda_3 r^{-1} \\
&+ \mu_3 r + \mu_{\mathbf{1}} \mathbf{r} - \mu_{\mathbf{1}} \mathbf{r}] \\
&= r^{n-[x(2)+x(3)]} [\mu_1 + \mu_2 + \mu_3 + r(\mu_1 + \mu_2 + \mu_3) + \mu_1 - \mu_1 r] \\
&= r^{n-[x(2)+x(3)]} [1 + \mu_1 (1 - r)] \\
&> r^{n-[x(1)+x(2)+x(3)]}
\end{aligned}$$

So, $r^{n-[x(1)+x(2)+x(3)]}$ is subharmonic on ∂_1. Hence their maximum $f_n(x)$ is subharmonic on ∂_1. Similar arguments prove that $f_n(x)$ is subharmonic on ∂_2 and ∂_3.
For $\mathbf{x} \in \partial_{\mathbf{1,2}}$ let $x(1) = 0, x(2) = 0, x(3) > 0$.

- $r_{1,2}^{n-[x(1)+x(2)]}$

$$\begin{aligned}
\mathbb{E}_x[h(X)] &= \lambda_1 r_{1,2}^{n-1} + \mu_1 r_{1,2}^{n} + \lambda_2 r_{1,2}^{n-1} + \mu_2 r_{1,2}^{n} + \lambda_3 r_{1,2}^{n} + \mu_3 r_{1,2}^{n} \\
&= r_{1,2}^{n}[\lambda_1 r_{1,2}^{-1} + \mu_1 + \lambda_2 r_{1,2}^{-1} + \mu_2 + \lambda_3 + \mu_3 + \boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} - \boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}] \\
&= r_{1,2}^{n}[1 + (\mu_1 - \lambda_1) + (\mu_2 - \lambda_2)] \\
&> r_{1,2}^{n-[x(1)+x(2)]}
\end{aligned}$$

where we have used the stability assumption. So, $r_{1,2}^{n-[x(1)+x(2)]}$ is subharmonic on $\partial_{1,2}$.

- $\rho_1^{n-x(1)}$

$$\begin{aligned}
\mathbb{E}_x[h(X)] &= \lambda_1 \rho_1^{n-1} + \mu_1 \rho_1^{n} + \lambda_2 \rho_1^{n} + \mu_2 \rho_1^{n} + \lambda_3 \rho_1^{n} + \mu_3 \rho_1^{n} \\
&= \rho_1^{n}[\lambda_1 \rho_1^{-1} + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3] \\
&= \rho_1^{n}[\mu_1 + \mu_1 + \lambda_2 + \mu_2 + \lambda_3 + \mu_3 + \boldsymbol{\lambda_1} - \boldsymbol{\lambda_1}] \\
&= \rho_1^{n}[1 + (\mu_1 - \lambda_1)] \\
&> \rho_1^{n-x(1)}
\end{aligned}$$

where we have used $\lambda_i + \mu_i = 1$ for $i = 1, 2, 3$ and the stability assumption. So, $\rho_1^{n-x(1)}$ is subharmonic on $\partial_{1,2}$.

- $r^{n-[x(1)+x(2)+x(3)]}$

$$\begin{aligned}
\mathbb{E}_x[h(X)] &= \lambda_1 r^{n-[1+x(3)]} + \mu_1 r^{n-x(3)} + \lambda_2 r^{n-[1+x(3)]} + \mu_2 r^{n-x(3)} + \lambda_3 r^{n-[x(3)+1]} \\
&+ \mu_3 r^{n-[x(3)-1]} \\
&= r^{n-x(3)}[\lambda_1 r^{-1} + \mu_1 + \lambda_2 r^{-1} + \mu_2 + \lambda_3 r^{-1} + \mu_3 r \\
&+ \boldsymbol{\mu_1 r} + \boldsymbol{\mu_2 r} - \boldsymbol{\mu_1 r} - \boldsymbol{\mu_2 r}] \\
&= r^{n-x(3)}[1 + \mu_1(1-r) + \mu_2(1-r)] \\
&> r^{n-[x(1)+x(2)+x(3)]}
\end{aligned}$$

So, $r^{n-[x(1)+x(2)+x(3)]}$ is subharmonic on $\partial_{1,2}$. Hence their maximum $f_n(x)$ is subharmonic on $\partial_{1,2}$. Similar arguments prove that $f_n(x)$ is subharmonic on $\partial_{1,3}$ and $\partial_{2,3}$. Therefore $f_n(x)$ is a subharmonic function of X on $A_n - \partial A_n$. □

Proposition 3.9.

$$f_n(x) - \rho_1^n \leq P_x(\tau_n < \tau_0)$$

Proof. By previous calculations we know that $f_n(x)$ is a subharmonic function of X. Therefore $f_n(X_k)$ is submartingale. By optional sampling theorem applied to $f_n(X_k)$ at the bounded stopping time $\tau_n \wedge \tau_0$ we have:

$$\begin{aligned} f_n(x) &\leq \mathbb{E}_x[f_n(X_k)_{(\tau_n \wedge \tau_0)}] \\ &= \mathbb{E}_x[f_n(x_{\tau_n})\mathbb{1}_{\{\tau_n < \tau_0\}}] + \mathbb{E}_x[f_n(0)\mathbb{1}_{\{\tau_n > \tau_0\}}] \end{aligned}$$

Since $f_n(x) \leq 1$ on ∂A_n and $f_n(0) = \rho_1^n$:

$$\begin{aligned} f_n(x) &\leq P_x(\tau_n < \tau_0) + \rho_1^n \\ f_n(x) - \rho_1^n &\leq P_x(\tau_n < \tau_0) \end{aligned}$$

□

3.4 Completion of error analysis

We can now give a proof of our main Theorem 3.1:

Proof of Theorem 3.1. The processes X and $\bar{X}$ have the same dynamics up to time σ_1. This implies:

$$|P_{x_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)| \leq P_{x_n}(\sigma_1 < \tau_n < \tau_0) + P_{T_n(x_n)}(\sigma_1 < \tau < \infty).$$

This and Propositions 3.1 and 3.7 imply that there exists $N_0 > 0$ such that

$$|P_{x_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)| \leq 2\boldsymbol{h}_0(x_n) + 2c_{11}n\rho_1^n$$

for $n > N_0$. This and the lower bound on $P_x(\tau_n < \tau_0)$ given in Proposition 3.9 imply

$$\frac{|P_{x_n}(\tau_n < \tau_0) - P_{T_n(x_n)}(\tau < \infty)|}{P_{x_n}(\tau_n < \tau_0)} \leq \frac{2\boldsymbol{h}_0(x_n) + 2c_{11}n\rho_1^n}{f_n(x_n) - \rho_1^n}.$$

Comparing the components of $\boldsymbol{h}_0$ (see (3.37) and (3.12)) and f_n (see 3.82 we note that both $\boldsymbol{h}_0$ and ρ_1^n decay exponentially faster than $f_n(x_n)$ if $x(1) > 0$. This implies (3.1). $\square$

CHAPTER 4

A NUMERICAL EXAMPLE AND AN APPLICATION TO FINANCE

4.1 A numerical example

In this chapter, we present an example that demonstrates the numerical performance of the approximation algorithm.

For paramater values we take:

$$\lambda_1 = 0.13, \ \ \lambda_2 = 0.1, \ \ \lambda_3 = 0.12 \qquad (4.1)$$
$$\mu_1 = 0.21, \ \ \mu_2 = 0.19, \ \ \mu_3 = 0.25.$$

For these parameter values we have:

$$\rho_1 = 0.6190 > \rho_2 = 0.5263 > \rho_3 = 0.48$$

and

$$r^2/\rho_2 = 0.5509, \ \ r^2/\rho_3 = 0.6040,$$
$$r_{1,2}^2/\rho_2 = 0.6282, \ \ r_{1,3}^2/\rho_3 = 0.6154.$$

Therefore, these parameter values satisfy all of the assumptions made in Chapter 1.

Recall our probability of interest $\mathbb{P}(\tau_n < \tau_0)$. The approximation formula that we developed for this probability is given in Proposition 2.5 where the coefficients c_1, c_2 and c_3 need to be chosen for the parameter values above. We would like to choose these coefficients so that $\boldsymbol{h}_r$ is as close to 1 as possible on ∂B. This can be done by a calculation similar to the one given in the proof of Proposition 2.5, which gives:

$$c_1 = 4.5367, \quad c_2 = 20.7079, \quad c_3 = 16.3512.$$

The graph of $\boldsymbol{h}_r$ on $\partial_B = \{y(1) = y(2) + y(3), y(2), y(3) \geq 0\}$ for these choices of c and the parameter values listed in (4.1) is shown in Figure 4.1. Different colors corresponds to different values of the function.

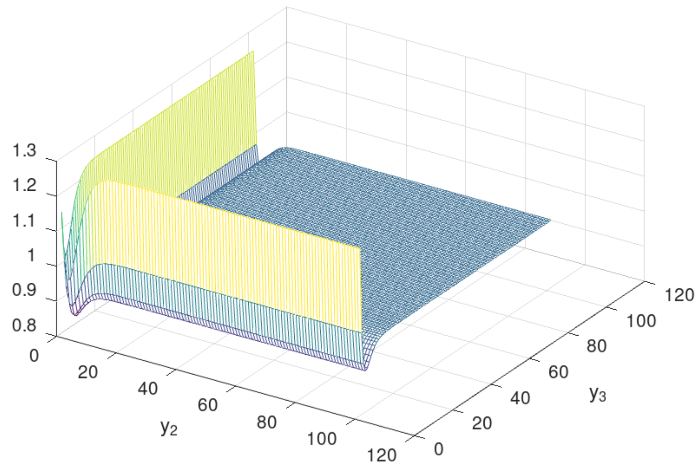


Figure 4.1: The graph of h_r on ∂B

We see from this figure that $\boldsymbol{h}_r$ varies between approximately 1.3 and 0.8 along the y_2 and y_3 axes and quickly converges to 1 for $y_2, y_3 > 0$. More precisely we have:

$$\max_{y \in \partial B} h_r(y) = 1.2857, \min_{y \in \partial B} h_r(y) = 0.8535.$$

The functions $y \mapsto \mathbb{P}_y(\tau < \infty)$ and h_r are both ∂B determined Y-harmonic functions. This and the above display imply

$$\frac{1}{1.2857}\boldsymbol{h}_r(y) \leq \mathbb{P}_y(\tau < \infty) \leq \frac{1}{0.8535}\boldsymbol{h}_r(y) \tag{4.2}$$

for $y \in B$. Note that we cannot directly compute the probability $\mathbb{P}_y(\tau < \infty)$. We use the function $\boldsymbol{h}_r$ to approximate this probability. In particular, $\boldsymbol{h}_r$ approximates $\mathbb{P}_y(\tau < \infty)$ with relative error bounded by 0.2857.

By Theorem 3.1 we know that for $x(1) > 0$, $\mathbb{P}_{T_n(x)}(\tau < \infty)$ approximates $\mathbb{P}_x(\tau_n < \tau_0)$ with vanishing relative error. Then by (4.2) $h_r(T_n(x))$ approximates $\mathbb{P}_x(\tau_n < \tau_0)$ with relative error bounded by 0.2857 for n large.

Define

$$p_n^{(0)}(x) = \begin{cases} 1, & x \in \partial A_n \\ 0, & x \in A_n - \partial A_n. \end{cases} \tag{4.3}$$

then we recursively define $p_n^{(k)}$ as

$$p_n^{(k+1)}(x) = \mathbb{E}_x[p_n^{(k)}(X_1)], x \in A_n. \tag{4.4}$$

The Markov property of X implies that:

$$\mathbb{P}_x(\tau_n < \tau_0 \leq k) = p_n^{(k)}(x).$$

Letting $k \to \infty$ in the above display gives:

$$\mathbb{P}_x(\tau_n < \tau_0) = \lim_{k\to\infty} p_n^{(k)}(x).$$

Note that $p_n^{(k)}$ can be computed recursively starting from (4.3) and iterating (4.4) k times. This iteration can be done computationally for small values of n. If we choose k large enough, this gives a very precise computation of $\mathbb{P}_x(\tau_n < \tau_0)$. In the computations below we will use the result of this computation as the exact value of $\mathbb{P}_x(\tau_n < \tau_0)$.

Since $p_n(x)$ decays exponentially in n, to get more easily interpretable graphs we will plot p_n and its approximation by $\boldsymbol{h}_r$ in $\log$ scale. For this purpose define

$$V_n(x) = -\log(\mathbb{P}_x(\tau_n < \tau_0))/n,$$
$$W_n(x) = -\log(\boldsymbol{h}_r(T_n(x)))/n.$$

Figure 4.2 shows the graphs of V_n (computed using the iteration procedure outline above) and W_n for $x(1) = 3$, $n = 60$ and the parameter values listed in (4.1). We note that both functions qualitatively look similar.

Figure 4.3 shows the relative error $(W_n(x) - V_n(x))/V_n(x)$ for the same parameter values.

We see from this figure that W_n provides an excellent approximation of V_n; around the x_2 and x_3 axes the relative error remains between -0.2 and 0.3 and away from the axes it is mostly near 0.

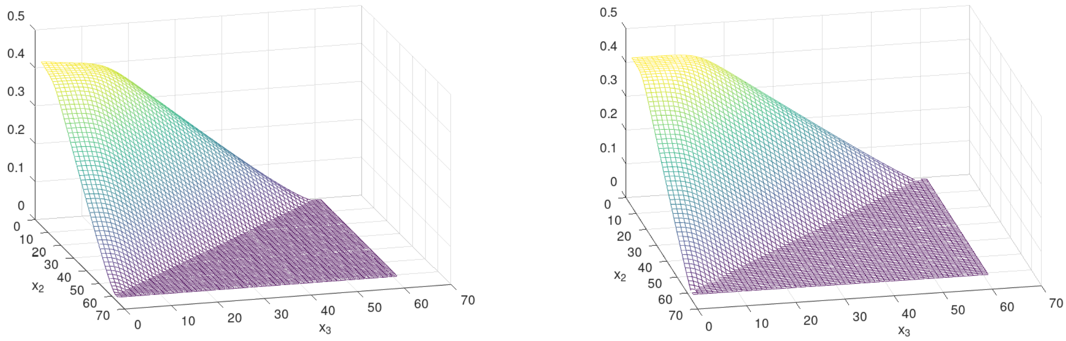


Figure 4.2: Graphs of $-\log(\mathbb{P}_x(\tau_n < \tau_0))/n$ and $-\log(\boldsymbol{h}_r(T_n(x)))/n$ for $x(1) = 3$, $n = 60$ and the parameter values listed in (4.1)

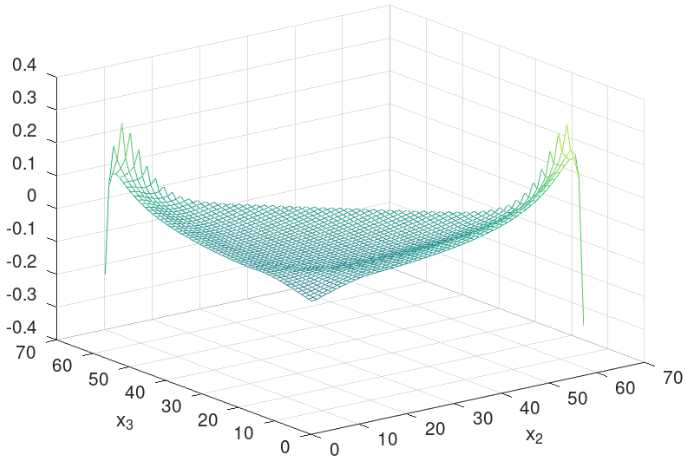


Figure 4.3: The relative error $(V_n - W_n)/V_n$ for $n = 60$ and $x(1) = 3$

Let us also give a numerical comparison of the actual probability values for a particular value of x: for $x = (3, 1, 1)$, $p_n(x)$ (as computed with the iterative procedure (4.4)) turns out to be 1.2766×10^{-11}. The approximation of the same probability by $\boldsymbol{h}_r$ is: 1.3380×10^{-11} which corresponds to a relative error of 0.04. If we take the computation of the expectation in (4.4) for a single x value as one computational step, the total iteration of (4.4) to compute p_n takes approximately around 3.8×10^7 steps. While the computation of $\boldsymbol{h}_r$ takes approximately only 2 computational steps. For moderately large values of n, the iteration (4.4) becomes no longer feasible. For example, for $n = 10^4$, it is no longer possible to compute p_n via iterating (4.4). On the other hand, the evaluation of $\boldsymbol{h}_r$ is essentially independent of n and happens practically instantly. A numerical example: for $n = 10^4$, and $x = (9900, 10, 10)$, $\boldsymbol{h}_r$ yields the approximate value 2.5648×10^{-20}.

4.2 Possible application to finance

As the global financial system becomes more complex, multidimensional models have started to attract more interest. A growing financial system requires modeling of systems of companies or financial networks; see, for example, [13, 30, 1, 4]. In modeling of systems of companies, constraint conditions can be added to the model such as "no short-selling" or "dividend payments" [36, 43, 24]. A possible application area of constrained processes and escape probabilities of the type studied in this thesis is the modeling of non-performing loans (NPL) portfolio of several financial intermediaries, such as commercial banks. In the following paragraphs we elaborate on this possible application.

Non-performing loans are defined as the loans in which the borrower has failed to make scheduled payments of principal or interest for a specified period of time; although it depends on the country, this time in general corresponds to 90 days or more. The total amount of NPL is crucial for a bank's healthy operation. As in the 2008 financial crisis, which in part was a result of defaults in mortgage loans, high amounts of non-performing loans could lead to systemic risks which are damaging to a country's economy [31, 25]. One of the ways banks use in handling NPL is selling a certain amount of them to asset management companies [7]. Let us consider a banking system consisting of, for example, three different banks (e.g. three big state-owned banks in Turkey). We can model the total amount of non-performing loans portfolio of these banks as a process in $\mathbb{R}^3_+$. In this case, our model has the following characteristics:

1. Each jump of the process correspond to a single period (e.g., a period of a month), leading to a discrete-time process.

2. Each coordinate axis corresponds to the non-performing loans portfolio of each bank.

3. Stability. Here,stability means that the banks make an effort (by selling from their NPL portfolios regularly) to keep their NPL porfolios small over time, on average.

Within this framework, a natural question is: what is the probability that the total NPL portfolio in the system reaches a very high level in a given time horizon. Based on this probability, policy makers can consider when and how much capital injection should be made to these banks.

Evidently, the dynamics of the above model is different from the dynamics studied in the present thesis. Nonetheless, the underlying process is constrained and the probability of interest is of the type p_n studied in the present work. A natural direction for future research would be a precise formulation of the above model and attempting to extend the results of the present thesis to the formulated model.

CHAPTER 5

REVIEW OF LITERATURE

There is a wide literature on the analysis of p_n or similar quantities, for a comprehensive analysis of literature; see, for example [37, 38]. Lots of research [41], [44], [29], [21], [12], [14], [40], [22], [20], [3], [16], [9], [8], [26], [11], [32], [34], [18], [35], [5], [6], [37], [39], [42], [43] can be found in the literature related to the calculation of this probability. The most common approaches to the problem have been using simulation and large deviations analysis (LDA) [37]. Stability condition on the constrained random walk X suggests the event $\{\tau_n < \tau_0\}$ rarely happens and its probability decreases exponentially as the n increases. Large deviations analysis is used in the study of rare events and their probabilities with exponentially decay rates. Therefore this makes it a natural tool in studying of such probabilities. However, for moderate sample sizes or finite systems, large deviations analysis has potential to provide imperfect approximations, resulting in inaccurate predictions of rare events probabilities. In order to obtain more accurate estimates than the ones provided with large deviations analysis, simulation techniques with variance reduction, like importance sampling (IS), are generally implemented.

[22] introduces an alternative approach for the large deviation analysis of Jackson networks. The technique used in the paper relies on altering the probability measure using an exponential martingale and examining the associated fluid limits. This paper also calculates an explicit formula for the large deviation rate function $L(x, v)$. As an exit time problem application, [22] calculates the following limit:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[\tau_n] = -\lim_{n\to\infty} \frac{1}{n} \log P(\tau_n < \tau_0) = \min_{1\le i\le N} \log \frac{\mu_i}{\nu_i}$$

where μ_i's are the service rates and ν_i's are the overall arrival rates.

Another approach to the approximation of p_n is using simulation in order to obtain more accurate estimates. However, simulation techniques such as Basic Monte Carlo Sampling necessitates larger sample paths. As p_n is a rare event probability, this makes Basic Monte Carlo Sampling inefficient for this type of approximations. Therefore, one can use importance sampling technique in which the demanded samples are reweighed so that it would be easier to sample from the emphasized sets. Large deviation analysis complemented with IS approach proposes optimal approximations in dealing with rare event probabilities. In importance sampling it is of crucial importance to choose a good proposal distribution used in reweighing of rare events (for the construction of new simulation measure). LDA provides an insight to select an efficient proposal distribution which would minimize the variance of the IS estimator. In his work [41] defines an asymptotically optimal change of measure of IS estimator for the problem of approximating the error probabilities of the sequential probability ratio test. [2] analyses that asymptotically efficient optimal change of measure for the simulation of single queue with arbitrary arrival and departure rates comes from the exponential distortion of aforementioned rates by a parameter θ_0. For a single queue with arrival rate λ and departure rate μ, this parameter has the following condition:

$$\left(\frac{\lambda}{\lambda+\theta_0}\right)\left(\frac{\mu}{\mu-\theta_0}\right)=1$$

Using the above defined parameter to twist the arrival and departure rates works as switching λ and μ.

[29] uses large deviations analysis to approximate the rare events' probability in an open Jackson network. For a two dimensional constrained random walk with two tandem queues, [29] evaluates the efficiency of IS estimator based on LDA. For importance sampling, this work uses a static change of measure (basically switching the arrival rate (λ) with the smallest departure rate ($\mu_i' s$)) suggested by large deviations. It has been shown that it may not be the optimal IS change of measure for constrained multidimensional random walks. The defined change of measure is adapted according to the boundaries; allowing it to be state dependant.

[20] further studied the estimator proposed by [29]. Asymptotic performance of this estimator is analyzed in [20] and it was shown that static change of measure performs poorly across the exit boundary and hence asymptotically inefficient especially for

when the the arrival rate is small and service rates are closely equal. Necessary and sufficient conditions for asymptotic efficiency are further explained.

As can be understood from the papers [20], [29], for two or more dimensional constrained random walks (e.g. two dimensional tandem Jackson network) the importance sampling change of measure proposed by the large deviations analysis fails to be asymptotically optimal especially for the boundaries. The papers [32], [33], [16], proposes a theoretical framework in order to build asymptotically efficient optimal IS algorithms for the simulations of rare events probabilities with starting point $x = 0$ in two tandem queuing networks. The idea here is to use sub-solutions to the Isaacs equation (specifically Hamilton Jacobi Bellmann equaiton) stemming from the limit analysis of the IS estimator and obtain the boundary conditions that will eventually lead to optimal IS schemes. The IS estimator defined in [16] is the following:

$$\hat{p_n} = 1_{A_n} \prod_{k=0}^{T_n - 1} \frac{\Theta(Y(k+1))}{\bar{\Theta}^n(Y(k+1)) \mid X^n k)}$$

where Y is a iid random variable with distribution Θ and $\bar{\Theta}^n(. \mid .)$ is a characterization of state-dependent change of measure. The sub-solutions to the following Isaacs equation are used to build up asymptotically optimal importance sampling estimators.

$$0 = \sup_{\bar{\Theta} \in \mathbb{P}^+(\mathbb{V})} \inf_{\theta \in \mathbb{P}^+(\mathbb{V})} \left[\langle DW(x) \ , \ \mathbb{F}(\theta) \rangle + \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}(v_i)}{\Theta(v_i)} + R(\theta || \Theta) \right]$$

By implementing the subsolutions to the Isaacs equation approach, the papers [15], [18] and [35] further studies the problem for more dimensional cases, different hitting boundaries and broader dynamics. Comments on further studies [17], [10], [19], [27], [28], [23] can be reached from [38].

[37], [38] introduces an affine transformation approach to calculate the probability p_n for two or more tandem queues. Affine transformation approach introduced in these papers is also implemented in the current thesis. Approximation formulas for p_n are constructed and it has been shown that this approximation has an exponentially decaying relative error for two tandem queues. The idea in affine transformation approach, as explained for our case three dimensional parallel walk, is to observe X as Y where these two random walks are the same except for the first coordinates

are reversed. The approach is to move the origin of the coordinate system in X to the exit boundary and removing some of the boundaries results in the computation of $P(y < \infty)$. This probability can later than be computed with the construction of harmonic functions of Y. An explicit formula is provided for a two tandem case as follows:

$$\rho_2^{y(1)-y(2)} + \rho_1^{y(1)-y(2)} \rho_1^{y(2)} \frac{\mu_2 - \lambda}{\mu_2 - \mu_1} + \rho_2^{y(1)-y(2)} \rho_1^{y(2)} \frac{\mu_2 - \lambda}{\mu_1 - \mu_2} \tag{5.1}$$

where λ, μ_i are arrival and departure rates respectively and $\rho_i = \lambda/\mu_i$.

The studies [5], [6], [42], [43] dealing with the problem of calculating p_n for different dynamics on X also implements the affine transformation approach defined in [37], [38]. [5], [6] treats the problem for two queues working in tandem and the defined random walk X has a Markov modulated scheme. By construction of the harmonic and superharmonic functions of the process Y, derived random walk coming from the transformation, approximation formulas for the desired probability are obtained and it has been shown that this approximation has a bounded relative error.

[42], [43] implements the similar ideas provided by [37], [38] for a two dimensional constrained simple random walk. In the current thesis, we try to extend the results given in [42], [43] to the three dimensional case. The dynamics of affine transformation given in these studies are shown in Figure 5.1. Similarly, harmonic functions of Y are built and their specific linear combination is used to approximate the desired probability. An exact formula is also constructed here under specific conditions ($r^2 = \rho_1\rho_2$). The error analysis provided in these works contains the implementation of explicit subharmonic functions of X and subsolutions to a limit Hamilton Jacobi Bellmann equation.

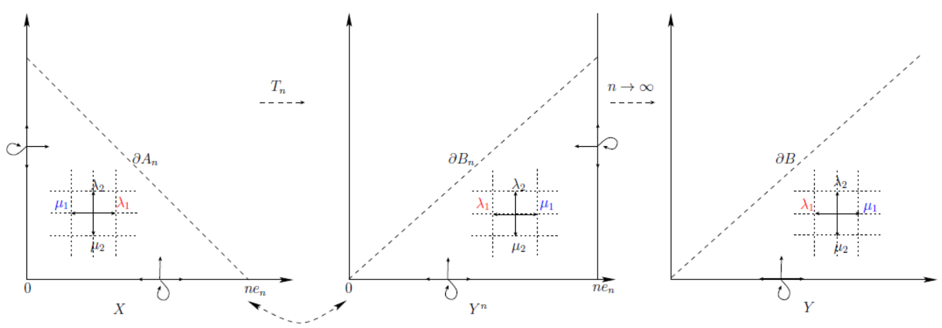


Figure 5.1: Affine transformation for two dimensional parallel walk

[39] is a recent paper on the subject with which this thesis has a great connection. In this paper, formulas approximating the probability $P_x(\tau_n < \tau_0)$ for a d dimensional constrained random walk X, with d tandem queues (see Figure 5.2), are developed. The affine transformation of X reduces the problem to the calculation of the probability $P_y(\tau < \infty)$. This probability is shown to be explicitly written in terms of λ and μ_i's and can be derived with the solutions to the harmonic systems associated with the constructed Y harmonic functions. This novel explicit formula is given in Equation 5.2. A harmonic system is defined to be a $\{2, 3, 4, ..., d\}$ regular graph with a set of equations/restraints. Points on a characteristic surface of Y are represented as nodes on the graph and edges between the points denotes the conjugacy relation. This approach also used in this thesis. 2.1 in Chapter 2 is the graph of a harmonic system for the three dimensional constrained simple random walk in our case.

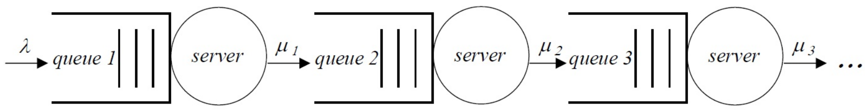


Figure 5.2: d tandem queues

$$P_y(\tau < \infty) = \sum_{d=1}^{\mathbf{d}} \Big[\prod_{l=d+1}^{\mathbf{d}} \frac{\mu_l - \lambda}{\mu_l - \mu_d} \Big] h_d^*(y) \tag{5.2}$$

where h_d^*'s are ∂B-determined Y-harmonic functions. This equation is in fact a generalization of Equation 5.1.

For the approximation analysis in [39], supermartingales are constructed and used to bound the probabilities given in relative error. The rationale in the error analysis is in brief as follows: it is shown that the incident $\{\tau_n < \tau_0\}$ mostly coincide with $\{\tau < \infty\}$. For the convergence analysis, the upper bounds on the probabilities of events are computed through the construction of Y-superharmonic functions and the related supermartingales. The paper uses superharmonic functions for the convergence analysis as they have a much simpler forms compared to the Y-harmonic functions employed in $P_y(\tau < \infty)$. The ideas provided in this paper are utilized in the current thesis as mentioned in Chapter 2 and Chapter 3.

CHAPTER 6

CONCLUSION

Constrained random walks arise in queuing theory and other fields as models of systems of objects, information or customers. Systems comprised of multiple components lead to multidimensional constrained random walks, e.g., the banking system considered in Chapter 4. In this thesis, we model three queues working in parallel. We have a three dimensional simple random walk X constrained to remain on the positive orthant. Setting the hitting time τ_n as the first time when the sum of the components of X equals n, we approximate the probability $P_x(\tau_n < \tau_0)$.
For the approximation of $P_x(\tau_n < \tau_0)$, we use an affine transformation of X and observe the system from the exit point $(n, 0, 0)$. The resulting process Y is indeed the same process as X except for the first coordinates. With the transformation, the problem turns out to be the approximation of the probability $P_y(\tau < \infty)$. We provide a convergence analysis showing that $P_y(\tau < \infty)$ approximates our desired probability $P_x(\tau_n < \tau_0)$ with an exponentially diminishing relative error for X is away from the constraining boundary $x(1) = 0$. The convergence analysis is based on the construction of superharmonic and subharmonic functions of X. In order to approximate the probability $P_y(\tau < \infty)$, we construct Y harmonic functions by using a four node harmonic system, points on which correspond to the roots on the characteristic surface associated with Y. These roots are used in the construction of Y harmonic functions using all four nodes on the graph of the harmonic system. As in [38, 43, 6], we also use harmonic systems with pair of nodes to construct additional Y-harmonic functions. Furthermore, as in these previous works, another harmonic function comes from the intersection of characteristic surfaces. Finally we provide a numerical example in order to show the that the function constructed for $P_y(\tau < \infty)$ approximates

the probability $P_x(\tau_n < \tau_0)$ quite well. An explicit formula for the two dimensional parallel walk case exists with an additional condition of $r^2 = \rho_1\rho_2$ [43]. We have not derived a similar formula in the case of three dimensions, such a derivation can be considered in future research. Instead of an exact computation, we approximate this probability by taking an appropriate linear combination of the Y harmonic functions constructed from solutions of harmonic systems mentioned above. Further construction of Y harmonic functions could help in order to obtain better approximations of the probability. Moreover, for a suitable combination of such functions one could probably diminish the error for the approximation of this probability. How many functions do we need or how can we optimally combine them remains as questions for future work.

The work [39] provides an exact formulation in terms of the ratios of arrival rate and departure rates for the probability $P_y(\tau < \infty)$ for a d dimensional tandem walk. A similar generalization to d dimensions of the results in this thesis also remain for future work.

6.1 Comparison with the previous studies

This thesis can be regarded as an extension of [42, 43]. [42, 43] treat the two dimensional simple walk case for the same problem of computing the probability $p_n = P(\tau_n < \tau_0)$. Both works and [5, 39] use the same approach in the calculation of p_n; using an affine transformation of the constrained random walk X and obtaining a similar process Y, and approximating p_n with the probability $P_y(\tau < \infty)$. Furthermore, $P_y(\tau < \infty)$ is computed by constructing harmonic functions / harmonic systems and their solutions as well as conjugate points of the characteristic surfaces. Adding an extra dimension to the problem makes the calculations further complicated and requires further adaptations. In this part, we compare our work with the two dimensional case in detail. For the comparison of two dimensional cases for tandem walk, parallel walk and Markov modulated dynamics we can refer the reader to [42, 5].
The number of constraining boundaries increases as we add one more dimension to the problem. This leads to extra characteristic surfaces used for the construction of harmonic functions. In the two dimensional case, harmonic functions are of type

$[(\beta, \alpha), y] = \beta^{y(1)-y(2)}\alpha^{y(2)}$ whereas in our case they are of the form $[(\beta, \alpha_2, \alpha_3), y] = \beta^{y(1)-(y(2)+y(3))}\alpha_2^{y(2)}\alpha_3^{y(3)}$. From the solutions of characteristic polynomials, we obtain our four node harmonic system, the idea of which is based on [39] and new compared to [42]. The $\boldsymbol{h}_r$ function used for the approximation of $P_y(\tau < \infty)$ is constructed from the points coming from the solutions of characteristic polynomials and the conjugate points on the characteristic surfaces. For the construction of $\boldsymbol{h}_r$, we implement the arguments provided in [37, 39] which treats the 2 or more dimensional tandem walks. By using harmonic system, we obtain harmonic functions from all of four nodes and pair of nodes.

It is possible to write an explicit formula for $P_y(\tau < \infty)$ with an additional condition for the two dimensional case. Obviously, no explicit formula exists for the three dimension. $P_y(\tau < \infty)$ can only be approximated with $\boldsymbol{h}_r$ with a bounded relative error.

The most crucial differences from the two dimensional case stems from the error analysis and the upper bounds on the probabilities given in Chapter 3. Upper bound on these probabilities are obtained through the large deviations analysis and subsolutions of the limit HJB equation. In our case, upper bounds are obtained based on the construction of superharmonic functions and their corresponding supermartingales. One critical difference here is that the superharmonic harmonic functions constructed for $P_x(\sigma_1 < \tau_n < \tau_0)$ depend on the order of $r_{1,2,3}$ and $r_{1,3}$ since 3.6 doesn't resolve how $r_{1,2,3}$ and $r_{1,3}$ compare with each other. Different functions for each stage are constructed in Subsections 3.1.2, 3.1.1. Such a distinction is not necessary for two dimensions since there is a single assumption of $\rho_2 \leq r \leq \rho_1$.

In the construction of superharmonic functions, the use of time truncation allows us to obtain meaningful bounds. To truncate time for the X process, we use a general result from [42, 43]. For the truncation of time for the Y process we generalize the argument given in [42, 43] to three dimensions: this consists of finding an upper bound on the moment generating function.

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