CONSTRUCTION AND OPTIMALITY OF EXTENDED MOST BALANCED GROUP DIVISIBLE TREATMENT INCOMPLETE BLOCK DESIGNS

A PH.D. THESIS
IN
STATISTICS

MIDDLE EAST TECHNICAL UNIVERSITY

BY

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SEPTEMBER, 1989

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DEDICATED TO:

MY MOTHER

AND

THE MEMORY OF MY FATHER

ABSTRACT

CONSTRUCTION AND OPTIMALITY OF EXTENDED MOST BALANCED GROUP DIVISIBLE TREATMENT INCOMPLETE BLOCK DESIGNS

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162 PAGES, SEPTEMBER 1989

The Balanced Treatment Incomplete Block (BTIB) Designs were proposed by Bechhofer and Tamhane (1981) for the problem of comparing $v \geq 2$ test-treatments with a control in "b" blocks of size "k" each. The requirement of complete symmetry of information matrix (called **M** matrix), however, imposes severe restriction on the class of such designs and, as a consequence, there are a number of design parameters for which BTIB designs do not exist. A new class of designs, henceforth called **Extended Most Balanced Group Divisible Treatment Incomplete Block (E MB GD TIB) Designs**, is therefore introduced here which, upto some extent, relaxes the condition of complete symmetry of the information matrix and hence, leads to the study of a larger class, which includes BTIB designs as a sub-class. The symmetry of the information matrix is, however, not affected much and still the elements of the leading diagonal remain equal. Thus conventional idea of A-optimality (minimizing the trace of M^{-1} matrix) remain intact and A-optimal designs can, therefore, be obtained easily.

The minimal complete class of generator designs (MCCGD), (a finite set of elementary designs which are used in the construction of all other admissible and non-equivalent designs in the class) are constructed for $k = 2,3,4$ and $v = 4,6,8,10$. The A-optimal designs for all these k and v values are also obtained.

In addition, very powerful and practical computer algorithms, programs/subroutines are designed, which can run very efficiently to provide all the designs as well as the A-optimal designs for any given (k,v). Also, programs are developed which help much in finding minimal complete class of generator designs and can provide a complete structure for any design in the class.

KEY WORDS: BTIB Designs, Treatment-control Comparison, Reinforced Designs, Group-divisible Designs, A-optimal Designs, Optimal Designs.

ÖZET

GELİŞTİRİLMİŞ EN DENGELİ GRUPLANDIRILABİLİR MUAMELELİ TAM OLMAYAN BLOK TASARIMLARININ HAZIRLANMASI VE OPTİMALLİĞİ

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162 Sayfa, Eylül 1989

Dengeli muameleli tam olmayan Blok (BTIB) tasarımları, Bechoffer ve Tamhane (1981) tarafından $v > 2$ test-muamelesinin "b" sayıdaki blokta bir kontrol ile mukayesesi problemi için önerilmişti. Bununla beraber bilgi matrisinin (M matrisi) tam simetriye sahip olması koşulu, bu sınıftaki tasarımlar üzerine katı kasıtlamalar getirmektedir ve sonuç olarak da bazı tasarım parametreleri için BTIB tasarımları bulmak olanaksızdır. Bundan böyle "Geliştirilmiş En Dengeli Gruplandırılabilir Muameleli Tam Olmayan Blok Tasarımları" (E MB GD TIB) diye adlandırılan yeni sınıf tasarımlar burada takdim olunacaktır ki, bir dereceye kadar bu tasarımlar bilgi matrisinin tam simetri koşulunu aramayacak ve dolayısiyle, BTIB tasarımlarını da bir alt sınıf olarak içeren daha geniş bir sınıftaki tasarımların incelenmesine olanak verecektir. Maamafih,

bilgi matrisinin simetrisi çok etkilenmemekte ve baştaki köşegen elemanları eşit olarak kalmaktadır. Böylece, alışılagelmiş olan A-optimal olma (M^{-1} matrisi'nin izninin minimum olması) fikri korunmuştur ve A-optimal tasarımlarını kolayca elde etmek mümkün olacaktır.

En küçük tam sınıf oluşturan jeneratör tasarımlar (MCCGD), (bunlar sonlu sayıda olup diğer bütün kabul edilebilir ve aynı sınıfta eşdeğeri bulunmayan tasarımların oluşturulmasında kullanılan elemanter tasarımlardır) k=2,3,4 ve v=4,6,8,10 için oluşturulmuştur. Bütün bu k ve v değerleri için A-optimal tasarımlar da elde edilmiştir.

Ayrıca, çok güçlü ve kullanışlı bilgisayar algoritmaları, programları/alt-programları hazırlanmıştır, ki bunlar verilen herhangi (k,v) için A-optimal tasarımların yanında diğer bütün tasarımlarında elde edilmesinde çok randımanlı olarak kullanılabilecektir. Bunlara ilaveten, en küçük tam sınıf jeneratör tasarımların bulunmasında çok faydalı olacak olan programlar da mevcuttur ve bunlar bu sınıftaki herhangi bir tasırım için bütün yapıyı verebileceklerdir.

Anahtar Kelimeler: BTIB Tasarımlar, Muamele-kontrol Mukayeseleri, Takviyeli Tasarımlar, Gruplara Ayrılabilen Tasarımlar, A-optimal Tasarımlar, Optimal Tasarımlar.

ACKNOWLEDGEMENT

I would like to express my sincere gratitude to Dr. Fuat Ozkan for his patient supervision and understanding throughout this study. I am deeply indebted to Dr. T. Erkan Ture, who encouraged me to work in this area, provided me a lot of reading material and allowed me to use some of his results.

I owe my thanks to my family particularly to my wife for her patience, help, and a long period of living alone to look after my old parents and children while I was away for my studies in Turkey.

I am also thankful to all members of the Department of Statistics, Middle East Technical University, for their cordial behavior , encouragement and help during my studies.

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LIST OF SYMBOLS

b	Number of blocks
k_j	Size of j th block
n	Total experimental size
y_{iju}	Response of u th plot in j th block treated by i th treatment
γ	Mean response
α_i	i th treatment effect
β_j	j th block effect
e_{iju}	Error term
v	Number of treatments
ϕ	Incidence matrix for treatments
ψ	Incidence matrix for blocks
y	Vector of responses
J_n	nx1 vector of unity
α	Vector of treatment effects
β	Vector of block effects
e	Vector of errors
r_i	Total number of times i th treatment is repeated
r_{ij}	Number of times i th treatment is repeated in j th block
M	Information matrix
$\hat{\alpha}$	Vector of estimates of treatment effects
Q	Vector of adjusted treatment total
m_{it}	(i,t) th element of vector M

λ_{il}	Number of times i th and l th treatment appear together in a block over the whole design
λ_0	Number of times a test-treatment appears with control in a block in a design
λ_1	Number of times a test-treatment appears with another test-treatment in a block in a design
D	Represents a design
λ_2	Number of times a test-treatment appears with another test-treatment (of second associates) in a block
m_1	Control appears "1" times in "m_1" blocks
r	Number of times each test-treatment is repeated
r_0	Number of times the control is repeated
N	Number of TT-pairs
M_j	Number of TT-pairs in j th BB combination
b_j	Number of blocks in j th BB combination
R	Incidence matrix of the design
S	Design matrix
γ	Eigen values of "S" matrix
$\underset{\sim}{m}$	Vector of allocation of the control

CHAPTER 1.

1.1 INTRODUCTION:

In many areas of Science it is desirable to compare simultaneously several test-treatments (or varieties) with a standard or existing one, called the control. Also, many practical situations require the blocking of experimental units in order to cut down on bias and improve the precision of the experiment. The most suitable designs are the Randomized Block Designs which require a block size equal to the number of treatments (including the control) involved. If the total number of treatment is large, then a large block is required to accommodate all the treatments. This create practical difficulties in handling the experiment, as the size of experiment increases considerably. Incomplete block designs are therefore suggested, in which size of the block is less than the number of treatments. Balanced Incomplete Block (BIB) Designs are well known designs of this type, but they are not appropriate for comparing test-treatments with the control, as they assign equal weight for all treatment comparisons, including the control; whereas, more efficiency is required for the treatment-control comparison. One way out from this problem is to construct BIB designs and add equal number of control to each block. Such a design is called augmented or reinforced design and is more efficient than BIB designs for comparing treatments with a control, but it is less efficient for pair wise comparisons between treatments. Such designs are studied by Das(1958), Constantine(1983) and many other authors.

A new general class of designs is proposed by Bechhofer and Tamhane (1981), that is appropriate for the above problem. They are called Balanced Treatment Incomplete Block (BTIB) Designs and are balanced with respect to treatment other than the control. They study the structure of these designs and give the method of their construction. A lot of work has been done in this area since then. Ture (1982) studied in detail the construction of these designs for a variety of parameter range and also discussed the A-optimality of such designs. Bechhofer and Tamhane (1983) focus their attention on the problem of optimal allocation of experimental units among the test-treatments and the control to minimize the total size of the experiment. Majumdar and Notz (1983) established optimal properties of some BTIB designs. Hedayat and Majumdar (1984) have an extensive study of A-optimal BTIB designs in blocks of size 2, through a combination of theoretical results and numerical investigations. Hedayat and Majumdar (1985) investigated the optimality of augmented BIB designs and were successful in solving the problem analytically. Stufken (1987) extended the results obtained by Hedayat and Majumdar (1985), and discussed the A-optimality of R- (reinforced) and S- (Step) type designs.

Majumdar (1981) shows that one needs to consider only binary designs, in which each treatment appears at most once in a block, in order to search for A-optimal designs. Since treatments are equally replicated in binary BTIB designs, it follows that in many cases an exactly A-optimal design will not exist. In such a case one might try to construct a design which is as close to a BTIB design as possible. Here by being close to a BTIB design it is meant that a design is as equally replicated as possible, and each treatment-control pair in a block should occur (over the whole design) as equally as possible and the same should be true for the repetition of all treatment-treatment pairs.

This leads to the study of Most Balanced designs, the topic of research in the following pages of this dissertation.

1.2 PRELIMINARIES:

1.2.1 GENERAL BLOCK DESIGNS:

Consider "b" blocks of more or less homogeneous experimental units or plots. Suppose j th block consists of k_j plots. The design is said to be an incomplete block design if the number of treatments "v", is greater than the number of plots in a block. There are $n = \sum_{j=1}^{b} k_j$ experimental units. The usual model is

$$y_{iju} = \gamma + \alpha_i + \beta_j + e_{iju}, \quad \begin{array}{l} i = 1,2,\ldots\ldots,v \\ j = 1,2,\ldots\ldots,b \\ u = 1,2,\ldots\ldots,k_j \end{array}$$

where i th treatament has been applied to u th plot belonging to j th block.

y_{iju} is the response from such a plot

α_i is the i th treatment effect

β_j is the j th block effect

e_{iju} is the residual of the plot

Consider the incidence matrices for treatments and blocks,

$$\phi = (\phi_1, \phi_2, \ldots\ldots\ldots, \phi_v)$$

and $\psi = (\psi_1, \psi_2, \ldots\ldots\ldots, \psi_b)$

where, $\phi_i = (\phi_{i1}, \phi_{i2}, \ldots\ldots, \phi_{in})'$

and $\psi_j = (\psi_{j1}, \psi_{j2}, \ldots\ldots, \psi_{jn})'$

where, $\phi_{iu} = \begin{cases} 1 & \text{, if i th tretament alloted to u th plot} \\ 0 & \text{, otherwise} \end{cases}$

and, $\psi_{ju} = \begin{cases} 1 & \text{, if plot u recieving treatment i is in block j} \\ 0 & \text{, otherwise} \end{cases}$

The above linear model can be written as

$$y = \gamma . J_n + \phi . \alpha + \beta . \psi + e$$

where, y is nx1 vector of observations

J_n is nx1 vector of unity

α is vx1 vector of treatment effects

β is bx1 vector of block effects

e is nx1 vector of residuals

The grand total is $G = J_n' y$

Treatment total is $T = (T_1, T_2, \ldots\ldots, T_v)' = \phi' . y$

Block totals is $B = (B_1, B_2, \ldots\ldots, B_b)' = \psi' . y$

The normal equations of the least-square estimation are

$$\begin{bmatrix} J_n' \\ \phi' \\ \psi' \end{bmatrix} . \begin{bmatrix} J_n & \phi & \psi \end{bmatrix} . \begin{bmatrix} \hat{\gamma} \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} J_n' \\ \phi' \\ \psi' \end{bmatrix} . y$$

that is $\begin{bmatrix} n & r' & k' \\ r & D_r & R \\ k & R' & D_k \end{bmatrix} . \begin{bmatrix} \hat{\gamma} \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} G \\ T \\ B \end{bmatrix}$

where, $r = (r_1, r_2, \ldots\ldots\ldots, r_v)'$

r_i is the total number of times i th treatment is repeated in the design.

$k = (k_1, k_2, \ldots\ldots\ldots k_b)'$

$D_r = \text{diag}\ (r_i)$ is a vxv diagonal matrix of r_i's and off diagonal elements zeros.

$D_k = \text{diag}\ (k_j)$ is a bxb diagonal matrix of k_j's and off diagonal elements zeros.

$R = (r_{ij})$ is a vxb matrix of elements,

r_{ij}= number of times i th treatment appears in j th block There are $v + b + 1$ linear equations as a whole, but they are linearly dependent. Indeed,

$$n \hat{\gamma} + r' \hat{\alpha} + k' \hat{\beta} = J_v' (r \hat{\gamma} + D_r \hat{\alpha} + R \hat{\beta})$$
$$= J_b' (k \hat{\gamma} + R' \hat{\alpha} + D_k \hat{\beta})$$

Thus we consider only two equations,

$$r \hat{\gamma} + D_r \hat{\alpha} + R \hat{\beta} = T \qquad 1.2.1$$

$$k \hat{\gamma} + R' \hat{\alpha} + D_k \hat{\beta} = B \qquad 1.2.2$$

Premultiply eq.1.2.2 by $R\ D_k^{-1}$ and subtract from eq.1.2.1 to obtain,

$$(D_r - R\ D_k^{-1} R') \hat{\alpha} = T - R\ D_k^{-1} B \equiv Q = (Q_i) \ \text{(say)}$$

where,

$$Q_i = T_i - \sum_{j=1}^{b} r_{ij} \cdot \frac{B_j}{k_j}$$

is called the adjusted (adjusted for blocks) treatment totals, and $\sum_{i=1}^{v} Q_i = 0$.

Let, $M = D_r - R\ D_k^{-1} R' = (m_{il})$; $i,l = 1,2,\ldots\ldots,v$.

Then,

$$M \hat{\alpha} = Q \qquad 1.2.3$$

1.2.2 INFORMATION MATRIX:

In eq.(3), above , $M = (m_{il})$ is called Information Matrix for estimating α. Here,

$$m_{ii} = r_i - \sum_{j=1}^{b} r_{ij}^2 / k_j$$

and,

$$m_{it} = -\sum_{j=1}^{b} (r_{ij} \times r_{tj}) / k_j \quad , i \neq t;\ i,t = 1,2,\ldots,v$$

M^{-1} is the variance-covariance matrix, whose diagonal elements give the variance of the estimates and off diagonal elements give the covariances.

1.2.3 PROPER AND CONNECTED DESIGNS AND ESTIMABILITY:

In general all treatment effects cannot be estimated. However, it is easy to show that if a linear function $a'\alpha$ is estimable, then this must be a treatment contrast e.g. $\alpha_i - \alpha_t$ for all i and t.

A design is called a proper design if all blocks have the same number of plots k, i.e. $k_1 = k_2 = \ldots\ldots\ldots = k_b = k$. Bechhofer and Tamhane (1981) show that for a proper design with parameters b, k and v (total v+1 treatments including the control), the information matrix for estimating all contrasts $\alpha_0 - \alpha_i$ (i= 1,2,.. .....,v) is a vxv symmetric non-negative definite matrix $M = (m_{it})$ such that,

$$m_{it} = \begin{cases} r_i - \frac{1}{k} \sum_{j=1}^{b} r_{ij}^2 & , \text{ if } i = t \\ - \frac{1}{k} \lambda_{it} & , \text{ if } i \neq t \end{cases}$$

where,

λ_{it} = number of times i th and t th treatments appears together in a block over the whole design.

In equireplicate design each treatment is repeated equal number of times over the whole design i.e. $r_1 = r_2 = ... = r_v = r$

In binary design a treatment appears at most once in a block i.e. $r_{ij} = 0$ or 1. Therefore, $r_{ij}^2 = r_{ij}$, and the information matrix $M = (m_{it})$ is given by,

$$m_{it} = \begin{cases} r_i - \frac{1}{k} \sum_{j=1}^{b} r_{ij} & , \quad i = t \\ - \frac{1}{k} \lambda_{it} & , \quad i \neq t \end{cases}$$

A treatment and a block are said to be associated if the treatment occurs in the block. A design is connected if all the treatments and block are associated with each other. For a connected design every treatment contrast is estimable and

$$\text{rank} (M) = \text{number of treatments} - 1$$

1.3 BALANCED TREATMENT INCOMPLETE BLOCK (BTIB) DESIGNS:

1.3.1 DEFINITION:

Consider an incomplete block design in "v" treatments and a control denoted by "0" in "b" blocks of size "k" ($v \geq k$) each

The design is a BTIB design if,

$$\lambda_{0i} = \lambda_0 , \quad i = 1,2,.......,v$$

and

$$\lambda_{it} = \lambda_1 , \quad i,t = 1,2,.......,v \; ; \; i \neq t$$

Hence, a BTIB design is such that each test-treatment appears with control (in a block) same number of times over the design, and any pair of test-treatments appears together (in a block) the same number of times over the design.

1.3.2 INFORMATION MATRIX:

The information matrix of a BTIB design is given by (see eq. A.1 of Bechhofer and Tamhane 1981),

$$m_{it} = \begin{cases} \frac{1}{k} \{ \lambda_0 + (v - 1).\lambda_1 \} & , i = t, \; i = 1,2,....,v \\ - \frac{1}{k} \lambda_1 , & i \neq t; \; i,t = 1,2,.........,v \end{cases}$$

So the information matrix is completely symmetric. Since, M^{-1} is the variance-covariance matrix of the vector of estimates ($\hat{\alpha}_0 - \hat{\alpha}_1, \ldots\ldots, \hat{\alpha}_0 - \hat{\alpha}_v$), a BTIB design provides BLUE 's $\hat{\alpha}_0 - \hat{\alpha}_i$ (i = 1,2,.....,v), where,

$$\hat{\alpha}_0 - \hat{\alpha}_i = \frac{-\lambda_1 Q_0 - \lambda_0 Q_i}{\lambda_0 (\lambda_0 + v\lambda_1)} \quad , \; i = 1,2,\ldots\ldots,v$$

where,

$$Q_i = k.T_i - \sum_{j=1}^{b} r_{ij}.B_j$$

also,

$$V(\hat{\alpha}_0 - \hat{\alpha}_i) = \frac{k(\lambda_0 + \lambda_1)}{\lambda_0 (\lambda_0 + v\lambda_1)} \sigma^2 = \tau^2 \sigma^2$$

$$\text{Corr.}\{\hat{\alpha}_0 - \hat{\alpha}_i, \hat{\alpha}_0 - \hat{\alpha}_t\} = \frac{\lambda_1}{\lambda_0 + \lambda_1} = \rho \quad , \quad i \neq t \quad i,t = 1,2,\ldots,v$$

$$\text{Cov.}\{\hat{\alpha}_0 - \hat{\alpha}_i, \hat{\alpha}_0 - \hat{\alpha}_t\} = \frac{k\lambda_1}{\lambda_0 (\lambda_0 + v\lambda_1)} \quad , \quad i \neq t \quad i,t = 1,2,\ldots,v$$

1.3.3 MINIMAL COMPLETE CLASS OF GENERATOR DESIGNS(MCCGD):

For given (v,k), a generator design is a BTIB design no proper subset of whose blocks forms a BTIB design, and no block of which contains only one of v+1 treatments.

Generator designs are important for the

construction of BTIB designs, as any design is either a generator design or a union of copies of generator designs. In fact, if for given (v,k) there are "n" generator designs D_i (i = 1,2,.......,n) where D_i has parameters (b_i, $\lambda_0^{(i)}$, $\lambda_1^{(i)}$), i = 1,2,......,n; then the BTIB design $D = \bigcup_{i=1}^{n} f_i.D_i$ obtained by taking union of f_i replications of D_i has parameters (b, λ_0, λ_1) given by,

$$b = \sum_{i=1}^{n} f_i\, b_i \ , \quad \lambda_0 = \sum_{i=1}^{n} f_i\, \lambda_0^{(i)}, \quad \lambda_1 = \sum_{i=1}^{n} f_i\, \lambda_1^{(i)}$$

Notz and Tamhane (1981) show that for a given (v,k) there exists only finitely many generator designs.

For given (v,k) a design D_2 with parameters (b_2, $\lambda_0^{(2)}$, $\lambda_1^{(2)}$) is inadmissible with respect to D_1 with parameters (b_1, $\lambda_0^{(1)}$, $\lambda_1^{(1)}$) iff $b_1 \le b_2$, $\tau_1^2 \le \tau_2^2$, $\rho_1 \ge \rho_2$ with at least one inequality strict.The D_1 and D_2 are equivalent if $b_1 = b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} = \lambda_1^{(2)}$.

A smallest set of generator designs with the property that any admissible design can either be constructed from the set or it is equivalent to a design which is constructed from the set, is called the Minimal Complete Class of Generator Designs (MCCGD).

1.4 PARTIALLY BALANCED TREATMENT INCOMPLETE BLOCK DESIGNS:

1.4.1 PARTIALLY BALANCED SCHEME:

Extensive studies have been done in the area of finding BTIB designs and catalog of these designs for values of v, b, and k in the practical range $2 \leq k \leq v \leq 10$ and $v \leq b \leq 50$, has been published by Hedayat and Majumdar (1984, 1985). However, there are good many values of v, b, and k in the practical range where no BTIB design exists. This is due to the severe restriction of complete symmetry of information matrix (i.e. all diagonal elements are equal and all off diagonal elements are equal). In other words, if the restrictions that,

$$\lambda_{0i} = \lambda_0 , \qquad i = 1,2,\ldots\ldots,v \quad \text{and}$$

$$\lambda_{it} = \lambda_1 , \quad i,t = 1,2,\ldots\ldots,v \; ; \; i \neq t$$

can be relaxed in some way, there is hope of finding a more general class of designs having members existing even for those values of v, b, and k for which BTIB design do not exist. The following idea is therefore, proposed.

Take, $\lambda_{0i} = \lambda_0 \quad , \quad i = 1,2,\ldots\ldots,v$

$$\lambda_{it} = \begin{cases} \lambda_1 , & \text{for some pairs i \& t, } i \neq t \\ \lambda_2 , & \text{for some other pairs i \& t, } i \neq t \end{cases}$$

Note that the information matrix of such a design is still symmetrical, but not completely (all diagonal elements are equal but off diagonal elements are different).

Before going to discuss in detail the construction and existence of such designs, in next chapter, it is necessary to revise some old and to give some new preliminary definitions.

DEFINITION 1: (BIB designs)

An arrangement of "v" treatments in "b" blocks of "k" experimental units each is called a Balanced Incomplete Block (BIB) Design if it satisfies the following three conditions:

(i) each block contains k different treatments.

(ii) each treatment occurs in r blocks, and

(iii) any two treatments occurs togather in λ blocks

DEFINITION 2: (PBIB Designs)

Given v treatments, a relation satisfying the following conditions is said to be an association scheme with m associate classes:

(i) any two treatments are either first or second or,........., m th associates.

(ii) any treatment has n_i i th associates, i = 1,2,........,m.

(iii) if two treatments are i th associates, then the number of treatments that are j th associates of the first and k th associates of the second is p^{i}_{jk} $(i,j,k = 1,2,........,m)$.

A design is said to be partially balanced incomplete block design if the following conditions are satisfied:

(i) each block contains k different treatments.

(ii) each treatment occurs in r blocks.

(iii) any two treatments that are i th associates occurs together in λ_i blocks, $i = 1,2,........,m$.

<u>DEFINITION 3:</u> (GD PBIB Designs)

This is a special case of PBIB designs with $m = 2$. Let there be $v = m.n$ treatments, divided into "m" groups of "n" treatments each in the following way.

T R E A T M E N T S

GROUP					
1	1	2	3	....	n
2	n+1	n+2	n+3	....	2n
⋮	⋮	⋮	⋮		⋮
m	(m-1)n+1	(m-1)n+2	(m-1)n+3	....	mn

Treatments belonging to the same group are first associates, and treatments belonging to different groups are the second associates. This association scheme is called a Group Divisible association leading to group-divisible partially balanced incomplete block designs(GD PBIBD). So, any two treatments that are in the same group occur together in λ_1 blocks and any two treatments that are in different groups occur together in λ_2 blocks.

DEFINITION 4: (BTIB Designs)

A design in v+1 treatments 0,1,2,..........,v, (0 is the control) in b blocks of size k each ($v \geq k$) is a balanced treatment incomplete block design if,

$$\lambda_{0i} = \lambda_0 , \quad i = 1,2,........,v$$
$$\lambda_{it} = \lambda_1 , \quad i,t = 1,2,.........,v \ ; \ i \neq t$$

DEFINITION 5: (PBTIB Designs)

Consider v test-treatments 1,2,......,v and a control 0. Apply the association scheme described in Definition 2 of partially incomplete block designs on test-treatemnts only. A design is partially balanced treatment incomplete block design if any two treatments that are i th associate occur together in λ_i blocks, i = 1,2,.......,m.

DEFINITION 6: (GD TIB Designs)

Let v = m.n , and v+1 treatments 0,1,2,......,v, can be divided in m+1 groups 0,1,2.......,m, such that group 0 contains control treatment only and the remaining m groups are as given in

Definition 3 of GD PBIB designs. Two test-treatments, if they are in the same group are first associates, and if they are in different groups are second associates. A design is group divisible treatment incomplete block design if,

$$\lambda_{0i} = \lambda_0 \quad , \quad i = 1,2,\ldots\ldots\ldots,v$$

$$\lambda_{it} = \begin{cases} \lambda_1 \text{ , i \& t are first associates, } i \neq t \\ \lambda_2 \text{ , i \& t are second associates, } i \neq t \end{cases}$$

1.4.2 MOST BALANCED SCHEME:

DEFINITION 1: (MB GD IB Designs)

If a design is GD PBIB design with 2 groups and $\lambda_2 = \lambda_1 \pm 1$ it is called Most Balanced Group Divisible Incomplete Block Design (MB GD IBD). Furthermore, a design is binary MB GD IB design if each treatment has the same number of replications and appears in each block at most once (Cheng 1978).

DEFINITION 2: (MB GD TIB Designs)

A GD TIB design in Definition 6 of Section 1.4.1 is Most Balanced Group Divisible Treatment Incomplete Block Design (MB GD TIBD) if $m = 2$ and $\lambda_2 = \lambda_1 \pm 1$, and it is binary if each test-treatment is equi-replicated and occurs in each block at most once. Note that no restriction is imposed upon the occurrence of control in blocks.

DEFINITION 3: (E MB GD TIB Designs)

An extended family of MB GD TIB Designs is defined as a family of designs with m = 2 and $\lambda_2 = \lambda_1 + \delta$, where $\delta = -1,0,1$. In this way it contains all BIB designs, BTIB designs and MB GD TIB designs.

EXAMPLE 1: The following design is a member of extended family with k = 3, v = 4, b = 10

BLOCK		1	2	3	4	5	6	7	8	9	10
PLOT	1	0	0	0	0	0	0	1	1	1	2
	2	1	1	1	2	2	3	2	2	3	3
	3	2	3	4	3	4	4	3	4	4	4

For this allocation,

$$r_0 = r_1 = r_2 = r_3 = r_4 = r = 6$$

$$\lambda_{01} = \lambda_{02} = \lambda_{03} = \lambda_{04} = \lambda_0 = 3$$

$$\lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = \lambda_{34} = \lambda_1 = 3$$

Since, $\lambda_0 = \lambda_1$ this is a BIB design.

EXAMPLE 2: Let k = 3, v = 4, b = 6

BLOCK		1	2	3	4	5	6
PLOT	1	0	0	0	0	0	0
	2	1	1	1	2	2	3
	3	2	3	4	3	4	4

Here, $r_0 = 6$, $r_i = r = 3$, $i = 1,2,3,4$

$\lambda_{0i} = \lambda_0 = 3$, $i = 1,2,3,4$

$\lambda_{it} = \lambda_1 = 1$, $i<t$, $i,t = 1,2,3,4$

This is a BTIB design and hence, a member of the family

EXAMPLE 3: Let $k = 3$, $v = 4$, $b = 8$

BLOCK		1	2	3	4	5	6	7	8
PLOT	1	0	0	0	0	0	0	1	1
	2	1	2	2	2	0	0	2	3
	3	4	3	4	4	1	3	3	4

This is a MB GD TIB design with the following allocation scheme.

$m = 2$, $n = 2$

group 0 contains control

group 1 contains treatments 1 and 2

group 2 contains treatments 3 and 4

Hence, (1,2) and (3,4) are first associates and (1,3), (1,4,), (2,3), (2,4) are second associates.

Here, $r_0 = 8$, $r_i = r = 4$, $i = 1,2,3,4$

$\lambda_{0i} = \lambda_0 = 3$, $i = 1,2,3,4$

$$\lambda_{it} = \begin{cases} \lambda_1 = 1, & i \neq t \text{ are first associates} \\ \lambda_2 = 2, & i \neq t \text{ are second associates} \end{cases}$$

DEFINITION 4: (Basic Blocks)

A block that contains control is called a "basic block" (BB). A BB is identified by the number of TC-pairs in it. When k = 4, for example, there are two different types of BB 's.

Type 1: $\begin{bmatrix} 0 \\ 0 \\ x \\ y \end{bmatrix}$ referred as 4-BB, since it contains 4 TC-pairs viz. (0,x), (0,y), (0,x), (0,y), where x, y are test-treatments.

Type 2: $\begin{bmatrix} 0 \\ x \\ y \\ z \end{bmatrix}$ referred as 3-BB, since it contains 3 TC-pairs viz. (0,x), (0,y), (0,z), where x, y, z are test-treatments.

DEFINITION 5: (Basic Block Combinations)

Given (v,k) and an E MB GD TIB design with parameters $(b, \lambda_0, \lambda_1, \lambda_2)$, the total number of TC-pairs are $v\lambda_0$ which can be allocated to different possible combinations of BBs, where each such combination must have $v\lambda_0$ such pairs.

A BB combination can be represented by an ordered pair. For example, if k = 4, v = 4, a BB combination is represented by an orderd pair (U, V), which means that there are U of 4-BBs and V of 3-BBs in the combination.

CHAPTER 2

2.1 THEORETICAL RESULTS:

2.1.1 THEORETICAL FACTS:

consider a design D with (k, v, b) and

λ_{0i} = number of times treatment i and control are paired in a block.

λ_{it} = number of times treatment i and treatment t ($i \neq t$) are paired in a block.

Let control appears "l" times in "m_l" blocks

FACT 2.1

Given (k,v) and an E MB GD TIB design with parameters b, λ_0, λ_1, λ_2, the control appears a total of $v.\lambda_0$ times with the test-treatments (TC-pairs) i.e.

$$\sum_{i=1}^{v} \lambda_{0i} = v \lambda_0$$

FACT 2.2

Given (k,v) and an E MB GD TIB design with parameters b, λ_0, λ_1, λ_2, the number of times a test-treatment is paired with other test-treatments is equal to

$$\frac{1}{2}\left[(v - 2) \lambda_1 + v \lambda_2 \right]$$

Consider the following association scheme,

group 1 contains treatments 1,2,......,v/2

group 2 contains treatments (v/2)+1,(v/2)+2,.......,v

So,

number of times a test-treatment is paired with other test-treatments of the same group (i.e.first associates) = $(\frac{v}{2} -1) \lambda_1$

number of times a test-treatment is paired with other test-treatments of different group (i.e. second associates) = $(v/2) \lambda_2$

The sum of these two gives the result.

FACT 2.3

Given (k,v) and an E MB GD TIB design with parameters b, λ_0, λ_1, λ_2, the total number of Treatment- Treatment (test) pairs (TT-pairs) is equal to

$$\frac{v}{4}\left[(v - 2) \lambda_1 + v \lambda_2 \right]$$

Actually,

$$\sum_{\substack{i,t \\ \text{first assoc.}}} \lambda_{it} = \text{no. of TT-pairs (first associate type)}$$

$$= 2.\binom{v/2}{2} \lambda_1 = \frac{1}{4} v (v - 2) \lambda_1$$

$$\sum_{\substack{i,t \\ \text{second} \\ \text{assoc.}}} \lambda_{it} = \text{no. of TT-pairs (second associate type)}$$

$$= \binom{v/2}{1} \binom{v/2}{1} \lambda_2 = \frac{1}{4} v^2 \lambda_2$$

The sum of these two gives the result.

FACT 2.4

A binary E MB GD TIB design is equi-replicate in terms of test-treatments, with the common replicate size "r" given by,

$$r = (b\, k - r_0)/v \qquad 2.1.1$$

FACT 2.5

For a design with parameters k, v, b,

(i) $r_0 = \sum_{l=1}^{k-1} l\, m_l$ 2.1.2

(ii) $\sum_{i=1}^{v} \lambda_{0i} = r_0 (k - 1) - \sum_{l=1}^{k-1} l (l - 1) m_l$ 2.1.3

(iii) $\sum_{\substack{i,t=1 \\ i \neq t}}^{v} \lambda_{it} = \frac{1}{2} b\, k\, (k - 1) - \frac{1}{2} \sum_{l=1}^{k-1} l\, (l-1)\, m_l - \sum_{i=1}^{v} \lambda_{0i}$

2.1.4

For proof see Ture (1982).

EXAMPLE: Consider the following MB GD TIB design with the following association scheme, group 0 contains control

group 1 contains treatments 1,2

group 2 contains treatments 3,4

BLOCK		1	2	3	4	5	6
PLOT	1	0	0	1	1	1	2
	2	1	3	2	2	3	3
	3	2	4	3	4	4	4

Here, $k = 3,\ v = 4,\ b = 6$

$\lambda_0 = 1,\ \lambda_1 = 3,\ \lambda_2 = 2$

$m_1 = 2,\ m_2 = 0$

See that,

$$r_0 = 2 = \sum_{l=1}^{k} l\ m_l \qquad \text{(FACT 2.5(i) verified)}$$

$$\sum_{i=1}^{4} \lambda_{0i} = 4 \qquad \text{(FACT 2.1 and FACT 2.5(ii) verified)}$$

$$\sum_{\substack{i,t=1 \\ i\neq t}}^{4} \lambda_{it} = 14 \qquad \text{(FACT 2.2 and FACT 2.5(iii) verified)}$$

<u>FACT 2.6</u> (Ture 1982)

For $v = 4$, $k = 4$ the BB combinations are given by (see Definition 5 of section 1.4.2).

(i) ($U = \lambda_0 - 3j + 3$, $V = 4j - 4$), $j = 1,2,\ldots$ $\ldots\ldots, [\lambda_0/3] + 1$, if λ_0 is even with $M_j = \lambda_0 + 9j - 9$ and $b_j = \lambda_0 + j - 1$, where $[x]$ is the largest integer smaller than x.

(ii) $(U = \lambda_0 - 3j , V = 4j), j = 1,2,....,[\lambda_0/3]$ if $\lambda_0 (> 1)$ is odd with $M_j = \lambda_0 + 9j$ and $b_j = \lambda_0 + j$

Where M_j is the number of TT-pairs in the j th BB combination and b_j is the number of blocks it contains.

FACT 2.7 (Ture 1982)

Consider all possible BB combinations for a specified $(v, k, \lambda_0, \lambda_1, \lambda_2)$, then

$$b \geq b_{j_0}$$

where j_0 is the index of the BB combination with b_{j_0} BBs for which $| N - M_{j_0} |$ is closest to zero; N is the total number of TT-pairs in the design and M_{j_0} is the number of TT-pairs in the j_0 th BB combination.

2.1.2 NON-EXISTENCE OF DESIGNS:

THEOREM: An Extended Most Balanced Group Divisible Treatment Incomplete Block Design with parameters k, v, b, r, λ_0, λ_1, and λ_2 is non-existent if,

$$b < \begin{cases} 2, & \text{when } r = \lambda_1 > \lambda_2 \\ v - 1, & \text{when } \lambda_1 < \lambda_2, \lambda_1 < r, r = \lambda_1 + \dfrac{v(\lambda_2 - \lambda_1)}{2} \\ v, & \text{when } r > \lambda_1 \geq \lambda_2, \text{ or } \\ & \lambda_1 < \lambda_2, \quad r > \lambda_1 + \dfrac{v(\lambda_2 - \lambda_1)}{2} \end{cases}$$

PROOF:

Let $\underset{(v+1\times b)}{R} = (r_{ij})$ be the incidence matrix of the design, where r_{ij} = no. of times i th treatment is replicated in j th block; $i = 0,1,2,\ldots\ldots,v$; $j = 1,2,\ldots\ldots,b$.

Let, $S = R\,R' = (s_{it})$, with

$$s_{it} = \begin{cases} \sum_{j=1}^{b} r_{0j}^2 = s_{00}, & i = t = 0 \\ \lambda_0 & , i = 0,\ t = 1,2,\ldots\ldots,v \\ r & , i = t;\ i,t = 1,2,\ldots,v \\ \lambda_1 & , i \text{ and } t \text{ are first associates} \\ \lambda_2 & , i \text{ and } t \text{ are second associates} \end{cases}$$

Note that $r \geq \lambda_1, \lambda_2$

Let, $\nu_0, \nu_1, \ldots\ldots, \nu_v$ are v+1 eigen values of S, given by the following equation,

$$| \, S - \nu I \, | = 0$$

where,

$$\left[S - \nu I\right] = \left[\begin{array}{c|ccc|ccc} s_{00} - \nu & \lambda_0 & \ldots & \lambda_0 & \lambda_0 & \ldots & \lambda_0 \\ \hline \lambda_0 & r - \nu & \ldots & \lambda_1 & \lambda_2 & \ldots & \lambda_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_0 & \lambda_1 & \ldots & r - \nu & \lambda_2 & \ldots & \lambda_2 \\ \hline \lambda_0 & \lambda_2 & \ldots & \lambda_2 & r - \nu & \ldots & \lambda_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_0 & \lambda_2 & \ldots & \lambda_2 & \lambda_1 & \ldots & r - \nu \end{array}\right]$$

The equation $| S - \nu I | = 0$ reduces to the following polynomial equation,

$$(r - \nu - \lambda_1)^{v-2}\left[r - \nu + (\tfrac{v}{2} - 1)\lambda_1 - \tfrac{v}{2}\lambda_2\right]\left[(s_{00} - \nu)\left\{r - \nu + (\tfrac{v}{2} - 1)\lambda_1 + \tfrac{v}{2}\lambda_2\right\} - v\lambda_0^2\right] = 0$$

Solution of this equation gives the following v + 1 roots

$$\nu_0, \nu_1 = \frac{1}{2}\left[\left\{ r + (\tfrac{v}{2} - 1)\lambda_1 + \tfrac{v}{2}\lambda_2 + s_{00}\right\} \pm \left[\left\{\left(r + (\tfrac{v}{2} - 1)\lambda_1 + \tfrac{v}{2}\lambda_2\right) - s_{00}\right\}^2 + 4 v \lambda_0^2\right]^{1/2}\right]$$

At least one of these two roots is positive.

$$\nu_2, \ldots\ldots\ldots, \nu_{v-1} = r - \lambda_1 = \mu \quad (\text{ say })$$

$$\nu_v = r + (\tfrac{v}{2} - 1)\lambda_1 - \tfrac{v}{2}\lambda_2 = \mu + \tfrac{v}{2}(\lambda_1 - \lambda_2) = \mu^* \quad (\text{ say })$$

Now, $\text{rank}(R) = \text{rank}(S) \leq b$

Whereas, the rank(S) is equal to the number of positive eigen values of S, which can be obtained by considering the following possibilities.

$$\mu \begin{cases} = 0, \text{ if } r = \lambda_1 \\ > 0, \text{ if } r > \lambda_1 \end{cases}$$

Note that,

$\mu^* = 0$, if $\mu = 0$ (i.e. $r = \lambda_1$) and $\lambda_1 = \lambda_2$, or

if $\mu > 0$ (i.e. $r > \lambda_1$), $\lambda_1 < \lambda_2$, and

$$\mu = \tfrac{v}{2}(\lambda_2 - \lambda_1).$$

$\mu^* > 0$, if $\mu = 0$ (i.e. $r = \lambda_1$) and $\lambda_1 > \lambda_2$, or

if $\mu > 0$ (i.e. $r > \lambda_1$) and $\lambda_1 = \lambda_2$, or

if $\mu > 0$ (i.e. $r > \lambda_1$) and $\lambda_1 > \lambda_2$, or

if $\mu > 0$ (i.e. $r > \lambda_1$), $\lambda_1 < \lambda_2$, and

$$\mu > \frac{v}{2} (\lambda_2 - \lambda_1)$$

The following are,therefore, true

(i) rank(R) $\leq$ 2, if $\mu = 0$, $\mu^* = 0$ (i.e. if $r = \lambda_1 = \lambda_2$), hence $b \geq 1$

(ii) rank(R) = 2 or 3, if $\mu = 0$, $\mu^* > 0$ (i.e. if $r = \lambda_1 > \lambda_2$), hence $b \geq 2$

(iii) rank(R)= $v - 1$ or v, if $\mu > 0$, $\mu^* = 0$ (i.e. if $\lambda_1 < \lambda_2$, $\lambda_1 < r$, and $r = \lambda_1 + \frac{v}{2} (\lambda_2 - \lambda_1)$, hence $b \geq v - 1$

(iv) rank(R) = v or $v + 1$, if $\mu > 0$, $\mu^* > 0$ (i.e. $r > \lambda_1 \geq \lambda_2$ or $\lambda_1 < \lambda_2$,and $r > \lambda_1 + \frac{v}{2} (\lambda_2 - \lambda_1)$, hence $b \geq v$

Possibility (i) is trivial. The converse of possibilities (ii), (iii), and (iv) give the conditions required for the non-existence of the design in the theorem.Hence, the theorem is proved.

<u>REMARK</u>: Although the theorem is completely valid to discard a design as non-existent, it does not provide sufficient conditions for the existence of a design i.e. a design may not be constructed even if its non-existence is not proved. However, the theorem is extremely useful as it eliminates the useless effort of trying to construct a

design which is actually non-existent. It is our experience that out of hundreds of designs studied there are very few designs which disobey the condition of non-existence but even the construction of them is not possible.

The usefulness of this theorem is explained by the following examples.

EXAMPLE 1: Consider a BTIB design with parameters

$k = 4,\ v = 4,\ b = 4,\ r = 3,\ \lambda_0 = 3,\ \lambda_1 = 1,\ \lambda_2 = 1$

The following relation holds,

$$r > \lambda_1 = \lambda_2$$

According to the possibility (iv), b should be greater than or equal to v for the possible existence of the design. Since, it is so, the design may exists.

EXAMPLE 2: Consider a MB GD TIB design with,

$k = 3,\ v = 4,\ b = 8,\ r = 4,\ \lambda_0 = 3,\ \lambda_1 = 1, \lambda_2 = 2$

Here, $\lambda_1 < \lambda_2$ and $r > \lambda_1 + \frac{v}{2}(\lambda_2 - \lambda_1) = 3$.

This implies that $b \geq v = 4$ for the possible existence.

The design may exists.

EXAMPLE 3: Consider a BIB design with parameters,

$k = 4,\ v = 6,\ b = 4,\ r = 2,\ \lambda_0 = 1, \lambda_1 = 1,\ \lambda_2 = 1$.

Since, $r > \lambda_1 = \lambda_2$ implies that $b \geq v = 6$ for the possible existence, the design cannot exists.

EXAMPLE 4: Let us have a MB GD TIB design with,

$k = 3$, $v = 8$, $b = 7$, $r = 2$, $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 0$

The relation $r > \lambda_1 > \lambda_2$ gives the condition $b \geq 8$ for the possible existence. This is not true here, because $b = 7$. Hence, the design will not exists.

2.1.3 INFORMATION MATRIX AND RELATED MEASURES:

The information matrix of a GD TIB design with parameters k, v, b, λ_0, λ_1, and λ_2 is given by,

$$\underset{(v\mathrm{x}v)}{M} = (m_{it})$$

where,

$$m_{it} = \begin{cases} \frac{1}{k} \left(\lambda_0 + \frac{v-2}{2} \lambda_1 + \frac{v}{2} \lambda_2 \right), & i = t \\ -\frac{1}{k} \lambda_1 , & i \text{ and } t \text{ are first associates} \\ -\frac{1}{k} \lambda_2 , & i \text{ and } t \text{ are second associates} \end{cases}$$

Let, $\alpha = \frac{1}{k} \{ \lambda_0 + \frac{v-2}{2} + \frac{v}{2} \lambda_2 \}$, $\beta = -\frac{1}{k} \lambda_1$, $\gamma = -\frac{1}{k} \lambda_2$

So, the information matrix M can be written as

$$M = \left[\begin{array}{c|c} P & Q \\ \hline Q & P \end{array} \right]$$

where,

$$\underset{(v/2\mathrm{x}v/2)}{P} = \begin{bmatrix} \alpha & \beta & \ldots\ldots\beta \\ \beta & \alpha & \ldots\ldots\alpha \\ \vdots & \vdots & \vdots \\ \alpha & \alpha & \ldots\ldots\beta \end{bmatrix}, \text{ and } \underset{(v/2\mathrm{x}v/2)}{Q} = \gamma . \begin{bmatrix} 1 & 1\ldots\ldots1 \\ 1 & 1\ldots\ldots1 \\ \vdots & \vdots \quad \vdots \\ 1 & 1\ldots\ldots1 \end{bmatrix} = \gamma . J$$

The variance-covariance matrix for estimating $(\hat{\alpha}_0 - \hat{\alpha}_i)$, $i = 1,2,\ldots\ldots,v$ is given by

$$V = \sigma^2 \quad M^{-1}$$

where,

$$M^{-1} = \left[\begin{array}{c|c} P^{-1} (I + Q S^{-1} Q P^{-1}) & - P^{-1} Q S^{-1} \\ \hline - S^{-1} Q P^{-1} & S^{-1} \end{array}\right]$$

where, $S = P - Q P^{-1} Q$.

The four elements in M^{-1} are as follows,

(i) $P^{-1}(I + Q S^{-1}Q P^{-1}) = (p_{ij})$, where

$$p_{ij} = \begin{cases} A \{ \alpha + (\frac{v}{2} - 2) \beta \} + E , & \text{if } i = j \\ - \beta A + E & , \text{if } i \neq j \end{cases}$$

(ii),(iii) $- P^{-1}Q S^{-1} = - S^{-1}Q P^{-1} = -\gamma (\alpha - \beta)(d - e)ADJ$

(iv) $S^{-1} = D (q_{ij})$, where

$$q_{ij} = \begin{cases} d + (\frac{v}{2} - 2) e & , \text{if } i = j \\ - e & , \text{if } i \neq j \end{cases}$$

Where,

$$A = (\alpha - \beta)^{-1} \{ \alpha + (\frac{v}{2} - 1) \beta \}^{-1}$$

$$D = (d - e)^{-1} \{ d + (\frac{v}{2} - 1) e \}^{-1}$$

$$E = \frac{v}{2} \gamma^2 (\alpha - \beta)^2 (d - e) A^2 D$$

$$d = \alpha - \frac{v}{2} \gamma^2 (\alpha - \beta) A$$

$$e = \beta - \frac{v}{2} \gamma^2 (\alpha - \beta) A$$

The diagonal elements of V, gives the variance of the estimates.

$$V(\hat{\alpha}_0 - \hat{\alpha}_i) = \frac{\sigma^2 k \{(\lambda_0 + \lambda_1 + \frac{v}{2}\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_2) - \frac{v}{2}(\frac{v}{2} - 1)\lambda_2^2\}}{\lambda_0(\lambda_0 + v\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_1 + \frac{v}{2}\lambda_2)}$$

$$= \eta^2 \sigma^2 \quad (\text{ say }) \quad , \quad \text{for } i = 1,2,\ldots\ldots,v$$

The off diagonal elements of V give the covariance terms

$$\text{cov}\{(\hat{\alpha}_0 - \hat{\alpha}_i),(\hat{\alpha}_0 - \hat{\alpha}_t)\} = \begin{cases} \dfrac{\sigma^2 k \{\lambda_1 + (\lambda_0 + \frac{v}{2}\lambda_2) + \frac{v}{2}\lambda_2^2\}}{\lambda_0(\lambda_0 + v\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_1 + \frac{v}{2}\lambda_2)}, \\ \quad i \ \& \ t \text{ are first associates} \\ \dfrac{\sigma^2 k \lambda_2}{\lambda_0(\lambda_0 + v\lambda_2)}, \\ \quad i \ \& \ t \text{ are second associates} \end{cases}$$

Furthermore, the correlation between estimates are the following,

$$\text{corr.}\{(\hat{\alpha}_0 - \hat{\alpha}_i), (\hat{\alpha}_0 - \hat{\alpha}_t)\} = \begin{cases} \dfrac{\lambda_1(\lambda_0 + \frac{v}{2}\lambda_2) + \frac{v}{2}\lambda_2^2}{(\lambda_0 + \lambda_1 + \frac{v}{2}\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_2) - \frac{v}{2}(\frac{v}{2} - 1)\lambda_2^2} \\ = \rho_1 \text{ (say)}, i \ \& \ t \text{ are first assoc.} \\ \dfrac{\lambda_2(\lambda_0 + \frac{v}{2}\lambda_2) + \frac{v}{2}\lambda_2^2}{(\lambda_0 + \lambda_1 + \frac{v}{2}\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_2) - \frac{v}{2}(\frac{v}{2} - 1)\lambda_2^2} \\ = \rho_2 \text{ (say)}, i \ \& \ t \text{ are second assoc.} \end{cases}$$

and trace is given by,

$$\text{Tr}(\ M^{-1}\ ) = \frac{vk\{(\lambda_0 + \lambda_1 + \frac{v}{2}\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_2) - \frac{v}{2}(\frac{v}{2} - 1)\lambda_2^2\}}{\lambda_0(\lambda_0 + v\lambda_2)(\lambda_0 + \frac{v}{2}\lambda_1 + \frac{v}{2}\lambda_2)}$$

2.1.4 INADMISSIBLE AND EQUIVALENT DESIGNS:

Having defined the terms η^2, ρ_1 and ρ_2 in the Section 2.1.3 and for fixed λ_0, it can be verified that η^2 is a decreasing and ρ_1 and ρ_2 are increasing functions of λ_1 and λ_2.

Consider two designs D_1 and D_2 with the following parameters,

$$D_1 \equiv (b_1, \lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)})$$

$$D_2 \equiv (b_2, \lambda_0^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)})$$

DEFINITION 1: D_1 and D_2 are equivalent if,

$$b_1 = b_2, \lambda_0^{(1)} = \lambda_0^{(2)}, \lambda_1^{(1)} = \lambda_1^{(2)}, \lambda_2^{(1)} = \lambda_2^{(2)}$$

DEFINITION 2: D_2 is strongly inadmissible with respect to D_1 if.

$$b_1 \leq b_2, \eta_1^2 \leq \eta_2^2, \rho_1^{(1)} \geq \rho_1^{(2)}, \rho_2^{(1)} \geq \rho_2^{(2)}$$

with at least one inequality strict.

Since, η^2 is decreasing and ρ_1 and ρ_2 are increasing functions of λ_1 and λ_2, the above conditions for inadmissibility of D_2 with respect to D_1 can be written as,

$$b_1 \leq b_2, \lambda_0^{(1)} = \lambda_0^{(2)}, \lambda_1^{(1)} \geq \lambda_1^{(2)}, \lambda_2^{(1)} \geq \lambda_2^{(2)}$$

with at least one inequality strict.

A design which is inadmissible can be discarded because there exists a better design (i.e. with less variance of estimates) with lesser number of blocks.

Furthermore, only one design can be selected from each class of equivalent designs.

2.1.5 SOME LEMMAS:

LEMMA 2.1 For given (k,v) a group divisible treatment incomplete block design with $(b, \lambda_0, \lambda_1, \lambda_2)$ satisfy the following inequality,

$$\frac{4v\lambda_0 + v(v-2)\lambda_1 + v^2\lambda_2}{2k(k-1)} \leq b \leq \frac{4v\lambda_0 + v(v-2)\lambda_1 + v^2\lambda_2}{4(k-1)}$$

Furthermore, the lower inequality is an equality iff the design is completely binary (i.e. $r_{ij} = 0$ or 1, $0 \leq i \leq v$, $1 \leq j \leq b$).

PROOF:

Consider information matrix M given in Section 1.2.4 and Section 2.1.3. One can write,

$$m_{00} = r_0 - \frac{1}{k}\sum_{j=1}^{b} r_{0j}^2 = \frac{v\lambda_0}{k}$$

which gives,

$$k\,r_0 = v\lambda_0 + \sum_{j=1}^{b} r_{0j}^2 \qquad 2.1.5$$

and also,

$$m_{ii} = r_i - \frac{1}{k}\sum_{j=1}^{b} r_{ij}^2 = \frac{1}{k}\{\lambda_0 + \frac{v-2}{2}\lambda_1 + \frac{v}{2}\lambda_2\}$$

which gives,

$$k\, r_i = \lambda_0 + \frac{v-2}{2}\lambda_1 + \frac{v}{2}\lambda_2 + \sum_{j=1}^{b} r_{ij}^2 \qquad 2.1.6$$

Adding 2.1.5 and 2.1.6,

$$k\sum_{i=0}^{v} r_i = k^2 b = 2v\lambda_0 + \frac{v(v-2)}{2}\lambda_1 + \frac{v^2}{2}\lambda_2 + \sum_{i=0}^{v}\sum_{j=1}^{b} r_{ij}^2 \qquad 2.1.7$$

it can be easily verified that the minimum value of $\sum_{i=0}^{v}\sum_{j=1}^{b} r_{ij}^2$ is bk and the maximum value is $b(k^2 - 2k + 2)$.

Substituting these two values of $\sum_{i=0}^{v}\sum_{j=1}^{b} r_{ij}^2$ in eq.2.1.7 give the lower and the upper bounds on b in the Lemma.

REMARK: For $(k=3, v=4)$ an E MB GD TIB design with parameters $(b, \lambda_0, \lambda_1, \lambda_2 = \lambda_1 + \delta)$, where $\delta = -1, 0, 1$, satisfy the following,

$$b \geq \frac{2}{3}(2\lambda_0 + \lambda_1 + 2\lambda_2)$$

LEMMA 2.2 For given (k,v) consider a group divisible treatment incomplete block design with $(b, \lambda_0, \lambda_1, \lambda_2)$. If the design is binary in terms of test-treatments then the test-treatments are replicated equally in the design with a common replicate size,

$$r = \frac{2\lambda_0 + (v-2)\lambda_1 + v\lambda_2}{2(k-1)}$$

PROOF: For a design that is binary in terms of test-treatments, eq. 2.1.6 gives,

$$Kr_i - r_i = \lambda_0 + \frac{v-2}{2}\lambda_1 + \frac{v}{2}\lambda_2, \quad \left[\sum_{j=1}^{b} r_{ij}^2 = \sum_{j=1}^{b} r_{ij} = r_i\right]$$

So, $$r_i = \frac{2\lambda_0 + (v-2)\lambda_1 + v\lambda_2}{2(k-1)}, \quad i = 1,2,\dots\dots\dots,v$$

It, therefore, follows that,

$$r_1 = r_2 = \dots\dots = r_v = r = \frac{2\lambda_0 + (v-2)\lambda_1 + v\lambda_2}{2(k-1)}$$

REMARK: For $(k=3, v=4)$ and a E MB GD TIB design with parameters $(b, \lambda_0, \lambda_1, \lambda_2 = \lambda_1 + \delta)$, where $\delta = -1,0,1$, Lemma 2.2 gives,

$$r = \frac{1}{2}(\lambda_0 + \lambda_1 + 2\lambda_2)$$

LEMMA 2.3 For any E MB GD TIB design for $k = 3$, $v \geq 3$, we have,

$$b \geq \frac{v\lambda_0}{2}$$

PROOF: Let r_0 units are allocated to the control and $\underset{\sim}{m} = (m_1, m_2)$ represents the allocation vector (i.e.control appears "l"(l = 1,2) times in "m_l" blo cks in the design). Then,

$$\sum_{l=1}^{2} m_l = m_1 + m_2 \leq b \qquad 2.1.8$$

and

$$\sum_{l=1}^{2} l m_l = m_1 + 2m_2 = r_0 \qquad 2.1.9$$

and

$$\sum_{l=1}^{2} l(l-1)m_l = 2\ m_2 \qquad 2.1.10$$

Eqs. 2.1.8 and 2.1.9 gives,

$$r_0 - m_2 \leq b \qquad 2.1.11$$

Using FACT 2.1 and FACT 2.5(ii),

$$v\lambda_0 = \sum_{i=1}^{v} \lambda_{oi} = 2(r_0 - m_2)$$

Using 2.1.11 gives,

$$b \geq \frac{v\lambda_0}{2}$$

REMARK: for any E MB GD TIB design for k = 3, v = 4, we have,

$$b \geq 2\lambda_0$$

LEMMA 2.4 In a binary GD TIB design with $k > 2$ odd, the quantities $v\lambda_0$ and $\lambda_0 + \frac{v-2}{2}\lambda_1 + \frac{v}{2}\lambda_2$ are even integers.

PROOF: Recall that,

$$r_0 = \sum_{l=1}^{k-1} l m_l \ ,$$

$$\sum_{j=1}^{b} r_{0j}^2 = \sum_{l=1}^{k-1} l^2 m_l \ ,$$

$$\sum_{j=1}^{b} r_{ij}^2 = r_i$$

From eq. 2.1.5, we can obtain the following relation,

$$v\lambda_0 = \sum_{l=1}^{k-1} l(k-l)m_l$$

Now, when k is odd, l(k-l) is even. Therefore, $v\lambda_0$ is also even. Furthermore, from eq. 2.1.6, we can get,

$$\lambda_0 + \frac{v-2}{2}\lambda_1 + \frac{v}{2}\lambda_2 = (k-1)r_i$$

So, when k is odd, k-1 is even and hence, $(k-1)r_i$ is even i.e. $\lambda_0 + \frac{v-2}{2}\lambda_1 + \frac{v}{2}\lambda_2$ is even.

REMARK: For an E MB GD TIB design with $(b, \lambda_0, \lambda_1, \lambda_1 + \delta)$, where $\delta = -1, 0, 1$ and for $k = 3$ and $v = 4$, the quantity, $\lambda_0 + \lambda_1 + 2\lambda_2$ is even.

LEMMA 2.5 In a GD TIB design with $k = 3$, $v = 4$, if $\lambda_0 = 0$, then $\lambda_1 = \lambda_2$.

PROOF: In such a design (1,2),(3,4) are first associates, whereas, (1,3),(1,4),(2,3),(2,4) are second associates.

In any block (e.g. 1,2,3 or 1,3,4 etc.) of such a design, we have,

number of TT-pairs of first associates type = 1

and, number of TT-pairs of se-cond associates type = 2

Therefore, in b blocks,

total number of TT-pairs of first associates type = b

total number of TT-pairs of second associates type = 2b

Using the relations in FACT 2.3, one can write,

$$b = \frac{1}{4} .4 .2 .\lambda_1 \quad \Rightarrow \quad b = 2\lambda_1$$

and,
$$2b = \frac{1}{4} .4^2 . \lambda_2 \quad \Rightarrow \quad b = 2\lambda_2$$

Hence, $\lambda_1 = \lambda_2$.

<u>REMARK</u>: In an E MB GD TIB design for $k = 3$, $v = 3$ and with $(b, \lambda_0 = 0, \lambda_1, \lambda_2 = \lambda_1 + \delta)$, where $\delta = -1,0,1$, the only designs which can exist are those for which $\delta = 0$ (i.e. $\lambda_1 = \lambda_2$).

<u>LEMMA 2.6</u> In a GD TIB design with $k = 3$, $v = 4$, if $\lambda_0 = 1$, then $\lambda_2 \neq \lambda_1$.

<u>PROOF</u>: In such a design (1,2),(3,4) are first associates and (1,3),(1,4),(2,3),(2,4) are second associates. If $\lambda_0 = 1$, the following two situations may arise.

(i) two blocks contain the control once in each block, and each of the remaining b - 2 blocks, contain test-treataments only.

BLOCK		1	2	3 .... b
PLOT	1	0	0	
	2	1	2	test-treatments only
	3	3	4	
		B_1		B_2

number of TT-pairs of first associates type in B_1 = 0

number of TT-pairs of first associates type in B_2 = b-2
(see Lemma 2.5)

total number of TT-pairs of first associates type = b-2

number of TT-pairs of second associates type in B_1=2

number of TT-pairs of second associates type in B_2=2(b-2)
(see Lemma 2.5)

total number of TT-pairs of second associates type = 2b-2

Using the relations in FACT 2.3, we can write,

$$b - 2 = 2\lambda_1$$

and,
$$2b - 2 = 4\lambda_2$$

Hence,

$$\lambda_2 = \lambda_1 + \frac{1}{2}, \quad \text{i.e} \quad \lambda_2 \neq \lambda_1$$

(ii) Same situation as in (i) above but, the following configuration,

BLOCK		1	2	3 .... b
PLOT	1	0	0	
	2	1	3	test-treatments only
	3	2	4	
		B_1		B_2

The following are, therefore, true:

total number of TT-pairs of first associates type = b

total number of TT-pairs of second associates type=2(b-2)

Using FACT 2.3, we can write,

$$b = 2\lambda_1,$$

and, $$2(b-2) = 4\lambda_1$$

Hence, $\lambda_2 = \lambda_1 - 1$ i.e. $\lambda_2 \neq \lambda_1$

The proof of the Lemma is, therefore, complete.

LEMMA 2.7 In an GD TIB design with k = 4,v = 4 and $(b,\lambda_0, \lambda_1, \lambda_2 = \lambda_1 + \delta)$, $\delta = -1,0,1$, the following are true,

(i) if $\lambda_0 = 0 \bmod(3)$, then $\lambda_2 = \lambda_1$

(ii) if $\lambda_0 = 1 \bmod(3)$, then $\lambda_2 = \lambda_1 + 1$

(iii) if $\lambda_0 = 2 \bmod(3)$, then $\lambda_2 = \lambda_1 - 1$

PROOF: Let the design contain j th BB combination. Then by FACT 2.6 there are $U = \lambda_0 - 3j + 3$ 4-BBs and $V = 4j - 4$ 3-BBs ($j = 1,2,\ldots\ldots.,[\lambda_0/3]+1$) if λ_0 is even, or $U = \lambda_0 - 3j$ 4-BBs and $V = 4j$ 3-BBs ($j = 1,2,\ldots\ldots.,[\lambda_0/3]$) if λ_0 is odd.

Let there are "q" (an integer) 4-BBs in which x & y are first associates (see Definition 4 of Section 1.4.2). Hence, in the rest U - q 4-BBs, x & y are second associates.

The following facts are, therefore, noted.

number of TT-pairs of first associates type in"q"4-BBs= q number of TT-pairs of second associates type in"q"4-BBs=0

number of TT-pairs of first associate type in"U-q"4-BBs=0

number of TT-pairs of second associates type in "U - q" 4-BBs = U - q

number of TT-pairs of first associates type in"V"3-BBs= V

number of TT-pairs of second associates type in "V" 3-BBs = 2 v

number of TT-pairs of first associates type in $b - b_j$ remaining blocks = $2(b - b_j)$

number of TT-pairs of second associates type in $b - b_j$ remaining blocks = $4(b - b_j)$

where, $b_j = U + V$. So, the

total number of TT-pairs of first associates type in the design = $q + V + 2(b - b_j)$

total number of TT-pairs of second associates type in the design = $U - q + 2V + 4(b - b_j)$

Using FACT 2.3, one can get,

$$q = \begin{cases} \frac{\lambda_0}{3} - j + 1 - \frac{4}{3}\delta, & \text{if } \lambda_0 \text{ is even} \\ \frac{\lambda_0}{3} - j - \frac{4}{3}\delta, & \text{if } \lambda_0 \text{ is odd} \end{cases}$$

Consider the following cases,

<u>CASE 1</u>: $\lambda_0 = 0 \bmod(3) = 3m$, m= 0,1,2,..........

$$q = \begin{cases} m - j + 1 - \frac{4}{3}\delta, & \text{if } \lambda_0 \text{ is even} \\ m - j - \frac{4}{3}\delta, & \text{if } \lambda_0 \text{ is odd} \end{cases}$$

Now, q is integer only when $\delta = 0$, i.e. when $\lambda_2 = \lambda_1$. Part (i) of the Lemma is, therefore, proved.

CASE 2: $\lambda_0 = 1 \bmod(3) = 3m + 1$, $m = 0,1,2,\ldots\ldots.$

$$q = \begin{cases} m - j - \frac{4}{3}(\delta - 1), & \text{if } \lambda_0 \text{ is even} \\ m - j - \frac{1}{3}(4\delta - 1), & \text{if } \lambda_0 \text{ is odd} \end{cases}$$

The q is integer only for $\delta = 1$, i.e., for $\lambda_2 = \lambda_1 + 1$. This proves part (ii) of the Lemma.

CASE 3: $\lambda_0 = 2 \bmod(3) = 3m + 2$, $m = 0,1,2,\ldots\ldots$

$$q = \begin{cases} m - j - \frac{1}{3}(4\delta - 5), & \text{if } \lambda_0 \text{ is even} \\ m - j - \frac{2}{3}(2\delta - 1), & \text{if } \lambda_0 \text{ is odd} \end{cases}$$

Here, q is integer if $\delta = -1$, therefore, $\lambda_2 = \lambda_1 - 1$. This proves part (iii) of the Lemma. This completes the proof of Lemma 2.7

2.2 METHOD OF CONSTRUCTION:

consider a design with the parameters k, v, b and allocation of r_0 units to the control.

From FACT 2.1,

$$\lambda_0 = \sum_{i=1}^{v} \lambda_{0i} / v \qquad 2.2.1$$

where, $\sum_{i=1}^{v} \lambda_{0i}$ is given by eq. 2.1.3

Also from FACT 2.3,

$$\lambda_1 + \frac{v}{v-2}\lambda_2 = \frac{4}{v(v-2)} \sum_{i\neq t} \lambda_{it} \qquad 2.2.2$$

where, $\sum_{i\neq t} \lambda_{it}$ is given by eq. 2.1.4.

For the construction of the design the following algorithm is used.

1. check that the value of r calculated from eq. 2.1.1 is an integer.
2. select vector $\underline{m} = (m_1, m_2, \ldots\ldots, m_{k-1})$ which satisfy eq.2.1.2, and compute $\sum \lambda_{0i}$ and $\sum \lambda_{it}$ from eqs. 2.1.3 and 2.1.4 respectively.
3. check whether λ_0 from eq. 2.2.1 is an integer.
4. try to find some integer values of λ_1 and λ_2 which satisfy eq. 2.2.2.
5. verify that the design is not non-existent by applying Theorem in Section 2.1.2.

If all the conditions mentioned above are satisfied a design with these parameters may be constructed. It is, however, warned that the satisfaction of these conditions does not imply the actual existence of the design. Trial to actually construct the design are always required. Usually, with few trials it becomes clear that whether or not the treatments can be allocated to the plots according to the plan suggested by the parameters.

__EXAMPLE__: Consider a design with the following parameters and follow the algorithm.

$$k = 4,\ v = 8,\ b = 23,\ r_0 = 36$$

1. $r = (bk - r_0)/v = 7$ (an integer)
2. let $\underline{m} = (m_1, m_2, m_3) = (11, 11, 1)$. So,

$$\sum_{l=1}^{a} l m_l = 36 \quad \text{(eq. 2.1.2 satisfied)}$$

and,
$$\sum_{l=1}^{a} l(l-1) m_l = 28$$

Hence, from eq. 2.1.3 and eq. 2.1.4, we have,

$$\sum_{i=1}^{a} \lambda_{0i} = 80 \quad , \quad \text{and} \quad \sum_{i,t=1}^{a} \lambda_{it} = 44$$

3. from eq. 2.2.1 $\lambda_0 = 80/8 = 10$ (an integer)
4. the values $\lambda_1 = 1$ and $\lambda_2 = 2$, satisfy eq 2.2.2
5. note that, $\lambda_1 < \lambda_2$ and $r > \lambda_1 + \dfrac{v(\lambda_2 - \lambda_1)}{2}$.

   According to the Theorem of non-existence in Section 2.1.2, the design is non-existent if $b < v$. Since, $b>v$, the design may, therefore, exist .

In fact, the design exists, with the following structure.

BLOCK	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
PLOT 1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	2	2	2	3	3	4	0	0	0	0	0	0	0	0	0	0	0	0
3	2	3	4	5	6	3	4	7	4	5	6	1	2	2	2	3	3	3	4	4	4	5	0
4	5	6	7	8	7	8	6	8	5	7	8	8	5	6	7	7	6	8	5	7	8	6	1

CHAPTER 3

3.1 MINIMAL COMPLETE CLASS OF GENERATOR DESIGNS (MCCGD) FOR K = 2, V = 4 :

THEOREM 3.1

For k = 2 and v = 4 the minimal complete class of generator designs is given in Table 3.1.

TABLE 3.1 MCCGD FOR k = 2, v = 4 .

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	2	0	1	0	0	1	0	1 3 2 4
D_2	4	0	2	0	0	0	1	1 1 2 2 3 4 3 4
D_3	4	4	1	4	1	0	0	0 0 0 0 1 2 3 4

PROOF:

The above mentioned designs D_1, D_2, D_3 are the only possible generator designs and since they are distinct and neither of them is strongly inadmissible with respect to the other one, they constitute the minimal complete set (see Section 1.3.3) for k = 2,v = 4.

Each of the following theorems has the similar proof as given in the case of Theorem 3.1.

3.2 MCCGD FOR k = 2, v = 6 :

THEOREM 3.2

For k = 2, v = 6 the minimal complete class of generator designs is given in Table 3.2.

TABLE 3.2 MCCGD FOR k = 2, v = 6.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	6	0	2	0	0	1	0	1 1 2 4 4 5 2 3 3 5 6 6
D_2	6	6	1	6	1	0	0	0 0 0 0 0 0 1 2 3 4 5 6
D_3	9	0	3	0	0	0	1	1 1 1 2 2 2 3 3 3 4 5 6 4 5 6 4 5 6

3.3 MCCGD FOR k = 2, v = 8 :

THEOREM 3.3

For k = 2 and v = 8 the minimal complete class of generator designs is given in Table 3.3.

TABLE 3.3 MCCGD FOR k = 2, v = 8.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	8	8	1	8	1	0	0	0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8
D_2	12	0	3	0	0	1	0	1 1 1 2 2 3 5 5 6 6 7 2 3 4 3 4 4 6 8 7 8 8
D_3	16	0	4	0	0	0	1	1 1 1 1 2 2 2 2 3 3 3 3 4 5 6 7 8 5 6 7 8 5 6 7 8 5 4 4 4 6 7 8

3.4 MCCGD FOR k = 2, v = 10 :

THEOREM 3.4

For k = 2 and v = 10 the minimal complete class of generator designs is given in Table 3.4.

TABLE 3.4 MCCGD FOR k = 2, v = 10

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	10	10	1	10	1	0	0	0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8 9 10
D_2	20	0	4	0	0	1	0	1 1 1 1 2 2 2 3 3 4 6 6 6 2 3 4 5 3 4 5 4 5 5 7 8 9 6 7 7 7 8 8 9 10 8 9 10 9 10 10

TABLE 3.4 (CONTINUED)

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_3	25	0	5	0	0	0	1	1 1 1 1 1 2 2 2 2 6 7 8 9 10 6 7 8 9 --- 2 3 3 3 3 3 4 4 4 10 6 7 8 9 10 6 7 8 --- 4 4 5 5 5 5 5 9 10 6 7 8 9 10

CHAPTER 4

4.1 GENERAL METHOD OF PROVING A CLASS OF GENERATOR DESIGNS TO BE MCCGD :

Let us suppose that for a given (v,k), $\{ D_1, D_2, \ldots\ldots, D_n \}$ is a set of generator designs. Also, consider an arbitrary design "D" with Parameters (b, λ_0, λ_1, λ_2), where $\lambda_2 = \lambda_1 + \delta$, $\delta = -1,0,1$. To show that the set of generator designs is MCCGD, we show that there exists a design D^* with parameters (b^*, λ_0^*, λ_1^*, λ_2^*), such that,

$$D^* = \bigcup_{i=1}^{n} f_i \, D_i \, .$$

$\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1$, $\lambda_2^* = \lambda_2$ and $b^* \leq b$

where,

$$\lambda_0^* = \sum_{i=1}^{n} f_i \lambda_0^{(i)}, \ \lambda_1^* = \sum_{i=1}^{n} f_i \lambda_1^{(i)}, \ \lambda_2^* = \sum_{i=1}^{n} f_i \lambda_2^{(i)}, \ b^* = \sum_{i=1}^{n} f_i b_i$$

b_i, $\lambda_0^{(i)}$, $\lambda_1^{(i)}$, $\lambda_2^{(i)}$ are the parameters of generator design D_i (i = 1,2,......., n).

The design D is, therefore, either equivalent to or strongly inadmissible with respect to D^* (Section 1.3.3 and Section 2.1.4). The proof is completed by finally noting that the set $\{ D_1, D_2, \ldots\ldots\ldots, D_n \}$ consists of non-equivalent generator designs.

In the next section, we give MCCGD for E MB GD TIB design for k = 3, v = 4 and prove the minimal complete nature of the class. In other sections MCCGD are given for k = 3 and v = 6, 8, 10, without the accompanying proofs which are very long and tedious, but similar to the one given in the next section.

4.2 MCCGD FOR k = 3, v = 4 :

THEOREM 4.2

For k = 3 and v = 4 the minimal complete class of generator designs is given in Table 4.1.

TABLE 4.2 MCCGD FOR k = 3, v = 4.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	2	2	1	2 0	1	1	0	0 0 1 3 2 4
D_2	4	0	3	0 0	0	2	2	1 1 1 2 2 2 3 3 3 4 4 4
D_3	4	8	1	0 4	2	0	0	0 0 0 0 0 0 0 0 1 2 3 4

TABLE 4.2 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_4	4	4	2	4 0	2	0	1	0 0 0 0 1 1 2 2 3 4 3 4
D_5	6	6	3	2 2	2	2	1	0 0 0 0 1 2 1 1 0 0 3 3 2 2 3 4 4 4
D_6	7	5	4	3 1	2	2	2	0 0 0 0 1 1 2 1 1 2 0 2 3 3 2 3 3 4 4 4 4
D_7	8	8	4	4 2	3	1	2	0 0 0 0 0 0 1 1 1 2 2 2 0 0 2 3 4 3 4 4 1 3 3 4
D_8	11	9	6	3 3	3	3	3	0 0 0 0 0 0 1 1 1 1 2 3 0 0 0 2 2 2 4 4 4 1 2 3 3 3 4 1 2 3 3 4 4

PROOF:

Before going in the detail of the proof, note that $\lambda_0 - 3\lambda_1$ and $3\lambda_1 - \lambda_0$ are even integers (Remark of LEMMA 2.4). This implies that λ_1 is even if λ_0 is even, and λ_1 is odd if λ_0 is odd.

CASE 1. $\lambda_0 = 0 \bmod(3)$.

(a) $\lambda_2 = \lambda_1 - 1$

Note that $\lambda_1 \geq 1$

(i) let $D^* = f_1\ D_1 \cup f_3\ D_3 \cup f_4\ D_4$ with,

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1 + 2}{2}, \quad f_4 = \lambda_1 - 1$$

Note the following,

- the quantity $\lambda_0 - 3\lambda_1 + 2$ is even (Remark in Lemma 2.4), therefore, f_3 is an integer.
- $f_3 \geq 0$ if $\lambda_1 \leq \lambda_0/3$
- $b^* = f_1.2 + f_3.4 + f_4.4 = 2.\lambda_0$

Therefore, $b \geq b^*$ (Remark in Lemma 2.3)

Hence, D is either equivalent (if $b = b^*$) or S-inadmissible (if $b > b^*$) with respect to D^* for,

$$\lambda_0 = 0 \bmod(3),\ \lambda_2 = \lambda_1 - 1,\ \lambda_1 \leq \lambda_0/3.$$

(ii) let $D^* = f_1\ D_1 \cup f_2\ D_2 \cup f_4\ D_4 \cup f_6\ D_6$, with

$$f_1 = \frac{\lambda_0}{3}, \quad f_2 = \frac{3\lambda_1 - \lambda_0 - 6}{6}, \quad f_4 = \frac{\lambda_0 - 3}{3}, \quad f_6 = 1.$$

Note that,

- quantities λ_0 and $\lambda_0 - 3$ are both divisible by 3. Hence, f_1 and f_4 are integers.
- $f_2 \geq 0$, if $\lambda_1 \geq \frac{\lambda_0}{3} + 2$.
- $\lambda_0 = 3 \bmod(6)$ implies that $\lambda_1 = 1 \bmod(2)$ for f_2 to be integer, and $\lambda_0 = 0 \bmod(6) > 0$ implies that

$\lambda_1 = 0 \mod(2)$ for f_2 to be integer.

This includes all possible values of λ_0 and λ_1 for the design except $\lambda_0 = 0$, $\lambda_1 = 0 \mod(2)$.

□ $b^* = f_1.2 + f_2.4 + f_4.4 + 1.7 = \frac{2}{3}(2\lambda_0 + 3\lambda_1) - 1$

□ Now, by Remark of Lemma 2.1

$$b \geq \frac{2}{3}(2\lambda_0 + 3\lambda_1) - \frac{4}{3}$$

It is easy to see that $\frac{2}{3}(2\lambda_0 + 3\lambda_1)$ is an integer. Therefore, right hand side of the above inequality is not integer. The next higher integer is the same as b^*. Hence,

$$b \geq b^*$$

The design D is , therefore, equivalent or S-inadmissible with respect to D^*, except in the following case,

□ $\lambda_0 = 0$ and $\lambda_1 = 0 \mod(2) \geq 2$

In such a case Lemma 2.5 is applicable and hence, the only designs exist are those for which $\lambda_2 = \lambda_1$. So, No design with the above mentioned parameter values and $\lambda_2 = \lambda_1 - 1$ exists. So, the possibility of $\lambda_0 = 0$ is ruled out.

Hence, D is equivalent or S-admissible with respect to D^* for,

$$\lambda_0 = 0 \mod(3),\ \lambda_2 = \lambda_1 - 1,\ \lambda_1 \geq \frac{\lambda_0}{3} + 2.$$

Only one case when $\lambda_1 = 1 + \lambda_0/3$ is also not covered here. It can be shown by using Remark of Lemma 2.2

that r is not integer ($r = \lambda_0 + 1/2$) in such a case and hence, there do not exist designs with $\lambda_1 = 1 + \frac{\lambda_0}{3}$ and $\lambda_2 = \lambda_1 - 1$.

(b) $\underline{\lambda_2 = \lambda_1}$

(i) let $D^* = f_1 D_1 \cup f_3 D_3 \cup f_4 D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1}{2}, \quad f_4 = \lambda_1$$

Note the following,

- $\lambda_0 - 3\lambda_1$ is even by Lemma 2.4. Therefore, f_3 is integer.
- $\lambda_1 \le \lambda_0/3$ in order to have $f_3 \ge 0$.
- $b^* = f_1.2 + f_3.4 + f_4.4 = 2\lambda_0$

Therefore, $b \ge b^*$ by Lemma 2.3.

(ii) let $D^* = f_1 D_1 \cup f_2 D_2 \cup f_4 D_4$, with

$$f_1 = \lambda_0/3, \quad f_2 = \frac{3\lambda_1 - \lambda_0}{6}, \quad f_4 = \lambda_0/3$$

Note the following,

- $\lambda_1 \ge \lambda_0/3$ in order to have $f_2 \ge 0$.
- $\lambda_0 = 0 \bmod(6)$ implies that $\lambda_1 = 0 \bmod(2)$ for f_2 to be integer, and $\lambda_0 = 3 \bmod(6)$ implies that $\lambda_1 = 1 \bmod(2)$ for f_2 to be integer.
- By Lemma 2.1, b^* requires the minimum number of blocks because D_1, D_2, and D_4 all are completely binary. Hence, $b \ge b^*$.

By (i) and (ii) above it is clear that D is either equivalent to or S-inadmissible with respect to a designwhich is union of copies of designs from Table 4.2 for,

$$\lambda_0 = 0 \bmod(3) \text{ and } \lambda_2 = \lambda_1 .$$

(c) $\underline{\lambda_2 = \lambda_1 + 1}$

(i) let $D^* = f_1D_1 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1 - 2}{2}, \quad f_4 = \lambda_1 + 1$$

Note the following,

- $\lambda_0 - 3\lambda_1 - 2$ is even by Lemma 2.4.It implies that f_3 is integer.
- $f_3 \geq 0$ iff $\lambda_1 \leq \frac{\lambda_0 - 2}{3}$. But $\frac{\lambda_0 - 2}{3}$ is not integer, therefore,

$$\lambda_1 \leq \frac{\lambda_0 - 3}{3} \quad \text{(next lower integer)}$$

- $b^* = 2\lambda_0$

Therefore, $b \geq b^*$ (Lemma 2.3)

(ii) let $D^* = f_1D_1 \cup f_2D_2$, with,

$$f_1 = \lambda_0, \quad f_2 = \frac{\lambda_1 - \lambda_0}{2}$$

Note the following,

- $\lambda_1 - \lambda_0$ is even by Lemma 2.4 $\Rightarrow$ f_3 is integer.
- $f_2 \geq 0 \Rightarrow \lambda_1 \geq \lambda_0$
- D^* contains minimum number of blocks because D_1 and D_2 are completely binary (Lemma 2.1).

Hence, $b \geq b^*$.

(iii) let $D^* = f_1D_1 \cup f_2D_2 \cup f_4D_4 \cup f_7D_7$, with

$$f_1 = \frac{\lambda_0 - 3}{3},\ f_2 = \frac{3\lambda_1 - \lambda_0}{6},\ f_4 = \frac{\lambda_0 - 3}{3},\ f_7 = 1$$

Note the following,

- $\lambda_0 - 3$ is divisible by 3 $\Rightarrow$ f_1 and f_4 are integers
- $f_2 \geq 0$ iff $\lambda_1 \geq \lambda_0/3$
- $\lambda_0 = 3 \bmod(6) \Rightarrow \lambda_1 = 1 \bmod(2)$ for f_2 to be integer $\lambda_0 = 0 \bmod(6) > 0 \Rightarrow \lambda_1 = 0 \bmod(2)$ for f_2 to be integer.
- the cases $\lambda_0 = 0$, $\lambda_1 \geq 0$ are covered in (ii) above.
- $b^* = \frac{2}{3}(2\lambda_0 + 3\lambda_1 + 2)$

Whereas, $b \geq \frac{2}{3}(2\lambda_0 + 3\lambda_1 + 2)$ by Lemma 2.1.

Therefore, $b \geq b^*$

Hence, D is either equivalent or S-inadmissible with respect to a design which is a union of copies of designs from Table 4.1 for,

$$\lambda_0 = 0 \bmod(3),\ \lambda_2 = \lambda_1 + 1$$

So, (a),(b) and (c) covers all possibilities for,

$$\lambda_0 = 0 \bmod(3).$$

CASE 2. $\lambda_0 = 1 \bmod(3)$

(a) $\lambda_2 = \lambda_1 - 1$

Note that $\lambda_1 \geq 1$

(i) let $D^* = f_1D_1 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1 + 2}{2}, \quad f_4 = \lambda_1 - 1$$

Note the following,

- $\lambda_0 - 3\lambda_1 + 2$ is even (Lemma 2.4) $\Rightarrow$ f_3 is integer
- $f_3 \geq 0$ iff $\lambda_1 \leq \lambda_0/3$

  Since $\lambda_0/3$ is not integer, $\lambda_1 \leq \dfrac{\lambda_0 - 1}{3}$
- $b^* = 2\lambda_0$

Hence, $b \geq b^*$ (Lemma 2.3)

(ii) let $D^* = f_1D_1 \cup f_2D_2 \cup f_4D_4$, with

$$f_1 = \frac{\lambda_0 + 2}{3}, \quad f_2 = \frac{3\lambda_1 - \lambda_0 - 2}{6}, \quad f_4 = \frac{\lambda_0 - 1}{3}$$

Note the following,

- Since, $\lambda_0 + 2$ and $\lambda_0 - 1$ are both divisible by 3, f_1 and f_4 are integers.
- $f_2 \geq 0$ iff $\lambda_1 \leq \dfrac{\lambda_0 + 2}{3}$

□ $\lambda_0 = 1 \bmod(6)$ and $\lambda_1 = 1 \bmod(2)$ $\Rightarrow$ f_2 is integer.

$\lambda_0 = 4 \bmod(6)$ and $\lambda_1 = 0 \bmod(2) \geq 2$ $\Rightarrow$ f_2 is integer.

□ b^* requires minimum number of blocks because D_1, D_2 and D_4 all are completely binary (Lemma 2.1).

Hence, $b \geq b^*$.

Cases (i) and (ii) covers the whole range for,

$$\lambda_0 = 1 \bmod(3) \text{ and } \lambda_2 = \lambda_1 - 1.$$

(b) $\underline{\lambda_2 = \lambda_1}$

(i) let $D^* = f_1D_1 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1}{2}, \quad f_4 = \lambda_1$$

Note the following,

□ $\lambda_0 - 3\lambda_1$ is even (Lemma 2.4) $\Rightarrow$ f_3 is integer.

□ $f_3 \geq 0$ iff $\lambda_1 \leq \dfrac{\lambda_0 - 1}{3}$ (in fact $\lambda_1 \leq \lambda_0/3$, but $\lambda_0/3$ is not integer)

□ $b^* = 2\lambda_0$

Hence, $b \geq b^*$ (Lemma 2.3)

(ii) let $D^* = f_1D_1 \cup f_2D_2 \cup f_4D_4 \cup f_7D_7$, with

$$f_1 = \frac{\lambda_0 - 1}{3}, \quad f_2 = \frac{3\lambda_1 - \lambda_0 - 2}{6}, \quad f_4 = \frac{\lambda_0 - 4}{3}, f_7 = 1$$

Note the following,

- Since, $\lambda_0 - 1$ nd $\lambda_0 - 4$ are divisible by 3, f_1 and f_4 are integers.
- $f_2 \geq 0$ iff $\lambda_1 \geq \frac{\lambda_0 + 2}{3}$
- $\lambda_0 = 1 \bmod(6) > 1$, $\lambda_1 = 1 \bmod(2) > 1$ for f_2 to be integer.
- $\lambda_0 = 4 \bmod(6)$, $\lambda_1 = 0 \bmod(2)$ for f_2 to be integer.
- The case when $\lambda_0 = 1$ is not covered above. But it is clear from Lemma 2.6 that for $\lambda_0 = 1$ a design with $\lambda_2 = \lambda_1$ does not exists.
- $b^* = \frac{2}{3}(2\lambda_0 + 3\lambda_1 + 1)$.

  This is the next higher integer than the lower bound $\frac{2}{3}(2\lambda_0 + 3\lambda_1)$ on b given by Lemma 2.1.

Hence, $b \geq b^*$.

Cases (i),(ii) covers the whole range for,

$$\lambda_0 = 1 \bmod(3) \text{ and } \lambda_2 = \lambda_1$$

(c) $\underline{\lambda_2 = \lambda_1 + 1}$

(i) let $D^* = f_1 D_1 \cup f_3 D_3 \cup f_4 D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1 - 2}{2}, \quad f_4 = \lambda_1 + 1$$

Note the following,

- f_3 is integer because $\lambda_0 - 3\lambda_1 - 2$ is even (Lemma 2.4)

□ $f_3 \geq 0$ iff $\lambda_1 \leq \frac{\lambda_0 - 4}{3}$ (an integer)

□ $b^* = 2\lambda_0$

Hence, $b \geq b^*$ (Lemma 2.3)

(ii) let $D^* = f_1D_1 \cup f_2D_2 \cup f_4D_4 \cup f_6D_6$, with

$$f_1 = \frac{\lambda_0 - 4}{3},\ f_2 = \frac{3\lambda_1 - \lambda_0 - 2}{6},\ f_4 = \frac{\lambda_0 - 1}{3},\ f_6 = 1$$

Note that,

□ $\lambda_0 - 4$ and $\lambda_0 - 1$ are both divisible by 3, hence, f_1 and f_4 are integers.

□ $f_2 \geq 0$ iff $\lambda_1 \geq \frac{\lambda_0 + 2}{3}$

□ $\lambda_0 = 1 \bmod(6)$, $\lambda_1 = 1 \bmod(2)$ for f_2 to be integer
$\lambda_0 = 4 \bmod(6)$, $\lambda_1 = 0 \bmod(2) \geq 2$ for f_2 to be integer.

□ $b^* = \frac{2}{3}(2\lambda_0 + 3\lambda_1 - 8)$
b^* is smaller than the lower bound $\frac{2}{3}(2\lambda_0 + 3\lambda_1 + 2)$ on b given by Lemma 2.1.

Hence, $b \geq b^*$

The case which is not covered above is when $\lambda_1 = \frac{\lambda_0 - 1}{3}$.

In such a case $r = \lambda_0 + 1/2$ (from Lemma 2.2) which is not integer, and therefore, such a design does not exists.

All the possibilities are, therefore, covered by (i),(ii) above for,

$$\lambda_0 = 1 \bmod(3) \text{ and } \lambda_2 = \lambda_1 + 1$$

<u>CASE 3.</u> $\lambda_0 = 2 \bmod(3)$

(a) <u>$\lambda_2 = \lambda_1 - 1$</u>

Note that $\lambda_1 \geq 1$

(i) let $D^* = f_2D_2 \cup f_5D_5$, with

$$f_2 = \frac{\lambda_1 - 2}{2}, \quad f_5 = 1$$

Note the following,

- $f_2 \geq 0$ and integer if $\lambda_1 = 0 \bmod(2) \geq 2$
- $\lambda_0 = 2$
- $b^* = 2(\lambda_1 + 1)$
- $b \geq 2\lambda_1 + 4/3$ (Lemma 2.1)

  This lower bound on b is not an integer. therefore, the next higher possible integer is $2(\lambda_1 + 2)$ which is larger than b^*.

Hence, $b > b^*$

(ii) let $D^* = f_1D_1 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1 + 2}{2}, \quad f_4 = \lambda_1 - 1$$

Note the following,

- $\lambda_0 - 3\lambda_1 + 2$ is even by Lemma 2.4, i.e. f_3 is integer.

- $f_3 \geq 0$ iff $\lambda_1 \leq \frac{\lambda_0 - 2}{3}$
- $b^* = 2\lambda_0$

Therefore, $b \geq b^*$ (Lemma 2.3)

(iii) let $D^* = f_1 D_1 \cup f_2 D_2 \cup f_4 D_4 \cup f_7 D_7$, with

$$f_1 = \frac{\lambda_0 + 1}{3}, f_2 = \frac{3\lambda_1 - \lambda_0 - 4}{6}, f_4 = \frac{\lambda_0 - 5}{3}, f_7 = 1$$

Note the following,

- $\lambda_0 + 1$ and $\lambda_0 - 5$ are divisible by 3, i.e. f_1 and f_4 are integers.
- $f_2 \geq 0$ iff $\lambda_1 \geq \frac{\lambda_0 + 4}{3}$
- $\lambda_0 = 2 \bmod(6) > 2$, $\lambda_1 = 0 \bmod(2) > 2$ for f_2 to be integer.

  $\lambda_0 = 5 \bmod(6)$, $\lambda_1 = 1 \bmod(2) > 1$ for f_2 to be integer.
- A case not covered above is for $\lambda_0 = 2$ and $\lambda_1 = 0 \bmod(2)$.This is already covered in (i) above.
- $b^* = \frac{2}{3}(2\lambda_0 + 3\lambda_1 - 1)$
- $b \geq \frac{2}{3}(2\lambda_0 + 3\lambda_1 - 2)$ by Lemma 2.1. This lower bound is not integer and b^* is smaller than the next higher integer $\frac{2}{3}(2\lambda_0 + 3\lambda_1 + 2)$.

Hence, $b > b^*$

The case for $\lambda_1 = \frac{\lambda_0 + 1}{3}$ can be discarded as

$r = \lambda_0 - 1/2$ (Lemma 2.2) is not an integer and such designs do not exist.

This covers all the possibilities for,

$$\lambda_0 = 2 \bmod(3) \text{ and } \lambda_2 = \lambda_1 - 1$$

(b) $\underline{\lambda_2 = \lambda_1}$

(i) let $D^* = f_1D_1 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = \lambda_1, \quad f_3 = \frac{\lambda_0 - 3\lambda_1}{2}, \quad f_4 = \lambda_1$$

Note that,

- $\lambda_0 - 3\lambda_1$ is even, therefore, f_3 is integer
- $f_3 \geq 0$ iff $\lambda_1 \leq \dfrac{\lambda_0 - 2}{3}$ (an integer)
- $b^* = 2\lambda_0$

Hence, $b \geq b^*$ (Lemma 2.3)

(iii) let $D^* = f_1D_1 \cup f_2D_2 \cup f_4D_4 \cup f_6D_6$, with

$$f_1 = \frac{\lambda_0 - 2}{3}, \quad f_2 = \frac{3\lambda_1 - \lambda_0 - 4}{6}, \quad f_4 = \frac{\lambda_0 - 2}{3}, \quad f_6 = 1$$

Note that,

- $\lambda_0 - 2$ is divisible by 3, hence, f_1 and f_4 are integers.
- $f_2 \geq 0$ iff $\lambda_1 \geq \dfrac{\lambda_0 + 4}{3}$

□ $\lambda_0 = 2 \bmod(6)$, $\lambda_1 = 0 \bmod(2) > 0$ for f_2 to be integer.

$\lambda_0 = 5 \bmod(6)$, $\lambda_1 = 1 \bmod(3) > 1$ for f_2 to be integer.

□ $b^* = \frac{2}{3}\,(\,2\lambda_0 + 3\lambda_1 + \frac{1}{2}\,)$

□ $b \geq \frac{2}{3}\,(\,2\lambda_0 + 3\lambda_1\,)$ by Lemma 2.1, but the lower bound is not an integer. The next higher integer than this lower bound is $\frac{2}{3}\,(\,2\lambda_0 + 3\lambda_1 + 2)$ which is larger then b^*.

Hence, $b \geq b^*$

The uncovered case for $\lambda_1 = \dfrac{\lambda_0 + 1}{3}$ gives $r = \lambda_0 + 1/2$ (Lemma 2.2) which is not an integer and, therefore, the designs do not exist.

All the possibilities has,therefore, been considered for,

$$\lambda_0 = 2 \bmod(3),\ \lambda_2 = \lambda_1$$

(c) $\underline{\lambda_2 = \lambda_1 + 1}$

(i) let $D^* = f_1D_1 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = \lambda_1,\ f_3 = \frac{\lambda_0 - 3\lambda_1 - 2}{2},\ f_4 = \lambda_1 + 1$$

Note the following,

□ $\lambda_0 - 3\lambda_1 - 2$ is even (Lemma 2.4), i.e. f_3 is integer.

□ $f_3 \geq 0$ iff $\lambda_1 \leq \dfrac{\lambda_0 - 2}{3}$

$b^* = 2\lambda_0$

therefore, $b \geq b^*$ (Lemma 2.3)

(ii) let $D^* = f_1 D_1 \cup f_2 D_2 \cup f_4 D_4$, with

$$f_1 = \frac{\lambda_0 - 2}{3}, \quad f_2 = \frac{3\lambda_1 - \lambda_0 + 2}{6}, \quad f_4 = \frac{\lambda_0 + 1}{3}$$

Note that,

- $\lambda_0 - 2$ and $\lambda_0 + 1$ are divisible by 3, i.e. f_1 and f_4 are integers.
- $f_2 \geq 0$ iff $\lambda_1 \geq \dfrac{\lambda_0 - 2}{3}$
- $\lambda_0 = 2 \bmod(6)$, $\lambda_1 = 0 \bmod(2)$ for f_2 to be integer.
  $\lambda_0 = 5 \bmod(6)$, $\lambda_1 = 1 \bmod(2)$ for f_2 to be integer.
- $b^* = \frac{2}{3}(2\lambda_0 + 3\lambda_1 + 2)$
- $b \geq \frac{2}{3}(2\lambda_0 + 3\lambda_1 + 2)$ (Lemma 2.1)

Hence, $b \geq b^*$

The cases (i),(ii) cover all the possibilities for,

$$\lambda_0 = 2 \bmod(3), \lambda_2 = \lambda_1 + 1$$

CASE 1, CASE 2, and CASE 3 cover all the possible values of λ_0 (= 0,1,2,3,.......), of λ_1 (= 0,1,2,3,.......) and of λ_2 ($= \lambda_1 + \delta$, $\delta = -1, 0, 1$) and in each case it is shown that an arbitrary design D is either equivalent or S-inadmissible with respect to a design D^* which is the union of copies of generator designs from Table 4.2. Furthermore, it can be noted that none of the designs in Table 4.2 is equivalent to any other design in the table, and all the designs in Table 4.2, except the design D_8 , are used in the construction of design D for all possible combinations of parameters λ_0, λ_1, λ_2. The design D_8 in Table 4.2 is a generator design and since it is not equivalent or S-inadmissible with respect to the union of any number of copies of any combination of other generator designs, it has to be included in the list.

It is, therefore, concluded that the designs in Table 4.2 constitute a minimal complete class of generator designs for k = 3, v = 4. Hence, the Theorem is proved.

4.3 MCCGD FOR k = 3, v = 6 :

THEOREM 4.3

For k = 3 and v = 6 the minimal complete class of generator designs is given in Table 4.3.

TABLE 4.3 MCCGD FOR k = 3, v = 6.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	2	0	1	0 0	0	1	0	1 4 2 5 3 6

<u>TABLE 4.3 (CONTINUED).</u>

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_2	6	12	1	0 6	2	0	0	0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6
D_3	6	6	2	6 0	2	1	0	0 0 0 0 0 0 1 1 2 4 4 5 2 3 3 5 6 6
D_4	7	3	3	3 0	1	1	1	0 0 0 1 1 2 2 1 3 5 3 4 3 4 2 4 6 5 6 6 5
D_5	9	9	3	9 0	3	0	1	0 0 0 0 0 0 0 0 0 1 1 1 2 2 2 3 3 3 4 5 6 4 5 6 4 5 6
D_6	10	0	5	0 0	0	2	2	1 1 1 1 1 2 2 2 3 2 2 3 3 4 3 3 4 5 6 5 4 6 5 4 5 6 6 / 4 5 6
D_7	12	6	5	6 0	2	1	2	0 0 0 0 0 0 1 1 1 1 1 2 2 3 3 2 3 5 4 5 4 6 5 6 6 4 6 / 2 2 3 3 4 4 5 5 6
D_8	15	3	7	3 0	1	2	3	0 0 0 1 1 1 1 1 1 1 2 3 2 2 3 3 4 5 4 5 6 4 6 5 6 5 6 / 2 2 2 2 3 3 3 3 4 5 4 4 4 5 6 6 5 6
D_9	15	9	6	3 3	2	2	2	0 0 0 0 0 0 1 1 1 1 1 2 0 0 0 2 3 4 2 3 3 4 5 6 4 5 6 / 1 2 2 2 3 3 5 3 4 5 4 4 6 6 5 6 5 6

TABLE 4.3 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_{10}	18	0	9	0 0	0	3	4	1 1 1 1 1 1 1 1 1 2 2 2 3 3 3 4 4 5 4 5 6 4 5 6 5 6 6 2 2 2 2 2 2 3 3 3 3 3 3 4 4 5 4 4 5 4 5 6 5 6 6 5 6 6

4.4 MCCGD FOR k = 3, v = 8 :

THEOREM 4.4

For k = 3 and v = 8 the minimal complete class of generator designs is given in Table 4.4.

TABLE 4.4 MCCGD FOR k = 3, v = 8.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	8	16	1	0 8	2	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8
D_2	12	4	4	4 0	1	1	1	0 0 0 0 1 1 1 2 2 1 3 5 7 3 4 6 3 4 2 4 6 8 5 8 7 7 6 2 3 4 5 6 5 8 8 7
D_3	12	12	3	12 0	3	1	0	0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 5 5 5 2 3 4 3 4 4 6 7 8 0 0 0 6 6 7 7 8 8

TABLE 4.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_4	16	16	4	160	4	0	1	015 016 017 018 025 026 027 028 035 036 037 038 045 046 047 048
D_5	18	14	5	102	3	1	1	014 056 024 027 028 036 034 038 067 078 001 005 126 137 158 235 457 468
D_6	19	9	6	71	2	2	1	012 013 023 056 057 068 078 004 124 134 156 178 234 257 268 358 367 458 467
D_7	23	13	7	111	3	1	2	015 016 017 026 027 028 035 037 038 046 058 004 125 136 178 148 237 248 256 345 368 467 457
D_8	23	21	6	191	5	1	1	014 015 016 017 018 024 025 026 027 028 034 035 036 037 038 045 046 057 067 008 123 568 478

TABLE 4.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
D_9	24	0	9	0	0	2	3	1	1	1	1	1	1	1	1	1
				0				2	2	3	3	4	4	5	5	6
								5	6	7	8	6	7	8	7	8
								2	2	2	2	2	2	2	3	3
								3	3	4	4	6	5	7	4	4
								5	6	7	8	7	8	8	5	8
								3	3	3	4	4	4			
								5	6	6	5	5	7			
								7	7	8	6	6	8			
D_{10}	25	11	8	5	2	2	2	0	0	0	0	0	0	0	0	1
				3				4	4	5	5	6	0	0	0	2
								6	7	7	8	8	1	2	3	5
								1	1	1	1	1	1	2	2	2
								2	3	3	4	4	7	3	3	6
								6	5	6	7	8	8	7	8	7
								2	2	3	3	3	5	5		
								4	4	4	4	7	6	6		
								5	8	5	6	8	7	8		
D_{11}	30	10	10	6	2	2	3	0	0	0	0	0	0	0	0	1
				2				1	1	2	2	3	3	0	0	2
								5	6	7	5	6	7	4	8	7
								1	1	1	1	1	1	1	2	2
								2	3	3	4	4	6	7	3	3
								8	5	6	5	8	7	8	5	6
								2	2	2	2	3	3	3	3	4
								4	4	5	6	4	4	5	7	5
								7	6	8	8	7	8	8	8	6
								4	4	5						
								5	6	6						
								7	8	7						
D_{12}	32	0	12	0	0	4	3	1	1	1	1	1	1	1	1	1
				0				2	2	2	2	3	3	3	4	4
								3	4	5	6	7	8	4	5	6
								1	1	1	2	2	2	2	2	2
								5	6	7	3	3	3	4	4	5
								7	8	8	4	5	6	7	8	8
								2	2	3	3	3	3	3	4	4
								6	7	4	4	5	5	6	5	5
								7	8	5	8	6	7	8	6	8

TABLE 4.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								4 5 5 5 6 6 6 6 7 7 7 7 8 8 8
D_{13}	38	10	13	6 2	2	4	3	0 0 0 0 0 0 0 0 1 1 1 2 5 5 6 0 0 2 2 3 3 6 7 7 4 8 3 1 1 1 1 1 1 1 1 1 2 2 3 4 4 5 5 6 6 3 4 4 8 5 6 7 7 8 1 2 2 2 2 2 2 2 2 7 3 4 4 5 6 6 5 7 8 4 5 7 6 7 8 8 8 3 3 3 3 3 3 3 4 4 4 4 5 5 5 6 7 5 6 6 6 7 8 7 8 8 8 7 4 5 7 6 8 8

4.5 MCCGD FOR k = 3, v = 10 :

THEOREM 4.5

For k = 3 and v = 10 the minimal complete class of generator designs is given in Table 4.5.

TABLE 4.5 MCCGD FOR k = 3, v = 10 .

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	10	20	1	0 10	2	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8 9

TABLE 4.5 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								0 0 10
D_2	14	12	3	8 2	2	1	0	0 0 0 0 0 0 0 0 0 1 1 2 3 6 6 7 8 0 2 3 4 4 7 8 9 9 5 0 1 2 6 7 0 4 3 9 8 10 5 5 10 10
D_3	20	20	4	20 0	4	1	0	0 0 0 0 0 0 0 0 0 1 1 1 1 2 2 2 3 3 2 3 4 5 3 4 5 4 5 0 0 0 0 0 0 0 0 0 4 6 6 6 6 7 7 7 8 5 7 8 9 10 8 9 10 9 0 0 8 9 10 10
D_4	25	5	7	5 0	1	2	1	0 0 0 0 0 1 1 1 1 1 3 6 8 5 2 3 4 5 2 4 7 9 10 3 4 5 6 1 1 2 2 2 2 2 3 3 7 9 3 4 4 7 8 5 6 8 10 5 5 6 9 10 8 9 3 4 4 5 6 6 6 7 7 9 7 7 8 8 10 8 10 9 10 9 10
D_5	25	15	6	15 0	3	1	1	0 0 0 0 0 0 0 0 0 1 1 1 2 2 2 3 3 4 3 5 8 4 7 8 5 9 6 0 0 0 0 0 0 1 1 1 4 5 6 6 7 7 2 4 6 10 10 9 10 8 9 10 9 7 2 2 3 3 4 5 8 3 5 4 7 5 6 9 6 9 8 10 7 8 10

TABLE 4.5 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_6	25	25	5	25 0	5	0	1	0 0 0 0 0 0 0 0 0 1 1 1 1 1 2 2 2 2 6 7 8 9 10 6 7 8 9
								0 0 0 0 0 0 0 0 0 2 3 3 3 3 3 4 4 4 10 6 7 8 9 10 6 7 8
								0 0 0 0 0 0 0 4 4 5 5 5 5 5 9 10 6 7 8 9 10
D_7	30	0	9	0 0	0	2	2	1 1 1 1 1 1 1 1 1 2 2 2 3 4 5 6 7 9 3 6 7 8 5 8 9 10 10
								2 2 2 2 2 2 2 3 3 3 4 4 5 5 7 8 4 4 9 6 10 7 9 8 10 5 8
								3 3 3 3 4 4 4 5 5 5 6 6 7 6 7 9 6 6 10 7 9 10 8 9 10 7 10
								5 6 7 8 8 8 9 10 9
D_8	30	10	8	10 0	2	1	2	0 0 0 0 0 0 0 0 0 1 1 2 2 3 3 4 4 5 6 7 8 9 10 6 7 8 9
								0 1 1 1 1 1 1 2 2 5 2 3 4 5 7 9 3 4 10 10 8 9 6 8 10 7 6
								2 2 2 3 3 3 3 4 4 5 6 8 4 5 6 8 5 6 7 9 10 10 9 7 9 8 10
								4 5 5 7 7 6 9 10 8
D_9	32	16	8	14 1	3	2	1	0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 5 5 5 2 3 4 3 4 4 6 7 8

TABLE 4.5 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
								0	0	0	0	0	0	1	1	1
								6	6	7	7	9	0	2	3	4
								9	8	9	8	10	10	5	5	6
								1	1	2	2	2	2	3	3	3
								7	8	3	4	5	8	4	7	8
								9	10	6	7	9	10	5	10	9
								4	4	6	6	6				
								5	8	7	7	9				
								10	9	8	10	10				
D_{10}	32	26	7	24	5	1	1	0	0	0	0	0	0	0	0	0
				1				1	1	1	1	1	2	2	2	2
								3	4	5	8	9	4	5	7	8
								0	0	0	0	0	0	0	0	0
								2	3	3	3	3	4	4	4	5
								9	5	7	9	10	6	7	9	6
								0	0	0	0	0	0	0	1	1
								5	6	6	6	7	8	0	2	6
								7	8	9	10	8	10	10	10	7
								2	3	4	5	7				
								3	4	5	8	9				
								6	8	10	9	10				
D_{11}	37	11	10	9	2	2	2	0	0	0	0	0	0	0	0	0
				1				1	1	2	2	3	4	6	7	8
								3	5	4	7	5	6	9	8	9
								0	1	1	1	1	1	1	1	1
								0	2	2	3	4	4	5	6	6
								10	8	8	10	9	9	10	7	7
								2	2	2	2	2	2	3	3	3
								3	3	4	5	5	7	4	4	5
								6	6	10	9	9	10	8	8	10
								3	3	4	4	4	5	5	6	7
								7	7	5	5	6	6	6	9	8
								9	9	7	7	10	8	8	10	10
								8								
								9								
								10								

TABLE 4.5 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_{12}	37	21	9	19 1	4	1	2	0 0 0 0 0 0 0 0 0 1 1 1 1 2 2 2 2 3 6 7 8 9 6 8 9 10 7
								0 0 0 0 0 0 0 0 0 3 3 3 4 4 4 4 5 5 8 9 10 6 7 8 5 6 7
								0 0 1 1 1 1 1 2 2 5 0 2 3 4 5 7 3 4 9 10 7 6 9 10 8 7 6
								2 2 3 3 3 4 4 5 5 5 7 4 5 6 7 9 6 8 8 9 8 9 10 10 10 7 10
								6 8 9
D_{13}	44	32	10	28 2	6	1	2	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 6 7 8 9 10 6 6 7 8
								0 0 0 0 0 0 0 0 0 2 2 2 3 3 3 3 3 3 9 10 7 6 7 8 9 10 8
								0 0 0 0 0 0 0 0 0 4 4 4 4 4 4 5 5 5 6 7 8 9 10 9 6 7 8
								0 0 0 1 1 1 1 2 2 5 0 0 2 3 4 7 3 4 9 5 10 8 9 5 10 6 10
								2 3 3 4 5 5 6 7 5 4 5 6 6 8 9 8 9 7 10 8 7 10 10 9
D_{14}	49	17	13	13 2	3	2	3	0 0 0 0 0 0 0 0 0 1 1 1 2 2 2 3 3 3 6 7 8 7 8 9 8 9 10
								0 0 0 0 0 0 1 1 1 4 4 4 5 0 0 2 2 3 6 7 9 6 5 10 9 10 6

TABLE 4.5 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								1 1 1 1 1 1 1 2 2 3 4 4 5 5 6 8 3 3 7 8 9 10 7 9 10 6 7
								2 2 2 2 2 2 3 3 3 4 4 5 5 6 8 4 4 5 8 9 10 6 7 10 10 6 8
								3 3 3 4 4 4 4 5 5 5 7 9 5 5 6 7 6 7 9 8 10 7 10 8 10 8 9
								5 6 6 7 8 7 9 8 9 10 10 9
D_{15}	56	18	15	12 3	3	3	3	0 0 0 0 0 0 0 0 0 1 2 3 4 4 4 8 5 5 7 7 7 8 5 10 9 9 8
								0 0 0 0 0 0 1 1 1 6 6 6 0 0 0 2 2 2 10 9 10 1 2 3 6 8 9
								1 1 1 1 1 1 1 1 1 3 3 3 4 4 4 5 5 5 10 6 8 7 6 8 7 9 10
								1 2 2 2 2 2 2 2 2 9 3 3 3 4 4 4 5 5 10 9 10 7 10 7 9 6 8
								2 2 3 3 3 3 3 3 3 5 8 4 4 4 5 5 5 8 6 10 6 8 9 7 6 10 9
								4 4 4 5 6 6 6 6 7 5 5 6 8 7 7 7 8 8 7 10 9 9 8 8 9 10 10
								7 7 9 9 10 10

CHAPTER 5

5.1 MCCGD FOR $k = 4$, $v = 4$:

THEOREM 5.1

For $k = 4$ and $v = 4$ the minimal complete class of generator designs is given in Table 5.1.

TABLE 5.1 MCCGD FOR $k = 4$, $v = 4$.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	1	0	1	0 0 0	0	1	1	1 2 3 4
D_2	2	4	1	0 2 0	2	1	0	0 0 0 0 1 3 2 4
D_3	4	8	2	0 4 0	4	0	1	0 0 0 0 0 0 0 0 1 1 2 2 3 4 3 4
D_4	4	4	3	4 0 0	3	2	2	0 0 0 0 1 1 1 2 2 2 3 3 3 4 4 4

PROOF:

The procedure for proving the class to be minimal class is same as discussed in Section 4.1. The lower bound on B is, however, obtained by using the

FACT 2.6 and FACT 2.7, i.e. by finding an index j_0 which minimizes $| N - M_{j_0} |$, where N is the total number of TT-pairs in the design and M_{j_0} is the number of TT-pairs in the j_0 th BB combination with b_{j_0} blocks in it.

CASE 1: $\lambda_0 = 0 \bmod(3)$

According to the Lemma 2.7, $\lambda_2 = \lambda_1$, so there is no need to consider the cases when $\lambda_2 = \lambda_1 \pm 1$. Further note that, $N = 6\lambda_1$ (FACT 2.3).

(a) Consider $3\lambda_1 - 2\lambda_0 \geq 0$.

Let $D^* = f_1D_1 \cup f_3D_3$, with

$$f_1 = \frac{3\lambda_1 - 2\lambda_0}{3}, \quad f_3 = \lambda_0/3$$

Since $\lambda_0 = 0 \bmod(3)$, f_1 and f_3 are both integers. Furthermore,

$$b^* = \frac{3\lambda_1 + 2\lambda_0}{3}$$

By Lemma 2.1,

$$b \geq \frac{3\lambda_1 + 2\lambda_0}{3} = b^*$$

Hence, D is either equivalent or S-inadmissible with respect to D^*.

(b) Consider $3\lambda_1 - 2\lambda_0 < 0$.

Note that $6\lambda_1 - \lambda_0 = L \bmod(9)$, $L = 0,3,6$.

We will take these cases separately.

(i) $6\lambda_1 - \lambda_0 = 0 \bmod(9) = 9m$, $m = 0,1,2\ldots\ldots$

Note that $N = \lambda_0 + 9m$ (FACT 2.3)

Let $D^* = f_2D_2 \cup f_3D_3 \cup f_4D_4$, with

$$f_2 = \frac{2\lambda_0 - 3\lambda_1}{9}, \quad f_3 = \frac{6\lambda_1 - \lambda_0}{9}, \quad f_4 = f_2$$

Note that f_2, f_3 are non-negative integers.

so, $b^* = \lambda_0 + m$

take,

$$j_0 = \begin{cases} m & , \text{ if } \lambda_0 \text{ is odd} \\ m+1 & , \text{ if } \lambda_0 \text{ is even} \end{cases}$$

So, $M_{j_0} = \lambda_0 + 9m$ (FACT 2.6)

and $b_{j_0} = \lambda_0 + m$

Therefore,

$$b \geq b_{j_0} = b^* \qquad (\text{ FACT 2.7 })$$

(ii) $6\lambda_1 - \lambda_0 = 3 \bmod(9) = 9m + 3$, $m = 0,1,2\ldots.$

Note that, $N = \lambda_0 + 9m + 3$ (FACT 2.3)

Take,

$$j_0 = \begin{cases} m & , \text{ if } \lambda_0 \text{ is odd} \\ m+1 & , \text{ if } \lambda_0 \text{ is even} \end{cases}$$

So, $M_{j_0} = \lambda_0 + 9m$ is closest to N, with $b_{j_0} = \lambda_0 + m$

But $b \geq b_{j_0} = \lambda_0 + m$,

Now consider D^* with ($b^*, \lambda_0^*, \lambda_1^*, \lambda_2^*$), where $\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1 + 1$, and $\lambda_2^* = \lambda_2$. Then

$$6\lambda_1^* - \lambda_0^* = 6\lambda_1 - \lambda_0 + 6 = 0 \bmod(9)$$

Therefore, from Subsection b(i) above, $b^* = \lambda_0 + m$.

Hence, D is with ($b \geq \lambda_0 + m, \lambda_0, \lambda_1, \lambda_2$) and D^* is with($b^* = \lambda_0 + m, \lambda_0, \lambda_1 + 1, \lambda_1 + 1$). That means any D with λ_0, λ_1, λ_2 and $6\lambda_1 - \lambda_0 = 3 \bmod(9)$ is S-inadmissible with respect to D^*.

(iii) $6\lambda_1 - \lambda_0 = 6 \bmod(9) = 9m + 6$, $m = 0,1,2,...$

Note that $N = 6\lambda_1 = \lambda_0 + 9m + 6$ (FACT 2.3)

Let, $D^* = f_1D_1 \cup f_2D_2 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = 1,\ f_2 = f_4 = \frac{2\lambda_0 - 3\lambda_1 + 3}{9},\ f_3 = \frac{6\lambda_1 - \lambda_0 - 6}{9}$$

Note that f_2, f_3, and f_4 are non-negative integers.

Now, $b^* = \lambda_0 + m + 1$.

Take,

$$j_0 = \begin{cases} m+1, & \text{if } \lambda_0 \text{ is odd} \\ m+2, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

So, $M_{j_0} = \lambda_0 + 9m + 9$, which is closest to N, and

$b_{j_0} = \lambda_0 + m + 1$ (FACT 2.6)

Now, $b \geq b_{j_0} = b^*$ (FACT 2.7)

<u>CASE 2 :</u> $\lambda_0 = 1 \bmod(3)$

According to Lemma 2.7, $\lambda_2 = \lambda_1 + 1$ and therefore, $N = 6\lambda_1 + 4$ (FACT 2.3)

(a) $3\lambda_1 - 2\lambda_0 \geq -8$. Therefore, $N = 4\lambda_0 + 6m - 12$.

Let $D^* = f_1 D_1 \cup f_3 D_3 \cup f_4 D_4$, with

$$f_1 = \frac{3\lambda_1 - 2\lambda_0 + 8}{3}, \quad f_3 = 1, \quad f_4 = \frac{\lambda_0 - 4}{3}$$

Note that, f_1 and f_4 are non-negative integers.

So, $$b^* = \frac{4\lambda_0 - 4}{3} + m = 5m$$

Take,

$$j_0 = \begin{cases} \dfrac{\lambda_0 - 1}{3} + m, & \text{if } \lambda_0 \text{ is odd} \\ \dfrac{\lambda_0 + 2}{3} + m, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

So,

$M_{j_0} = 4\lambda_0 + 9m - 3$ is closest to N and

$b_{j_0} = 5m + 1$ (FACT 2.6)

Therefore, $b \geq b_{j_0} > b^*$ (FACT 2.7)

(b) $3\lambda_1 - 2\lambda_0 < -8$

Consider the cases when,

$$6\lambda_1 - \lambda_0 = L \bmod(9),\ L = 2,5,8.$$

(i) $6\lambda_1 - \lambda_0 = 2 \bmod(9) = 9m + 2,\ m = 0,1,2,....$

Then, $N = \lambda_0 + 9m + 6$

Let, $D^* = f_1D_1 \cup f_2D_2 \cup f_3D_3 \cup f_4D_4$, with

$$f_1 = 1, f_2 = \frac{2\lambda_0 - 3\lambda_1 - 5}{9}, f_3 = \frac{2\lambda_0 - 3\lambda_1 + 4}{9}, f_4 = \frac{6\lambda_1 - \lambda_0 - 2}{9}$$

Note that, f_2, f_3, f_4 are non-negative integers.

So, $b^* = \lambda_0 + m + 1.$

Take,

$$j_0 = \begin{cases} m + 1, & \text{if } \lambda_0 \text{ is odd} \\ m + 2, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

Therefore, $M_{j_0} = \lambda_0 + 9m + 9$ is closest to N, with

$b_{j_0} = \lambda_0 + m + 1$ (FACT 2.6)

So,

$b \geq b_{j_0} = b^*$ (FACT 2.7)

(ii) $6\lambda_1 - \lambda_0 = 5 \bmod(9) = 9m + 5,\ m = 0,1,2,...$

Note that, $N = \lambda_0 + 9m + 9.$

Take, $D^* = f_2D_2 \cup f_3D_3 \cup f_4D_4$, with

$$f_2 = \frac{2\lambda_0 - 3\lambda_1 - 8}{9},\ f_3 = \frac{2\lambda_0 - 3\lambda_1 + 1}{9},\ f_4 = \frac{6\lambda_1 - \lambda_0 + 4}{9}$$

It can be seen that f_2, f_3, f_4 are non-negative integers.

Also, $b^* = \lambda_0 + m + 1$

Take,

$$j_0 = \begin{cases} m + 1, \text{ if } \lambda_0 \text{ is odd} \\ m + 2, \text{ if } \lambda_0 \text{ is even} \end{cases}$$

Then, $M_{j_0} = \lambda_0 + 9m + 9$ is equal to N with

$b_{j_0} = \lambda_0 + m + 1$ (FACT 2.6)

So, $b \geq b_{j_0} = b^*$ (FACT 2.7)

(iii) $6\lambda_1 - \lambda_0 = 8 \bmod(9) = 9m + 8$, $m = 0,1,2,..$

Take, j_0 as in Subsection (ii) above.

So, $b \geq b_{j_0} = \lambda_0 + m + 1$

Consider, D^* with (b^*, $\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1 + 1$, $\lambda_2^* = \lambda_2$)

Now, $6\lambda_1^* - \lambda_0^* = 6\lambda_1 - \lambda_0 + 6 = 9m + 14 = 5 \bmod(9)$

Therefore, by Subsection (ii) above.

$$b^* = \lambda_0 + m + 1$$

Hence, D is with ($b \geq \lambda_0 + m + 1$, λ_0, λ_1, λ_2) and D^* is with ($b^* = \lambda_0 + m + 1$, λ_0, $\lambda_1 + 1$, $\lambda_1 + 1$). Therefore, any D with λ_0, λ_1, λ_2 and $6\lambda_1 - \lambda_0 = 5 \bmod(9)$ is S-inadmissible with respect to D^*.

<u>CASE 3 :</u> $\lambda_0 = 2 \bmod(3)$

According to Lemma 2.7, $\lambda_2 = \lambda_1 - 1$, and therefore, $N = 6\lambda_1 - 4$ (FACT 2.3)

(a) $3\lambda_1 - 2\lambda_0 \geq -1$

$N = 4\lambda_0 + 6m - 6$

Let, $D^* = f_1 D_1 \cup f_2 D_2 \cup f_4 D_4$, with

$$f_1 = \frac{3\lambda_1 - 2\lambda_0 + 1}{3}, \quad f_2 = 1, \quad f_4 = \frac{\lambda_0 - 2}{3}$$

Note that, f_1 and f_4 are non-negative integers.

So,

$$b^* = \frac{4\lambda_0 + 3m - 2}{3}$$

Take,

$$j_0 = \begin{cases} \dfrac{\lambda_0 - 2}{3} + m, & \text{if } \lambda_0 \text{ is odd} \\ \dfrac{\lambda_0 + 1}{3} + m, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

So, $M_{j_0} = 4\lambda_0 + 9m - 6$, and

$$b_{j_0} = \frac{4\lambda_0 + 3m - 2}{3}$$

So,

$$b \geq b_{j_0} = b^*$$

(b) $3\lambda_1 - 2\lambda_0 < -1$

Consider $6\lambda_1 - \lambda_0 = L \bmod(9)$, $L = 1,4,7$.

(i) $6\lambda_1 - \lambda_0 = 1 \bmod(9) \geq 10$ ($= 9m + 1$, $m = 0,1,2,.....$)

Note that, $N = \lambda_0 + 9m + 6$

Let, $D^* = f_1 D_1 \cup f_2 D_2 \cup f_3 D_3 \cup f_4 D_4$, with

$$f_1 = 1, f_2 = \frac{2\lambda_0 - 3\lambda_1 + 11}{9}, f_3 = \frac{2\lambda_0 - 3\lambda_1 + 2}{9}, \quad f_4 = \frac{6\lambda_1 - \lambda_1 - 10}{9}$$

So, $b^* = \lambda_0 + m$

Take,

$$j_0 = \begin{cases} m + 1, & \text{if } \lambda_0 \text{ is odd} \\ m + 2, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

So, $M_{j_0} = \lambda_0 + 9m + 9$ is closest to N, and

$b_{j_0} = \lambda_0 + m + 1$

Therefore,

$$b \geq b_{j_0} > b^*$$

(ii) $6\lambda_1 - \lambda_0 = 4 \bmod(9) = 9m + 4, \ m = 0,1,2,...$

Therefore, $N = \lambda_0 + 9m - 3$

Let, $D^* = f_2D_2 \cup f_3D_3 \cup f_4D_4$, with

$$f_2 = \frac{2\lambda_0 - 3\lambda_1 + 8}{9}, f_3 = \frac{2\lambda_0 - 3\lambda_1 - 1}{9}, f_4 = \frac{6\lambda_1 - \lambda_0 - 4}{9}$$

This gives, $b^* = \lambda_0 + m$

Take,

$$j_0 = \begin{cases} m & , \text{if } \lambda_0 \text{ is odd} \\ m + 1, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

So, $M_{j_0} = \lambda_0 + 9m$, with

$b_{j_0} = \lambda_0 + m$

Therefore, $b \geq b_{j_0} = b^*$.

(iii) $6\lambda_1 - \lambda_0 = 7 \bmod(9) = 9m + 7$, $m = 0,1,2,\ldots$

Note that, $N = \lambda_0 + 9m + 3$

Take,

$$j_0 = \begin{cases} m, & \text{if } \lambda_0 \text{ is odd} \\ m+1, & \text{if } \lambda_0 \text{ is even} \end{cases}$$

So that, $M_{j_0} = \lambda_0 + 9m$ and $b_{j_0} = \lambda_0 + m$

Therefore, $b \geq \lambda_0 + m$

Let, D^* is a design with $(b^*, \lambda_0^* = \lambda_0, \lambda_1^* = \lambda_1 + 1, \lambda_2^* = \lambda_2)$

Here, $6\lambda_1^* - \lambda_0^* = 6\lambda_1 - \lambda_0 + 6 = 4 \bmod(9)$

This case is covered in b(ii) above, and hence,

$$b^* = \lambda_0 + m$$

Therefore, D is with $(b \geq \lambda_0 + m, \lambda_0, \lambda_1, \lambda_2)$ and

D^* is with $(b^* = \lambda_0 + m, \lambda_0, \lambda_1 + 1, \lambda_2^*)$

Hence, D has smaller λ_1 value than the same for D^*.

Therefore, any design D for which $6\lambda_1 - \lambda_0 = 7 \bmod(9)$, is S-inadmissible with respect to D^*.

(iv) $6\lambda_1 - \lambda_0 = 1$

Therefore, $\lambda_0 = 1 \bmod(6) = 6m + 1$, $m = 0,1,2,\ldots\ldots\ldots$

From Lemma 2.2,

$$r = \frac{\lambda_0 + 3\lambda_1 - 2}{3} = 2m + \lambda_1 - 1/3,$$

which is not an integer.

Therefore, such designs cannot exist.

Note that $6\lambda_1 - \lambda_0$ is not allowed to take all possible values in the above cases. It will be shown below that for all such uncovered cases any design D will be S-inadmissible with respect to a design D^*, which is union of copies of designs from Table 5.1.

CASE 1: Let λ_0 is even.

From FACT 2.7 , $b \geq b_{j_0} = \lambda_0 + j_0 - 1$, $j = 1,2,\ldots \ldots\ldots,[\lambda_0/3]+1$

Therefore, $b \geq \lambda_0$

(i) Let $\lambda_0 = 0 \bmod(6)$ and $6\lambda_1 - \lambda_0 < 0 \Rightarrow \lambda_1 < \lambda_0/6$

Let, $D^* = \frac{\lambda_0}{6} D_2 \cup \frac{\lambda_0}{6} D_3$

So, $b^* = \lambda_0, \lambda_0^* = \lambda_0, \lambda_1^* = \lambda_0/6$

Therefore, D^* is with $(b^* = \lambda_0, \lambda_0^* = \lambda_0, \lambda_1^* = \lambda_0/6, \lambda_2^*)$

Whereas, D is with $(b \geq \lambda_0, \lambda_0, \lambda_1 < \lambda_0/6, \lambda_2)$

So, D has smaller λ_1 than λ_1^* of D^*. Hence, D is S-inadmissible with respect to D^*.

(ii) Let $\lambda_0 = 2 \bmod(6)$, $6\lambda_1 - \lambda_0 < 4 \Rightarrow$
$\lambda_1 < \frac{\lambda_0 + 4}{6}$.

Let, $D^* = \frac{\lambda_0 + 4}{6} D_2 \cup \frac{\lambda_0 - 2}{6} D_3$

So, $b^* = \lambda_0, \lambda_0^* = \lambda_0$, and $\lambda_1^* = \frac{\lambda_0 + 4}{6}$

So, D^* is with $(b^* = \lambda_0, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 + 4}{6}, \lambda_2^*)$

and D is with $(b \geq \lambda_0, \lambda_0, \lambda_1 < \frac{\lambda_0 + 4}{6}, \lambda_2)$

Hence, D is S-inadmissible with respect to D^*.

(iii) Let $\lambda_0 = 4 \bmod(6)$, $6\lambda_1 - \lambda_0 < -4 \Rightarrow$

$\lambda_1 < \frac{\lambda_0 - 4}{6}$

Let, $D^* = \frac{\lambda_0 - 4}{6} D_2 \cup \frac{\lambda_0 + 2}{6} D_3$

Therefore, $b^* = \lambda_0, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 - 4}{6}$

So, D^* is with $(b^* = \lambda_0, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 - 4}{6}, \lambda_2^*)$

and D is with $(b \geq \lambda_0, \lambda_0, \lambda_1 < \frac{\lambda_0 - 4}{6}, \lambda_2)$

Hence, D is S-inadmissible with respect to D^*.

<u>CASE 2:</u> Let λ_0 is odd

From FACT 2.7, $b \geq b_{j_0} = \lambda_0 + j_0$, $j_0 = 1,2,\ldots,[\lambda_0/3]$

Therefore, $b \geq \lambda_0 + 1$.

(i) Let $\lambda_0 = 1 \bmod(6)$ and $6\lambda_1 - \lambda_0 < 5 \Rightarrow$

$\lambda_1 < \frac{\lambda_0 + 5}{6}$

Let, $D^* = \frac{\lambda_0 - 7}{6} D_2 \cup \frac{\lambda_0 - 1}{6} D_3 \cup D_4$

Then, $b^* = \lambda_0 + 1, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 + 5}{6}$

So, D^* is with $(b^* = \lambda_0 + 1, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 + 5}{6}, \lambda_2^*)$

and D is with $(b \geq \lambda_0 + 1, \lambda_0, \lambda_1 < \frac{\lambda_0 + 5}{6}, \lambda_2)$

Hence, D is S-inadmissible with respect to D^*, for $6\lambda_1 - \lambda_0 < 5$.

(ii) Let $\lambda_0 = 3 \bmod(6)$ and $6\lambda_1 - \lambda_0 < 9 \Rightarrow \lambda_1 < \frac{\lambda_0 + 9}{6}$.

Let, $D^* = \frac{\lambda_0 - 3}{6} D_2 \cup \frac{\lambda_0 - 3}{6} D_3 \cup D_4$

So, $b^* = \lambda_0 + 1, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 + 9}{6}$, i.e.,

D^* is with $(b^* = \lambda_0 + 1, \lambda_0^* = \lambda_0, \lambda_2^* = \frac{\lambda_0 + 9}{6}, \lambda_2^*)$

and D is with $(b \geq \lambda_0 + 1, \lambda_0, \lambda_1 < \frac{\lambda_0 + 5}{6}, \lambda_2)$

So, D is S-inadmissible with respect to D^* for $6\lambda_1 - \lambda_0 < 9$.

(iii) Let $\lambda_0 = 5 \bmod(6)$ and $6\lambda_1 - \lambda_0 < 13 \Rightarrow \lambda_1 < \frac{\lambda_0 + 13}{6}$

Let $D^* = \frac{\lambda_0 + 1}{6} D_2 \cup \frac{\lambda_0 - 5}{6} D_3 \cup D_4$

So, $b^* = \lambda_0 + 1, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 + 13}{6}$

So, D^* is with $(b^* = \lambda_0 + 1, \lambda_0^* = \lambda_0, \lambda_1^* = \frac{\lambda_0 + 13}{6}, \lambda_2^*)$

and D is with $(b \geq \lambda_0 + 1, \lambda_0, \lambda_1 < \frac{\lambda_0 + 13}{6}, \lambda_2)$

Hence, D is S-inadmissible with respect to D^* for $6\lambda_1 - \lambda_0 < 13$.

This makes the three cases considered before exhaustive. Hence proof of the theorem is complete.

5.2 MCCGD FOR $k = 4$, $v = 6$:

THEOREM 5.2

For $k = 4$ and $v = 6$ the minimal complete class of generator designs is given in Table 5.2

TABLE 5.2 MCCGD FOR $k = 4$, $v = 6$.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	2	2	1	2 0 0	1	1	0	0 0 1 4 2 5 3 6
D_2	6	0	4	0 0 0	0	3	2	1 1 1 1 2 3 2 2 2 4 4 4 3 3 3 5 5 5 4 5 6 6 6 6
D_3	6	18	1	0 0 6	3	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6
D_4	6	12	2	0 6 0	4	1	0	0 0 0 0 0 0 0 0 0 0 0 0 1 1 2 4 4 5 2 3 3 5 6 6
D_5	7	4	4	4 0 0	2	2	2	0 0 0 0 1 1 2 1 1 2 4 2 3 3 2 3 3 5 4 4 5 6 5 4 6 5 6 6

TABLE 5.2 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_6	7	10	3	4 3 0	4	1	1	0 0 0 0 0 0 0 1 1 2 4 0 0 0 2 3 3 5 1 2 3 6 5 4 6 4 5 6
D_7	9	0	6	0 0 0	0	3	4	1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 3 4 4 5 4 4 5 4 4 5 5 6 6 5 6 6 5 6 6
D_8	9	6	5	0 3 0	2	2	3	0 0 0 1 1 1 1 2 2 0 0 0 2 2 3 3 3 3 1 2 3 4 5 4 5 4 4 4 6 5 5 6 6 6 5 6
D_9	9	18	3	0 9 0	6	0	1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 2 2 2 3 3 3 4 5 6 4 5 6 4 5 6
D_{10}	10	10	5	10 0 0	5	2	2	0 0 0 0 0 0 0 0 0 1 1 1 1 1 2 2 2 3 2 2 3 3 4 3 3 4 5 5 6 4 6 5 4 5 6 6 .... 0 4 5 6
D_{11}	12	6	7	6 0 0	3	3	4	0 0 0 0 0 0 1 1 1 1 1 1 2 2 3 2 2 3 2 3 4 3 4 5 4 5 4 5 4 5 6 6 6 6 6 6 .... 1 2 2 3 3 3 5 4 4 6 5 5
D_{12}	12	12	6	6 3 0	5	2	3	0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 0 0 0 2 3 5 3 4 4 1 2 3 4 6 6 5 6 5 4 5 6 ....

TABLE 5.2 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								1 1 2 2 3 3 5 4 4 6 5 6
D_{13}	12	12	6	12 0 0	6	3	2	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 2 3 4 5 3 4 5 3 3 4 5 6 6 6 5 6 0 0 0 3 3 4 4 4 5 6 5 6
D_{14}	12	18	5	6 6 0	7	1	2	0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 0 0 0 2 3 5 3 4 4 1 2 3 6 5 6 4 6 5 4 5 6 0 0 0 0 0 0 1 2 3 4 5 6
D_{15}	15	12	8	12 0 0	6	3	4	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 4 4 3 3 4 6 5 4 6 5 6 4 5 6 0 0 0 1 1 2 2 3 3 2 3 3 5 4 5 4 5 4 6 5 6 5 6 6
D_{16}	15	18	7	12 3 0	8	2	3	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 4 5 3 3 4 4 5 5 6 6 6 4 6 6 0 0 0 0 0 0 2 3 3 0 0 0 5 4 4 1 2 3 6 5 5 4 5 6
D_{17}	18	18	9	18 0 0	9	3	4	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 2 2 2 3 3 3 4 4 5 4 5 6 4 5 6 5 6 6

TABLE 5.2 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								0 0 0 0 0 0 0 0 0 2 2 2 2 2 2 3 3 3 3 3 3 4 4 5 4 4 5 4 5 6 5 6 6 5 6 6

5.3 MCCGD FOR k = 4, v = 8 :

THEOREM 5.3

For k = 4 and v = 8 the minimal complete class of generator designs is given in Table 5.3.

TABLE 5.3 MCCGD FOR k = 4, v = 8 .

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	2	0	1	0 0 0	0	1	0	1 5 2 6 3 7 4 8
D_2	8	24	1	0 0 8	3	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8
D_3	10	8	4	0 4 0	2	2	1	0 0 0 0 1 1 1 2 5 0 0 0 0 2 2 3 3 6 1 2 3 4 3 4 4 4 7 5 6 7 8 8 7 6 5 8 .... 5 6 7 8

TABLE 5.3 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
D_4	12	0	6	0	0	2	3	1	1	1	1	1	1	2	2	2
				0				2	2	3	3	4	4	3	3	4
				0				5	7	5	6	5	6	5	7	5
								6	8	7	8	8	7	6	8	7
								2	3	3						
								4	4	4						
								6	5	6						
								8	8	7						
D_5	12	16	4	8	5	1	1	0	0	0	0	0	0	0	0	0
				4				1	1	1	2	2	3	4	6	0
				0				2	3	4	3	5	5	5	7	1
								7	8	6	4	8	6	7	8	5
								0	0	0						
								0	0	0						
								2	3	4						
								6	7	8						
D_6	12	24	3	0	6	1	0	0	0	0	0	0	0	0	0	0
				12				0	0	0	0	0	0	0	0	0
				0				1	1	1	2	2	3	5	5	5
								2	3	4	3	4	4	6	7	8
								0	0	0						
								0	0	0						
								6	6	7						
								7	8	8						
D_7	15	12	6	8	4	2	2	0	0	0	0	0	0	0	0	0
				2				1	1	2	2	3	3	4	4	0
				0				5	6	5	6	5	6	5	6	1
								7	8	8	7	7	8	8	7	2
								0	1	1	1	2	2			
								0	2	3	4	3	4			
								3	3	5	7	7	5			
								4	4	6	8	8	6			
D_8	16	8	7	8	3	2	3	0	0	0	0	0	0	0	0	1
				0				1	1	1	2	2	3	3	4	3
				0				2	4	5	3	7	4	5	6	6
								6	8	7	5	8	7	6	8	7
								1	1	1	2	2	2	3		
								2	3	4	4	3	4	4		
								5	7	5	6	6	5	5		
								8	8	6	7	8	7	8		

TABLE 5.3 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
D_9	16	32	4	0 16 0	8	0	1	0 0 1 5	0 0 1 6	0 0 1 7	0 0 1 8	0 0 2 5	0 0 2 6	0 0 2 7	0 0 2 8	0 0 3 5
								0 0 3 6	0 0 3 7	0 0 3 8	0 0 4 5	0 0 4 6	0 0 4 7	0 0 4 8		
D_{10}	19	12	8	3 3 1	3	3	3	0 1 5 8	0 2 6 8	0 3 7 8	0 0 1 7	0 0 2 5	0 0 3 6	0 0 0 4	1 2 3 7	1 2 4 8
								1 2 6 7	1 3 4 8	1 3 5 6	1 4 5 6	2 3 5 8	2 4 5 7	2 3 4 6	3 4 5 7	4 6 7 8
								5 6 7 8								
D_{11}	19	20	7	16 2 0	7	2	2	0 1 3 5	0 1 4 6	0 1 5 7	0 1 6 8	0 1 7 8	0 2 3 7	0 2 4 8	0 2 5 6	0 2 5 8
								0 2 6 7	0 3 6 8	0 3 6 7	0 3 5 8	0 4 5 6	0 4 5 7	0 4 7 8	0 0 1 2	0 0 3 4
								1 2 3 4								
D_{12}	19	28	6	8 10 0	8	2	1	0 1 2 3	0 1 2 4	0 1 3 5	0 1 4 6	0 2 3 4	0 2 5 7	0 3 4 8	0 6 7 8	0 0 1 7
								0 0 1 8	0 0 2 6	0 0 2 8	0 0 3 6	0 0 3 7	0 0 4 5	0 0 4 7	0 0 5 6	0 0 5 8

TABLE 5.3 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								5 6 7 8
D_{13}	20	8	9	0 4 0	2	3	4	0 0 0 0 1 1 1 1 1 0 0 0 0 2 2 2 3 3 1 2 3 4 4 5 7 5 6 5 6 7 8 6 7 8 8 7 1 1 1 2 2 2 2 2 3 3 4 4 3 3 3 4 6 4 6 5 7 5 4 5 5 7 5 8 6 8 6 7 8 8 8 7 3 4 4 5 6 6 8 7
D_{14}	20	16	8	16 0 0	6	2	3	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 4 5 7 3 4 5 6 7 8 5 8 8 7 5 6 0 0 0 0 0 0 0 1 1 2 3 3 3 3 4 4 3 4 6 4 4 5 6 5 6 5 6 8 6 8 7 7 7 8 6 7 2 2 3 4 5 7 8 8
D_{15}	23	36	7	11 11 1	10	1	2	0 0 0 0 0 0 0 0 0 1 1 1 1 1 2 2 2 3 2 3 4 5 6 3 4 7 4 5 6 7 8 7 8 6 8 5 0 0 0 0 0 0 0 0 0 3 4 0 0 0 0 0 0 0 5 6 1 2 2 2 3 3 3 7 8 8 5 6 7 7 6 8 0 0 0 0 0 0 0 0 0 0 4 4 4 5 0 5 7 8 6 1

TABLE 5.3 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
D_{16}	23	44	6	3 19 1	11	1	1	0	0	0	0	0	0	0	0	0
								1	1	6	0	0	0	0	0	0
								2	4	7	1	1	1	2	2	2
								3	5	8	6	7	8	4	5	6
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								2	2	3	3	3	3	3	4	4
								7	8	4	5	6	7	8	6	7
								0	0	0	0	0				
								0	0	0	0	0				
								4	5	5	5	0				
								8	6	7	8	1				
D_{17}	24	24	9	24 0 0	9	2	3	0	0	0	0	0	0	0	0	0
								1	1	1	1	1	1	1	1	1
								2	2	3	3	4	4	5	5	7
								7	8	5	6	6	8	6	7	8
								0	0	0	0	0	0	0	0	0
								2	2	2	2	2	2	2	3	3
								3	3	4	4	5	5	6	4	4
								7	8	5	6	6	8	7	7	5
								0	0	0	0	0	0			
								3	3	3	4	4	4			
								5	6	6	5	6	7			
								8	8	7	7	8	8			
D_{18}	30	40	10	22 6 2	12	2	3	0	0	0	0	0	0	0	0	0
								1	1	1	1	1	1	1	1	1
								2	2	3	3	4	4	5	6	7
								5	8	5	6	8	6	7	7	8
								0	0	0	0	0	0	0	0	0
								2	2	2	2	2	2	2	3	3
								3	3	4	4	5	6	7	4	4
								7	8	5	6	6	7	8	5	7
								0	0	0	0	0	0	0	0	0
								3	3	4	4	0	0	0	0	0
								5	6	5	6	3	3	4	4	5
								8	8	7	8	6	7	8	7	6
								0	0	0						
								0	0	0						
								5	0	0						
								8	1	2						

TABLE 5.3 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_{19}	32	32	12	32 0 0	12	4	3	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 2 2 2 2 3 3 3 3 4 5 6 7 8 5 6 7 8 5 0 0 0 0 0 0 0 0 0 1 1 1 2 2 2 2 2 2 4 4 4 3 3 3 3 4 4 6 7 8 5 6 7 8 5 6 0 0 0 0 0 0 0 0 0 2 2 3 3 3 3 5 5 5 4 4 4 4 4 4 6 6 6 7 8 5 6 7 8 7 7 8 0 0 0 0 0 5 5 5 6 6 6 7 7 7 7 8 8 8 8 8

5.4 MCCGD FOR k = 4, v = 10 :

THEOREM 5.4

For k = 4 and v = 10 the minimal complete class of generator designs is given in Table 5.4

TABLE 5.4 MCCGD FOR k = 4, v = 10.

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_1	10	30	1	0 0 10	3	0	0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8 9 0 0 0 10

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
D_2	13	12	4	6	3	1	1	0	0	0	0	0	0	0	0	0
				3				1	2	3	4	5	6	0	0	0
								9	7	8	7	8	7	1	3	5
				0				10	8	10	9	9	10	2	4	6
								1	1	2	2					
								3	4	3	4					
								5	6	6	5					
								7	8	9	10					
D_3	14	26	3	4	5	1	0	0	0	0	0	0	0	0	0	0
				8				1	1	6	6	0	0	0	0	0
								2	4	7	9	2	2	3	3	7
				2				3	5	8	10	4	5	4	5	9
								0	0	0	0	0				
								0	0	0	0	0				
								7	8	8	0	0				
								10	9	10	1	6				
D_4	15	0	6	0	0	2	2	1	1	1	1	1	1	2	2	2
				0				2	2	3	4	4	5	3	4	6
								3	5	6	7	6	7	4	5	8
				0				8	9	9	8	10	10	7	6	10
								2	3	3	3	4	6			
								7	5	4	5	5	7			
								9	6	9	8	8	8			
								10	7	10	10	9	9			
D_5	15	10	5	0	2	2	1	0	0	0	0	0	1	1	1	1
				5				0	0	0	0	0	2	3	4	6
								1	3	5	7	9	3	5	5	8
				0				2	4	6	8	10	4	9	7	10
								2	2	2	3	4	7			
								3	4	6	6	6	8			
								5	5	7	7	8	9			
								8	10	9	10	9	10			
D_6	19	26	5	8	6	1	1	0	0	0	0	0	0	0	0	0
				9				1	1	2	2	3	3	4	4	0
								7	8	5	8	5	6	6	7	1
				0				10	9	9	10	10	9	10	9	5
								0	0	0	0	0	0	0	0	1
								0	0	0	0	0	0	0	0	2
								1	2	2	3	3	4	4	9	3
								6	6	7	7	8	8	5	10	4

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								5
								6
								7
								8
D_7	20	40	4	0	8	1	0	0 0 0 0 0 0 0 0 0
				20				0 0 0 0 0 0 0 0 0
				0				1 1 1 1 2 2 2 3 3
								2 3 4 5 3 4 5 4 5
								0 0 0 0 0 0 0 0 0
								0 0 0 0 0 0 0 0 0
								4 6 6 6 6 7 7 7 8
								5 7 8 9 10 8 9 10 9
								0 0
								0 0
								8 9
								10 10
D_8	20	10	7	10	3	2	2	0 0 0 0 0 0 0 0 0
				0				1 1 1 2 2 2 3 3 3
				0				4 4 8 5 5 8 6 6 9
								7 10 9 7 9 10 7 8 10
								0 1 1 1 1 2 2 3 4
								4 2 2 3 5 3 4 4 7
								5 3 6 5 6 4 6 5 8
								6 7 10 8 9 9 8 10 9
								5 6
								7 7
								8 9
								10 10
D_9	20	20	6	10	5	2	1	0 0 0 0 0 0 0 0 0
				5				1 1 1 2 2 3 4 5 6
				0				2 3 4 3 7 4 5 6 7
								10 8 9 5 9 10 8 9 8
								0 0 0 0 0 0 1 1 2
								6 0 0 0 0 0 2 3 3
								7 1 2 3 4 5 4 5 4
								10 6 8 9 7 10 5 7 6
								6 7
								8 8
								9 9
								10 10

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
D_{10}	25	50	5	0 25 0	10	0	1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 2 2 2 2 6 7 8 9 10 6 7 8 9
								0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 3 3 3 3 3 4 4 4 10 6 7 8 9 10 6 7 8
								0 0 0 0 0 0 0 0 0 0 0 0 0 0 4 4 5 5 5 5 5 9 10 6 7 8 9 10
D_{11}	25	40	6	10 15 0	9	1	1	0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 3 5 6 2 4 7 4 5 4 5 6 8 3 9 8 6 9 8 10 7 10
								0 0 0 0 0 0 0 0 0 7 0 0 0 0 0 0 0 0 9 1 1 1 2 2 2 3 3 10 5 6 10 7 8 10 6 7
								0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 4 4 4 5 6 8 9 5 7 10 8 8 9
D_{12}	25	20	8	20 0 0	6	2	2	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 3 3 4 4 5 9 3 3 4 8 9 7 10 6 10 8 10 7
								0 0 0 0 0 0 0 0 0 2 2 2 3 3 4 4 5 5 4 5 6 5 6 5 6 6 8 9 10 9 7 7 8 8 10 9
								0 0 1 1 3 3 7 6 7 2 2 4 4 8 7 8 5 6 5 6 9 9 10 7 8 9 10 10
D_{13}	25	30	7	20 5 0	8	2	1	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 4 5 9 3 4 6 3 4 5 7 8 10 10 8 7

<u>**TABLE 5.4 (CONTINUED).**</u>

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
								0	0	0	0	0	0	0	0	0
								2	3	3	3	4	5	6	6	6
								5	4	4	7	5	6	8	8	9
								9	5	6	9	10	7	9	10	10
								0	0	0	0	0	0	0		
								7	7	0	0	0	0	0		
								8	8	1	2	3	4	7		
								9	10	6	5	8	9	10		
D_{14}	26	14	9	12	4	2	3	0	0	0	0	0	0	0	0	0
				1				1	1	1	1	2	2	2	2	3
				0				3	4	5	7	3	4	5	7	4
								6	8	10	9	10	6	9	8	7
								0	0	0	0	1	1	1	1	1
								3	4	5	0	2	2	3	4	5
								5	6	6	9	6	8	6	9	7
								8	8	7	10	7	10	9	10	8
								2	2	2	3	3	3	4	4	
								3	4	5	4	5	6	5	5	
								7	8	6	7	8	8	6	7	
								9	9	10	10	9	10	9	10	
D_{15}	27	18	9	14	5	3	2	0	0	0	0	0	0	0	0	0
				2				1	1	1	1	1	2	2	2	3
				0				2	2	3	4	6	3	5	6	4
								3	4	5	7	10	4	7	10	9
								0	0	0	0	0	0	0	1	1
								3	4	6	6	7	0	0	2	3
								6	5	7	8	8	5	8	5	7
								7	8	9	9	10	10	9	8	8
								1	1	2	2	2	3	3	4	7
								4	5	3	4	7	4	6	5	8
								9	6	5	6	9	5	8	6	9
								10	9	9	8	10	10	10	7	10
D_{16}	30	40	8	20	10	1	2	0	0	0	0	0	0	0	0	0
				10				1	1	1	1	1	1	2	2	2
				0				2	3	4	5	6	8	3	4	5
								7	9	9	10	8	10	6	10	8
								0	0	0	0	0	0	0	0	0
								2	2	3	3	3	3	4	4	4
								6	9	4	5	6	7	5	6	7
								9	10	8	7	10	10	6	7	8

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								0 0 0 0 0 0 0 0 0 5 5 0 0 0 0 0 0 0 7 8 1 2 3 4 5 1 2 9 9 6 7 8 9 10 7 8
								0 0 0 0 0 0 3 4 5 9 10 6
D_{17}	30	30	9	30 0 0	9	2	2	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 10 10 9
								0 0 0 0 0 0 0 0 0 2 2 2 2 2 2 2 3 3 3 3 4 4 5 5 6 4 4 7 10 9 10 8 8 9 6 8
								0 0 0 0 0 0 0 0 0 3 3 3 4 4 4 5 6 6 5 5 6 5 5 7 6 7 8 9 9 10 6 7 10 7 8 10
								0 0 0 7 7 8 8 9 9 9 10 10
D_{18}	31	24	10	22 1 0	7	2	3	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 2 2 2 3 3 4 5 6 6 3 4 7 8 10 8 7 9 9 9 8
								0 0 0 0 0 0 0 0 0 2 2 2 2 3 3 3 3 4 4 5 6 6 4 5 7 7 5 10 10 7 8 6 9 8 10 9
								0 0 0 0 0 1 1 1 2 4 4 5 8 0 2 4 5 3 7 9 7 9 5 6 7 8 7 8 10 10 10 6 10 9 10 9
								2 3 3 4 5 4 5 5 8 6 6 6 9 10 8 7
D_{19}	32	58	7	7 24 1	12	1	1	0 0 0 0 0 0 0 0 0 1 1 1 2 3 4 5 0 0 2 4 6 6 7 8 8 1 1 3 5 7 9 10 10 9 8 9

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								1	2	2	2	2	2	3	3	3
								10	4	5	7	8	10	4	5	6
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								3	3	4	4	4	5	5	5	6
								8	9	6	7	9	6	7	10	8
								0	0	0	0	0				
								0	0	0	0	0				
								6	7	7	9	0				
								10	8	9	10	1				
D_{20}	32	48	8	17	11	2	1	0	0	0	0	0	0	0	0	0
				14				1	1	1	1	1	2	2	2	3
				1				2	2	3	4	5	3	5	5	4
								3	4	6	8	7	4	6	9	5
								0	0	0	0	0	0	0	0	0
								3	4	6	6	6	7	7	7	0
								5	6	8	8	9	8	8	9	1
								10	7	9	10	10	9	10	10	5
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								1	1	2	2	2	3	3	3	4
								9	10	7	8	10	7	8	9	5
								0	0	0	0	0				
								0	0	0	0	0				
								4	4	5	6	0				
								9	10	8	7	6				
D_{21}	32	28	10	24	8	3	2	0	0	0	0	0	0	0	0	0
				2				1	1	1	1	1	1	1	1	2
				0				2	2	2	3	3	4	5	6	3
								3	4	9	5	10	7	9	8	4
								0	0	0	0	0	0	0	0	0
								2	2	2	2	3	3	3	4	4
								3	4	5	6	4	5	6	5	6
								7	5	6	10	5	8	8	10	9
								0	0	0	0	0	0	0	0	1
								4	5	6	6	7	8	0	0	4
								7	6	7	9	8	9	7	8	6
								8	7	10	10	9	10	9	10	8

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
								1	2	2	3	3				
								5	5	7	4	6				
								7	8	8	9	7				
								10	9	10	10	9				
D_{22}	34	16	12	8	4	3	4	0	0	0	0	0	0	0	0	0
				4				1	1	1	1	2	4	5	8	0
				0				2	3	4	5	3	7	6	9	2
								7	8	10	6	10	9	10	10	6
								0	0	1	1	1	1	1	1	1
								0	0	2	2	3	3	4	4	5
								4	5	6	7	6	7	6	8	7
								8	9	8	10	10	9	9	9	8
								1	2	2	2	2	2	2	2	3
								5	3	3	4	4	4	5	5	4
								9	5	8	6	7	9	6	7	5
								10	9	9	8	10	10	9	8	8
								3	3	3	3	4	5			
								4	4	6	6	5	7			
								5	6	7	8	6	8			
								10	7	9	10	7	10			
D_{23}	36	34	11	32	10	2	3	0	0	0	0	0	0	0	0	0
				1				1	1	1	1	1	1	1	1	1
				0				2	3	3	4	4	5	5	6	7
								6	7	8	8	10	6	9	10	9
								0	0	0	0	0	0	0	0	0
								1	2	2	2	2	2	2	2	2
								8	3	3	4	4	5	5	6	7
								9	6	9	7	8	9	10	9	8
								0	0	0	0	0	0	0	0	0
								2	3	3	3	3	3	3	4	4
								8	4	4	5	5	6	7	5	6
								10	9	10	7	8	10	10	6	7
								0	0	0	0	0	0	1	3	4
								4	4	5	5	5	0	2	6	5
								6	7	6	7	8	9	7	8	9
								8	9	7	8	10	10	10	9	10
D_{24}	37	58	9	14	13	1	2	0	0	0	0	0	0	0	0	0
				22				1	1	1	1	1	2	2	2	2
				0				2	3	4	7	8	3	4	6	8
								5	6	10	9	10	7	10	10	9

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
								0	0	0	0	0	0	0	0	0
								3	3	3	4	4	0	0	0	0
								4	5	9	5	6	1	1	1	1
								8	9	10	7	9	6	7	8	9
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								2	2	2	2	3	3	3	3	4
								6	7	8	9	6	7	8	10	6
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								4	4	4	5	5	5	5	5	7
								7	8	9	6	8	9	10	10	10
								5								
								6								
								7								
								8								
D_{25}	37	48	10	27	12	2	2	0	0	0	0	0	0	0	0	0
				9				1	1	1	1	1	1	1	1	1
				1				2	2	3	3	4	4	5	5	6
								9	9	10	10	8	8	6	7	7
								0	0	0	0	0	0	0	0	0
								2	2	2	2	2	2	3	3	3
								3	3	4	4	7	7	6	6	4
								5	5	6	6	10	10	8	8	7
								0	0	0	0	0	0	0	0	0
								3	4	4	5	5	6	6	7	7
								4	5	5	8	8	9	9	8	8
								7	9	9	10	10	10	10	9	9
								0	0	0	0	0	0	0	0	0
								0	0	0	0	0	0	0	0	0
								2	2	3	3	4	4	5	5	6
								8	8	9	9	10	10	6	7	7
								0								
								0								
								0								
								1								
D_{26}	37	38	11	34	11	3	2	0	0	0	0	0	0	0	0	0
				2				1	1	1	1	1	1	1	1	1
				0				2	2	2	3	3	4	4	5	5
								3	4	5	7	9	7	10	8	10

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE
								0 0 0 0 0 0 0 0 0 1 1 2 2 2 2 2 2 2 6 6 3 3 4 5 6 6 8 8 9 4 5 7 10 7 8 9
								0 0 0 0 0 0 0 0 0 2 3 3 3 3 3 3 4 4 9 4 4 5 6 6 8 5 5 8 5 8 7 10 10 9 8 9
								0 0 0 0 0 0 0 0 0 4 4 5 5 6 7 7 0 0 6 6 6 6 7 8 9 7 8 9 10 7 9 8 10 10 9 10
								7 8 9 10
D_{27}	42	48	12	34 7 0	13	2	3	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 2 2 3 3 4 4 5 5 7 6 10 6 8 7 9 8 9 9
								0 0 0 0 0 0 0 0 0 1 1 2 2 2 2 2 2 2 7 8 3 3 4 4 5 5 6 10 10 6 9 8 10 7 10 8
								0 0 0 0 0 0 0 0 0 2 2 3 3 3 3 3 3 3 7 8 4 4 5 5 7 8 9 9 9 6 7 9 10 8 10 10
								0 0 0 0 0 0 0 0 0 4 4 4 4 4 5 6 0 0 5 5 6 7 8 6 9 1 2 6 10 9 10 9 7 10 6 7
								0 0 0 0 0 5 0 0 0 0 0 6 3 4 5 5 6 7 7 8 8 9 10 8
D_{28}	54	56	16	48 4 0	16	3	4	0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 2 2 2 3 3 3 4 4 4 6 8 9 6 9 10 7 9 10

TABLE 5.4 (CONTINUED).

DESIGN	b	r_0	r	m	λ_0	λ_1	λ_2	STRUCTURE								
								0	0	0	0	0	0	0	0	0
								1	1	1	1	1	1	1	2	2
								5	5	5	6	6	7	8	3	3
								7	8	10	7	8	9	10	7	9
								0	0	0	0	0	0	0	0	0
								2	2	2	2	2	2	2	2	2
								3	4	4	4	5	5	5	6	7
								10	6	7	8	6	8	9	10	9
								0	0	0	0	0	0	0	0	0
								2	2	3	3	3	3	3	3	3
								7	8	4	4	4	5	5	5	6
								10	10	6	8	10	7	9	10	7
								0	0	0	0	0	0	0	0	0
								3	3	3	4	4	4	4	4	4
								6	7	8	5	5	5	6	6	7
								8	8	9	7	9	10	9	10	8
								0	0	0	0	0	0	0	5	7
								4	5	5	0	0	0	0	6	8
								8	6	6	5	6	7	9	9	9
								9	7	8	8	9	10	10	10	10

CHAPTER 6.

6.1 A-OPTIMAL DESIGNS :

Consider an E GD TIB design with parameters (k, v, b, λ_0, λ_1, λ_2). Then from Section 2.1.3, the trace of the inverse of information matrix M is given as

$$Tr(M^{-1}) = \frac{vk\left[(\lambda_0+\lambda_1+\frac{v}{2}\lambda_2)(\lambda_0+\frac{v}{2}\lambda_2) - \frac{v}{2}(\frac{v}{2}-1)\lambda_2^2\right]}{\lambda_0(\lambda_0+v\lambda_2)(\lambda_0+\frac{v}{2}\lambda_1+\frac{v}{2}\lambda_2)}$$

The design, among all the E GD MB TIB designs for given (k, v, b), which minimizes the $Tr(M^{-1})$, given above, is called A-optimal. In fact, such a design, if exists, has a least average variance of the BLUE's of treatemnt contrasts { $\alpha_0 - \alpha_i$}, i = 1,2,.....,v (the sum of diagonal elements of the variance-covariance matrix of the estimates).

Majumdar (1981) and Majumdar and Notz (1983) showed that if there is a design that minimizes $Tr(M^{-1})$ then it is binary in the test-treatments and the number of units allocated to the control are less than half of the the total units in the design, i.e.

$$r_0 < \frac{b\,k}{2}$$

The actual minimization of the $Tr(M^{-1})$, given above, is done for k = 2, 3, 4 and v = 4, 6, 8, 10, and for each b < 50. The A-optimal designs are obtained and given in the following tables with their possible structure through the corresponding (for that k and v) MCCGD given in the previous chapter.

6.2 A-OPTIMAL E MB GD TIB DESIGNS FOR k = 2 :

TABLE 6.2 A-OPTIMAL E MB GD TIB DESIGNS FOR k = 2.

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
2	4	4	1	0	0	D_3 (Table 3.1)
		6	1	1	0	$D_1 \cup D_3$
		8	1	0	1	$D_2 \cup D_3$
		10	2	1	0	$D_1 \cup 2D_3$
		12	2	0	1	$D_2 \cup 2D_3$
		14	2	1	1	$D_1 \cup D_2 \cup 2D_3$
		16	3	0	1	$D_1 \cup 3D_3$
		18	3	1	1	$D_1 \cup D_2 \cup 3D_3$
		20	3	2	1	$2D_1 \cup D_2 \cup 3D_3$
		22	4	1	1	$D_1 \cup D_2 \cup 4D_3$
		24	4	2	1	$2D_1 \cup D_2 \cup 4D_3$
		26	4	1	2	$D_1 \cup 2D_2 \cup 4D_3$
		28	5	2	1	$2D_1 \cup D_2 \cup 5D_3$
		30	5	1	2	$D_1 \cup 2D_2 \cup 5D_3$

TABLE 6.2 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
2	4	32	5	2	2	$2D_1 \cup 2D_2 \cup 5D_3$
		34	6	1	2	$D_1 \cup 2D_2 \cup 6D_3$
		36	6	2	2	$2D_1 \cup 2D_2 \cup 6D_3$
		38	6	3	2	$3D_1 \cup 2D_2 \cup 6D_3$
		40	7	2	2	$2D_1 \cup 2D_2 \cup 7D_3$
		42	7	3	2	$3D_1 \cup 2D_2 \cup 7D_3$
		44	8	2	2	$2D_1 \cup 2D_2 \cup 8D_3$
		46	8	3	2	$3D_1 \cup 2D_2 \cup 8D_3$
		48	8	2	3	$2D_1 \cup 3D_2 \cup 8D_3$
		50	8	3	3	$3D_1 \cup 3D_2 \cup 8D_3$
2	6	6	1	0	0	D_1 (Table 3.2)
		9	0	0	1	D_3
		12	1	1	0	$D_1 \cup D_2$
		15	1	0	1	$D_2 \cup D_3$
		18	2	1	0	$D_1 \cup 2D_2$
		21	2	0	1	$2D_2 \cup D_3$
		24	3	1	0	$D_1 \cup 3D_2$
		27	3	0	1	$3D_2 \cup D_3$
		30	4	1	0	$D_1 \cup 4D_2$
		33	3	1	1	$D_1 \cup 3D_2 \cup D_3$
		36	5	1	0	$D_1 \cup 5D_2$
		39	4	1	1	$D_1 \cup 4D_2 \cup D_3$
		42	3	1	2	$D_1 \cup 3D_2 \cup 2D_3$

TABLE 6.2 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
2	6	45	5	1	1	$D_1 \cup 5D_2 \cup D_3$
		48	4	1	2	$D_1 \cup 4D_2 \cup 2D_3$
2	8	8	1	0	0	D_1 (Table 3.3)
		12	0	1	0	D_2
		16	2	0	0	$2D_1$
		20	1	1	0	$D_1 \cup D_2$
		24	1	0	1	$D_1 \cup D_3$
		28	2	1	0	$2D_1 \cup D_2$
		32	2	0	1	$2D_1 \cup D_3$
		36	3	1	0	$3D_1 \cup D_2$
		40	3	0	1	$3D_1 \cup D_3$
		44	2	1	1	$2D_1 \cup D_2 \cup D_3$
		48	4	0	1	$4D_1 \cup D_3$
2	10	10	1	0	0	D_1 (Table 3.4)
		20	2	0	0	$2D_1$
		25	0	0	1	D_3
		30	1	1	0	$D_1 \cup D_2$
		35	1	0	1	$D_1 \cup D_3$
		40	2	1	0	$2D_1 \cup D_2$
		45	2	0	1	$2D_1 \cup D_3$
		50	3	1	0	$3D_1 \cup D_2$

6.3 A-OPTIMAL E MB GD TIB DESIGNS FOR k = 3 :

TABLE 6.3 A-OPTIMAL E MB GD TIB DESIGNS FOR k = 3.

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
3	4	2	1	1	0	D_1 (Table 4.2)
		4	2	0	1	D_4
		6	3	1	1	$D_1 \cup D_4$
		7	2	2	2	D_6
		8	4	2	1	$2D_1 \cup D_4$
		9	3	3	2	$D_1 \cup D_6$
		10	5	1	2	$D_1 \cup 2D_4$
		11	4	2	3	$D_4 \cup D_6$
		12	6	2	2	$2D_1 \cup 2D_4$
		13	5	3	3	$D_1 \cup D_4 \cup D_6$
		14	7	3	2	$3D_1 \cup 2D_4$
		15	6	4	3	$2D_1 \cup D_4 \cup D_6$

TABLE 6.3 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
3	4	16	8	2	3	$2D_1 \cup 3D_4$
		17	7	3	4	$D_1 \cup 2D_4 \cup D_6$
		18	9	3	3	$3D_1 \cup 3D_4$
		19	8	4	4	$2D_1 \cup 2D_4 \cup D_6$
		20	10	4	3	$4D_1 \cup 3D_4$
		21	9	5	4	$3D_1 \cup 2D_4 \cup D_6$
		22	11	3	4	$3D_1 \cup 4D_4$
		23	10	4	5	$2D_1 \cup 3D_4 \cup D_6$
		24	12	4	4	$4D_1 \cup 4D_4$
		25	11	5	5	$3D_1 \cup 3D_4 \cup D_6$
		26	13	5	4	$5D_1 \cup 4D_4$
		27	12	6	5	$4D_1 \cup 3D_4 \cup D_6$
		28	14	4	5	$4D_1 \cup 5D_4$
		29	13	5	6	$3D_1 \cup 4D_4 \cup D_6$
		30	15	5	5	$5D_1 \cup 5D_4$
		31	14	6	6	$4D_1 \cup 4D_4 \cup D_6$
		32	16	6	5	$6D_1 \cup 5D_4$
		33	15	7	6	$5D_1 \cup 4D_4 \cup D_6$
		34	17	5	6	$5D_1 \cup 6D_4$
		35	16	6	7	$4D_1 \cup 5D_4 \cup D_6$
		36	18	6	6	$6D_1 \cup 6D_4$
		37	17	7	7	$5D_1 \cup 5D_4 \cup D_6$

TABLE 6.3 (CONTINUED).

k	v	b	λ'_0	λ_1	λ_2	STRUCTURE
3	4	38	19	7	6	$7D_1 \cup 6D_4$
		39	18	8	7	$6D_1 \cup 5D_4 \cup D_6$
		40	20	6	7	$6D_1 \cup 7D_4$
		41	19	7	8	$5D_1 \cup 6D_4 \cup D_6$
		42	21	7	7	$7D_1 \cup 7D_4$
		43	20	8	8	$6D_1 \cup 6D_4 \cup D_6$
		44	22	8	7	$8D_1 \cup 7D_4$
		45	21	9	8	$7D_1 \cup 6D_4 \cup D_6$
		46	23	7	8	$7D_1 \cup 8D_4$
		47	22	8	9	$6D_1 \cup 7D_4 \cup D_6$
		48	24	8	8	$8D_1 \cup 8D_4$
		49	23	9	9	$7D_1 \cup 7D_4 \cup D_6$
		50	25	9	8	$9D_1 \cup 8D_4$
3	6	2	0	1	0	D_1 (Table 4.3)
		6	2	1	0	D_3
		7	1	1	1	D_4
		9	3	0	1	D_5
		10	0	2	2	D_6
		11	3	1	1	$D_1 \cup D_5$
		12	2	1	2	D_7
		13	3	2	1	$2D_1 \cup D_5$
		14	2	2	2	$2D_4$

TABLE 6.3 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
3	6	15	5	1	1	$D_3 \cup D_5$
		16	4	1	2	$D_4 \cup D_5$
		17	5	2	1	$D_1 \cup D_3 \cup D_5$
		18	4	2	2	$D_3 \cup D_7$
		19	3	2	3	$D_4 \cup D_7$
		20	6	1	2	$D_1 \cup 2D_5$
		21	7	2	1	$2D_3 \cup D_5$
		22	6	2	2	$2D_1 \cup 2D_5$
		23	5	2	3	$2D_4 \cup D_5$
		24	8	1	2	$D_3 \cup 2D_5$
		25	5	3	3	$D_1 \cup 2D_4 \cup D_5$
		26	8	2	2	$D_1 \cup D_3 \cup 2D_5$
		27	7	2	3	$D_1 \cup D_4 \cup 2D_5$
		28	8	3	2	$2D_1 \cup D_3 \cup 2D_5$
		29	7	3	3	$2D_1 \cup D_4 \cup 2D_5$
		30	10	2	2	$2D_3 \cup 2D_5$
		31	9	2	3	$2D_1 \cup 3D_5$
		32	10	3	2	$D_1 \cup 2D_3 \cup 2D_5$
		33	9	3	3	$3D_1 \cup 3D_5$
		34	8	3	4	$D_1 \cup 2D_4 \cup 2D_5$
		35	11	2	3	$D_1 \cup D_3 \cup 3D_5$
		36	12	3	2	$3D_3 \cup 2D_5$
		37	11	3	3	$2D_1 \cup D_3 \cup 3D_5$

TABLE 6.3 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
3	6	38	10	3	4	$2D_1 \cup D_4 \cup 3D_5$
		39	11	4	3	$3D_1 \cup D_3 \cup 3D_5$
		40	10	4	4	$3D_1 \cup D_4 \cup 3D_5$
		41	13	3	3	$D_1 \cup 2D_3 \cup 3D_5$
		42	12	3	4	$3D_1 \cup 4D_5$
		43	13	4	3	$2D_1 \cup 2D_3 \cup 3D_5$
		44	12	4	4	$4D_1 \cup 4D_5$
		45	15	3	3	$3D_3 \cup 3D_5$
		46	14	3	4	$2D_1 \cup D_3 \cup 4D_5$
		47	15	4	3	$D_1 \cup 3D_3 \cup 3D_5$
		48	14	4	4	$3D_1 \cup D_3 \cup 4D_5$
		49	13	4	5	$3D_1 \cup D_4 \cup 4D_5$
		50	14	5	4	$4D_1 \cup D_3 \cup 4D_5$
3	8	8	2	0	0	D_1 (Table 4.4)
		12	3	1	0	D_3
		16	4	0	1	D_4
		18	3	1	1	D_5
		19	2	2	1	D_6
		20	5	1	0	$D_1 \cup D_3$
		23	5	1	1	D_8
		24	4	2	1	$D_2 \cup D_3$
		25	2	2	2	D_{10}
		28	5	1	2	$D_2 \cup D_4$

TABLE 6.3 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
3	8	30	6	2	1	$D_3 \cup D_5$
		31	3	3	2	$D_2 \cup D_6$
		32	8	0	1	$2D_1 \cup D_4$
		34	7	1	2	$D_4 \cup D_5$
		35	6	2	2	$D_2 \cup D_8$
		36	5	3	2	$2D_2 \cup D_3$
		37	3	3	3	$D_2 \cup D_{10}$
		38	2	4	3	D_{13}
		39	9	1	2	$D_4 \cup D_8$
		40	8	2	2	$D_2 \cup D_3 \cup D_4$
		42	7	3	2	$D_6 \cup D_8$
		43	4	4	3	$2D_2 \cup D_6$
		44	11	1	2	$D_3 \cup 2D_4$
		46	10	2	2	$2D_8$
		47	9	3	2	$2D_3 \cup D_7$
		48	6	4	3	$3D_2 \cup D_3$
		49	4	4	4	$2D_2 \cup D_{10}$
		50	3	5	4	$D_2 \cup D_{13}$
3	10	10	2	0	0	D_1 (Table 4.5)
		14	2	1	0	D_2
		20	4	1	0	D_3
		25	3	1	1	D_5

TABLE 6.3 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
3	10	30	2	1	2	D_8
		32	5	1	1	D_{10}
		35	7	0	1	$D_1 \cup D_6$
		37	4	1	2	D_{12}
		39	5	2	1	$D_2 \cup D_5$
		40	8	1	0	$2D_1 \cup D_3$
		44	6	1	2	D_{13}
		45	7	2	1	$D_3 \cup D_5$
		49	3	2	3	D_{14}
		50	8	1	2	$D_5 \cup D_6$

6.4 A-OPTIMAL E MB GD TIB DESIGNS FOR k = 4.

TABLE 6.4 A-OPTIMAL E MB GD TIB DESIGNS FOR k = 4 .

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	4	1	0	1	1	D_1 (Table 5.1)
		2	2	1	0	D_2
		3	2	2	1	$D_1 \cup D_2$
		4	3	2	2	D_4
		5	3	3	3	$D_1 \cup D_4$
		6	5	3	2	$D_2 \cup D_4$
		7	5	4	3	$D_1 \cup D_2 \cup D_4$
		8	6	4	4	$2D_4$
		9	7	3	4	$D_1 \cup D_3 \cup D_4$
		10	8	5	4	$D_2 \cup 2D_4$
		11	8	6	5	$D_1 \cup D_2 \cup 2D_4$
		12	9	6	6	$3D_4$
		13	10	5	6	$D_1 \cup D_3 \cup 2D_4$
		14	11	7	6	$D_2 \cup 3D_4$
		15	11	8	7	$D_1 \cup D_2 \cup 3D_4$
		16	12	8	8	$4D_4$
		17	13	7	8	$D_1 \cup D_3 \cup 3D_4$
		18	14	9	8	$D_2 \cup 4D_4$
		19	14	10	9	$D_1 \cup D_2 \cup 4D_4$
		20	15	10	10	$5D_4$
		21	16	9	10	$D_1 \cup D_3 \cup 4D_4$

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	4	22	17	11	10	$D_2 \cup 5D_4$
		23	17	12	11	$D_1 \cup D_2 \cup 5D_4$
		24	18	12	12	$6D_4$
		25	19	11	12	$D_1 \cup D_3 \cup 5D_4$
		26	20	13	12	$D_2 \cup 6D_4$
		27	20	14	13	$D_1 \cup D_2 \cup 6D_4$
		28	21	14	14	$7D_4$
		29	22	13	14	$D_1 \cup D_3 \cup 6D_4$
		30	23	15	14	$D_2 \cup 7D_4$
		31	23	16	15	$D_1 \cup D_2 \cup 7D_4$
		32	24	16	16	$8D_4$
		33	25	15	16	$D_1 \cup D_3 \cup 7D_4$
		34	26	17	16	$D_2 \cup 8D_4$
		35	26	18	17	$D_1 \cup D_2 \cup 8D_4$
		36	27	18	18	$9D_4$
		37	28	17	18	$D_1 \cup D_3 \cup 8D_4$
		38	29	19	18	$D_2 \cup 9D_4$
		39	29	20	19	$D_1 \cup D_2 \cup 9D_4$
		40	30	20	20	$10D_4$
		41	31	19	20	$D_1 \cup D_3 \cup 9D_4$
		42	32	21	20	$D_2 \cup 10D_4$
		43	32	22	21	$D_1 \cup D_2 \cup 10D_4$
		44	33	22	22	$11D_4$

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	4	45	34	21	22	$D_1 \cup D_3 \cup 10D_4$
		46	35	23	22	$D_2 \cup 11D_4$
		47	35	24	23	$D_1 \cup D_2 \cup 11D_4$
		48	36	24	24	$12D_4$
		49	37	23	24	$D_1 \cup D_3 \cup 11D_4$
		50	38	25	24	$D_2 \cup 12D_4$
4	6	2	1	1	0	D_1 (Table 5.2)
		7	4	1	1	D_6
		9	5	2	1	$D_1 \cup D_6$
		10	5	2	2	D_{10}
		11	7	1	1	$D_1 \cup D_9$
		12	6	3	2	D_{13}
		13	8	2	1	$2D_1 \cup D_9$
		14	6	3	3	$D_1 \cup D_{12}$
		15	8	2	3	D_{16}
		16	9	3	2	$2D_1 \cup D_{14}$
		17	9	3	3	$D_1 \cup D_{16}$
		18	9	3	4	D_{17}
		19	10	4	3	$2D_1 \cup D_{16}$
		20	10	4	4	$D_1 \cup D_{17}$
		21	12	3	3	$D_1 \cup D_6 \cup D_{16}$
		22	11	5	4	$2D_1 \cup D_{17}$
		23	13	4	3	$2D_1 \cup D_6 \cup D_{14}$

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	6	24	13	4	4	$D_1 \cup D_6 \cup D_{16}$
		25	13	4	5	$D_6 \cup D_{17}$
		26	14	5	4	$D_6 \cup D_{16}$
		27	14	5	5	$D_1 \cup D_6 \cup D_{17}$
		28	14	5	6	$D_{10} \cup D_{17}$
		29	15	6	5	$2D_1 \cup D_6 \cup D_{17}$
		30	15	6	6	$D_1 \cup D_{10} \cup D_{17}$
		31	17	5	5	$D_1 \cup 2D_6 \cup D_{16}$
		32	16	7	6	$2D_1 \cup D_{10} \cup D_{17}$
		33	18	6	5	$2D_1 \cup 2D_5 \cup D_{15}$
		34	18	6	6	$D_1 \cup 2D_6 \cup D_{17}$
		35	18	6	7	$D_6 \cup D_{10} \cup D_{17}$
		36	19	7	6	$2D_1 \cup 2D_6 \cup D_{17}$
		37	19	7	7	$D_1 \cup D_6 \cup D_{10} \cup D_{17}$
		38	19	7	8	$2D_{10} \cup D_{17}$
		39	20	8	7	$2D_1 \cup D_6 \cup D_{10} \cup D_{17}$
		40	20	8	8	$D_1 \cup 2D_{10} \cup D_{17}$
		41	22	7	7	$D_1 \cup 3D_6 \cup D_{17}$
		42	21	9	8	$2D_1 \cup 2D_{10} \cup D_{17}$
		43	23	8	7	$2D_1 \cup 3D_6 \cup D_{17}$
		44	23	8	8	$D_1 \cup 2D_6 \cup D_{10} \cup D_{17}$
		45	23	8	9	$D_6 \cup 2D_{10} \cup D_{17}$

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	6	46	24	9	8	$2D_1 \cup 2D_6 \cup D_{10} \cup D_{17}$
		47	24	9	9	$D_1 \cup D_6 \cup 2D_{10} \cup D_{17}$
		48	24	9	10	$3D_{10} \cup D_{17}$
		49	25	10	9	$2D_1 \cup D_6 \cup 2D_{10} \cup D_{17}$
		50	25	10	10	$D_1 \cup 3D_{10} \cup D_{17}$
4	8	2	0	1	0	D_1 (Table 5.3)
		10	2	2	1	D_3
		12	5	1	1	D_5
		14	5	2	1	$D_1 \cup D_5$
		15	4	2	2	D_7
		16	3	2	3	D_8
		17	4	3	2	$D_1 \cup D_7$
		19	7	2	2	D_{11}
		20	6	2	3	D_{14}
		21	7	3	2	$D_1 \cup D_{11}$
		22	6	3	3	$D_1 \cup D_{14}$
		23	10	1	2	D_{15}
		24	9	2	3	D_{17}
		26	9	3	3	$D_1 \cup D_{17}$
		27	4	4	5	$D_4 \cup D_7$
		28	9	4	3	$2D_1 \cup D_{17}$
		29	4	5	5	$D_1 \cup D_4 \cup D_7$
		30	12	2	3	D_{18}

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	8	31	12	3	3	$D_5 \cup D_{11}$
		32	12	4	3	D_{19}
		33	7	5	5	$D_1 \cup D_4 \cup D_{11}$
		34	11	4	4	$D_1 \cup D_5 \cup D_{14}$
		35	10	4	5	$D_7 \cup D_{14}$
		36	14	3	4	$D_5 \cup D_{17}$
		37	10	5	5	$D_1 \cup D_7 \cup D_{14}$
		38	14	4	4	$D_1 \cup D_5 \cup D_{17}$
		39	13	4	5	$D_7 \cup D_{17}$
		40	14	5	4	$2D_1 \cup D_5 \cup D_{17}$
		41	13	5	5	$D_1 \cup D_7 \cup D_{17}$
		42	12	5	6	$D_1 \cup D_8 \cup D_{17}$
		43	16	4	5	$D_{11} \cup D_{17}$
		44	17	5	4	$D_5 \cup D_{19}$
		45	16	5	5	$D_1 \cup D_{11} \cup D_{17}$
		46	15	5	6	$D_1 \cup D_{14} \cup D_{17}$
		47	16	6	5	$D_7 \cup D_{19}$
		48	15	6	6	$2D_1 \cup D_{14} \cup D_{17}$
		49	10	7	8	$D_1 \cup D_4 \cup D_7 \cup D_{14}$
		50	18	5	6	$D_1 \cup 2D_{17}$
4	10	10	3	0	0	D_1 (Table 5.4)
		13	3	1	1	D_2

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	10	14	5	1	0	D_3
		15	2	2	1	D_5
		20	5	2	1	D_9
		25	6	2	2	D_{12}
		26	4	2	3	D_{14}
		27	5	3	2	D_{15}
		28	3	3	3	$D_2 \cup D_4$
		30	9	2	2	D_{17}
		31	7	2	3	D_{18}
		32	8	3	2	D_{21}
		33	6	3	3	$D_2 \cup D_8$
		34	4	3	4	D_{22}
		36	10	2	3	D_{23}
		37	11	3	2	D_{26}
		38	9	3	3	$D_2 \cup D_{12}$
		39	7	3	4	$D_2 \cup D_{14}$
		40	8	4	3	$D_2 \cup D_{15}$
		41	4	4	5	$D_4 \cup D_{14}$
		42	13	2	3	D_{27}
		43	12	3	3	$D_2 \cup D_{17}$
		44	10	3	4	$D_2 \cup D_{18}$
		45	11	4	3	$D_2 \cup D_{21}$

TABLE 6.4 (CONTINUED).

k	v	b	λ_0	λ_1	λ_2	STRUCTURE
4	10	46	7	4	5	$D_4 \cup D_{18}$
		47	8	5	4	$D_4 \cup D_{21}$
		48	6	5	5	$D_2 \cup D_4 \cup D_8$
		49	13	3	4	$D_2 \cup D_{23}$
		50	12	4	4	$D_8 \cup D_{17}$

CHAPTER 7

7.1 COMPUTER ALGORITHM FOR FINDING ALL BINARY E MB GD TIB DESIGNS :

The program " DSGN " (See Appendix-A) uses an algorithm based on the discussion in Section 2.2 and generates all possibily existing, admissible,and binary E MB GD TIB designs for a given (v,k). The details are as follows.

INPUT: An input disk file named " PARAMS " (abbreviation of parameters of the design) is created and saved by the user with the following ent-ries.

K (represents the block size k) in the first column.

V (represents the number of test-treatments v) in the next three columns, right justified.

MAXB (represents the maximum number of blocks for which the designs are required, i.e. $1 \leq b \leq$ MAXB) in the next three columns, right justified.

ALGORITHM:

Step 1. Generate a value B [B = 1(1)MAXB]

Step 2. Call Subroutine " VECTOR " which generates the vector M [represents the vector $\underset{\sim}{m}$ = (m_1, m_2, ...,m_{k-1})].

If all possible vectors M have been generated then go to Step 1 to generate the next value of B, otherwise continue.

Step 3. Compute R (represents r, the replications of test-treatments) and check whether it is an integer.

If not an integer then go to Step 2 to generate next possible vector M, otherwise continue.

Step 4. Compute LD0 (represents λ_0) and check whether it is an integer.

If not an integer then go to Step 2, otherwise continue.

Step 5. Compute LD1 (represents λ_1) and check whether it is an integer.

If not an integer then go to Step 2, otherwise continue.

Step 6 Compute LD2 (represents λ_2)

Step 7 Call Subroutine " EXIST " which checks for the non-existence of the design.

If the design is non-existent go to Step 2, otherwise continue.

Step 8 Call Subroutine " DTYP " which decides the type of design as BIB, BTIB, or MBTIB designs.

Step 9 Call Subroutine " ADMSB " (see remark below) which checks whether the design is admissible or not (see Section 2.1.4).

If inadmissible go to Step 2, otherwise continue.

Step 10 Call Subroutine " OPTIM " which finds the optimal design. Then go to Step 2.

The program will automatically stop when the B values are exhausted (i.e. when B > MAXB) by giving a massage " JOB COMPLETED ! ".

<u>OUTPUT:</u>

1. All designs are stored in a disk file (Created by the program) named as " DESIGN " .

NOTE: The file "DESIGN" can be written (properly and automatically formatted) on the printer by running a program "PRT1" given in Appendix-D.

The sample output of this file is given in Appendix-G.

2. All optimal designs classified in BIB, BTIB, or MBTIB designs are stored in a disk file (created by the program) named as "OPTIMAL".

NOTE: The file "OPTIMAL" can be printed (properly and automatically formatted) by running a program "PRT2" given in Appendix-E.

A sample output of this file is given in Appendix-H.

<u>REMARK:</u> In order to check the admissibility of a design a matrix named "ADM" is created by the program. The i th row of this matrix corresponds to $\lambda_0 = i - 1$. The parameters b, λ_1, λ_2 are entered into the

(λ_0 + 1) th row of this matrix. So, "ADM" looks as follows (for example).

	b	λ_1	λ_2	
row 1	...	...	...	(λ_0 = 0)
2	...	...	...	(λ_0 = 1)
⋮	⋮	⋮	⋮	⋮
8	7	2	1	(λ_0 = 8)
⋮	⋮	⋮	⋮	⋮

Hence, the program knows that for $\lambda_0 = 8$ (for example) a design with b = 7, $\lambda_1 = 2$, $\lambda_2 = 1$ is already available. So, if the program find a candidate of the form (b, $\lambda_0 = 8$, $\lambda_1 = 2$, $\lambda_2 = 1$) with $b \geq 7$ it discard it treating as equivalent or inadmissible design.

In this way all equivalent or inadmissible designs can be discarded.

7.2 COMPUTER ALGORITHM/PROGRAM FOR FINDING A POSSIBLE SET OF DESIGNS TO BE MCCGD, AND THE STRUCTURED CONSTRUCTION OF ALL DESIGNS :

The program "SEARCH" (given in Appendix-B), helps in finding a MCCGDs for a given (v,k).

INPUT:1. Disk file "PARAMS" created by the user and updated by the program "DSGN" (Appendix-A)

2. Disk file "DESIGN" created by the program "DSGN" (Appendix-A)

OUTPUT:Disk file "GD" created by the program "SEARCH" contains the possible candidates for the MCCGDs for the (v,k).

REMARK: Each design in the file "GD" is tried to be constructed manually. The design which cannot be constructed (a little trial and error is required) should be deleted from the file "GD" by the user. In this way the file "GD" is updated and then saved containing only the MCCGDs.

The program "STRUCT" (given in Appendix-C) is used to find the structure of a design by finding the possible union of replicates (copies) of generator designs which is equivalent to the design structured.

INPUT:1. Disk file "PARAMS" updated by the program "DSGN" (Appendix-A).

2. Disk file "DESIGN" created by the program "DSGN" (Appendix-A).

3. Disk file "GD" updated by the user as MCCGDs.

OUTPUT:Disk file "STRC" created by the program "STRUCT" contains the parameters of the designs structured followed by the labels of the generator designs used and the number of copies of them required.

NOTE: The file "STRC" can be written on a printer (properly and automatically formatted) by running the program "PRT3" given in Appendix-F. A sample output of the file is given in Appendix-I.

REMARK: All the program given here are written in standard FORTRAN and are run on Burroughs large system main frame of the Middle East Technical University.

CHAPTER 8

CONCLUSIONS:

An Extended Class of Treatment Incomplete Block Designs is introduced. This class contains all binary Balanced Incomplete Block (BIB) designs, Balanced Treatment Incomplete Block (BTIB) designs, and the Most Balanced Group Divisible Treatment Incomplete Block (MB GD TIB) designs. The minimal complete classes of generator designs (MCCGDs) are constructed for the practical range of k (the size of blocks) and v (number of test-treatments) values and the minimal complete nature of such classes is proved through case by case enumeration.A general non-existence theorem for GD TIB designs is given and the usefulness of the theorem is explained through examples. A general method/ algorithm for the construction of all admissible E MB GD TIB designs is suggested and applied inorder to construct the designs for various parameters values.

The A-optimality criterion is applied in order to obtain optimal designs for a given (v,k,b). Elaborate computer programs/subroutines in standard FORTRAN are prepared which are not only useful in finding the MCCGD but also suggest possible structure of these designs. The program sorts out the A-optimal designs as well. Some trial and error is required in finding the actual structure of these designs.These efforts are, however, minimized through the use of the computer program provided here.

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APPENDICES:

APPENDIX-A: (PROGRAM " DSGN ")

```
FILE   7(KIND=DISK,TITLE="PARAMS",FILETYPE=7)
FILE   8(KIND=DISK,TITLE="ADMSB",FILETYPE=7)
FILE   9(KIND=DISK,TITLE="DESIGN",FILETYPE=7)
FILE   10(KIND=DISK,TITLE="OPTIMAL",FILETYPE=7)
       INTEGER V,U,W,STD,STO,CTN,PB,PM(8),B,R0,R,TCP,TTP,S3,
      *        M(8),ADM(100,12)
       REAL DSG(1700,20),OPT(320,20)
       READ(7,2000)K,V,MAXB,MLD0,LB,U,W,STD,STO,CTN,PB,
      *(PM(J),J=1,K-1)
       CTN=CTN+1
       IF(LB.GT.MAXB) GO TO 1800
       IF(LB.NE.0) GO TO 1900
       DO 10 ROW = 1,100
       DO 10 COL = 1,12
   10  ADM(ROW,COL)=-1
       PB=1
       STD=1
       STO=1
       MINB=1
       W=1
   50  DO 200 B = MINB,MAXB
       PRINT/,"DESIGN IN PROCESS B=",B
       PB=B
       CALL VECTOR(K,B,R0,PM,M,I,N,S3,MAXL,&100,&700,&1700)
  100  IF(R0.GT.(K-1)*B) GO TO 600
       IF(MOD(B*K-R0,V).EQ.0) GO TO 400
       GO TO 600
  150  LB=PB
       IF(OPT(1+(K+7)*(STO-1),4*W-3).EQ.0) GO TO 200
       W=W+1
       IF(W.EQ.6) GO TO 300
  200  CONTINUE
       PRINT/,"JOB COMPLETED !"
       GO TO 1700
  300  W=1
       STO=STO+1
       GO TO 200
  400  R=(B*K-R0)/V
  500  TCP=(K-1)*R0-S3
       IF(MOD(TCP,V).EQ.0) GO TO 900
  600  CALL VECT(K,B,R0,PM,M,I,N,S3,MAXL,&100,&700,&1700)
  700  DO 800 J = 1,K-1
       PM(J)=0
  800  CONTINUE
       GO TO 150
```

```
  900 LD0=TCP/V
      IF(LD0.LT.MAXL) GO TO 600
      IF(LD0.GT.MLD0) MLD0=LDO
      TTP=K*(K-1)*B/2-S3/2-TCP
      N1=4*TTP
      N2=2*V*(V-1)
      IF(MOD(N1,N2).EQ.0) GO TO 910
      IF(MOD(N1-V**2,N2).EQ.0) GO TO 920
      IF(MOD(N1+V**2,N2).EQ.0) GO TO 930
      GO TO 600
  910 LD1=N1/N2
      LD2=LD1
      GO TO 1200
  920 LD1=(N1-V**2)/N2
      LD2=LD1+1
      GO TO 1200
  930 LD1=(N1+V**2)/N2
      LD2=LD1-1
 1200 IF(LD2.LT.0.OR.LD1.GT.R) GO TO 600
      CALL EXIST(V,B,R,LD1,LD2,&1300,&600)
 1300 IF(LD1.EQ.LD2) GO TO 1320
      IF(LD1.EQ.LD2+1.OR.LD1.EQ.LD2-1) GO TO 1310
      DSN="   PB"
      GO TO 1360
 1310 DSN="MBTIB"
      GO TO 1360
 1320 IF(LD0.EQ.LD1) GO TO 1340
 1330 DSN=" BTIB"
      GO TO 1360
 1340 IF(R.EQ.R0) GO TO 1350
      GO TO 1330
 1350 DSN="  BIB"
 1360 IF(DSN.EQ."   PB") GO TO 600
      IF(LD0.EQ.0) GO TO 1370
      NUM=(LD0+LD1+V*LD2/2.)*(LD0+V*LD2/2.)-V*(V-2)*LD2**2
     *    /4.
      DEN=LD0*(LD0+V*LD2)*(LD0+V*LD1/2.+V*LD2/2.)
      TRC=V*K*NUM/DEN
      GO TO 1380
 1370 TRC=-1
 1380 CALL ADMSB(K,B,AD,LD0,LD1,LD2,DSN,ADM,M,PM,&1400,&600)
 1400 U=U+1
      LB=B
      DSG(1+(K+8)*(STD-1),U)=B
      DSG(2+(K+8)*(STD-1),U)=R0
      DSG(3+(K+8)*(STD-1),U)=R
      DO 1410 ROW = 1,K-1
 1410 DSG(ROW+3+(K+8)*(STD-1),U)=M(ROW)
      DSG((K+8)*STD-5,U)=LD0
      DSG((K+8)*STD-4,U)=LD1
      DSG((K+8)*STD-3,U)=LD2
```

```
       DSG((K+8)*STD-2,U)=TRC
      DSG((K+8)*STD-1,U)=DSN
      DSG((K+8)*STD,U)=AD
      IF(U.EQ.20) GO TO 1600
 1500 IF(OPT(1+(K+7)*(STO-1),4*W-3).EQ.0) GO TO 1510
      IF(R0.GT.INT(B*K/2.+.5)) GO TO 600
 1510 CALL OPTIM(K,W,STO,TRC,DSN,M,OPT,B,R0,R,LD0,LD1,LD2)
      GO TO 600
 1600 U=0
      STD=STD+1
      GO TO 1500
 1700 DO 1750 J = 1,K-1
 1750 PM(J)=M(J)
      CALL SAVE(K,V,MAXB,MLD0,LB,U,W,STD,STO,CTN,B,PM,ADM,DSG,
     *          OPT)
 1800 STOP
 1900 CALL RINIT(K,MLD0,STD,STO,ADM,DSG,OPT)
      MINB=PB
      GO TO 50
 2000 FORMAT(I1,2(X,I2),X,I3,2(X,I2),X,I1,X,I3,X,I2,X,I3,X,I2,
     *       8(X,I3))
      END
      SUBROUTINE VECTOR(K,B,R0,PM,M,I,N,S3,MAXL,*,*,*)
      INTEGER S,R0,PM(8),M(8),S3,S1,S2
      DO 100 I = 1,K-1
      M(I)=PM(I)
  100 CONTINUE
      I=0
  110 IF(I.EQ.0) GO TO 120
      M(K-1)=-1
      GO TO 150
  120 M(K-1)=M(K-1)-1
  150 I=K-1
      N=B
      IF(K.EQ.2) GO TO 250
      DO 200 J = 1,K-2
      N=N-M(J)
  200 CONTINUE
      ENTRY VECT(K,B,R0,PM,M,I,N,S3,MAXL,*,*,*)
  250 M(I)=M(I)+1
      IF(M(I).GT.N) GO TO 700
      IF(TIME(2).GT.1400) RETURN3
      S1=0
      S2=0
      S3=0
      MAXL=0
      DO 300 L = 1,K-1
      S1=S1+M(L)
      S2=S2+L*M(L)
      S3=S3+L*(L-1)*M(L)
      IF(M(L).GT.0) MAXL=L
  300 CONTINUE
      IF(S1.GT.B) GO TO 400
      R0=S2
```

```
      RETURN1
400  I=I-1
     IF(I.EQ.0) GO TO 650
     IF(I.EQ.1) GO TO 600
     N=B
     DO 450 J = 1,I-1
     N=N-M(J)
450  CONTINUE
     IF(M(I).EQ.N) GO TO 550
500  M(I)=M(I)+1
     GO TO 110
550  IF(I.EQ.1) GO TO 650
     M(I)=0
     GO TO 400
600  IF(M(I).EQ.B) GO TO 650
     GO TO 500
650  RETURN2
700  DO 750 J = 1,K-1
     PM(J)=M(J)
750  CONTINUE
     GO TO 400
     END
     SUBROUTINE EXIST(V,B,R,LD1,LD2,*,*)
     INTEGER V,B,R
     IF(R.LT.LD1.OR.R.LT.V*(LD2-LD1)/2+LD1) GO TO 300
     IF(B.GE.V) GO TO 250
     IF(R.EQ.LD1) GO TO 200
     IF(LD1.LT.LD2) GO TO 100
     GO TO 300
100  IF(R.EQ.LD1+V*(LD2-LD1)/2) GO TO 150
     GO TO 300
150  IF(B.EQ.V-1) GO TO 250
     GO TO 300
200  IF(LD1.EQ.LD2) GO TO 250
     IF(B.GE.2) GO TO 250
     GO TO 300
250  RETURN1
300  RETURN2
     END
     SUBROUTINE ADMSB(K,B,AD,LD0,LD1,LD2,DSN,ADM,M,PM,*,*)
     INTEGER B,ADM(100,12),M(8),PM(8)
     J=10
     AD="  NAD"
400  IF(B.EQ.ADM(LD0+1,J).AND.LD1.EQ.ADM(LD0+1,J+1).AND.LD2.
    *   EQ.ADM(LD0+1,J-2)) GO TO 850
     IF(B.EQ.ADM(LD0+1,J).AND.LD1.LE.ADM(LD0+1,J+1).AND.LD2.
    *   LE.ADM(LD0+1,J-2)) GO TO 750
     IF(LD1.LE.ADM(LD0+1,J+1).AND.LD2.LE.ADM(LD0+1,J+2))
    *   GO TO 750
450  ADM(LD0+1,J)=B
     ADM(LD0+1,J+1)=LD1
     ADM(LD0+1,J+2)=LD2
     IF(J.EQ.10) GO TO 500
```

```
      RETURN1
500  AD="  ADM"
     INDC=2
550  IF(DSN.EQ."  BIB") GO TO 700
     IF(DSN.EQ." BTIB") GO TO 650
     J=7
600  IF(INDC.EQ.1) GO TO 400
     GO TO 450
650  J=4
     GO TO 600
700  J=1
     GO TO 600
750  IF(J.EQ.10) GO TO 800
     RETURN2
800  INDC=1
     GO TO 550
850  NN=0
     DO 900 JJ = 1,K-1
     IF(M(JJ).EQ.PM(JJ)) NN=NN+1
900  CONTINUE
     IF(NN.EQ.K-1) RETURN2
     GO TO 450
     END
     SUBROUTINE OPTIM(K,W,STO,TRC,DSN,M,OPT,B,R0,R,LD0,LD1,
    *                 LD2)
     INTEGER W,STO,M(8),B,R0,R,ROW
     REAL OPT(320,20)
     J=1
100  IF(TRC.LT.OPT((K+7)*STO-1,J+4*(W-1))) GO TO 400
150  IF(J.EQ.1) GO TO 200
     RETURN
200  IF(DSN.EQ."  BIB") GO TO 350
     IF(DSN.EQ." BTIB") GO TO 300
     IF(DSN.EQ."MBTIB") GO TO 250
     IF(J.EQ.1) RETURN
     GO TO 100
250  J=4
     GO TO 100
300  J=3
     GO TO 100
350  J=2
     GO TO 100
400  IF(TRC.LT.0) TRC=999.9
     OPT(1+(K+7)*(STO-1),J+4*(W-1))=B
     OPT(2+(K+7)*(STO-1),J+4*(W-1))=R0
     OPT(3+(K+7)*(STO-1),J+4*(W-1))=R
     DO 450 ROW = 1,K-1
     OPT(ROW+3+(K+7)*(STO-1),J+4*(W-1))=M(ROW)
450  CONTINUE
     OPT((K+7)*STO-4,J+4*(W-1))=LD0
     OPT((K+7)*STO-3,J+4*(W-1))=LD1
     OPT((K+7)*STO-2,J+4*(W-1))=LD2
```

```
      OPT((K+7)*STO-1,J+4*(W-1))=TRC
      OPT((K+7)*STO,J+4*(W-1))=DSN
      GO TO 150
      END
      SUBROUTINE SAVE(K,V,MAXB,MLD0,LB,U,W,STD,STO,CTN,PB,PM,
     *                ADM,DSG,OPT)
      INTEGER V,U,W,STD,STO,CTN,PB,PM(8),ADM(100,12),ROW,COL,
     *        ST
      REAL DSG(1700,20),OPT(320,20)
      REWIND(7)
      WRITE(7,500)K,V,MAXB,MLD0,LB,U,W,STD,STO,CTN,PB,(PM(J),
     *            J=1,K-1)
      LOCK(7)
      REWIND(8)
      WRITE(8,600)((ADM(ROW,COL),COL=1,12),ROW=1,MLD0+1)
      LOCK(8)
      REWIND(9)
      DO 200 ST = 1,STD
      WRITE(9,700)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW+1
     *              ,K+5)
      WRITE(9,750)(DSG((K+8)*ST-2,COL),COL=1,20)
      WRITE(9,800)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW=
     *              K+7,K+8)
  200 CONTINUE
      LOCK(9)
      REWIND(10)
      DO 400 ST = 1,STO
      WRITE(10,700(((OPT(ROW+(K+7)*(ST-1),COL),COL=1,20),ROW=
     *               1,K+5)
      WRITE(10,750)(OPT((K+7)*ST-1,COL),COL=1,20)
      WRITE(10,800)(OPT((K+7)*ST,COL),COL=1,20)
  400 CONTINUE
      LOCK(10)
      RETURN
  500 FORMAT(I1,2(X,I2),X,I3,2(X,I2),X,I1,X,I3,X,I2,X,I3,X,I2
     *       ,8(X,I3))
  600 FORMAT(4(3(X,I3),2X))
  700 FORMAT(20(X,F5.1))
  750 FORMAT(20(X,F5.2))
  800 FORMAT(20(X,A5))
      END
      SUBROUTINE RINIT(K,MLD0,STD,STO,ADM,DSG,OPT)
      INTEGER STD,STO,ADM(100,12),ROW,COL,ST
      REAL DSG(1700,20),OPT(320,20)
      READ(8,850)((ADM(ROW,COL),COL=1,12),ROW=1,MLD0+1)
      DO 100 ROW = MLD0+2,100
      DO 100 COL = 1,12
      ADM(ROW,COL)=-1
  100 CONTINUE
      DO 200 ST =1,STD
      READ(9,900)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW=1,
     *             K+5)
```

```
      READ(9,920)(DSG((K+8)*ST-2,COL),COL=1,20)
      READ(9,950)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW=
     *              K+7,K+8)
  200 CONTINUE
      DO 300 ST = 1,STO
      READ(10,900)((OPT(ROW+(K+7)*(ST-1),COL),COL=1,20),ROW=1
     *               ,K+5)
      READ(10,920)(OPT((K+7)*ST-1,COL),COL=1,20)
      READ(10,950)(OPT((K+7)*ST,COL),COL=1,20)
  300 CONTINUE
      RETURN
  850 FORMAT(4(3(X,I3),2X))
  900 FORMAT(20(X,F5.1))
  920 FORMAT(20(X,F5.2))
  950 FORMAT(20(X,A5))
      END
```

APPENDIX-B: (PROGRAM "SEARCH")

```
FILE  7(KIND=DISK,TITLE="PARAMS",FILETYPE=7)
FILE  9(KIND=DISK,TITLE="DESIGN",FILETYPE=7)
FILE  10(KIND=DISK,TITLE="GD",FILETYPE=7)
      INTEGER V,STD,ST,STS,ROW,COL,CL,B,DB,DL0,DL1,DL2,BK(100
     *        ),LD0(100),LD1(100),LD2(100)
      REAL DSG(1700,20)
      READ(7,2000)K,V,STD
      DO 100 ST = 1,STD
      READ(9,2500)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW=1
     *               ,K+5)
      READ(9,3000)(DSG(K+6+(K+8)*(ST-1),COL),COL=1,20)
      READ(9,3500)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW+K
     *               +7,k+8)
  100 CONTINUE
      MAXJ=1
      BK(1)=DSG(1,1)
      LD0(1)=DSG(K+3,1)
      LD1(1)=DSG(K+4,1)
      LD2(1)=DSG(K+5,1)
      DO 870 ST = 1,STD
      DO 870 COL = 1,20
      B=DSG(1+(K+8)*(ST-1),COL)
      IF(B.EQ.999.OR.B.EQ.0) GO TO 180
      L0=DSG(K+3+(K+8)*(ST-1),COL)
      L1=DSG(K+4+(K+8)*(ST-1),COL)
      L2=DSG(K+5+(K+8)*(ST-1),COL)
      DO 150 J = 0,MAXJ-1
      IF(B.NE.BK(MAXJ-J)) GO TO 160
      IF(L0.EQ.LD0(MAXJ-J).AND.L1.EQ.LD1(MAXJ-J).AND.L2.EQ.
     *   LD2(MAXJ-J)) GO TO 870
```

```
 150 CONTINUE
 160 DO 850 J = 1,MAXJ
     IF(B-BK(J).GE.DSG(1,1)) GO TO 500
     IF(ST.EQ.1.AND.COL.EQ.1) GO TO 870
     GO TO 750
 180 DO 200 J = 1,MAXJ
 200 WRITE(10,4000)BK(J),LD0(J),LD1(J),LD2(J)
     LOCK(10)
     PRINT/,"JOB COMPLETED !"
     STOP
 500 IF(L0.GE.LD0(J).AND.L1.GE.LD1(J).AND.L2.GE.LD2(J))
    *   GO TO 550
     GO TO 850
 550 DB=B-BK(J)
     DL0=L0-LD0(J)
     DL1=L1-LD1(J)
     DL2=L2-LD2(J)
     STS=ST
 600 IF(DB.GT.DSG(1+(K+8)*(STS-1),1)) GO TO 650
     IF(STS.EQ.1) GO TO 650
     STS=STS-1
     GO TO 600
 650 DO 700 CL = 1,20
     IF(DB.EQ.DSG(1+(K+8)*(STS-1),CL)) GO TO 800
     IF(DB.LT.DSG(1+(K+8)*(STS-1),CL)) GO TO 850
 700 CONTINUE
 720 STS=STS+1
     IF(DB.EQ.DSG(1+(K+8)*(STS-1),1)) GO TO 650
     GO TO 850
 750 MAXJ=MAXJ+1
     BK(MAXJ)=B
     LD0(MAXJ)=L0
     LD1(MAXJ)=L1
     LD2(MAXJ)=L2
     GO TO 870
 800 IF(DL0.EQ.DSG(K+3+(K+8)*(STS-1),CL).AND.DL1.EQ.DSG(K+4+
    *    (K+8)*(STS-1),CL).AND.DL2.EQ.DSG(K+5+(K+8)*(STS-1),
    *   CL)) GO TO 870
     GO TO 700
 850 CONTINUE
     GO TO 750
 870 CONTINUE
     GO TO 180
2000 FORMAT(I1,X,I2,16X,I3)
2500 FORMAT(20(X,F5.1))
3000 FORMAT(20(X,F5.2))
3500 FORMAT(20(X,A5))
4000 FORMAT(4(X,I3))
     END
```

APPENDIX-C: (PROGRAM "STRUCT")

```
FILE  7(KIND=DISK,TITLE="PARAMS",FILETYPE=7)
FILE  9(KIND=DISK,TITLE="DESIGN",FILETYPE=7)
FILE  10(KIND=DISK,TITLE="GD",FILETYPE=7)
FILE  11(KIND=DISK,TITLE="STRC",FILETYPE=7)
      INTEGER V,STD,ST,STS,ROW,COL,CL,B,DB,DL0,DL1,DL2,BK(100
     *        ),LD0(100),LD1(100),LD2(100),STR(10000,20)
      REAL DSG(1700,20)
      READ(7,2000)K,V,STD
      DO 100 ST = 1,STD
      READ(9,2500)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW=1
     *              ,K+5)
      READ(9,3000)(DSG(K+6+(K+8)*(ST-1),COL),COL=1,20)
      READ(9,3500)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20),ROW=K
     *              +7,K+8)
  100 CONTINUE
      N=0
      DO 150 J = 1,1000
      READ(10,8050,END=200)BK(J),LD0(J),LD1(J),LD2(J)
      N=N+1
  150 CONTINUE
  200 DO 250 ST = 1,100
      DO 250 ROW = 1,N+4
      DO 250 COL = 1,20
      STR(ROW+(N+4)*(ST-1),COL)=999
  250 CONTINUE
      DO 500 ST = 1,STD
      DO 450 COL = 1,20
      B=DSG(1+(K+8)*(ST-1),COL)
      IF(B.GE.999.OR.B.EQ.0) GO TO 560
      L0=DSG(K+3+(K+8)*(ST-1),COL)
      L1=DSG(K+4+(K+8)*(ST-1),COL)
      L2=DSG(K+5+(K+8)*(ST-1),COL)
      STR(1+(N+4)*(ST-1),COL)=B
      STR(2+(N+4)*(ST-1),COL)=L0
      STR(3+(N+4)*(ST-1),COL)=L1
      STR(4+(N+4)*(ST-1),COL)=L2
      IF(B.GT.BK(N)) GO TO 350
      DO 300 J = 1,N
      IF(B.EQ.BK(J)) GO TO 1000
  300 CONTINUE
  350 DO 400 J = 1,N
      DB=B-BK(J)
      DL0=L0-LD0(J)
      DL1=L1-LD1(J)
      DL2=L2-LD2(J)
      STS=ST
      IF(DB.GE.DSG(1,1)) GO TO 600
  400 CONTINUE
  450 CONTINUE
```

```
 500 CONTINUE
 560 REWIND(11)
     DO 550 ST = 1,STD
     WRITE(11,9050)((STR(ROW+(N+4)*(ST-1),COL),COL=1,20),ROW
    *                =1,N+4)
 550 CONTINUE
     LOCK(11)
     PRINT/,"JOB COMPLETED !"
     STOP
 600 IF(DL0.LT.0.OR.DL1.LT.0.OR.DL2.LT.0) GO TO 400
 650 IF(DB.GT.DSG(1+(K+8)*(STS-1),1)) GO TO 700
     IF(STS.EQ.1) GO TO 700
     STS=STS-1
     GO TO 650
 700 DO 700 CL = 1,20
     IF(DB.EQ.DSG(1+(K+8)*(STS-1),CL)) GO TO 800
     IF(DB.LT.DSG(1+(K+8)*(STS-1),CL)) GO TO 400
 750 CONTINUE
     STS=STS+1
     IF(DB.EQ.DSG(1+(K+8)*(STS-1),1)) GO TO 700
     GO TO 400
 800 IF(DL0.EQ.DSG(K+3+(k+8)*(STS-1),CL).AND.DL1.EQ.DSG(K+4+
    *   (K+8)*(STS-1),CL.AND.DL2.EQ.DSG(K+5+(K+8)*(STS-1),CL
    *    )) GO TO 850
     GO TO 750
 850 STR(J+4+(N+4)*(ST-1),COL)=1
     DO 900 JJ = 1,N
     IF(STR(JJ+4+(N+4)*(STS-1),CL).EQ.999) GO TO 900
     IF(STR(JJ+4+(N+4)*(ST-1),COL).EQ.999) GO TO 950
     STR(JJ+4+(N+4)*(ST-1),COL)=STR(JJ+4+(N+4)*(ST-1),COL)+
    *   STR(JJ+4+(N+2)*(STS-1),CL)
 900 CONTINUE
     GO TO 450
 950 STR(JJ+4+(N+4)*(ST-1),COL)=STR(JJ+4+(N+4)*(STS-1),CL)
     GO TO 900
1000 IF(L0.EQ.LD0(J).AND.L1.EQ.LD1(J).AND.L2.EQ.LD2(J))
    *   GO TO 1050
     GO TO 300
1050 STR(J+4+(N+4)*(ST-1),COL)=1
     GO TO 450
2000 FORMAT(I1,X,I2,16X,I3)
2500 FORMAT(20(X,F5.1))
3000 FORMAT(20(X,F5.2))
3500 FORMAT(20(X,A5))
8050 FORMAT(4(X,I3))
9050 FORMAT(20(X,I3))
     END
```

APPENDIX-D: (PROGRAM "PRT1")

```
FILE  7(KIND=DISK,TITLE="PARAMS",FILETYPE=7)
FILE  8(KIND=DISK,TITLE="DESIGN",FILETYPE=7)
FILE  9(KIND=PRINTER)
      INTEGER V,STD,ST,ROW,COL
      REAL HD(17),DSG(1700,20)
      READ(7,500)K,V,STD
      WRITE(9,600)K,V
      HD(1)="  BI"
      HD(2)=" ROI"
      HD(3)="  RI"
      HD(K+3)="LD0I"
      HD(K+4)="LD1I"
      HD(K+5)="LD2I"
      HD(K+6)="TRCI"
      HD(K+7)="DSNI"
      HD(K+8)="ADMI"
      DO 100 ST = 1,STD
      READ(8,700,END=105)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20
     *                         ),ROW+1,K+5)
      READ(8,750,END=105)(DSG(K+6+(K+8)*(ST-1),COL),COL=1,20)
      READ(8,800,END=105)((DSG(ROW+(K+8)*(ST-1),COL),COL=1,20
     *                         ),ROW=K+7,K+8)
  100 CONTINUE
  105 IF(DSG(1+(K+8)*(STD-1),1).EQ.0) STD=STD-1
      MROW=K+6
      DO 120 ST = 1,STD
      IF(ST.EQ.STD) MROW=1
      DO 120 ROW = MROW,K+6
      DO 120 COL = 1,20
      IF(ROW.EQ.K+6) GO TO 130
      IF(DSG(ROW+(K+8)*(ST-1),COL).LE.0) DSG(ROW+(K+8)*(ST-1)
     *   ,COL)=999999.
  120 CONTINUE
      GO TO 150
  130 IF(DSG(ROW+(K+8)*(ST-1),COL).LE.0) DSG(ROW+(K+8)*(ST-1)
     *   ,COL)=999999.
      GO TO 120
  150 DO 300 ST = 1,STD
      IF(DSG(1+(K+8)*(ST-1),1).EQ.999999..OR.DSG(1+(K+8)*(ST-
     *   1),1).EQ.0) GO TO 320
      DO 200 ROW = 1,K+8
      IF(ROW.LE.2.OR.ROW.EQ.4.OR.ROW.EQ.K+3.OR.ROW.EQ.K+6.OR.
     *   ROW.EQ.K+7) WRITE(9,900)
      IF(ROW.LE.3.OR.ROW.EQ.K+3.OR.ROW.EQ.K+4.OR.ROW.EQ.K+5)
     *   WRITE(9,1000)(HD(ROW),(DSG(ROW+(K+8)*(ST-1),COL),COL
     *   =1,20,HD(ROW))
      IF(ROW.GE.4.AND.ROW.LE.K+2) WRITE(9,1100)(ROW-3,(DSG(RO
     *   W+(K+8)*(ST-1),COL),COL=1,20),ROW-3)
```

```
      IF(ROW.EQ.K+6) WRITE(9,1200)(HD(ROW),(DSG(ROW+(K+8)*(ST
     *   -1),COL).COL=1,20),HD(ROW))
      IF(ROW.GT.K+6) WRITE(9,1300,END=320)(HD(ROW),(DSG(ROW+(
     *   K+8)*(ST-1),COL),COL=1,20),HD(ROW))
  200 CONTINUE
      WRITE(9,900)
      WRITE(9,1400)
      IF(DSG(1+(K+8)*(ST-1),20).EQ.999999.) GO TO 320
      PRINT/,"DESIGN IN PROCESS B=",INT(DSG(1+(K+8)*(ST-1),20
     *       ))
  300 CONTINUE
  320 PRINT/,"JOB COMPLETED !"
  350 STOP
  500 FORMAT(I1,X,I2,16X,I3)
  600 FORMAT(10X,"K = ",I2/10X,"-------"/10X,"V = ",I2/10X,"--
     *       ----"/10X,"A L L  D E S I G N S"/10X,21("-"))
  700 FORMAT(20(X,F5.1))
  750 FORMAT(20(X,F5.2))
  800 FORMAT(20(X,A5))
  900 FORMAT(X,"+---+",20("-----+"),"---+")
 1000 FORMAT(X,"I",A4,20(I5,"I"),A4)
 1100 FORMAT(X,"IM",I2,"I",20(I5,"I"),"M",I2,"I")
 1200 FORMAT(X,"I",A4,20(F5.2,"I"),A4)
 1300 FORMAT(X,"I",A4,20(A5,"I"),A4)
 1400 FORMAT(/)
      END
```

APPENDIX-E: (PROGRAM "PRT2")

```
FILE  7(KIND=DISK,TITLE="PARAMS",FILETYPE=7)
FILE  8(KIND=DISK,TITLE="OPTIMAL",FILETYPE=7)
FILE  9(KIND=PRINTER
      INTEGER V,STO,ST,ROW,COL
      REAL HD(16),OPT(320,20)
      READ(7,500)K,V,STO
      WRITE(9,600)K,V
      HD(1)="  BI"
      HD(2)=" R0I"
      HD(3)="  RI"
      HD(K+3)="LD0I"
      HD(K+4)="LD1I"
      HD(K+5)="LD2I"
      HD(K+6)="TRCI"
      HD(K+7)="DSNI"
      DO 100 ST = 1,STO
      READ(8,700,END=105)((OPT(ROW+(K+7)*(ST-1),COL),COL=1,20
     *                     ),ROW=1,K+ 5)
      READ(8,750,END=105)(OPT(K+6+(K+7)*(ST-1),COL),COL=1,20
      READ(8,800,END=105)(OPT(K+7+(K+7)*(ST-1),COL),COL=1,20)
```

```
  100 CONTINUE
  105 IF(OPT(1+(K+7)*(STO-1),1).EQ.0) STO=STO-1
      DO 120 ST = 1,STO
      DO 120 ROW = 1,K+6
      DO 120 COL = 1,20
      IF(ROW.EQ.K+6) GO TO 130
      IF(OPT(ROW+(K+7)*(ST-1),COL).GE.0) OPT(ROW+(K+7)*(ST-1)
     *    ,COL)=999999.
  120 CONTINUE
      GO TO 150
  130 IF(OPT(ROW+(K+7)*(ST-1),COL).LE.0) OPT(ROW+(K+7)*(ST-1)
     *    ,COL)=999999.
      GO TO 120
  150 DO 300 ST = 1,STO
      IF(OPT(1+(K+7)*(ST-1),1).EQ.999999.) GO TO 320
      WRITE(9,900)
      WRITE(9,650)
      DO 200 ROW = 1,K+7
      IF(ROW.LE.2.OR.ROW.EQ.4.OR.ROW.EQ.K+3.OR.ROW.EQ.K+6.OR.
     *    ROW.EQ.K+7) WRITE(9,900)
      IF(ROW.LE.3.OR.ROW.EQ.K+3.OR.ROW.EQ.K+4.OR.ROW.EQ.K+5)
     *  WRITE(9,1000)(HD(ROW,(OPT(ROW+(K+7)*(ST-1),COL),COL=1
     *  ,20),HD(ROW))
      IF(ROW.GE.4.AND.ROW.LE.K+2)WRITE(9,1100)(ROW-3,(OPT(ROW
     *    +(K+7)*(ST-1),COL),COL=1,20),ROW-3)
      IF(ROW.EQ.K+6) WRITE(9,1200)(HD(ROW),(OPT(ROW+(K+7)*(ST
     *    -1),COL),COL=1,20),HD(ROW))
      IF(ROW.GT.K+6)WRITE(9,1300,END=320)(HD(ROW),(OPT(ROW+(K
     *    +7)*(ST-1),COL),COL=1,20),HD(ROW))
  200 CONTINUE
      WRITE(9,900)
      WRITE(9,1400)
      IF(OPT(1+(K+7)*(ST-1),17).EQ.999999.) GO TO 320
      PRINT/,"DESIGN IN PROCESS B=",INT(OPT(1+(K+7)*(ST-1),17
     *        ))
  300 CONTINUE
  320 PRINT/,"JOB COMPLETED !"
  350 STOP
  500 FORMAT(I1,X,I2,20X,I2)
  600 FORMAT(10X,"K = ",I2/10X,"------"/10X,"V = "I2/10X,"---
     *    ---"/10X,"O P T I M A L  D E S I G N S"/10X,29("-"))
  650 FORMAT(X,"IDSGI",5(" ALL I BIB I BTIBIMBTIBI"),"DSGI")
  700 FORMAT(20(X,F5.1))
  750 FORMAT(20(X,F5.2))
  800 FORMAT(20(X,A5))
  900 FORMAT(X,"+---+",20("-----+"),"---+")
 1000 FORMAT(X,"I",A4,20(I5,"I"),A4)
 1100 FORMAT(X,"IM",I2,"I",20(I5,"I"),"M",I2,"I")
 1200 FORMAT(X,"I",A4,20(F5.2,"I"),A4)
 1300 FORMAT(X,"I",A4,20(A5,"I"),A4)
 1400 FOEMAT(/)
      END
```

APPENDIX-F: (PROGRAM "PRT3")

```
FILE  7(KIND=DISK,TITLE="PARAMS",FILETYPE=7)
FILE  8(KIND=DISK,TITLE="GD",FILETYPE=7)
FILE  9(KIND=DISK,TITLE="STRC",FILETYPE=7)
FILE  10(KIND=PRINTER)
      INTEGER V,STD,BK(100),LD0(100),LD1(100),LD2(100),ROW,CO
*        L,ST,STR(1000,20)
      READ(7,8000)K,V,STD
      DO 100 J = 1,100
      READ(8,8050,END=150)BK(J),LD0(J),LD1(J),LD2(J)
      N=N+1
  100 CONTINUE
  150 DO 200 ST = 1,STD
      READ(9,9050)((STR(ROW+(N+4)*(ST-1),COL),COL=1,20),ROW=1
     *                ,N+4)
  200 CONTINUE
      HD(1)="  B "
      HD(2)=" LD0"
      HD(3)=" LD1"
      HD(4)=" LD2"
      DO 250 ST = 1,STD
      DO 250 ROW = 1,N+4
      DO 250 COL = 1,20
      IF(STR(ROW+(N+4)*(ST-1),COL).EQ.999)STR(ROW+(N+4)*(ST-1
     *    ),COL)=9999
  250 CONTINUE
      WRITE(10,9100)K,V
      WRITE(10,9150)
      WRITE(10,9160)
      WRITE(10,9150)
      DO 300 J = 1,N
      WRITE(10,9200)J,BK(J),LD0(J),LD1(J),LD2(J)
  300 CONTINUE
      WRITE(10,9150)
      WRITE(10,9250)
      DO 350 ST = 1,STD
      IF(STR(1+(N+4)*(ST-1),1),1).EQ.9999) GO TO 360
      WRITE(10,9300)
      DO 400 ROW = 1,4
      WRITE(10,9350)(HD(ROW,(STR(ROW+(N+4)*(ST-1),COL),COL=1,
     *                 20),HD(ROW))
      IF(ROW.EQ.1.OR.ROW.EQ.4) WRITE(10,9300)
  400 CONTINUE
      DO 450 ROW = 1,N
      WRITE(10,9360)(STR(ROW+4+(N+4)*(ST-1),COL),COL=1,20),RO
     *                 W)
  450 CONTINUE
      WRITE(10,9300)
      WRITE(10,9400)
```

```
      IF(STR(1+(N+4)*(ST-1),20).EQ.9999) GO TO 360
      PRINT/,"DESIGN IN PROCESS B=",STR(1+(N+4)*(ST-1),20)
  350 CONTINUE
  360 PRINT/,"JOB COMPLETED !"
      STOP
 8000 FORMAT((I1,X,I2,16X,I3))
 8050 FORMAT(4(X,I3))
 9050 FORMAT(20(X,I3))
 9100 FORMAT(10X,"K = ",I2/10X,"------"/10X,"V = ",I2/10X,"--
     *       ----"//10X,"GENERATOR DESIGNS"/)
 9150 FORMAT(10X,"+----+",4("---+"))
 9160 FORMAT(10X,"IDSGNI B ILD0ILD1ILD2I")
 9200 FORMAT(10X,"ID",I3,"I",4(I3,"I"))
 9250 FORMAT(//10X,"S T R U C T U R E"/10X,17("-")/)
 9300 FORMAT(X,"+----+",20("---+"),"----+")
 9350 FORMAT(X,"I",A4,"I",20(I3,"I"),A4,"I")
 9360 FORMAT(X,"ID",I3,"I",20(I3,"I"),"D",I3,"I")
 9400 FORMAT(/)
      END
```

APPENDIX-G: ALL BINARY E MB GD TIB DESIGNS FOR k = 4, v = 4, b ≤ 10.

K = 4

V = 4

A L L D E S I G N S

B	1	2	2	3	3	4	4	4	B
R0 R	0 1	0 2	4 1	0 3	4 2	0 4	12 1	4 3	R0 R
M 1 M 2 M 3	0 0 0	0 0 0	0 2 0	0 0 0	0 2 0	0 0 0	0 0 4	0 2 0	M 1 M 2 M 3
LD0 LD1 LD2	0 1 1	0 2 2	2 1 0	0 3 3	2 2 1	0 4 4	3 0 0	2 3 2	LD0 LD1 LD2
TRC	*****	*****	6.00	*****	3.67	*****	5.33	3.07	TRC
DSN	BTIB	BTIB	MBTIB	BTIB	MBTIB	BTIB	BTIB	MBTIB	DSN

B	4	4	4	5	5	5	5	5	B
R0 R	8 2	8 2	4 3	0 5	4 4	8 3	8 3	4 4	R0 R
M 1 M 2 M 3	0 4 0	2 0 2	4 0 0	0 0 0	0 2 0	0 4 0	3 1 1	4 0 0	M 1 M 2 M 3
LD0 LD1 LD2	4 0 1	3 1 1	3 2 2	0 5 5	2 4 3	4 1 2	4 1 2	3 3 3	LD0 LD1 LD2
TRC	2.83	3.05	2.42	*****	2.79	2.13	2.13	2.13	TRC
DSN	MBTIB	BTIB	BTIB	BTIB	MBTIB	MBTIB	MBTIB	BIB	DSN

B	6	6	6	6	6	6	6	6	B
RO R	0 6	4 5	16 2	8 4	12 3	8 4	12 3	8 4	RO R
M 1 M 2 M 3	0 0 0	0 2 0	0 2 4	0 4 0	0 6 0	2 0 2	2 2 2	3 1 1	M 1 M 2 M 3
LD0 LD1 LD2	0 6 6	2 5 4	5 1 0	4 2 3	6 1 1	3 3 3	5 2 1	4 2 3	LD0 LD1 LD2
TRC	*****	2.62	2.74	1.82	1.87	2.13	1.97	1.82	TRC
DSN	BTIB	MBTIB	MBTIB	MBTIB	BTIB	BTIB	MBTIB	MBTIB	DSN

B	6	6	7	7	7	7	7	7	B
RO R	4 5	8 4	0 7	4 6	8 5	12 4	8 5	12 4	RO R
M 1 M 2 M 3	4 0 0	4 2 0	0 0 0	0 2 0	0 4 0	0 6 0	3 1 1	3 3 1	M 1 M 2 M 3
LD0 LD1 LD2	3 4 4	5 3 2	0 7 7	2 6 5	4 3 4	6 2 2	4 3 4	6 2 2	LD0 LD1 LD2
TRC	1.96	1.64	*****	2.52	1.64	1.52	1.64	1.52	TRC
DSN	BTIB	MBTIB	BTIB	MBTIB	MBTIB	BTIB	MBTIB	BTIB	DSN

B	7	7	8	8	8	8	8	8	B
RO R	4 6	8 5	0 8	4 7	8 6	20 3	12 5	16 4	RO R
M 1 M 2 M 3	4 0 0	4 2 0	0 0 0	0 2 0	0 4 0	0 4 4	0 6 0	0 8 0	M 1 M 2 M 3
LD0 LD1 LD2	3 5 5	5 4 3	0 8 8	2 7 6	4 4 5	7 0 1	6 3 3	8 2 1	LD0 LD1 LD2
TRC	1.86	1.46	*****	2.44	1.53	1.82	1.33	1.40	TRC
DSN	BTIB	MBTIB	BTIB	MBTIB	MBTIB	MBTIB	BTIB	MBTIB	DSN

B	8	8	8	8	8	8	8	8	B
RO	16	8	12	4	8	12	12	8	RO
R	4	6	5	7	6	5	5	6	R
M 1	2	3	3	4	4	4	6	8	M 1
M 2	4	1	3	0	2	4	0	0	M 2
M 3	2	1	1	0	0	0	2	0	M 3
LD0	7	4	6	3	5	7	6	6	LD0
LD1	1	4	3	6	5	2	3	4	LD1
LD2	2	5	3	6	4	3	3	4	LD2
TRC	1.45	1.53	1.33	1.78	1.34	1.25	1.33	1.21	TRC
DSN	MBTIB	MBTIB	BTIB	BTIB	MBTIB	MBTIB	BTIB	BTIB	DSN

B	9	9	9	9	9	9	9	9	B
RO	0	4	8	16	8	16	4	8	RO
R	9	8	7	5	7	5	8	7	R
M 1	0	0	0	0	3	3	4	4	M 1
M 2	0	2	4	8	1	5	0	2	M 2
M 3	0	0	0	0	1	1	0	0	M 3
LD0	0	2	4	8	4	8	3	5	LD0
LD1	9	8	5	3	5	3	7	6	LD1
LD2	9	7	6	2	6	2	7	5	LD2
TRC	*****	2.38	1.45	1.19	1.45	1.19	1.72	1.26	TRC
DSN	BTIB	MBTIB	MBTIB	MBTIB	MBTIB	MBTIB	BTIB	MBTIB	DSN

B	9	9	9	10	10	10	10	10	B
RO	12	12	8	0	4	8	24	16	RO
R	6	6	7	10	9	8	4	6	R
M 1	4	7	8	0	0	0	0	0	M 1
M 2	4	1	0	0	2	4	6	8	M 2
M 3	0	1	0	0	0	0	4	0	M 3
LD0	7	7	6	0	2	4	9	8	LD0
LD1	3	3	5	10	9	6	1	4	LD1
LD2	4	4	5	10	8	7	1	3	LD2
TRC	1.13	1.13	1.13	*****	2.34	1.39	1.37	1.06	TRC
DSN	MBTIB	MBTIB	BTIB	BTIB	MBTIB	MBTIB	BTIB	MBTIB	DSN

B	10	10	10	10	10	10	10	10	B
R0 R	20 5	20 5	8 8	16 6	4 9	8 8	12 7	16 6	R0 R
M 1 M 2 M 3	0 10 0	2 6 2	3 1 1	3 5 1	4 0 0	4 2 0	4 4 0	4 6 0	M 1 M 2 M 3
LD0 LD1 LD2	10 1 2	9 2 2	4 6 7	8 4 3	3 8 8	5 7 6	7 4 5	9 3 3	LD0 LD1 LD2
TRC	1.12	1.15	1.39	1.06	1.68	1.20	1.04	1.02	TRC
DSN	MBTIB	BTIB	MBTIB	MBTIB	BTIB	MBTIB	MBTIB	BTIB	DSN

B	10	10	10	10	*****	*****	*****	*****	B
R0 R	16 6	12 7	8 8	12 7	***** *****	***** *****	***** *****	***** *****	R0 R
M 1 M 2 M 3	6 2 2	7 1 1	8 0 0	8 2 0	***** ***** *****	***** ***** *****	***** ***** *****	***** ***** *****	M 1 M 2 M 3
LD0 LD1 LD2	8 4 3	7 4 5	6 6 6	8 5 4	***** ***** *****	***** ***** *****	***** ***** *****	***** ***** *****	LD0 LD1 LD2
TRC	1.06	1.04	1.07	0.97	*****	*****	*****	*****	TRC
DSN	MBTIB	MBTIB	BIB	MBTIB					DSN

APPENDIX-H: OPTIMAL E MB GD TIB DESIGNS FOR k = 4, v = 4, b ≤ 10.

K = 4

V = 4

O P T I M A L D E S I G N S

DSG	ALL	BIB	BTIB	MBTIB	ALL	BIB	BTIB	MBTIB	DSG
B	1	*****	1	*****	2	*****	2	2	B
R0 R	0 1	***** *****	0 1	***** *****	4 1	***** *****	0 2	4 1	R0 R
M 1 M 2 M 3	0 0 0	***** ***** *****	0 0 0	***** ***** *****	0 2 0	***** ***** *****	0 0 0	0 2 0	M 1 M 2 M 3
LD0 LD1 LD2	0 1 1	***** ***** *****	0 1 1	***** ***** *****	2 1 0	***** ***** *****	0 2 2	2 1 0	LD0 LD1 LD2
TRC	*****	*****	*****	*****	6.00	*****	*****	6.00	TRC
DSN	BTIB		BTIB		MBTIB		BTIB	MBTIB	DSN

DSG	ALL	BIB	BTIB	MBTIB	ALL	BIB	BTIB	MBTIB	DSG
B	3	*****	3	3	4	*****	4	4	B
R0 R	4 2	***** *****	0 3	4 2	4 3	***** *****	4 3	8 2	R0 R
M 1 M 2 M 3	0 2 0	***** ***** *****	0 0 0	0 2 0	4 0 0	***** ***** *****	4 0 0	0 4 0	M 1 M 2 M 3
LD0 LD1 LD2	2 2 1	***** ***** *****	0 3 3	2 2 1	3 2 2	***** ***** *****	3 2 2	4 0 1	LD0 LD1 LD2
TRC	3.67	*****	*****	3.67	2.42	*****	2.42	2.83	TRC
DSN	MBTIB		BTIB	MBTIB	BTIB		BTIB	MBTIB	DSN

DSG	ALL	BIB	BTIB	MBTIB	ALL	BIB	BTIB	MBTIB	DSG
B	5	5	5	5	6	*****	6	6	B
R0 R	8 3	4 4	0 5	8 3	8 4	***** *****	12 3	8 4	R0 R
M 1 M 2 M 3	0 4 0	4 0 0	0 0 0	0 4 0	4 2 0	***** ***** *****	0 6 0	4 2 0	M 1 M 2 M 3
LD0 LD1 LD2	4 1 2	3 3 3	0 5 5	4 1 2	5 3 2	***** ***** *****	6 1 1	5 3 2	LD0 LD1 LD2
TRC	2.13	2.13	*****	2.13	1.64	*****	1.87	1.64	TRC
DSN	MBTIB	BIB	BTIB	MBTIB	MBTIB		BTIB	MBTIB	DSN

DSG	ALL	BIB	BTIB	MBTIB	ALL	BIB	BTIB	MBTIB	DSG
B	7	*****	7	7	8	*****	8	8	B
R0 R	8 5	***** *****	12 4	8 5	8 6	***** *****	8 6	12 5	R0 R
M 1 M 2 M 3	4 2 0	***** ***** *****	0 6 0	4 2 0	8 0 0	***** ***** *****	8 0 0	4 4 0	M 1 M 2 M 3
LD0 LD1 LD2	5 4 3	***** ***** *****	6 2 2	5 4 3	6 4 4	***** ***** *****	6 4 4	7 2 3	LD0 LD1 LD2
TRC	1.46	*****	1.52	1.46	1.21	*****	1.21	1.25	TRC
DSN	MBTIB		BTIB	MBTIB	BTIB		BTIB	MBTIB	DSN

DSG	ALL	BIB	BTIB	MBTIB	ALL	BIB	BTIB	MBTIB	DSG
B	9	*****	9	9	10	10	10	10	B
R0 R	12 6	***** *****	8 7	12 6	12 7	8 8	16 6	12 7	R0 R
M 1 M 2 M 3	4 4 0	***** ***** *****	8 0 0	4 4 0	8 2 0	8 0 0	4 6 0	8 2 0	M 1 M 2 M 3
LD0 LD1 LD2	7 3 4	***** ***** *****	6 5 5	7 3 4	8 5 4	6 6 6	9 3 3	8 5 4	LD0 LD1 LD2
TRC	1.13	*****	1.13	1.13	0.97	1.07	1.02	0.97	TRC
DSN	MBTIB		BTIB	MBTIB	MBTIB	BIB	BTIB	MBTIB	DSN

APPENDIX-I: STRUCTURE OF E MB GD TIB DESIGNS FOR k = 4, v = 4, b ≤ 10.

K = 4

V = 4

GENERATOR DESIGNS

DSGN	B	LD0	LD1	LD2
D 1	1	0	1	1
D 2	2	2	1	0
D 3	4	4	0	1
D 4	4	3	2	2

S T R U C T U R E

B	1	2	2	3	3	4	4	4	4	4	4	5	B
LD0	0	0	2	0	2	0	3	2	4	3	3	0	LD0
LD1	1	2	1	3	2	4	0	3	0	1	2	5	LD1
LD2	1	2	0	3	1	4	0	2	1	1	2	5	LD2
D 1	1	2	***	3	1	4	***	2	***	***	***	5	D 1
D 2	***	***	1	***	1	***	***	1	***	***	***	***	D 2
D 3	***	***	***	***	***	***	***	***	1	***	***	***	D 3
D 4	***	***	***	***	***	***	***	***	***	***	1	***	D 4

B	5	5	5	5	6	6	6	6	6	6	6	6	B
LD0	2	4	4	3	0	2	5	4	6	3	5	4	LD0
LD1	4	1	1	3	6	5	1	2	1	3	2	2	LD1
LD2	3	2	2	3	6	4	0	3	1	3	1	3	LD2
D 1	3	1	1	1	6	4	***	2	***	***	***	2	D 1
D 2	1	***	***	***	***	1	***	***	1	***	***	***	D 2
D 3	***	1	1	***	***	***	***	1	1	***	***	1	D 3
D 4	***	***	***	1	***	***	***	***	***	***	***	***	D 4

B	6	6	7	7	7	7	7	7	7	7	8	8	B
LD0	3	5	0	2	4	6	4	6	3	5	0	2	LD0
LD1	4	3	7	6	3	2	3	2	5	4	8	7	LD1
LD2	4	2	7	5	4	2	4	2	5	3	8	6	LD2
D 1	2	***	7	5	3	1	3	1	3	1	8	6	D 1
D 2	***	1	***	1	***	1	***	1	***	1	***	1	D 2
D 3	***	***	***	***	1	1	1	1	***	***	***	***	D 3
D 4	1	1	***	***	***	***	***	***	1	1	***	***	D 4

B	8	8	8	8	8	8	8	8	8	8	8	8	B
LD0	4	7	6	8	7	4	6	3	5	7	6	6	LD0
LD1	4	0	3	2	1	4	3	6	5	2	3	4	LD1
LD2	5	1	3	1	2	5	3	6	4	3	3	4	LD2
D 1	4	***	2	***	***	4	2	4	2	***	2	***	D 1
D 2	***	***	1	2	***	***	1	***	1	***	1	***	D 2
D 3	1	***	1	1	***	1	1	***	***	1	1	***	D 3
D 4	***	***	***	***	***	***	***	1	1	1	***	2	D 4

B	9	9	9	9	9	9	9	9	9	9	9	10	B
LD0	0	2	4	8	4	8	3	5	7	7	6	0	LD0
LD1	9	8	5	3	5	3	7	6	3	3	5	10	LD1
LD2	9	7	6	2	6	2	7	5	4	4	5	10	LD2
D 1	9	7	5	1	5	1	5	3	1	1	1	10	D 1
D 2	***	1	***	2	***	2	***	1	***	***	***	***	D 2
D 3	***	***	1	1	1	1	***	***	1	1	***	***	D 3
D 4	***	***	***	***	***	***	1	1	1	1	2	***	D 4

B	10	10	10	10	10	10	10	10	10	10	10	10	B
LD0	2	4	9	8	10	9	4	8	3	5	7	9	LD0
LD1	9	6	1	4	1	2	6	4	8	7	4	3	LD1
LD2	8	7	1	3	2	2	7	3	8	6	5	3	LD2
D 1	8	6	***	2	***	***	6	2	6	4	2	***	D 1
D 2	1	***	***	2	1	***	***	2	***	1	***	1	D 2
D 3	***	1	***	1	2	***	1	1	***	***	1	1	D 3
D 4	***	***	***	***	***	***	***	***	1	1	1	1	D 4

B	10	10	10	10	***	***	***	***	***	***	***	***	B
LD0	8	7	6	8	***	***	***	***	***	***	***	***	LD0
LD1	4	4	6	5	***	***	***	***	***	***	***	***	LD1
LD2	3	5	6	4	***	***	***	***	***	***	***	***	LD2
D 1	2	2	2	***	***	***	***	***	***	***	***	***	D 1
D 2	2	***	***	1	***	***	***	***	***	***	***	***	D 2
D 3	1	1	***	***	***	***	***	***	***	***	***	***	D 3
D 4	***	1	2	2	***	***	***	***	***	***	***	***	D 4

NOTE:The designs which are not equivalent to any combination of generator designs are actually inadmissible with respect to some design, in the table, with the same number of blocks.

V I T A

The author was born in India in 1951 and latter emigrated with his family to Pakistan. He received the Master's Degree in Statistics in 1972 from University of Karachi and joined the same University as a Lecturer. In 1974 he joined a newly growing University in a remote area of the country. He was married in 1977 and has a son and two daughters. In 1983, he got a chance to join Middle East Technical University as a Ph.D. student and subsequently as a staff member also.