INVENTORY PROCESS DYNAMICS OF A MARKET MAKING MODEL WITH REGIME SWITCH

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

NAZLAN BELEMİR ALKAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
FINANCIAL MATHEMATICS

JANUARY 2025

Approval of the thesis:

INVENTORY PROCESS DYNAMICS OF A MARKET MAKING MODEL WITH REGIME SWITCH

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ABSTRACT

INVENTORY PROCESS DYNAMICS OF A MARKET MAKING MODEL WITH REGIME SWITCH

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January 2025, 42 pages

We study a version of the Avellaneda Stoikov (AS) market-making model, which optimizes the bid and ask spread of the market maker, with regime switch as proposed in the preprint “Short Term Market Changes and Market Making with Inventory” by Kim et al. As in the ordinary AS model, under a structural assumption for the value function, the Hamilton Jacobi Bellman equation for the value function reduces to an ordinary differential equation. We note that, as opposed to the ordinary AS model, the resulting ODE is nonlinear and, therefore, cannot be solved via matrix exponentials. We numerically solve the ODE via finite differences and study the dynamics of the inventory process and their dependence on model parameters. We note that the inventory process of the model with regime switch behaves like a process with regime switch where in each regime, the inventory process has dynamics similar to the inventory process of an AS model with no regime switch.

Keywords: Market Making Model, Stochastic Optimal Control, Regime Switching, Inventory Management

ÖZ

REJİM DEĞİŞİMİ İLE PİYASA YAPICI MODELİN ENVANTER SÜRECİ DİNAMİKLERİ

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Ocak 2025, 42 sayfa

Bu tezde bir piyasa yapıcının alış ve satış marjlarının optimize edildiği Avellaneda-Stoikov (AS) piyasa yapıcılığı modelinin, Kim vd.'nin "Kısa Vadeli Piyasa Değişiklikleri ve Envanter ile Piyasa Yapıcılığı" başlıklı ön baskıda çalıştıkları rejim değişimli versiyonunu inceliyoruz. Olağan AS modelinde olduğu gibi, değer fonksiyonu için yapılan yapısal bir varsayım altında, değer fonksiyonu için Hamilton-Jacobi-Bellman denklemi bir adi diferansiyel denkleme (ADD) indirgenir. Ancak, olağan AS modelinden farklı olarak, ortaya çıkan ADD doğrusal değildir ve bu nedenle matris üstel fonksiyonları ile çözülemez. ADD'yi sonlu farklar yöntemiyle sayısal olarak çözüyoruz ve envanter sürecinin dinamiklerini ve bu dinamiklerin model parametrelerine olan bağımlılığını inceliyoruz. Rejim değişimi içeren modelin envanter sürecinin, her rejimde rejim değişimi olmayan bir AS modelindeki envanter sürecine benzer dinamiklere sahip olduğu rejim değişimli bir süreç gibi davrandığını gözlemliyoruz.

Anahtar Kelimeler: Piyasa Yapıcılığı Modeli, Stokastik Optimal Kontrol, Rejim Değişimi, Envanter Yönetimi

To my mom & dad

ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor, Prof. Dr. Ali Devin Sezer, for his guidance and help. I am also thankful to my co-advisor, Dr. Fatma Başoğlu Kabran, for her support and feedback.

I thank my Examining Committee Members, Assoc. Prof. Dr. Kamil Demirberk Ünlü and Assist. Prof. Dr. Büşra Zeynep Temoçin, for reading my thesis and their feedback.

This thesis was supported by the TUBITAK project 122F043, and I am grateful for the opportunity to be part of this project, which enriched my academic journey.

I would like to extend my heartfelt appreciation to Damla, whose support and friendship have been a source of strength and motivation. Special thanks to Hakan for his collaboration and inspiration. I am also grateful to the members of the Executive Math Club—Ekin, Berna, and Erdem—for the positive environment they created and their inspiring contributions. I am incredibly grateful to İpek, whose kindness and thoughtful words have been a constant source of comfort.

I would also like to express my sincere gratitude to Alperen and Ece, dear friends whose conversations have enriched my perspective and provided invaluable support. Their friendship, encouragement, and viewpoints have been a source of inspiration.

To my dearest best friend, Sabiha, thank you for being my rock, confidant, and source of unwavering positivity. Your belief in me has been a constant reminder of the strength of our bond.

I am deeply thankful to the love of my life, Emirhan, for his encouragement, endless patience, and belief in me even during moments of self-doubt. His presence has been my greatest source of motivation and comfort. Emirhan, your love and support have made every challenge more manageable, and I cannot thank you enough for always being by my side.

Finally, I owe my deepest gratitude to my mom, dad, and brother for their unconditional love, encouragement, and sacrifices. To my parents, thank you for teaching me the values of perseverance, hard work, and kindness, and for always cheering me on. To my brother, Emirhan, your warmth and humor have brought light to even the most stressful days, and I am grateful for your support. Without the strength and love of my family, this journey would not have been possible.

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CHAPTER 1

INTRODUCTION

Market makers are companies such as banks that quote buy and sell (bid/ask) prices for assets in financial markets, this ensures the liquidity of these assets in these markets [4], for ease of reference we will also use the term "dealer" to refer to a market maker. A well-known model of market making is the Avellaneda-Stoikov (AS) model [4]. This model and its solution are presented in [12, Chapter 11] and the presentation of the AS model in this thesis will be based on this reference. In the simplest version of the AS model only a single asset is considered. The mid price of the asset is taken to be a Brownian motion with volatility $\sigma > 0$ and hence independent from the actions of the dealer. The dealer chooses the bid/ask spread (i.e., how much the bid and ask prices deviate from the mid price) to optimize the expected utility of his terminal cash and asset position. The buy/sell orders are assumed to arrive one by one following jump processes whose intensities are determined by the bid/ask spread. Assuming a constant absolute risk aversion (CARA) utility determined by a constant $\gamma > 0$ and exponentially decaying jump intensities (determined by the spreads) the Hamilton Jacobi Bellman (HJB) can be reduced to a linear ordinary differential equation. This reduction is accomplished by guessing (and then verifying) that the value function has a product form. This gives a fairly explicit solution of the model. We review this solution, based on its presentation in [12], in Chapter 2.

An important component of the AS model is the dynamics of the inventory process, which is the amount of the underlying asset that is being held by the dealer. The market maker would like to keep this process as close to 0 as possible to avoid market and liquidity risk. In this thesis, we will be focusing on how the model parameters impact the dynamics of the inventory process in the AS model.

The assumptions of the AS model may seem unrealistic. For example, it is often observed that asset prices don't have a constant volatility. Therefore, it is of interest to generalize the AS model so that it can exhibit more realistic behavior. One way to accomplish this is by adding a regime switching dynamic into the model. In other words, an external finite state Markov process is incorporated into the model whose current state determines the model parameters. Such a generalization of the AS model was introduced and studied in the preprint [16]. Our goals in this thesis are 1) the identification of the optimal controls in this regime switch model and 2) how the model parameters in the AS model with regime switch influence the dynamics of the inventory process.

Chapter 3 presents the AS model with regime switch studied in [16]. The original AS model has the following parameters: γ and σ mentioned above, and k and C which appear in the function connecting the bid/ask spread to the arrival rates of buy and sell orders. As already noted above, as compared to the ordinary AS model presented in Chapter 2, the novel feature of this model is a Jump process N taking values in $\{1, 2\}$ and jump rates λ_i, $i = 1, 2$. It is assumed that the AS model parameters σ_i, k_i, and C_i depend on the state process N. In both models, the total inventory size is assumed to be bounded below by $-Q$ and above by Q. This means that if the inventory reaches the level Q (-Q) the market makers stop buying (selling).

Let q denote the inventory variable, it is integer valued and is between Q and $-Q$, $y \in \mathbb{R}$ is the mid price variable and finally let $t \in [0, T]$ denote time. The solution method of the ordinary AS model is as follows:

1. one guesses that the value function of the problem is of the form $V(x, y, q, t) = -\exp(-\gamma(x + qy))u(q, t)^{-\gamma/k})$.
2. substituting this form into the HJB equation reduces it to a system of linear ODE for the function $(q, \cdot)$.

The work [16] tries to implement a similar approach starting with the guess that the value function is of the form $V(x, y, q, t, i) = -\exp(-\gamma(x+qy))u(i, q, t)^{-\gamma/k_i}$. Substituting this form of the value function into the HJB equation associated with the AS model with regime switch [16] arrives again at a linear ODE system. To the best of our understanding, this reduction is incorrect. In Section 3.2 we do the same substitution and carry out the reduction and find that the reduction does indeed work but ends with a nonlinear ODE system (see (3.9), (3.11) and (3.13)). This computation and observation are the first contribution of this thesis. In the same subsection the discrete versions (based on replacing time derivatives with their corresponding forward differences) of the same ODE are also given. These are used in the computations of the chapters that follow.

We begin our numerical study of the dependence of inventory process dynamics on model parameters in Chapter 4. To carry out this study we proceed as follows:

1. we fix parameter values $\sigma_i, C_i, k_i, \lambda_i$ $i = 1, 2$ and γ and use the discretized ODE to compute the value function. Then formulas (3.6) and (3.7) give the jump intensities of the inventory process.
2. We then compute the value function of the ordinary model (without regime switch) for two sets of parameter values: $\{\sigma_1, C_1, k_1, \gamma\}$ and $\{\sigma_2, C_2, k_2, \gamma\}$
3. We compare the jump intensity of the inventory process of the model with regime switch with the jump intensities of the ordinary model with the above parameter values.

Upon this comparison we observe the following: the jump rates of the inventory process of the regime switch model is like a linear combination of the inventory process jump rates of the models with no regime switch (see Figure 4.1 and 4.2). This suggests that the dependence of the jump rates of the inventory process of the model with regime switch on the parameters σ_i, C_i, k_i and γ is similar to the dependence of the inventory jump rates in the ordinary model without regime switch. The dependence of jump intensities on model parameters for the ordinary model was studied in [2]. Based on the discussion above we expect a similar situation in the modulated setup. Nonetheless, for completeness, we provide a numerical study of the dependence of the jump intensities of the inventory process on the parameters σ_i, C_i, k_i and γ in Chapter 5.

Then the most novel parameter in the analysis of the dependence of jump rate intensities on model parameters in the regime switch framework is the λ_i, i.e., the jump intensities of the regime process. This dependence is studied in Section 4.1 again by comparing the jump intensities of the regime switch model with the jump intensities of the ordinary model.

In Chapter 6 we study the distribution of $q(T)$ as a function of the model paramaters. Since this distribution is not explicitly available we use simulation. Conclusion of the thesis comments on possible future work.

1.1 Literature Review

The main market model that we will study is the Avellaneda Stoikov model [4], this model and its modification with regime switch will be discussed in detail in the following chapters.

The market making problem is closely related to the problem of optimal trade execution. The AS model in particular is closely related to the optimal trade execution model studied by Almgren and Chriss in [1]; see [12, Chapter 2] for a detailed presentation of this model. Earlier work preceding [1] is [15] which studies the problem of trade execution taking into account transaction costs. [11] studies a similar problem, in their model the bid-ask spread is explained by how different traders have differing levels of information about the underlying stock. There is a wide literature on optimal trade execution; for a further literature review and references we refer the reader to [12].

As also discussed in detail in the following chapters of this thesis the AS market making model can be solved explicitly. Guéant, Lehalle, and Fernandez-Tapia (2013) [13] study the asymptotics of the resulting formulas for $T \to \infty$; they additionally develop more computationally tractable approximations of the optimal quotes. [12, Section 11.3] discusses possible extensions of the AS model: these extensions are all based on the parallels between optimal trade execution and market making:

> The AS model can be generalized in several ways. A group of researchers driven by Cartea and Jaimungal proposed many optimal trading models,

with various features, including a drift in the price process , a short-term α, some form of adverse selection, etc. see [5, 7, 6, 8, 9]. Some of these features can be added to enrich the original Avellaneda-Stoikov model, without much change in the mathematics.

For a more detailed discussion of these possible extensions we refer the reader to [13]. [12, Subsection 11.3.2] presents an extension of the AS model to a market maker selling/buying multiple assets.

Fodra and Labadie (2012) [10] studies an extension of the AS market making model where the mid-price S_t evolves as a jump diffusion process with drift. It also incorporates a penalty term for invetory remaining at terminal time. Their solution method is similar to the solution of the AS model and starts with an assumption on the form of the value function. Incorporating regime-switching dynamics into a model means to allow the parameters of the model to depend on the state of an auxilary process. Models with regime switch and their analysis go at least back to [14]. The work [17] studies an extension of the AS market making model with regime-switch. The midprice process is assumed to be an Ito process with drift and jump where model parameters are allowed to depend on the regime process. It also incorporates traders with varying levels of information. They solve their model by replacing some of the nonlinear terms in the HJB equation with linear approximations which yield equations that can be solved analytically. The preprint [16] studies a direct extension of the AS model with regime switch. As already discussed above, the subject of the present thesis is this model which is presented and studied in detail in the following chapters.

CHAPTER 2

AVELLANEDA-STOIKOV MARKET MAKING MODEL

In this chapter, we will review the mathematical framework [12] of the Avellaneda-Stoikov market making model. The following review is based on the presentation of this model in [12, Chapter 11].

2.1 Framework

The Avellaneda-Stoikov market making model [4] studies a market maker operating on a single asset. The market price (or reference price) of this asset $(S_t)_t$ is modeled by a process of Brownian dynamics as follows:

$$dS_t = \sigma dW_t. \tag{2.1}$$

The model's assumption is that there is one unit of the asset traded at a time of transaction, while the liquidity provider can propose the bid and ask quotes continuously. The stochastic processes $(S_t^b)_t$ and $(S_t^a)_t$ denote the bid and ask quotes, respectively.

Transactions occur at random times. Furthermore, the time of the trade depends on the bid and ask prices which the market maker quoted.

How many assets are bought and sold can be denoted by processes $(N_t^b)_t$ and $(N_t^a)_t$, respectively. The inventory of a market maker is modeled by the process $(q_t)_t$ which is defined by

$$q_t = N_t^b - N_t^a. \tag{2.2}$$

The main assumption of the Avellaneda - Stoikov model is that the poisson intensities $(\lambda_t^b)_t$ and $(\lambda_t^a)_t$ are as follows:

$$\lambda_t^b = \Lambda^b(\delta_t^b) \quad \text{and} \quad \lambda_t^a = \Lambda^a(\delta_t^a), \tag{2.3}$$

where

$$\delta_t^b = S_t - S_t^b \quad \text{and} \quad \delta_t^a = S_t^a - S_t, \tag{2.4}$$

and where Λ^b and Λ^a are assumed to be:

$$\Lambda^a(\delta) = \Lambda^b(\delta) = Ce^{-k\delta}. \tag{2.5}$$

Avellaneda and Stoikov [4] derived the partial differential equations presenting optimal quotes for δ^b and δ^a.

As C represents the asset's liquidity and k represents the price sensitivity of the market participants, the following case is considered specifically on the Avellaneda - Stoikov model which is slightly modified by Guéant et. al. [12] in order to solve the problem precisely.

$$\begin{aligned} \lambda_t^b &= \Lambda^b(\delta_t^b)\mathbf{1}_{q_t<Q} \\ \lambda_t^a &= \Lambda^a(\delta_t^a)\mathbf{1}_{q_t>-Q}, \end{aligned} \tag{2.6}$$

Here, $Q \in \mathbb{N}^*$ denotes the maximum acceptable inventory either long or short. Therefore, the market maker pauses quoting a bid and an ask price, when $q_t = Q$ and $q_t = -Q$, respectively.

The process $(X_t)_t$ represents the number of inventories that is in the market maker's account, and holds the following equality:

$$dX_t = S_t^a dN_t^a - S_t^b dN_t^b = (S_t + \delta_t^a)dN_t^a - (S_t - \delta_t^b)dN_t^b. \tag{2.7}$$

To define the optimization problem that the market maker faces, the process operates within a given time horizon T. Within this framework, the aim of to market maker is to optimize their trading strategy in such a way that they maximize the given expected Constant Absolute Risk Aversion (CARA) utility criterion

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T S_T - \ell(q_T))\right)\right], \tag{2.8}$$

carried out over $(\delta_t^b)_t \in \mathcal{A}$ and $(\delta_t^a)_t \in \mathcal{A}$, where the set of admissible controls $\mathcal{A}$, is defined as the collection of predictable processes that are bounded from below.

The risk-liquidity premium function $\ell(q_T)$ is negligible in this market making model [12] and [13], since the inventory process has the mean reverting around 0.

2.2 HJB Equation and the Solution of the Optimization Problem

Guéant et. al. [13] exhibits the HJB equation of the optimization problem. The equation is as follows:

$$\begin{aligned}
0 &= \partial_t u(t,x,q,S) + \frac{1}{2}\sigma^2 \partial_S^2 u(t,x,q,S) \\
&+ \mathbf{1}_{q<Q} \sup_{\delta^b} \Lambda^b(\delta^b)[u(t,x-S+\delta^b,q+1,S) - u(t,x,q,S)] \\
&+ \mathbf{1}_{q>-Q} \sup_{\delta^a} \Lambda^a(\delta^a)[u(t,x+S+\delta^a,q-1,S) - u(t,x,q,S)].
\end{aligned} \tag{2.9}$$

Here, $q \in \{-Q,\dots,Q\}$, and $(t,S,x) \in [0,T] \times \mathbb{R}^2$, and the terminal condition of $u(T,x,q,S)$ is,

$$u(T,x,q,S) = -\exp(-\gamma(x + qS - \ell(q))). \tag{2.10}$$

The ℓ function here is the inventory risk premium function. As in [12], we assume ℓ to be 0.

In [13], Guéant et al. transformed the equations into a system of ODEs by reducing their dimensionalities. Their results will be given in the following steps.

They came out the following exponential intensity functions by the change of variables,

$$u(t,x,q,S) = -\exp(-\gamma(x+qS))v_q(t)e^{-\frac{k}{\gamma}S}. \tag{2.11}$$

As a result, the HJB equation simplifies to the following form:

$$\frac{d}{dt}v_q(t) = \frac{k}{2}\gamma\sigma^2 q^2 v_q(t) - C\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{\gamma}{k}\right)}(\mathbf{1}_{q>-Q}v_{q-1}(t) + \mathbf{1}_{q<Q}v_{q+1}(t)), \tag{2.12}$$

with the terminal condition

$$u(T,x,q,S) = -\exp(-\gamma(x+qS-\ell(q))). \tag{2.13}$$

Since the problem is reduced to a linear system of ODEs, it can be solved subsequently.

If we denote by M the matrix

$$M = \begin{pmatrix} \alpha Q^2 & -\eta & 0 & \cdots & 0 \\ -\eta & \alpha(Q-1)^2 & -\eta & \cdots & 0 \\ 0 & -\eta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha(Q-1)^2 & -\eta \\ 0 & \cdots & 0 & -\eta & \alpha Q^2 \end{pmatrix},$$

where $\alpha = \frac{k}{2}\gamma\sigma^2$ and $\eta = C\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{\gamma}{k}\right)}$, then

$$(v_{-Q}(t),\ldots,v_Q(t)) = \left(e^{-k\ell(-Q)},\ldots,e^{-k\ell(Q)}\right)\exp(-M(T-t))$$

defines the unique solution of Eq. (2.9) with the appropriate terminal condition.

The optimal controls $(\delta_t^{b*})_t$ and $(\delta_t^{a*})_t$ are given by

$$\delta_t^{b*} = \delta^b(t,q_t) \quad \text{and} \quad \delta_t^{a*} = \delta^a(t,q_t),$$

where

$$\delta^b(t,q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q+1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right), \quad q<Q,$$

$$\delta^a(t,q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q-1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right), \quad q>-Q.$$

CHAPTER 3

THE AS MODEL WITH REGIME SWITCH

Below we present the model studied in [16], which is an extension of the Avellaneda and Stoikov market making model with dynamic parameters like trading intensity and volatility determined by a jump process. This switching mechanism represents real-world market conditions since trading volume and volatility have a high level of correlation. [16] Their study makes a contribution for market microstructure research by suggesting an optimal trading strategy for market makers which justifies short-term market changes, emphasizing inventory management at the beginning of a trading session and asset value maximization at the end.

The jump process is denoted by N; to keep the presentation simple we will assume that N takes only two values. The jump rate of N at state i is denoted by λ_i.

The mid-market stock price y_t in Regime i follows:

$$dy_t = \sigma_i dW_t,$$

where W_t is a standard Brownian motion, and σ_i are positive constants with $\sigma_H > \sigma_L$.

The cumulative shares purchased (L_t) and sold (U_t) up to time t determine the market maker's cash amount x_t:

$$dx_t = -(y_t - B_t^i)dL_t + (y_t + A_t^i)dU_t,$$

where B_t^i and A_t^i are the distances between the mid-market price and the bid/ask prices, respectively:

$$\begin{cases} B_t^i = y_t - \text{bid}_t^i, \\ A_t^i = \text{ask}_t^i - y_t. \end{cases}$$

The processes L_t and U_t are modeled as non-homogeneous Poisson processes with intensities $I^i(B_t^i)$ and $I^i(A_t^i)$, respectively:

$$dL_t = \begin{cases} 1 & \text{with probability } I^i(B_t^i)dt, \\ 0 & \text{with probability } 1 - I^i(B_t^i)dt, \end{cases}$$

$$dU_t = \begin{cases} 1 & \text{with probability } I^i(A_t^i)dt, \\ 0 & \text{with probability } 1 - I^i(A_t^i)dt. \end{cases}$$

The trading intensities $I^i(B_t^i)$ and $I^i(A_t^i)$ depend on bid-ask spreads. The market maker aims to maximize the expected constant absolute risk aversion (CARA) utility of terminal wealth, with inventory limits Q and $-Q$.

3.1 Solution proposed in the preprint

In [16], Kim et al. provide the following solution of the HJB equation considering the regime switching dynamic. They first define the value function in high-volatility and low-volatility regimes, $V_i (i = H, L)$, respectively. Furthermore, the optimal bid and ask distances are defined to be B_i and $A_i (i = H, L)$ in high and low regimes, respectively.

For $q = Q$,

$$\begin{aligned} &V_t^i + \frac{1}{2}\sigma_i^2 V_{yy}^i + \max_{A_t^i} C_i \exp(-k_i A_t^i) \left[V^i(x + y + A_t^i, y, q - 1, t) - V^i(x, y, q, t)\right] \\ &\quad + \lambda_{i,t}(V^j - V^i) = 0, \end{aligned} \tag{3.1}$$

For $|q| < Q$,

$$\begin{aligned} &V_t^i + \frac{1}{2}\sigma_i^2 V_{yy}^i + \max_{B_t^i} C_i \exp(-k_i B_t^i) \left[V^i(x - y + B_t^i, y, q + 1, t) - V^i(x, y, q, t)\right] \\ &\quad + \max_{A_t^i} C_i \exp(-k_i A_t^i) \left[V^i(x + y + A_t^i, y, q - 1, t) - V^i(x, y, q, t)\right] \\ &\quad + \lambda_{i,t}(V^j - V^i) = 0, \end{aligned} \tag{3.2}$$

For $q = -Q$,

$$V_t^i + \frac{1}{2}\sigma_i^2 V_{yy}^i + \max_{B_t^i} C_i \exp(-k_i B_t^i) \left[V^i(x - y + B_t^i, y, q+1, t) - V^i(x, y, q, t)\right]$$
$$+ \lambda_{i,t}(V^j - V^i) = 0, \tag{3.3}$$

with the terminal condition

$$V^i(x, y, q, T) = -\exp\left(-\gamma(x + qy)\right), \quad \forall q \in \{-Q, \dots, Q\}. \tag{3.4}$$

The function $V^i(x, y, q, t)$ is,

$$V^i(x, y, q, t) = -\exp\left(-\gamma(x + qy)\right) \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i} \tag{3.5}$$

The optimal bid and ask distances B_t^i and A_t^i are:

$$\begin{aligned} B_t^i &= \frac{1}{\gamma}\log\left(1 + \frac{\gamma}{k_i}\right) + \frac{1}{k_i}\log\left(\frac{u_q^i(t)}{u_{q+1}^i(t)}\right) \\ A_t^i &= \frac{1}{\gamma}\log\left(1 + \frac{\gamma}{k_i}\right) + \frac{1}{k_i}\log\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right) \end{aligned} \tag{3.6}$$

The optimal bid-ask spread $\delta_t^i = B_t^i + A_t^i$ is then

$$\delta_t^i = \frac{2}{\gamma}\log\left(1 + \frac{\gamma}{k_i}\right) + \frac{1}{k_i}\log\left(\frac{\left(u_{q+1}^i(t)\right)^2}{u_{q+1}^i(t)u_{q-1}^i(t)}\right).$$

The jump intensities of the inventory process are computed from A and B with the formula

$$\lambda_t^{b,i} = C_i e^{-k_i B_t^i}. \ \lambda_t^{a,i} = C_i e^{-k_i A_t^i}. \tag{3.7}$$

Putting (3.5),(3.6) into the non-linear PDE, Kim et al. came up with the following:

For $q = Q$:

$$\dot{u}_q^i(t) - \left(\lambda_i - \frac{\gamma^2\sigma^2 Q^2}{2}\right) u_q^i(t) - C_i\frac{\gamma}{k_i}\left(1 + \frac{\gamma}{k_i}\right)^{-\frac{1-k_i/\gamma}{\gamma}} u_{q-1}^i(t) + \lambda_{i,t}u_q^j(t) = 0$$

For $|q| < Q$:

$$\dot{u}^i_q(t) - \left(\lambda_i - \frac{\gamma^2\sigma^2Q^2}{2}\right)u^i_q(t) \\ + C_i\frac{\gamma}{k_i}\left(1+\frac{\gamma}{k_i}\right)^{-\frac{1-k_i/\gamma}{\gamma}}\left(u^i_{q+1}(t)+u^i_{q-1}(t)\right) + \lambda_{i,t}u^j_q(t) = 0$$

For $q = -Q$:

$$\dot{u}^i_q(t) - \left(\lambda_i - \frac{\gamma^2\sigma^2Q^2}{2}\right)u^i_q(t) + C_i\frac{\gamma}{k_i}\left(1+\frac{\gamma}{k_i}\right)^{-\frac{1-k_i/\gamma}{\gamma}}u^i_{q+1}(t) + \lambda_{i,t}u^j_q(t) = 0$$

Subject to the terminal condition: $u_q(T) = 1, \quad \forall q \in \{-Q, \ldots, Q\}$. After the calculation, Kim et al. approximate the equations using forward differences depending on the variable t. However, there is an error in the above derivation: upon substituting V of the form (3.4) in the HJB equation, one doesn't get the linear equation above. In the next section of our study, we redo this computation. The form of V given in (3.4) continues to work in the regime switch framework but leads to nonlinear equation.

3.2 Reduction to ODE

In this section we substitute the V^i of the form (3.4) into the HJB equation and see whether a reduction to an ODE is possible. In the subsections below, we carry out the calculations for the cases $q = Q$, $|q| < Q$ and $q = -Q$.

3.2.1 The case $q = Q$

For $q = Q$,

$$V^i_t + \frac{1}{2}\sigma_i^2V^i_{yy} + \max_{A^i_t} C_i\exp(-k_iA^i_t)\left[V^i(x+y+A^i_t, y, q-1, t) - V^i(x,y,q,t)\right] \\ + \lambda_{i,t}(V^j - V^i) = 0, \tag{3.8}$$

Firstly, let us compute some terms to clarify:

$$V_t^i(x,y,q,t) = \frac{\partial V^i}{\partial t} = \frac{\partial}{\partial t}\left[-e^{(-\gamma(x+qy))} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]$$
$$= \left[-e^{(-\gamma(x+qy))}\right] \cdot (-\frac{\gamma}{k_i}) \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \cdot \dot{u}_q^i(t)$$

$$V_{yy}^i(x,y,q,t) = \frac{\partial^2 V^i}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left[-e^{(-\gamma(x+qy))} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]$$
$$= \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}} \cdot \left[-\gamma^2 \cdot q^2 \cdot e^{(-\gamma(x+qy))}\right]$$

$$V^i(x+y+A_t^i, y, q-1, t) = \left[-e^{(-\gamma(x+y+A_t^i+(q-1)y))}\right] \cdot \left\{u_{q-1}^i(t)\right\}^{-\frac{\gamma}{k_i}}$$

Then, Eq. (3.8) becomes:

$$= e^{-\gamma(x+qy)} \cdot \frac{\gamma}{k_i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \cdot \dot{u}_q^i(t) - \frac{1}{2}\sigma_i^2 \cdot e^{-\gamma(x+qy)} \cdot \gamma^2 \cdot q^2 \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i}$$
$$+ \max_{A_t^i} C_i \cdot e^{-k_i \cdot A_i}\left(-e^{-\gamma(x+A_t^i+qy)} \cdot \left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} + e^{-\gamma(x+qy)} \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i}\right)$$
$$+\lambda_{i,t}\left(-e^{-\gamma(x+qy)} \cdot \left\{u_q^j(t)\right\}^{-\gamma/k_j} + e^{-\gamma(x+qy)}\left\{u_q^i(t)\right\}^{-\gamma/k_i}\right) = 0$$

All terms have $e^{-\gamma(x+qy)}$ factor; they cancel out.

Then the equation becomes,

$$= \frac{\gamma}{k_i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \cdot \dot{u}_q^i(t) - \frac{1}{2}\sigma_i^2 \cdot \gamma^2 \cdot q^2 \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i}$$
$$+C_i \max_{A_t^i}\left(-e^{-(\gamma+k_i)A_t^i}\left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} + e^{-k_i A_t^i}\left\{u_q^i(t)\right\}^{-\gamma/k_i}\right)$$
$$+\lambda_{i,t}\left(-\left\{u_j^i(t)\right\}^{-\gamma/k_j} + \left\{u_q^i(t)\right\}^{-\gamma/k_i}\right) = 0$$

Maximization problem:

$$\max_{A_t^i}\left(-e^{-(\gamma+k_i)A_t^i} \cdot \left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} + e^{-k_i A_t^i} \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i}\right).$$

The objective function is,

$$f(A_t^i) = -e^{-(\gamma+k_i)A_t^i}\left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} + e^{-k_i A_t^i}\left\{u_q^i(t)\right\}^{-\gamma/k_i}$$

Differentiating $f(A_t^i)$ with respect to A_t^i ,

$$\frac{\partial f}{\partial A_t^i} = \frac{\partial}{\partial A_t^i}\left(-e^{-(\gamma+k_i)A_t^i}\left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} + e^{-k_iA_t^i}\left\{u_q^i(t)\right\}^{-\gamma/k_i}\right)$$

$$\frac{\partial f}{\partial A_t^i} = (\gamma+k_i)e^{-(\gamma+k_i)A_t^i}\left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} - k_ie^{-k_iA_t^i}\left\{u_q^i(t)\right\}^{-\gamma/k_i}$$

Equate the derivative to zero,

$$(\gamma+k_i)e^{-(\gamma+k_i)A_t^i}\left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i} = k_ie^{-k_iA_t^i}\left\{u_q^i(t)\right\}^{-\gamma/k_i}$$

Dividing each side by $e^{-k_iA_t^i}$ and arranging the terms,

$$\frac{\gamma+k_i}{k_i} = e^{\gamma A_t^i}\cdot\frac{\left\{u_q^i(t)\right\}^{-\gamma/k_i}}{\left\{u_{q-1}^i(t)\right\}^{-\gamma/k_i}}$$

$$\frac{\gamma+k_i}{k_i} = e^{\gamma A_t^i}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{-\gamma/k_i}$$

Taking the natural logarithm of both sides,

$$\ln\left(\frac{\gamma+k_i}{k_i}\right) = \gamma\cdot A_t^i - \frac{\gamma}{k_i}\cdot\ln\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right).$$

Leaving A_t^i alone, the maximizer is

$$A_t^i = \frac{1}{\gamma}\cdot\ln\left(1+\frac{\gamma}{k_i}\right) + \frac{1}{k_i}\cdot\ln\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right).$$

Let us compute the term $\max\limits_{A_t^i} e^{-(\gamma+k_i)A_t^i}$

$$\begin{aligned}\max_{A_t^i} e^{-(\gamma+k_i)A_t^i} &= \left[\left(1+\frac{\gamma}{k_i}\right)^{\frac{1}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{\frac{1}{k_i}}\right]^{-(\gamma+k_i)} \\ &= \left(1+\frac{\gamma}{k_i}\right)^{-\frac{(\gamma+k_i)}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{-\frac{(\gamma+k_i)}{k_i}}\end{aligned}$$

Then, let us compute the term $\max\limits_{A_t^i} e^{-k_iA_t^i}$

$$\begin{aligned}\max_{A_t^i} e^{-k_iA_t^i} &= \left[\left(1+\frac{\gamma}{k_i}\right)^{\frac{1}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{\frac{1}{k_i}}\right]^{-k_i} \\ &= \left(1+\frac{\gamma}{k_i}\right)^{\frac{-k_i}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{-1}\end{aligned}$$

Inserting them into the equation,

$$
\begin{aligned}
&= \frac{\gamma}{k_i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \cdot \dot{u}_q^i(t) - \frac{1}{2}\sigma_i^2 \cdot \gamma^2 \cdot q^2 \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i} \\
&\quad -C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot \left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{-\frac{\gamma}{k_i}-1} \cdot \left\{u_{q-1}^i(t)\right\}^{-\frac{\gamma}{k_i}} \\
&\quad +C_i \cdot \left[\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot \frac{u_{q-1}^i(t)}{u_q^i(t)}\right] \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}} \\
&\quad +\lambda_{i,t}\left(-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}} + \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right) = 0
\end{aligned}
$$

Rearranging the similar terms,

$$
\begin{aligned}
&\frac{\gamma}{k_i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \cdot \dot{u}_q^i(t) - \frac{1}{2}\sigma_i^2 \cdot \gamma^2 \cdot q^2 \cdot \left\{u_q^i(t)\right\}^{-\gamma/k_i} \\
&\quad -C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \cdot \left\{u_{q-1}^i(t)\right\} \\
&\quad +C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q-1}^i(t) \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1} \\
&\quad +\lambda_{i,t}\left(-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}} + \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right) = 0
\end{aligned}
$$

Divide each side by $\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}$

$$
\begin{aligned}
&\frac{\gamma}{k_i} \cdot \dot{u}_q^i(t) - \frac{1}{2}\sigma_i^2 \cdot \gamma^2 \cdot q^2 \cdot \left\{u_q^i(t)\right\} \\
&\quad -C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q-1}^i(t) \\
&\quad +C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q-1}^i(t) \\
&\quad +\lambda_{i,t}\left(\frac{-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}}}{\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}} + \left\{u_q^i(t)\right\}\right) = 0
\end{aligned}
\tag{3.9}
$$

Forward difference approximation For numerical computation, we now discretize (3.9) using the following forward difference scheme:

$$
\dot{u}_q^i(t) \approx \frac{u_q^i(t+\Delta t) - u_q^i(t)}{\Delta t}.
$$

Substituting this into the equation gives:

$$\frac{\gamma}{k_i} \cdot \frac{u_q^i(t+\Delta t) - u_q^i(t)}{\Delta t} - \frac{1}{2}\sigma_i^2\gamma^2 q^2 \cdot \{u_q^i(t)\} - C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q-1}^i(t)$$
$$+C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q-1}^i(t) + \lambda_{i,t} \cdot \left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\}\right) = 0.$$

Rearranging for $u_q^i(t+\Delta t)$, we get:

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \left[\frac{1}{2}\sigma_i^2\gamma^2 q^2 \cdot \{u_q^i(t)\} + C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q-1}^i(t)\right.$$
$$\left. -C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q-1}^i(t) - \lambda_{i,t} \cdot \left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\}\right)\right].$$

$$X_i(q) = \frac{1}{2}\sigma_i^2\gamma^2 q^2, Y_i = C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}, Z_i = C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1}.$$

Using these definitions, the equation becomes

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \left[X_i(q) \cdot u_q^i(t) + Z_i \cdot u_{q-1}^i(t) - Y_i \cdot u_{q-1}^i(t)\right.$$
$$\left. -\lambda_{i,t} \cdot \left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\}\right)\right].$$

3.2.2 The case $|q| < Q$

For $|q| < Q$ the HJB equation is

$$V_t^i + \frac{1}{2}\sigma_i^2 V_{yy}^i + \max_{B_t^i} C_i \exp(-k_i B_t^i)\left[V^i(x-y+B_t^i, y, q+1, t) - V^i(x,y,q,t)\right] +$$
$$\max_{A_t^i} C_i \exp(-k_i A_t^i)\left[V^i(x+y+A_t^i, y, q-1, t) - V^i(x,y,q,t)\right]$$
$$+ \lambda_{i,t}(V^j - V^i) = 0, \tag{3.10}$$

$$V^i(x-y+B_t^i, y, q+1, t) = \left[-e^{(-\gamma(x-y+B_t^i+(q+1)y))}\right] \cdot \left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}$$
$$= \left[-e^{(-\gamma(x+B_t^i+qy))}\right] \cdot \left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}$$

As we insert $V^i(x-y+B_t^i, y, q+1, t)$, V_t^i and V_{yy}^i which we computed before, we get,

$$\left[-e^{(-\gamma(x+qy))}\right]\cdot(-\frac{\gamma}{k_i})\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}\cdot\dot{u}_q^i(t)+\frac{1}{2}\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\cdot\left[-\sigma_i^2\gamma^2q^2e^{(-\gamma(x+qy))}\right]$$
$$+\max_{B_t^i} C_i e^{-k_iB_t^i}\left[\left(-e^{(-\gamma(x+B_t^i+qy))}\right)\cdot\left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}-\left(-e^{(-\gamma(x+qy))}\right)\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]$$
$$+\max_{A_t^i} C_i e^{-k_iA_t^i}\left[\left(-e^{(-\gamma(x+A_t^i+qy))}\right)\cdot\left\{u_{q-1}^i(t)\right\}^{-\frac{\gamma}{k_i}}-\left(-e^{(-\gamma(x+qy))}\right)\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]$$
$$+\lambda_{i,t}\left[\left(-e^{(-\gamma(x+qy))}\right)\cdot\left\{u_q^j(t)\right\}^{-\gamma/k_j}+\left(e^{(-\gamma(x+qy))}\right)\left\{u_q^i(t)\right\}^{-\gamma/k_i}\right]=0$$

All terms have $e^{-\gamma(x+qy)}$ factor; they cancel out.

Then the equation becomes,

$$\frac{\gamma}{k_i}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}\cdot\dot{u}_q^i(t)-\frac{1}{2}\sigma_i^2\gamma^2q^2\left\{u_q^i(t)\right\}^{-\gamma/k_i}$$

$$+\max_{B_t^i} C_i e^{-k_iB_t^i}\left[\left(-e^{(-\gamma B_t^i)}\right)\cdot\left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}+\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]$$
$$+\max_{A_t^i} C_i e^{-k_iA_t^i}\left[\left(-e^{(-\gamma A_t^i)}\right)\cdot\left\{u_{q-1}^i(t)\right\}^{-\frac{\gamma}{k_i}}+\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]$$
$$+\lambda_{i,t}\left[-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}}+\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]=0$$

$$\frac{\gamma}{k_i}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}\cdot\dot{u}_q^i(t)-\frac{1}{2}\sigma_i^2\gamma^2q^2\left\{u_q^i(t)\right\}^{-\gamma/k_i}$$

$$+C_i\max_{B_t^i}\left(-e^{-(\gamma+k_i)B_t^i}\left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}+e^{-k_iB_t^i}\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right)$$
$$+C_i\max_{A_t^i}\left(-e^{-(\gamma+k_i)A_t^i}\left\{u_{q-1}^i(t)\right\}^{-\frac{\gamma}{k_i}}+e^{-k_iA_t^i}\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right)$$
$$+\lambda_{i,t}\left[-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}}+\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]=0$$

The maximization problem on B_t^i:

$$\max_{B_t^i}\left(-e^{-(\gamma+k_i)B_t^i}\cdot\left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}+e^{-k_iB_t^i}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right).$$

Let the objective function be:

$$f(B_t^i) = -e^{-(\gamma+k_i)B_t^i} \cdot \left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}} + e^{-k_i B_t^i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}.$$

Differentiate $f(B_t^i)$ with respect to B_t^i:

$$\begin{aligned}\frac{\partial f}{\partial B_t^i} &= \frac{\partial}{\partial B_t^i}\left(-e^{-(\gamma+k_i)B_t^i} \cdot \left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}} + e^{-k_i B_t^i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right) \\ &= (\gamma+k_i) \cdot e^{-(\gamma+k_i)B_t^i} \cdot \left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}} - k_i \cdot e^{-k_i B_t^i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}.\end{aligned}$$

To find the critical points, set $\frac{\partial f}{\partial B_t^i} = 0$:

$$(\gamma+k_i) \cdot e^{-(\gamma+k_i)B_t^i} \cdot \left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}} = k_i \cdot e^{-k_i B_t^i} \cdot \left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}.$$

Divide through by $e^{-k_i B_t^i}$ and rearrange terms:

$$\frac{\gamma+k_i}{k_i} = e^{\gamma B_t^i} \cdot \frac{\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}}{\left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}}.$$

Simplify the fraction:

$$\frac{\gamma+k_i}{k_i} = e^{\gamma B_t^i} \cdot \left(\frac{u_q^i(t)}{u_{q+1}^i(t)}\right)^{-\frac{\gamma}{k_i}}.$$

Take the natural logarithm of both sides:

$$\ln\left(\frac{\gamma+k_i}{k_i}\right) = \gamma \cdot B_t^i - \frac{\gamma}{k_i} \cdot \ln\left(\frac{u_q^i(t)}{u_{q+1}^i(t)}\right).$$

Rearrange to isolate the maximizer B_t^i:

$$B_t^i = \frac{1}{\gamma} \cdot \ln\left(1 + \frac{\gamma}{k_i}\right) + \frac{1}{k_i} \cdot \ln\left(\frac{u_q^i(t)}{u_{q+1}^i(t)}\right).$$

Substituting the expressions for A_t^i and B_t^i into the terms involving $e^{-(\gamma+k_i)A_t^i}$, $e^{-k_iA_t^i}$, $e^{-(\gamma+k_i)B_t^i}$, and $e^{-k_iB_t^i}$, we get:

$$
\begin{aligned}
&\frac{\gamma}{k_i}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}\cdot\dot{u}_q^i(t)-\frac{1}{2}\sigma_i^2\gamma^2q^2\left\{u_q^i(t)\right\}^{-\gamma/k_i}\\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-\frac{\gamma+k_i}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q+1}^i(t)}\right)^{-\frac{\gamma+k_i}{k_i}}\cdot\left\{u_{q+1}^i(t)\right\}^{-\frac{\gamma}{k_i}}\right.\\
&\quad\left.+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q+1}^i(t)}\right)^{-1}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]\\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-\frac{\gamma+k_i}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{-\frac{\gamma+k_i}{k_i}}\cdot\left\{u_{q-1}^i(t)\right\}^{-\frac{\gamma}{k_i}}\right.\\
&\quad\left.+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot\left(\frac{u_q^i(t)}{u_{q-1}^i(t)}\right)^{-1}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]\\
&+\lambda_{i,t}\cdot\left[-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}}+\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]=0.
\end{aligned}
$$

Rearranging the terms;

$$
\begin{aligned}
&\frac{\gamma}{k_i}\cdot\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}\cdot\dot{u}_q^i(t)-\frac{1}{2}\sigma_i^2\gamma^2q^2\left\{u_q^i(t)\right\}^{-\gamma/k_i}\\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-1-\frac{k_i}{\gamma}}\cdot\left(u_q^i(t)\right)^{-1-\frac{\gamma}{k_i}}\cdot\left\{u_{q+1}^i(t)\right\}\right.\\
&\quad\left.+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot\left(u_{q+1}^i(t)\right)\cdot\left\{u_q^i(t)\right\}^{-1-\frac{\gamma}{k_i}}\right]\\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-\frac{\gamma+k_i}{\gamma}}\cdot\left(u_q^i(t)\right)^{-1-\frac{\gamma}{k_i}}\cdot\left\{u_{q-1}^i(t)\right\}\right.\\
&\quad\left.+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot\left(u_{q-1}^i(t)\right)\cdot\left\{u_q^i(t)\right\}^{-1-\frac{\gamma}{k_i}}\right]\\
&+\lambda_{i,t}\cdot\left[-\left\{u_q^j(t)\right\}^{-\frac{\gamma}{k_j}}+\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}}\right]=0.
\end{aligned}
$$

Divide each side by $\left\{u_q^i(t)\right\}^{-\frac{\gamma}{k_i}-1}$

$$\begin{aligned}
&\frac{\gamma}{k_i}\cdot\dot{u}_q^i(t)-\frac{1}{2}\sigma_i^2\gamma^2q^2\left\{u_q^i(t)\right\} \\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-1-\frac{k_i}{\gamma}}\cdot\left\{u_{q+1}^i(t)\right\}+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot\left(u_{q+1}^i(t)\right)\right] \\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-1-\frac{k_i}{\gamma}}\cdot\left\{u_{q-1}^i(t)\right\}+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot\left(u_{q-1}^i(t)\right)\right] \\
&+\lambda_{i,t}\cdot\left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}}+\{u_q^i(t)\}\right)=0.
\end{aligned} \tag{3.11}$$

Forward difference approximation The time derivative $\dot{u}_q^i(t)$ is approximated using the forward difference method:

$$\dot{u}_q^i(t)\approx\frac{u_q^i(t+\Delta t)-u_q^i(t)}{\Delta t}.$$

Substituting this into the equation gives:

$$\begin{aligned}
&\frac{\gamma}{k_i}\cdot\frac{u_q^i(t+\Delta t)-u_q^i(t)}{\Delta t}-\frac{1}{2}\sigma_i^2\gamma^2q^2\cdot\{u_q^i(t)\} \\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1}\cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1}+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot u_{q+1}^i(t)\right] \\
&+C_i\cdot\left[-\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1}\cdot u_{q-1}^i(t)^{-\frac{\gamma}{k_i}-1}+\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}\cdot u_{q-1}^i(t)\right] \\
&+\lambda_{i,t}\cdot\left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}}+\{u_q^i(t)\}\right)=0.
\end{aligned}$$

Rearranging for $u_q^i(t+\Delta t)$, we get:

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \Bigg[\frac{1}{2}\sigma_i^2\gamma^2 q^2 \cdot \{u_q^i(t)\}$$

$$+ C_i \cdot \left(\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} - \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q+1}^i(t) \right)$$

$$+ C_i \cdot \left(\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q-1}^i(t)^{-\frac{\gamma}{k_i}-1} - \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q-1}^i(t) \right)$$

$$- \lambda_{i,t} \cdot \left(- \frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \right) \Bigg].$$

The variables $X_i(q)$, Y_i, and Z_i are defined as:

$$X_i(q) = \frac{1}{2}\sigma_i^2\gamma^2 q^2, Y_i = C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}}, Z_i = C_i \cdot \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1}.$$

Using these definitions, the final discretized equation for $u_q^i(t+\Delta t)$ is:

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \Bigg[X_i(q) \cdot u_q^i(t) + Z_i \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} - Y_i \cdot u_{q+1}^i(t)$$

$$+ Z_i \cdot u_{q-1}^i(t)^{-\frac{\gamma}{k_i}-1} - Y_i \cdot u_{q-1}^i(t) - \lambda_{i,t} \cdot \left(- \frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \right) \Bigg].$$

3.2.3 The case $q = -Q$

The HJB equation for this case is

$$V_t^i + \frac{1}{2}\sigma_i^2 V_{yy}^i + \max_{B_t^i} C_i \exp(-k_i B_t^i) \left[V^i(x-y+B_t^i, y, q+1, t) - V^i(x,y,q,t) \right]$$
$$+ \lambda_{i,t}(V^j - V^i) = 0, \tag{3.12}$$

By following the similar steps on the solution paths of Eq. (3.8) and Eq. (3.10), Eq. (3.12) becomes,

$$\begin{aligned}
&\frac{\gamma}{k_i} \cdot \dot{u}_q^i(t) - \frac{1}{2}\sigma_i^2\gamma^2 q^2 \left\{u_q^i(t)\right\} \\
&+ C_i \cdot \left[-\left(1+\frac{\gamma}{k_i}\right)^{-1-\frac{k_i}{\gamma}} \cdot \left\{u_{q+1}^i(t)\right\} + \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot \left(u_{q+1}^i(t)\right) \right] \\
&+ \lambda_{i,t} \cdot \left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \right) = 0.
\end{aligned} \tag{3.13}$$

Forward difference approximation As before we use the approximation

$$\dot{u}_q^i(t) \approx \frac{u_q^i(t+\Delta t) - u_q^i(t)}{\Delta t}.$$

Substituting this into the equation gives:

$$\begin{aligned}
&\frac{\gamma}{k_i} \cdot \frac{u_q^i(t+\Delta t) - u_q^i(t)}{\Delta t} - \frac{1}{2}\sigma_i^2\gamma^2 q^2 \cdot \{u_q^i(t)\} \\
&+ C_i \cdot \left[\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} - \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q+1}^i(t) \right] \\
&- \lambda_{i,t} \cdot \left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \right) = 0.
\end{aligned}$$

Rearranging for $u_q^i(t+\Delta t)$, we get:

$$\begin{aligned}
&u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \left[\frac{1}{2}\sigma_i^2\gamma^2 q^2 \cdot \{u_q^i(t)\} \right. \\
&+ C_i \cdot \left(\left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}-1} \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} - \left(1+\frac{\gamma}{k_i}\right)^{-\frac{k_i}{\gamma}} \cdot u_{q+1}^i(t) \right) \\
&\left. - \lambda_{i,t} \cdot \left(-\frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \right) \right].
\end{aligned}$$

Substituting the variables $X_i(q)$, Y_i, and Z_i , the final equation becomes:

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \Bigg[X_i(q) \cdot u_q^i(t) + Z_i \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} \\ - Y_i \cdot u_{q+1}^i(t) - \lambda_{i,t} \cdot \Bigg(- \frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \Bigg) \Bigg].$$

3.2.4 Forward Difference Approximations

For ease of reference in numerical calculations we collect all of the forward difference approximations below:

For $q = Q$,

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \Bigg[X_i(q) \cdot u_q^i(t) + +Z_i \cdot u_{q-1}^i(t)^{-\frac{\gamma}{k_i}-1} - Y_i \cdot u_{q-1}^i(t) \\ - \lambda_{i,t} \cdot \Bigg(- \frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \Bigg) \Bigg]. \tag{3.14}$$

For $|q| < Q$,

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \Bigg[X_i(q) \cdot u_q^i(t) + Z_i \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} - Y_i \cdot u_{q+1}^i(t) \\ + Z_i \cdot u_{q-1}^i(t)^{-\frac{\gamma}{k_i}-1} - Y_i \cdot u_{q-1}^i(t) - \lambda_{i,t} \cdot \Bigg(- \frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \Bigg) \Bigg]. \tag{3.15}$$

For $q = -Q$,

$$u_q^i(t+\Delta t) = u_q^i(t) + \Delta t \cdot \frac{k_i}{\gamma} \cdot \Bigg[X_i(q) \cdot u_q^i(t) + Z_i \cdot u_{q+1}^i(t)^{-\frac{\gamma}{k_i}-1} - Y_i \cdot u_{q+1}^i(t) \\ - \lambda_{i,t} \cdot \Bigg(- \frac{\{u_q^j(t)\}^{-\frac{\gamma}{k_j}}}{\{u_q^i(t)\}^{-\frac{\gamma}{k_i}-1}} + \{u_q^i(t)\} \Bigg) \Bigg]. \tag{3.16}$$

CHAPTER 4

INVENTORY PROCESS DYNAMICS IN THE REGIME SWITCH MODEL

Our goal is to understand the behavior of the jump probabilities of the inventory process. A natural way to do this is to compare them with the jump probabilities in the case of no regime switch. As noted in the introduction, we will do the comparison numerically using the following steps:

1. we fix parameter values $\sigma_i, C_i, k_i, \lambda_i$ $i = 1, 2$ and γ and use the discretized ODE to compute the value function. Then formulas (3.6) and (3.7) give the jump intensities of the inventory process.
2. We then compute the value function of the ordinary model (without regime switch) for two sets of parameter values: $\{\sigma_1, C_1, k_1, \gamma\}$ and $\{\sigma_2, C_2, k_2, \gamma\}$
3. We compare the jump intensity of the inventory process of the model with regime switch with the jump intensities of the ordinary model with the above parameter values.

The parameter values to be used in the numerical computation are as follows:

$$\sigma_1, k_1, C_1, \lambda_1 = (1, 1, 3, 1), \tag{4.1}$$

$$\sigma_2, k_2, C_2, \lambda_2 = (2, 2, 1, 1), \tag{4.2}$$

γ is taken to be 1.

Let $p_t^{i,a}$ the ask probabilities of the inventory process in regime i of the model with regime switch. Let $q_t^{1,a}$ ($q_t^{2,a}$) denote the ask probabilities of the ordinary model (with no regime switch) with parameter values listed in (4.1) and (4.2). Figure 4.1 shows the graphs of $p_t^{1,a}$, $q_t^{1,a}$ and $q_t^{2,a}$ for $t = 0.7$ and Figure 4.2 shows the graphs of $p_t^{2,a}$, $q_t^{1,a}$ and $q_t^{2,a}$ for $t = 0.7$. We give only the ask probabilities since the model is symmetric in ask and bid and we have $p_t^{i,b}(q) = p_t^{i,a}(-q)$.

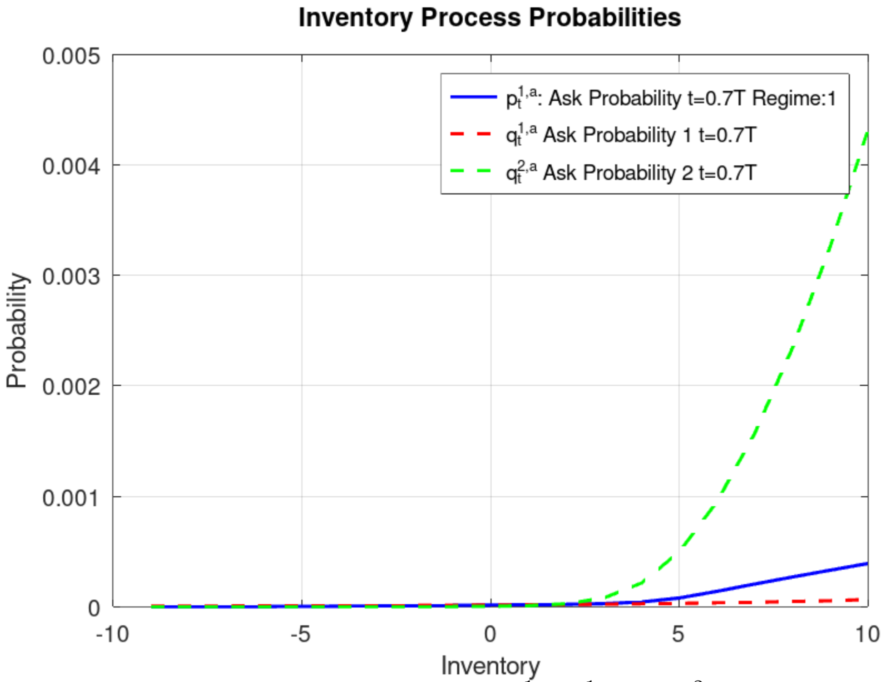

Figure 4.1: Graphs of $p_t^{1,a}$, $q_t^{1,a}$ and $q_t^{2,a}$

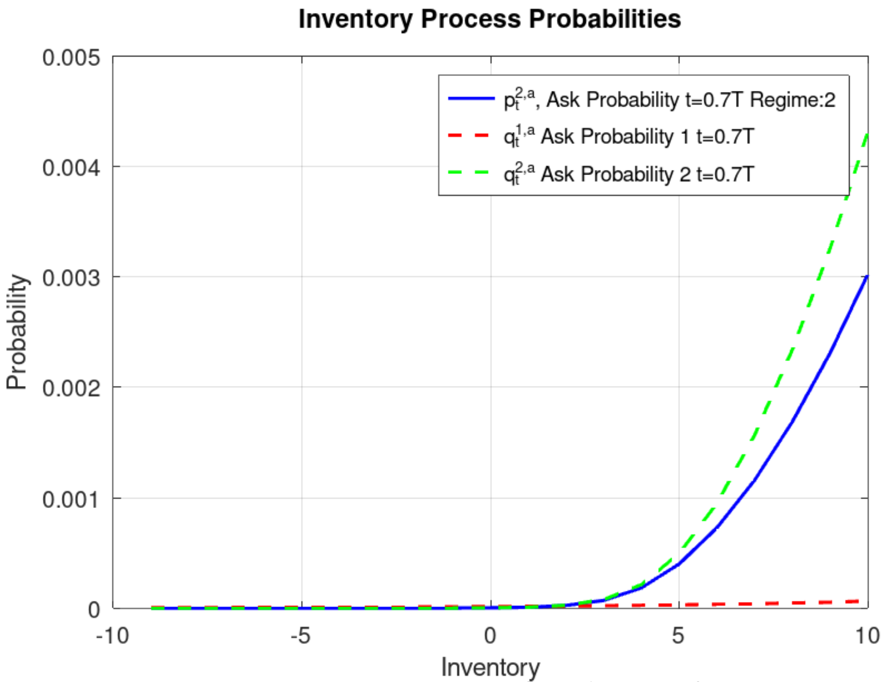

Figure 4.2: Graphs of $p_t^{2,a}$, $q_t^{1,a}$ and $q_t^{2,a}$

Looking at these figures we see that the regime switch jump probability $p_t^{i,a}$ lies between and behaves similarly to the jump probabilities $q_t^{i,a}$ of the ordinary model with no regime switch computed separately for the parameter values listed in (4.1) and (4.2). When we repeated the same computation with different parameter values we saw that this behavior persists across parameter values. This suggests that the dependence on the parameters γ, σ_i, C_i, k_i of the jump probabilities $p_t^{i,a}$ of the regime switch model is similar to the dependence on γ, σ, C, k of the jump probabilities of the ordinary model. This is further explored numerically in the next chapter.

4.1 Impact of λ on inventory process dynamics

Now let us consider the impact of the regime switch rates λ_1 and λ_2 on inventory process dynamics. In the market making model with regime switch, the inventory process itself is a regime switching process, with jump probabilities $p_t^{i,a}$ and $p_t^{i,b}$ in state i. The parameters λ_1 and λ_2 directly serve as the regime switch intensities for the inventory process as well. Therefore, these parameters are directly parameters of the inventory process.

λ_1 and λ_2 may also influence the dynamics of the inventory process indirectly by influencing the jump probabilities $p_t^{i,a}$ and $p_t^{i,b}$. In Figures 4.3 and 4.4 we consider this influence. These figures show $p_t^{1,a}$, $q_t^{1,a}$ and $q_t^{2,a}$ where parameter values are as in (4.1) and (4.2) except for λ. In Figure 4.3, $p_t^{1,a}$ is computed with $\lambda_1 = 1$ and $\lambda_1 = 3$ and $\lambda_2 = 1$ fixed. This figure suggests that λ_1 has a limited impact on $p_t^{1,a}$ slightly pushing it up towards $q_t^{a,2}$.. In Figure 4.4, $p_t^{1,a}$ is computed with $\lambda_2 = 1$ and $\lambda_2 = 3$ and $\lambda_1 = 1$ fixed. This figure suggests that λ_2 has no visible impact on $p_t^{1,a}$ at least for the parameter values considered in this example.

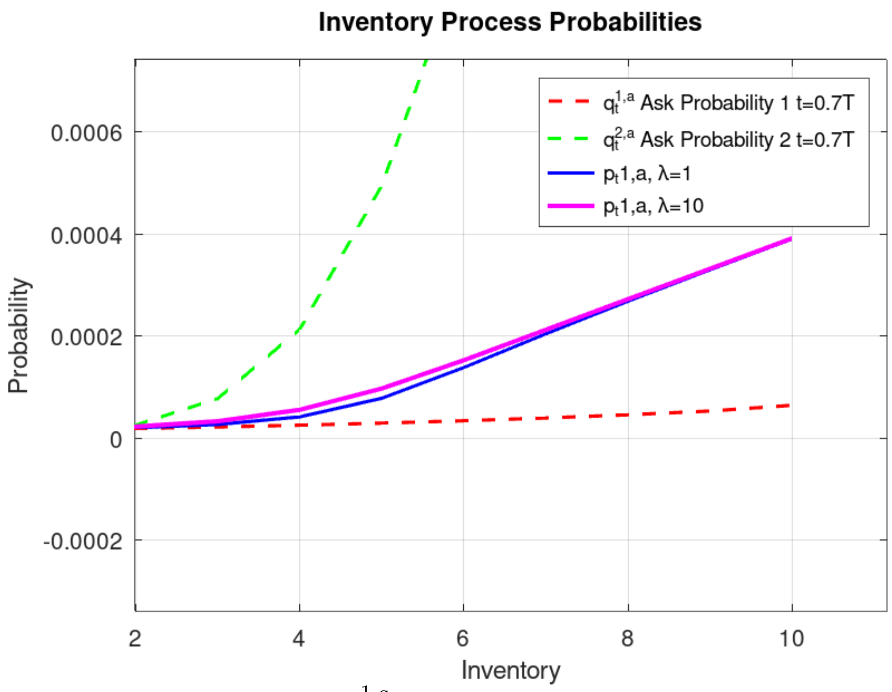

Figure 4.3: Graph of $p_t^{1,a}$, for $\lambda_2 = 1$ fixed and $\lambda_1 = 1$ and $\lambda_1 = 3$

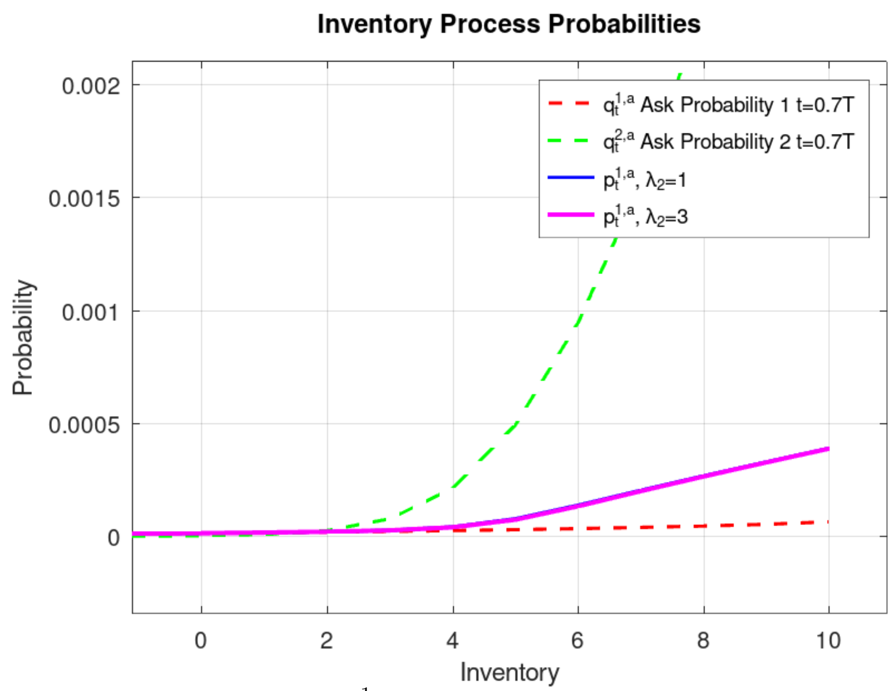

Figure 4.4: Graph of $p_t^{1,a}$, for $\lambda_1 = 1$ fixed and $\lambda_2 = 1$ and $\lambda_2 = 3$

CHAPTER 5

DEPENDENCE OF THE INVENTORY DYNAMICS ON THE ORDINARY MODEL PARAMETERS

The goal of this chapter is to see how the jump probabilities of the inventory process change with model parameters γ, σ_i, C_i and k_i.

For parameter values not depending on i, we will use the following baseline values:

- $Q = 25$, the maximum authorized inventory,
- $T = 1$, the terminal time,
- $\gamma = 0.1 \cdot 10^{-1}$, the risk aversion parameter,
- $\Delta t = \frac{1}{1000}$, the time increment,

Baseline values for parameters depending on i are as follows: [4]

$$\begin{aligned} \lambda_1 = 30, \quad \lambda_2 = 1, \\ C_1, k_1, \sigma_1 = (1; 1.5; 0.5), \\ C_2, k_2, \sigma_2 = (0.5; 3.0; 0.1). \end{aligned} \tag{5.1}$$

For each of the parameters γ, σ and C and k we will fix the rest of the parameter values as above and vary the chosen parameter.

After substituting these parameters into Eqs. (3.14), (3.15), and (3.16) and generating their plots, the results are obtained as follows, represented by the following graphics:

- Effect of C on the model when parameters varying around the baseline values.
- Effect of γ on the model when parameters varying around the baseline values.
- Effect of k on the model when parameters varying around the baseline values.

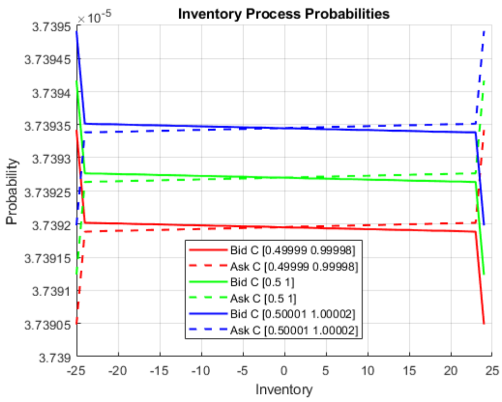

Figure 5.1: Effect of C on the Model

Figure 5.1 illustrates the inventory process probabilities for different ranges of C, showing bid and ask probabilities as functions of inventory levels, while other parameters remain as in the baseline parameters as in 5.1. The probabilities are nearly constant across inventory levels, with sharp changes at the extremes ($q = \pm 25$). For $C_i = [0.49999\ \ 0.99998]$ (red dashed lines), the probabilities are the lowest, while for $C_i = [0.50001\ \ 1.00002]$ (blue dashed lines), they are the highest. The probabilities for $C_i = [0.5\ \ 1]$ (green solid lines) lie between these two ranges. Across all ranges of C, the bid and ask probabilities are symmetric around the inventory axis ($q = 0$), reflecting balanced market behavior. The sharp drops at the inventory extremes indicate higher sensitivity or adjustments at these points. Overall, while the magnitude of probabilities varies slightly with C, the general shape of the remain the same.

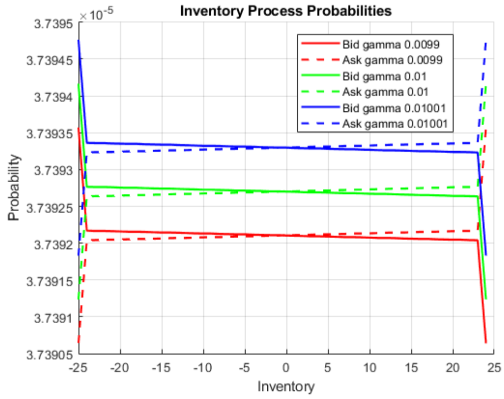

Figure 5.2: Effect of γ on the Model

Figure 5.2 compares the inventory process probabilities for different values of γ (0.0099, 0.01, and 0.01001), while keeping the other parameters fixed at their baseline values as in 5.1. The probabilities are plotted against inventory levels, ranging from -25 to 25. For $\gamma = 0.0099$ (red lines), the probabilities are the lowest, with a range of approximately 3.7392×10^{-5} to 3.73945×10^{-5}. For $\gamma = 0.01$ (green lines), the probabilities are slightly higher, and for $\gamma = 0.01001$ (blue lines), the probabilities are the highest, ranging from 3.7393×10^{-5} to 3.7395×10^{-5}. Across all values of γ, the bid and ask probabilities are symmetric around the inventory axis ($q = 0$) and remain nearly constant across most inventory levels, with sharp drops at the extremes ($q = \pm 25$). The increase in γ leads to a slight upward shift in probabilities, reflecting the impact of higher risk aversion on the inventory process. However, the overall shape of the jump probabilities remains the same for different γ values.

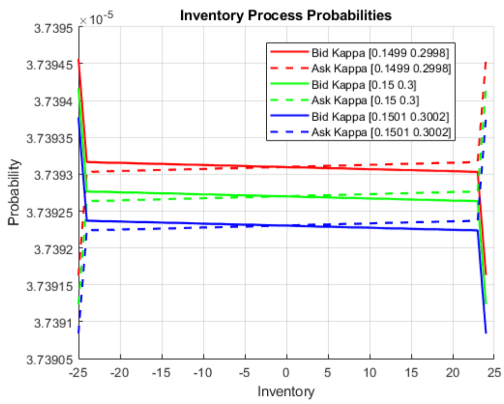

Figure 5.3: Effect of k on the Model

Figure 5.3 compares the inventory process probabilities for different values of k $k_i = ([0.1499\ 0.2998],\ [0.15\ 0.3],$ and $[0.1501\ 0.3002])$, while keeping the other parameters fixed at their baseline values as in 5.1. For $k_i = [0.1499\ 0.2998]$ (red lines), the probabilities are the highest, ranging from approximately 3.7394×10^{-5} to 3.7395×10^{-5}. For $k_i = [0.15\ 0.3]$ (green lines), the probabilities are slightly lower, and for $k_i = [0.1501\ 0.3002]$ (blue lines), the probabilities are the lowest, ranging from 3.7392×10^{-5} to 3.73935×10^{-5}. Across all values of k, the bid and ask probabilities are symmetric around the inventory axis ($q = 0$) and remain nearly constant across most inventory levels, with sharp drops at the extremes ($q = \pm 25$). The decrease in k leads to a slight upward shift in probabilities, reflecting the impact of lower price sensitivity on the inventory process. As with the other parameters, the shape of the jump probabilities remain same for different k values.

To summarize: the inventory process jump probabilities are sensitive to small changes in the parameters C, γ, and k, as shown in the graphs. For C and γ, an increase in their values results in higher probabilities, with the blue lines (representing the highest values of C and γ) consistently showing the largest probabilities, while the red lines (representing the lowest values) show the smallest probabilities. For k, the effect is opposite: higher values of k result in lower probabilities, as seen in the blue lines for $k_i = [0.1501\ 0.3002]$, which are the lowest, and the red lines for $k_i = [0.1499\ 0.2998]$, which are the highest.

CHAPTER 6

DISTRIBUTION OF THE TERMINAL POSITION

We would like to examine the distribution of $q(T)$, the marketmaker's position in the underlying asset at terminal time T, and how this distribution depends on model parameters. We will do this through a simulation study.

Recall that the ask and bid jump rates of q are symmetric around the origin. This suggests $\mathbb{E}[q(T)] = 0$.

10 simulated sample paths of q and one simulated path of the regime process are shown in Figure 6.1. The parameter values used in the simulation are:
$k = (1, 2), \quad C = (150, 300), \quad \lambda = (30, 30), \quad \sigma = (1, 2), \quad \gamma = 0.6, \quad T = 0.2.$

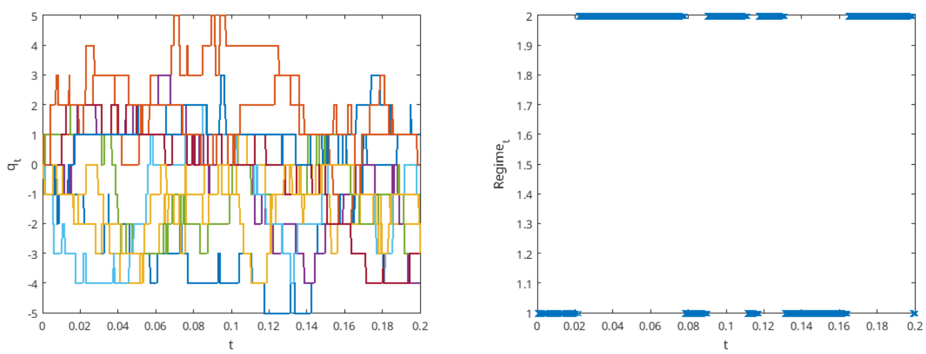

Figure 6.1: 10 sample paths of q and one simulated path of the regime process

The histogram of $q(T)$ constructed from 4000 sample paths with the same model parameters is shown in Figure 6.2. The sample mean of $q(T)$ over the simulated sample paths is -7×10^{-3}. We note that this and the shape of the histogram given in Figure 6.2 are compatible with $\mathbb{E}[q(T)] = 0$ and this is independent of model parameters.

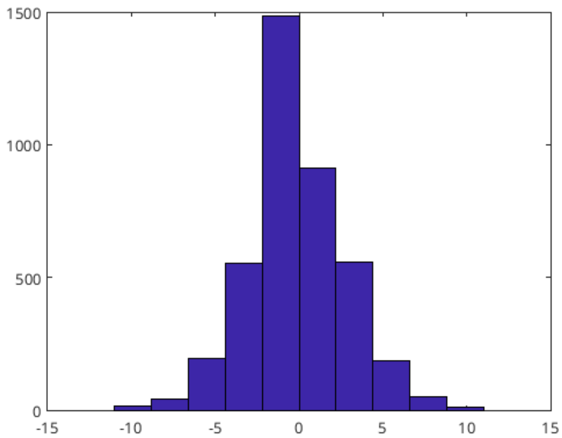

Figure 6.2: Histogram of $q(T)$ constructed from 4000 sample paths.

Therefore, a natural way to study how the distribution of $q(T)$ changes with model parameters is to study how the standard deviation of $q(T)$ changes with model parameters. This is done in the figures below. Since the model is symmetric with respect to the regimes, it is sufficient to study the dependence of $\sigma(q(T))$ with respect to only σ_1, C_1, and k_1; the dependence on σ_2, C_2, and k_2 will be entirely symmetric.

In what follows, let $q_i(T)$ denote the terminal position of the ordinary model with model parameters set to the model parameters in regime i.

Figure 6.3 shows how the standard deviation $\sigma(q(T))$ of $q(T)$ varies with σ_1. We note that, similar to the jump probabilities, $\sigma(q(T))$ lies between $\sigma(q_1(T))$ and $\sigma(q_2(T))$. In Figure 6.3, $\sigma(q_2(T))$ remains approximately constant since the only parameter that is being varied is σ_1 and $\sigma(q_2(T))$ is independent of σ_1. We further note that both $\sigma(q(T))$ and $\sigma(q_1(T))$ are decreasing in σ_1.

Figure 6.4 shows how $\sigma(q(T))$ changes with γ. Once again $\sigma(q(T))$ of the regime-switching model lies between $\sigma(q_1(T))$ and $\sigma(q_2(T))$. All three graphs show a decreasing trend as γ increases. Recall that γ is the risk aversion parameter; the more the risk averse the trader is, less the standard deviation of the terminal position.

Figure 6.5 shows how the standard deviation $\sigma(q(T))$ of $q(T)$ varies with C_1. As before $\sigma(q(T))$ lies between $\sigma(q_1(T))$ and $\sigma(q_2(T))$. When C_1 increases, both $\sigma(q(T))$ and $\sigma(q_1(T))$ increase, while $\sigma(q_2(T))$ remains constant (since q_2 is independent of C_1). Recall that C_1 determines the jump rates of the sell and buy orders. Increasing

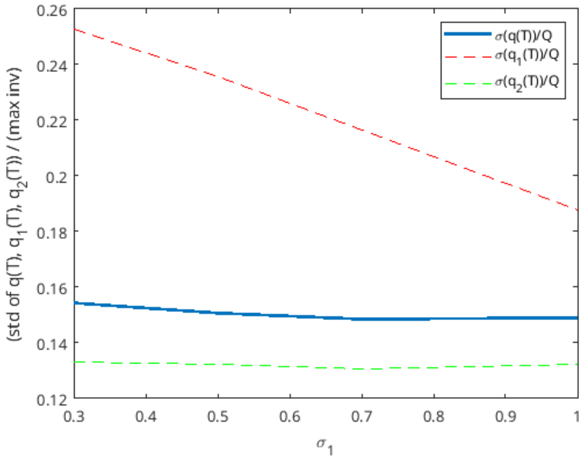

Figure 6.3: $\sigma(q(T))$ as a function of σ_1.

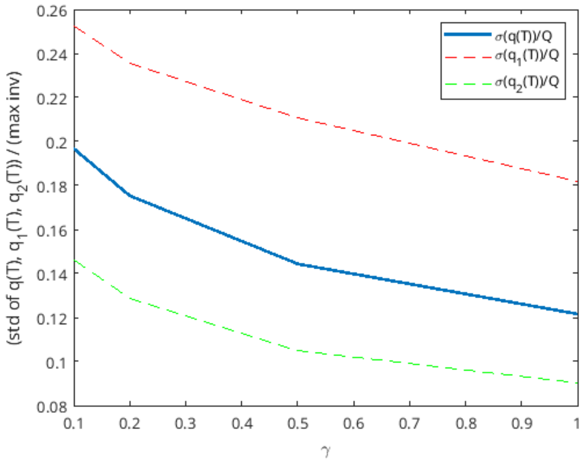

Figure 6.4: $\sigma(q(T))$ as a function of γ.

this jump rate leads to higher values of $\sigma(q(T))$.

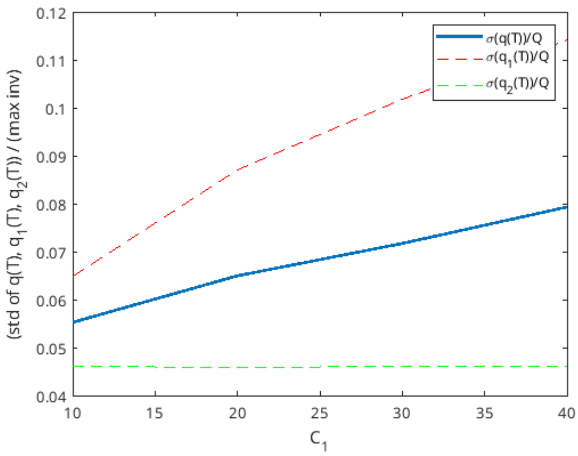

Figure 6.5: $\sigma(q(T))$ as a function of C_1.

Figure 6.6 illustrates how the standard deviation $\sigma(q(T))$ of $q(T)$ is affected by changes in k_1. $\sigma(q(T))$ lies once again between $\sigma(q_1(T))$ and $\sigma(q_2(T))$. As k_1 increases, both $\sigma(q(T))$ and $\sigma(q_1(T))$ decrease, while $\sigma(q_2(T))$ remains constant (since $q_2(T)$ is indepedent of k_1). Recall that the buy and sell order jump rates decrease when k_1 increases; this explains the monotone decreasing behavior of $\sigma(q(T)$ and $\sigma(q_1(T))$ in k_1.

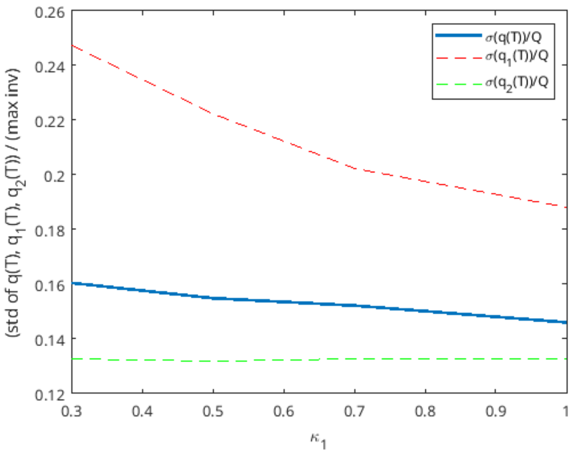

Figure 6.6: $\sigma(q(T))$ as a function of k_1.

Finally, Figure 6.7 shows how $\sigma(q(T))$ changes with λ_1. Note that in this figure both $\sigma(q_1(T))$ and $\sigma(q_2(T))$ are approximately constant because the non-modulated model doesn't depend on λ_1. As λ_1 increases $\sigma(q(T))$ decreases toward $\sigma(q_2(T))$. This makes sense since higher values of λ_1 corresponds to higher jump rates from the first state to the second, which means that the model spends more time in the second state, which increases the influence of this state on $\sigma(q(T))$.

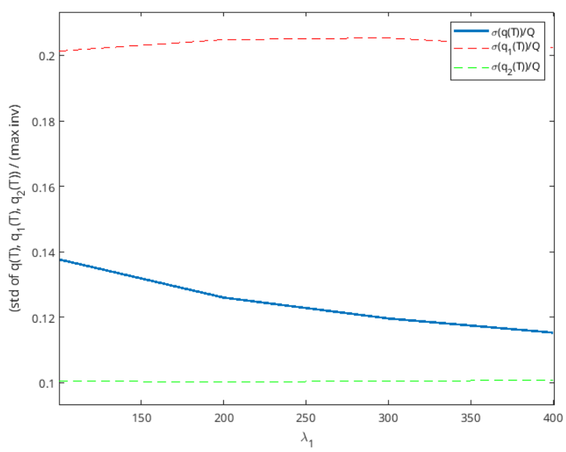

Figure 6.7: $\sigma(q(T))$ as a function of λ_1.

CHAPTER 7

CONCLUSION

In this thesis, we studied a version of the AS market making model, presented in [16], with regime switch with two regimes. We noted that the reduction to ODE that is possible in the ordinary AS model is also possible in the model with regime switch. However, this reduction leads to a nonlinear ODE, as opposed to the ordinary case where the reduced ODE is linear. We numerically solved the ODE to study the dependence of the dynamics of the inventory process to model parameters. We noted that the inventory process behaves itself like a process with regime switch, the regime process of the model also determining the regime of the inventory process. In each regime the jump probabilities have a shape similar to the jump probabilities of the ordinary AS market making model and lie between the jump probabilities of two ordinary AS models where the first one has parameters of the first regime and the second one has the parameters of the second regime. We also noted that the jump probabilities of the regime switching model respond to the model parameters C_i, γ and k_i in a way that is similar to how the jump probabilities of the ordinary model respond to the same parameters.

In the present thesis, we focused on two regimes; we think that the case of more than two regimes would lead to results similar to those provided in the current thesis. This can be checked in future work. Secondly, we focused on the case $\ell = 0$; the case $\ell \neq 0$ was considered in [3] for the ordinary model. Future work can also study this generalization in the regime switch model. Finally, it would be interesting to try to derive some theoretical results that explain the observations made in the current thesis using numerical calculations.

REFERENCES

[1] R. Almgren and N. Chriss, Optimal execution of portfolio transactions, Journal of Risk, 3, pp. 5–40, 2001.

[2] I. Atceken, Dynamics of the inventory process in the avellenada-stoikov market making model, Term project report, Graduate School of Applied Mathematics, Middle East Technical University, 2020.

[3] G. D. Atesagaoglu, Impact of risk liquidity premium on inventory process in a market making model, Masters thesis, Graduate School of Applied Mathematics, Middle East Technical University, 2024.

[4] M. Avellaneda and S. Stoikov, High-frequency trading in a limit order book, Quantitative Finance, 8(3), pp. 217–224, 2008.

[5] Á. Cartea, R. Donnelly, and S. Jaimungal, Algorithmic trading with model uncertainty, SIAM Journal on Financial Mathematics, 8(1), pp. 635–671, 2017.

[6] Á. Cartea and S. Jaimungal, Order-flow and liquidity provision, Available at SSRN 2553154, 2015.

[7] Á. Cartea and S. Jaimungal, Risk metrics and fine tuning of high-frequency trading strategies, Mathematical Finance, 25(3), pp. 576–611, 2015.

[8] Á. Cartea, S. Jaimungal, and D. Kinzebulatov, Algorithmic trading with learning, International Journal of Theoretical and Applied Finance, 19(04), p. 1650028, 2016.

[9] Á. Cartea, S. Jaimungal, and J. Penalva, *Algorithmic and high-frequency trading*, Cambridge University Press, 2015.

[10] P. Fodra and M. Labadie, High-frequency market-making with inventory constraints and directional bets, arXiv preprint arXiv:1206.4810, 2012.

[11] L. R. Glosten and P. R. Milgrom, Bid, ask and transaction prices in a specialist market with heterogeneously informed traders, Journal of financial economics, 14(1), pp. 71–100, 1985.

[12] O. Guéant, *The Financial Mathematics of Market Liquidity: From optimal execution to market making*, volume 33, CRC Press, 2016.

[13] O. Guéant, C.-A. Lehalle, and J. Fernandez-Tapia, Dealing with the inventory risk: a solution to the market making problem, Mathematics and financial economics, 7, pp. 477–507, 2013.

[14] J. D. Hamilton, A new approach to the economic analysis of nonstationary time series and the business cycle, Econometrica: Journal of the econometric society, pp. 357–384, 1989.

[15] T. Ho and H. R. Stoll, Optimal dealer pricing under transactions and return uncertainty, Journal of Financial economics, 9(1), pp. 47–73, 1981.

[16] J. G. Kim, S. Beatson, B.-G. Jang, H.-S. Lee, and S. Park, Short-term market changes and market making with inventory, Available at SSRN 3639715, 2020.

[17] Q.-Q. Yang, W.-K. Ching, J.-W. Gu, and T.-K. Siu, Market-making strategy with asymmetric information and regime-switching, Journal of Economic Dynamics and Control, 90, pp. 408–433, 2018.