A DYNAMIC THEORY FOR POLARIZABLE AND MAGNETIZABLE MAGNETO-ELECTRO THERMO-VISCOELASTIC ANISOTROPIC SOLIDS WITH THERMAL AND ELECTRICAL CONDUCTION

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by

YAŞAR ERSOY

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Major Supervisor

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Chairman of Major Department

Examining Committee in Charge:

Assoc. Prof. Dr. Suha Selamoğlu

Assoc. Prof. Dr. Erhan Kıral :

Assoc. Prof. Dr. Erdoğan Karahan

Assoc. Prof. Dr. Ali İ. Üsseli

..............................:

..............................:

Committee Chairman
Prof. Dr. Murat Dikmen

ABSTRACT

A DYNAMIC THEORY FOR POLARIZABLE AND MAGNETIZABLE MAGNETO-ELECTRO THERMO- VISCOELASTIC ANISOTROPIC SOLIDS WITH THERMAL AND ELECTRICAL CONDUCTION

ERSOY, Yaşar
Ph.D. in E.S.
Supervisor: Assoc.Prof.Dr.Erhan Kıral
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A dynamic theory for polarizable and magnetizable magneto-electro thermo-viscoelastic anisotropic solids with thermal and electrical conduction is developed for time-dependent electromagnetic fields. The first part of this thesis is concerned with the several formulations of Maxwell's equations, and the interactions between the electromagnetic fields and the deformable continua. Using the balance laws of nonrelativistic classical continuum mechanics, the balance equations and the boundary conditions have been formulated, and the constitutive equations for linear anisotropic materials having magnetic symmetry have been derived.

Since the governing equations are highly nonlinear and very complicated, they have been linearized in the second part. Thus, the governing equations are decomposed into two groups. The first group is the same as that of rigid body electrodynamics, and the second group is the one accounting for the interactions of the electromagnetic fields with thermo-viscoelastic continuum through linearized equations. Further, the theory developed for a general anisotropy is applied to special cases.

In the last part of the present research, an application is given for a special case of the linearized theory. Propagation of magneto-mechanical waves in magneto-visco-elastic, electrically conductive, isotropic solids in an externally uniform primary magnetic field is investigated. The phase velocities and the attenuations per wavelenght have been obtained, both analytically and numerically. Some interesting behavior of the phase velocities and the attenuations of these waves are numerically detected for certain frequencies and strong magnetic field.

Key words: anisotropy, attenuation, conduction, deformation, magnetizable material, magneto-electro thermodynamics, polarizable material, phase velocity, viscoelasticity.

ÖZET

POLARİZE VE MAGNETİZE OLAN, ISI VE ELEKTRİĞİ İLETEN MAGNETO-ELECTRO TERMO-VİZKOELASTİK ANİZOTROP KATI CİSMİN DİNAMİK TEORİSİ

ERSOY, Yaşar
Doktora Tezi, Müh.Bil.Bölümü
Tez Yöneticisi: Doç.Dr.Erhan Kıral
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Bu tezde, ısı ve elektriği ileten, polarize ve magnetize olan magneto-elektro termo-vizkoelastik anizotrop katı cismin zamana bağlı elektromagnetik alan içerisindeki dinamik teorisi geliştirilmiştir. Bu araştırmanın ilk kısmı, Maxwell denklemlerinin çeşitli formülasyonları, elektromagnetik alanların şekil değiştiren ortamla olan karşılıklı etkileşmeleriyle ilgilidir. Relativistik olmayan klâsik sürekli ortam mekaniğinin denge denklemleri kullanılarak, karşılıklı etkileşmelerle ilgili denge denklemleri, sınır koşulları ve magnetik simetriye sahip, lineer anizotrop cisimlerin bünye denklemleri türetilmiştir.

Etkileşmeyi yöneten denklemler nonlineer ve çok karışık olduğundan araştırmanın ikinci kısmında, bütün denklemler lineerleştirilmiştir. Böylece, denklemler iki gruba ayrıştırılmış olup, ilk gruptakiler rijit cisimlerin elektrodinamiğindekinin aynısı, ikinciler ise elektromagnetik alanın termo-vizkoelastik ortamla karşılıklı etkileşmesini belirliyen lineer denklemlerdir. Daha sonra, genel anizotrop cisimlerle ilgili teori bazı özel durumlara uygulanmıştır.

Bu tezin son kısmı lineer teorinin özel bir durumunun uygulanışı ile ilgilidir. Üniform magnetik alanın içerisinde, elektriği ileten izotrop katı cismin içerisinde yayılan magneto-mekanik dalgalar incelenmiştir. Yayılan dalgaların faz hızları ve birim dalga boyundaki zayıflamaları analitik ve de sayısal olarak elde edilmiştir. Büyük magnetik alan içerisinde yayılan belirli frekanstaki dalgaların faz hızlarında ve zayıflamalarında ilginç davranışlar sayısal olarak saptanmıştır.

Anahtar sözcükler: anizotropi, faz hızı, iletken, magnetize olan cisim, magneto-elektro termodinamik, polarize olan cisim, şekil değiştirme, vizkoelastisite, zayıflama.

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NOMENCLATURE

1. Symbolic Characters:

$\{\,,\ ,\}$	Set of quantities in the bracket
$[\![\quad]\!]$	Values of (.) from positive and negative sides of a surface (or a line)
$(*)\big\|_{(\cdot)}$	Value of the quantity (*) evaluated at specific values of the variables (.)
$(\cdot)^t$	Transpose of the enclosed quantity (.)
$(\cdot)_r$	Real part of (.)
$(\ )_i$	Imaginary part of (.)
$(\ )^A$	Quantities in Amperian formulation
$(\cdot)^B$	Quantities in Boffi formulation
$(\cdot)^C$	Quantities in Chu formulation
$(\cdot)^M$	Quantities in Minkowski formulation
$(\cdot)^o$	Quantities in other formulation
$(\dot{*})$	Time rate of the enclosed quantity (*)
∇	del operator in 3-dimensional space
∇^2	Laplance operator in 3-dim. space
$\Box$	D'Alembert operator in 3-dim. space
$\bar{\Box}$	D'Alembert operator in the rigid body state
$\Box\!\!\mid$	Four-dimen. del operator
$O(10^n)$	Means of order 10^n
$\oint_{\partial\mathcal{C}}$	Line integral of the closed line
$\oint_{\partial\mathcal{V}}$	Surface integral of the closed surface
$\int_{\mathcal{V}}$	Volume integral of the volume
$d(\cdot)$	Differential element of the enclosed quantity (.)
$(\cdot)^{(e)}$	Quantities associated with elasticity
$(\cdot)^{(g)}$	Quantities associated with temperature gradient
$(\cdot)^{(m)}$	Quantities associated with magnetization
$(\cdot)^{(P)}$	Quantities associated with polarization
$(\cdot)^{(\theta)}$	Quantities associated with temperature
$(\cdot)^{(v)}$	Quantities associated with viscoelasticity

$(\cdot)^{(ep)}$	Quantities associated with electroelasticity
$(\cdot)^{(em)}$	Quantities associated with magnetoelasticity
$(\cdot)^{(e\theta)}$	Quantities associated with thermoelasticity
$(\cdot)^{(pm)}$	Quantities associated with magnetoelectricity
$(\cdot)^{(m\theta)}$	Quantities associated with thermomagnetism
$(\cdot)^{(p\theta)}$	Quantities associated with thermoelectricity
$\Vert\ \Vert$	Matrix
$(\cdot)^{-1}$	Inverse of the enclosed quantity (.)
$(\cdot)\times(*)$	Vectorial product of the quantities $(\cdot)$ and $(*)$
$(*)\cdot(*)$	Dot product of the quantities $(*)$ and $(*)$
$(\cdot)\oplus(*)$	Direct sum of the quantities $(\cdot)$ and $(*)$
$(\cdot)\otimes(*)$	Outer product of the quantities $(\cdot)$ and $(*)$

2. Upper and Lower Case Latin and Greek Characters

$\underset{\sim}{A}^{(p)}(=\bar{\underset{\sim}{A}}^{(p)}+\underset{\sim}{a}^{(p)})$	Electrical vector potential
$\underset{\sim}{A}^{(m)}(=\bar{\underset{\sim}{A}}^{(m)}+\underset{\sim}{a}^{(p)})$	Magnetical vector potential
a_{ij}	Entries of matrix $\Vert A_{ij}\Vert$
$\underset{\sim}{a}$	Acceleration of a particle
a	Time shift constant
$\alpha;\ \underset{\sim}{\alpha};\ \bar{\alpha};\ \underset{=}{\alpha};\ \alpha_1;\ \alpha_2$	Attenuation per wavelength
$\underset{\sim}{B}(x,t)\ ;\ \underset{\sim}{B}'$	Magnetic induction in S and S' frames
b_α	A four-vector
$\mathcal{B}_0;\ \mathcal{R}_R;\ \mathcal{B}_t$	Initial, reference and present configurations of body
$\beta;\ \bar{\beta};\ \underset{=}{\beta};\ \beta_1;\ \beta_2\ \ldots$	Dimensionless numbers
$C_{KL}\ ;\ c_{kl}$	Green and Cauchy deformation tensors
$c=1/\sqrt{\epsilon_0\mu_0}$	Speed of light in vacuum (3×10^8 ms^{-1})
c_0	Characteristic wave speed
$\hat{c}_1;\ \hat{c}_2;\ldots;\hat{c}_{11};\ \hat{c}_1';\ldots\hat{c}_5'$	Material constants for isotropic solids
$\hat{\underset{\sim}{C}}^{(\cdot)};\ \hat{\underset{\approx}{C}}^{(\cdot)};\ \hat{\underset{\sim}{J}}^{(\cdot)};\ \hat{\underset{\approx}{J}}$	Material constants associated with (.)
curl	curl at a fixed spatial point $\underset{\sim}{x}$ at time t
$\partial\mathcal{C}(t)$	Surface enclosed by the contour $\mathcal{C}$ at t
$\underset{\sim}{D}(x,t);\ \underset{\sim}{D}'$	Electrical displacement in S and S'frames
$\mathcal{D}$	A positive quantity
Δ_{ij}	Cofactor of the matrix $\Vert\Lambda_{kl}\Vert$

dA ; da	Area element in $\mathcal{B}_R$ and $\mathcal{B}_t$
dS; ds	Line element in $\mathcal{B}_R$ and $\mathcal{B}_t$
dV ; dv	Volume element in $\mathcal{B}_R$ and $\mathcal{B}_t$
$\det \lVert \cdot \rVert$	Determinant of the matrix $\lVert \cdot \rVert$
$d\phi$	Infinitesimal angle
Div ; div	Divergence in four-dim. and 3-dim. spaces
$\frac{d(\cdot)}{dt}$; $\frac{\partial(\cdot)}{\partial t}$	Material and partial derivative of (.) with respect to
$\delta_{\alpha\beta}$	Kronecker delta in four-dim. space
δ_{KL} ; δ_{kl}	Kronecker delta in $\mathcal{B}_R$ and $\mathcal{B}_t$
δ_{kK}	Shifter
δ ; $\bar{\delta}$; $\underset{\sim}{\delta}$; δ_1; δ_2	Dimensionless number
E_{KL} ; e_{kl}	Lagrangian and Eulearian strain tensors
$\tilde{e}_{KL}$; $\tilde{\underset{\sim}{E}}_{KL}$	Infinitesimal strain tensor
$\underset{\sim}{E}(\underset{\sim}{x},t)$ $(=\bar{\underset{\sim}{E}}+\underset{\sim}{e})$	Electric field
$\underset{\sim}{E}'$	Effective value of $\underset{\sim}{E}$ measured on a moving frame
E_3	3-dim. Euclidean space
$\underset{\sim}{\mathcal{E}}'$	Effective perturbed electric field
${}^{o}\underset{\sim}{E}$	Stationary or uniform electric field
$\underset{\sim}{e}$; $\tilde{\underset{\sim}{e}}$	Perturbed electric field
	Amplitude of harmonic electric field
$\underset{\sim}{\mathcal{E}}^{*}$ $(=\bar{\underset{\sim}{\mathcal{E}}}+\underset{\sim}{\mathcal{E}}')$	Chu electric field in the rest frame of reference
ε	Internal energy per unit mass
$\epsilon_{\alpha\beta\gamma\delta}$	Alternating tensor in four-dim. space
ϵ_{KLM} ; ϵ_{klm}	Alternating tensor in $\mathcal{B}_R$ and $\mathcal{B}_t$
ϵ_0	Permitivity of vacuum
exp.	Exponential
$\eta\,(=\bar{\eta}+\eta')$	Entropy per unit mass
${}^{o}\eta$	Stationary or uniform entropy
$\theta(\underset{\sim}{x},t)$ $(=\bar{\theta}+\theta')$	Absolute temperature change
θ_0	Initial temperature change
θ'; $\tilde{\theta}$	Perturbed temperature change
F_{kK} ; $\underset{\sim}{F}$	Deformation gradient tensor.
$\underset{\sim}{f}$	Total body force per unit mass
$\underset{\sim}{f}^{(L)}$	Lorentz force
$\underset{\sim}{f}^{(M)}$; $\underset{\sim}{f}^{(em)}$	Mechanical and electromagnetic body forces

$\underset{\sim}{f}^{(p)}$; $\underset{\sim}{f}^{(m)}$	Lorentz force due to polarization and magnetization
f_α	A four-force
f_j	Space part of the e.m. force in four-dim. formu.
G_{KL} ; g_{kl}	Metric tensor in $\mathcal{B}_R$ and $\mathcal{B}_t$
G_K	Material measure of temperature gradient
$\underset{\sim}{G}^{(\cdot)}$; $\underset{\approx}{G}^{(\cdot)}$	Material property tensor
G	Centre of mass of rigid body
g_k	Temperature gradient
$\underset{\sim}{g}$	Electromagnetic momentum
γ ; $\bar{\gamma}$	Parameter in Lorentz group of transformations
grad	Gradient in 3-dim. space
γ_{ij} ; $\hat{\gamma}^*_{ji}$	Magneto-electric material property tensor
$\{\gamma\}$	Set of independent electromagnetic variables
$\Gamma(t)$; $\gamma(t)$	Line of discontinuity in $\mathcal{B}_R$ and $\mathcal{B}_t$
$\underset{\sim}{H}(\underset{\sim}{x},t)$ $(= \underset{\sim}{\bar{H}} + \underset{\sim}{h})$	Magnetic field
$\underset{\sim}{H}'$	Effective value of $\underset{\sim}{H}$ measured on a moving frame
$\underset{\sim}{\mathcal{H}}$ $(= \underset{\sim}{\bar{\mathcal{H}}} + \underset{\sim}{\mathcal{H}}')$	Chu magnetic field in the rest frame of reference
$\underset{\sim}{{}_{\circ}H}$	Stationary or uniform magnetic field
$\underset{\sim}{\mathcal{H}}'$	Effective perturbed magnetic field
$\underset{\sim}{h}$; $\tilde{h}$	Perturbed magnetic field
$\underset{\sim}{h}^*$	Amplitude of harmonic magnetic field
$\underset{\sim}{I}$	Identity tensor in 3-dim. space
I_{KL}	Moment of inertia
$\underset{\sim}{I}_K$; $\underset{\sim}{i}_k$	Base vectors along the coordinate axes $\underset{\sim}{X}$ and $\underset{\sim}{x}$
i	Imaginary number ($\sqrt{-1}$)
$\mathcal{I}m\,(\cdot)$	Imaginary part of (.)
$\mathcal{J}^{(f)}_\alpha$	A free current four-vector
$\underset{\sim}{J}^{(f)}(\underset{\sim}{x},t)$ $(= \bar{J}^{(f)} + \underset{\sim}{j}^{(f)})$	Free current vector
$\underset{\sim}{J}$ $(= \bar{J} + \underset{\sim}{j})$	Conduction current vector
$\underset{\sim}{\mathcal{J}}^{(f)'}$	Effective value of $\underset{\sim}{J}^{(f)}$ measured on a moving frame
$\underset{\sim}{J}^{(m)}$; $\underset{\sim}{J}^{(p)}$	Magnetization and polarization current densities in the Chu formulation
$\underset{\sim}{\hat{J}}^{(m)}$; $\underset{\sim}{\hat{J}}^{(p)}$	Magnetization and polarization current densities in the Amperian formulation

$\underset{\sim}{\check{J}}^{(m)}$; $\underset{\sim}{\check{J}}^{(p)}$	Magnetization and polarization current densities in the Boffi formulation
J	Jacobian
$\underset{\sim}{\hat{K}}^{(\cdot)}$; $\underset{\sim}{\hat{\underline{K}}}^{(\cdot)}$	Material property tensor
$\underset{\sim}{K}$; $\underset{\sim}{k}$	Unit tangent vector in $\mathcal{B}_R$ and $\mathcal{B}_t$
$\underset{\sim}{\mathbb{k}}\ (=\underset{\sim}{\mathbb{k}}_r+\underset{\sim}{\mathbb{k}}_i)$	Wave vector
$\mathbb{k}^*$; $\mathbb{k}_0$	Dimensionless and reference wave number
$\underset{\sim}{K}^{(f)}$; $K^{(s)}$	Free surface current and charge densities
$\mathbb{K}_{\alpha\beta}$	Electromagnetic field tensor in four-dim. space
$\underset{\sim}{\hat{\xi}}^{(\cdot)}$	Material property tensor in nonlinear constitutive equations
$\hat{\xi}_{\alpha\beta\gamma\delta}$	Generalized susceptibility tensor in four-dim. space
ξ_{CH} ; $\bar{\xi}_{CH}$; $\underline{\xi}_{CH}$ ξ_1 ; ξ_2 ; $\cdots$; ξ_6	Dimensionless number
$\underset{\sim}{l}$	Distance between monopoles
$\underset{\sim}{\ell}\ (=\bar{\underset{\sim}{\ell}}+\underset{\sim}{\ell}')$	Total body couple in the Chu formulation
$\underset{\sim}{\ell}^{(m)}$; $\underset{\sim}{\ell}^{(p)}$	Body couple due to magnetization and polarization
Λ_{ij}	Entries of the matrix $\lVert\Lambda_{ij}\rVert$
$\lVert\Lambda_{ij}^{(1)}\rVert$; $\lVert\Lambda_{ij}^{(2)}\rVert$	Submatrix of $\lVert\Lambda_{kl}\rVert$
$\hat{\lambda}$; $\hat{\underline{\lambda}}$	Elastic and viscous Lamé constant
$\lVert\Lambda_{\alpha\beta}\rVert$; $\lVert\bar{\Lambda}_{\alpha\beta}\rVert$	Lorentz and Galilean transformation matrix
$\lambda\!\!\lambda$	Wavelenght
λ	Parameter
$\underset{\sim}{M}(\underset{\sim}{x},t)\ (=\bar{\underset{\sim}{M}}+\underset{\sim}{m})$	Magnetization per unit volume
$\underset{\sim}{\check{M}}$	Magnetization per unit mass
$\underset{\sim}{\tilde{M}}'$	Magnetization in the rest frame of reference
${}_0\underset{\sim}{M}$	Stationary or uniform magnetic field
$\underset{\sim}{m}$; $\underset{\sim}{\tilde{m}}$	Perturbed magnetization
$\underset{\sim}{\mathcal{M}}$	The Chu magnetization in the rest frame of reference
$\underset{\sim}{\mathfrak{M}}$	Material measure of magnetization per unit mass
$\mathfrak{M}$	Total mass
$\mathbb{M}_\alpha$	Magnetization four-vector
$\{\mathfrak{M}\}$; $\{\mathfrak{M}'\}$	Magnetic point group
$\hat{\mu}$; $\hat{\underline{\mu}}$	Elastic and viscous Lamé constant
μ_0	Permeability of vacumm

$\underset{\sim}{N}$; $\underset{\sim}{n}$	Normal unit vector in $\mathcal{B}_R$ and $\mathcal{B}_t$
n	Particle density
ν_c ; ν_H ; $\bar{\nu}_H$; $\underline{\nu}_H$	Dimensionless number
P, p	A material point in $\mathcal{B}_R$ and $\mathcal{B}_t$
$\underset{\sim}{P}(x,t)$ $(=\underset{\sim}{\bar{P}}+\underset{\sim}{p})$	Polarization per unit volume
$\underset{\sim}{\check{P}}$	Polarization per unit mass
$\underset{\sim}{P}'$	Polarization in the rest frame of reference
${}_{o}\underset{\sim}{P}$	Stationary or uniform polarization
$\underset{\sim}{\mathcal{P}}$	The Chu polarization in the rest frame of reference
$\underset{\sim}{p}$; $\underset{\sim}{\tilde{P}}$	Perturbed polarization
$\mathbb{P}_{\alpha\beta}$	Polarization tensor and vector in four-dim. space
$\underset{\sim}{\Pi}$	Material measure of polarization per unit mass
$\rho(x,t)$ $(=\bar{\rho}+\rho')$	Mass density at point x and time t
$\rho^{(m)}$; $\rho^{(p)}$	Magnetization and polarization charge densities in the Chu formulation
$\beta^{(m)}$; $\beta^{(p)}$	Magnetization and polarization charge densities in the Amperian formulation
$\check{\rho}^{(m)}$; $\check{\rho}^{(p)}$	Magnetization and polarization charge densities in the Boffi formulation
$\rho^{(f)}$ $(=\bar{\rho}^{(f)}+\rho'^{(f)})$	Free charge density
$\underset{\sim}{Q}(t)$	Matrix of a orthogonal transformation
$\underset{\sim}{q}(x,t)$ $(=\underset{\sim}{\bar{q}}+\underset{\sim}{q}')$	Energy flux vector at point x and t
$q^{(m)}$; $q^{(p)}$	Magnetic and polarization charges in the Chu formulation
$\underset{\sim}{\overset{+}{q}}$; $\underset{\sim}{\mathfrak{q}}$	True heat flux and extra energy flux vectors
R_{kK}	Finite rotation tensor
r_E ; r_θ	Energy supply per unit mass due to electromagnetic and other sources
r_η	Entropy supply per unit mass
$\mathcal{R}$	Time inversion operator mapping t to $-t$
$\mathbb{R}$	Dimensionless number
$Re(\cdot)$	Real part of (.)
$\underset{\sim}{S}$	Sum of the entropy flux through the moving surface $\partial V-\sigma$
$S(X_K,t)=0$; $s(x_k,t)=0$	Surface in $\mathcal{B}_R$ and $\mathcal{B}_t$
S , S'	Laboratory and stationary frames of reference
$\underset{\sim}{S}^i$	Element of conventional crystallographic point

	group in 3-dim. space
$S_{\mu\nu}$	Four-dim. energy-momentum tensor
$\{S\}$	Point group describing symmetry properties of a classical crystal class
$t_{kl}\ (=\bar{t}+t')$	Actual stress tensor
$t^{(T)}$	Total stress tensor
${}^{(D)}\underset{\sim}{t}\ ,\ {}^{(N)}\underset{\sim}{t}$	Dissipative and nondissipative part of
t	Time
$\mathfrak{t}$	Shifted time
$\underset{\sim}{\tau}\ (=\bar{\underset{\sim}{\tau}}+\underset{\sim}{\tau}')$	Maxwell's stress tensor in the Chu formulation
$\underset{\sim}{\tau}^{(m)}\ ;\ \underset{\sim}{\tau}^{0}$	Maxwell's stress tensor in material and vacuum
T_0	Initial temperature
$\bar{T}$	Temperature in the rigid body state
T	Final temperatere
$\mathfrak{T}_{\alpha\beta}$	Electromagnetic field tensor in four-dim. space
$\hat{\underset{\sim}{\Sigma}}^{(\cdot)}\ ;\ \hat{\underset{\approx}{\Sigma}}^{(\cdot)}\ ;\ \hat{\underset{\sim}{\Im}}^{(\cdot)}\ ;\ \hat{\underset{\approx}{\Im}}^{(\cdot)}$	Material property tensor
$\hat{\sigma}\ (\hat{\sigma}^{(p)})\ ;\ \hat{\sigma}^{(m)}$	Electrical and magnetical conductivity
$\Sigma(t)\ ;\ \sigma(t)$	Discontinuity surface in $\mathcal{B}_R$ and $\mathcal{B}_t$
$\underset{\sim}{U}$	Symmetric positive definite matrix
$\underset{\sim}{u}(\underset{\sim}{x},t)$	Displacement vector at point $\underset{\sim}{x}$ and time t
$\bar{\underset{\sim}{u}}$	Displacement vector of rigid body
$\hat{\underset{\sim}{u}}$	Total displacement
$\tilde{\underset{\sim}{u}}$	Displacement vector from the reference configuration to the present one
$\underset{\sim}{u}^*$	Amplidude of harmonic displacement
$\underset{\sim}{V}$	Symmetric positive definite matrix
$\underset{\sim}{V}\ ;\ \underset{\sim}{V}^{e}\ ;\ \underset{\sim}{V}^{g}$	Phase, energy and group velocity
V_0	Velocity of light in the material of electric susceptibility is zero
$V_p\ ;\ V_s$	Phase velocity of elastic P and S wave
$\tilde{V}\ ;\ \bar{V}\ ;\ \underset{\sim}{V}\ ;\ V_1^*\ ;\ V_2^*$	Dimensionless velocity
$v(\underset{\sim}{x},t)$	Velocity of a particle at point $\underset{\sim}{x}$ and time t
$\underset{\sim}{v}\ ;\ \bar{\underset{\sim}{v}}$	Velocity of frame of reference in Lorentz group of transformations
$\underset{\sim}{v}^*$	Velocity of discontinuity surface $\sigma(t)$ or $\gamma(t)$ line
$\mathcal{V}_0+\partial\mathcal{V}_0\ ;\ \mathcal{V}+\partial\mathcal{V}\ ;\ v+\partial v$	Region occupied by the body at time $t=0$, $t=t_0$

	and $\not{t} = t$
Φ ; $\boldsymbol{\Phi}$	Any scalar or vector field
$\Phi^{(p)} (= \bar{\Phi}^{(p)} + \varphi^{(p)})$	Electrical scalar potential
$\Phi^{(m)} (= \bar{\Phi}^{(m)} + \varphi^{(m)})$	Magnetical scalar potential
Ψ	Helmholtz free energy
Ψ^{*}	A column vector
φ	Angle between $\boldsymbol{k}$ and ${}_{o}\boldsymbol{H}$
${}_{o}\underset{\sim}{X}$; $\underset{\sim}{X}$, $\underset{\sim}{x}$	Coordinates of a point in initial, material and spatial description
$\bar{\underset{\sim}{x}}$	Another spatial frame of reference
$\bar{\underset{\sim}{X}}$	Coordinates of centre of mass
$\underset{\sim}{x}^{+}$; $\underset{\sim}{x}^{-}$	Coordinates of (+)ve and (-)ve charge of a dipole
x_{μ}	Coordinates of a point in four-dim. space
$x_{k,K}$; X_{Kk}	Deformation and inverse of deformation gradient
$\underset{\sim}{\hat{\chi}}^{(p)}$; $\underset{\sim}{\hat{\chi}}^{(m)}$	Electrical and magnetical susceptibility tensor
$\hat{\chi}$; $\hat{\bar{\chi}}$	Material property constant
$\underset{\sim}{Y}$	Coordinates of a particle in moving system of coordinate
$\underset{\sim}{\Omega}$	Angular velocity of a particle with respect to G
$W (= \bar{W} + W')$	Electromagnetic energy per unit volume
ω ; ω^{*} ; ω_{o}	Angular, dimensionless and reference frequency
ω_{kl}	Antisymmetric part of rate of deformation grad.
$\bar{\omega}$; $\underline{\omega}$	Dimensionless number
$\bar{\omega}_{\nu}$; $\underline{\omega}_{\nu}$	Elastic-viscous parameter
$\hat{\zeta}_{\alpha\beta\delta}$	Generalized conductivity tensor
d_{kl}	Symmetric part of rate of deformation gradient

CHAPTER 1

INTRODUCTION

The goal of this thesis is the formulation of nonrelativistic, macroscopic governing equations characterizing the dynamic response of viscoelestic anisotropic solids to the simultaneous actions of mechanical, electromagnetic and thermal effects.

Special effects produced by the interactions of electromagnetic and thermal fields on the deformable continua have had a revival of interest in the last two decades, so that the literature is quite extensive. However, properly invariant, macroscopic and microscopic nonlinear theories of electroelasticity (elastic dielectrics), magnetoelasticity, electromagnetothermoelasticity and electrodynamics of deformable continua are still evolving research topics.

The strict mathematical description of the nonlinear theory of elastic dielectrics given by Toupin [1] is applicable only to static cases. This theory makes use of the concept of a free energy which depends on polarization and deformation gradient, and by means of a variational principle, the form of constitutive relations is determined. Eringen's and Grindlay's works [2,3], which are concerned with the static theory of elastic dielectrics undergoing finite deformations. depend upon the principle of virtual work, and the constitutive relations are also obtained therein.

Later, the dynamic theory of elastic dielectrics presented by Toupin [4], which is based on the balance equations, is adequate to predict numerious experimental effects, such as piezoelectricity, photoelasticity and the Faraday effect. The dielectrics is assumed, however, to be a perfect insulator and no thermal effects are considered.

Parallel to Toupin's work [1], Tiersten [5] considered the macroscopic behavior of a magnetically saturated insulator undergoing large deformation. His equations are derived by means of systematic and consistent application of the laws of continuum physics to a well defined model motivated by microscopic consideration. He also verifies his formulation by a variational principle [6]. The phenomenological macroscopic theory of magnetoelasticity presented by Brown [7] is successful in accounting for many experimental results. In [6,7] magnetization gradient is included in addition to deformation gradient and magnetization as a constitutive variable since they are particularly interested in various ferrits under the saturated state.

Further, Midlin [8] and Şuhubi [9] present variational principles for the description of the nonlinear behavior of elastic dielectrics in static equilibrium when polarization gradient is a supplemantary constitutive variable. The inclusion of polarization gradient in the constitutive relations is needed to explain the surface energy phenomenon.

The static nonlinear theory of both polarizable and magnetizable thermo-elastic solids conducting both electricity and heat is presented by Jordan and Eringen [10]. An electrically polarizable, finitely deformable heat conducting continuum is considered, using different approach, by Tiersten [11] and Tiersten and Tsai [12]. Afterwards, Pao and Yeh [13] give a macroscopic, linear static theory of magnetoelastic interactions and Hutter and Pao [14] investigate a macroscopic dynamic theory of a magnetizable elastic solid with thermal and electrical conduction by means of the balance laws of classical continuum mechanics.

On the other hand, Grot and Eringen [15], Bragg [16], and Boulanger and Mayne [17,18] present relativistic approaches to the interacting continuum. The results in the first two works reduce, in the nonrelativistic approximation, to Toupin's work of elastic dielectrics [4]. It is not, however, clear as to whether Grot and Eringen's theory agrees relativistically with Bragg's and Boulanger and Mayne's results.

Penfield and Haus's monograph [19], which is a very compressive treatment of the several formulations of electromagnetism, deals with the Maxwell equations, body forces, body couples and the energy supplies of electromagnetic origin from both the relativistic and nonrelativistic points of view, leaving out the constitutive equations, therefore, the equations presented in [19] are not complete.

The anisotropic materials considered in, e.g., [1-19] are the classical crystals which are assumed to be centrosymmetric. However, there are theoretical and experimental evidences that the magnetic symmetry of the crystals must be taken into account to make clear certain physical phenomena, such as magneto-electricity, piezomagnetism and pyromagnetism [20-25], and non-centrosymmetric materials are also to be considered in dealing with piezoelectricity, pyroelectricity, etc. Therefore, the present theory, besides being dynamic, through its governing equations contributes the following to those of the existing theories: i) The material is <u>both</u> polarizable and magnetizable with <u>thermal</u> and <u>electrical</u> conduction, ii) the material possesses magnetic symmetry instead of classical symmetry, and iii) the material is mechanically dissipative (Kelvin-Voigt type viscoelastic). Polarization and magnetization gradients and their time rates are not taken to be constitutive variables since the exchange and electromagnetic hysteretic effects are assumed to be negligible.

As for the applications given in the relevant research works, the theories have been applied to various situations of electroelasticity, magnetoelasticity and magneto-thermoelasticity. For example, the resulting equations of isotropic dielectrics in [2] are applied to the uniform extension of an internally charged circular cylinder; the linearized equations of Toupin [4] are used to investigate plane wave problems in elastic dielectrics. The special cases of the decomposed equations of Pao and Yeh's [13], and Hutter and Pao's works [14] are, respectively, employed in the buckling of an elastic plate in a uniform magnetic field and the propagation of magneto-mechanical waves in soft ferromagnetic isotropic materials [26].

There is a considerable number of research works studying, in particular, the propagation of electromagnetic, magneto-mechanical and magneto-thermomechanical waves through rigid or deformable isotropic or anisotropic materials [26-40,54,55]. To mention a few, Mckenzie [27] and Potekin [28] present the propagation of electromagnetic waves in moving isotropic and stationary anisotropic rigid materials, respectively. Birss and Shrubsall [29] and Fuch [30] deal with the propagation of electromagnetic plane waves in magneto-electric crystals, and Smith and Rivlin [31], and Ersoy and Kıral [32] discuss the propagation of electromagnetic waves in deformable anisotropic materials due to deformation.

In these investigations there are no external electric or magnetic field affecting the propagation of electromagnetic and mechanical waves. In this view, Knopoff [33] studies the effect of the earth magnetic field on elastic waves in the conducting core of the earth. The effect of the angle between the uniform magnetic field and the direction of propagation on the coupled plane waves is investigated by Chadwick [34]. In view of the Minkowski formulation of Maxwell's equations, Dunkin and Eringen [35] discuss the coupling of electromagnetic and elastic waves in a moving medium. Kaliski [36], Paria [37] and Parkus [38], while initiating the study of magneto-thermoelastic plane waves, investigate the interactions between electromagnetic and thermoelastic fields. Later, Wilson [39] and Parushothama [40] reinvestigate the problem of plane waves in the presence of uniform thermal and magnetic fields in different orientations. More recently, Tokuoka and Kobayashi [54], and Saito and Tokuoka [55] present the propagation of mechanico-electromagnetic waves in isotropic and anisotropic elastic dielectric crystals in a uniform magnetic field, respectively.

None of these previous works contains the propagation of the electro-magnetomechanical waves in mechanically dispersive media under an external primary magnetic or electric field. Besides, numerical results are not presented with the exception of Hutter's work [26], where the phase velocities and the attenuations of the magneto-elastic waves through electrically conductive unbounded solids are given. Together with electrical conductivity, internal friction also affects the propagation of waves in a rather significant manner. Dispersion due to electrical conductivity and viscosity of the solids is an important phenomenon because it governs the change of the shape of a pulse as it propagates through a medium.

In the last part of this dissertation, the propagation of magneto-mechanical waves in electrically conductive magnetizable viscoelastic solids is investigated both analytically and numerically.

Our treatment of electrodynamics and the interactions of electromagnetic and thermal fields with deformable continua is nonrelativistic. However, the relativistic formulation of electromagnetism is given in Appendix A. First, electrodynamics is treated in its general form (Lorentz invariant), but a later stage, all the terms compared with the linear terms and containing $(v/c)^2$ are neglected. The balance laws of classical continuum mechanics are Galilean invariant and those of electrodynamics are Lorentz invariant. Since the first two terms are retained in the transformations of the electromagnetic field variables under the Lorentz group, the formulation is regarded as a linear approximation of the relativistic theory of electromagnetism. Consequently, our combined set of equations is neither Lorentz nor Galilean invariant.

This thesis consists of three parts. The first part (Chapters 2-5) is a fairly general investigation of the governing equations of the interacting continua having magnetic symmetry. The second part (Chapters 6-7) is devoted to linearization of the derived equations and the discussion of special cases. In the third part (Chapter 8), the propagation of magneto-mechanical waves in an unbounded dispersive medium is discussed both analytically and numerically.

More specifically, the basic concepts and the general balance equations of continuum physics in global and local forms are summarized in Chapter 2.

The subject of the electrodynamics of moving media has always been a contraversial one [19,41-43]. Apart from the Minkowski formulation of Maxwell's equations, there exists a variety of other forms which are motivated by some particular models for polarization and magnetization. In Chapter 3, several versions of Maxwell's equations for moving media with the associated boundary conditions have been obtained from one set of the global Maxwell's equations without introducing any model. This procedure to set up several formulations of Maxwell's equations is different from those given in [19,43]. The electromagnetic body forces, body couples and the Maxwell's stress tensor are introduced in terms of the Chu variables of electrodynamics. The Chu formulation is adopted since a polarizable material, in a nonrelativistic motion, is distinguished from a magnetizable one. This is not the case in the Minkowski formulation.

In Chapter 4, the basic equations of the interacting continua are established from the balance laws of continuum physics. The pertinent principles are the conservation of mass, charge and energy, the balance of momentum and moment of momentum, the Maxwell equations and the entropy inequality. In the balance of moment of momentum, the body couples are taken into account although the surface couples are assumed to be absent. As a consequence, the actual stress tensor is asymmetric. These balance equations are inadequate in number to determine the unknowns.

Chapter 5 is devoted to the nonlinear constitutive theory of <u>both</u> polarizable and magnetizable magneto-electro thermo-viscoelastic anisotropic solids possessing <u>magnetic</u> symmetry with <u>thermal</u> and <u>electrical</u> conduction. It has been shown that there exists a thermodynamic potential from which one can determine the nondissipative part of the stress tensor, the electric and magnetic fields and the entropy by means of differential operations. The dissipative part of the stess tensor, the heat flux and conduction current vectors are not derivable from a potential, but they are restricted by the inequality. In deriving the constitutive equations explicitly, it is assumed that the material is linear.

The governing equations for the considered linear interaction phenomena are still nonlinear and complicated. To illustrate the physical implications of the theory there may be two alternatives: One is to try a numerical technique of solutions of the nonlinear partial differential equations and the other is to linearize the equations on the basis of a sequence of consistent approximation. The latter is followed in Chapter 6. Following mainly the linearization process of Hutter and Pao [14], all the governing equations are decomposed into two groups: The first group is associated with the rigid body motions of both polarizable and magnetizable material within the electromagnetic and thermal fields and the second group encompasses the perturbed quantities from the rigid body state due to infinitesimal deformations. Thus, the motion of the body is viewed as the superposition of the infinitesimal deformation on the global rigid body motions. This decomposition is different from that employed by Toupin [4] and Tiersten [5] where the infinitesimal deformation is superimposed on the finite static deformation.

Chapter 7 deals with special cases, such as the materials constrained from rigid body motions, the materials which are thermally and electrically nonconductive, etc. An alternative linearization process is also proposed.

The resulting equations of Chapter 7 can be solved for certain problems. The simplest one is the propagation of electro-magneto thermo-mechanical waves in an unbounded medium. The establishment of relations between the physical properties of the material and the acoustic wave propagation has numerious applications in engineering, for example: establishing the degree of fatigue of a material, testing the crystal structure of metals, and especially, alloys, detecting trace impurities in ultra-pure materials and producting materials with exactly determined mechanical or electro-magneto mechanical parameters.

Chapter 8 deals with the propagation of magneto-mechanical waves through magnetizable, viscoelastic isotropic solids with electrical conduction in a primary, uniform magnetic field in an arbitrary direction. The influences of the primary magnetic field on the phase velocities and the attenuations are discussed both analytically and numerically. All the modes of the propagating waves are dispersive and dissipative due to the electrical conductivity and the viscosity of the medium. The coupled modes of magneto-mechanical waves depend upon the direction of the primary magnetic field. The properties of all sorts of waves for the variation of frequency, applied magnetic field and the magneto-mechanical parameters have been studied such that the phase velocities and the attenuations are plotted .

Chapter 9 is devoted to the conclusions of the thesis.

In the dissertation all the symbols are defined where first used, and the considered physical quantities are to be measured in SI units. Many of the other symbols may be quickly identified from the list in the nomenclature. Similarly, the figures may also be recognized from the list of figures.

A computer program written in FORTRAN IV language (IBM 370/145) with double precision complex algebra is developed for the determination of the roots of the secular equation.

PART: I

GENERAL FORMULATION

CHAPTER 2

DEFORMATION AND MOTION

This chapter is a review of the basic concepts of classical continuum mechanics, and the general balance laws of continuous media. The review is not concerned with the theory of polar and nonlocal continua. For such media, see Eringen [44,45].

2.1. Coordinates, Motion, Deformation and Strain Measures

Within the scope of continuum physics bodies are considered to be composed of particles with mass and charge having translational and rotational motions and stretch. Thus all the physical phenomena are regarded as the result of the motion of these particles under a variety of external effects.

We consider a body $\mathcal{B}$ which is a smooth manifold of material particles denoted by P. At time $t=0$, the body $\mathcal{B}$ occupies the region $\mathcal{V}_0+\partial\mathcal{V}_0$ in the Euclidean space E_3, where $\mathcal{V}_0$ is the volume of this configuration and $\partial\mathcal{V}_0$ is its boundary. This configuration denoted as $\mathcal{B}_0$ is called the "initial configuration". At time t_0 (t_0 may be equal to zero), the body occupies the region $\mathcal{V}+\partial\mathcal{V}$ in the same space and this configuration abbreviated as $\mathcal{B}_R$ is called the "reference configuration".

The coordinates of the material point P in $\mathcal{B}_R$ are X_K ($K=1,2,3$) and are called "material" (or Lagrangian) coordinates, Fig.(2.1). A particle p occupies the spatial place $\mathbf{x}$ at time $t \geq t_0$, and the coordinates x_k ($k = 1,2,3$) are called the "spatial" (or Eulerian) coordinates. The configuration $\mathcal{B}_t$ is called the "present configuration".

The motion of the body is characterized by the time evolution of the position of every material point. Mathematically, the motion of the body is the continuous mapping

$$\underset{\sim}{x} = \hat{\underset{\sim}{x}}(\underset{\sim}{X}, t) \quad ; \quad x_k = \hat{x}_k(X_K, t) \quad . \tag{2.1}$$

With (2.1), the region $\mathcal{V} + \partial\mathcal{V}$ is mapped into the region $v + \partial v$.

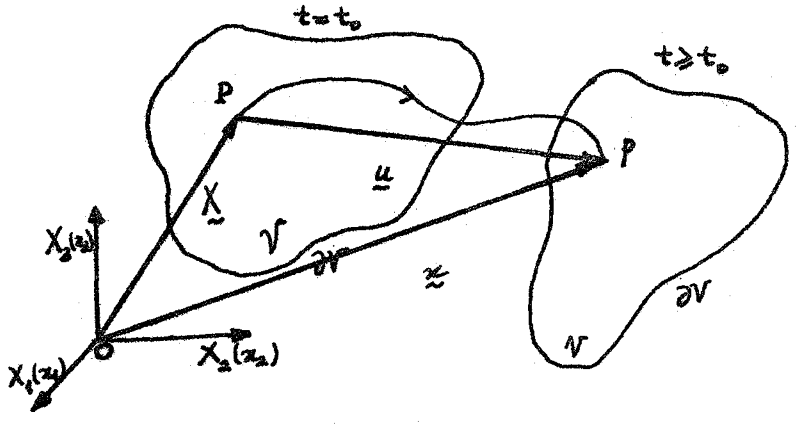


Fig. 2.1 Motion of a Material Point.

Impenetrability of the matter requires that (2.1) have a non-vanishing Jacobian, except perhaps at some singular points, lines or surfaces. Moreover, the unique inverse of (2.1) exists in the neighborhood of the spatial point $\underset{\sim}{x}$:

$$X_K = \hat{X}_K(\underset{\sim}{x}, t) \tag{2.2}$$

at time t . The quantities referred to the spatial coordinates will be denoted by small Latin kernel letters and their components by small Latin indices. Since Cartesian coordinate systems are used in this study, there is no difference between the covariant, contravariant, and mixed components of tensors.

A differential vector element $d\underset{\sim}{X}$ at P and $d\underset{\sim}{x}$ at p are expressed as

$$d\underset{\sim}{X} = \frac{\partial \underset{\sim}{X}}{\partial X_K} dX_K = \underset{\sim}{I}_K \, dX_K \tag{2.3}$$

and

$$d\underset{\sim}{x} = \frac{\partial \hat{\underset{\sim}{x}}}{\partial x_k} dx_k = \underset{\sim}{i}_k \, dx_k \tag{2.4}$$

where $\underset{\sim}{I}_K$ and $\underset{\sim}{i}_k$ are unit base vectors along the coordinate axes X_K and x_k respectively. Upon the repeated indices the summation is understood.

$$G_{KL}(\underset{\sim}{X}) \equiv \underset{\sim}{I}_K \cdot \underset{\sim}{I}_L \qquad \text{and} \qquad g_{kl}(\underset{\sim}{x}) \equiv \underset{\sim}{i}_k \cdot \underset{\sim}{i}_l \tag{2.5}$$

are in general "metric tensors" in the material and spatial frames of reference respectively. Since a common Cartesian coordinate system is used as the reference frame, the metric tensors are simply equal to the Kroneker deltas, i.e., $G_{KL} = \delta_{KL}$, $g_{kl} = \delta_{kl}$. The symbol $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$. Also, $i_k \cdot I_K = \delta_{kK}$ is known as the "shifter" and used to shift the components of a vector from a material frame of reference to the spatial frame of reference or vice versa.

In continuum mechanics, deformation gradients play a central role. These are defined by

$$x_{k,K} \equiv \frac{\partial x_k}{\partial X_K} \quad ; \quad X_{K,k} \equiv \frac{\partial X_K}{\partial x_k} \tag{2.6}$$

or sometimes dyatic notation $\underset{\sim}{F}$ and $\underset{\sim}{F}^{-1}$ is used for the tensors corresponding to $(2.6)_1$ and $(2.6)_2$, respectively, e.g.

$$\underset{\sim}{F} \equiv \nabla \underset{\sim}{x} \qquad \text{or} \qquad F_{kK} \equiv \frac{\partial x_k}{\partial X_K} \tag{2.7}$$

where a comma denotes partial differentiation. The usual vector operators of differentiation with respect to the spatial coordinates are expressed as $(.)_{,k}$, $(.)_{k,k}$ and $\epsilon_{ijk}(.)_{k,j}$ and the corresponding operators with respect to the material coordinates are $(.)_{,K}$, $(.)_{K,K}$ and $\epsilon_{IJK}(.)_{K,J}$ respectively. ϵ_{ijk} and ϵ_{IJK} are the alternating tensors in the spatial and material configurations respectively. ϵ_{ijk} (or ϵ_{IJK}) is zero if any two indices are the same, +1 if the indices are an even permutation of 123 and -1 if the indices are an odd permutation of 123.

The deformation gradients (2.6) satisfy the nine linear equations

$$x_{k,K} X_{K,l} = \delta_{kl} \quad ; \quad x_{k,L} X_{K,k} = \delta_{KL} \quad . \tag{2.8}$$

The solution of one of (2.8) gives one set of deformation gradient (2.6) interms of the other, e.g.

$$X_{K,k} = \frac{1}{2} J \epsilon_{KLM} \epsilon_{klm} x_{l,L} x_{m,M} \tag{2.9}$$

where

$$J \equiv \frac{1}{3!} \epsilon_{KLM} \epsilon_{klm} x_{k,K} x_{l,L} x_{m,M} \equiv \det x_{k,K} \tag{2.10}$$

and $\mathcal{J}$ being Jacobian. $\mathcal{J}$ may be assumed to be positive without loss of generality.

The squares of the arc length, $(ds)^2$ in $\mathcal{B}_t$ and $(dS)^2$ in $\mathcal{B}_R$, are given by

$$(ds)^2 = \delta_{kl}\, dx_k\, dx_l \qquad ; \qquad (dS)^2 = \delta_{KL}\, dX_K\, dX_L \quad . \tag{2.11}$$

Whenever $(ds)^2 = (dS)^2$ for all material points, then the body undergoes a "rigid body" motion.

"Green" and "Cauchy" deformation tensors are, respectively,

$$C_{KL}(\underset{\sim}{X},t) = \delta_{kl}\, x_{k,K}\, x_{l,L}$$

and (2.12)

$$c_{kl}(\underset{\sim}{x},t) = \delta_{KL}\, X_{K,k}\, X_{L,l}$$

and they can be used as measures for the local deformation in a neighborhood of points P and p. Both of these quantities are symmetric, and positive definite.

E_{KL} and e_{kl} are, respectively, the "Lagrangian" and "Eulerian" strain tensors defined by

$$E_{KL} = E_{LK} \equiv \tfrac{1}{2}\left(C_{KL}(\underset{\sim}{X},t) - \delta_{KL} \right)$$

and (2.13)

$$e_{kl} = e_{lk} \equiv \tfrac{1}{2}\left(\delta_{kl} - c_{kl}(\underset{\sim}{x},t) \right) \quad .$$

Therefore E_{KL} and e_{kl} may also be used as a measure of local deformation. If they vanish, the deformation is locally rigid.

The displacement vector $\underset{\sim}{u}$ is defined as a vector that extends from a material point P in $\mathcal{B}_R$ to the same material point p in the deformed body $\mathcal{B}_t$, Fig.2.1 . The deformation tensors in terms of the displacement vector $\underset{\sim}{u}$ are

$$C_{KL} = \delta_{KL} + 2E_{KL} = \delta_{KL} + u_{K,L} + u_{L,K} + u_{M,K}\, u_{M,L}$$

$$c_{kl} = \delta_{kl} - 2e_{kl} = \delta_{kl} - u_{k,l} - u_{l,k} + u_{m,k}\, u_{m,l} \quad . \tag{2.14}$$

The displacement gradients and the deformation tensors are relater by

$$x_{k,K} = R_{kL}\, C_{LK}^{1/2} = R_{lk}\, c_{kl}^{-1/2}$$

$$X_{K,k} = R_{Kl}^{-1}\, c_{lk}^{1/2} = R_{Lk}^{-1}\, C_{KL}^{-1/2} \tag{2.15}$$

where $\underset{\sim}{R}$ is a finite rotation tensor, and $\underset{\sim}{R}^{-1}$ represents the inverse rotation tensor. In matrix notation, any invertible linear transformation $\underset{\sim}{F}$ has two multiplicative decompositions

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{\mathbb{U}} = \underset{\sim}{\mathbb{V}}\,\underset{\sim}{R} \tag{2.16}$$

where $\underset{\sim}{R}$ is orthogonal, i.e., $\underset{\sim}{R}\,\underset{\sim}{R}^t = \underset{\sim}{R}^t\underset{\sim}{R} = \underset{\sim}{I}$ and $\underset{\sim}{\mathbb{U}}$ and $\underset{\sim}{\mathbb{V}}$ are symmetric and positive definite matrices. In this expression and the following the superscript t denotes the transpose of the quantity. The following relations hold

$$\underset{\sim}{\mathbb{U}}^2 = \underset{\sim}{F}^t\underset{\sim}{F} \quad ; \quad \underset{\sim}{\mathbb{V}}^2 = \underset{\sim}{F}\,\underset{\sim}{F}^t \tag{2.17}$$

$$\underset{\sim}{\mathbb{V}} = \underset{\sim}{R}\,\underset{\sim}{U}\,\underset{\sim}{R}^t \quad ; \quad \underset{\sim}{\mathbb{V}}^2 = \underset{\sim}{R}\,\underset{\sim}{U}^2\,\underset{\sim}{R}^t \quad . \tag{2.18}$$

When we identify $\underset{\sim}{F}$ by $x_{k,K}$, then $\underset{\sim}{R}$ is the rotation tensor R_{kK} and $\underset{\sim}{\mathbb{U}}$ and $\underset{\sim}{\mathbb{V}}$ are identified as $\mathbb{U}_{KL} = C_{KL}^{-1/2}$ and $\mathbb{V}_{kl} = c_{kl}^{-1/2}$, and they are sometimes called "right" and "left" stretch tensors respectively.

The elements of area can be represented by an axial vector. The element of area dA_K in B_e and da_k in B_t is related by

$$da_k = J\,X_{K,k}\,dA_K \quad . \tag{2.19}$$

Similarly, the change of volume is calculated from

$$dv = J\,dV \quad . \tag{2.20}$$

2.2. Kinematics, Time Rates of Tensors

Material time rate of a vector $\underset{\sim}{\Phi}(x,t)$ (or tensor) is

$$\begin{aligned} \dot{\Phi}_k = \frac{d\Phi_k(x,t)}{dt} &= \frac{\partial \Phi_k}{\partial t} + \Phi_{k,l}\,\frac{\partial x_l}{\partial t} \\ &= \dot{\Phi}_k = \frac{\partial \Phi_K}{\partial t} \end{aligned} \tag{2.21}$$

and is called the "material derivative" of Φ_k. The dot over a symbol denotes the material time derivative, whether the description is material or spatial, because of

$$\underset{\sim}{\Phi}[x(\underset{\sim}{X},t),t] = \underset{\sim}{\Phi}(\underset{\sim}{X},t) \quad . \tag{2.22}$$

The first term at the right hand side of (2.21) is called the "local" or "nonstationary rate" and the second term is the "convective time rate".

The velocity is the time rate of the position vector and given by

$$\underset{\sim}{v} \equiv \frac{\partial x_k(\underset{\sim}{X},t)}{\partial t} \underset{\sim}{i}_k \qquad (2.23)$$

The acceleration vector $\underset{\sim}{a}$ is the material time rate of the velocity vector

$$\underset{\sim}{a}(x,t) \equiv \frac{dv_k}{dt} \underset{\sim}{i}_k \quad \left(a_k \equiv \frac{\partial v_k}{\partial t} + v_{k,\ell} v_\ell \right) \qquad (2.24)$$

The material derivative of the displacement gradients and the differential element are, respectively,

$$\frac{d}{dt}(x_{k,K}) = v_{k,\ell}\, x_{\ell,K} \;\; ; \;\; \frac{d}{dt}(X_{K,k}) = -v_{\ell,k}\, X_{K,\ell} \qquad (2.25)$$

and

$$\frac{d}{dt}(dx_k) = v_{k,\ell}\, dx_\ell \qquad (2.26)$$

To describe the local motion of a deformable body the gradient of the velocity field $\operatorname{grad} \underset{\sim}{v}$ is introduced. The velocity gradient can further be separated into a symmetric part $\underset{\sim}{d}$ and a skew symmetric part $\underset{\sim}{\omega}$ as

$$\operatorname{grad} \underset{\sim}{v} = \underset{\sim}{d} + \underset{\sim}{\omega} \qquad (2.27)$$

where

$$\begin{aligned} d_{k\ell} &\equiv \tfrac{1}{2}(v_{k,\ell} + v_{\ell,k}) \equiv v_{(k,\ell)} \\ \omega_{k\ell} &\equiv \tfrac{1}{2}(v_{k,\ell} - v_{\ell,k}) \equiv v_{[k,\ell]} \end{aligned} \qquad (2.28)$$

Parantheses are used to denote the symmetric part of the indexed quantities and brackets are used to denote the anti-symmetric part.

The rates of the Lagrangian and the Eulerian strains are, respectively, given by

$$\dot{E}_{KL} = \tfrac{1}{2}\dot{C}_{KL} = d_{k\ell}\, x_{k,K}\, x_{\ell,L} \qquad (2.29)$$

and

$$\dot{e}_{k\ell} = -\tfrac{1}{2}\dot{c}_{k\ell} = d_{k\ell} - e_{mk} v_{m,\ell} - e_{m\ell} v_{m,k}$$

2.3. Balance Laws of Continuum Physics

The material derivative of any field Φ over a material volume $\mathcal{V}$ enclosed by a surface $\partial\mathcal{V}$ is given by

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_{\mathcal{V}} \rho\phi\, dv = \int_{\mathcal{V}} \frac{\partial(\rho\phi)}{\partial t}\, dv + \int_{\partial\mathcal{V}} \rho\phi\, \underset{\sim}{v}\cdot d\underset{\sim}{a} \quad . \qquad (2.30)$$

In continuum physics discontinuities sweeping a material manifold are of common occurrence, for example, shock and acceleration waves and electromagnetic fields [46]. The case of a material volume $\mathcal{V}$ enclosed by a surface $\partial\mathcal{V}$ and intersected by a discontinuity surface $\sigma(t)$ moving with velocity $\underset{\sim}{v}^*$ is expressed in Fig. 2.2.

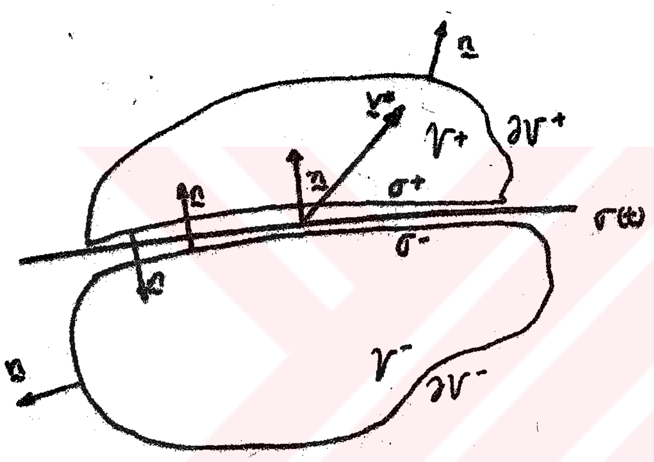


Fig.2.2 Discontinuity Surface.

Applying (2.30) to the two volumes $\mathcal{V}^+$ and $\mathcal{V}^-$ bounded by $\partial\mathcal{V}^+ + \sigma^+$ and $\partial\mathcal{V}^- + \sigma^-$ respectively, and adding the resulting two equations while σ^+ and σ^- approach $\sigma(t)$, one obtains

$$\frac{d}{dt}\int_{\mathcal{V}-\sigma} \rho\phi\, dv = \int_{\mathcal{V}-\sigma} \frac{\partial}{\partial t}(\rho\phi)\, dv + \int_{\partial\mathcal{V}-\sigma} \rho\phi\, \underset{\sim}{v}\cdot\underset{\sim}{n}\, da - \int_{\sigma} [\![\rho\phi\, \underset{\sim}{v}^*]\!]\cdot\underset{\sim}{n}\, da \quad . \qquad (2.31)$$

In Eq.(2.31) $\underset{\sim}{n}$ is a unit normal vector to the surface and a bold face bracket indicates the jump of the enclosed quantity across the discontinuity surface $\sigma(t)$, i.e.

$$[\![(\cdot)]\!] \equiv (\cdot)^+ - (\cdot)^- \qquad (2.32)$$

Here $(.)^+$ and $(.)^-$ are the values of $(.)$ from the positive and negative sides of $\underset{\sim}{n}$ of $\sigma(t)$.

By means of the Green-Gauss theorem , i.e.

$$\oint_{\partial V-\sigma} \Phi \cdot n \, da = \int_{V-\sigma} \operatorname{div} \Phi \, dv + \int_{\sigma(t)} [\![\Phi]\!] \cdot n \, da \tag{2.33}$$

the second integral on the right hand side of (2.31) may be converted to a volume integral. Thus

$$\begin{aligned}\frac{d}{dt}\int_{V-\sigma} \rho\phi \, dv = & \int_{V-\sigma} [\frac{\partial}{\partial t}(\rho\phi) + \operatorname{div}(\rho\phi v)] \, dv \\ & + \int_{\sigma(t)} [\![\rho\phi (v - v^*)]\!] \cdot n \, da \quad .\end{aligned} \tag{2.34}$$

A similar argument can be extended to material surface $\mathcal{S}$ enclosed by the line $\partial\mathcal{S}$ and intersected by a discontinuity line $\gamma(t)$ which is moving with velocity v^* Fig.2.3.

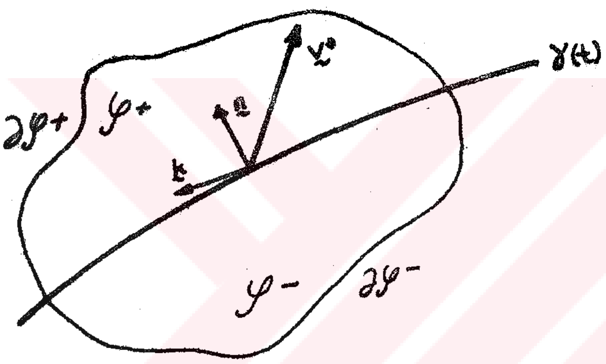


Fig.2.3 Discontinuity Line

In Eq.(2.30) if Φ is identified as a vector q and dv as a area element da , then it follows that

$$\frac{d}{dt}\int_{\mathcal{S}(t)} q \cdot da = \int_{\mathcal{S}(t)} \left(\frac{\partial q}{\partial t} + v \operatorname{div} q \right) . da + \int_{\partial\mathcal{S}(t)} (q \times v) . k \, ds \tag{2.35}$$

to the surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$, adding the resulting two equations while γ^+ and γ^- approach $\gamma(t)$ and using Stokes' theorem, i.e.

$$\oint_{\partial\mathcal{S}-\gamma} \Phi . k \, ds = \int_{\mathcal{S}-\gamma} (\operatorname{curl} \Phi) . n \, da + \int_{\gamma(t)} [\![\Phi]\!] \cdot k \, ds \tag{2.36}$$

in the second integral on the right hand side of (2.35), one obtains

$$\frac{d}{dt}\int_{\mathscr{S}-\gamma} \underset{\sim}{q}\cdot\underset{\sim}{n}\, da = \int_{\mathscr{S}-\gamma}\left[\frac{\partial \underset{\sim}{q}}{\partial t} + \operatorname{curl}(\underset{\sim}{q}\times\underset{\sim}{v}) + \underset{\sim}{v}\operatorname{div}\underset{\sim}{q}\right]\cdot\underset{\sim}{n}\, da + \int_{\gamma(t)} [\![\underset{\sim}{q}\times(\underset{\sim}{v}-\underset{\sim}{v}^*)]\!]\cdot\underset{\sim}{k}\, ds \quad . \tag{2.37}$$

Now, the balance laws of continuum physics are expressed in the following forms:

$$\frac{d}{dt}\int_{\mathscr{S}-\gamma} \underset{\sim}{q}\cdot\underset{\sim}{n}\, da = \oint_{\partial\mathscr{S}-\gamma} \underset{\sim}{h}\cdot\underset{\sim}{k}\, ds + \int_{\mathscr{S}-\gamma} \underset{\sim}{f}\cdot\underset{\sim}{n}\, da \tag{2.38}$$

$$\frac{d}{dt}\int_{\mathscr{V}-\sigma} \varphi\, dv = \oint_{\partial\mathscr{V}-\sigma} \hat{\underset{\sim}{\tau}}\cdot\underset{\sim}{n}\, da + \int_{\mathscr{V}-\sigma} \hat{g}\, dv \tag{2.39}$$

where $\hat{\underset{\sim}{q}}$, $\underset{\sim}{h}$ and $\underset{\sim}{f}$ are vector fields and φ, $\hat{\underset{\sim}{\tau}}$ and $\hat{g}$ are tensor fields.

By means of (2.37) and (2.31), one can convert (2.38) and (2.39) into

$$\int_{\mathscr{S}-\gamma}\left[\frac{\partial \underset{\sim}{q}}{\partial t} + \operatorname{curl}(\underset{\sim}{q}\times\underset{\sim}{v}) + \underset{\sim}{v}\operatorname{div}\underset{\sim}{q} - \operatorname{curl}\underset{\sim}{h} - \underset{\sim}{f}\right]\cdot\underset{\sim}{n}\, da + \int_{\gamma(t)} [\![\underset{\sim}{q}\times(\underset{\sim}{v}-\underset{\sim}{v}^*) - \underset{\sim}{h}]\!]\cdot\underset{\sim}{k}\, ds = 0 \tag{2.40}$$

and
$$\int_{\mathscr{V}-\sigma}\left[\frac{\partial \varphi}{\partial t} + \operatorname{div}(\varphi\underset{\sim}{v}) - \operatorname{div}\hat{\underset{\sim}{\tau}} - \hat{g}\right] dv + \int_{\sigma(t)} [\![\varphi(\underset{\sim}{v}-\underset{\sim}{v}^*) - \hat{\underset{\sim}{\tau}}]\!]\cdot\underset{\sim}{n}\, da = 0 \quad . \tag{2.41}$$

2.4. Master Laws for Local Balance

Assuming Eq.(2.40) to be valid for every area and line elements, one obtains

$$\frac{\partial \underset{\sim}{q}}{\partial t} + \operatorname{curl}(\underset{\sim}{q}\times\underset{\sim}{v}) + \underset{\sim}{v}\operatorname{div}\underset{\sim}{q} - \operatorname{curl}\underset{\sim}{h} - \underset{\sim}{f} = 0 \qquad \text{in } \mathscr{S}-\gamma \tag{2.42}$$

and

$$[\![\underset{\sim}{q}\times(\underset{\sim}{v}-\underset{\sim}{v}^*) - \underset{\sim}{h}]\!]\cdot\underset{\sim}{k} = 0 \qquad \text{on } \gamma(t) \tag{2.43}$$

Similarly, if Eq.(2.41) is assumed to be valid for every volume and surface elements, one obtains

$$\frac{\partial \varphi}{\partial t} + \operatorname{div}(\varphi\underset{\sim}{v}) - \operatorname{div}\hat{\underset{\sim}{\tau}} - \hat{g} = 0 \qquad \text{in } \mathscr{V}-\sigma \tag{2.44}$$

and
$$[\![\varphi(\underset{\sim}{v}-\underset{\sim}{v}^*) - \hat{\underset{\sim}{\tau}}]\!]\cdot\underset{\sim}{n} = 0 \qquad \text{on } \sigma(t)\,. \tag{2.45}$$

CHAPTER 3

FUNDAMENTALS OF ELECTROMAGNETISM

This chapter deals with the different approaches to the theory of electromagnetism and the several formulations of Maxwell's equations. Also the electromagnetic body forces, body forces, body couples and Maxwell's stress tensor are introduced.

3.1. Approaches to Theory of Electromagnetism

The concepts of electric field $\underset{\sim}{E}$, magnetic field $\underset{\sim}{H}$ (or magnetic induction $\underset{\sim}{B}$), free electric current density $\underset{\sim}{J}^{(f)}$ and free charge density $\rho^{(f)}$ each has a clear distict meaning if these fields are not time varying. In the case of time varying fields, $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are no longer independent, but are tied together by Maxwell's equations and their meanings come even more blurred if material is in motion.

The approaches are classified as

a) historical or logical sequence,

b) macroscopic or microscopic formulation,

c) relativistic or nonrelativistic treatment.

3.1.a) Historical or Logical Sequence:

In historical sequence of the theory of electromagnetism, one first considers electrostatics, magnetostatics, slowly varying direct currents, alternating currents and electromagnetic waves. Consequently, the subjects of electrostatics and magnetostatics are distinguished from each other and from the whole subject of electromagnetic theory by the following requirements:

i) All quantities do not vary with time,

ii) There is no motion of charges for electrostatics, and there is constant current present for magnetostatics.

Hence, under these conditions, Maxwell's equations, which are used in the logical sequence, split into two groups of independent equations, one of which contains terms relating to the electric field only, and the other terms relating to the magnetic field only. In this way, there are similarities between the fundamental problems of electrostatics and magnetostatics. In logical sequence, one directly starts with Maxwell's equations to cover the theory of electromagnetism.

3.1.b) Macroscopic or Microscopic Formulation:

A wide range of electromagnetic phenomena may be accounted for without introducing the microscopic nature of matter and the discrete nature of electric charges. In this approach, known as the phenomenological theory of electromagnetism, electric and magnetic properties of a substance are described by the material property tensors, permitivity $\hat{\underset{\sim}{\epsilon}}$, permeability $\hat{\underset{\sim}{\mu}}$, and electrical conductivity $\hat{\underset{\sim}{\sigma}}^{(e)}$. Charges and currents are assumed to be distributed continuously in space and are described by charge and current densities.

On the other hand, the microscopic formulation is based on the discrete nature of the electric charge. The microscopic equations predict, in detail, the behavior of the particles and their fields, but the macroscopic equations predict the average behavior of the same particles and fields.

The first attempt is ascribed to Lorentz who used an averaging procedure to obtain the macroscopic equations from the microscopic equations of the electron theory [47]. Subsequently, there have been many attempts to derive the macroscopic equations by some kind of averaging process. Depending upon the degree of approximation, different sets of electromagnetic equations are derived, all having the name of Maxwell's equations [19,41-43]. For instance, De Groot and Suttorp [48] derived Maxwell's equations from the microscopic equations for the electromagnetic fields in the presence of point charges using the principles of statistical mechanics.

3.1.c) Relativistic or Nonrelativistic Treatment:

Let $\underset{\sim}{x}$ and t be the spatial and time coordinates measured by an observer in the frame S, and $\underset{\sim}{x}'$ and t' be the spatial and time coordinates measured by another observer in the reference frame S' which moves with uniform velocity $\underset{\sim}{v}$ relative to S.

According to the concepts of absolute space and absolute time, the coordinates are related by

$$\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{v}t \qquad ; \qquad t' = t \quad . \tag{3.1}$$

The pair of transformation (3.1) is called the Galilean transformation under which the laws of classical mechanics are invariant.

According to the theory of relativity, the coordinates are related by [see, Eq.(A.5)]

$$\begin{aligned} \underset{\sim}{x}' &= \underset{\sim}{x} - \gamma \underset{\sim}{v} t + (\gamma - 1)\frac{\underset{\sim}{x}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\ t' &= \gamma\left(t - \frac{\underset{\sim}{x}\cdot\underset{\sim}{v}}{c^2}\right) \end{aligned} \tag{3.2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.3}$$

and c is the speed of light in vacuum. The pair of transformation under which the laws of electrodynamics are invariant*. The electromagnetic field variables of the Minkowski formulation in the frames S and S' are related by [see, Eq.(A.32)]

$$\begin{aligned} \underset{\sim}{E}' &= \gamma(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}) + (1-\gamma)\frac{\underset{\sim}{E}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\ \underset{\sim}{B}' &= \gamma(\underset{\sim}{B} - \frac{1}{c^2}\underset{\sim}{v}\times\underset{\sim}{E}) + (1-\gamma)\frac{\underset{\sim}{B}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\ \underset{\sim}{D}' &= \gamma(\underset{\sim}{D} + \frac{1}{c^2}\underset{\sim}{v}\times\underset{\sim}{H}) + (1-\gamma)\frac{\underset{\sim}{D}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\ \underset{\sim}{H}' &= \gamma(\underset{\sim}{H} - \underset{\sim}{v}\times\underset{\sim}{D}) + (1-\gamma)\frac{\underset{\sim}{H}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\ \underset{\sim}{J}'^{(f)} &= \underset{\sim}{J}^{(f)} - \gamma\underset{\sim}{v}\rho^{(f)} + (\gamma-1)\frac{\underset{\sim}{J}^{(f)}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\ \rho'^{(f)} &= \gamma(\rho^{(f)} - \frac{1}{c^2}\underset{\sim}{J}^{(f)}\cdot\underset{\sim}{v}) \end{aligned} \tag{3.4}$$

according to the Lorentz group of transformations, where $\underset{\sim}{D}$ is the electric displacement. It should be noted that the Galilean transformation (3.1) is found by taking the speed

*Four dimensional formulation of mechanics and electrodynamics are appropriate in the mathematical formulation, but some of the physical insight may be lost [49].

of light c to be equal to infinity in (3.2).

The particles considered in modern physics have a speed comparable to the speed of light in vacuum, but the bulk material used in engineering never reach this speed. For this reason the nonrelativistic theory of electromagnetism may be used in the solutions of problems of bulk materials. Hence, in this thesis, we will make use of the macroscopic, nonrelativistic theory of electromagnetism following the logical sequence. That is, equations of mechanical origin are taken to be Galilean invariant and those of electromagnetic origin are approximated as the linear terms with respect to $(\frac{v}{c})^2$ in expansion of (3.4). Thus the combined set of equations of electrodynamics and mechanics are neither Galilean nor Lorentz invariant.

3.2. Several Formulations of Maxwell's Equations for Moving Materials

There are disagreements concerning the formulations of electromagnetism in the literature. Definitions, names and number of electromagnetic field vectors needed to describe the electromagnetic phenomena are different. However, all of them give the same results in free space (vacuum) where the macroscopic fields are measured, in the presence of charged particles. Therefore, the well established theory of electromagnetism is that of free space.

The electromagnetic phenomena occuring in a material are described by the following set of vectors and a scalar: $\underset{\sim}{E}(\underset{\sim}{x},t)$, $\underset{\sim}{B}(\underset{\sim}{x},t)$, $\underset{\sim}{D}(\underset{\sim}{x},t)$, $\underset{\sim}{H}(\underset{\sim}{x},t)$, $\underset{\sim}{J}^{(f)}(\underset{\sim}{x},t)$, $\underset{\sim}{P}(\underset{\sim}{x},t)$, $\underset{\sim}{M}(\underset{\sim}{x},t)$ and $\rho^{(f)}(\underset{\sim}{x},t)$. The last two vectors are called electric and magnetic polarizations per unit volume respectively or, simply, polarization and magnetization. In the formulations of Maxwell's equations, one vector from electricity ($\underset{\sim}{P}$ or $\underset{\sim}{D}$ or $\underset{\sim}{E}$) and one from magnetism ($\underset{\sim}{M}$ or $\underset{\sim}{B}$ or $\underset{\sim}{H}$) are excluded and known as auxilary electromagnetic fields. This choice depends upon certain formulations of electrodynamics. The apparent forms of the governing equations of the electrodynamics and the meaning of the electromagnetic fields are different even if the same name and symbol are used. Furthermore, the recognition of this difference is important since the polarization and magnetization will affect other dynamical quantities, such as body forces, body couples and electromagnetic energy supplies.

In this section, the governing equations of apparently different formulations and the relationships between them are discussed.

3.2.a) Maxwell's Equations for Moving Media

There has been a revival of interest in the formulations of governing equations of electrodynamics in the presence of moving materials in the last two decades [19,41-43].

There are apparently different approaches to formulate Maxwell's equations in the presence of moving materials. The first one is given correctly in the famous work of Minkowski which is based on the special theory of relativity [49]. In this formulation, the Maxwell's equations are covariant under the Lorentz group of transformations and they are used for moving and deformable bodies. However, the constitutive equations are different in two frames of reference which are moving at constant velocity relative to each other. Later on O'Dell [50] and Post [51] formed the linear constitutive equations for anisotropic rigid materials which are now covariant under the Lorentz group of transformations [see, Eq(A.1)]

The second type of the formulations depends upon the models chosen for polarization and magnetization. In 1953, Chu [41] and later Boffi [42] developed new formulations with significant nonrelativistic differences in the electromagnetic force expressions. The Chu formulation is based on two generalized charge densities and current densities while the Boffi formulation is the relativistic modification of the Amperian model of magnetization of materials. The model for polarization is the dipole model of electric charges which is the same for these three formulations. The model of magnetization for the Chu formulation is again the dipole model of magnetic charges however, for others the circuit-current model is used. In fact these two models for magnetization are different from each other. Meanwhile, Corsteiu [56] established that the magnetic current and the existence of magnetic charge depend essentially on the rotation of material and spatial variation of magnetic permeability respectively. In all these formulations, electromagnetic body forces, body couples and energy supplies are expressed in terms of their own fields and they appear different in form. Experiments have not been successful in resolving this contraversy, because many of the differences involve small relativistic effects and, moreover, the associated quantities are not accessible to direct measurements.

Later on, Tai [43], and Penfield and Haus [19] compared extensively the various formulations. The latter authors, in view of the special theory of relativity, modified the incomplete electromagnetic forces in these formulations so that they become equivalent.

Maxwell's equations are given below in the integral form without taking any model and assumed to be valid for any arbitrary moving frame.

Faraday's law:

$$\int_{\partial\mathcal{S}-\gamma} \underset{\sim}{E}' \cdot d\underset{\sim}{s} + \frac{d}{dt}\int_{\mathcal{S}-\gamma} \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \cdot d\underset{\sim}{a} = 0 \tag{3.5}$$

Ampére's law (modified by Maxwell):

$$\int_{\partial\mathcal{S}-\gamma} \underset{\sim}{H}' \cdot d\underset{\sim}{s} - \frac{d}{dt}\int_{\mathcal{S}-\gamma} (\epsilon_0 \underset{\sim}{E} + \underset{\sim}{P}) \cdot d\underset{\sim}{a} - \int_{\mathcal{S}-\gamma} \underset{\sim}{J}^{(f)} \cdot d\underset{\sim}{a} - \int_{\partial\mathcal{S}-\gamma} \underset{\sim}{K}^{(f)} \cdot d\underset{\sim}{s} = 0 \tag{3.6}$$

Gauss' law:

$$\int_{\partial V-\sigma} (\epsilon_0 \underset{\sim}{E} + \underset{\sim}{P}) \cdot d\underset{\sim}{a} - \int_{V-\sigma} \rho^{(f)} dv - \int_{\partial V-\sigma} \omega^{(f)} n \cdot d\underset{\sim}{a} = 0 \tag{3.7}$$

Conservation of magnetic flux:

$$\int_{\partial V-\sigma} \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \cdot d\underset{\sim}{a} = 0 \tag{3.8}$$

where

$\underset{\sim}{E}'$, $\underset{\sim}{H}'$, $\underset{\sim}{J}^{(f)}$: effective values of $\underset{\sim}{E}$, $\underset{\sim}{H}$ and $\underset{\sim}{J}^{(f)}$ measured on a moving frame relative velocity of which is $\underset{\sim}{v}$,

$\underset{\sim}{K}^{(f)}$:free surface current density,

$\omega^{(f)}$:free surface charge density per unit area,

ϵ_0 , μ_0 :permitivity and permeability of vacuum respectively.

In addition to these laws, one has the conservation of electric charge

$$\frac{d}{dt}\int_{V-\sigma} \rho^{(f)} dv + \int_{\partial V-\sigma} \underset{\sim}{J}'^{(f)} \cdot d\underset{\sim}{a} = 0 \tag{3.9}$$

The forms of Eqs.(3.5,6) and (3.7-9) are similar to those of (2.39) and (2.40) respectively. The local balance laws and the boundary conditions of the electromagnetic fields are obtained by assuming the integrals in (3.5-9) to be valid for every part of the material. Following the derivations of local equations introduced in Chapter 2, local forms of the equations of electrodynamics and the associated boundary conditions are obtained. They are

$$\left.\begin{aligned} &\mu_0 \frac{\partial}{\partial t}(\underset{\sim}{H}+\underset{\sim}{M}) + \operatorname{curl}[\underset{\sim}{E}' + \mu_0(\underset{\sim}{H}+\underset{\sim}{M})\times\underset{\sim}{v}] = 0 \\ &\frac{\partial}{\partial t}(\epsilon_0 \underset{\sim}{E}+\underset{\sim}{P}) + \operatorname{div}(\epsilon_0\underset{\sim}{E}+\underset{\sim}{P})\underset{\sim}{v} - \operatorname{curl}[\underset{\sim}{H}' - (\epsilon_0\underset{\sim}{E}+\underset{\sim}{P})\times\underset{\sim}{v}] \\ &\qquad + \underset{\sim}{J}^{(f)} = 0 \end{aligned}\right\} \text{in } \mathcal{V}-\gamma \tag{3.10}$$

$$\left.\begin{aligned} &\operatorname{div}(\epsilon_0\underset{\sim}{E}+\underset{\sim}{P}) - \rho^{(f)} = 0 \;;\quad \mu_0 \operatorname{div}(\underset{\sim}{H}+\underset{\sim}{M}) = 0 \\ &\frac{\partial \rho^{(f)}}{\partial t} + \operatorname{div}(\rho^{(f)}\underset{\sim}{v} + \underset{\sim}{J}'^{(f)}) = 0 \end{aligned}\right\} \text{in } \mathcal{V}-\sigma$$

and

$$\left.\begin{aligned} &[\![(\underset{\sim}{H}+\underset{\sim}{M})\times(\underset{\sim}{v}-\underset{\sim}{v}^*) + \underset{\sim}{E}']\!]\cdot\underset{\sim}{k} = 0 \\ &[\![(\epsilon_0\underset{\sim}{E}+\underset{\sim}{P})\times(\underset{\sim}{v}-\underset{\sim}{v}^*) - \underset{\sim}{H}' + \underset{\sim}{K}^{(f)}]\!]\cdot\underset{\sim}{k} = 0 \end{aligned}\right\} \text{on } \gamma(t) \tag{3.11}$$

$$\left.\begin{aligned} &[\![\epsilon_0\underset{\sim}{E}+\underset{\sim}{P}]\!]\cdot\underset{\sim}{n} - \omega^{(f)} = 0 \;;\; \mu_0[\![\underset{\sim}{H}+\underset{\sim}{M}]\!]\cdot\underset{\sim}{n} = 0 \\ &[\![\rho^{(f)}(\underset{\sim}{v}-\underset{\sim}{v}^*) + \underset{\sim}{J}'^{(f)}]\!]\cdot\underset{\sim}{n} = 0 \end{aligned}\right\} \text{on } \sigma(t)\,.$$

Assuming that the discontinuity surface is a material interface, i.e., $\underset{\sim}{v} = \underset{\sim}{v}^*$, one obtains

$$[\![\underset{\sim}{E}']\!]\cdot\underset{\sim}{k} = 0 \;;\quad [\![-\underset{\sim}{H}' + \underset{\sim}{K}^{(f)}]\!]\cdot\underset{\sim}{k} = 0 \qquad \text{on } \gamma(t) \tag{3.12}$$

$$[\![\underset{\sim}{J}']\!]\cdot\underset{\sim}{n} = 0 \qquad \text{on } \sigma(t)\,.$$

For a stationary surface, $\underset{\sim}{v}^* = 0$:

$$[\![\mu_0(\underset{\sim}{H}+\underset{\sim}{M})\times\underset{\sim}{v} + \underset{\sim}{E}']\!]\cdot\underset{\sim}{k} = 0 \;;\; [\![(\epsilon_0\underset{\sim}{E}+\underset{\sim}{P})\times\underset{\sim}{v} - \underset{\sim}{H}' + \underset{\sim}{K}^{(f)}]\!]\cdot\underset{\sim}{k} = 0 \quad \text{on } \gamma(t)$$

$$[\![\rho^{(f)} \underset{\sim}{v} + \underset{\sim}{J}^{(f)}]\!] \cdot \underset{\sim}{n} = 0 \qquad \text{on } \sigma(t) \quad . \tag{3.13}$$

3.2.b) Several Formulations of Maxwell's Equations

Several formulations of Maxwell's equations and boundary conditions, namely, Minkowski, Ampére, Boffi, Chu and other formulations are obtained by introducing new field variables. For the specification of field quantities in different formulations, a superscript will be used. In this section, the superscripts M , A , B , C denote the associated quantity in the Minkowski, Ampére, Boffi and Chu formulations respectively.

i) <u>The Minkowski formulation</u>:

The oldest and most well known formulation of electromagnetic theory was given by Minkowski [49]. Redefining new electromagnetic variables as

$$\begin{aligned} &\underset{\sim}{B}^M \equiv \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \quad ; \quad \underset{\sim}{D}^M \equiv \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P} \\ &\underset{\sim}{E}^M \equiv \underset{\sim}{E}' + \underset{\sim}{B}^M \times \underset{\sim}{v} \quad ; \quad \underset{\sim}{H}^M \equiv \underset{\sim}{H}' - \underset{\sim}{D}^M \times \underset{\sim}{v} \quad ; \quad \underset{\sim}{J}^{(f)} \equiv \underset{\sim}{J}^{(f)} + \operatorname{div} \underset{\sim}{D}^M \underset{\sim}{v} \end{aligned} \tag{3.14}$$

and substituting these into (3.10) and (3.11), one obtains

$$\operatorname{curl} \underset{\sim}{E}^M + \frac{\partial \underset{\sim}{B}}{\partial t} = 0 \quad ; \quad \operatorname{curl} \underset{\sim}{H}^M - \frac{\partial \underset{\sim}{D}^M}{\partial t} = \underset{\sim}{J}^{(f)} \qquad \text{in } \mathcal{V} - \sigma$$

$$\left. \begin{aligned} &\operatorname{div} \underset{\sim}{D}^M = \rho^{(f)} \quad ; \quad \operatorname{div} \underset{\sim}{B}^M = 0 \\ &\frac{\partial \rho^{(f)}}{\partial t} + \operatorname{div} \underset{\sim}{J}^{(f)} = 0 \end{aligned} \right\} \quad \text{in } \mathcal{V} - \sigma \tag{3.15}$$

and

$$[\![\underset{\sim}{E}^M - \underset{\sim}{B}^M \times \underset{\sim}{v}^*]\!] \cdot \underset{\sim}{k} = 0 \quad ; \quad [\![\underset{\sim}{H}^M + \underset{\sim}{D}^M \times \underset{\sim}{v}^* - \underset{\sim}{K}^{(f)}]\!] \cdot \underset{\sim}{k} = 0 \qquad \text{on } \gamma(t)$$

$$\left. \begin{aligned} &[\![\underset{\sim}{D}^M]\!] \cdot \underset{\sim}{n} - \omega^{(f)} = 0 \quad ; \quad [\![\underset{\sim}{B}^M]\!] \cdot \underset{\sim}{n} = 0 \\ &[\![\underset{\sim}{J}^{(f)} - \rho^{(f)} \underset{\sim}{v}^*]\!] \cdot \underset{\sim}{n} = 0 \end{aligned} \right\} \quad \text{on } \sigma(t) \tag{3.16}$$

respectively.Eqs.$(3.15)_1$ and $(3.15)_4$ are not completely independent of each other, because of the mathematical identity

$\operatorname{div} \operatorname{curl} \underset{\sim}{\Phi} \equiv 0$, for any vector $\underset{\sim}{\Phi}$. Therefore $(3.15)_4$ exists as an auxilary condition in the solution of $(3.15)_1$. Neither $(3.15)_2$ nor $(3.15)_3$ is completely independent, because of $(3.15)_5$.

If one assumes $\underset{\sim}{v} = \underset{\sim}{v}^*$, the boundary conditions $(3.16)_{1,2,5}$ become

$$[\![\underset{\sim}{E}^M - \underset{\sim}{B}^M \times \underset{\sim}{v}]\!] \cdot \underset{\sim}{k} = 0 \quad ; [\![\underset{\sim}{H}^M + \underset{\sim}{D}^M \times \underset{\sim}{v} - \underset{\sim}{K}^{(f)}]\!] \cdot \underset{\sim}{k} = 0 \quad \text{on} \quad \gamma(t)$$

$$[\![\underset{\sim}{J}^{(f)} - \rho^{(f)} \underset{\sim}{v}]\!] \cdot \underset{\sim}{n} = 0 \qquad \text{on} \quad \sigma(t) \tag{3.17}$$

and the remaining equations are the same. Now, if the velocity vector is continuous on $\sigma(t)$ and $\gamma(t)$ and its normal component and the surface charge density vanish, then one has

$$[\![\underset{\sim}{E}^M]\!] \cdot \underset{\sim}{k} = 0 \quad ; [\![\underset{\sim}{H}^M - \underset{\sim}{K}^{(f)}]\!] \cdot \underset{\sim}{k} = 0 \qquad \text{on} \quad \gamma(t)$$

$$[\![\underset{\sim}{J}^{(f)}]\!] \cdot \underset{\sim}{n} = 0 \qquad \text{on} \quad \sigma(t) \tag{3.18}$$

and the remaining equations are the same. These boundary conditions are the same as that of the field variables in stationary media. Otherwise, the boundary conditions of electromagnetic fields would be different in stationary and moving media.

The field vectors $\underset{\sim}{E}^M$, $\underset{\sim}{B}^M$, $\underset{\sim}{D}^M$, $\underset{\sim}{H}^M$ and $\underset{\sim}{J}^{(f)}$ are related through constitutive equations. In most general cases, each of the constitutive relations must also depend upon the set of independent thermo-mechanical variables associated with the thermo-mechanical behavior of the system. These dependencies have the general functional form

$$\underset{\sim}{D} = \hat{\underset{\sim}{D}} \left(\{\gamma\}, \{\chi\} \right) \tag{3.19}$$

where $\{\gamma\}$, $\{\chi\}$ represent the set of independent electromagnetic and thermo-mechanical variables respectively. A similar functional form is valid for $\underset{\sim}{H}$ and $\underset{\sim}{J}^{(f)}$. The constitutive equations for magneto-electro thermo-viscoelastic anisotropic materials will be handled in Chapter 5. The constitutive equations for linear anisotropic magneto-electric material is given in Appendix A [see, Eqs.(A.25-31)].

For the free space, the constitutive equations are

$$\underset{\sim}{D}^M = \epsilon_0 \underset{\sim}{E}^M \quad ; \quad \underset{\sim}{H}^M = \frac{1}{\mu_0} \underset{\sim}{B}^M \quad . \qquad (3.20)$$

Despite the fact that the constitutive equations in the Minkowski formulation change when the material is in motion, (3.20) is the same if the velocity of the material is small compared to the speed of light.

When (3.20) is substituted into $(3.15)_{2,3}$, then the Maxwell equations can be solved simultaneosly to determine $\underset{\sim}{E}^M$ and $\underset{\sim}{B}^M$ if the sources $\underset{\sim}{J}^{(f)}$ and $\rho^{(f)}$ are prescribed.

ii) <u>Amperian and Boffi formulations</u>:

The Amperian formulation of Maxwell's equations depends upon the dipole model for polarization and the electric circuit model for magnetization. Following Ampére's original idea, Lorentz formulated the Maxwell equations for moving media. By adding to the Lorentz formulation the correction introduced by the relativistic treatment of the circuit model, one may derive the equations in the Amperian formulation.

The Boffi formulation is the extension of the Amperian formulation of electrodynamics for moving media [42,19]. The models chosen for polarization and magnetization of materials are the same in both formulations, but in the Amperian formulation an equivalent polarization charge occurs when the current loop moves in a direction normal to its area vector. The equivalent polarization is in a direction perpendicular to both the velocity and the magnetizatization vector, being equal to $-\frac{1}{c^2} \underset{\sim}{M} \times \underset{\sim}{v}$. Therefore, both formulations are identical when the material is not in motion, i.e., $\underset{\sim}{v} = 0$.

<u>Amperian formulation</u>:

Introducing

$$\begin{aligned} &\underset{\sim}{B}^A \equiv \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \quad ; \quad \underset{\sim}{H} \equiv \underset{\sim}{H}' - \epsilon_0 \underset{\sim}{E} \times \underset{\sim}{v} \\ &\underset{\sim}{E}^A \equiv \underset{\sim}{E}' + \mu_0 (\underset{\sim}{H} + \underset{\sim}{M}) \times \underset{\sim}{v} \end{aligned} \qquad (3.21)$$

and subtituting $\underset{\sim}{P} - \frac{\underset{\sim}{M} \times \underset{\sim}{v}}{c^2}$ instead of $\underset{\sim}{P}$ in (3.10), and neglecting the terms of order $(v/c)^2$ in the resulting equations, one obtains

$$\frac{\partial \underset{\sim}{B}^A}{\partial t} + \operatorname{curl} \underset{\sim}{E}^A = 0 \qquad \text{in} \quad \mathcal{S} - \gamma$$

$$\frac{1}{\mu_0}\,\mathrm{curl}\,\mathbf{B}^A - \epsilon_0 \frac{\partial \mathbf{E}^A}{\partial t} = \mathbf{J}^{(f)} + \hat{\mathbf{J}}^{(p)} + \hat{\mathbf{J}}^{(m)} \tag{3.22}$$

$$\epsilon_0\, \mathrm{div}\, \mathbf{E}^A = \rho^{(f)} + \hat{\rho}^{(p)} \; ; \quad \mathrm{div}\, \mathbf{B}^A = 0 \qquad \text{in} \quad V-\sigma$$

where

$$\hat{\rho}^{(p)} \equiv -\,\mathrm{div}\left(\mathbf{P}^A - \frac{1}{c^2}\mathbf{M}\times\mathbf{v}\right)$$

$$\hat{\mathbf{J}}^{(p)} \equiv \frac{\partial}{\partial t}\left(\mathbf{P}^A - \frac{1}{c^2}\mathbf{M}\times\mathbf{v}\right) + \mathrm{curl}\left(\mathbf{P}^A\times\mathbf{v}\right) \tag{3.23}$$

and

$$\hat{\mathbf{J}}^{(m)} \equiv \mathrm{curl}\,\mathbf{M}^A$$

and are called polarization charge density, polarization and magnetization current densities respectively.

The boundary conditions now become

$$\left.\begin{aligned} &[\![\mathbf{E}^A - \mathbf{B}^A\times\mathbf{v}^*]\!]\cdot\mathbf{k} = 0 \\ &[\![\tfrac{1}{\mu_0}\mathbf{B}^A - \mathbf{M}^A - \mathbf{P}^A\times\mathbf{v} + (\epsilon_0\mathbf{E}^A + \mathbf{P}^A)\times\mathbf{v}^* - \mathbf{K}^{(f)}]\!]\cdot\mathbf{k} = 0 \end{aligned}\right\} \text{on} \quad \gamma(t) \tag{3.24}$$

$$[\![\epsilon_0\mathbf{E}^A + \mathbf{P}^A - \frac{\mathbf{M}^A\times\mathbf{v}}{c^2}]\!]\cdot\mathbf{n} - \omega^{(f)} = 0 \; ; \quad [\![\mathbf{B}^A]\!]\cdot\mathbf{n} = 0 \qquad \text{on} \quad \sigma(t)$$

When the velocity of discontinuity line $\gamma(t)$ is equal to the velocity of the material, or if the velocity vector $\mathbf{v}$ is continuous on $\sigma(t)$ and $\gamma(t)$ and its normal component vanishes, then simpler boundary conditions follow. Obviously, the boundary conditions in the stationary and moving media are different from each other.

Boffi formulation:

The Boffi formulation of electrodynamics is similar to the Amperian one, in that both use $\mathbf{E}$ and $\mathbf{B}$ as the field variables and $\mathbf{P}$ and $\mathbf{M}$ as the material variables. The definations of $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{E}$ are the same as in (3.21) with superscript B instead of A , but there is no equivalent polarization. Boffi interprets the term $-\mathbf{P}^A + \frac{\mathbf{M}^A\times\mathbf{v}}{c^2}$ as the polarization by the electric and magnetic fields, and the term $\mathbf{M}^A + \mathbf{P}^A\times\mathbf{v}$ as magnetization. Thus, the Maxwell equations and the boundary conditions in the Boffi formulation are the following

$$\left.\begin{aligned} &\frac{\partial \mathbf{B}^A}{\partial t} + \mathrm{curl}\,\mathbf{B}^B = 0 \\ &\frac{1}{\mu_0}\,\mathrm{curl}\,\mathbf{B}^B - \epsilon_0\frac{\partial \mathbf{E}^B}{\partial t} = \mathbf{J}^{(f)} + \mathbf{J}^{(p)} + \mathbf{J}^{(m)} \end{aligned}\right\} \text{in} \quad \mathcal{V}-\gamma \tag{3.25}$$

$$div\ \epsilon_0 \underset{\sim}{E}^B = \rho^{(f)} + \overset{\vee}{\rho}^{(P)} \quad ; \quad div\ \underset{\sim}{B}^B = 0 \qquad \text{in} \quad V - \Gamma$$

and

$$\left.\begin{aligned} & [\![\underset{\sim}{E}^B - \underset{\sim}{B}^B \times \underset{\sim}{v}^*]\!] \cdot \underset{\sim}{k} = 0 \\ & [\![\tfrac{1}{\mu_0} \underset{\sim}{B}^B - \underset{\sim}{M}^B + (\epsilon_0 \underset{\sim}{E}^B + \underset{\sim}{P}^B) \times \underset{\sim}{v}^* - \underset{\sim}{K}^{(f)}]\!] \cdot \underset{\sim}{k} = 0 \end{aligned}\right\} \text{ on } \quad \gamma(t) \qquad (3.26)$$

$$[\![\epsilon_0 \underset{\sim}{E}^B + \underset{\sim}{P}^B]\!] \cdot \underset{\sim}{n} - \omega^{(f)} = 0 \quad ; \quad [\![\underset{\sim}{B}^B]\!] \cdot \underset{\sim}{n} = 0 \qquad \text{on } \Gamma(t)$$

where

$$\overset{\vee}{\rho}^{(P)} \equiv - div\ \underset{\sim}{P}^B \quad ; \quad \overset{\vee}{\underset{\sim}{J}}^{(P)} \equiv \frac{\partial \underset{\sim}{P}^B}{\partial t} \quad ; \quad \overset{\vee}{\underset{\sim}{J}}^{(m)} \equiv \text{Curl}\ \underset{\sim}{M}^B \qquad (3.27)$$

and are called the polarization charge density, polarization and magnetization current densities respectively. If the velocity of the discontinuity line $\gamma(t)$ is equal to the material velocity, then $\underset{\sim}{v}^*$ is replaced by $\underset{\sim}{v}$ in (3.26). It should be noted that the equations in the Boffi formulation coincide with those in Amperian, when $\underset{\sim}{v} = 0$ in the latter. Finally, it can be shown that the field equations in both the Boffi and Minkowski formulations reduce to the same set of equations in the quasistatic approximations.

iii) <u>Chu formulation</u>:

In the Chu formulation of electrodynamics [41] the following quantities are assumed to describe the electromagnetic state of a material completely:

a) The free charge distribution is represented by the volume and surface charge densities,

b) The free current distribution is represented by the volume and surface current densities,

c) The electric polarization is represented by the dipole moment density,

d) The magnetization is represented by the magnetic dipole moment density.

The model for polarization is the same as the Amperian and Boffi formulations. However, the model for magnetization is quite different. In the Chu formulation, it is assumed that the magnetic charges may exist in pairs.

Introducing

$$\underset{\sim}{E}^c \equiv \underset{\sim}{E}' + \mu_0 \underset{\sim}{H} \times \underset{\sim}{v} \quad ; \quad \underset{\sim}{H}^c \equiv \underset{\sim}{H}' - \epsilon_0 \underset{\sim}{E} \times \underset{\sim}{v} \qquad (3.28)$$

into (3.10,11), one then obtains

$$\left.\begin{aligned} \operatorname{curl} \underset{\sim}{E}^c + \mu_0 \frac{\partial \underset{\sim}{H}^c}{\partial t} &= -\underset{\sim}{J}^{(m)} \\ \operatorname{curl} \underset{\sim}{H}^c - \epsilon_0 \frac{\partial \underset{\sim}{E}^c}{\partial t} &= \underset{\sim}{J}^{(f)} + \underset{\sim}{J}^{(p)} \end{aligned}\right\} \quad \text{in} \quad \mathcal{V}-\gamma \qquad (3.29)$$

$$\epsilon_0 \operatorname{div} \underset{\sim}{E}^c = \rho^{(f)} + \rho^{(p)} \; ; \; \mu_0 \operatorname{div} \underset{\sim}{H}^c = \rho^{(m)} \qquad \text{in} \quad \mathcal{V}-\sigma$$

and

$$\left.\begin{aligned} [\![\underset{\sim}{E}^c + \mu_0 \underset{\sim}{M}^c \times \underset{\sim}{v} - \mu_0 (\underset{\sim}{H}^c + \underset{\sim}{M}^c) \times \underset{\sim}{v}^*]\!] \cdot \underset{\sim}{k} &= 0 \\ [\![\underset{\sim}{H}^c - \underset{\sim}{P}^c \times \underset{\sim}{v} + (\epsilon_0 \underset{\sim}{E}^c + \underset{\sim}{P}^c) \times \underset{\sim}{v}^* - \underset{\sim}{K}^{(f)}]\!] \cdot \underset{\sim}{k} &= 0 \end{aligned}\right\} \quad \text{on} \quad \gamma(t) \qquad (3.30)$$

$$[\![\epsilon_0 \underset{\sim}{E}^c + \underset{\sim}{P}^c]\!] \cdot \underset{\sim}{n} - \omega^{(f)} = 0 \; ; \; \mu_0 [\![\underset{\sim}{H}^c + \underset{\sim}{M}^c]\!] \cdot \underset{\sim}{n} = 0 \quad \text{on} \quad \sigma(t)$$

where

$$\begin{aligned} \underset{\sim}{J}^{(p)} &\equiv \frac{\partial \underset{\sim}{P}^c}{\partial t} + \operatorname{curl} (\underset{\sim}{P}^c \times \underset{\sim}{v}) \\ \underset{\sim}{J}^{(m)} &\equiv \frac{\partial \mu_0 \underset{\sim}{M}^c}{\partial t} + \operatorname{curl} (\mu_0 \underset{\sim}{M}^c \times \underset{\sim}{v}) \\ \rho^{(p)} &\equiv -\operatorname{div} \underset{\sim}{P}^c \quad ; \quad \rho^{(m)} \equiv -\operatorname{div} \mu_0 \underset{\sim}{M}^c \end{aligned} \qquad (3.31)$$

and are called the polarization and magnetization current densities, and the polarization and magnetization charge densities respectively.

If one assumes that $\underset{\sim}{v}^* = \underset{\sim}{v}$, then $(3.30)_{1,2}$ now become

$$\left.\begin{aligned} [\![\underset{\sim}{E}^c - \mu_0 \underset{\sim}{H}^c \times \underset{\sim}{v}]\!] \cdot \underset{\sim}{k} &= 0 \\ [\![\underset{\sim}{H}^c + \epsilon_0 \underset{\sim}{E}^c \times \underset{\sim}{v} - \underset{\sim}{K}^{(f)}]\!] \cdot \underset{\sim}{k} &= 0 \end{aligned}\right\} \quad \text{on} \quad \gamma(t) \qquad (3.32)$$

respectively.Whenever the normal component of the velocity and the surface charge density vanish, then the boundary conditions are the same as that of the stationary media. It should be noted that the Chu and Amperian formulations agree if $\underset{\sim}{M} = 0$.

iv) Other formulations:

In addition to the previous formulations, one may express Maxwell's equations in terms of the variables $\underset{\sim}{D}$, $\underset{\sim}{H}$ $\underset{\sim}{P}$, $\underset{\sim}{M}$ and $\underset{\sim}{v}$, or $\underset{\sim}{D}$, $\underset{\sim}{B}$, $\underset{\sim}{P}$, $\underset{\sim}{M}$ and $\underset{\sim}{v}$ without giving any physical interpretations to the space and time derivatives of these quantities. For example, if one introduces

$$\underset{\sim}{E}^{0} \equiv \underset{\sim}{E}' + \mu_0 \cdot (\underset{\sim}{H} + \underset{\sim}{M}) \times \underset{\sim}{v}$$

$$\underset{\sim}{D}^{0} \equiv \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P} \qquad (3.33)$$

and

$$H^{0} \equiv \underset{\sim}{H}' - \underset{\sim}{D} \times \underset{\sim}{v}$$

then Maxwell's equations and the boundary conditions now become

$$\left.\begin{aligned} &\mathrm{curl}\, \underset{\sim}{D}^{0} + \frac{1}{c^2}\frac{\partial \underset{\sim}{H}^{0}}{\partial t} + \frac{1}{c}\frac{\partial \underset{\sim}{M}^{0}}{\partial t} - \mathrm{curl}\, \underset{\sim}{P}^{0} = 0 \\ &\mathrm{curl}\, \underset{\sim}{H}^{0} - \frac{\partial \underset{\sim}{D}^{0}}{\partial t} = \underset{\sim}{J}^{(t)} \end{aligned}\right\} \text{in } \mathcal{V}-\gamma \qquad (3.34)$$

$$\mathrm{div}\, \underset{\sim}{D}^{0} = \rho^{(t)} \; ; \; \mathrm{div}\, \mu_0 \underset{\sim}{H}^{0} + \mathrm{div}\, \mu_0 \underset{\sim}{M}^{0} = 0 \quad \text{in } \mathcal{V}-\sigma$$

$$\left.\begin{aligned} &[\![\underset{\sim}{D}^{0} - \epsilon_0 \underset{\sim}{P}^{0} - \tfrac{1}{c^2}(\underset{\sim}{H}^{0} + \underset{\sim}{M}^{0}) \times \underset{\sim}{v}^{*}]\!] \cdot \underset{\sim}{k} = 0 \\ &[\![\underset{\sim}{H}^{0} + \underset{\sim}{D}^{0} \times \underset{\sim}{v}^{*} - \underset{\sim}{K}^{(t)}]\!] \cdot \underset{\sim}{k} = 0 \end{aligned}\right\} \text{on } \gamma(t) \qquad (3.35)$$

$$[\![\underset{\sim}{D}^{0}]\!] \cdot \underset{\sim}{n} - \omega^{(t)} = 0 \; ; \; \mu_0 [\![\underset{\sim}{H}^{0} + \underset{\sim}{M}^{0}]\!] \cdot \underset{\sim}{n} = 0 \quad \text{on } \sigma(t) .$$

There are, of course, other possibilities to introduce new electromagnetic fields and obtain the associated Maxwell's equations.

3.2.c) Relationships Between the Electromagnetic Fields in Different Formulations:

The relationships can be obtained by comparing the fields in two different sets of the Maxwell's equations.

Comparing Eq.(3.15) with (3.22) one finds the relationships between the Minkowski and Amperian variables:

$$\underset{\sim}{E}^{M} = \underset{\sim}{E}^{A} \quad ; \quad \underset{\sim}{B}^{M} = \underset{\sim}{B}^{A}$$

$$\underset{\sim}{D}^{M} - \epsilon_0 \underset{\sim}{E}^{M} = \underset{\sim}{P}^{A} - \frac{1}{c^2} \underset{\sim}{M}^{A} \times \underset{\sim}{v} \quad ; \quad \frac{1}{\mu_0} \underset{\sim}{B}^{M} - \underset{\sim}{H}^{M} = \underset{\sim}{M}^{A} + \underset{\sim}{P}^{A} \times \underset{\sim}{v} \qquad (3.36)$$

where the terms of order $(v/c)^2$ are neglected.

Comparing Eq.(3.15) with (3.25) the relationships between the Minkowski and Boffi variables are obtained similarly.

They are

$$\underset{\sim}{E}^B = \underset{\sim}{E}^M \quad ; \quad \underset{\sim}{B}^B = \underset{\sim}{B}^M$$

$$\underset{\sim}{P}^B = \underset{\sim}{D}^M - \epsilon_0 \underset{\sim}{E}^M \quad ; \quad \underset{\sim}{M}^B = \frac{1}{\mu_0} \underset{\sim}{B}^M - \underset{\sim}{H}^M \quad . \tag{3.37}$$

Similarly, the Amperian and Boffi variables are related by

$$\underset{\sim}{P}^B = \underset{\sim}{P}^A - \frac{1}{c^2} \underset{\sim}{M}^A \times \underset{\sim}{V} \quad ; \quad \underset{\sim}{M}^B = \underset{\sim}{M}^A + \underset{\sim}{P}^A \times \underset{\sim}{V} \tag{3.38}$$

and the other two variables are the same.

Comparing Eq.(3.29) with (3.15) and (3.29) with (3.25) one gets the relationships between the Chu and Minkowski variables, and the Chu and Boffi variables respectively:

$$\underset{\sim}{E}^M = \underset{\sim}{E}^C + \mu_0 \underset{\sim}{M}^C \times \underset{\sim}{V} \quad ; \quad \underset{\sim}{H}^M = \underset{\sim}{H}^C - \underset{\sim}{P}^C \times \underset{\sim}{V}$$

$$\underset{\sim}{D}^M - \epsilon_0 \underset{\sim}{E}^M = \underset{\sim}{P}^C - \frac{1}{c^2} \underset{\sim}{M}^C \times \underset{\sim}{V} \quad ; \quad \frac{1}{\mu_0} \underset{\sim}{B}^M - \underset{\sim}{H}^M = \underset{\sim}{M}^C + \underset{\sim}{P}^C \times \underset{\sim}{V} \tag{3.39}$$

and

$$\underset{\sim}{E}^B = \underset{\sim}{E}^C + \mu_0 \underset{\sim}{M}^C \times \underset{\sim}{V} \quad ; \quad \underset{\sim}{B}^B = \mu_0 (\underset{\sim}{H}^C + \underset{\sim}{M}^C)$$

$$\underset{\sim}{P}^B = \underset{\sim}{P}^C - \frac{1}{c^2} \underset{\sim}{M}^C \times \underset{\sim}{V} \quad ; \quad \underset{\sim}{M}^B = \underset{\sim}{M}^C + \underset{\sim}{P}^C \times \underset{\sim}{V} \quad . \tag{3.40}$$

In a similar manner, one can express the Chu variables in terms of the Amperian variables, or the Boffi and Minkowski variables in terms of the Chu variables.

3.2.d) Comments on the Different Formulations:

Although different sets of electromagnetic variables are used and different sets of Maxwell equations are obtained in the above formulations, there are certain similarities and differences between them.

<u>Similarites:</u>

i) In free space ($\underset{\sim}{D} = \epsilon_0 \underset{\sim}{E}$, $\underset{\sim}{B} = \mu_0 \underset{\sim}{H}$ in the Minkowski formulation, $\underset{\sim}{P} = 0$ and $\underset{\sim}{M} = 0$ in the other formulations), all the formulations reduce essentially to the same form.

ii) In the presence of moving and/or deforming continuum, all of them involve four electromagnetic variables and the same $\underset{\sim}{J}^{(f)}$ and $\rho^{(f)}$. Eventhough they recognize the same $\underset{\sim}{J}^{(f)}$ and $\rho^{(f)}$ inside moving and/or deforming media, the force which is exerted on current inside the material is different, but the overall forces on the material are the same in all formulations provided that compatible constitutive equations are used.

iii) All the formulations have four Maxwell's equations (two vectors and two scalars) which contain the same information, although they are different in form, and require two constitutive equations to describe nonconductors.

iv) All of them possess total energy-momentum tensor [see, Eq.(A.38)], but they are apparently different.

v) The Maxwell equations obtained are not sufficient in number to determine the unknowns.

<u>Differences</u>:

i) The variables used to describe electromagnetic fields inside materials are, in fact, different eventhough the same names and symbols are used.

ii) The forms of the Maxwell equations, constitutive equations and the transformations of electromagnetic fields under the Lorentz group are different.

iii) The boundary conditions for the electromagnetic fields are not similar.

iv) The form of ponderomotive Lorentz forces, body couples and energy supplies (or the energy-momentum tensors in four dim. formulation) inside the material and their interpretations are different.

The models associated with the Chu, Amperian and Boffi formulations allow one to use physical reasoning. Therefore one can interpret the polarization and magnetization charges and currents, body forces, body couples etc., but these interpretations can not be made in the Minkowski formulation. The Chu model for magnetization is easiar to interpret, because the magnetic charge is the dual of the electric charge in the sense that it is acted upon by a magnetic field rather than an electric field; and it recieves a force at right angles to its motion in an electric field rather than a magnetic field.

The magnetic dipole model used in the Chu formulation is a suitable representation for magnetism in materials with a domain structure. It has also been applied successfully to interpret the complex gyromagnetic effect (see, for example [7], Sec.7.8 and [19], Sec.4.9). On the other hand, the Chu model is not adequate to describe the microscopic nature of the material, because it is intended to interpret the macroscopic behavior.

The Maxwell equations in the Minkowski, Amperian and Boffi formulations are expressed in terms of a scalar and a vector potentials [50,57]. Hutter and Pao ([14], p.93) state that this is not possible for the Chu formulation. Introducing two scalar and two vector potentials, we are able to express them. Although the introduction of two additional potentials in the Chu formulation seems to be a disadvantage, fortunately they satisfy the same type of differential equations.

3.3. The Chu Formulation of Maxwell Equations in terms of Potentials

One can express the Maxwell equations (3.29) in terms of two scalar and two vector potentials. If one introduces

$$\underset{\sim}{E} = -\operatorname{grad}\phi^{(p)} - \frac{\partial \underset{\sim}{A}^{(p)}}{\partial t} - \frac{1}{\epsilon_0}\operatorname{curl}\underset{\sim}{A}^{(m)}$$

and

$$\underset{\sim}{H} = \frac{1}{\mu_0}\operatorname{curl}\underset{\sim}{A}^{(p)} - \operatorname{grad}\phi^{(m)} - \frac{\partial \underset{\sim}{A}^{(m)}}{\partial t} \tag{3.41}$$

the Maxwell's equations can then be written as

$$\Box\phi^{(p)} = -\frac{1}{\epsilon_0}\left(\rho^{(f)} + \rho^{(p)}\right) ; \quad \Box\phi^{(m)} = -\frac{1}{\mu_0}\rho^{(m)}$$

$$\Box\underset{\sim}{A}^{(p)} = -\mu_0\left(\underset{\sim}{J}^{(f)} + \underset{\sim}{J}^{(p)}\right) ; \quad \Box\underset{\sim}{A}^{(m)} = -\epsilon_0\underset{\sim}{J}^{(m)} \tag{3.42}$$

where the scalar potentials $\phi^{(p)}$ and $\phi^{(m)}$, and the vector potentials $\underset{\sim}{A}^{(p)}$ and $\underset{\sim}{A}^{(m)}$ satisfy the Lorentz condition

$$\operatorname{div}\underset{\sim}{A}^{(p)} + \frac{1}{c^2}\frac{\partial\phi^{(p)}}{\partial t} = 0 ; \quad \operatorname{div}\underset{\sim}{A}^{(m)} + \frac{1}{c^2}\frac{\partial\phi^{(m)}}{\partial t} = 0 \quad . \tag{3.43}$$

In Eq.(3.42), the operator $\Box$ is called the D'Alembert operator defined by

$$\Box \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad . \tag{3.44}$$

It should be noted that all the potentials satisfy the same type of differential equations with different source terms.

3.4. The Body Forces, Body Couples and the Maxwell Stress Tensor

A dynamic theory of electromagnetism of moving media requires the specifications of body forces, body couples and energy supplies of electromagnetic origin. There are apparently different expressions for these quantities since there exist several formulations of electrodynamics. The body forces and the body couples emerge from the basic assumptions which are motivated from the models of polarization and magnetization.

The body forces, body couples and the Maxwell stress tensors are expressed below in view of the Chu models for polarization and magnetization.

3.4.a) The Body Forces and the Body Couples

When a charged particle q moves with velocity $\underset{\sim}{v}$ in vacuum in which both an electric field $\underset{\sim}{E}$ and a magnetic field $\underset{\sim}{H}$ (or $\underset{\sim}{B}$) exist, the expression

$$\underset{\sim}{f}^{(L)} = q\,(\underset{\sim}{E} + \underset{\sim}{v} \times \mu_0 \underset{\sim}{H}) = q\,(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \qquad (3.45)$$

is called the Coulomb-Lorentz (or simply the Lorentz) force acting on the charged particle. This expression is unique and used in the definitions of the electric field $\underset{\sim}{E}$ and magnetic field $\underset{\sim}{H}$ (or $\underset{\sim}{B}$), and all the formulations of electrodynamics are in agreement with this expression.

Now, various expressions for the ponderomotive Lorentz force, which gives surface force at the bounding surface of the material as well as volume force within the material, have been proposed. These expressions fall into three categories:

i) Static or quasi static electric (magnetic) fields (see, e.g. [7,10,13,58-63]).

In electrostatics (magnetostatics), the force acting on electric (magnetic) charge (or current) in an electric (magnetic) field can be expressed in terms of volume or surface integrals of the electric (magnetic) field. For example, Birss [58] obtained the macroscopic total electric (magnetic) ponderomotive force and torque expressions from the Lorentz force expression in terms of surface integrals. Brown [7] gives ponderomotive force and torque expressions in terms of volume and surface integrals and transforms them into several forms by means of vector identities.

ii) Nonrelativistic motion of polarizable and/or magnetizable rigid or deformable materials (see,e.g. [12-14,19,41,64]).

The body forces and the body couples contain the terms associated with the motion of material besides the quantities in the static case. These forces and couples depend upon the models for polarization and magnetization. The Chu model is the simplest one and has certain physical significance even if a magnetic monopole has not been observed yet ([57]Sec.6-12) and [65-67].

iii) Relativistic motion of the material from the microscopic and macroscopic points of view (see, e.g. [15-19,68,69]).

In the relativistic theory, energy and momentum is a second order tensor in four-dim. Minkowski space [see,Eq. (A.38)]. Force is an expression as the three entries in the same row or column of the energy-momentum tensor.

Assuming each dipole as a doublet of monopoles upon each of which the Lorentz force acts, the body force and body couple are then evaluated. A similar calculation can be made when magnetization is modeled by a current loop model.

In addition to the force acting on a free charge density $\rho^{(f)}$ and current density $\underset{\sim}{J}^{(f)}$, Chu [41] suggested that there are electric and magnetic Lorentz forces per unit volume due to the polarization and magnetization currents respectively.

$$\begin{aligned} \rho \underset{\sim}{f}^{(p)} &= \rho^{(p)} \underset{\sim}{E} + \underset{\sim}{J}^{(p)} \times \mu_0 \underset{\sim}{H} \\ \rho \underset{\sim}{f}^{(m)} &= \rho^{(m)} \mu_0 \underset{\sim}{H} - \underset{\sim}{J}^{(m)} \times \epsilon_0 \underset{\sim}{E} \end{aligned} \tag{3.46}$$

where $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are the Chu variables.

If one now considers an electric dipole, then he can assume that the electric Lorentz force acts upon each monopole (positive or negative bound charge) of the dipole. Assuming that the center of mass of the dipole has the coordinate and the distance between the monopoles is $\underset{\sim}{1}$, then the positive and negative charges are supposed to be displaced relative to each other due to some external agencies (electric field, magnetic field, deformation, etc.). The positions of the polarized charges $+q^{(p)}$ and $-q^{(p)}$ are, respectively,

$$\underset{\sim}{x}^{+} = \underset{\sim}{x} + \frac{1}{2}\underset{\sim}{l} \quad ; \quad \underset{\sim}{x}^{-} = \underset{\sim}{x} - \frac{1}{2}\underset{\sim}{l} \quad . \tag{3.47}$$

Since the positions of the positive and negative charges are different from each other, the electric and magnetic fields at these points are different. If the Taylor series expansions are used and the first two terms are considered, then summing over all dipoles in unit volume gives the body force due to electric (and/or magnetic) dipoles. Similarly the body couple is determined by evaluating the total torque which is the sum of the torques exerted by the positive and negative charges about the center of mass of the particle in a unit volume.

Thus, the force acting on the electric dipole per unit volume is

$$\rho f^{(P)} = n q^{(P)} \underset{\sim}{l} \cdot \operatorname{grad} \underset{\sim}{E} + n q^{(P)} \underset{\sim}{v} \times (\underset{\sim}{l} \cdot \operatorname{grad} \mu_0 \underset{\sim}{H}) + n q^{(P)} \frac{d\underset{\sim}{l}}{dt} \times \mu_0 \underset{\sim}{H} \tag{3.48}$$

where n is the number of particles per unit volume. Using the conservation of mass and (2.21) one can show that

$$n q^{(P)} \frac{d\underset{\sim}{l}}{dt} = n \frac{d}{dt}\left(\frac{n q^{(P)} \underset{\sim}{l}}{n}\right) = \frac{\partial}{\partial t}(n q^{(P)} \underset{\sim}{l}) + \operatorname{div}(\underset{\sim}{v}\, n q^{(P)} \underset{\sim}{l}) . \tag{3.49}$$

Defining polarization per unit volume $\underset{\sim}{P}$ by

$$\underset{\sim}{P} \equiv \lim_{\underset{\sim}{l} \to 0} n q^{(P)} \underset{\sim}{l} \tag{3.50}$$

and introducing this into (3.49) and the resulting equation in (3.48), one obtains

$$\begin{aligned} \rho f^{(P)} = {} & \underset{\sim}{P} \cdot \operatorname{grad} \underset{\sim}{E} + \underset{\sim}{v} \times (\underset{\sim}{P} \cdot \operatorname{grad} \mu_0 \underset{\sim}{H}) \\ & + \left[\frac{\partial \underset{\sim}{P}}{\partial t} + \operatorname{div}(\underset{\sim}{v} \otimes \underset{\sim}{P})\right] \times \mu_0 \underset{\sim}{H} \quad . \end{aligned} \tag{3.51}$$

Finally, if one introduces polarization per unit mass $\check{\underset{\sim}{P}} = \frac{\underset{\sim}{P}}{\rho}$ then follows

$$\begin{aligned} \rho \overset{*}{\underset{\sim}{P}} & \equiv \rho \frac{d}{dt}\left(\frac{\underset{\sim}{P}}{\rho}\right) = \frac{d\underset{\sim}{P}}{dt} - \frac{1}{\rho}\frac{d\rho}{dt}\underset{\sim}{P} \\ & = \frac{\partial \underset{\sim}{P}}{\partial t} + \operatorname{div}(\underset{\sim}{v} \otimes \underset{\sim}{P}) \quad . \end{aligned} \tag{3.52}$$

Substituting this into (3.51), one obtains

$$\rho \underset{\sim}{f}^{(p)} = \underset{\sim}{P} \cdot \text{grad}\, \underset{\sim}{E} + \underset{\sim}{v} \times (\underset{\sim}{P} \cdot \text{grad}\, \mu_0 \underset{\sim}{H}) + \rho\, \check{\dot{\underset{\sim}{P}}} \times \mu_0 \underset{\sim}{H} \quad (3.53)$$

as the ponderomotive Lorentz force due to polarization.

Of course, in the Chu model for magnetization there is no essential difference in the derivation of the body force due to magnetization. It is found to be

$$\rho \underset{\sim}{f}^{(m)} = \mu_0 \underset{\sim}{M} \cdot \text{grad}\, \underset{\sim}{H} + \underset{\sim}{v} \times (\mu_0 \underset{\sim}{M} \cdot \text{grad}\, \epsilon_0 \underset{\sim}{E}) + \rho \mu_0\, \check{\dot{\underset{\sim}{M}}} \times \epsilon_0 \underset{\sim}{E} \quad (3.54)$$

where

$$\underset{\sim}{M} \equiv \lim_{\underset{\sim}{l} \to 0} q^{(m)} n \underset{\sim}{l} \quad ; \quad \check{\underset{\sim}{M}} \equiv \frac{\underset{\sim}{M}}{\rho} \quad . \quad (3.55)$$

The total body force per unit volume is then

$$\rho \underset{\sim}{f}^{(em)} = \rho^{(f)} \underset{\sim}{E} + \underset{\sim}{J}^{(f)} \times \mu_0 \underset{\sim}{H} + \rho \underset{\sim}{f}^{(p)} + \rho \underset{\sim}{f}^{(m)} \quad . \quad (3.56)$$

These expressions are obtained in a different manner by Penfield and Haus [19]. They are in agreement with (3.53) and (3.54) if the relativistic effects are ignored.

If one restricts himself to the quasi static electric or magnetic field system, Eqs.(3.53,54) reduce respectively to

$$\rho \underset{\sim}{f}^{(p)} = \underset{\sim}{P} \cdot \text{grad}\, \epsilon_0 \underset{\sim}{E} \quad ; \quad \rho \underset{\sim}{f}^{(m)} = \mu_0 \underset{\sim}{M} \cdot \text{grad}\, \underset{\sim}{H} \quad . \quad (3.57)$$

In an analogous manner the body couples are determined. Since the position of the particle coincides with the centre of mass of the electric (magnetic) dipole, the couple acting upon the particle can be determined by evaluating the torque exerted on the positive and negative charges about the centre of mass of the dipoles.

The electric couples on the positive and negative charges are, respectively,

$$\rho \underset{\sim}{l}^{+(p)} = \frac{1}{2} \underset{\sim}{l} \times \rho \underset{\sim}{f}^{+(p)} = \frac{1}{2} \underset{\sim}{l} \times \left[q^{(p)} \underset{\sim}{E}(\underset{\sim}{x} + \frac{\underset{\sim}{l}}{2}, t) + q^{(p)} (\underset{\sim}{v} + \frac{1}{2} \frac{d\underset{\sim}{l}}{dt}) \times \mu_0 \underset{\sim}{H}(\underset{\sim}{x} + \frac{\underset{\sim}{l}}{2}, t) \right] \quad (3.58)$$

$$\rho \underset{\sim}{l}^{-(p)} = \frac{1}{2} \underset{\sim}{l} \times \rho \underset{\sim}{f}^{-(p)} = \frac{1}{2} \underset{\sim}{l} \times \left[-q^{(p)} \underset{\sim}{E}(\underset{\sim}{x} - \frac{\underset{\sim}{l}}{2}, t) - q^{(p)} (\underset{\sim}{v} - \frac{1}{2} \frac{d\underset{\sim}{l}}{dt}) \times \mu_0 \underset{\sim}{H}(\underset{\sim}{x} - \frac{\underset{\sim}{l}}{2}, t) \right] .$$

Multiplying Eq.(3.58) by $n(\underset{\sim}{x}, t)$ and adding the couples and using the defination of $\underset{\sim}{P}$, one obtains

$$\rho \underset{\sim}{\ell}^{(p)} = \underset{\sim}{P} \times (\underset{\sim}{E} + \underset{\sim}{v} \times \mu_0 \underset{\sim}{H}) \quad . \tag{3.59}$$

Similarly, the couple per unit volume due to the magnetization is found to be

$$\rho \underset{\sim}{\ell}^{(m)} = \mu_0 \underset{\sim}{M} \times (\underset{\sim}{H} - \underset{\sim}{v} \times \epsilon_0 \underset{\sim}{E}) \quad . \tag{3.60}$$

Then, the total body couple per unit volume is the sum of these two couples, i.e.,

$$\rho \underset{\sim}{\ell} = \rho \underset{\sim}{\ell}^{(p)} + \rho \underset{\sim}{\ell}^{(m)} \quad . \tag{3.61}$$

Using the transformations of the Chu variables in the non-relativistic approximation (A.52), the total body couple is rewritten as

$$\rho \underset{\sim}{\ell} = \underset{\sim}{\mathcal{P}} \times \underset{\sim}{\mathcal{E}} + \mu_0 \underset{\sim}{\mathcal{M}} \times \underset{\sim}{\mathcal{H}} \tag{3.62}$$

where

$$\begin{aligned} &\underset{\sim}{\mathcal{P}} \cong \underset{\sim}{P} \quad ; \quad \underset{\sim}{\mathcal{M}} \cong \underset{\sim}{M} \\ &\underset{\sim}{\mathcal{E}} \cong \underset{\sim}{E} + \underset{\sim}{v} \times \mu_0 \underset{\sim}{H} \quad ; \quad \underset{\sim}{\mathcal{H}} \cong \underset{\sim}{H} - \underset{\sim}{v} \times \epsilon_0 \underset{\sim}{E} \quad . \end{aligned} \tag{3.63}$$

3.4.b) The Maxwell Stress Tensor

A second order tensor defined by

$$\tau_{ij} = \tau_{ij}^{o} + \tau_{ij}^{(m)} \tag{3.64}$$

where $\tau_{ij}^{o} = \epsilon_0 E_i E_j + \mu_0 H_i H_j - W \delta_{ij} = \tau_{ji}^{o}$

and $\tau_{ij}^{(m)} = \mathcal{P}_i \mathcal{E}_j + \mu_0 \mathcal{M}_i \mathcal{H}_j$ (3.65)

is known as the Maxwell stress tensor in the Chu formulation. In Eq.(3.65), τ_{ij}^{o} and $\tau_{ij}^{(m)}$ are, respectively, the Maxwell stress tensors in free space and in the material, and W is the electromagnetic energy density per unit volume defined by

$$W = \frac{1}{2} (\epsilon_0 E_k E_k + \mu_0 H_k H_k) \quad . \tag{3.66}$$

In free space, i.e., $\underset{\sim}{\mathcal{P}} = 0$ and $\underset{\sim}{\mathcal{M}} = 0$ the Maxwell stress tensor is simply equal to τ_{ij}^{o}. Thus, the Maxwell stress tensor also exists outside the material media. Moreover, since τ_{ij}^{o} is symmetric, the antisymmetric part of (3.64) is

$$\tau_{[ij]} = \tau^{(m)}_{[ij]} \quad . \tag{3.67}$$

That is, the antisymmetric part of the Maxwell stress tensor τ_{ij} is equal to the total body couple derived in (3.62).

Taking the divergence of (3.64) and using $(3.29)_{3,4}$, one obtains

$$\begin{aligned} \tau_{ji,j} = {} & \rho^{(f)} E_i + P_j E_{i,j} + \mu_0 M_j H_{i,j} + \epsilon_{ijk} \dot{x}_j \left(P_m \mu_0 H_{k,m} - \frac{1}{c^2} M_m E_{k,m}\right) \\ & + \epsilon_{ijk} \left[(\dot{x}_j P_m)_{,m} \mu_0 H_k - (\dot{x}_j \mu_0 M_m)_{,m} \epsilon_0 E_k\right] \\ & + (E_{i,j} - E_{j,i}) \epsilon_0 E_j + (H_{i,j} - H_{j,i}) \mu_0 H_j \quad . \end{aligned} \tag{3.68}$$

Multiplying Eq.$(3.29)_1$ by $\epsilon_0 \underline{E}$ and $(3.29)_2$ by $\mu_0 \underline{H}$ vectorially and adding the resulting equations, one has

$$\begin{aligned} \epsilon_0 (E_{i,j} - E_{j,i}) E_j + \mu_0 (H_{i,j} - H_{j,i}) H_j = {} & g_i \\ & + \epsilon_{ijk} \left\{ J_j^{(f)} \mu_0 H_k + \frac{\partial P_j}{\partial t} \mu_0 H_k - \frac{1}{c^2} \frac{\partial M_j}{\partial t} E_k \right. \\ & \left. + (P_j \dot{x}_m - P_m \dot{x}_j)_{,m} \mu_0 H_k - \frac{1}{c^2} (M_j \dot{x}_m - M_m \dot{x}_j)_{,m} E_k \right\} \end{aligned} \tag{3.69}$$

where

$$g_i \equiv \frac{1}{c^2} \epsilon_{ijk} \frac{\partial}{\partial t} (E_j H_k) \tag{3.70}$$

and is known as the electromagnetic field momentum.

Substituting (3.69) into (3.68) and after rearranging

$$\begin{aligned} \tau_{ji,j} = {} & \rho^{(f)} E_i + \epsilon_{ijk} J_j^{(f)} \mu_0 H_k + P_j E_{i,j} + \mu_0 M_j H_{i,j} \\ & + \epsilon_{ijk} \left[\frac{\partial P_j}{\partial t} \mu_0 H_k + (P_j \dot{x}_m)_{,m} \mu_0 H_k - \frac{1}{c^2} \frac{\partial M_j}{\partial t} E_k \right. \\ & \left. - \frac{1}{c^2} (M_j \dot{x}_m)_{,m} E_k \right] + \epsilon_{ikm} \dot{x}_k \left(P_j \mu_0 H_{m,j} - \frac{1}{c^2} M_j E_{m,j}\right) \\ & + g_i \end{aligned} \tag{3.71}$$

is obtained. Comparing Eq.(3.71) with (3.56), one arrives at

$$\rho f_i^{(em)} = \tau_{ji,j} - g_i \quad . \tag{3.72}$$

The body forces, body couples and the Maxwell stress tensors in the other formulations can be obtained using the relationships between the Chu variables and the others given in Section 3.2.c.

CHAPTER 4

BALANCE EQUATIONS AND BOUNDARY CONDITIONS OF INTERACTING CONTINUA

The present chapter is devoted to the formulations of the global balance laws and entropy inequality of the interacting continua and their local forms with the associated jump conditions on any discontinuity surface sweeping the body.

4.1. Interactions of Electromagnetic Fields and Deformable Continua

It is well known that the electric and magnetic fields are coupled even if the material is not deformable. We first summarize the means through which these interactions occur in rigid materials.

4.1.a) Interactions of Electric and Magnetic Fields:

The electric and magnetic fields are coupled to each other because of the following reasons:

i) Conduction current

In an electrically conducting medium, even if the fields are assumed to be independent of time, the electric field generates the conduction current, then this current becomes the source of the magnetic field $\underset{\sim}{H}$ (i.e., $\underset{\sim}{J} = \hat{\sigma}\underset{\sim}{E}$ and $\operatorname{curl}\underset{\sim}{H} = \underset{\sim}{J}$). Therefore the source of the magnetic field depends on the electric field, but the source of the electric field is seen to be independent of the magnetic field.

ii) Time dependent electric and magnetic fields

If the electric and magnetic fields vary with time, then the Maxwell's equations must be solved simultaneously. Hence the time dependence of the fields makes the electric and magnetic fields coupled.

iii) Motion of the material

If a polarizable or a magnetizable material is in motion, the distinction between the concepts of electric and magnetic fields become blurred. An electric field (or magnetic field) in one frame of reference S may not correspond to an electric field (or magnetic field) in another frame of reference S' which is in relative motion with respect to S. The electric (magnetic) field may transform into a combination of the electric (magnetic) and magnetic (electric) fields, see Eq.(3.4).

iv) The magneto-electric material

Referring to (A.27) or (A.28) a material becomes polarizable (magnetizable) in the magnetic (electric) field even if in the rest frame, because of the constitutive nature of the material.

4.1.b) Interaction of Electromagnetic Fields with Deformable Continua:

A number of author have developed theories for the interactions of electromagnetic fields with deformable continua, such as electroelasticity (elastic dielelectrics), magnetoelasticity and electrodynamics of deformable media. Broadly speaking, the theories may be classified according to

i) linear versus nonlinear,
ii) static versus dynamic,
iii) quasi static electric field versus quasi static magnetic field (polarizable or magnetizable material),
iv) thermally and electrically nonconductor versus conductor,
v) relativistic versus nonrelativistic,
vi) macroscopic versus microscopic or semi-microscopic,
vii) polar versus nonpolar,
viii) local versus nonlocal,
ix) classical crystals versus magnetic crystals.

The equations governing the interaction for the cases stated in the above categories are usually obtained either using variational principles or the balance laws of continuum physics.

The theory presented in this thesis is nonlinear, nonpolar, local, dynamic and macroscopic theory; and the material is assumed to be both polarizable and magnetizable with thermal and electrical conduction, and the material possesses magnetic symmetry. Moreover, the theory is not concerned with the relativistic effects.

4.2. Global Balance Laws

In continuum physics the following balance laws are postulated to be valid, irrespective of material constitution and geometry: i) Conservation of mass, ii) Balance of momentum, iii) Balance of moment of momentum, iv) Conservation of energy, v) Entropy inequality, vi) Conservation of charge, vii) Faraday's law, viii) Ampére's law and ix) Gauss' law for electric and magnetic charges.

We have already discussed the last four laws in Chapter 3. Electromagnetic and thermal fields interacting with deformable continua are to be incorporated with the balance laws (ii-v). In the presence of electromagnetic fields, electromagnetic body forces, body couples and energy supplies must be considered in addition to mechanical and thermodynamical ones.

i) Conservation of mass:

The global equation of conservation of mass is expressed by the form

$$\int_{\mathcal{V}-\Sigma} \rho_0 \, dV = \int_{v-\sigma} \rho \, dv \qquad \text{or} \qquad \frac{d}{dt}\int_{v-\sigma} \rho \, dv = 0 \tag{4.1}$$

where ρ_0 and ρ are the mass densities in the reference and present configurations respectively.

ii) Balance of momentum:

The global balance law of the momentum is given by

$$\frac{d}{dt}\int_{v-\sigma} \rho \dot{x}_i \, dv = \int_{\partial v-\sigma} t_{i(n)} \, da + \int_{v-\sigma} \rho\left(f_i^{(em)} + f_i^{(M)}\right) dv \tag{4.2}$$

where $\dot{x}_i$ is the velocity of the particle, $t_{i(n)}$ the actual stress vector associated with the outward normal n to the surface $\partial v-\sigma$ which encloses the volume $v-\sigma$. Furthermore, $\rho f_i^{(em)}$ and $\rho f_i^{(M)}$ are the electromagnetic and mechanical body forces per unit volume respectively.

If an infinitesimal tetrahedron with three sides parallel to coordinate surfaces is considered, and the balance of momentum (4.2) is invoked, it follows that the stress vector $\underset{\sim}{t}_{(n)}$ depends linearly on the unit normal $\underset{\sim}{n}$, i.e.,

$$t_{k(\underline{n})} = t_{ik}\, n_i \qquad . \tag{4.3}$$

Upon substituting (4.3) into (4.2) and using (3.72), one obtains

$$\frac{d}{dt}\int_{V-\sigma} \rho \dot{x}_i \, dv = \int_{\partial V-\sigma} t_{ki}\, n_k \, da + \int_{V-\sigma} (\tau_{ki,k} - g_i + \rho f_i^{(M)})\, dv$$

and after rearranging, (4.2) reduces to

$$\frac{d}{dt}\int_{V-\sigma} \rho \dot{x}_i \, dv = \int_{\partial V-\sigma} t_{ki}^{(T)}\, n_k \, da + \int_{V-\sigma} (\rho f_i^{(M)} - g_i)\, dv \tag{4.4}$$

where we introduce

$$t_{ki}^{(T)} = t_{ki} + \tau_{ki} \tag{4.5}$$

which is called the total stress tensor to be the sum of the actual stress and the Maxwell stress tensor.

iii) <u>Balance of moment of momentum</u>:

Taking the moment of (4.4) about a fixed point (origin of the reference frame), the global balance of moment of momentum becomes

$$\frac{d}{dt}\int_{V-\sigma} \epsilon_{ijk}\, x_j\, \rho \dot{x}_k \, dv = \int_{\partial V-\sigma} \epsilon_{ijk}\, x_j\, t_{lk}^{(t)}\, n_l \, da + \int_{V-\sigma} \epsilon_{ijk}\, x_j\, (\rho f_k^{(M)} - g_k)\, dv . \tag{4.6}$$

In deriving (4.6), surface couples of both mechanical and electromagnetic origins are excluded since the material is assumed to be apolar and the exchange effects are regarded to be small.

Note that the same result is obtained alternatively with the aid of ([4] , Eq.(2.3)).

iv) <u>Conservation of energy</u>:

The global energy balance law of the interacting continua is

$$\begin{aligned} \frac{d}{dt}\int_{V-\sigma} [\rho(\tfrac{1}{2}\dot{x}_k \dot{x}_k + \varepsilon) + W]\, dv = & \int_{\partial V-\sigma} (t_{lk}^{(T)} + \dot{x}_l\, g_k)\, \dot{x}_k\, n_l \, da \\ & - \int_{\partial V-\sigma} (\overset{*}{q}_k + q_k)\, n_k \, da + \int_{V-\sigma} \rho\, (\dot{x}_k f_k^{(M)} + r_\theta)\, dv \end{aligned} \tag{4.7}$$

where $\frac{1}{2}\dot{x}_k \dot{x}_k$ and ε are the kinetic and internal energy per unit mass respectively. The first integral at the right hand side of (4.7) is the time rate of work done by the total stress vector, and the electromagnetic momentum flow. $\overset{+}{\underset{\sim}{q}}$ and $\underset{\sim}{q}$ are, respectively, the true heat flux vector and the extra energy flux of unspecific nature which has to be determined by a constitutive equation. Thus $\underset{\sim}{q} = \overset{+}{\underset{\sim}{q}} + \underset{\sim}{q}$ is the energy flux throughtout the moving surface $\partial \mathcal{V} - \tilde{\sigma}$. Moreover, the first term in the argument of the last integral is the power exerted by $\rho \underset{\sim}{f}^{(M)}$, and ρr_{θ} is the energy supply per unit volume due to the sources other than electromagnetic and mechanical origins.

v) Entropy inequality:

The time rate of change of the total entropy is never less than the sum of the flux $\underset{\sim}{s}'$ of entropy through the moving surface $\partial \mathcal{V} - \sigma$ of the body and the entropy supplied by the body sources

$$\frac{d}{dt} \int_{\mathcal{V} - \sigma} \rho \eta \, dv \geq - \int_{\partial \mathcal{V} - \sigma} s_k n_k \, da + \int_{\mathcal{V} - \sigma} \rho r_{\eta} \, dv \quad . \qquad (4.8)$$

This postulate is assumed to be true for all independent processes, and the specific form of (4.8) depends on the process. In a simple thermodynamic process, $s_k = q_k / \theta$ and $r_{\eta} = \frac{r_{\theta}}{\theta}$ can be written, where θ is the absolute temperature. The entropy inequality (4.8) thus reads

$$\frac{d}{dt} \int_{\mathcal{V} - \sigma} \rho \eta \, dv \geq - \int_{\partial \mathcal{V} - \sigma} \frac{q_k}{\theta} n_k \, da + \int_{\mathcal{V} - \sigma} \frac{\rho r_{\theta}}{\theta} \, dv \quad . \qquad (4.9)$$

Hence, the processes described by the conservations of mass and energy, the balance laws of momentum and moment of momentum, and the Maxwell equations are subject to this so called Clausius-Duhem inequality. This inequality imposes restrictions upon the constitutive equations to be derived in the next chapter.

4.3. Local Balance Laws and Boundary Conditions

Since the balance laws (4.1,4,6,7,9) have the form (2.9), the integral theorem (2.44,45) forms the basis for the derivation of local balance laws are assumed to be valid for every element of the body, the local balance laws together with the associated boundary conditions are obtained as follows.

i) Conservation of mass:

Comparing $(4.1)_2$ with (2.39) one identifies $\varphi=\rho$, $\hat{\underset{\sim}{\mathcal{C}}}=0$, $\hat{g}=0$. The local form of $(4.1)_2$ according to (2.44) and (2.45) is given by

$$\frac{\partial\rho}{\partial t}+(\rho\dot{x}_k)_{,k}=0 \qquad \text{in } \mathcal{V}-\sigma \qquad (4.10)$$

$$[\![\rho(\dot{x}_k-\dot{x}_k^*)]\!]n_k=0 \qquad \text{on } \sigma(t) \qquad (4.11)$$

where $\dot{\underset{\sim}{x}}^*$ is the velocity of the discontinuity surface $\sigma(t)$. Thus $\dot{\underset{\sim}{x}}-\dot{\underset{\sim}{x}}^*$ is the relative velocity of $\sigma(t)$ with respect to the particle. If $\dot{\underset{\sim}{x}}=\dot{\underset{\sim}{x}}^*$, i.e., the particle velocity is equal to that of $\sigma(t)$, (4.11) vanishes.

ii) Balance of momentum:

Now comparing (4.4) with (2.39) one identifies $\underset{\sim}{\varphi}=\rho\dot{\underset{\sim}{x}}$, $\hat{\underset{\sim}{\mathcal{C}}}=\underset{\sim}{t}^{(T)}$, $\hat{\underset{\sim}{g}}=-\underset{\sim}{g}+\rho\underset{\sim}{f}^{(M)}$. Therefore, the local form of (4.4) according to (2.44) and (2.45) is given by

$$\frac{d}{dt}(\rho\dot{x}_i)=t_{ki,k}+(\tau_{ki,k}-g_i)+\rho f_i^{(M)} \qquad \text{in } \mathcal{V}-\sigma \qquad (4.12)$$

$$[\![\rho\dot{x}_i(\dot{x}_k-\dot{x}_k^*)-t_{ki}^{(T)}]\!]n_k=0 \qquad \text{on } \sigma(t). \qquad (4.13)$$

Upon using (4.10) and (3.72), Eq.(4.12) is written alternatively as

$$\rho\ddot{x}_i=t_{ki,k}+\rho\left(f_i^{(em)}+f_i^{(M)}\right) \qquad \text{in } \mathcal{V}-\sigma. \qquad (4.14)$$

If $\dot{\underset{\sim}{x}}=\dot{\underset{\sim}{x}}^*$ is assumed, (4.13) becomes

$$[\![t_{ki}^{(T)}]\!]n_k=0 \qquad \text{on } \sigma(t). \qquad (4.15)$$

iii) Balance of moment of momentum:

From (4.6) and (2.39) it is clear that $\underset{\sim}{\varphi}=\underset{\sim}{x}\times\rho\dot{\underset{\sim}{x}}$, $\hat{\underset{\sim}{\mathcal{C}}}=\underset{\sim}{x}\times\underset{\sim}{t}^{(T)}$, $\hat{\underset{\sim}{g}}=\underset{\sim}{x}\times(-\underset{\sim}{g}+\rho\underset{\sim}{f}^{(M)})$. The local form of (4.6) becomes

$$\epsilon_{ijk}x_j\left[\frac{d}{dt}(\rho\dot{x}_k)-t_{\ell k,\ell}-\rho\left(f_k^{(em)}+f_k^{(M)}\right)\right]=\epsilon_{ijk}t_{jk}^{(T)} \qquad \text{in } \mathcal{V}-\sigma \qquad (4.16)$$

$$\epsilon_{ijk}[\![x_j x_k(\dot{x}_\ell-\dot{x}_\ell^*)-x_j t_{\ell k}^{(T)}]\!]n_\ell=0 \qquad \text{on } \sigma(t). \qquad (4.17)$$

Imposing the restrictions coming from (4.10) and (4.14) onto (4.16), one obtains

$$\epsilon_{ijk}\, t^{(T)}_{jk} = 0 \qquad \text{or} \qquad t_{[jk]} = -\tau_{[jk]} \quad \text{in} \quad \mathcal{V}-\sigma \,. \tag{4.18}$$

This means that the total stress tensor is symmetric, but the actual stress tensor is not. From (4.17) it is clear that

$$\epsilon_{ijk}\, x_j \,[\![\rho \dot{x}_k (\dot{x}_l - \dot{x}^*_l) - t^{(T)}_{lk}]\!]\, n_l = 0 \quad \text{on} \quad \sigma(t) \tag{4.19}$$

and if $\dot{\underset{\sim}{x}} = \dot{\underset{\sim}{x}}^*$ is substituted, the equation reduces to

$$\epsilon_{ijk}\, x_j \,[\![t^{(T)}_{lk}]\!]\, n_l = 0 \qquad \text{on} \quad \sigma(t)\,. \tag{4.20}$$

This is nothing but the boundary condition obtained in the balance of momentum.

iv) <u>Conservation of energy</u>:

The form of (4.7) is similar to that of (2.39). First, adding and substracting the term $\dot{x}_k \frac{\partial g_k}{\partial t}$ to (4.7) and using (4.10), one now obtains the local form of the conservation of energy and the boundary condition as

$$\dot{x}_k \Big[\rho \ddot{x}_k - t_{ik,i} - \Big(\tau_{ik,i} - \frac{\partial g_k}{\partial t}\Big) - \rho f^{(M)}_k \Big] + \rho \dot{\varepsilon} + \frac{\partial W}{\partial t} + (W \dot{x}_i)_{,i}$$
$$- t^{(T)}_{ik}\, \dot{x}_{k,i} - (\dot{x}_i\, \dot{x}_k\, g_k)_{,i} - \dot{x}_k \frac{\partial g_k}{\partial t} + q_{i,i} - \rho r_\theta = 0$$
$$\text{in} \quad \mathcal{V}-\sigma \tag{4.21}$$

$$[\![(\tfrac{1}{2}\rho\, \dot{x}_k \dot{x}_k + \rho\varepsilon + W)(\dot{x}_i - \dot{x}^*_i) - (t^{(T)}_{ik} + \dot{x}_i g_k)\dot{x}_k + q_i]\!]\, n_i = 0$$
$$\text{on} \quad \sigma(t). \tag{4.22}$$

Upon substituting (4.12) into (4.21), one arrives at

$$\rho\dot{\varepsilon} + \frac{\partial W}{\partial t} + (W\dot{x}_i)_{,i} - t^{(T)}_{ik}\, \dot{x}_{k,i} - (\dot{x}_i\, \dot{x}_k\, g_k)_{,i} - \dot{x}_k \frac{\partial g_k}{\partial t}$$
$$+ q_{i,i} - \rho r_\theta = 0 \qquad \text{in} \quad \mathcal{V}-\sigma \,. \tag{4.23}$$

Next, multiplying $(3.29)_1$ and $(3.29)_2$ by $\underset{\sim}{H}$ and $\underset{\sim}{E}$ respectively and using the vector identity $(\nabla \times \underset{\sim}{A}) \cdot \underset{\sim}{B} - (\nabla \times \underset{\sim}{B}) \cdot \underset{\sim}{A} = \nabla \cdot (\underset{\sim}{A} \times \underset{\sim}{B})$ and (3.68,72) there follows

$$\epsilon_{ijk}\,(E_j H_k)_{,i} = -\frac{\partial W}{\partial t} - J_i^{(f)} E_i - \rho \overset{*}{\mathcal{P}_i} E_i - \rho \mu_0 \overset{*}{\mathcal{M}_i} H_i$$
$$- (\dot{x}_i P_k)_{,k} E_i - (\dot{x}_i \mu_0 M_k)_{,k} H_i \quad . \tag{4.24}$$

Now making use of (3.67), Eq.(4.24) reduces to

$$\frac{\partial W}{\partial t} = (\tau_{ij}^{(m)} \dot{x}_j - \epsilon_{ijk}\, E_j H_k)_{,i} - \Pi \tag{4.25}$$

where

$$\Pi = J_i^{(f)} E_i + \rho \overset{*}{\mathcal{P}_i} E_i + \rho \mu_0 \overset{*}{\mathcal{M}_i} H_i + \dot{x}_i (P_k E_{i,k} + \mu_0 M_k H_{i,k}) . \tag{4.26}$$

Introducing (4.25) into (4.23) and neglecting the terms of order $(\frac{v}{c})^2$, one finds

$$\rho \dot{\varepsilon} - t_{ik} \dot{x}_{k,i} + q_{i,i} - \rho r_\theta - \Pi + f_k^{(em)} \dot{x}_k = 0 \quad \text{in } V - \sigma . \tag{4.27}$$

In view of (4.26) and (3.72) and using the vector identity $\underset{\sim}{A} \cdot \underset{\sim}{B} \times \underset{\sim}{C} = \underset{\sim}{B} \cdot \underset{\sim}{C} \times \underset{\sim}{A} = \underset{\sim}{C} \cdot \underset{\sim}{A} \times \underset{\sim}{B}$ one is now ready to write (4.27) as

$$\rho \dot{\varepsilon} = t_{ik} \dot{x}_{k,i} + J_i^{(f)} \mathcal{E}_i + \rho \overset{*}{\mathcal{P}_i} \mathcal{E}_i + \rho \mu_0 \overset{*}{\mathcal{M}_i} \mathcal{H}_i - q_{i,i} + \rho r_\theta \quad \text{in } V - \sigma . \tag{4.28}$$

This result is in agreement with the one given by Hutter and Pao ([14], p.92) stated without derivation.

Neglecting the terms of order $(\frac{\dot{x}}{c})^2$ in (4.22), one writes

$$[\![(\tfrac{1}{2} \rho \dot{x}_k \dot{x}_k + \rho \varepsilon + W)(\dot{x}_i - \dot{x}_i^*) - t_{ik}^{(T)} \dot{x}_k + q_i]\!] n_i = 0 \quad \text{on } \sigma(t) \tag{4.29}$$

and if $\dot{\underset{\sim}{x}} = \dot{\underset{\sim}{x}}^*$ is assumed, the last equation reduces to

$$[\![t_{ik}^{(T)} \dot{x}_k - q_i]\!] n_i = 0 \quad \text{on } \sigma(t) . \tag{4.30}$$

v) <u>Entropy inequality:</u>

Substituting $\varphi = \rho \eta$, $\hat{\underset{\sim}{\tau}} = -\frac{1}{\theta} \underset{\sim}{q}$ and $\hat{g} = \rho \frac{r_\theta}{\theta}$ in (2.39) and making use of (4.10), one obtains

$$\rho \dot{\eta} \geq -(\tfrac{1}{\theta} q_i)_{,i} + \tfrac{1}{\theta} \rho r_\theta \quad \text{in } V - \sigma \tag{4.31}$$

and

$$[\![\rho \eta (\dot{x}_i - \dot{x}_i^*) + \tfrac{1}{\theta} q_i]\!] n_i \geq 0 \quad \text{on } \sigma(t) . \tag{4.32}$$

If one assumes that $\dot{\underset{\sim}{x}} = \dot{\underset{\sim}{x}}^*$, this reduces to

$$[\![\frac{1}{\theta} q_i]\!] n_i \geq 0 \qquad \text{on } \sigma(t) \,. \quad (4.33)$$

If ρr_θ is eliminated between (4.28) and (4.31), one obtains

$$\left.\begin{aligned} -\rho(\dot{\varepsilon} - \theta\dot{\eta}) + t_{ik}\dot{x}_{k,i} + \mathcal{J}_i^{(f)} \mathcal{E}_i + \rho \overset{*}{\dot{\mathcal{P}}_i} \mathcal{E}_i \\ + \rho\mu_0 \overset{*}{\dot{\mathcal{M}}_i} \mathcal{H}_i - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \end{aligned}\right\} \text{in } \mathcal{V} - \sigma \,. \quad (4.34)$$

For the thermally conductive material, it is advantageous to define Helmholtz free energy density by means of Legendré transformation

$$\Psi \equiv \varepsilon - \theta\eta \qquad (4.35)$$

The inequality (4.34) now takes the form

$$\left.\begin{aligned} -\rho(\dot{\Psi} + \eta\dot{\theta}) + t_{ik}\dot{x}_{k,i} + \mathcal{J}_i^{(f)} \mathcal{E}_i + \rho \overset{*}{\dot{\mathcal{P}}_i} \mathcal{E}_i \\ + \rho\mu_0 \overset{*}{\dot{\mathcal{M}}_i} \mathcal{H}_i - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \end{aligned}\right\} \text{in } \mathcal{V} - \sigma \,. \quad (4.36)$$

These basic balance equations and boundary conditions are to be supplemented by constitutive equations.

CHAPTER 5

CONSTITUTIVE EQUATIONS FOR POLARIZABLE AND MAGNETIZABLE MAGNETO-ELECTRO THERMO-VISCOELASTIC ANISOTROPIC SOLIDS HAVING MAGNETIC SYMMETRY WITH THERMAL AND ELECTRICAL CONDUCTION

The basic equations considered in the two previous chapters are valid for all types of media irrespective of their internal constitutions. The number of equations is inadequate for the determination of the unknowns except for some trivial situations. To them equations characterizing the material properties must be added.

Our objective in this chapter is to derive the basic constitutive relations of a relatively simple theory of magneto-electro thermo-viscoelastic anisotropic solids with thermal and electrical conduction undergoing finite deformation.

5.1. Résumé of the Fundamental Equations

The local form of the balance laws is listed below.

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + (\rho \dot{x}_k)_{,k} = 0$$

Balance of linear momentum:

$$\rho \ddot{x}_i - t_{ki,k} - \rho f_i = 0$$

Balance of moment of momentum:

$$\epsilon_{ijk} \, t^{(T)}_{kj} = 0$$

Balance of energy:

$$\rho \dot{\Psi} = t_{ik} \dot{x}_{k,i} + J_i^{(\mathcal{E})} \mathcal{E}_i + \rho \dot{\mathcal{P}}_i \, \mathcal{E}_i + \rho \mu_0 \dot{\mathcal{M}}_i \mathcal{H}_i$$
$$- \rho (\eta \theta)^{\cdot} - q_{i,i} + \rho r_\theta \qquad (5.1)$$

in $\mathcal{V}-\sigma$

Entropy inequality:

$$\rho\dot{\eta} + \left(\frac{1}{\theta} q_i\right)_{,i} - \frac{1}{\theta}\rho r_\theta \geqslant 0$$

Maxwell's equations:

$$\epsilon_0 E_{i,i} = \rho^{(f)} + \rho^{(p)} \quad ; \mu_0 H_{i,i} = \rho^{(m)}$$

$$\left.\begin{aligned} &\epsilon_{ijk} E_{k,j} + \mu_0 \frac{\partial H_i}{\partial t} = -J_i^{(m)} \\ &\epsilon_{ijk} H_{k,j} - \epsilon_0 \frac{\partial E_i}{\partial t} = J_i^{(f)} + J_i^{(p)} \end{aligned}\right\} \text{ in } \mathcal{S}-\gamma \qquad (5.2)$$

where

$$\rho^{(p)} \equiv - P_{i,i} \quad ; \quad \rho^{(m)} \equiv -\mu_0 M_{i,i} \qquad (5.3)$$

$$\begin{aligned} &J_i^{(p)} \equiv \frac{\partial P_i}{\partial t} + \epsilon_{ijk}\epsilon_{kmn}(P_m \dot{x}_n)_{,j} \\ &J_i^{(m)} \equiv \frac{\partial \mu_0 M_i}{\partial t} + \epsilon_{ijk}\epsilon_{kmn}(\mu_0 M_m \dot{x}_n)_{,j} \end{aligned} \qquad (5.4)$$

The charge and current densities in (5.3,4) satisfy the following continuity equations which are not independent of the Maxwell equations $(5.1)_{6,7}$ and (5.2)

$$\frac{\partial\rho^{(f)}}{\partial t} + J_{i,i}^{(f)} = 0 \quad ; \quad \frac{\partial\rho^{(p)}}{\partial t} + J_{i,i}^{(p)} = 0 \quad ; \quad \frac{\partial\rho^{(m)}}{\partial t} + J_{i,i}^{(m)} = 0 \quad . \qquad (5.5)$$

For the latter use, assuming that the velocities of the discontinuity surfaces and lines are equal to the velocity of the material particle and that the surface charge and the surface current densities vanish, one rewrites the boundary conditions (3.30) and (4.13,22,32) as

$$\left.\begin{aligned} &[\![t_{ki}^{(T)}]\!] n_k = 0 \quad ; \quad [\![t_{ik}^{(T)} \dot{x}_k - (\overset{+}{q}_i + q_i)]\!] n_i = 0 \\ &[\![\tfrac{1}{\theta} q_i]\!] n_i \geqslant 0 \\ &[\![\epsilon_0 E_i + P_i]\!] n_i = 0 \quad ; \quad \mu_0 [\![H_i + M_i]\!] n_i = 0 \end{aligned}\right\} \text{ on } \sigma(t) \qquad (5.6)$$

and

$$[\![\mathcal{E}_i]\!] k_i = 0 \; ; \; [\![\mathcal{H}_i]\!] k_i = 0 \qquad \text{on } \sigma(t). \tag{5.7}$$

Given mechanical force $f_i^{(M)}$, Eqs.(5.1-5) constitute 15 independent equations and one inequality for the determination of 35 unknowns: ρ, x_k, t_{kl}, q_k, E_k, H_k, P_k, M_k, $J_k^{(f)}$, $\rho^{(f)}$, ψ, θ and η. Thus, the problem is grossly underdetermined. Twenty additional equations must be provided to make the problem determinate.

5.2. Constitutive Relations for Magneto-electro Thermo-visco-elastic Solids with Thermal and Electrical Conduction

In this thesis, it is assumed that the constitutive equations for stress, electric and magnetic fields, entropy, electric current and energy flux vectors to be functions of deformation gradient, time rate of deformation gradient, polarization and magnetization per unit mass, temperature and temperature gradient. Instead of polarization and magnetization per unit volume, which varies due to deformation, those of per unit mass are taken as the constitutive variables. To these arguments, polarization and magnetization gradients can be added as is done partially in the works of Tiersten [5], Brown [7], Mindlin [8] and Şuhubi [9]. Moreover, if one is interested in the hysteretic effects, time rates of polarization and magnetization must be included (see, e.g. Coleman [70], Toupin and Rivlin [71]). The gradients and the time rates of polarization and magnetization are not taken into account since the exchange and hysteretic effects are excluded from the present research.

According to the axiom of equipresence, at the outset, all the constitutive responses are to be considered to depend on the same list of constitutive variables until the contrary is deduced. Thus, one has

$$\begin{aligned}
t_{kl}(\underset{\sim}{x},t) &= \hat{t}_{kl}(x_{m,K}\,,\,\dot{x}_{m,K}\,,\,\check{\mathcal{P}}_m\,,\,\mu_0\check{\mathcal{M}}_m\,,\,\theta,\,g_k\,,\,X_K) \\
\mathcal{E}_k(\underset{\sim}{x},t) &= \hat{\mathcal{E}}_k(\;\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\;) \\
\mathcal{H}_k(\underset{\sim}{x},t) &= \hat{\mathcal{H}}_k(\;\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\;) \qquad (5.8) \\
\mathcal{J}_k(\underset{\sim}{x},t) &= \hat{\mathcal{J}}_k(\;\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\quad\text{"}\;).
\end{aligned}$$

$$
\begin{aligned}
q_k(\underset{\sim}{x},t) &= \hat{q}_k\,(x_{m,K}\,,\dot{x}_{m,K}\,,\check{P}_m\,,\mu_o\check{M}_m\,,\theta,\,g_k,\,X_K\,) \\
\eta\,(\underset{\sim}{x},t) &= \hat{\eta}\;\;(\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;) \\
\Psi(\underset{\sim}{x},t) &= \hat{\Psi}\;\;(\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;\;''\;\;).
\end{aligned}
$$

In Eq.(5.8) $\hat{\underset{\sim}{t}}$ is tensor valued and $\hat{\underset{\sim}{\mathcal{E}}}$, $\hat{\underset{\sim}{\mathcal{H}}}$, $\hat{\underset{\sim}{J}}$, $\hat{\underset{\sim}{q}}$ are vector valued, and $\hat{\eta}$, $\hat{\Psi}$ are scalar valued functions of their arguments. Eq.(5.8) can also be written as

$$
\begin{aligned}
\underset{\sim}{t}(\underset{\sim}{x},t) &= \hat{\underset{\sim}{t}}\,(\underset{\sim}{F},\,\dot{\underset{\sim}{F}},\,\tfrac{1}{\rho}\underset{\sim}{P}\,,\,\tfrac{\mu_o}{\rho}\underset{\sim}{M}\,,\,\theta,\underset{\sim}{g},\underset{\sim}{X}\,) \\
\underset{\sim}{\mathcal{E}}(\underset{\sim}{x},t) &= \hat{\underset{\sim}{\mathcal{E}}}\;(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;) \\
\underset{\sim}{\mathcal{H}}(\underset{\sim}{x},t) &= \hat{\underset{\sim}{\mathcal{H}}}\;(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;) \\
\underset{\sim}{J}(\underset{\sim}{x},t) &= \hat{\underset{\sim}{J}}\;(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;) \\
\underset{\sim}{q}(\underset{\sim}{x},t) &= \hat{\underset{\sim}{q}}\;(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;) \\
\eta(\underset{\sim}{x},t) &= \hat{\eta}\;(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;) \\
\Psi(\underset{\sim}{x},t) &= \hat{\Psi}\;(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;)
\end{aligned}
\tag{5.9}
$$

where $\dot{\underset{\sim}{F}}$ is the material time derivative of $\underset{\sim}{F}$ defined by (2.7). These constitutive functionals are assumed to be differentiable with respect to their arguments.

According to the axiom of objectivity, the constitutive functions (5.8) or (5.9) must be form invariant under the orthogonal group of transformations. Therefore, an admissible process must remain admissible after a change of the frame of reference of the form

$$
\bar{x}\,(\underset{\sim}{X},\bar{t}\,) = \underset{\sim}{Q}(t)\;\underset{\sim}{x}\,(\underset{\sim}{X},t)\;+b(t) \tag{5.10}
$$

where $\underset{\sim}{Q}(t)$ is a proper orthogonal transformation, $\underset{\sim}{b}(t)$ is a translation and $\bar{t}$ is obtained from t by a constant shift of time:

$$
\underset{\sim}{Q}\,\underset{\sim}{Q}^t = \underset{\sim}{Q}^t\underset{\sim}{Q} = \underset{\sim}{I}\;\;;\;\;\det\underset{\sim}{Q} = 1\;\;;\;\;\bar{t} = t-a \tag{5.11}
$$

where "a" is a scalar constant.

The scalars θ , η and Ψ are unaffected by a change of frame, but the other quantities transform as follows:

$$\underset{\sim}{E}^* = \underset{\sim}{Q}\,\underset{\sim}{E} \quad ; \quad \check{\underset{\sim}{\mathcal{P}}}^* = \underset{\sim}{Q}\,\check{\underset{\sim}{\mathcal{P}}} \tag{5.12}$$
$$\check{\underset{\sim}{\mathcal{M}}}^* = \underset{\sim}{Q}\,\check{\underset{\sim}{\mathcal{M}}} \quad ; \quad \underset{\sim}{g}^* = \underset{\sim}{Q}\,\underset{\sim}{g}$$

and

$$\underset{\sim}{t}^* = \underset{\sim}{Q}\,\underset{\sim}{t}\,\underset{\sim}{Q}^t \quad ; \quad \underset{\sim}{\mathcal{E}}^* = \underset{\sim}{Q}\,\underset{\sim}{\mathcal{E}} \quad ; \quad \underset{\sim}{\mathcal{H}}^* = \underset{\sim}{Q}\,\underset{\sim}{\mathcal{H}}$$
$$\underset{\sim}{J}^* = \underset{\sim}{Q}\,\underset{\sim}{J} \quad ; \quad \underset{\sim}{q}^* = \underset{\sim}{Q}\,\underset{\sim}{q} \tag{5.13}$$

Thus, the axiom of objectivity is satisfied if

$$\begin{aligned}
\underset{\sim}{Q}\,\hat{\underset{\sim}{t}}(\underset{\sim}{F}, \dot{\underset{\sim}{F}}, \check{\underset{\sim}{\mathcal{P}}}, \mu_0\check{\underset{\sim}{\mathcal{M}}}, \theta, \underset{\sim}{g})\,\underset{\sim}{Q}^t &= \hat{\underset{\sim}{t}}(\underset{\sim}{F}^*, \dot{\underset{\sim}{F}}^*, \check{\underset{\sim}{\mathcal{P}}}^*, \mu_0\check{\underset{\sim}{\mathcal{M}}}^*, \theta, \underset{\sim}{g}^*)\\
\underset{\sim}{Q}\,\hat{\underset{\sim}{\mathcal{E}}}(\text{" " " " " "}) &= \hat{\underset{\sim}{\mathcal{E}}}(\text{" " " " " "})\\
\underset{\sim}{Q}\,\hat{\underset{\sim}{\mathcal{H}}}(\text{" " " " " "}) &= \hat{\underset{\sim}{\mathcal{H}}}(\text{" " " " " "})\\
\underset{\sim}{Q}\,\hat{\underset{\sim}{J}}(\text{" " " " " "}) &= \hat{\underset{\sim}{J}}(\text{" " " " " "})\\
\underset{\sim}{Q}\,\hat{\underset{\sim}{q}}(\text{" " " " " "}) &= \hat{\underset{\sim}{q}}(\text{" " " " " "})\\
\hat{\eta}(\text{" " " " " "}) &= \hat{\eta}(\text{" " " " " "})\\
\hat{\Psi}(\text{" " " " " "}) &= \hat{\Psi}(\text{" " " " " "}).
\end{aligned} \tag{5.14}$$

In writing Eq.(5.14), the material is assumed to be homogeneous. Restrictions imposed by (5.14) yield

$$\begin{aligned}
t_{kl} &= \hat{t}_{KL}(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \underset{\sim}{\Pi}, \mu_0\underset{\sim}{\mathfrak{M}}, \theta, \underset{\sim}{G})\,X_{K,k}\,X_{L,l}\\
\mathcal{E}_k &= \hat{\mathcal{E}}_K(\text{" " " " " "})\,X_{K,k}\\
\mathcal{H}_k &= \hat{\mathcal{H}}_K(\text{" " " " " "})\,X_{K,k}\\
J_k &= \hat{J}_K(\text{" " " " " "})\,X_{K,k}\\
q_k &= \hat{q}_K(\text{" " " " " "})\,X_{K,k}\\
\eta &= \hat{\eta}(\text{" " " " " "})\\
\Psi &= \hat{\Psi}(\text{" " " " " "})
\end{aligned} \tag{5.15}$$

where

$$
\begin{aligned}
\underline{t}_{KL} &= \hat{t}_{k\ell}\, x_{k,K}\, x_{\ell,L} \\
\underline{\mathcal{E}}_K &= \hat{\mathcal{E}}_k\, x_{k,K} \quad ; \quad \underline{\mathcal{H}}_K = \hat{\mathcal{H}}_k\, x_{k,K} \qquad (5.16)\\
\underline{J}_K &= \hat{J}_k\, x_{k,K} \quad ; \quad \underline{q} = \hat{q}_k\, x_{k,K}
\end{aligned}
$$

and

$$
\begin{aligned}
C_{KL} &= x_{k,K}\, x_{\ell,L} \quad ; \quad \dot{C}_{KL} = 2 d_{k\ell}\, x_{k,K}\, x_{\ell,L} \\
\Pi_K &= \check{P}_k\, x_{k,K} \; ; \; M_K = \check{\mathcal{M}}_k\, x_{k,K} \; ; \; G_K = g_k\, x_{k,K}\,. \qquad (5.17)
\end{aligned}
$$

The expressions in (5.17) are objective.

A more convenient form of (5.15) is obtained if one introduces quantities underlined by tilde as

$$
\begin{aligned}
&\underline{t}_{KL} = J^{-1}\, \underset{\sim}{\hat{t}}_{MN}\, C_{MK}\, C_{NL} \quad ; \quad \underline{\mathcal{E}}_K = J^{-1}\underset{\sim}{\hat{\mathcal{E}}}_L\, C_{LK} \\
&\underline{\mathcal{H}}_K = J^{-1}\underset{\sim}{\hat{\mathcal{H}}}_L\, C_{LK} \; ; \; \underline{J}_K = J^{-1}\underset{\sim}{\hat{J}}_L\, C_{LK} \; ; \; \underline{q}_K = J^{-1}\underset{\sim}{\hat{q}}_L\, C_{LK} \qquad (5.18)
\end{aligned}
$$

where J^{-1} is the inverse of Jacobian J defined earlier (2.10). Upon substituting (5.18) into (5.15), one obtains

$$
\begin{aligned}
t_{k\ell} &= J^{-1}\underset{\sim}{\hat{t}}_{KL}\,(\underset{\sim}{C}, \underset{\sim}{\dot{C}}, \underset{\sim}{\Pi}, \mu_0\underset{\sim}{M}, \theta, \underset{\sim}{G})\, x_{k,K}\, x_{\ell,L} \\
\mathcal{E}_k &= J^{-1}\underset{\sim}{\hat{\mathcal{E}}}_K\,(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;)\, x_{k,K} \\
\mathcal{H}_k &= J^{-1}\underset{\sim}{\hat{\mathcal{H}}}_K\,(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;)\, x_{k,K} \\
J_k &= J^{-1}\underset{\sim}{\hat{J}}_K\,(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;)\, x_{k,K} \qquad (5.19)\\
q_k &= J^{-1}\underset{\sim}{\hat{q}}_K\,(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;)\, x_{k,K} \\
\eta &= \hat{\eta}\,(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;) \\
\Psi &= \hat{\Psi}\,(\;''\;\;''\;\;''\;\;''\;\;''\;\;''\;)\,.
\end{aligned}
$$

The axiom of admissibility requires that (5.19) must be consistent with the principles of the conservation of mass and energy, the balance of momentum and the moment of momentum, the Clasius-Duhem inequality and the Maxwell equations.

Upon substituting Eq.(5.19) into (4.36), one has

$$
-\rho\left(\frac{\partial\hat{\Psi}}{\partial C_{KL}}\dot{C}_{KL} + \frac{\partial\hat{\Psi}}{\partial\dot{C}_{KL}}\ddot{C}_{KL} + \frac{\partial\hat{\Psi}}{\partial\Pi_K}\dot{\Pi}_K + \frac{\partial\hat{\Psi}}{\partial M_K}\dot{M}_K + \frac{\partial\hat{\Psi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\Psi}}{\partial G_K}\dot{G}_K\right)
$$

(continued)

$$-\rho\hat{\eta}\dot{\theta}+\frac{\rho}{\rho_0}\Big[\hat{\underset{\sim}{t}}_{KL}\,x_{l,L}\,v_{l,k}+\frac{\rho}{\rho_0}\hat{\underset{\sim}{\mathcal{E}}}_K\hat{\underset{\sim}{J}}_L\,x_{k,K}+\rho\hat{\underset{\sim}{\mathcal{E}}}_K(\mathring{\Pi}_L X_{L,k}+\Pi_L\dot{X}_{L,k})+\mu_0\rho\hat{\underset{\sim}{\mathcal{H}}}_K(\dot{M}_L X_{L,k}+M_L\dot{X}_{L,k})-\frac{1}{\theta}\hat{\underset{\sim}{q}}_K G_L X_{L,k}\Big]x_{k,K}\geq 0\,. \tag{5.20}$$

Next, using (2.8) and $(2.12)_1$, one obtains

$$\underset{\sim}{\mathcal{E}}_K\,x_{k,K}=\underset{\sim}{\mathcal{E}}_K\,X_{L,k}\,C_{LK} \quad . \tag{5.21}$$

By inserting Eq.(5.21) into (5.20), and using (2.28) there follows

$$\begin{aligned}&-\rho\frac{\partial\hat{\Psi}}{\partial C_{KL}}\dot{C}_{KL}-\rho\frac{\partial\hat{\Psi}}{\partial\dot{C}_{KL}}\ddot{C}_{KL}-\rho\frac{\partial\hat{\Psi}}{\partial\Pi_K}\mathring{\Pi}_K+\frac{\rho^2}{\rho_0}\hat{\underset{\sim}{\mathcal{E}}}_K\mathring{\Pi}_K-\rho\frac{\partial\hat{\Psi}}{\partial M_K}\dot{M}_K\\&+\frac{\rho^2}{\rho_0}\mu_0\hat{\underset{\sim}{\mathcal{H}}}_K\dot{M}_K-\rho\frac{\partial\hat{\Psi}}{\partial\theta}\dot{\theta}-\rho\hat{\eta}\dot{\theta}-\frac{\partial\hat{\Psi}}{\partial G_K}\dot{G}_K+\frac{\rho}{\rho_0}\hat{\underset{\sim}{t}}_{KL}\,x_{k,K}\,x_{l,L}\,v_{l,k}\\&+\left(\frac{\rho}{\rho_0}\right)^2\hat{\underset{\sim}{\mathcal{E}}}_K\hat{\underset{\sim}{J}}_L C_{KL}-\frac{\rho^2}{\rho_0}\left(\hat{\underset{\sim}{\mathcal{E}}}_K\Pi_L+\mu_0\hat{\underset{\sim}{\mathcal{H}}}_K M_L\right)X_{L,l}\,x_{k,K}\,v_{l,k}\\&-\frac{1}{\theta}\frac{\rho}{\rho_0}\hat{\underset{\sim}{q}}_K G_K\geq 0 \quad .\end{aligned} \tag{5.22}$$

Now making use of $(2.31)_1$, Eq.(5.22) is rearranged as

$$\begin{aligned}&\rho\left(\frac{\partial\hat{\Psi}}{\partial\theta}+\hat{\eta}\right)\dot{\theta}+\rho\left(-\frac{\partial\hat{\Psi}}{\partial\Pi_K}+\frac{\rho}{\rho_0}\hat{\underset{\sim}{\mathcal{E}}}_K\right)\mathring{\Pi}_K+\rho\left(-\frac{\partial\hat{\Psi}}{\partial M_K}+\frac{\rho}{\rho_0}\mu_0\hat{\underset{\sim}{\mathcal{H}}}_K\right)\dot{M}_K\\&-\rho\frac{\partial\hat{\Psi}}{\partial G_K}\dot{G}_K-\rho\frac{\partial\hat{\Psi}}{\partial\dot{C}_{KL}}\ddot{C}_{KL}-\frac{2\rho}{\rho_0}\left(\rho_0\frac{\partial\hat{\Psi}}{\partial C_{KL}}-\frac{1}{2}\hat{\underset{\sim}{t}}_{KL}\right)x_{k,K}\,x_{l,L}\,v_{l,k}\\&-\frac{\rho^2}{\rho_0}\left(\hat{\underset{\sim}{\mathcal{E}}}_K\Pi_L+\mu_0\hat{\underset{\sim}{\mathcal{H}}}_K M_L\right)X_{L,l}\,x_{k,K}\,v_{l,k}+\left(\frac{\rho}{\rho_0}\right)^2\hat{\underset{\sim}{\mathcal{E}}}_K\hat{\underset{\sim}{J}}_L C_{KL}\\&-\frac{\rho}{\rho_0}\frac{1}{\theta}\hat{\underset{\sim}{q}}_K G_K\geq 0 \quad .\end{aligned} \tag{5.23}$$

The actual stress tensor can now be split into two parts:

$$\underset{\sim}{t}_{KL}\equiv{}_{(ND)}t_{KL}+{}_{(D)}t_{KL} \tag{5.24}$$

where

$$ {}_{(ND)}\underset{\sim}{t}_{KL} = {}_{(ND)}\hat{\underset{\sim}{t}}_{KL}\,(\underset{\sim}{C}, \underset{\sim}{\Pi}, \mu_o \underset{\sim}{\mathcal{M}}, \theta) \tag{5.25} $$

and

$$ {}_{(D)}\underset{\sim}{t}_{KL} = {}_{(D)}\hat{\underset{\sim}{t}}_{KL}\,(\underset{\sim}{C}, \dot{\underset{\sim}{C}}, \underset{\sim}{\Pi}, \mu_o \underset{\sim}{\mathcal{M}}, \theta, \underset{\sim}{G}) \quad . \tag{5.26} $$

These are called the nondissipative (reversible) and dissipative (irreversible) parts of the actual stress tensor respectively.

As a result of (5.24), Eq.(5.23) is rewritten in an equivalent form

$$ \begin{aligned} &\rho\left(\frac{\partial\hat{\Psi}}{\partial\theta} + \hat{\eta}\right)\dot{\theta} + \rho\left(-\frac{\partial\hat{\Psi}}{\partial\Pi_K} + \frac{\rho}{\rho_o}\hat{\underset{\sim}{\mathcal{E}}}_K\right)\dot{\Pi}_K + \rho\left(-\frac{\partial\hat{\Psi}}{\partial\mathcal{M}_K} + \frac{\rho}{\rho_o}\hat{\underset{\sim}{\mathcal{H}}}_K\right)\dot{\mathcal{M}}_K \\ &- \rho\frac{\partial\hat{\Psi}}{\partial G_K}\dot{G}_K - \rho\frac{\partial\hat{\Psi}}{\partial\dot{C}_{KL}}\ddot{C}_{KL} + \frac{\rho}{2\rho_o}\Big(-2\rho_o\frac{\partial\hat{\Psi}}{\partial C_{KL}} + {}_{(ND)}\underset{\sim}{t}_{KL} \\ &- \rho\hat{\underset{\sim}{\mathcal{E}}}_K \Pi_M X_{L,m} X_{M,m} - \rho\mu_o\hat{\underset{\sim}{\mathcal{H}}}_K \mathcal{M}_M X_{L,m} X_{M,m}\Big)\dot{C}_{KL} \\ &+ \frac{\rho}{\rho_o}\Big({}_{(D)}\hat{\underset{\sim}{t}}_{KL}\, x_{k,K}\, x_{\ell,L}\, v_{\ell,k} + \frac{\rho}{\rho_o}\hat{\underset{\sim}{\mathcal{E}}}_K \hat{\underset{\sim}{\mathcal{J}}}_L C_{KL} - \frac{1}{\theta}\hat{q}_K G_K\Big) \geqslant 0 \;. \end{aligned} \tag{5.27} $$

This inequality implies that

$$ \frac{\partial\hat{\Psi}}{\partial G_K} = 0 \quad ; \quad \frac{\partial\hat{\Psi}}{\partial\dot{C}_{KL}} = 0 \tag{5.28} $$

$$ \begin{aligned} &\hat{\eta} = -\frac{\partial\hat{\Psi}}{\partial\theta} \quad ; \quad \hat{\underset{\sim}{\mathcal{E}}}_K = \frac{\rho}{\rho_o}\frac{\partial\hat{\Psi}}{\partial\Pi_K} \quad ; \quad \mu_o\hat{\underset{\sim}{\mathcal{H}}}_K = \frac{\rho}{\rho_o}\frac{\partial\hat{\Psi}}{\partial\mathcal{M}_K} \\ &{}_{(ND)}\hat{\underset{\sim}{t}}_{KL} = 2\rho_o\frac{\partial\hat{\Psi}}{\partial C_{KL}} + \rho\left(\hat{\underset{\sim}{\mathcal{E}}}_K \Pi_M + \mu_o\hat{\underset{\sim}{\mathcal{H}}}_K \mathcal{M}_M\right) X_{M,m} X_{L,m} \end{aligned} \tag{5.29} $$

and

$$ {}_{(D)}\hat{\underset{\sim}{t}}_{KL}\, x_{k,K}\, x_{\ell,L}\, v_{\ell,k} + \frac{\rho}{\rho_o}\hat{\underset{\sim}{\mathcal{E}}}_K \hat{\underset{\sim}{\mathcal{J}}}_L C_{KL} - \frac{1}{\theta}\hat{q}_K G_K \geqslant 0 \,. \tag{5.30} $$

From (5.28), it is easily seen that $\Psi = \hat{\Psi}(\underset{\sim}{C}, \underset{\sim}{\Pi}, \mu_o\underset{\sim}{\mathcal{M}}, \theta)$, that is, Ψ does not depent on the rate of deformation and temperature gradient.

Now making use of (5.19) and (5.25), Eq.(5.29) reduces to

$$ \eta = -\frac{\partial\hat{\Psi}}{\partial\theta} \quad ; \quad \mathcal{E}_k = \frac{\partial\hat{\Psi}}{\partial\Pi_K} x_{k,K} \quad ; \quad \mu_o\mathcal{H}_k = \frac{\partial\hat{\Psi}}{\partial\mathcal{M}_K} x_{k,K} \tag{5.31} $$

$$_{(ND)}t_{kl} = \rho \frac{\partial \hat{\Psi}}{\partial E_{KL}} x_{k,K} x_{l,L} + \mathcal{E}_k \mathcal{P}_l + \mu_0 \mathcal{H}_k \mathcal{M}_l$$

where $(2.15)_1$ has been used. The entropy, electric and magnetic fields and the nondissipative part of the actual stress are expressed once the Helmholtz free energy Ψ is specified.

From $(5.31)_4$, it follows that

$$_{(ND)}t_{[kl]} = \rho \frac{\partial \hat{\Psi}}{\partial E_{KL}} x_{[k,K} x_{l],L} + \mathcal{E}_{[k} \mathcal{P}_{l]} + \mu_0 \mathcal{H}_{[k} \mathcal{M}_{l]} \quad (5.32)$$

and the first term at the right hand side of (5.32) vanishes since $E_{KL} = E_{LK}$. Imposing the restrictions coming from (4.28) together with (3.67) into (5.32) and (5.24), one finds

$$_{(D)}t_{[kl]} = 0 \quad (5.33)$$

that is, $_{(D)}\underset{\sim}{t}$ is a symmetric tensor. Henceforth, one has

$$_{(D)}\underset{\sim}{t}_{KL} x_{k,K} x_{l,L} v_{l,k} = {}_{(D)}\underset{\sim}{\hat{t}}_{KL} \frac{1}{2}(v_{k,l}+v_{l,k}) x_{k,K} x_{l,L} = {}_{(D)}\underset{\sim}{\hat{t}}_{KL} \dot{E}_{KL} . \quad (5.34)$$

Using (5.34) and $(2.15)_1$, the inequality (5.30) now becomes

$$\emptyset \equiv {}_{(D)}\hat{t}_{KL} \dot{E}_{KL} + \hat{\mathcal{E}}_k X_{K,k} (2E_{KL} + \delta_{KL}) \underset{\sim}{\hat{J}}_L - \frac{1}{\theta} \underset{\sim}{\hat{q}}_K G_K \geqslant 0 . \quad (5.35)$$

To proceed further, one needs the Helmholtz energy explicitly.

5.3. Polynomial Approximation of the Free Energy

Letting the free energy Ψ be an analytic function of the components E_{KL}, Π_K, $\mu_0 \mathcal{M}_K$ and the scalar θ and expanding $\Psi = \hat{\Psi}(E_{KL}, \Pi_K, \mu_0 \mathcal{M}_K, \theta)$ at the point of natural state into the Taylor series, one obtains the constitutive equations (5.32) explicitly. In the natural state of the body, it is assumed that $E_{KL} = 0$, $\Pi_K = 0$, $\mu_0 \mathcal{M}_K = 0$ and $\theta = 0$. Thus, one has

$$\Psi(\cdot) = \hat{\Psi}_0 + \hat{\Sigma}^{(e)}_{KL} E_{KL} + \hat{\Sigma}^{(p)}_{K} \Pi_K + \hat{\Sigma}^{(m)}_{K} \mathcal{M}_K + \hat{\Sigma}'^{(\theta)} \theta$$

$$+ \frac{1}{2\rho_0} \hat{\Sigma}^{(e)}_{KLMN} E_{KL} E_{MN} + \frac{1}{2}\rho_0 \hat{\Sigma}^{(p)}_{KL} \Pi_K \Pi_L + \frac{1}{2}\rho_0 \hat{\Sigma}^{(m)}_{KL} \mathcal{M}_K \mathcal{M}_L$$

(continued)

$$
\begin{aligned}
&+\frac{1}{2}\hat{\Sigma}^{(\theta)}\theta^2+\hat{\Sigma}^{(ep)}_{KLM}E_{KL}\Pi_M+\hat{\Sigma}^{(em)}_{KLM}E_{KL}\mathcal{M}_L \\
&+\hat{\Sigma}^{(e\theta)}_{KL}E_{KL}\theta+\rho_0\hat{\Sigma}^{(pm)}_{KL}\Pi_K\mathcal{M}_L+\hat{\Sigma}^{(p\theta)}_{K}\Pi_K\theta+\hat{\Sigma}^{(m\theta)}_{K}\mathcal{M}_K\theta \\
&+\frac{1}{6\rho_0}\hat{\xi}^{(e)}_{KLMNPQ}E_{KL}E_{MN}E_{PQ}+\frac{1}{6}\hat{\xi}^{(p)}_{KLM}\Pi_K\Pi_L\Pi_M \\
&+\frac{1}{6}\hat{\xi}^{(m)}_{KLM}\mathcal{M}_K\mathcal{M}_L\mathcal{M}_M+\frac{1}{6}\hat{\xi}^{(\theta)}\theta^3+\hat{\xi}^{(ep)}_{KLMNP}E_{KL}E_{MN}\Pi_P \qquad (5.36)\\
&+\hat{\xi}^{(ep)}_{KLMN}E_{KL}\Pi_M\Pi_N+\hat{\xi}^{(em)}_{KLMNP}E_{KL}E_{MN}\mathcal{M}_P \\
&+\hat{\xi}^{(em)}_{KLMN}E_{KL}\mathcal{M}_M\mathcal{M}_N+\hat{\xi}^{(e\theta)}_{KLMN}E_{KL}E_{MN}\theta+\hat{\xi}^{(e\theta)}_{KL}E_{KL}\theta^2 \\
&+\hat{\xi}^{(pm)}_{KLM}\Pi_K\Pi_L\mathcal{M}_M+\hat{\xi}^{(mp)}_{KLM}\Pi_K\mathcal{M}_L\mathcal{M}_M+\hat{\xi}^{(p\theta)}_{KL}\Pi_K\Pi_L\theta \\
&+\hat{\xi}^{(p\theta)}_{K}\Pi_K\theta^2+\hat{\xi}^{(m\theta)}_{KL}\mathcal{M}_K\mathcal{M}_L\theta+\hat{\xi}^{(m\theta)}_{K}\mathcal{M}_K\theta^2 \\
&+\hat{\xi}^{(epm)}_{KLMN}E_{KL}\Pi_M\mathcal{M}_N+\hat{\xi}^{(ep\theta)}_{KLM}E_{KL}\Pi_M\theta \\
&+\hat{\xi}^{(em\theta)}_{KLM}E_{KL}\mathcal{M}_M\theta+\hat{\xi}^{(pm\theta)}_{KL}\Pi_K\mathcal{M}_L\theta+\cdots\cdots\cdots
\end{aligned}
$$

where

$$
\begin{aligned}
&\hat{\Psi}=\hat{\Psi}(0,0,0,0)\ ;\ \hat{\Sigma}^{(e)}_{KL}=\left(\frac{\partial\hat{\Psi}}{\partial E_{KL}}\right)_0\ ;\ \hat{\Sigma}^{(p)}_{K}=\left(\frac{\partial\hat{\Psi}}{\partial\Pi_K}\right)_0 \\
&\hat{\Sigma}^{(m)}_{K}=\left(\frac{\partial\hat{\Psi}}{\partial\mathcal{M}_K}\right)_0\ ;\ \hat{\Sigma}^{(\theta)'}=\left(\frac{\partial\hat{\Psi}}{\partial\theta}\right)_0\ ;\ \hat{\Sigma}^{(e)}_{KLMN}=\rho_0\left(\frac{\partial^2\hat{\Psi}}{\partial E_{KL}\partial E_{MN}}\right)_0 \\
&\hat{\Sigma}^{(p)}_{KL}=\frac{1}{\rho_0}\left(\frac{\partial^2\hat{\Psi}}{\partial\Pi_K\partial\Pi_L}\right)_0\ ;\ \hat{\Sigma}^{(m)}_{KL}=\frac{1}{\rho_0}\left(\frac{\partial^2\hat{\Psi}}{\partial\mathcal{M}_K\partial\mathcal{M}_L}\right)_0 \qquad (5.37)\\
&\hat{\Sigma}^{(\theta)}=\left(\frac{\partial^2\hat{\Psi}}{\partial\theta^2}\right)_0\ ;\ \hat{\Sigma}^{(ep)}_{KLN}=\left(\frac{\partial^2\hat{\Psi}}{\partial E_{KL}\partial\Pi_N}\right)_0\ ;\ \hat{\Sigma}^{(em)}_{KLN}=\left(\frac{\partial^2\hat{\Psi}}{\partial E_{KL}\partial\mathcal{M}_N}\right)_0 \\
&\hat{\Sigma}^{(e\theta)}_{KL}=\left(\frac{\partial^2\hat{\Psi}}{\partial E_{KL}\partial\theta}\right)_0\ ;\ \hat{\Sigma}^{(pm)}_{KL}=\frac{1}{\rho_0}\left(\frac{\partial^2\hat{\Psi}}{\partial\Pi_K\partial\mathcal{M}_L}\right)_0
\end{aligned}
$$

$$\hat{\Sigma}_K^{(p\theta)} = \left(\frac{\partial^2 \hat{\Psi}}{\partial \Pi_K \partial \theta}\right)_0 \quad ; \quad \hat{\Sigma}_K^{(m\theta)} = \left(\frac{\partial^2 \hat{\Psi}}{\partial \mathcal{M}_K \partial \theta}\right)_0$$

and

$$\hat{\xi}_{KLMNPQ}^{(e)} = \rho_0 \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial E_{MN} \partial E_{PQ}}\right)_0 \quad ; \quad \hat{\xi}_{KLM}^{(p)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \Pi_K \partial \Pi_L \partial \Pi_M}\right)_0$$

$$\hat{\xi}_{KLM}^{(m)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \mathcal{M}_K \partial \mathcal{M}_L \partial \mathcal{M}_M}\right)_0 \quad ; \quad \hat{\xi}^{(\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \theta^3}\right)_0$$

$$\hat{\xi}_{KLMNP}^{(ep)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial E_{MN} \partial \Pi_P}\right)_0 \quad ; \quad \hat{\xi}_{KLMN}^{(ep)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial \Pi_M \partial \Pi_N}\right)_0$$

$$\hat{\xi}_{KLMN}^{(e\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial E_{MN} \partial \theta}\right)_0 \quad ; \quad \hat{\xi}_{KL}^{(e\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial \theta^2}\right)_0$$

$$\hat{\xi}_{KLM}^{(pm)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \Pi_K \partial \Pi_L \partial \mathcal{M}_M}\right)_0 \quad ; \quad \hat{\xi}_{KLN}^{(mp)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \Pi_K \partial \mathcal{M}_L \partial \mathcal{M}_N}\right)_0$$

$$\hat{\xi}_{KL}^{(p\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \Pi_K \partial \Pi_L \partial \theta}\right)_0 \quad ; \quad \hat{\xi}_{K}^{(p\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \Pi_K \partial \theta^2}\right)_0 \tag{5.38}$$

$$\hat{\xi}_{KL}^{(m\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \mathcal{M}_K \partial \mathcal{M}_L \partial \theta}\right)_0 \quad ; \quad \hat{\xi}_{K}^{(m\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \mathcal{M}_K \partial \theta^2}\right)_0$$

$$\hat{\xi}_{KLMNP}^{(em)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial E_{MN} \partial \mathcal{M}_P}\right)_0 \quad ; \quad \hat{\xi}_{KLMN}^{(em)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial \mathcal{M}_M \partial \mathcal{M}_N}\right)_0$$

$$\hat{\xi}_{KLMN}^{(epm)} = \left(\frac{\partial^3 \Psi}{\partial E_{KL} \partial \Pi_M \partial \mathcal{M}_N}\right)_0 \quad ; \quad \hat{\xi}_{KLM}^{(ep\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial \Pi_M \partial \theta}\right)_0$$

$$\hat{\xi}_{KLM}^{(em\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial E_{KL} \partial \mathcal{M}_M \partial \theta}\right)_0 \quad ; \quad \hat{\xi}_{KL}^{(pm\theta)} = \left(\frac{\partial^3 \hat{\Psi}}{\partial \Pi_K \partial \mathcal{M}_L \partial \theta}\right)_0 .$$

In Eq.(5.38), the symbol $(.)_o$ denotes the derivative evaluated at the point $E_{KL}=\Pi_K=\mu_o \mathcal{M}_K=\theta=0$. The material property tensors $\hat{\underset{\sim}{\Sigma}}^{(\cdot)}$ and $\hat{\underset{\sim}{\xi}}^{(\cdot)}$ have certain intrinsic symmetries to be discussed later.

In view of (5.38) and (5.31), one obtains anisotropic, nonlinear constitutive equations for electric and magnetic fields, entropy and the nondissipative part of the stress. The linearized constitutive equations are obtained if the terms of order up to and including two in the energy expansion (5.36) are retained. Moreover, it is assumed that in the natural state, the free energy Ψ_o vanishes. Thus, the free energy now becomes

$$\begin{aligned}\Psi(\cdot) = {} & \hat{\Sigma}^{(e)}_{KL} E_{KL} + \hat{\Sigma}^{(p)}_{K} \Pi_K + \hat{\Sigma}^{(m)}_{K} \mathcal{M}_K + \hat{\Sigma}^{\prime(\theta)} \theta + \frac{1}{2\rho_o} \hat{\Sigma}^{(e)}_{KLMN} E_{KL} E_{MN} \\ & + \frac{1}{2}\rho_o \hat{\Sigma}^{(p)}_{KL} \Pi_K \Pi_L + \frac{1}{2}\rho_o \hat{\Sigma}^{(m)}_{KL} \mathcal{M}_K \mathcal{M}_L \\ & + \frac{1}{2}\hat{\Sigma}^{(\theta)} \theta^2 + \hat{\Sigma}^{(ep)}_{KLM} E_{KL} \Pi_M + \hat{\Sigma}^{(em)}_{KLM} E_{KL} \mathcal{M}_M \\ & + \hat{\Sigma}^{(e\theta)}_{KL} E_{KL}\theta + \rho_o \hat{\Sigma}^{(pm)}_{KL} \Pi_K \mathcal{M}_L + \hat{\Sigma}^{(p\theta)}_{K} \Pi_K \theta + \hat{\Sigma}^{(m\theta)}_{K} \mathcal{M}_K \theta\end{aligned} \tag{5.39}$$

where the material property tensors have the following intrinsic symmetries:

$$\begin{aligned} & \hat{\Sigma}^{(e)}_{KL} = \hat{\Sigma}^{(e)}_{LK} \quad ; \quad \hat{\Sigma}^{(e)}_{KLMN} = \hat{\Sigma}^{(e)}_{LKMN} = \hat{\Sigma}^{(e)}_{KLNM} = \hat{\Sigma}^{(e)}_{MNKL} \\ & \hat{\Sigma}^{(p)}_{KL} = \hat{\Sigma}^{(p)}_{LK} \quad ; \quad \hat{\Sigma}^{(m)}_{KL} = \hat{\Sigma}^{(m)}_{LK} \quad ; \quad \hat{\Sigma}^{(e\theta)}_{KL} = \hat{\Sigma}^{(e\theta)}_{LK} \\ & \hat{\Sigma}^{(ep)}_{KLM} = \hat{\Sigma}^{(ep)}_{LKM} \quad ; \quad \hat{\Sigma}^{(em)}_{KLM} = \hat{\Sigma}^{(em)}_{LKM} \ . \end{aligned} \tag{5.40}$$

From Eq.(5.31), if in the natural state of the body the initial electric field, magnetic field, entropy and the initial stress vanish, then the material property tensors are taken to be

$$\hat{\Sigma}^{(p)}_{K} = \hat{\Sigma}^{(m)}_{K} = \hat{\Sigma}^{\prime(\theta)} = \hat{\Sigma}^{(e)}_{KL} = 0 \tag{5.41}$$

in the energy expression (5.39). Thus, the free energy is further simplified to be

$$\Psi = \frac{1}{2\rho_0}\hat{\sum}^{(e)}_{KLMN} E_{KL}E_{MN} + \frac{1}{2}\rho_0 \hat{\sum}^{(p)}_{KL} \Pi_K \Pi_L + \frac{1}{2}\rho_0 \hat{\sum}^{(m)}_{KL} \mathcal{M}_K \mathcal{M}_L$$
$$+ \frac{1}{2}\hat{\sum}^{(\theta)} \theta^2 + \hat{\sum}^{(e\theta)}_{KL} E_{KL}\theta + \rho_0 \hat{\sum}^{(pm)}_{KL} \Pi_K \mathcal{M}_L \tag{5.42}$$
$$+ \hat{\sum}^{(ep)}_{KLM} E_{KL}\Pi_M + \hat{\sum}^{(em)}_{KLM} E_{KL}\mathcal{M}_M + \hat{\sum}^{(p\theta)}_{K} \Pi_K\theta + \hat{\sum}^{(m\theta)}_{K} \mathcal{M}_K\theta$$

of which different coefficients are often named as the "elastic", "electric anisotropy", "magnetic anisotropy", "thermal", "thermo-elastic", "magneto-electric", "piezoelectric", "piezomagnetic", "pyroelectric" and "pyromagnetic" constants. There are theoretical and experimental evidences supporting the interaction phenomena due to these material property tensors, e.g., Landau and Lifshitz [72], Dzyaloshinskii [22], Astrov [23], O'Dell [21], Bhagavantam [20] and Nye [73].

The magneto-electric effect which is due to the term $\hat{\sum}^{(pm)}_{KL}\Pi_K\mathcal{M}_L$ occours in magnetic crystals only. The possibility of the occurence of this effect has already been suggested by Landau and Lifshitz [72]. Later, Dzyaloshinskii [22] proposed a potential for isothermal rigid magneto-electric materials which is the special case of (5.42). The magneto-electric effect has also been confirmed experimentally by Astrov [23], Al'shin and Astrov [24] and O'Dell [21].

The piezomagnetic effect which is due to the term $\hat{\sum}^{(em)}_{KLM}E_{KL}\mathcal{M}_M$ occurs also in certain magnetic crystals. This effect has actually been discovered and experimentally measured by Borovik-Romonov [25] in CoF_2 and MnF_2 crystals.

The pyromagnetic effect is concerned with the changes in the magnetic moment in a magnetic crystal caused by the introduction of a temperature change. This effect is taken into account through the term $\hat{\sum}^{(m\theta)}_{K}\mathcal{M}_K\theta$. Thirty one of the 90 magnetic crystal classes are potentially capable of exhibiting this effect and its converse, magneto-caloric effect [20].

These three interaction phenomena, namely magneto-electricity, piezomagnetism and pyromagnetism are characterized by time-antisymmetric tensors (or c-tensors) since magnetic moment is time-antisymmetric. However, the material property tensors

$\hat{\sum}^{(e)}_{KLMN}$, $\hat{\sum}^{(p)}_{KL}$, $\hat{\sum}^{(m)}_{KL}$, $\hat{\sum}^{(\theta)}$, $\hat{\sum}^{(e\theta)}_{KL}$, $\hat{\sum}^{(ep)}_{KLM}$ and $\hat{\sum}^{(p\theta)}_{K}$ are i-tensors since the variables, except magnetization $\mathcal{M}$, are time-symmetric. One arrives at this conclusion according to the "product rule" which is analogous to the rule $(\pm).(\pm) = (+)$ and $(-).(+) = (-)$. That is, in a physically correct expression or a constitutive equation if both field tensors are i-tensors or c-tensors, the material property tensor is an i-tensor; and if one of the tensors is an i-tensor and the other a c-tensor, the material property tensor is a c-tensor.

The energy term $\hat{\sum}^{(ep)}_{KLM} \varepsilon_{KL} \Pi_M$ in (5.42) is associated with the piezoelectric effect. That is, when a stress is applied to certain crystals, an electric polarization is developed, or vice versa.

The pyroelectric effect is taken into account if one considers the energy term $\hat{\sum}^{(p\theta)}_{K} \Pi_K \theta$ in (5.42). Certain crystals have the property of developing an electric field when their temperature is changed.

The physical phenomena associated with the first five energy terms in (5.42) are well known in literature, since they exist in isotropic materials as well as anisotropic ones. Therefore, we do not need any discussion.

In addition to the assumption (5.41), there may be two additional assumptions: i) the materials belong to the 32 out of the 90 crystallographic magnetic point groups, ii) the materials belong to those of the 32 crystallographic conventional point groups which contain the inversion operator.

Because of assumption (i), the material property tensors in (5.42) are

$$\hat{\sum}^{(m\theta)}_{K} = \hat{\sum}^{(pm)}_{KL} = \hat{\sum}^{(em)}_{KLM} = 0 \quad . \tag{5.43}$$

Since $\mathcal{M}$ is an axial c-tensor and the Helmholtz free energy is a true (polar) i-tensor, these material property tensors have nonzero components if and only if the conventional symmetry elements $S^i \in \{S\}$ are in combination with the time-inversion element $\mathcal{R}$ $(\mathcal{R} : t \to -t)$, i.e., $\mathcal{R}S^i = \underline{S}^i \in \{\mathcal{M}\}$. Where $\{S\}$ and $\{\mathcal{M}\}$ are, respectively, conventional and magnetic point groups. Hence, the 90 crystallographic magnetic point groups are denoted as $\{\mathcal{M}\} = \{S\} \oplus \{\mathcal{M}'\}$ where $\{\mathcal{M}'\}$ is the 58 additional groups. There are theoretical and experimental evidences that the material tensors $\hat{\Sigma}_{K}^{(m\theta)}$, $\hat{\Sigma}_{KL}^{(pm)}$ and $\hat{\Sigma}_{KLM}^{(em)}$ have nonzero components for the materials the symmetry elements of which belong to $\{\mathcal{M}'\}$; for example, ferromagnetics, ferrimagnetics and certain antiferromagnetics [20,74].

Because of assumption (ii), the material property tensors of odd orders, and c-tensors of odd and even orders in (5.42) are zero, i.e.,

$$\hat{\Sigma}_{K}^{(e\theta)} = \hat{\Sigma}_{KLM}^{(ep)} = \hat{\Sigma}_{K}^{(m\theta)} = \hat{\Sigma}_{KLM}^{(em)} = \hat{\Sigma}_{KL}^{(pm)} = 0 \quad . \tag{5.44}$$

Tiersten [5], Midlin [8], Pao and Yeh [13], and Hutter and Pao [14] have been assumed that (5.44) is satisfied. This assumption is valid for the materials whose crystallographic point groups belong to $\{S\}$. Therefore, the constitutive equations based on the conventional 32 crystallographic point groups and those having the inversion operator (centrosymmetric crystals) do not comprise the general anisotropic materials.

5.4. Linear Constitutive Equations for Anisotropic Solids Having Magnetic Symmetry

The linear constitutive equations for magneto-electro thermo-viscoelastic anisotropic solids are obtained through (5.31) and (5.42). Taking the associated partial derivatives of (5.42) and substituting the resulting expressions into (5.31), one obtains

$$\eta(\underset{\sim}{x},t) = -\hat{\Sigma}^{(\theta)}\theta - \hat{\Sigma}_{KL}^{(e\theta)}E_{KL} - \frac{1}{\rho}\left(\hat{\Sigma}_{K}^{(p\theta)}P_k + \hat{\Sigma}_{K}^{(m\theta)}\mathcal{M}_k\right)x_{k,K}$$

$$\mathcal{E}_k(\underset{\sim}{x},t) = \left[\frac{\rho_0}{\rho}\left(\hat{\Sigma}_{KL}^{(p)}P_\ell + \hat{\Sigma}_{KL}^{(pm)}\mathcal{M}_\ell\right)x_{\ell,L} + \hat{\Sigma}_{MNK}^{(ep)}E_{MN} + \hat{\Sigma}_{K}^{(p\theta)}\theta\right]x_{k,K} \tag{5.45}$$

$$\mu_0\mathcal{H}_k(\underset{\sim}{x},t) = \left[\frac{\rho_0}{\rho}\left(\hat{\Sigma}_{LK}^{(pm)}P_\ell + \hat{\Sigma}_{KL}^{(m)}\mathcal{M}_\ell\right)x_{\ell,L} + \hat{\Sigma}_{MNK}^{(em)}E_{MN} + \hat{\Sigma}_{K}^{(m\theta)}\theta\right]x_{k,K}$$

and

$$_{(ND)}t_{k\ell}(\underset{\sim}{x},t) = \Big[\frac{\rho}{\rho_0}\hat{\Sigma}_{KLMN}^{(e)}E_{MN} + \left(\hat{\Sigma}_{KLM}^{(ep)}P_m + \hat{\Sigma}_{KLM}^{(em)}\mathcal{M}_m\right)x_{m,M}$$
$$+ \hat{\Sigma}_{KL}^{(e\theta)}\theta\Big]x_{k,K}x_{\ell,L} + \mathcal{E}_k P_\ell + \mu_0\mathcal{H}_k\mathcal{M}_\ell \quad .$$

In writing (5.45), Eqs.$(5.17)_{3,4}$ are used.

Next, substituting $(3.67)_{3,4}$ into Eq.(5.45) and solving the resulting equations simultaneously for E_k and H_k the equivalent forms of $(5.45)_{2,3}$ are obtained as

$$E_k(\underset{\sim}{x},t)=\frac{\rho_o}{\rho}\Big\{\Big[\Big(\hat{\Sigma}^{(p)}_{KL}\,P_\ell+\hat{\Sigma}^{(pm)}_{KL}\,M_\ell\Big)x_{\ell,L}+\hat{\Sigma}^{(ep)}_{MNK}E_{MN}$$

$$+\hat{\Sigma}^{(p\theta)}_{K}\,\theta\Big]x_{k,K}-\epsilon_{kmn}\,\dot{x}_m\Big(\hat{\Sigma}^{(pm)}_{LK}P_\ell+\hat{\Sigma}^{(m)}_{KL}M_\ell\Big)x_{\ell,L}\,x_{n,K}\Big\}$$

$$-\epsilon_{kmn}\,\dot{x}_m\Big(\hat{\Sigma}^{(em)}_{MNK}\,E_{MN}+\hat{\Sigma}^{(m\theta)}_{K}\,\theta\Big)x_{n,K} \tag{5.46}$$

$$\mu_o H_k(\underset{\sim}{x},t)=\frac{\rho_o}{\rho}\Big\{\Big[\Big(\hat{\Sigma}^{(pm)}_{LK}\,P_\ell+\hat{\Sigma}^{(m)}_{KL}\,M_\ell\Big)x_{\ell,L}+\hat{\Sigma}^{(em)}_{MNK}\,E_{MN}$$

$$+\hat{\Sigma}^{(m\theta)}_{K}\,\theta\Big]x_{k,K}+\frac{1}{c^2}\,\epsilon_{kmn}\,\dot{x}_m\Big(\hat{\Sigma}^{(p)}_{KL}\,P_\ell+\hat{\Sigma}^{(pm)}_{KL}\,M_\ell\Big)x_{\ell,L}\,x_{n,K}\Big\}$$

$$+\frac{1}{c^2}\,\epsilon_{kmn}\,\dot{x}_m\Big(\hat{\Sigma}^{(ep)}_{MNK}\,E_{MN}+\hat{\Sigma}^{(p\theta)}_{K}\,\theta\Big)x_{n,K}\quad .$$

Upon substituting $(5.45)_2$ into (5.35) and introducing $(5.17)_{3,4}$, one obtains

$$\mathcal{D}={}_{(D)}\hat{\underset{\sim}{t}}_{KL}\,\dot{E}_{KL}+\Big[\rho_o\Big(\hat{\Sigma}^{(p)}_{KM}\,\Pi_M+\hat{\Sigma}^{(pm)}_{KM}\,\mathcal{M}_M\Big)+\hat{\Sigma}^{(ep)}_{MNK}\,E_{MN}$$

$$+\hat{\Sigma}^{(p\theta)}_{K}\,\theta\Big](2E_{KL}+\delta_{KL})\,\hat{\underset{\sim}{J}}_L-\frac{1}{\theta}\,\hat{\underset{\sim}{q}}_K\,G_K\geqslant 0\quad . \tag{5.47}$$

The minimum value of (5.47) is equal to zero and this is possible if all the independent variables vanish, i.e.,

$$\mathcal{D}_{min}=0 \tag{5.48}$$

if

$$\underset{\sim}{E}=\dot{\underset{\sim}{E}}=\underset{\sim}{\Pi}=\mu_o\underset{\sim}{\mathcal{M}}=\theta=\underset{\sim}{G}=0\quad . \tag{5.49}$$

Minimizing Eq.(5.47), one can obtain information about the form of constitutive equations for ${}_{(D)}\hat{\underset{\sim}{t}}_{KL}$, $\hat{\underset{\sim}{J}}_K$ and $\hat{\underset{\sim}{q}}_K$. $\mathcal{D}$ is minimum if the following equations

$$\frac{\partial\mathcal{D}}{\partial\underset{\sim}{E}}\Big|_0=\frac{\partial\mathcal{D}}{\partial\dot{\underset{\sim}{E}}}\Big|_0=\frac{\partial\mathcal{D}}{\partial\underset{\sim}{\Pi}}\Big|_0=\frac{\partial\mathcal{D}}{\partial\underset{\sim}{\mathcal{M}}}\Big|_0=\frac{\partial\mathcal{D}}{\partial\theta}\Big|_0=\frac{\partial\mathcal{D}}{\partial\underset{\sim}{G}}\Big|_0=0 \tag{5.50}$$

are satisfied. In Eq.(5.50), $\frac{\partial\mathcal{D}}{\partial(\cdot)}\Big|_0$ is the partial derivative evaluated at point (5.49).

Taking the partial derivatives and using the condition (5.47,49), one writes

$$\frac{\partial \mathcal{D}}{\partial E_{KL}}\Big|_0 = \hat{\Sigma}^{(ep)}_{KLM}\,\hat{\underset{\sim}{J}}_M\Big|_0 = 0 \;;\quad \frac{\partial \mathcal{D}}{\partial \dot{E}_{KL}}\Big|_0 = {}_{(D)}\hat{\underset{\sim}{t}}_{KL}\Big|_0 = 0$$

$$\frac{\partial \mathcal{D}}{\partial \Pi_K}\Big|_0 = \hat{\Sigma}^{(p)}_{LK}\,\hat{\underset{\sim}{J}}_L\Big|_0 = 0 \;;\quad \frac{\partial \mathcal{D}}{\partial \mathcal{M}_K}\Big|_0 = \hat{\Sigma}^{(pm)}_{LK}\,\hat{\underset{\sim}{J}}_L = 0 \tag{5.51}$$

$$\frac{\partial \mathcal{D}}{\partial \theta}\Big|_0 = \hat{\Sigma}^{(p\theta)}_{K}\,\hat{\underset{\sim}{J}}_K\Big|_0 = 0 \;;\quad \frac{\partial \mathcal{D}}{\partial G_K}\Big|_0 = \frac{1}{\theta}\,\hat{\underset{\sim}{q}}_K\Big|_0 = 0 \;.$$

Eq.(5.52) is satisfied if and only if

$$\hat{\underset{\sim}{J}}_K(0,0,0,0,0,0) = \hat{\underset{\sim}{q}}_K(0,0,0,0,0,0) = {}_{(D)}\hat{\underset{\sim}{t}}_{KL}(0,0,0,0,0,0) = 0 \tag{5.52}$$

since the material property tensors are, in general, nonzero. Thus, constitutive equations for ${}_{(D)}\underset{\sim}{t}_{KL}$, $\underset{\sim}{J}_K$ and $\underset{\sim}{q}_K$ should have the form

$$\begin{aligned}
{}_{(D)}\underset{\sim}{t}_{KL} &= \hat{C}^{(e)}_{KLMN}\,E_{MN} + \hat{C}^{(v)}_{KLMN}\,\dot{E}_{MN} + \rho_0\,\hat{C}^{(p)}_{KLM}\,\Pi_M + \rho_0\,\hat{C}^{(m)}_{KLM}\,\mathcal{M}_M \\
&\quad + \hat{C}^{(\theta)}_{KL}\,\theta + \hat{C}^{(g)}_{KLM}\,G_M \\
\underset{\sim}{J}_K &= \hat{K}^{(e)}_{KLM}\,E_{LM} + \hat{K}^{(v)}_{KLM}\,\dot{E}_{LM} + \rho_0\,\hat{K}^{(p)}_{KL}\,\Pi_L + \rho_0\,\hat{K}^{(m)}_{KL}\,\mathcal{M}_L \\
&\quad + \hat{K}^{(\theta)}_{K}\,\theta + \hat{K}^{(g)}_{KL}\,G_L \\
\underset{\sim}{q}_K &= \hat{G}^{(e)}_{KLM}\,E_{LM} + \hat{G}^{(v)}_{KLM}\,\dot{E}_{LM} + \rho_0\,\hat{G}^{(p)}_{KL}\,\Pi_L + \rho_0\,\hat{G}^{(m)}_{KL}\,\mathcal{M}_L \\
&\quad + \hat{G}^{(\theta)}_{K}\,\theta + \hat{G}^{(g)}_{KL}\,G_L
\end{aligned} \tag{5.53}$$

where the material property tensors in (5.53) have the following intrinsic symmetry

$$\begin{aligned}
&\hat{C}^{(e)}_{KLMN} = \hat{C}^{(e)}_{LKMN} = \hat{C}^{(e)}_{KLNM} \;;\quad \hat{C}^{(v)}_{KLMN} = \hat{C}^{(v)}_{LKMN} = \hat{C}^{(v)}_{KLNM} \\
&\hat{C}^{(p)}_{KLM} = \hat{C}^{(p)}_{LKM} \;;\quad \hat{C}^{(m)}_{KLM} = \hat{C}^{(m)}_{LKM} \;;\quad \hat{C}^{(\theta)}_{KL} = \hat{C}^{(\theta)}_{LK} \\
&\hat{C}^{(g)}_{KLM} = \hat{C}^{(g)}_{LKM} \;;\quad \hat{K}^{(e)}_{KLM} = \hat{K}^{(e)}_{KML} \;;\quad \hat{K}^{(v)}_{KLM} = \hat{K}^{(v)}_{KML} \\
&\hat{G}^{(e)}_{KLM} = \hat{G}^{(e)}_{KML} \;;\quad \hat{G}^{(v)}_{KLM} = \hat{G}^{(v)}_{KML} \;.
\end{aligned} \tag{5.54}$$

The constitutive equations $(5.53)_2$ and $(5.53)_3$ are nothing, but the generalized Ohm law of electric conduction and Fourier law of heat conduction for the interacting continua for which the magnetic symmetry are taken into account.

Now making use of $(5.19)_{1,4,5}$ and (5.17) in (5.53), and the result $(5.45)_4$ together with (5.24), one obtains

$$
\begin{aligned}
t_{kl}(\underset{\sim}{x},t) = \Big\{ & \frac{\rho}{\rho_0}\Big[\underset{\approx}{\hat{C}}{}^{(e)}_{KLMN} E_{MN} + \hat{C}^{(v)}_{KLMN} \dot{E}_{MN} + \Big(\hat{\Sigma}^{(e\theta)}_{KL} + \frac{\rho}{\rho_0}\hat{\Sigma}^{(\theta)}_{KL}\Big)\theta \\
& + \underset{\approx}{\hat{C}}{}^{(p)}_{KLM} P_m + \underset{\approx}{\hat{C}}{}^{(m)}_{KLM} M_m + \frac{\rho}{\rho_0}\hat{C}^{(g)}_{KLM} g_m \Big] x_{m,M}\Big\} x_{k,K} x_{l,L} \\
& + \mathcal{E}_k P_l + \mu_0 \mathcal{H}_k \mathcal{M}_l \\
\mathcal{J}_k(\underset{\sim}{x},t) = \Big[& \frac{\rho}{\rho_0}\Big(\hat{K}^{(e)}_{KLM} E_{LM} + \hat{K}^{(v)}_{KLM} \dot{E}_{LM} + \hat{K}^{(\theta)}_{K}\theta\Big) \qquad (5.55) \\
& + \Big(\hat{K}^{(p)}_{KL} P_l + \hat{K}^{(m)}_{KL} M_l + \frac{\rho}{\rho_0}\hat{K}^{(g)}_{KL} g_l\Big) x_{l,L}\Big] x_{k,K} \\
q_k(\underset{\sim}{x},t) = \Big[& \frac{\rho}{\rho_0}\Big(\hat{G}^{(e)}_{KLM} E_{LM} + \hat{G}^{(v)}_{KLM} \dot{E}_{LM} + \hat{G}^{(\theta)}_{K}\theta\Big) \\
& + \Big(\hat{G}^{(p)}_{KL} P_l + \hat{G}^{(m)}_{KL} M_l + \frac{\rho}{\rho_0}\hat{G}^{(g)}_{KL} g_l\Big) x_{l,L}\Big] x_{k,K}
\end{aligned}
$$

where

$$\underset{\approx}{\hat{C}}{}^{(e)} = \underset{\sim}{\hat{\Sigma}}{}^{(e)} + \underset{\sim}{\hat{C}}{}^{(e)} \; ; \; \underset{\approx}{\hat{C}}{}^{(p)} = \underset{\sim}{\hat{\Sigma}}{}^{(ep)} + \underset{\sim}{\hat{C}}{}^{(p)} \; ; \; \underset{\approx}{\hat{C}}{}^{(m)} = \underset{\sim}{\hat{\Sigma}}{}^{(em)} + \underset{\sim}{\hat{C}}{}^{(m)} . \qquad (5.56)$$

For the materials belonging to the 32 out of the 90 crystallographic point groups, the material property tensors are

$$\hat{C}^{(m)}_{KLM} = \hat{K}^{(m)}_{KL} = \hat{G}^{(m)}_{KL} = 0 \qquad (5.57)$$

since $\mathcal{M}$ is an axial c-tensor, and ${}_{(D)}t_{kl}$, $\mathcal{J}_k$ and q_k are true i-tensors. Moreover, if the material is centrosymmetric, then all the i-tensors of odd order describing the properties of the materials belonging to $\{S\}$ also vanish. That is,

$$
\begin{aligned}
\underset{\approx}{\hat{C}}{}^{(p)}_{KLM} &= \underset{\approx}{\hat{C}}{}^{(m)}_{KLM} = \hat{C}^{(g)}_{KLM} = \hat{K}^{(e)}_{KLM} = \hat{K}^{(v)}_{KLM} = \hat{K}^{(\theta)}_{K} \\
&= \hat{G}^{(e)}_{KLM} = \hat{G}^{(v)}_{KLM} = \hat{G}^{(\theta)}_{K} = 0 \qquad (5.58)
\end{aligned}
$$

in addition to (5.57). Therefore, the constitutive equations for the classical crystals now become

$$
\begin{aligned}
\eta &= -\hat{\Sigma}^{(\theta)}\theta - \hat{\Sigma}^{(e\theta)}_{KL} E_{KL} \\
E_k &= \frac{\rho_0}{\rho}\left(\hat{\Sigma}^{(p)}_{KL} P_l x_{k,K} - \epsilon_{kmn}\dot{x}_m \hat{\Sigma}^{(m)}_{KL} M_l x_{n,K}\right) x_{l,L} \\
\mu_0 H_k &= \frac{\rho_0}{\rho}\left(\hat{\Sigma}^{(m)}_{KL} M_l x_{k,K} + \frac{1}{c^2}\epsilon_{kmn}\dot{x}_m \hat{\Sigma}^{(p)}_{KL} P_l x_{n,K}\right) x_{l,L} \\
t_{kl} &= \Big\{ \frac{\rho_0}{\rho}\left(\hat{C}^{(e)}_{KLMN} E_{MN} + \hat{C}^{(v)}_{KLMN}\dot{E}_{MN}\right) \\
&\quad + \left(\hat{\Sigma}^{(e\theta)}_{KL} + \frac{\rho}{\rho_0}\hat{C}^{(\theta)}_{KL}\right)\theta \Big\} x_{k,K} x_{l,L} + \mathcal{E}_k P_l + \mu_0 \mathcal{H}_k \mathcal{M}_l \qquad (5.59) \\
J_k &= \left(\hat{K}^{(p)}_{KL} P_l + \frac{\rho}{\rho_0}\hat{K}^{(g)}_{KL} g_l\right) x_{k,K} x_{l,L} \\
q_k &= \left(\hat{G}^{(p)}_{KL} P_l + \frac{\rho}{\rho_0}\hat{G}^{(g)}_{KL} g_l\right) x_{k,K} x_{l,L} .
\end{aligned}
$$

The constitutive equations derived by Tiersten [5], Mindlin [8], Wong and Grindlay [78], Pao and Yeh [13], and Hutter and Pao [14] are the special cases of (5.59), except the terms associated with the polarization or magnetization gradient therein.

Later, we will further reduce the constitutive equations to the case of isotropic materials.

Thus, the equations obtained from the basic principles and the constitutive equations presented in this chapter together with the boundary conditions must be solved simultaneously for prescribed initial conditions. For a particular material, first the nonvanishing components of the material property tensors of the interacting phenomena are to be determined.

PART : II

DECOMPOSITIONS, LINEARIZATION AND SPECIAL CASES

CHAPTER 6

DECOMPOSITION OF THE GOVERNING EQUATIONS

The dynamic equations of the magneto-electro thermo-viscoelastic solids as given in the previous two chapters are highly nonlinear and complicated. Apart from the usual nonlinearities due to finite deformation, there emerge difficulties coming from the Maxwell equations. Another difficulty concerns the boundary conditions that are expressed in deformed configuration which is not known a priori.

To render the equations amenable to analysis, the case of infinitesimal strains is considered in this chapter.

6.1. Fundamental Assumptions

The total deformation of the body from its initial configuration to the present one is assumed to be a state of infinitesimal fields superimposed on a rigid body motion. Electromagnetic and thermal fields are not assumed to be small in the course of this rigid body motion, but their corrections due to the strains and the time rate of strains are assumed to be small. In other words, the rigid body motion, if any, is assumed to have a dominant effect on the magnitudes of the electromagnetic and thermal fields.

Let the particles of the body in its initial configuration $\mathcal{B}_o$ be identified by the coordinates ${}_o\underset{\sim}{X}$, Fig.6.1. Through a rigid body motion, the body occupies an intermediate configuration $\mathcal{B}_R$ with coordinates $\underset{\sim}{X}$. Finally, the body is deformed and occupies the present configuration $\mathcal{B}_t$ with coordinates $\underset{\sim}{x}$. Thus the body is brought to the reference configuration from the initial configuration through a rigid body motion

$$X_K(t) = \hat{X}_K({}_o\underset{\sim}{X}, t) \quad , \quad \underset{\sim}{X} \in V-\Sigma \qquad (6.1)$$

where $V-\Sigma$ is the region occupied by the body in $\mathcal{B}_R$. Further the body is brought to the present configuration from the reference configuration through an infinitesimal motion

$$x_k = \hat{x}_k\ (\underset{\sim}{X}(t)\ , t\ )\qquad , \quad \underset{\sim}{x} \in V-\sigma \tag{6.2}$$

where $V-\sigma$ is the region occupied by the body in $\mathcal{B}_t$.

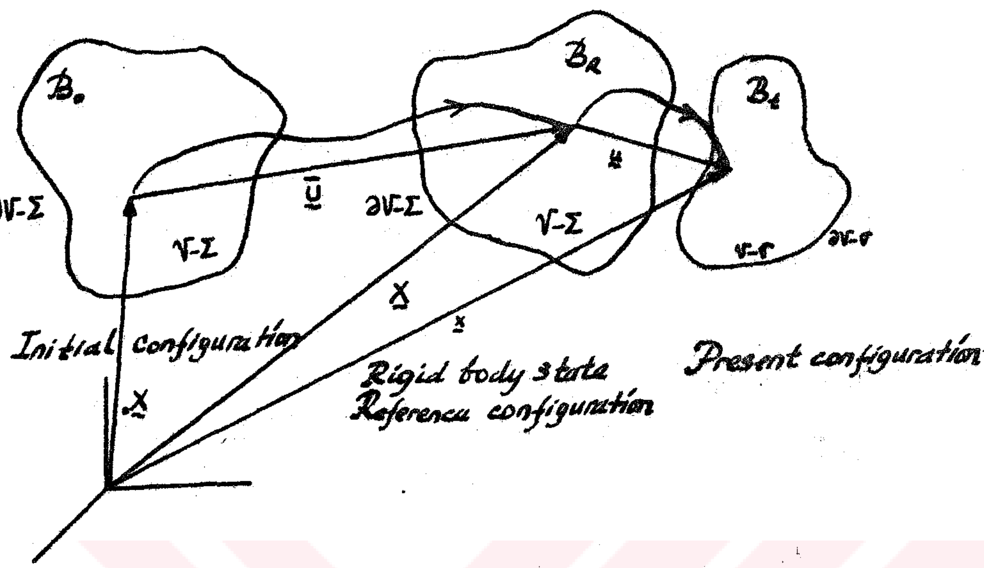


Fig.6.1 Initial, Rigid Body (Reference) and Present Configuration of Interacting Continua.

There are physical situations which do suggest a natural definition of the rigid body position of the deformed body. If, for example, the total displacements remain small in the course of motion, then the fields may be decomposed such that electromagnetic quantities are evaluated in a process of vanishing displacements plus corrections which are determined by a linear theory of deformations. This is the case for a polarizable and magnetizable body being supported such that it is in static equilibrium and stable.

Another situation occurs when large displacements are allowed, the major part of which is composed of a rigid body translation and rotation. In this case a displaced rigid body position may be suitably defined at least asymptotically in time.

Through the motions (6.1) and (6.2), the total displacement $\hat{\underset{\sim}{u}}$ of the particle is

$$\hat{\underset{\sim}{u}} = \bar{\underset{\sim}{u}} + \underset{\sim}{u} = (\underset{\sim}{X} - \underset{\circ}{\underset{\sim}{X}}\ ) + \underset{\sim}{u} \tag{6.3}$$

where $\bar{\underset{\sim}{u}}$ is being the displacement vector of the rigid body motion. At this stage we are only interested in relative displacement $\underset{\sim}{u}$ from the rigid body state to the present configuration as defined by

$$u_K(\underset{\sim}{x},t) = x_k\,\delta_{kK} - X_K \quad ; \quad u_k(\underset{\sim}{X},t) = x_k - X_K\,\delta_{kK} \quad . \qquad (6.4)$$

Assumption-1: Displacement gradients are small, i.e.,

$$\mathrm{norm}\left(\frac{\partial u_k}{\partial X_K}\right) = \mathrm{norm}\,(u_{k,K}) \ll 1 \qquad (6.5)$$

where the norm introduced may conveniently be defined as

$$\left(\mathrm{norm}\,(u_{k,K}(t))\right)^2 = \lim_{\tau\le t}\ \sup\ u_{k,K}(\tau)\, u_{k,K}(\tau) \quad .$$

This assumption does not suffice to furnish a theory linear in the displacement. Smallness of the displacement gradients of $\underset{\sim}{u}$ does not guarantee smallness of its material time derivative.

Assumption-2:The velocities of the particles are assumed to be small compared to a characteristic wave speed c_0 in the strained body, i.e.,

$$\frac{1}{c_0}\,\mathrm{norm}\left(\frac{du_i}{dt}\right) = \frac{1}{c_0}\,\mathrm{norm}\,(\dot{u}_i(\underset{\sim}{X},t)) \ll 1 \quad . \qquad (6.6)$$

Next, all the field variables in the present configuration are decomposed into two groups:

$$\begin{aligned} P_k &= \bar{P}_K\,\delta_{kK} + p_k \quad ; \quad M_k = \bar{M}_K\,\delta_{kK} + m_k \\ \theta &= \bar{\theta} + \theta' \quad ; \quad E_k = \bar{E}_K\,\delta_{kK} + e_k \qquad (6.7) \\ H_k &= \bar{H}_K\,\delta_{kK} + h_k \quad ; \quad \eta = \bar{\eta} + \eta' \quad ; \quad t_{kl} = \bar{t}_{KL}\,\delta_{kK}\delta_{lL} + t'_{kl} \\ &\ldots\ldots \end{aligned}$$

where the elements in the first group are indicated by an overhead bar and those in the second group by either a prime or lower case letter. The first group is identified with the fields of the body in its rigid body configuration and is called the "rigid body state". The second group stands for the "perturbation state" and is the corrections to account for the effect of deformation.

Assumption-3: The independent and dependent variables, their space and time derivatives in the perturbation state are small in magnitude when compared with those in the rigid body state, i.e.,

$$\text{norm}\left(\frac{p_k}{\bar{P}_K}\right), \quad \text{norm}\left(\frac{m_k}{\bar{M}_K}\right), \quad \text{norm}\left(\frac{e_k}{\bar{E}_K}\right), \ldots, \ll 1 \tag{6.8}$$

Hence, all the terms of the products of the two quantities in the perturbation state compared to the terms in the rigid body state will be neglected.

All derivatives with respect to the present configuration can be expressed in terms of those of the reference configuration. The following differentiable function $(\cdot)$ show this obviously.

$$\frac{\partial(\cdot)}{\partial x_i} = \frac{\partial(\cdot)}{\partial X_K}\frac{\partial X_K}{\partial x_i} = \left(\delta_{iK} - \frac{\partial u_K}{\partial x_i}\right)\frac{\partial(\cdot)}{\partial X_K} \tag{6.9}$$

where (6.4) is used. For example, if $(\cdot) = u_L$, it follows from (6.4) and (6.9) that

$$u_{K,i} = \delta_{iL}\frac{u_{K,L}}{1+u_{N,N}} \cong \delta_{iL}\, u_{K,L} \tag{6.10}$$

where quadratic and higher terms of displacement gradients are assumed to be small compared to unity. Substituting (6.10) into (6.9) one writes

$$\frac{\partial(\cdot)}{\partial x_i} \cong \left(\delta_{iK} - \delta_{iL}\frac{\partial u_K}{\partial X_L}\right)\frac{\partial(\cdot)}{\partial X_K} \;. \tag{6.11}$$

From (6.4), the velocity and the acceleration of a partical are, respectively,

$$v_k \equiv \dot{x}_k = \dot{X}_K\delta_{kK} + \dot{u}_k = \dot{X}_K\delta_{kK} + \frac{\partial u_k}{\partial t} + u_{k,K}\dot{X}_K \tag{6.12}$$

and

$$a_k \equiv \frac{dv_k}{dt} = \frac{\partial^2 u_k}{\partial t} + 2\dot{X}_K\frac{\partial u_{k,K}}{\partial t} + \dot{X}_K\dot{X}_L u_{k,KL} + (u_{k,K} + \delta_{kK})\ddot{X}_K \tag{6.13}$$

where $\quad \dot{X}_K \equiv \dfrac{dX_K}{dt} = \dfrac{\partial X_K}{\partial t}$.

For infinitesimal strains, the solution of (4.10) is given by

$$\rho = \bar{\rho}\,(1 - u_{K,K}) \tag{6.14}$$

where $\bar{\rho}$ is the mass density in the rigid body state or in the initial configuration.

The deformation gradient in terms of the displacement field is

$$x_{k,K} = \delta_{kK} + \frac{\partial u_k}{\partial X_K} = \delta_{kK} + u_{k,K} \quad . \tag{6.15}$$

From (6.5) and (6.15), it is clear that

$$x_{k,K}\, x_{\ell,L} = \delta_{kK}\,\delta_{\ell L} + \delta_{kK}\, u_{\ell,L} + \delta_{\ell L}\, u_{k,K} \quad . \tag{6.16}$$

Thus, the finite strain tensor reduces to

$$E_{KL} \cong \tilde{e}_{KL} = \frac{1}{2}\,(u_{K,L} + u_{L,K}) \tag{6.17}$$

where $\tilde{e}_{KL}$ is called the infinitesimal strain tensor.

Using (6.14) and (6.11), Eq.(4.14) takes the form

$$\bar{\rho}\,\dot{v}_I - t_{KI,K} - \rho f_I + t_{LI,K}\, u_{L,K} = 0 \tag{6.18}$$

where $\dot{v}_I$ is given by (6.12). The term $t_{LI,K}\, u_{L,K}$ will be retained until the constitutive equations are explicitly known.

6.2. Decompositions of the Energy Equations and the Entropy Inequality

The energy equation (4.28) and the entropy inequality (4.31) or their combination (4.36) are decomposed using the assumptions given in the previous section. Let $\bar{\eta}$ and $\bar{q}_K$ be the entropy and the energy flux associated with the rigid body state of the body at temperature $\bar{T}$. The body is initially at an ambient temperature T_0. After it has been polarized, magnetized and heated by the current and other sources, it changes from T_0 to T. The actual change of temperature, including polarization, magnetization and deformation effects, is $T - T_0$, where T is the temperature in the present configuration, i.e.,

$$T - T_0 = (\bar{T} - T_0) + (T - \bar{T}) \qquad \text{or} \qquad \theta = \bar{\theta} + \theta' \tag{6.19}$$

and

$$\frac{T - \bar{T}}{\bar{T}} << 1 \quad ; \quad T > 0 \quad . \tag{6.20}$$

By means of (4.35) and (5.31), the energy equation (4.28) can be expressed as

$$\rho\theta\dot{\eta} = {}_{(D)}t_{kl}\, d_{kl} + (-{}_{(ND)}t_{kl} + P_k \mathcal{E}_l + \mu_0 \mathcal{M}_k \mathcal{H}_l)\,\omega_{kl} + J_k \mathcal{E}_k - q_{k,k} + \rho r_\theta \tag{6.21}$$

where (5.24) and (2.27) are used.

Noting that ${}_{(ND)}t_{[kl} - P_k \mathcal{E}_l - \mu_0 \mathcal{M}_k \mathcal{H}_{l]} = 0$ and $\omega_{kl} = -\omega_{lk}$, (6.21) reduces to

$$\rho\theta\dot{\eta} = {}_{(D)}t_{kl}\, d_{lk} - J_k\, \mathcal{E}_k - q_{k,k} + \rho r_\theta\ . \tag{6.22}$$

According to Killing's theorem, the necessary and sufficient condition for the rigid motion is $d_{kl} = 0$. If this holds through a region of the body, (6.22) becomes

$$\bar{\rho}\,\bar{\theta}\,\dot{\bar{\eta}} = \bar{J}_K\, \bar{\mathcal{E}}_K - \bar{q}_{K,K} + \bar{\rho}\,\bar{r}_\theta \qquad . \tag{6.23}$$

By means of (6.12), one can express d_{kl} as

$$d_{kl} = \frac{1}{2}\left[\left(\frac{\partial u_{k,L}}{\partial t} + u_{k,KL}\,\dot{x}_K\right)\delta_{lL} + \left(\frac{\partial u_{l,L}}{\partial t} + u_{l,KL}\,\dot{x}_K\right)\delta_{kL}\right]. \tag{6.24}$$

Substituting (6.24) into (6.22) and using the symmetry of the dissipative part of the stress, there follows

$$\begin{aligned}\bar{\rho}\left[(\theta' - \bar{\theta}\, u_{K,K})\,\dot{\bar{\eta}} + \bar{\theta}\,\dot{\eta}'\right] = {}_{(D)}\bar{t}_{KL}\left(\frac{\partial}{\partial t}u_{K,L} + u_{K,ML}\,\dot{x}_M\right)\\ + J_K'\,\bar{\mathcal{E}}_K + \bar{J}_K\,\mathcal{E}_K' - q_{K,K}' - u_{K,L}\,\bar{q}_{K,L} + \bar{\rho}\,(r_\theta' - u_{K,K}\,\bar{r}_\theta)\end{aligned} \tag{6.25}$$

where ${}_{(D)}\bar{t}_{KL}$ is the value of ${}_{(D)}t_{kl}$ in the rigid body state.

Similarly, from (4.31), one decomposes the entropy inequality as

$$\bar{\rho}\,\dot{\bar{\eta}} \geqslant -\left(\frac{1}{\bar{\theta}}\,\bar{q}_K\right)_{,K} + \frac{1}{\bar{\theta}}\,\bar{\rho}\,\bar{r}_\theta \tag{6.26}$$

in the rigid body state,

$$\begin{aligned}\bar{\rho}\,(\dot{\eta}' - u_{K,K}\,\dot{\bar{\eta}}) \geqslant -\left(\frac{q_K'}{\bar{\theta}}\right)_{,K} + \left(\frac{\theta'}{\bar{\theta}}\,\bar{q}_K\right)_{,K} + u_{K,L}\left(\frac{\bar{q}_L}{\bar{\theta}}\right)_{,K}\\ + \frac{1}{\bar{\theta}}\,\bar{\rho}\,r_\theta' - \frac{\bar{\rho}\,\bar{r}_\theta}{\bar{\theta}}\left[\frac{\theta'}{\bar{\theta}} + u_{K,K}\left(1 - \frac{\theta'}{\bar{\theta}}\right)\right]\end{aligned} \tag{6.27}$$

in the perturbation state.

6.3. Decomposition of the Maxwell Equations

The Maxwell equations stated in the present configuration are now transformed to the reference configuration. Substituting (6.11) into Eqs.(5.2-4), the Maxwell equations in the rigid body state are obtained as

$$\epsilon_{IKL}\,\bar{E}_{L,K} + \mu_0 \frac{\partial \bar{H}_I}{\partial t} = -\bar{J}_I^{(m)}$$

$$\epsilon_{IKL}\,\bar{H}_{L,K} - \epsilon_0 \frac{\partial \bar{E}_I}{\partial t} = \bar{J}_I^{(f)} + \bar{J}_I^{(p)} \tag{6.28}$$

$$\epsilon_0\,\bar{E}_{K,K} = \bar{\rho}^{(f)} + \bar{\rho}^{(p)} \quad ; \quad \mu_0\,\bar{H}_{K,K} = \bar{\rho}^{(m)}$$

where

$$\bar{J}_K^{(p)} \equiv \frac{\partial \bar{P}_K}{\partial t} + \bar{P}_{K,L}\,\dot{X}_L - \bar{P}_{L,L}\,\dot{X}_K$$

$$\bar{J}_K^{(m)} \equiv \frac{\partial \mu_0 \bar{M}_K}{\partial t} + \mu_0\,\bar{M}_{K,L}\,\dot{X}_L - \mu_0\,\bar{M}_{L,L}\,\dot{X}_K \tag{6.29}$$

$$\bar{\rho}^{(p)} \equiv -\bar{P}_{K,K} \quad ; \quad \bar{\rho}^{(m)} \equiv \mu_0\,\bar{M}_{K,K} \quad .$$

The Maxwell equations in terms of potentials are similarly decomposed. Upon substituting

$$\Phi^{(p)} = \bar{\Phi}^{(p)} + \varphi^{(p)} \quad ; \quad \bar{\Phi}^{(m)} = \bar{\Phi}^{(m)} + \varphi^{(m)}$$

$$\underset{\sim}{A}^{(p)} = \underset{\sim}{\bar{A}}^{(p)} + \underset{\sim}{a}^{(p)} \quad ; \quad \underset{\sim}{A}^{(m)} = \underset{\sim}{\bar{A}}^{(m)} + \underset{\sim}{a}^{(m)} \tag{6.30}$$

into (3.41-44), there follows

$$\bar{E}_K = -\bar{\Phi}^{(p)}_{,K} - \frac{\partial \bar{A}_K^{(p)}}{\partial t} - \frac{1}{\epsilon_0}\,\epsilon_{KMN}\,\bar{A}^{(m)}_{N,M}$$

$$\bar{H}_K = \frac{1}{\mu_0}\,\epsilon_{KMN}\,\bar{A}^{(p)}_{N,M} - \bar{\Phi}^{(m)}_{,K} - \frac{\partial \bar{A}^{(m)}}{\partial t} \tag{6.31}$$

and

$$\Box\,\bar{\Phi}^{(p)} = -\frac{1}{\epsilon_0}\left(\bar{\rho}^{(f)} + \bar{\rho}^{(p)}\right) \quad ; \quad \Box\,\bar{\Phi}^{(m)} = -\frac{1}{\mu_0}\,\bar{\rho}^{(m)}$$

$$\Box\,\bar{A}_I^{(e)} = -\mu_0\left(\bar{J}_I^{(f)} + \bar{J}_I^{(p)}\right) \quad ; \quad \Box\,\bar{A}_I^{(m)} = -\epsilon_0\,\bar{J}_I^{(m)} \tag{6.32}$$

where the potentials satisfy the Lorentz condition

$$\bar{A}^{(p)}_{K,K} + \frac{1}{c^2}\frac{\partial \bar{\Phi}^{(p)}}{\partial t} = 0 \quad ; \quad \bar{A}^{(m)}_{K,K} + \frac{1}{c^2}\frac{\partial \bar{\Phi}}{\partial t} = 0 \tag{6.33}$$

and the operator $\bar{\Box}$ is the D'Alembert operator evaluated in the rigid body state, i.e.,

$$\bar{\Box} \equiv \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad . \tag{6.34}$$

In view of (B.5) and (B.11), the boundary conditions $(5.6)_{4,5}$ and (5.7) now become

$$\begin{aligned} &[\![\epsilon_0 \bar{E}_I + \bar{P}_I]\!] N_I = 0 \; ; [\![\bar{H}_I + \bar{M}_I]\!] N_I = 0 \quad \text{on} \quad \Sigma(t) \\ &[\![\bar{\mathcal{E}}_I]\!] K_I = 0 \quad ; \quad [\![\bar{\mathcal{H}}_I]\!] K_I = 0 \quad \text{on} \quad \Gamma(t) \end{aligned} \tag{6.35}$$

where

$$\bar{\mathcal{E}}_I = \bar{E}_I + \mu_0 \, \epsilon_{IKL} \dot{X}_K \bar{H}_L \quad ; \quad \bar{\mathcal{H}}_I = \bar{H}_I - \epsilon_0 \, \epsilon_{IKL} \dot{X}_K \bar{E}_L \, . \tag{6.36}$$

The Maxwell stress tensor (3.64) and the electromagnetic body force (3.56) in the rigid body state are now

$$\bar{\tau}_{IJ} = \epsilon_0 \bar{E}_I \bar{E}_J + \mu_0 \bar{H}_I \bar{H}_J - \bar{W} \delta_{IJ} + \bar{P}_I \bar{\mathcal{E}}_J + \mu_0 \bar{M}_I \bar{\mathcal{H}}_J \tag{6.37}$$

and

$$\begin{aligned} \bar{\rho} \bar{f}^{(em)}_I &= \bar{\tau}_{KI,K} - \frac{1}{c^2} \epsilon_{IKL} \frac{\partial}{\partial t} (\bar{E}_K \bar{H}_L) \\ &= \bar{\rho}^{(f)} \bar{E}_I + \mu_0 \, \epsilon_{IKL} \bar{J}^{(f)}_K \bar{H}_L + \bar{P}_K \bar{E}_{I,K} + \mu_0 \bar{M}_K \bar{H}_{I,K} \\ &\quad + \mu_0 \epsilon_{IKL} \Big[\frac{\partial \bar{P}_K}{\partial t} \bar{H}_L - \epsilon_0 \frac{\partial \bar{M}_K}{\partial t} \bar{E}_L \\ &\quad + \dot{X}_N (\bar{P}_{K,N} \bar{H}_L - \epsilon_0 \bar{M}_{K,N} \bar{E}_L) + \dot{X}_K (\bar{P}_N \bar{H}_{L,N} - \epsilon_0 \bar{M}_N \bar{E}_{L,N}) \Big] \end{aligned} \tag{6.38}$$

where

$$\bar{W} = \frac{1}{2} (\epsilon_0 \bar{E}_K \bar{E}_K + \mu_0 \bar{H}_K \bar{H}_K) \quad . \tag{6.39}$$

In vacuum, $\bar{P}_K = 0$ and $\bar{M}_K = 0$, therefore (6.37) becomes the Maxwell stress tensor

$$\bar{\tau}^{\circ}_{KL} = \epsilon_0 \bar{E}_K \bar{E}_L + \mu_0 \bar{H}_K \bar{H}_L - \bar{W} \delta_{KL} \tag{6.40}$$

that is, the first part of decomposed $\underset{\sim}{\tau}^{\circ}$ defined by $(3.65)_1$. From (3.67), it is clear that

$$\bar{\tau}_{[KL]} = \bar{\mathcal{P}}_{[K}\,\bar{\mathcal{E}}_{L]} + \mu_0\,\bar{\mathcal{M}}_{[K}\,\bar{\mathcal{H}}_{L]} \quad . \tag{6.41}$$

Eqs.(6.28,31,32,35,37,38) are in aqreement with those of the rigid body electrodynamics, e.g. ([41], Ch.9). These equations governing the rigid body state are still nonlinear, but simpler in the form compared to the original equations. However, if the rigid body body motion is prescribed they become linear, which would not be the case if the decomposition had not been made.

In Eq.(6.28), the velocity at any point $\dot{\underset{\sim}{X}}$ can be expressed in terms of the velocity of the centre of mass $\dot{\bar{\underset{\sim}{X}}}$ and the angular velocity $\underset{\sim}{\Omega}$ of the rigid body. These two kinematical quantities are governed by the Euler equations given in the next section.

Observing the restrictions coming from (6.5,6) and using (6.11), the Maxwell equations, boundary conditions, Maxwell stress tensor and the body force in the perturbation state are obtained. From (3.29), omitting the intermediate calculations, one writes

$$\begin{aligned}
&\epsilon_{IJK}\, e_{K,J} + \mu_0 \frac{\partial h_I}{\partial t} = -\, j_I^{(m)} + u_{K,L}\, \epsilon_{ILN}\, \bar{E}_{N,K} \\
&\epsilon_{IJK}\, h_{K,J} - \epsilon_0 \frac{\partial e_I}{\partial t} = j_I^{(f)} + j_I^{(m)} + u_{K,L}\, \epsilon_{ILN}\, \bar{H}_{N,K} \\
&\epsilon_0\, e_{K,K} = \rho^{\prime(f)} + \rho^{\prime(p)} + \epsilon_0\, u_{K,L}\, \bar{E}_{L,K} \\
&\mu_0\, h_{K,K} = \rho^{\prime(m)} + \mu_0\, u_{K,L}\, \bar{H}_{L,K}
\end{aligned} \tag{6.42}$$

where

$$\begin{aligned}
j_I^{(p)} &\equiv \frac{\partial P_I}{\partial t} + P_{I,K}\, \dot{X}_K - P_{L,L}\, \dot{X}_I + (\bar{P}_I\, \dot{u}_K - \bar{P}_K\, \dot{u}_I)_{,K} \\
&\quad + u_{K,L}\, (\bar{P}_{L,K}\, \dot{X}_I - \bar{P}_{I,K}\, \dot{X}_L) \\
j_I^{(m)} &\equiv \frac{\partial \mu_0 m_I}{\partial t} + \mu_0\, m_{I,K}\, \dot{X}_K - \mu_0\, m_{L,L}\, \dot{X}_I + \mu_0\, (\bar{M}_I \dot{u}_K - \bar{M}_K \dot{u}_I)_{,K} \\
&\quad + u_{K,L}\, \mu_0\, (\bar{M}_{L,K}\, \dot{X}_I - \bar{M}_{I,K}\, \dot{X}_L) \\
\rho^{\prime(p)} &\equiv -P_{K,K} + u_{K,L}\, \bar{P}_{L,K} \quad ; \quad \rho^{\prime(m)} \equiv -\mu_0\, m_{K,K} + \mu_0\, u_{K,L}\, \bar{M}_{L,K}
\end{aligned} \tag{6.43}$$

with

$$\dot{u}_I = \frac{\partial u_I}{\partial t} + u_{I,K}\,\dot{X}_K \quad . \tag{6.44}$$

The perturbed electromagnetic fields in terms of potentials are obtained as

$$\begin{aligned} e_I &= -\varphi^{(e)}_{,I} - \frac{\partial a^{(e)}_I}{\partial t} - \frac{1}{\epsilon_0}\,\epsilon_{IKL}\,a^{(m)}_{L,K} + u_{K,L}\,\bar{\Phi}^{(e)}_{,K} - \frac{1}{\epsilon_0}\,\epsilon_{IKL}\,u_{N,K}\,\bar{A}^{(m)}_{L,N} \\ h_I &= \frac{1}{\mu_0}\,\epsilon_{IKL}\,a^{(e)}_{L,K} - \varphi^{(m)}_{,I} - \frac{\partial a^{(m)}_I}{\partial t} + u_{K,I}\,\bar{\Phi}^{(m)}_{,K} - \frac{1}{\mu_0}\,\epsilon_{IKL}\,u_{N,K}\,\bar{A}^{(e)}_{L,N} \quad . \end{aligned} \tag{6.45}$$

When Eq.(6.45) is substituted into (6.42), the Maxwell's equations now become

$$\begin{aligned} \bar{\Box}\,\varphi^{(e)} &= -\frac{1}{\epsilon_0}\left(\rho'^{(f)} + \rho'^{(p)}\right) + 2\,\tilde{e}_{KL}\,\bar{\Phi}^{(e)}_{,KL} \\ \bar{\Box}\,\varphi^{(m)} &= -\frac{1}{\mu_0}\,\rho'^{(m)} + 2\,\tilde{e}_{KL}\,\bar{\Phi}^{(m)}_{,KL} \end{aligned} \tag{6.46}$$

and

$$\begin{aligned} \bar{\Box}\,a^{(e)}_I &= -\mu_0\left(j^{(f)}_I + j^{(p)}_I\right) + 2\,\tilde{e}_{KL}\,\bar{A}^{(e)}_{I,KL} \\ \bar{\Box}\,a^{(m)}_I &= -\epsilon_0\,j'^{(m)}_I + 2\,\tilde{e}_{KL}\,\bar{A}^{(m)}_{I,KL} \end{aligned} \tag{6.47}$$

where the potentials satisfy the Lorentz condition

$$\begin{aligned} a^{(e)}_{K,K} + \frac{1}{c^2}\,\frac{\partial \varphi^{(e)}}{\partial t} &= u_{K,L}\,\bar{A}^{(e)}_{L,K} \\ a^{(m)}_{K,K} + \frac{1}{c^2}\,\frac{\partial \varphi^{(m)}}{\partial t} &= u_{K,L}\,\bar{A}^{(m)}_{L,K} \quad . \end{aligned} \tag{6.48}$$

The boundary conditions in the perturbation state are

$$\left.\begin{aligned} &[\![\epsilon_0 e_I + P_I + (\epsilon_0 \bar{E}_I + \bar{P}_I)\,\tilde{e}_{KL} N_K N_L - (\epsilon_0 \bar{E}_K + \bar{P}_K)\,u_{I,N}]\!]\,N_I = 0 \\ &\mu_0[\![h_I + m_I + (\bar{H}_I + \bar{M}_I)\,\tilde{e}_{KL} N_K N_L - (\bar{H}_K + \bar{M}_K)\,u_{I,K}]\!]\,N_I = 0 \end{aligned}\right\} \text{on } \sigma(t) \tag{6.49}$$

$$\left.\begin{aligned} &[\![\mathcal{E}'_I - \bar{\mathcal{E}}_I\,\tilde{e}_{LN}\,K_L K_N + \bar{\mathcal{E}}_L\,u_{L,I}]\!]\,K_I = 0 \\ &[\![\mathcal{H}'_I - \bar{\mathcal{H}}_I\,\tilde{e}_{LN}\,K_L K_N + \bar{\mathcal{H}}_L\,u_{L,I}]\!]\,K_I = 0 \end{aligned}\right\} \text{on } \sigma(t) \tag{6.50}$$

where

$$\mathcal{E}'_I \equiv e_I + \mu_0 \epsilon_{IJK} (\dot{x}_J h_K + \dot{u}_J \bar{H}_K)$$
$$\mathcal{H}'_I \equiv h_I - \epsilon_0 \epsilon_{IJK} (\dot{x}_J e_K + \dot{u}_J \bar{E}_K) \quad . \tag{6.51}$$

The Maxwell's stress tensor and the body force are, respectively,

$$\begin{aligned} \tau'_{IJ} = & (\epsilon_0 \bar{E}_I + \bar{P}_I) e_J + \mu_0 (\bar{H}_I + \bar{M}_I) h_J + (\epsilon_0 e_I + P_I) \bar{E}_J \\ & + \mu_0 (h_I + m_I) \bar{H}_J - W' \delta_{IJ} + \mu_0 \epsilon_{JKL} \{ \dot{x}_K [\bar{P}_I h_L \\ & + \bar{H}_L P_I - \epsilon_0 (\bar{M}_I e_L + \bar{E}_L m_I)] + \dot{u}_K (\bar{P}_I \bar{H}_L - \epsilon_0 \bar{M}_I \bar{E}_L) \} \end{aligned} \tag{6.52}$$

and

$$\begin{aligned} \rho f_I'^{(em)} = & \bar{\rho}^{(f)} e_I + \rho'^{(f)} \bar{E}_I + \mu_0 \epsilon_{IJK} [\bar{J}_J^{(f)} h_K + j_J^{(f)} \bar{H}_K + \dot{u}_{L,L} (\bar{P}_J H_K \\ & - \epsilon_0 \bar{M}_J \bar{E}_K)] + e_{I,K} \bar{P}_K + P_K \bar{E}_{I,K} + \mu_0 (h_{I,K} \bar{M}_K + m_K \bar{H}_{I,K}) \\ & + \mu_0 \epsilon_{IJK} \{ \dot{x}_J (\bar{P}_N h_{K,N} + P_N \bar{H}_{K,N}) - \epsilon_0 \dot{x}_J (\bar{M}_N e_{K,N} + m_N \bar{E}_{K,N}) \\ & + \dot{u}_J (-\epsilon_0 \bar{M}_N \bar{E}_{K,N} + \bar{P}_N \bar{H}_{K,N}) + \frac{\partial \bar{P}_J}{\partial t} h_K + \frac{\partial P_J}{\partial t} \bar{H}_K \\ & - \epsilon_0 (\frac{\partial \bar{M}_J}{\partial t} e_K + \frac{\partial m_J}{\partial t} \bar{E}_K) + \dot{x}_N (P_{J,N} \bar{H}_K + h_K \bar{P}_{J,N}) \\ & - \epsilon_0 \dot{x}_N (m_{J,N} \bar{E}_K + e_K \bar{M}_{J,N}) + \dot{u}_N (\bar{H}_K \bar{P}_{J,N} - \bar{E}_K \bar{M}_{J,N}) \} \\ & - u_{L,N} \{ \bar{P}_N \bar{E}_{I,L} + \mu_0 \bar{M}_N \bar{H}_{I,L} + \mu_0 \epsilon_{IJK} [((\bar{P}_N + P_N) \dot{x}_J + \dot{u}_J \bar{P}_N) \bar{u}_{K,L} \\ & + \epsilon_0 ((\bar{M}_N + m_N) \dot{x}_J + \dot{u}_J \bar{M}_N) \bar{E}_{K,L} + ((\bar{P}_{J,L} + P_{J,L}) \dot{x}_N + \dot{u}_N \bar{P}_{J,L} \\ & + \dot{u}_{N,L} \bar{P}_J) \bar{H}_K + \epsilon_0 ((\bar{M}_{J,N} + m_{J,N}) \dot{x}_N + \dot{u}_N \bar{M}_{J,L} + \dot{u}_{N,L} \bar{M}_J) \bar{E}_K] \} \end{aligned} \tag{6.53}$$

where

$$W' = \epsilon_0 \bar{E}_K e_K + \mu_0 \bar{H}_K h_K \quad . \tag{6.54}$$

In vacuum, $\underset{\sim}{\bar{P}} = 0$, $\underset{\sim}{P} = 0$, $\underset{\sim}{\bar{M}} = 0$ and $\underset{\sim}{m} = 0$ are substituded in (6.52), from which one obtains

$$\tau^{o\prime}_{IJ} = \epsilon_o (\bar{E}_I e_J + e_I \bar{E}_J) + \mu_o (\bar{H}_I h_J + m_I \bar{H}_J) - W' \delta_{IJ} \quad (6.55)$$

The advantage of decomposition is that the equations governing the perturbed quantities are now linear, provided that the rigid body variables are known.

6.4. Kinetic Equations for the Rigid Body Motion

Our aim now is to give only a brief account of the basic concepts of rigid body mechanics when the body is in the electromagnetic fields.

To describe the motion of a rigid body under external forces and torques, two systems of coordinates are used: a "fixed" system $X_1 X_2 X_3$, and a "moving" system $Y_1 Y_2 Y_3$ which is supposed to be rigidly fixed in the body Fig.6.2. The origin of the moving system may conveniently be taken to coincide with the centre of mass of the body G. The position of the

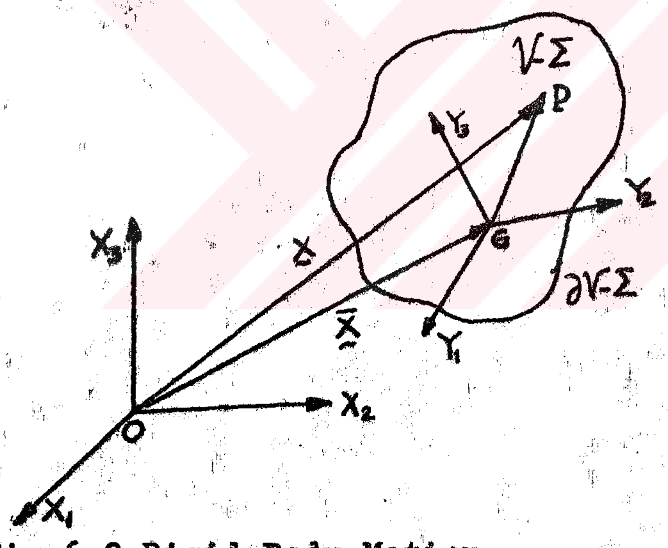


Fig.6.2 Rigid Body Motion.

body with respect to the system $X_1 X_2 X_3$ is completely determined if the position of the moving system is specified. Letting the origin G of the moving system have the radius vector $\underset{\sim}{\bar{X}}$, the orientation of the axes of that system relative to the fixed system is given by three independent angles which together with the three components of the vector $\underset{\sim}{\bar{X}}$ make six coordinates. Thus a rigid body is a mechanical system with six degrees of freedom.

Let $\underset{\sim}{Y}$ be the radius vector of an arbitrary point P in the moving system and $\underset{\sim}{X}$ the radius vector of the same point in the fixed system, Fig.6.2. Then the infinitesimal displacement $d\underset{\sim}{X}$ of P contains a displacement $d\bar{\underset{\sim}{X}}$, equal to that of the centre of mass, and a displacement $d\underset{\sim}{\phi} \times \underset{\sim}{Y}$ relative to the centre of mass resulting from a rotation through an infinitesimal angle $d\phi$, i.e.,

$$dX_K = d\bar{X}_K + \epsilon_{KLM}\, d\phi_L\, Y_M \quad . \tag{6.56}$$

Thus, the velocity $\dot{\underset{\sim}{X}}$ of any point in the rigid body relative to the fixed system is obtained from (6.56) as

$$\dot{X}_K = \dot{\bar{X}}_K + \epsilon_{KLM}\, \Omega_L\, Y_M \tag{6.57}$$

where $\dot{\bar{\underset{\sim}{X}}}$ is the velocity of the centre of mass of the body, sometimes called the "translational velocity" and the vector $\underset{\sim}{\Omega}$ is the "angular velocity". Naturally, both the magnitude and the direction of $\underset{\sim}{\Omega}$ may vary with time.

The number of equations of motion must be six. They are obtained from the principles of linear and angular momentums.

The total mass is defined by

$$\mathcal{M} \equiv \int_{V-\Sigma} \bar{\rho}\, dv \tag{6.58}$$

concentrated at G.

The linear momentum of a system of particles is equal to the momentum of a particle of mass equal to $\mathcal{M}$ moving with velocity $\dot{\bar{\underset{\sim}{X}}}$. So that the equation of translational motion of the centre of mass is obtained through

$$\frac{d}{dt}\int_{V-\Sigma} \bar{\rho}\, \dot{X}_K\, dv = \oint_{\partial V-\Sigma} \bar{t}_{IK}\, N_I\, da + \int_{V-\Sigma} \bar{\rho}\, \bar{f}_K^{(em)}\, dv \quad . \tag{6.59}$$

Upon using (6.38), (6.58) and the Green-Gauss theorem, one obtains

$$\frac{d}{dt}(\mathcal{M}\dot{\bar{X}}_K) = \oint_{\partial V-\Sigma} \bar{t}_{IK}^{(\tau)}\, N_I\, da - \int_{V-\Sigma} \frac{1}{c^2}\, \epsilon_{IKL} \frac{\partial}{\partial t}(\bar{E}_K \bar{H}_L)\, dv$$

or

$$\frac{d}{dt}(\mathcal{M}\dot{\bar{X}}_K) = \bar{\mathcal{F}}_K^{\varphi} + \bar{\mathcal{F}}_K^{\gamma} \tag{6.60}$$

where

$$
\begin{aligned}
\bar{\mathcal{F}}_K^{\mathscr{S}} &\equiv \oint_{\partial V-\Sigma} \bar{t}_{LK}^{(T)} N_L \, da = \oint_{\partial V-\Sigma} \bar{\tau}_{LK}^{\circ} N_L \, da \\
\bar{\mathcal{F}}_I^{\nu} &\equiv -\frac{1}{c^2} \epsilon_{IKL} \int_{V-\Sigma} \frac{\partial}{\partial t} (\bar{E}_K \bar{H}_L) dv \; .
\end{aligned} \tag{6.61}
$$

In Eq.(6.61), the surface integration is performed on a material surface just inside the boundary of the body and the boundary condition (4.15) is used.

The angular momentum of the particle about the fixed origin is as follows:

$$
\frac{d}{dt} \int_{V-\Sigma} \epsilon_{IJK} X_J \rho \dot{X}_K dv = \oint_{\partial V-\Sigma} \epsilon_{IJK} X_J \bar{t}_{LK} N_L \, da + \int_{V-\Sigma} \rho (\epsilon_{IJK} X_J \bar{f}_K^{(em)} + \bar{\ell}_I) dv \; . \tag{6.62}
$$

Substituting (6.38) into (6.62) and using

$$
X_I \bar{\tau}_{KL,K} = (X_I \bar{\tau}_{KL})_{,K} - \bar{\tau}_{IL} \tag{6.63}
$$

there follows

$$
\frac{d}{dt} \int_{V-\Sigma} \epsilon_{IJK} X_J \bar{\rho} \dot{X}_K dv = \bar{\mathcal{L}}^{\mathscr{S}} + \bar{\mathcal{L}}_I^{\nu} + \bar{\mathcal{L}}_I^{\mathscr{B}} \tag{6.64}
$$

where

$$
\begin{aligned}
\bar{\mathcal{L}}_I^{\mathscr{S}} &\equiv \oint_{\partial V-\Sigma} \epsilon_{IJK} X_J t_{LK}^{(T)} N_L \, da = \oint_{\partial V-\Sigma} \epsilon_{IJK} \bar{\tau}_{LK}^{\circ} N_L \, da \\
\bar{\mathcal{L}}_I^{\nu} &\equiv -\frac{1}{c^2} \int_{V-\Sigma} \epsilon_{IJK} \epsilon_{KLM} X_J \frac{\partial}{\partial t} (\bar{E}_L \bar{H}_M) dv \; ; \; \bar{\mathcal{L}}_I^{\mathscr{B}} \equiv \int_{V-\Sigma} (\rho \bar{\ell}_I - \epsilon_{IJK} \bar{\tau}_{JK}) dv .
\end{aligned} \tag{6.65}
$$

Eq.$(6.65)_3$ is called the intrinsic torque acting on the body, and due to Eqs.(3.62,66) and (6.41) it is easily seen that $\mathcal{L}_I^{\mathscr{B}} = 0$.

In order to establish the angular momentum about G, one substitutes $\underset{\sim}{X} = \underset{\sim}{Y}$ and $\dot{\underset{\sim}{X}} = \underset{\sim}{\Omega} \times \underset{\sim}{Y}$ into (6.64) and (6.65). Thus, the left hand side of (6.64) is

$$
\begin{aligned}
\frac{d}{dt} \int_{V-\Sigma} \epsilon_{IJK} Y_J \bar{\rho} \, \epsilon_{KLM} \Omega_L Y_M dv &= \frac{d}{dt} \int_{V-\Sigma} \bar{\rho} (Y_K Y_K \delta_{IJ} - Y_I Y_J) \Omega_J dv \\
&= \frac{d}{dt} (I_{IJ} \Omega_J)
\end{aligned} \tag{6.66}
$$

so that the equation of rotational motion becomes

$$\frac{d}{dt}(I_{LK}\,\Omega_K) = \bar{\mathcal{L}}^{\varphi}_{L} + \bar{\mathcal{L}}^{\nu}_{L} \tag{6.67}$$

where

$$I_{LK} \equiv \int_{V-\Sigma} \bar{\rho}\,[(X_M-\bar{X}_M)(X_M-\bar{X}_M)\delta_{LK} - (X_L-\bar{X}_L)(X_K-\bar{X}_K)]\,dV. \tag{6.68}$$

Eqs.(6.60) and (6.67) are the well known Euler's equations of motions. These equations are the same, in form, as those of Hutter and Pao's ([14],p.103), eventhough different formulations of electrodynamics have been used. However, $\bar{\mathcal{L}}^{B}_{I}$ in the Chu formulation vanishes that of the Amperian formulation does not.

The Maxwell equations (6.28) and the Euler equations of motions (6.64,67) are to be solved simultaneously. However, the number of equations is not sufficient; therefore, the constitutive equations associated with the rigid body electrodynamics are needed.

6.5. Decompositions of the Constitutive Equations

The constitutive equations for entropy, electric and magnetic fields, stress, electric current and heat flux vectors for linear anisotropic materials have been obtained in Chapter 5. Eventhough at most second order terms are retained in the polynomial approximation, the constitutive equations still contain kinematic nonlinearity.

In this section, the constitutive equations are separated into two groups, namely the constitutive equations in the rigid body state and those in the perturbation state.

Upon using (6.14,15), one decomposes Eq.$(5.45)_1$ as

$$\bar{\eta} = -\hat{\Sigma}^{(\theta)}\bar{\theta} - \frac{1}{\bar{\rho}}\left(\hat{\Sigma}^{(p\theta)}_{K}\bar{P}_K + \hat{\Sigma}^{(m\theta)}_{K}\bar{M}_K\right) \tag{6.69}$$

and

$$\eta' = -\hat{\Sigma}^{(\theta)}\theta' - \hat{\Sigma}^{(e\theta)}_{KL}\tilde{e}_{KL} - \frac{1}{\bar{\rho}}\Big[\hat{\Sigma}^{(p\theta)}_{K}(P_K + \tilde{e}_{LL}\bar{P}_K + u_{L,K}\bar{P}_L) + \hat{\Sigma}^{(m\theta)}_{K}(m_K + \tilde{e}_{LL}\bar{M}_K + u_{L,K}\bar{M}_L)\Big]. \tag{6.70}$$

Taking into account (6.14,15) in Eq.$(5.46)_1$, one obtains

$$\bar{E}_K = \hat{\Sigma}^{(p)}_{KL} \bar{P}_L + \hat{\Sigma}^{(pm)}_{KL} \bar{M}_L + \hat{\Sigma}^{(p\theta)}_{K} \bar{\theta}$$

$$- \epsilon_{KMN} \dot{X}_M \left(\hat{\Sigma}^{(pm)}_{LN} \bar{P}_L + \hat{\Sigma}^{(m)}_{NL} \bar{M}_L + \hat{\Sigma}^{(m\theta)}_{N} \bar{\theta} \right) \qquad (6.71)$$

and

$$e_K = \left(\hat{\Sigma}^{(p)}_{KL} - \epsilon_{KMN} \dot{X}_M \hat{\Sigma}^{(pm)}_{LN} \right) P_L + \left(\hat{\Sigma}^{(pm)}_{KL} - \epsilon_{KMN} \dot{X}_M \hat{\Sigma}^{(m)}_{NL} \right) m_L$$

$$+ \Big\{ \hat{\Sigma}^{(ep)}_{PQK} - \epsilon_{KLN} \dot{X}_L \hat{\Sigma}^{(em)}_{PQN} + \Big[\hat{\Xi}^{(p)}_{KQPM} - \epsilon_{KLN} \dot{X}_L \big(\hat{\Xi}^{(pm)}_{MQNP}$$

$$+ \tilde{e}_{SS} \hat{\underline{\Xi}}^{(pm)}_{MQNP} \big) \Big] \bar{P}_M + \Big[\hat{\Xi}^{(pm)}_{KQPM} - \epsilon_{KLN} \dot{X}_L \big(\hat{\Xi}^{(m)}_{QMNP} \qquad (6.72)$$

$$+ \tilde{e}_{SS} \hat{\underline{\Xi}}^{(m)}_{QMNP} \big) \Big] \bar{M}_M + \left(\hat{\Sigma}^{(p\theta)}_{Q} \delta_{KP} - \epsilon_{KLN} \dot{X}_L \hat{\Sigma}^{(m\theta)}_{Q} \delta_{PN} \right) \bar{\theta} \Big\} \times$$

$$\times u_{P,Q} + \left(\hat{\Sigma}^{(p\theta)}_{K} - \epsilon_{KMN} \dot{X}_M \hat{\Sigma}^{(m\theta)}_{N} \right) \theta'$$

$$- \epsilon_{KLN} \dot{u}_L \left(\hat{\Sigma}^{(pm)}_{MN} \bar{P}_M + \hat{\Sigma}^{(m)}_{NM} \bar{M}_M + \hat{\Sigma}^{(m\theta)}_{N} \bar{\theta} \right)$$

where

$$\hat{\Xi}^{(p)}_{KQPM} \equiv \hat{\Sigma}^{(p)}_{KQ} \delta_{PM} + \hat{\Sigma}^{(p)}_{QM} \delta_{KP} + \hat{\Sigma}^{(p)}_{KM} \delta_{PQ}$$

$$\hat{\Xi}^{(pm)}_{MQNP} \equiv \hat{\Sigma}^{(pm)}_{MQ} \delta_{NP} + \hat{\Sigma}^{(pm)}_{QN} \delta_{PM} + \hat{\Sigma}^{(pm)}_{MN} \delta_{PQ}$$

$$\hat{\underline{\Xi}}^{(pm)}_{MQNP} \equiv \hat{\Sigma}^{(pm)}_{MQ} \delta_{NP} + \hat{\Sigma}^{(pm)}_{QN} \delta_{PM} \qquad (6.73)$$

$$\hat{\Xi}^{(pm)}_{KQPM} \equiv \hat{\Sigma}^{(pm)}_{KQ} \delta_{PN} + \hat{\Sigma}^{(pm)}_{QM} \delta_{KP} + \hat{\Sigma}^{(pm)}_{KM} \delta_{PQ}$$

$$\hat{\Xi}^{(m)}_{QMNP} \equiv \hat{\Sigma}^{(m)}_{QM} \delta_{NP} + \hat{\Sigma}^{(m)}_{NQ} \delta_{PM} + \hat{\Sigma}^{(m)}_{NM} \delta_{PQ}$$

$$\hat{\underline{\Xi}}^{(m)}_{QMNP} \equiv \hat{\Sigma}^{(m)}_{QM} \delta_{NP} + \hat{\Sigma}^{(m)}_{NQ} \delta_{PM} \, .$$

In a similar manner, from $(5.46)_2$, one obtains

$$\mu_0 \bar{H}_K = \hat{\Sigma}^{(pm)}_{LK}\, \bar{P}_L + \hat{\Sigma}^{(m)}_{KL}\, \bar{M}_L + \hat{\Sigma}^{(m\theta)}_{K}\, \bar{\theta}$$

$$+ \frac{1}{c^2}\, \epsilon_{KMN}\, \dot{X}_M \left(\hat{\Sigma}^{(p)}_{NL}\, \bar{P}_L + \hat{\Sigma}^{(pm)}_{NL}\, \bar{M}_L + \hat{\Sigma}^{(p\theta)}_{N}\, \bar{\theta} \right) \qquad (6.74)$$

and

$$\mu_0 h_K = \left(\hat{\Sigma}^{(pm)}_{LK} + \frac{1}{c^2}\, \epsilon_{KMN}\, \dot{X}_M\, \hat{\Sigma}^{(p)}_{NL} \right) P_L + \left(\hat{\Sigma}^{(m)}_{KL} \right.$$

$$\left. + \frac{1}{c^2}\, \epsilon_{KMN}\, \dot{X}_M\, \hat{\Sigma}^{(pm)}_{NL} \right) m_L + \left\{ \hat{\Sigma}^{(em)}_{PQK} + \frac{1}{c^2}\, \epsilon_{KLN}\, \dot{X}_L\, \hat{\Sigma}^{(ep)}_{PQN} \right.$$

$$+ \left[\hat{\Xi}^{(pm)}_{QKPM} + \frac{1}{c^2}\, \epsilon_{KLN}\, \dot{X}_L \left(\hat{\Xi}^{(p)}_{QMNP} + \tilde{e}_{SS}\, \hat{\underline{\underline{\Xi}}}^{(p)}_{QMNP} \right) \right] \bar{P}_M \qquad (6.75)$$

$$+ \left[\hat{\Xi}^{(m)}_{KQPM} + \frac{1}{c^2}\, \epsilon_{KLN}\, \dot{X}_L \left(\hat{\Xi}^{(pm)}_{QMNP} + \tilde{e}_{SS}\, \hat{\underline{\underline{\Xi}}}^{(pm)}_{QMNP} \right) \right] \bar{M}_M$$

$$\left. + \left(\hat{\Sigma}^{(m\theta)}_{Q}\, \delta_{KP} + \frac{1}{c^2}\, \epsilon_{KLN}\, \dot{X}_L\, \hat{\Sigma}^{(p\theta)}_{Q}\, \delta_{NP} \right) \bar{\theta} \right\} u_{P,Q}$$

$$+ \left(\hat{\Sigma}^{(m\theta)}_{K} + \frac{1}{c^2}\, \epsilon_{KLN}\, \dot{X}_L\, \hat{\Sigma}^{(p\theta)}_{N} \right) \theta' + \frac{1}{c^2}\, \epsilon_{KLN}\, \dot{u}_L \left(\hat{\Sigma}^{(p)}_{NM}\, \bar{P}_M \right.$$

$$\left. + \hat{\Sigma}^{(pm)}_{NM}\, \bar{M}_M + \hat{\Sigma}^{(p\theta)}_{N}\, \bar{\theta} \right)$$

where

$$\hat{\Xi}^{(pm)}_{QKPM} \equiv \hat{\Sigma}^{(pm)}_{QK}\, \delta_{PM} + \hat{\Sigma}^{(pm)}_{MQ}\, \delta_{KP} + \hat{\Sigma}^{(pm)}_{MK}\, \delta_{KQ}$$

$$\hat{\Xi}^{(p)}_{QMNP} \equiv \hat{\Sigma}^{(p)}_{QM}\, \delta_{NP} + \hat{\Sigma}^{(p)}_{NQ}\, \delta_{PM} + \hat{\Sigma}^{(p)}_{NM}\, \delta_{PQ}$$

$$\hat{\underline{\underline{\Xi}}}^{(p)}_{QMNP} \equiv \hat{\Sigma}^{(p)}_{QM}\, \delta_{NP} + \hat{\Sigma}^{(p)}_{NQ}\, \delta_{PM} \qquad (6.76)$$

$$\hat{\Xi}^{(m)}_{KQPM} \equiv \hat{\Sigma}^{(m)}_{KQ}\, \delta_{PM} + \hat{\Sigma}^{(m)}_{QM}\, \delta_{KP} + \hat{\Sigma}^{(m)}_{KM}\, \delta_{PQ}$$

$$\hat{\Xi}^{(pm)}_{QMNP} \equiv \hat{\Sigma}^{(pm)}_{QM}\, \delta_{NP} + \hat{\Sigma}^{(pm)}_{NQ}\, \delta_{PM} + \hat{\Sigma}^{(pm)}_{NM}\, \delta_{PQ}$$

$$\hat{\underline{\underline{\Xi}}}^{(pm)}_{QMNP} \equiv \hat{\Sigma}^{(pm)}_{QM}\, \delta_{NP} + \hat{\Sigma}^{(pm)}_{NQ}\, \delta_{PM} \; .$$

Using (6.14,15) in Eq.(5.55)$_1$, one gets

$$t_{KL} = (\hat{\underline{C}}^{(p)}_{KLM} + \bar{E}_K \delta_{LM})\bar{P}_M + (\hat{\underline{C}}^{(m)}_{KLM} + \mu_0 \bar{\mathcal{H}}_K \delta_{LM})\bar{M}_M \tag{6.77}$$
$$+ \hat{\underline{C}}^{(\theta)}_{KL}\bar{\theta} + \hat{C}^{(g)}_{KLM}\bar{\theta}_{,M}$$

and

$$t'_{KL} = (\hat{\underline{C}}^{(p)}_{KLM} + \bar{E}_K\delta_{LM})p_M + (\hat{\underline{C}}^{(m)}_{KLM} + \mu_0\bar{\mathcal{H}}_K\delta_{LM})m_M + \tilde{e}'_K\bar{P}_L$$
$$+ \mu_0\tilde{h}'_K\bar{M}_L + \Big[\hat{\underline{C}}^{(e)}_{KLPQ} + \hat{C}^{(v)}_{KLPQ}\frac{\partial}{\partial t} + \hat{\underline{J}}^{(p)}_{KLQNP}\bar{P}_N$$
$$+ \hat{\underline{J}}^{(m)}_{KLQNP}\bar{M}_N + (\hat{\underline{J}}^{(\theta)}_{QLKP} - \hat{C}^{(\theta)}_{KL}\delta_{PQ})\bar{\theta} + (\hat{\underline{J}}^{(g)}_{QLMPK} \tag{6.78}$$
$$- \hat{C}^{(g)}_{KLM}\delta_{PQ})\bar{\theta}_{,M}\Big]u_{P,Q} + \hat{\underline{C}}^{(\theta)}_{KL}\theta' + \hat{C}^{(g)}_{KLM}\theta'_{,M}$$
$$+ \epsilon_{KMN}\dot{u}_M\left(\mu_0\bar{H}_N\bar{P}_L - \frac{1}{c^2}\bar{E}_N\bar{M}_L\right)$$

where

$$\hat{\underline{J}}^{(p)}_{KLQNP} \equiv \hat{\underline{C}}^{(p)}_{KLQ}\delta_{NP} + \hat{\underline{C}}^{(p)}_{KQN}\delta_{LP} + \hat{\underline{C}}^{(p)}_{QLN}\delta_{KP}$$
$$\hat{J}^{(m)}_{KLQNP} \equiv \hat{\underline{C}}^{(m)}_{KLQ}\delta_{NP} + \hat{\underline{C}}^{(m)}_{KQN}\delta_{LP} + \hat{\underline{C}}^{(m)}_{QLN}\delta_{KP} \tag{6.79}$$
$$\hat{\underline{J}}^{(\theta)}_{QLKP} \equiv \hat{\underline{C}}^{(\theta)}_{QL}\delta_{KP} + \hat{\underline{C}}^{(\theta)}_{KQ}\delta_{LP}$$
$$\hat{\underline{J}}^{(g)}_{QLMPK} \equiv \hat{C}^{(g)}_{QLM}\delta_{PK} + \hat{C}^{(g)}_{KQM}\delta_{LP}$$

and

$$\tilde{e}'_K = e_K + \mu_0\epsilon_{KLM}\dot{x}_L h_m \quad ; \quad \tilde{h}'_K = h_K - \epsilon_0\epsilon_{KLM}\dot{x}_L e_M . \tag{6.80}$$

One can similarly decompose the constitutive equations for for the conduction current and the heat flux vectors. From (5.55)$_2$ it follows that

$$\bar{J}_K = \hat{K}^{(p)}_{KL}\bar{P}_L + \hat{K}^{(m)}_{KL}\bar{M}_L + \hat{K}^{(\theta)}_K\bar{\theta} + \hat{K}^{(g)}_{KL}\bar{\theta}_{,L} \tag{6.81}$$

and

$$\dot{j}_K = \hat{K}^{(p)}_{KL} P_L + \hat{K}^{(m)}_{KL} m_L + \Big[\hat{K}^{(e)}_{KNM} + \hat{K}^{(v)}_{KNM}\frac{\partial}{\partial t}$$

$$+ \big(\underline{\hat{K}}^{(p)}_{ML}\bar{P}_L + \underline{\hat{K}}^{(m)}_{ML}\bar{M}_L + \underline{\hat{K}}^{(\theta)}_{M}\bar{\theta} + \underline{\hat{K}}^{(g)}_{ML}\bar{\theta}_{,L}\big)\delta_{KN}\Big] u_{N,M}$$

$$+ \hat{K}^{(\theta)}_{K}\theta' + \hat{K}^{(g)}_{KL}\theta'_{,L} \tag{6.82}$$

where

$$\begin{aligned}
\underline{\hat{K}}^{(p)}_{ML}\,\delta_{KN} &\equiv \hat{K}^{(p)}_{ML}\,\delta_{KN} + \hat{K}^{(p)}_{KM}\,\delta_{LN} \\
\underline{\hat{K}}^{(m)}_{ML}\,\delta_{KN} &\equiv \hat{K}^{(m)}_{ML}\,\delta_{KN} + \hat{K}^{(m)}_{KM}\,\delta_{LN} \\
\underline{\hat{K}}^{(\theta)}_{M}\,\delta_{KN} &\equiv \hat{K}^{(\theta)}_{M}\,\delta_{KN} - \hat{K}^{(\theta)}_{K}\,\delta_{NM} \\
\underline{\hat{K}}^{(g)}_{ML}\,\delta_{KN} &\equiv \hat{K}^{(g)}_{ML}\,\delta_{KN} - \hat{K}^{(g)}_{KL}\,\delta_{NM} \quad .
\end{aligned} \tag{6.83}$$

From $(5.55)_3$, one writes

$$\bar{q}_K = \hat{G}^{(p)}_{KL}\bar{P}_L + \hat{G}^{(m)}_{KL}\bar{M}_L + \hat{G}^{(\theta)}_{K}\bar{\theta} + \hat{G}^{(g)}_{KL}\bar{\theta}_{,L} \tag{6.84}$$

and

$$q'_K = \hat{G}^{(p)}_{KL} P_L + \hat{G}^{(m)}_{KL} m_L + \Big[\hat{G}^{(e)}_{KNM} + \hat{G}^{(v)}_{KNM}\frac{\partial}{\partial t}$$

$$+ \big(\underline{\hat{G}}^{(p)}_{ML}\bar{P}_L + \underline{\hat{G}}^{(m)}_{ML}\bar{M}_L + \underline{\hat{G}}^{(\theta)}_{M}\bar{\theta} + \underline{\hat{G}}^{(g)}_{ML}\bar{\theta}_{,L}\big)\delta_{KN}\Big] u_{N,M} \tag{6.85}$$

$$+ \hat{G}^{(\theta)}_{K}\theta' + G^{(g)}_{KL}\theta'_{,L}$$

where

$$\begin{aligned}
\underline{\hat{G}}^{(p)}_{ML}\,\delta_{KN} &\equiv \hat{G}^{(p)}_{ML}\,\delta_{KN} + \hat{G}^{(p)}_{KM}\,\delta_{LN} \\
\underline{\hat{G}}^{(m)}_{ML}\,\delta_{KN} &\equiv \hat{G}^{(m)}_{ML}\,\delta_{KN} + \hat{G}^{(m)}_{KM}\,\delta_{LN} \\
\underline{\hat{G}}^{(\theta)}_{M}\,\delta_{KN} &\equiv \hat{G}^{(\theta)}_{M}\,\delta_{KN} - \hat{G}^{(\theta)}_{K}\,\delta_{NM} \\
\underline{\hat{G}}^{(g)}_{ML}\,\delta_{KN} &\equiv \hat{G}^{(g)}_{ML}\,\delta_{KN} - \hat{G}^{(g)}_{KL}\,\delta_{NM} \quad .
\end{aligned} \tag{6.86}$$

Eqs.(6.81) and (6.84) stand for Ohm's law of electric conduction and Fourier's law of heat conduction in rigid body state.

The Maxwell's equations (6.28), the Euler equations of motions (6.64,67) together with the constitutive equations (6.81, 84) are sufficient for the solutions of the variables in the rigid body state.

6.6. Constitutive Equations for Special Symmetries

In the previous section, the constitutive equations have been decomposed into two groups without considering the symmetry of the materials. For these materials if either (5.43) and (5.57) or (5.44) and (5.58) are satisfied and/or the materials are isotropic, then the constitutive equations become simpler and the certain interaction phenomena disappear.

Imposing the restrictions coming from (5.43,44) and (5.57, 58) onto (6.69-86), one obtains the decomposed constitutive equations for the centrosymmetric materials without magnetic symmetry. The constitutive equations are

$$\begin{aligned}
\bar{\eta} &= -\hat{\Sigma}^{(\theta)}\bar{\theta} \\
\bar{E}_K &= \hat{\Sigma}^{(p)}_{KL}\bar{P}_L - \epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(m)}_{NL}\bar{M}_L \\
\mu_0\bar{H}_K &= \hat{\Sigma}^{(m)}_{KL}\bar{M}_L + \frac{1}{c^2}\epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(p)}_{NL}\bar{P}_L \\
\bar{t}_{KL} &= \bar{\mathcal{E}}_K\bar{P}_L + \mu_0\bar{\mathcal{H}}_K\bar{M}_L + \hat{\underline{C}}^{(\theta)}_{KL}\bar{\theta} \\
\bar{J}_K &= \hat{K}_{KL}\bar{P}_L + \hat{K}^{(q)}_{KL}\bar{\theta}_{,L} \\
\bar{q}_K &= \hat{G}^{(p)}_{KL}\bar{P}_L + \hat{G}^{(q)}_{KL}\bar{\theta}_{,L}
\end{aligned} \qquad (6.87)$$

in the rigid body state, and

$$\begin{aligned}
\eta' &= -\hat{\Sigma}^{(\theta)}\theta' - \hat{\Sigma}^{(e\theta)}_{KL}\tilde{e}_{KL} \\
e_K' &= \hat{\Sigma}^{(p)}_{KL}P_L - \epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(m)}_{NL}m_L + \Big[\hat{\Xi}^{(p)}_{KQPM}\bar{P}_M - \epsilon_{KLN}\dot{X}_L\big(\hat{\Xi}^{(m)}_{NMNP} \\
&\quad + \tilde{a}_{SS}\hat{\Xi}^{(m)}_{QMNP}\big)\bar{M}_M\Big]u_{P,Q} - \epsilon_{KLN}\dot{u}_L\hat{\Sigma}^{(m)}_{NM}\bar{M}_M
\end{aligned}$$

$$\mu_0 h_K = \frac{1}{c^2} \epsilon_{KMN} \overset{\circ}{X}_M \hat{\Sigma}^{(p)}_{NL} P_L + \hat{\Sigma}^{(m)}_{KL} m_L + \Big[\frac{1}{c^2} \epsilon_{KLN} \overset{\circ}{X}_L \big(\hat{\Xi}^{(p)}_{QMNP}$$

$$+ \tilde{e}_{SS} \underline{\hat{\Xi}}^{(p)}_{QMNP} \big) \bar{P}_M + \hat{\Xi}^{(m)}_{KQPM} \bar{M}_M \Big] u_{P,Q}$$

$$+ \frac{1}{c^2} \epsilon_{KLN} \dot{u}_L \hat{\Sigma}^{(p)}_{NM} \bar{P}_M$$

$$t'_{KL} = \Big[\underline{\hat{C}}^{(e)}_{KLPQ} + \hat{C}^{(v)}_{KLPQ} \frac{\partial}{\partial t} + \big(\underline{\hat{J}}^{(\theta)}_{QLKP} - \hat{C}^{(\theta)}_{KL} \delta_{PQ} \big) \bar{\theta} \Big] u_{P,Q}$$

$$+ \underline{\hat{C}}^{(\theta)}_{KL} \theta' + \tilde{e}'_K \bar{P}_L + \mu_0 \tilde{h}'_K \bar{M}_L \qquad (6.88)$$

$$+ \epsilon_{KMN} \dot{u}_M \big(\mu_0 \bar{H}_N \bar{P}_L - \frac{1}{c^2} \bar{E}_N \bar{M}_L \big)$$

$$j_K = \hat{K}^{(p)}_{KL} P_L + \big(\underline{\hat{K}}^{(p)}_{ML} \bar{P}_L + \underline{\hat{K}}^{(g)}_{ML} \bar{\theta}_{,L} \big) u_{K,M} + \hat{K}^{(g)}_{KL} \theta'_{,L}$$

$$q'_K = \hat{G}^{(p)}_{KL} P_L + \big(\underline{\hat{G}}^{(p)}_{ML} \bar{P}_L + \underline{\hat{G}}^{(g)}_{ML} \bar{\theta}_{,L} \big) u_{K,M} + \hat{G}^{(g)}_{KL} \theta'_{,L}$$

in the perturbation state. These constitutive equations are in agreement partially with those of Hutter and Pao's [14]. Thus, some of the interactions, such as piezoelectricity, pyroelectricity, piezomagnetism, etc. disappear in (6.87) and (6.88).

Now, making use of the result that all the even order material property tensors are expressed in terms of the Kronekar deltas, the constitutive equations above are further simplified [75]. Thus, one has

$$\underline{\hat{C}}^{(e)}_{KLMN} = \hat{c}_1 \delta_{KL} \delta_{MN} + \hat{c}_2 (\delta_{KM} \delta_{LN} + \delta_{KN} \delta_{LM}) + \hat{c}_3 (\delta_{KM} \delta_{LN} - \delta_{KN} \delta_{LM})$$

$$\hat{C}^{(v)}_{KLMN} = \hat{c}'_1 \delta_{KL} \delta_{MN} + \hat{c}'_2 (\delta_{KM} \delta_{LN} + \delta_{KN} \delta_{LM}) + \hat{c}'_3 (\delta_{KM} \delta_{LN} - \delta_{KN} \delta_{LM})$$

$$\hat{\Sigma}^{(\theta)} = -\hat{c}_7 \;;\; \hat{\Sigma}^{(e\theta)}_{KL} = \hat{c}'_6 \delta_{KL} \;;\; \hat{K}^{(p)}_{KL} = \hat{c}_8 \delta_{KL} \qquad (6.89)$$

$$\hat{K}^{(g)}_{KL} = \hat{c}_9 \delta_{KL} \;;\; \hat{G}^{(p)}_{KL} = \hat{c}_{10} \delta_{KL} \;;\; \hat{G}^{(g)}_{KL} = \hat{c}_{11} \delta_{KL} \;.$$

From (6.89) and (6.87,88), there follows that

$$
\begin{aligned}
&\bar{\eta} = \hat{c}_7\, \bar{\theta} \\
&\bar{E}_K = \hat{c}_4\, \bar{P}_K - \hat{c}_5\, \epsilon_{KLM}\, \dot{X}_L\, \bar{M}_M \\
&\mu_0 \bar{H}_K = \hat{c}_5 \bar{M}_K + \frac{1}{c^2}\hat{c}_4\, \epsilon_{KLM}\, \dot{X}_L\, \bar{P}_M \\
&\bar{t}_{KL} = \bar{\mathcal{E}}_K\, \bar{P}_L + \mu_0 \bar{\mathcal{H}}_K \bar{M}_L + \hat{c}_6\, \bar{\theta}\, \delta_{KL} \\
&\bar{J}_K = \hat{c}_8\, \bar{P}_K + \hat{c}_9\, \bar{\theta}_{,K} \\
&\bar{q}_K = \hat{c}_{10}\, \bar{P}_K + \hat{c}_{11}\, \bar{\theta}_{,K}
\end{aligned}
\tag{6.90}
$$

in the rigid body state, and

$$
\begin{aligned}
&\eta' = \hat{c}_7\, \theta' + \hat{c}_6\, \tilde{e}_{KK} \\
&e_K = \hat{c}_4 \left[P_K + (\tilde{e}_{NN}\, \delta_{KL} + 2\tilde{a}_{KL})\, \bar{P}_L \right] \\
&\qquad - \hat{c}_5\, \epsilon_{KLN} \left\{ \dot{X}_L \left[m_N + (\tilde{e}_{KK}\, \delta_{NM} + 2\tilde{a}_{NM})\, \bar{M}_M \right] + \dot{u}_L\, \bar{M}_N \right\} \\
&\mu_0 h_K = \hat{c}_5 \left[m_K + (\tilde{e}_{NN}\, \delta_{KL} + 2\tilde{a}_{KL})\, \bar{M}_L \right] \\
&\qquad + \frac{\hat{c}_4}{c^2}\, \epsilon_{KLN} \left\{ \dot{X}_L \left[P_N + (\tilde{e}_{RR}\, \delta_{NM} + 2\tilde{a}_{NM})\, \bar{P}_M \right] + \dot{u}_L\, \bar{P}_N \right\} \\
&t'_{KL} = \bar{\mathcal{E}}_K\, P_L + \mu_0 \bar{\mathcal{H}}_K\, m_L + \tilde{a}'_K\, \bar{P}_L + \mu_0 \tilde{h}'_K\, \bar{M}_L \\
&\qquad + \left(\hat{c}_1 + \hat{c}_1' \frac{\partial}{\partial t}\right) \tilde{e}_{MM}\, \delta_{KL} + 2\left(\hat{c}_2 + \hat{c}_2' \frac{\partial}{\partial t}\right) \tilde{e}_{KL} \\
&\qquad + \hat{c}_6\, (\theta'\, \delta_{KL} + \bar{\theta}\, u_{K,L}) + \epsilon_{KMN}\, \dot{u}_M \left(\mu_0 \bar{H}_N \bar{P}_L - \frac{1}{c^2} \bar{E}_N \bar{M}_L \right) \\
&j_K = \hat{c}_8\, (P_K + 2\tilde{a}_{KL}\, \bar{P}_L) + \hat{c}_9\, (\theta'_{,K} + u_{K,L}\, \bar{\theta}_{,L} - \tilde{e}_{LL}\, \bar{\theta}_{,K}) \\
&q'_K = \hat{c}_{10}\, (P_K + 2\tilde{a}_{KL}\, \bar{P}_L) + \hat{c}_{11}\, (\theta'_{,K} + u_{K,L}\, \bar{\theta}_{,L} - \tilde{e}_{LL}\, \bar{\theta}_{,K})
\end{aligned}
\tag{6.91}
$$

in the perturbation state. Using $(6.90)_{2,3}$ and $(6.91)_{2,3}$ in $(6.90)_4$ and $(6.91)_4$, one obtains

$$\bar{t}_{KL} = \hat{c}_4 \bar{P}_K \bar{P}_L + \hat{c}_5 \bar{M}_K \bar{M}_L + \hat{c}_6 \bar{\theta} \delta_{KL}$$

and

$$\begin{aligned} t'_{KL} = {} & \hat{c}_4 (\bar{P}_K P_L + P_K \bar{P}_L + \tilde{e}_{MM} \bar{P}_K \bar{P}_L + 2 \tilde{e}_{KM} \bar{P}_M \bar{P}_L) \\ & + \hat{c}_5 (\bar{M}_K m_L + m_K \bar{M}_L + \tilde{e}_{MM} \bar{M}_K \bar{M}_L + 2 \tilde{e}_{KM} \bar{M}_M \bar{M}_L) \\ & + \hat{c}_6 (\theta' \delta_{KL} + 2 \tilde{e}_{KL} \bar{\theta}) - c'_6 \bar{\theta} \tilde{e}_{MM} \delta_{KL} \\ & + (\hat{c}_1 + \hat{c}'_1 \frac{\partial}{\partial t}) \tilde{e}_{MM} \delta_{KL} + 2 (c_2 + c'_2 \frac{\partial}{\partial t}) \tilde{e}_{KL} \; . \end{aligned} \tag{6.92}$$

It should be noted that $\bar{t}_{[KL]} = 0$, but $t'_{[KL]} \neq 0$. Moreover, from (6.91), even infinitesimal deformation causes anisotropy eventhough the material is initially isotropic.

The special cases of the present governing equations are discussed in the next chapter.

CHAPTER 7

SPECIAL CASES

Chapters 2-5 have provided a dynamic theory for magneto-electro thermo-viscoelastic solids having thermal and electrical conductivity. It has been observed that the governing equations are highly nonlinear and very complicated. Later, in Chapter 6, we proposed a linearization process by decomposing the governing equations into two groups which are simpler than the original ones, but they still contain kinematic nonlinearity. On the other hand, in a specific problem, some of the considered variables may not occur, or they may have certain properties. In the following special cases, the governing equations can further be simplified:

i) Deformable material constrained from rigid body motions ($\dot{\underset{\sim}{X}} = 0$),

ii) Thermally and electrically nonconductive materials ($\underset{\sim}{J} = \underset{\sim}{q} = 0$),

iii) Omitting certain terms due to deformations (Certain terms in the perturbation state, such as $\bar{P}_K u_{K,L}$, $\bar{M}_K u_{K,L}$ are negligible compared to P_K , m_K respectively),

iv) Quasi static electric field system ($\underset{\sim}{M} = 0$, $\frac{\partial \underset{\sim}{H}}{\partial t} = 0$),

v) Quasi static magnetic field system ($\underset{\sim}{P} = 0$, $\frac{\partial \underset{\sim}{E}}{\partial t} = 0$),

vi) Pure thermo-viscoelasticity ($\underset{\sim}{P} = 0$, $\underset{\sim}{M} = 0$).

In this chapter, the cases (i-vi) are discussed and an alternative decomposition is proposed.

7.1. Material Constrained from Rigid Body Motions

If the material is constrained from the rigid body motions, one obtains the associated governing equations by substituting $\dot{\underset{\sim}{X}} = 0$. From Eqs.(6.12) and (6.13), there follows

$$v_k \equiv \dot{u}_k = \frac{\partial u_k}{\partial t}$$

and

$$a_k \equiv \frac{d v_k}{d t} = \frac{\partial v_k}{\partial t} + v_m v_{k,m} \cong \frac{\partial^2 u_k}{\partial t^2} \qquad (7.1)$$

The energy equation in the rigid body state and the entropy inequality remain unchanged; however, the former changes in the perturbation state. From (6.26), one obtains

$$\bar{\rho}\left[(\theta' - \bar{\theta}\, u_{K,K})\dot{\bar{\eta}} + \bar{\theta}\dot{\eta}'\right] = {}_{(D)}t_{KL}\,\dot{u}_{K,L} + j_K^{\prime(f)}\bar{\mathcal{E}}_K + \bar{J}_K^{(f)}\mathcal{E}'_K - q'_{K,K} - u_{K,L}\,\bar{q}_{K,L} + \rho\, r_\theta' \quad . \tag{7.2}$$

It follows from (6.29) and (6.32-35) that there is no apparent change in the form of the Maxwell's equations, but the current source terms in $(6.30)_{1,2}$ now become

$$\bar{J}_K^{(P)} = \frac{\partial \bar{P}_K}{\partial t} \quad ; \quad \bar{J}_K^{(m)} = \frac{\partial \mu_0 \bar{M}_K}{\partial t} \quad . \tag{7.3}$$

The boundary conditions (6.36) is the same, but (6.37) turns into

$$[\![\bar{E}_I]\!] K_I = 0 \quad ; \quad [\![\bar{H}_I]\!] K_I = 0 \quad . \tag{7.4}$$

From Eqs.(6.39,40), the Maxwell's stress tensor and the electromagnetic body force now change to

$$\bar{\tau}_{KL} = (\epsilon_0 \bar{E}_K + \bar{P}_K)\bar{E}_L + \mu_0(\bar{H}_K + \bar{M}_K)\bar{H}_L - \bar{W}\,\delta_{KL} \tag{7.5}$$

and

$$\rho \bar{f}_K^{(em)} = \bar{\rho}^{(f)}\bar{E}_K + \bar{P}_L\,\bar{E}_{K,L} + \mu_0 \bar{M}_L\,\bar{H}_{K,L} + \mu_0\,\epsilon_{KLM}\left[(\bar{J}_L^{(f)} + \bar{J}_L^{(P)})\bar{H}_M - \epsilon_0 c^2 \bar{J}_L^{(m)}\bar{E}_M\right] \tag{7.6}$$

respectively.

It is now observed that the form of the Maxwell equations in the perturbation state (6.44) and (6.47-50) do not change, but the current source terms $(6.45)_{1,2}$ reduce to

$$j_K^{(P)} = \frac{\partial p_K}{\partial t} + \left(\bar{P}_K \frac{\partial u_L}{\partial t} - \bar{P}_L \frac{\partial u_K}{\partial t}\right)_{,L}$$

and

$$j_K^{(m)} = \frac{\partial \mu_0 m_K}{\partial t} + \mu_0\left(\bar{M}_K \frac{\partial u_L}{\partial t} - \bar{M}_L \frac{\partial u_K}{\partial t}\right)_{,L} \quad . \tag{7.7}$$

It can be seen immediately that (6.50) is satisfied if

$$\tilde{\mathcal{E}}'_I = e_I + \mu_0\,\epsilon_{IJK}\,\dot{u}_J\,\bar{H}_K$$

and

$$\tilde{\mathcal{H}}'_I = h_I - \epsilon_0\,\epsilon_{IJK}\,\dot{u}_J\,\bar{E}_K \quad . \tag{7.8}$$

From Eqs.(6.52) and (6.53)

$$\tau'_{KL}=(\epsilon_0 \bar{E}_K+\bar{P}_K)e_L+\mu_0(\bar{H}_K+\bar{M}_K)h_L+(\epsilon_0 e_K+P_K)\bar{E}_L \qquad (7.9)$$
$$+\mu_0(h_K+m_K)\bar{H}_L-W'\delta_{KL}$$

and

$$\rho f_K^{\prime(em)}=\bar{\rho}^{(f)}e_K+\rho^{\prime(f)}\bar{E}_K+\mu_0\epsilon_{KLM}\Big[\bar{J}_L^{(f)}h_M+j_L^{(f)}\bar{H}_M$$
$$+\dot{u}_{N,N}(\bar{P}_L\bar{H}_M-\epsilon_0\bar{M}_L\bar{E}_M)\Big]+e_{K,L}\bar{P}_L+P_L\bar{E}_{I,L}$$
$$+\mu_0 h_{K,L}\bar{M}_L+\mu_0 m_L\bar{H}_{K,L}+\mu_0\epsilon_{KLM}\Big[\dot{u}_L(\bar{P}_N\bar{H}_{M,N}$$
$$-\epsilon_0\bar{M}_N\bar{E}_{M,N})+\dot{u}_N(\bar{H}_M\bar{P}_{L,N}-\bar{E}_M\bar{M}_{L,N}) \qquad (7.10)$$
$$+\frac{\partial\bar{P}_L}{\partial t}h_M+\frac{\partial P_L}{\partial t}\bar{H}_M-\epsilon_0\Big(\frac{\partial\bar{M}_L}{\partial t}e_M+\frac{\partial m_L}{\partial t}\bar{E}_M\Big)\Big]$$
$$-u_{L,N}\Big\{\bar{P}_N\bar{E}_{K,L}+\mu_0\bar{M}_N\bar{H}_{K,L}+\mu_0\epsilon_{KJM}\Big[\dot{u}_J(\bar{P}_N\bar{H}_{M,L}$$
$$+\epsilon_0\bar{M}_N\bar{E}_{M,L})+(\dot{u}_N\bar{P}_{J,L}+\dot{u}_{N,L}\bar{P}_J)\bar{H}_M+\epsilon_0(\dot{u}_N\bar{M}_{J,L}+\dot{u}_{N,L}\bar{M}_J)\bar{E}_M\Big\}$$

are found easily.

Provided that $\dot{\underset{\sim}{X}}=0$, the constitutive equations for the electric and magnetic fields and the stress tensor become much more simpler; however, those for the entropy, the electric and heat flux vectors do not alter. In this case, the constitutive equations for $\underset{\sim}{E}$, $\underset{\sim}{H}$ and $\underset{\sim}{t}$ in the rigid body and perturbation states are the following:

From Eqs.(6.71,74,77) and (6.72,75,78), one obtains

$$\bar{E}_K=\hat{\Sigma}^{(P)}_{KL}\bar{P}_L+\hat{\Sigma}^{(Pm)}_{KL}\bar{M}_L+\hat{\Sigma}^{(P\theta)}_{K}\bar{\theta}$$
$$\mu_0\bar{H}_K=\hat{\Sigma}^{(Pm)}_{LK}\bar{P}_L+\hat{\Sigma}^{(m)}_{KL}\bar{M}_L+\hat{\Sigma}^{(m\theta)}_{K}\bar{\theta}$$
$$\bar{t}_{KL}=(\hat{\underset{\sim}{C}}^{(P)}_{KLM}+\bar{E}_K\delta_{LM})\bar{P}_M+(\hat{\underset{\sim}{C}}^{(m)}_{KLM}+\mu_0\bar{H}_K\delta_{LM})\bar{M}_M \qquad (7.11)$$
$$+\hat{\underset{\sim}{C}}^{(\theta)}_{KL}\bar{\theta}+\hat{C}^{(g)}_{KLM}\bar{\theta}_{,M}$$

and

$$a_K = \hat{\Sigma}^{(p)}_{KL} P_L + \hat{\Sigma}^{(pm)}_{KL} m_L + \Big(\hat{\Sigma}^{(ep)}_{PQK} + \hat{\mathcal{J}}^{(p)}_{KQPM} \bar{P}_M + \hat{\mathcal{J}}^{(pm)}_{KQPM} \bar{M}_M$$
$$+ \hat{\Sigma}^{(p\theta)}_{Q} \delta_{KP} \bar{\theta} \Big) u_{P,Q} + \hat{\Sigma}^{(p\theta)}_{K} \theta' - \epsilon_{KLN} \dot{u}_L \Big(\hat{\Sigma}^{(pm)}_{MN} \bar{P}_M$$
$$+ \hat{\Sigma}^{(m)}_{NM} \bar{M}_M + \hat{\Sigma}^{(m\theta)}_{N} \bar{\theta} \Big)$$

$$\mu_0 h_K = \hat{\Sigma}^{(pm)}_{LK} P_L + \hat{\Sigma}^{(m)}_{KL} m_L + \Big(\hat{\Sigma}^{(em)}_{PQK} + \hat{\mathcal{J}}^{(pm)}_{QKPM} \bar{P}_M$$
$$+ \hat{\mathcal{J}}^{(m)}_{KQPM} + \hat{\Sigma}^{(m\theta)}_{Q} \delta_{KP} \bar{\theta} \Big) u_{P,Q} + \hat{\Sigma}^{(m\theta)}_{K} \theta' \tag{7.12}$$
$$+ \frac{1}{c^2} \epsilon_{KLN} \dot{u}_L \Big(\hat{\Sigma}^{(p)}_{NM} \bar{P}_M + \hat{\Sigma}^{(pm)}_{NM} \bar{M}_M + \hat{\Sigma}^{(p\theta)}_{N} \bar{\theta} \Big)$$

$$t'_{KL} = \Big(\hat{\underline{C}}^{(p)}_{KLM} + \bar{E}_K \delta_{LM} \Big) P_M + \Big(\hat{\underline{C}}^{(m)}_{KLM} + \mu_0 \bar{H}_K \delta_{LM} \Big) m_M + a_K \bar{P}_L$$
$$+ \mu_0 h_K \bar{M}_L + \Big[\hat{\underline{C}}^{(e)}_{KLPQ} + \hat{C}^{(v)}_{KLPQ} \frac{\partial}{\partial t} + \hat{\underline{\mathcal{J}}}^{(p)}_{KLQ} \delta_{NP} \bar{P}_N + \hat{\underline{\mathcal{J}}}^{(m)}_{KLQ} \delta_{NP} \bar{M}_N$$
$$+ \Big(\hat{\underline{\mathcal{J}}}^{(\theta)}_{QL} \delta_{KP} - \hat{C}^{(\theta)}_{KL} \delta_{PQ} \Big) \bar{\theta} + \Big(\hat{\underline{\mathcal{J}}}^{(g)}_{QLM} \delta_{PK} - \hat{C}^{(g)}_{KLM} \delta_{PQ} \Big) \bar{\theta}_{,M} \Big] u_{P,Q}$$
$$+ \hat{\underline{C}}_{KL} \theta' + \hat{C}^{(g)}_{KLM} \theta'_{,M} + \epsilon_{KMN} \dot{u}_M \Big(\mu_0 \bar{H}_N \bar{P}_L - \frac{1}{c^2} \bar{E}_N \bar{M}_L \Big) .$$

The constitutive equations $(7.11)_{1,2}$ are equivalent to those given by Dzyaloshinskii [22] if the temperature change is ignored.

Moreover, for the 32 out of the 90 magnetic crystals, the constitutive equations are further simplified. From $(6.87,88)_{2-4}$, one obtains

$$\bar{E}_K = \hat{\Sigma}^{(p)}_{KL} \bar{P}_L \quad ; \quad \mu_0 \bar{H}_K = \hat{\Sigma}^{(m)}_{KL} \bar{M}_L \tag{7.13}$$
$$\bar{t}_{KL} = \bar{E}_K \bar{P}_L + \mu_0 \bar{H}_K \bar{M}_L + \hat{\underline{C}}^{(\theta)}_{KL} \bar{\theta}$$

and

$$a_K = \hat{\Sigma}^{(p)}_{KL} P_L + \hat{\mathcal{J}}^{(p)}_{KLMN} \bar{P}_N u_{M,L} - \epsilon_{KLN} \dot{u}_L \hat{\Sigma}^{(m)}_{NM} \bar{M}_M$$
$$\mu_0 h_K = \hat{\Sigma}^{(m)}_{KL} m_L + \hat{\mathcal{J}}^{(m)}_{KLMN} \bar{M}_M u_{M,L} + \frac{1}{c^2} \epsilon_{KLN} \dot{u}_L \hat{\Sigma}^{(p)}_{NM} \bar{P}_M \tag{7.14}$$

$$t'_{KL} = e_K \bar{P}_L + \mu_0 h_K \bar{M}_L + \Big[\underset{\sim}{\hat{C}}{}^{(e)}_{KLPQ} + \hat{C}^{(v)}_{KLPQ} \frac{\partial}{\partial t} + \big(\underset{\sim}{\hat{J}}{}^{(\theta)}_{QLKP} - \hat{C}^{(\theta)}_{KL}\delta_{PQ}\big)\bar{\theta}\Big] u_{P,Q} + \underset{\sim}{\hat{C}}{}^{(\theta)}_{KL}\theta' + \epsilon_{KMN}\dot{u}_M\Big(\mu_0 \bar{H}_N \bar{P}_L - \frac{1}{c^2}\bar{E}_N \bar{P}_L\Big)$$

and the remainings of (6.87,88) are the same.

Some of the interactions disappear if the material is isotropic. By inserting $\underset{\sim}{\dot{X}} = 0$ into $(6.90,91)_{2,3}$, one obtains

$$\bar{E}_K = \hat{c}_4 \bar{P}_K \quad ; \mu_0 \bar{H}_K = \hat{c}_5 \bar{M}_K \tag{7.15}$$

$$\begin{aligned} e_K &= \hat{c}_4 \big[P_K + (\tilde{e}_{NN}\delta_{KL} + 2\tilde{e}_{KL})\bar{P}_L\big] - \hat{c}_5 \epsilon_{KLN}\dot{u}_L \bar{M}_N \\ \mu_0 h_K &= \hat{c}_5 \big[m_K + (\tilde{e}_{NN}\delta_{KL} + 2\tilde{e}_{KL})\bar{M}_L\big) + \frac{\hat{c}_4}{c^2}\epsilon_{KLM}\dot{u}_L \bar{P}_M \end{aligned} \tag{7.16}$$

and the remainings of (6.90,91) are the same. Although the anisotropy due to the rigid body motions in (7.15) disappears, that due to the infinitesimal deformation and motion is retained.

7.2. Thermally and Electrically Nonconductive Materials

If the material is thermally and electrically nonconductive, then the energy is dissipated only due to the viscosity of the elastic material. Once it is assumed that $\underset{\sim}{J} = 0$ and $\underset{\sim}{q} = 0$, Eq.(5.35) reduces to

$$\mathcal{D} \equiv {}_{(D)}\underset{\sim}{\hat{t}}_{KL}\,\dot{E}_{KL} \geq 0 \tag{7.17}$$

from which one obtains $(5.53)_1$. Thus, the constitutive equations for the dissipative part of the stress tensor does not change whether the material conducts electricity and heat or not.

Substituting $\bar{q}_K = 0$ and $\bar{J}_K = 0$, and $\rho^{(f)} = 0$ into Eqs.(6.23, 26,28,38), the energy, entropy inequality, Maxwell equations and the electromagnetic body force in the rigid body state are obtained.

Similarly, substituting $q_K = J_K = \rho^{(f)} = 0$ in the associated equations in the perturbation state, one obtains the governing equations. The terms containing these quantities in (6.25,27), $(6.42)_{2,3}$, $(6.46)_{1,3}$ and (6.53) are zero and the remaining equations in the rigid body and perturbation states are the same.

The equations obtained in this section are further simplified when the material is constrained from the rigid body motions and/or it has certain special symmetries.

7.3. Omitting Certain Terms due to Deformation

So far, in writing the governing equations one group of terms like $\bar{P}_K u_{K,L}$, $\bar{M}_K u_{K,L}$, $\bar{E}_K u_{K,L}$, etc. are consistently retained. Supposing that these terms are negligible compared to the perturbed quantities like p_K , m_K , e_K , etc. respectively, the governing equations in the rigid body state remain the same, although those in the perturbation state change considerably. The equations of motion (6.17) now become

$$\rho \dot{v}_K = t_{LK,L} + \rho f_K \quad . \tag{7.18}$$

The energy equation (6.25) and the entropy inequality (6.27), respectively, reduce to

$$\bar{\rho}(\theta' \dot{\bar{\eta}} + \bar{\theta} \dot{\eta}') = j_K^{(f)} \bar{\mathcal{E}}_K + \bar{J}_K^{(f)} \mathcal{E}'_K - q'_{K,K} + \bar{\rho} r_\theta' \tag{7.19}$$

and

$$\bar{\rho} \eta' \geq -\left(\frac{q'_K}{\bar{\theta}}\right)_{,K} + \left(\frac{\theta'}{\bar{\theta}} \bar{q}_K\right)_{,K} + \frac{1}{\bar{\theta}} \bar{\rho} \left(r_\theta' - \frac{\bar{r}_\theta \theta'}{\bar{\theta}}\right) . \tag{7.20}$$

From Eq.(6.42), the Maxwell equations are

$$\begin{aligned} &\epsilon_{IJK} e_{K,J} + \mu_0 \frac{\partial h_I}{\partial t} = -j_I^{(m)} \\ &\epsilon_{IJK} h_{K,J} - \epsilon_0 \frac{\partial e_I}{\partial t} = j_I^{(f)} + j_I^{(p)} \\ &\epsilon_0 e_{K,K} = \rho'^{(f)} + \rho'^{(p)} \quad ; \quad \mu_0 h_{K,K} = \rho'^{(m)} \end{aligned} \tag{7.21}$$

where

$$\begin{aligned} &j_I^{(p)} = \frac{\partial p_I}{\partial t} + p_{I,K} \dot{X}_K - p_{L,L} \dot{X}_I + \bar{P}_{I,K} \frac{\partial u_K}{\partial t} - \bar{P}_{K,K} \frac{\partial u_I}{\partial t} \\ &j_I^{(m)} = \frac{\partial \mu_0 m_I}{\partial t} + \mu_0 m_{I,K} \dot{X}_K - \mu_0 m_{L,L} \dot{X}_I + \mu_0 \bar{M}_{I,K} \frac{\partial u_K}{\partial t} - \mu_0 \bar{M}_{K,K} \frac{\partial u_I}{\partial t} \\ &\rho'^{(p)} = -p_{K,K} \quad ; \quad \rho'^{(m)} = -\mu_0 m_{K,K} \quad . \end{aligned} \tag{7.22}$$

From (6.45), there follows

$$e_I = -\varphi^{(e)}_{,I} - \frac{\partial a^{(e)}}{\partial t} - \frac{1}{\epsilon_0}\,\epsilon_{IKL}\, a^{(m)}_{L,K}$$
$$h_I = \frac{1}{\mu_0}\,\epsilon_{IKL}\, a^{(e)}_{L,K} - \varphi^{(m)}_{,I} - \frac{\partial a^{(m)}_I}{\partial t} \tag{7.23}$$

and if these are substituted into the Maxwell's equations, one obtains

$$\bar{\Box}\varphi^{(e)} = -\frac{1}{\epsilon_0}\left(\rho^{(f)} + \rho'^{(p)}\right) \;;\; \bar{\Box}\varphi^{(m)} = -\frac{1}{\mu_0}\rho'^{(m)}$$
$$\bar{\Box} a^{(e)}_I = -\mu_0\left(j^{(f)}_I + j'^{(p)}_I\right) \;;\; \bar{\Box} a^{(m)}_I = -\epsilon_0\, j'^{(m)}_I \tag{7.24}$$

where the potentials satisfy the Lorentz condition

$$a^{(e)}_{K,K} + \frac{1}{c^2}\frac{\partial \varphi^{(e)}}{\partial t} = 0 \;;\; a^{(m)}_{K,K} + \frac{1}{c^2}\frac{\partial \varphi^{(m)}}{\partial t} = 0\,. \tag{7.25}$$

Thus, the apparent inconsistency between the Maxwell's equations in the rigid body state and those in the perturbation state disappears.

Working through Eqs.(6.70,72,75,78,82,85), one now finds

$$\eta' = -\hat{\Sigma}^{(\theta)}\theta' - \hat{\Sigma}^{(e\theta)}_{KL}\tilde{e}_{KL} - \frac{1}{\bar{\rho}}\left(\hat{\Sigma}^{(p\theta)}_K P_K + \hat{\Sigma}^{(m\theta)}_K m_K\right)$$

$$e_K = \left(\hat{\Sigma}^{(p)}_{KL} - \epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(pm)}_{LN}\right)P_L + \left(\hat{\Sigma}^{(pm)}_{KL} - \epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(m)}_{NL}\right)m_L$$
$$+\left(\hat{\Sigma}^{(ep)}_{PQK} - \epsilon_{KLN}\dot{X}_L\hat{\Sigma}^{(em)}_{PQN}\right)u_{P,Q} + \left(\hat{\Sigma}^{(p\theta)}_K - \epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(m\theta)}_N\right)\theta'$$
$$-\epsilon_{KLN}\dot{u}_L\left(\hat{\Sigma}^{(pm)}_{MN}\bar{P}_M + \hat{\Sigma}^{(m)}_{NM}\bar{M}_M + \hat{\Sigma}^{(m\theta)}_N\bar{\theta}\right)$$

$$\mu_0 h_K = \left(\hat{\Sigma}^{(pm)}_{LK} + \frac{1}{c^2}\epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(p)}_{NL}\right)P_L + \left(\hat{\Sigma}^{(m)}_{KL} + \right.$$
$$\left.+\frac{1}{c^2}\epsilon_{KMN}\dot{X}_M\hat{\Sigma}^{(pm)}_{NL}\right)m_L + \left(\hat{\Sigma}^{(em)}_{PQK} + \frac{1}{c^2}\epsilon_{KLN}\dot{X}_L\hat{\Sigma}^{(ep)}_{PQN}\right)u_{P,Q}$$
$$+\left(\hat{\Sigma}^{(m\theta)}_K + \frac{1}{c^2}\epsilon_{KLN}\dot{X}_L\hat{\Sigma}^{(p\theta)}_N\right)\theta' + \frac{1}{c^2}\epsilon_{KLN}\dot{u}_L\left(\hat{\Sigma}^{(p)}_{NM}\bar{P}_M\right.$$
$$\left.+\hat{\Sigma}^{(pm)}_{NM}\bar{M}_M + \hat{\Sigma}^{(p\theta)}_N\bar{\theta}\right) \tag{7.26}$$

$$t'_{KL}=(\hat{\underline{C}}^{(p)}_{KLM}+\bar{\mathcal{E}}_K\delta_{LM})P_M+(\hat{\underline{C}}^{(m)}_{KLM}+\mu_0\bar{\mathcal{H}}_K\delta_{LM})m_M+e'_K\bar{P}_L$$

$$+\mu_0 h'_K\bar{M}_L+(\hat{\underline{C}}^{(e)}_{KLMN}+\hat{C}^{(v)}_{KLMN}\frac{\partial}{\partial t})\tilde{e}_{NM}+\hat{\underline{C}}^{(\theta)}_{KL}\theta'$$

$$+\hat{C}^{(g)}_{KLM}\theta'_{,M}+\epsilon_{KMN}\dot{u}_M(\mu_0\bar{H}_N\bar{P}_L-\frac{1}{c^2}\bar{E}_N\bar{M}_L)$$

$$j_K=\hat{K}^{(p)}_{KL}P_L+\hat{K}^{(m)}_{KL}m_L+(\hat{K}^{(e)}_{KNM}+\hat{K}^{(v)}_{KNM}\frac{\partial}{\partial t})\tilde{e}_{NM}$$

$$+\hat{K}^{(\theta)}_K\theta'+\hat{K}^{(g)}_{KL}\theta'_{,L}$$

$$q'_K=\hat{G}^{(p)}_{KL}P_L+\hat{G}^{(m)}_{KL}m_L+(\hat{G}^{(e)}_{KNM}+\hat{G}^{(v)}_{KNM}\frac{\partial}{\partial t})\tilde{e}_{NM}$$

$$+\hat{G}^{(\theta)}_K\theta'+\hat{G}^{(g)}_{KL}\theta'_{,L}\quad .$$

When the special cases stated earlier are introduced into these expressions further simplifications are achieved.

The governing equations for the quasi static electric field system are obtained by substituting $\underset{\sim}{M}=0$ and $\frac{\partial H}{\partial t}=0$ into the associated equations in Chapter 6. The derivations and the special cases of the governing equations have been obtained by Ersoy [76].

In a similar manner, the governing equations for the quasi static magnetic field system are obtained if $\underset{\sim}{P}=0$ and $\frac{\partial E}{\partial t}=0$ are substituted into the equations in Chapter 6. If the material considered does not possess the magnetic symmetry and the material is mechanically nondissipative (elastic, $\dot{E}_{KL}=0$), then the governing equations are in agreement with those of Hutter and Pao [14].

Moreover, if $E_{KL}=0$ is substituted into the governing equations of the quasi static field system and the material is assumed to be isotropic, then the equations of magneto-thermo-hydro-dynamics follow.

If $\underset{\sim}{P}=0$, $\underset{\sim}{M}=0$, hence, $\underset{\sim}{E}=0$, $\underset{\sim}{H}=0$ are substituted into the equations in the previous chapter, then the governing equations for pure thermo-viscoelasticity are obtained and these equations are in agreement with those of the Kelvin-Voigt type thermo-viscoelastic solids [46, Ch.9].

7.4. An Alternative Decomposition

To deduce the linear equations for the interacting continua, the infinitesimal dynamic deformation has been superimposed on the finite rigid body motions. In the decomposition all the relevant terms have been retained and the special cases of these equations have been explained in the previous sections. The proposed decomposition is appropriate for the effective rigid body motions and the infinitesimal deformations of the interacting continua. However, the linearization process, which is an infinitesimal dynamic deformation superimposed on static finite deformation, is appropriate for another class of problems and it has been consistenly applied by Toupin [4] and Tiersten [5] to their own theories. The aims of these two decompositions are different from each other.

We now assume that the body is constrained from the rigid body motions, $\dot{\underset{\sim}{\chi}}=0$, and all the quantities in the present configuration are in the form

$$\begin{aligned} &\underset{\sim}{P}={}_{o}\underset{\sim}{P}(\underset{\sim}{X})+\tilde{\underset{\sim}{P}}(\underset{\sim}{X},t) \quad ; \quad \underset{\sim}{M}={}_{o}\underset{\sim}{M}(\underset{\sim}{X})+\tilde{\underset{\sim}{m}}(\underset{\sim}{X},t) \\ &\underset{\sim}{E}={}_{o}\underset{\sim}{E}(\underset{\sim}{X})+\tilde{\underset{\sim}{e}}(\underset{\sim}{X},t) \quad ; \quad \underset{\sim}{H}={}_{o}\underset{\sim}{H}(\underset{\sim}{X})+\tilde{\underset{\sim}{h}}(\underset{\sim}{X},t) \qquad (7.27)\\ &\underset{\sim}{x}=\underset{\sim}{X}+\tilde{\underset{\sim}{u}}(\underset{\sim}{X},t) \quad , \text{ etc.} \end{aligned}$$

and

$$\mathrm{norm}\left(\frac{\tilde{\underset{\sim}{P}}}{{}_{o}\underset{\sim}{P}}\right) \; , \; \mathrm{norm}\left(\frac{\tilde{\underset{\sim}{m}}}{{}_{o}\underset{\sim}{M}}\right) \; ; \; \mathrm{norm}\left(\frac{\tilde{\underset{\sim}{e}}}{{}_{o}\underset{\sim}{E}}\right), \ldots\ldots \ll 1 . \qquad (7.28)$$

That is, the quantities with subscript zero at left are the primary fields and the quantities with overhead tilde are the perturbed fields. Thus, the product of the two perturbed fields is assumed to be negligible.

Upon substituting (7.27) into Eqs.(5.1-7) and using (7.28) and

$$\frac{\partial(\cdot)}{\partial x_i} \cong \left(\delta_{iK} - \delta_{iL}\frac{\partial \tilde{u}_K}{\partial X_L}\right)\frac{\partial(\cdot)}{\partial X_K} \quad ; \quad \rho = {}_o\rho\,(1 - \tilde{u}_{K,K}) \tag{7.29}$$

one obtains

$$\begin{aligned}
&{}_o t^{(T)}_{KL,K} = 0 \quad ; \quad \epsilon_{KLM}\,{}_o t^{(T)}_{LM} = 0 \\
&{}_o J^{(f)}_K\,{}_o E_K - {}_o q_{K,K} + {}_o\rho\,{}_o\gamma = 0 \quad ; \quad \left(\frac{1}{{}_o\theta}\,{}_o q_K\right)_{,K} - \frac{1}{{}_o\theta}\,{}_o\rho\,{}_o\gamma \geqslant 0 \\
&\epsilon_{KLM}\,{}_o E_{M,L} = 0 \quad ; \quad \epsilon_{KLM}\,{}_o H_{M,L} = {}_o J^{(f)}_K \\
&{}_o E_{K,K} = \frac{1}{\epsilon_o}\left({}_o\rho^{(f)} + {}_o\rho^{(p)}\right) \quad ; \quad {}_o H_{K,K} = \frac{1}{\mu_o}\,{}_o\rho^{(m)}
\end{aligned} \tag{7.30}$$

and

$$\begin{aligned}
&[\![{}_o t^{(T)}_{KL}]\!] N_K = 0 \quad ; \quad [\![{}_o q_K]\!] N_K = 0 \quad [\![\tfrac{1}{{}_o\theta}\,{}_o q_K]\!] N_K \geqslant 0 \quad \text{on } \Sigma(t) \\
&[\![\epsilon_o\,{}_o E_K + {}_o P_K]\!] N_K = 0 \quad ; \quad [\![{}_o H_K + {}_o M_K]\!] N_K = 0 \\
&[\![{}_o E_K]\!] K_K = 0 \quad ; \quad [\![{}_o H_K]\!] K_K = 0 \quad \text{on } \Gamma(t)
\end{aligned} \tag{7.31}$$

where

$${}_o\rho^{(p)} \equiv -{}_o P_{K,K} \quad ; \quad {}_o\rho^{(m)} = -\mu_o\,{}_o M_{K,K} \quad . \tag{7.32}$$

If the body is constrained from the rigid body motions, then the time-independent primary fields in the reference configuration must satisfy (7.30) with the associated constitutive equations.

The Maxwell's equations $(7.30)_{3-6}$ are equivalently expressed in terms of the potentials. From Eqs.(3.41-43), there follows

$$\begin{aligned}
&{}_o E_K = -{}_o\Phi^{(p)}_{,K} - \frac{1}{\epsilon_o}\,\epsilon_{KMN}\,{}_o A^{(m)}_{N,M} \\
&{}_o H_K = \frac{1}{\mu_o}\,\epsilon_{KLM}\,{}_o A^{(p)}_{M,M} - {}_o\Phi^{(m)}_{,K}
\end{aligned} \tag{7.33}$$

$$\begin{aligned}
&{}_o\Phi^{(p)}_{,KK} = -\frac{1}{\epsilon_o}\left({}_o\rho^{(f)} + {}_o\rho^{(p)}\right) \quad ; \quad {}_o\Phi^{(m)}_{,KK} = -\frac{1}{\mu_o}\,{}_o\rho^{(m)} \\
&{}_o A^{(e)}_{I,KK} = -\mu_o\,{}_o J^{(f)}_I \quad ; \quad {}_o A^{(m)}_{I,KK} = 0
\end{aligned} \tag{7.34}$$

and

$$ {}_oA^{(p)}_{K,K} = 0 \quad ; \quad {}_oA^{(m)}_{K,K} = 0 \; . \tag{7.35}$$

The Maxwell stress tensor (3.64) and the electromagnetic body force (3.56) reduce, respectively, to

$$ {}_o\tau_{IJ} = \epsilon_o\, {}_oE_I\, {}_oE_J + \mu_o\, {}_oH_I\, {}_oH_J - {}_oW\,\delta_{IJ} + {}_oP_I\, {}_oE_J + \mu_o\, {}_oM_I\, {}_oH_J $$

and (7.36)

$$ \rho_o f_I^{(em)} = {}_o\tau_{KL,K} = {}_o\rho^{(f)}\, {}_oE_I + \mu_o\, \epsilon_{IKL}\, {}_oJ_K^{(f)}\, {}_oH_L + {}_oP_K\, {}_oE_{I,K} + \mu_o\, {}_oM_K\, {}_oH_{I,K} $$

where

$$ {}_oW = \frac{1}{2}\left(\epsilon_o\, {}_oE_K\, {}_oE_K + \mu_o\, {}_oH_K\, {}_oH_K\right) . \tag{7.37}$$

Making use of the above assumptions, one obtains the constitutive equations. From $(5.45)_1$ and (5.46,55), there follows

$$ \begin{aligned}
\eta &= -\hat{\Sigma}^{(\theta)}\, {}_o\theta - \frac{1}{{}_o\rho}\left(\hat{\Sigma}^{(p\theta)}_K\, {}_oP_K + \hat{\Sigma}^{(m\theta)}_K\, {}_oM_K\right)\\
{}_oE_K &= \hat{\Sigma}^{(p)}_{KL}\, {}_oP_L + \hat{\Sigma}^{(pm)}_{KL}\, {}_oM_L + \hat{\Sigma}^{(p\theta)}_K\, {}_o\theta\\
\mu_o H_K &= \hat{\Sigma}^{(pm)}_{LK}\, {}_oP_L + \hat{\Sigma}^{(m)}_{KL}\, {}_oM_L + \hat{\Sigma}^{(m\theta)}_K\, {}_o\theta\\
{}_ot_{KL} &= (\hat{\underline{C}}^{(p)}_{KLM} + {}_oE_K\,\delta_{LM})\, {}_oP_M + (\hat{\underline{C}}^{(m)}_{KLM} + \mu_o\, {}_oH_K\,\delta_{LM})\, {}_oM_M\\
&\quad + \hat{\underline{C}}^{(\theta)}_{KL}\, {}_o\theta + \hat{\underline{C}}^{(g)}_{KLM}\, {}_o\theta_{,M}\\
{}_oJ_K &= \hat{K}^{(p)}_{KL}\, {}_oP_L + \hat{K}^{(m)}_{KL}\, {}_oM_L + \hat{K}_K\, {}_o\theta + \hat{K}^{(g)}_{KL}\, {}_o\theta_{,L}
\end{aligned} \tag{7.38}$$

and

$$ {}_oq_K = \hat{G}^{(p)}_{KL}\, {}_oP_L + \hat{G}^{(m)}_{KL}\, {}_oM_L + \hat{G}^{(\theta)}_K\, {}_o\theta + \hat{G}^{(g)}_{KL}\, {}_o\theta_{,L} \; . $$

The governing equations for the perturbed fields follow from the decomposed equations. From $(5.1)_2$, one obtains the equations of motion as

$$ {}_o\rho\, \dot{v}_I = \tilde{t}'_{KI,K} + {}_o\rho\, f_I^{(em)} - {}_ot_{LI,K}\, \tilde{u}_{L,K} \; . \tag{7.39}$$

In a similar manner, one obtains the energy equation and the entropy inequality.

From (5.2) and $(5.1)_{7,8}$, there follows the Maxwell equations in terms of the perturbed quantities

$$\epsilon_{IJK}\,\tilde{e}_{K,J} + \mu_0 \frac{\partial \tilde{h}_I}{\partial t} = -\tilde{j}_I^{(m)} + \tilde{u}_{K,L}\,\epsilon_{ILN}\,{}_\circ\mathcal{E}_{N,K}$$

$$\epsilon_{IJK}\,\tilde{h}_{K,J} - \epsilon_0 \frac{\partial \tilde{e}_I}{\partial t} = \tilde{j}_I^{(f)} + \tilde{j}_I^{(p)} + \tilde{u}_{K,L}\,\epsilon_{ILN}\,{}_\circ\mathcal{H}_{N,K}$$

$$\epsilon_0\,\tilde{e}_{K,K} = \tilde{\rho}^{(f)} + \tilde{\rho}^{(p)} + \epsilon_0\,\tilde{u}_{K,L}\,{}_\circ\mathcal{E}_{L,K} \tag{7.40}$$

$$\mu_0\,\tilde{h}_{K,K} = \tilde{\rho}^{(m)} + \mu_0\,\tilde{u}_{K,L}\,{}_\circ\mathcal{H}_{L,K}$$

where

$$\tilde{j}_I^{(p)} \equiv \frac{\partial \tilde{P}_I}{\partial t} + \left({}_\circ P_I \frac{\partial \tilde{u}_K}{\partial t} - {}_\circ P_K \frac{\partial \tilde{u}_I}{\partial t}\right)_{,K}$$

$$\tilde{j}_I^{(m)} \equiv \frac{\partial \mu_0 \tilde{M}_I}{\partial t} + \mu_0\left({}_\circ M_I \frac{\partial \tilde{u}_K}{\partial t} - {}_\circ M_K \frac{\partial \tilde{u}_I}{\partial t}\right)_{,K} \tag{7.41}$$

$$\tilde{\rho}^{(p)} \equiv -\tilde{P}_{K,K} + \tilde{u}_{K,L}\,{}_\circ P_{L,K}$$

$$\tilde{\rho}^{(m)} \equiv -\mu_0\,\tilde{M}_{K,K} + \mu_0\,\tilde{u}_{K,L}\,{}_\circ M_{L,K} \;.$$

The perturbed electric and magnetic fields in terms of the potentials are expressed as

$$\tilde{e}_I = -\tilde{\varphi}^{(e)}_{,I} - \frac{\partial \tilde{a}_I^{(e)}}{\partial t} - \frac{1}{\epsilon_0}\,\epsilon_{IKL}\,\tilde{a}^{(m)}_{L,K} + \tilde{u}_{K,L}\,{}_\circ\Phi_{,K} - \frac{1}{\epsilon_0}\,\epsilon_{IKL}\,\tilde{u}_{N,K}\,{}_\circ A^{(m)}_{L,N} \tag{7.42}$$

$$\tilde{h}_I = \frac{1}{\mu_0}\,\epsilon_{IKL}\,\tilde{a}^{(e)}_{L,K} - \tilde{\varphi}^{(m)}_{,I} + \frac{\partial \tilde{a}^{(m)}_I}{\partial t} + \tilde{u}_{K,I}\,{}_\circ\Phi^{(m)}_{,K} - \frac{1}{\mu_0}\,\epsilon_{IKL}\,\tilde{u}_{N,K}\,{}_\circ A^{(e)}_{L,N}$$

and the potentials satisfy the following equations

$$\bar{\Box}\,\tilde{\varphi}^{(e)} = -\frac{1}{\epsilon_0}\left(\tilde{\rho}^{(f)} + \tilde{\rho}^{(p)}\right) + 2\,\tilde{\underline{e}}_{KL}\,{}_\circ\Phi^{(e)}_{,KL}$$

$$\bar{\Box}\,\tilde{\varphi}^{(m)} = -\frac{1}{\mu_0}\,\tilde{\rho}^{(m)} + 2\,\tilde{\underline{e}}_{KL}\,{}_\circ\Phi^{(m)}_{,KL} \tag{7.43}$$

$$\bar{\Box}\,\tilde{a}^{(e)}_I = -\mu_0\left(\tilde{j}_I^{(f)} + \tilde{j}_I^{(p)}\right) + 2\,\tilde{\underline{e}}_{KL}\,{}_\circ A^{(e)}_{I,KL}$$

$$\bar{\Box}\,\tilde{a}^{(m)}_I = -\epsilon_0\,\tilde{j}_I^{(m)} + 2\,\tilde{\underline{e}}_{KL}\,{}_\circ A^{(m)}_{I,KL}$$

in which the potentials satisfy the Lorentz condition

$$\tilde{a}^{(e)}_{K,K} + \frac{1}{c^2}\frac{\partial \tilde{\varphi}^{(e)}}{\partial t} = \tilde{u}_{K,L}\,{}_\circ A^{(e)}_{L,K} \;;\quad \tilde{a}^{(m)}_{K,K} + \frac{1}{c^2}\frac{\partial \tilde{\varphi}^{(m)}}{\partial t} = \tilde{u}_{K,L}\,{}_\circ A^{(m)}_{L,K} . \tag{7.44}$$

Of course, Eqs.(7.42-44) are equivalent to (7.40). For given time-independent primary fields, the above equations are linear;

however, the electromagnetic fields, displacement gradient and the velocity are coupled.

It should be noted that in the decomposition of the governing equations, the term $\delta_{kL}\, u_{K,L} \frac{\partial(\cdot)}{\partial X_K}$ is retained in $(7.29)_1$ because of the interactions of the displacement field with the electromagnetic ones. However, Şuhubi [77] neglects such a differential term as is done in a state of pure elasticity. If this is the case, then the last terms at the right hand sides of (7.39,40,43-44) and the last two terms in (7.42) drop out so that the balance equations become much simpler.

The boundary conditions (5.6,7) now have forms similar to (6.49,50), if (6.51) is replaced by

$$\tilde{\mathcal{E}}'_I = \tilde{e}_I + \mu_0\, \epsilon_{IJK} \frac{\partial \tilde{u}_I}{\partial t}\, {}_\circ H_K \;;\quad \tilde{\mathcal{H}}'_I = \tilde{h}_I - \epsilon_0\, \epsilon_{IJK} \frac{\partial \tilde{u}}{\partial t}\, E_K \,. \tag{7.45}$$

The Maxwell stress tensor (3.64) and the body force (3.56), in terms of the perturbed variables, now become

$$\begin{aligned} \tilde{\tau}_{IJ} = {} & (\epsilon_0\, {}_\circ E_I + {}_\circ P_I)\, \tilde{e}_J + \mu_0 ({}_\circ H_I + {}_\circ M_I)\, \tilde{h}_J + (\epsilon_0\, \tilde{e}_I + \tilde{P}_I)\, {}_\circ E_J \\ & + \mu_0 (\tilde{h}_I + \tilde{m}_I)\, {}_\circ H_J - \tilde{W}\, \delta_{IJ} + \mu_0\, \epsilon_{JKL} \frac{\partial \tilde{u}}{\partial t} ({}_\circ P_I\, {}_\circ H_L - \epsilon_0\, {}_\circ M_I\, {}_\circ E_L) \end{aligned} \tag{7.46}$$

and

$$\begin{aligned} \rho \tilde{f}_I^{(em)} = {} & {}_\circ\rho^{(f)}\, \tilde{e}_I + \tilde{\rho}^{(f)}\, {}_\circ E_I + \mu_0\, \epsilon_{IJK} \Big[{}_\circ J_J^{(f)}\, \tilde{h}_K + \tilde{J}^{(f)}\, {}_\circ H_K + \frac{\partial \tilde{u}_{L,L}}{\partial t} ({}_\circ \tilde{P}_J\, {}_\circ H_K \\ & - \epsilon_0\, {}_\circ M_J\, {}_\circ E_K) \Big] + \tilde{e}_{I,K}\, {}_\circ P_K + \tilde{P}_K\, {}_\circ E_{I,K} + \mu_0\, \tilde{h}_{I,K}\, {}_\circ M_K \\ & + \mu_0\, \tilde{m}_K\, {}_\circ H_{I,K} + \mu_0\, \epsilon_{IJK} \Big[\frac{\partial \tilde{u}_J}{\partial t} ({}_\circ P_N\, {}_\circ H_{K,N} - \epsilon_0\, {}_\circ M_N\, {}_\circ E_{K,N}) \\ & + \frac{\partial \tilde{P}_J}{\partial t}\, {}_\circ H_K - \epsilon_0 \frac{\partial \tilde{m}_J}{\partial t}\, {}_\circ E_K + \frac{\partial \tilde{u}_N}{\partial t} ({}_\circ H_K\, {}_\circ P_{J,N} - {}_\circ E_K\, {}_\circ M_{J,N}) \Big] \\ & - \tilde{u}_{L,N} \Big[{}_\circ P_N\, {}_\circ E_{I,L} + \mu_0\, {}_\circ M_N\, {}_\circ H_{I,L} + \mu_0\, \epsilon_{IJK} \frac{\partial u_J}{\partial t}\, {}_\circ P_N\, {}_\circ H_{K,L} \\ & + \epsilon_0 \frac{\partial \tilde{u}_J}{\partial t}\, {}_\circ M_N\, {}_\circ E_{K,L} + \Big(\frac{\partial \tilde{u}_N}{\partial t}\, {}_\circ P_{J,L} + \frac{\partial \tilde{u}_{N,L}}{\partial t}\, {}_\circ P_J \Big)\, {}_\circ H_K \\ & + \epsilon_0 \Big(\frac{\partial \tilde{u}_N}{\partial t}\, {}_\circ M_{J,K} + \frac{\partial \tilde{u}_{N,L}}{\partial t}\, {}_\circ M_J \Big)\, {}_\circ E_K \Big] , \end{aligned} \tag{7.47}$$

respectively, where

$$\tilde{W} = -\left(\epsilon_0\, {}_\circ E_K\, \tilde{e}_K + \mu_0\, {}_\circ H_K\, \tilde{h}_K\right) . \tag{7.48}$$

Moreover, from $(5.45)_1$, (5.46,55) and (7.27,28), one establishes the constitutive equations for the perturbed fields as

$$\begin{aligned}
\tilde{\eta} = &-\hat{\Sigma}^{(\theta)}\tilde{\theta} - \hat{\Sigma}^{(e\theta)}_{KL}\,\underline{\tilde{e}}_{KL} - \frac{1}{{}_\circ\rho}\Big[\hat{\Sigma}^{(p\theta)}_{K}\left(\tilde{P}_K + \tilde{u}_{L,L}\,{}_\circ P_K + \tilde{u}_{L,K}\,{}_\circ P_L\right) \\
&+ \hat{\Sigma}^{(m\theta)}_{K}\left(\tilde{m}_K + \tilde{u}_{L,L}\,{}_\circ M_K + \tilde{u}_{L,K}\,{}_\circ M_L\right) \\
\tilde{\mathcal{E}}_K = &\ \hat{\Sigma}^{(p)}_{KL}\tilde{P}_L + \hat{\Sigma}^{(pm)}_{KL}\tilde{m}_L + \Big(\hat{\Sigma}^{(ep)}_{PQK} + \hat{\mathcal{J}}^{(p)}_{KQPM}\,{}_\circ P_M + \hat{\mathcal{J}}^{(pm)}_{KQPM}\,{}_\circ M_M \\
&+ \hat{\Sigma}^{(p\theta)}_{Q}\,\delta_{KP}\,{}_\circ\theta\Big)\tilde{u}_{P,Q} + \hat{\Sigma}^{(p\theta)}_{K}\tilde{\theta} - \epsilon_{KLN}\frac{\partial\tilde{u}_L}{\partial t}\left(\hat{\Sigma}^{(pm)}_{MN}\,{}_\circ P_M + \hat{\Sigma}^{(m)}_{NM}\,{}_\circ M_M + \hat{\Sigma}^{(m\theta)}_{N}\,{}_\circ\theta\right) \\
\mu_0\tilde{h}_K = &\ \hat{\Sigma}^{(pm)}_{LK}\tilde{P}_L + \hat{\Sigma}^{(m)}_{KL}\tilde{m}_L + \Big(\hat{\Sigma}^{(em)}_{PQK} + \hat{\mathcal{J}}^{(pm)}_{QKPM}\,{}_\circ P_M + \hat{\mathcal{J}}^{(m)}_{KQPM}\,{}_\circ M_M \\
&+ \hat{\Sigma}^{(m\theta)}_{Q}\,\delta_{KP}\,{}_\circ\theta\Big)\tilde{u}_{P,Q} + \hat{\Sigma}^{(m\theta)}_{K}\tilde{\theta} + \frac{1}{c^2}\epsilon_{KLN}\frac{\partial u_L}{\partial t}\left(\hat{\Sigma}^{(p)}_{NM}\,{}_\circ P_M + \hat{\Sigma}^{(pm)}_{NM}\,{}_\circ M_M + \hat{\Sigma}^{(p\theta)}_{N}\,{}_\circ\theta\right) \\
t_{KL} = &\left(\underline{\hat{C}}^{(p)}_{KLM} + \mathcal{E}_K\,\delta_{LM}\right)\tilde{P}_M + \left(\underline{\hat{C}}^{(m)}_{KLM} + \mu_0\,{}_\circ H_K\,\delta_{LM}\right)\tilde{m}_M \\
&+ \Big[\underline{\hat{C}}^{(e)}_{KLPQ} + \hat{C}^{(v)}_{KLPQ}\frac{\partial}{\partial t} + \underline{\hat{\mathcal{J}}}^{(p)}_{KLQ}\,\delta_{NP}\,{}_\circ P_N + \underline{\hat{\mathcal{J}}}^{(m)}_{KLQ}\,\delta_{NP}\,{}_\circ M_M \\
&+ \left(\underline{\hat{\mathcal{J}}}^{(\theta)}_{QL}\,\delta_{KP} - \hat{C}^{(\theta)}_{KL}\,\delta_{PQ}\right){}_\circ\theta + \left(\underline{\hat{\mathcal{J}}}^{(g)}_{QLM}\,\delta_{PK} - \hat{C}^{(g)}_{KLM}\,\delta_{PQ}\right){}_\circ\theta_{,M}\Big]\tilde{u}_{P,Q} \\
&+ \underline{\hat{C}}^{(\theta)}_{KL}\tilde{\theta} + \hat{C}^{(g)}_{KLM}\tilde{\theta}_{,L} + \tilde{\mathcal{E}}_K\,{}_\circ P_L + \mu_0\tilde{h}_K\,{}_\circ M_L \\
&+ \epsilon_{KMN}\frac{\partial\tilde{u}_M}{\partial t}\left(\mu_0\,{}_\circ H_N\,{}_\circ P_L - \frac{1}{c^2}\,{}_\circ\mathcal{E}_N\,{}_\circ M_L\right) \\
j_K = &\ \hat{K}^{(p)}_{KL}\tilde{P}_L + \hat{K}^{(m)}_{KL}\tilde{m}_L + \Big[\hat{K}^{(e)}_{KNM} + \hat{K}^{(v)}_{KNM}\frac{\partial}{\partial t} + \Big(\underline{\hat{K}}^{(p)}_{ML}\,{}_\circ P_L \\
&+ \underline{\hat{K}}^{(m)}_{ML}\,{}_\circ M_L + \underline{\hat{K}}^{(\theta)}_{M}\,{}_\circ\theta + \underline{\hat{K}}^{(g)}_{ML}\,{}_\circ\theta_{,L}\Big)\delta_{KN}\Big]\tilde{u}_{N,M} + \hat{K}^{(\theta)}_{K}\tilde{\theta} + \hat{K}^{(g)}_{KL}\tilde{\theta}_{,L}
\end{aligned} \tag{7.49}$$

and

$$\tilde{q}_K = \hat{G}^{(p)}_{KL}\tilde{P}_L + \hat{G}^{(m)}_{KL}\tilde{m}_L + [\hat{G}^{(e)}_{KNM} + \hat{G}^{(v)}_{KNM}\frac{\partial}{\partial t} + (\underline{\hat{G}}^{(p)}_{ML}\,{}_oP_L + \underline{\hat{G}}^{(m)}_{ML}\,{}_o\mathcal{M}_L + \underline{\hat{G}}^{(\theta)}_{M}\,{}_o\theta + \underline{\hat{G}}^{(g)}_{ML}\,{}_o\theta_{,L})\delta_{KN}]\tilde{u}_{N,M} + \hat{G}^{(\theta)}_{K}\tilde{\theta} + \hat{G}^{(g)}_{KL}\tilde{\theta}_{,L}\,.$$

The governing equations in this alternative decomposition can be directly obtained from the equations in Section 7.1 if one assumes that the quantities in the rigid body state are statimnary. Thus, the decomposition presented in this section is a special case of our decomposition process proposed in Chapter 6.

Considering the special material symmetries and the special cases taken into account in this chapter, the much simpler forms of the governing equations in this alternative decomposition are also obtained.

If one now assumes that the primary fields ${}_o\underline{P}$, $\underline{E}$, ${}_o\underline{\mathcal{M}}$, etc. in (7.27) are uniform, then the derivatives with respect to spatial coordinates and time vanish. In this case, Eq.(7.30) is automatically satisfied. The governing equations of the interacting continua become the following and from (3.39,40), one obtains

$$\begin{aligned}
&\rho\dot{v}_K = \tilde{t}'_{LK,L} + \rho f_K^{(em)} \\
&\epsilon_{IJK}\,\tilde{e}_{K,J} + \mu_o\frac{\partial\tilde{h}_I}{\partial t} = -\tilde{j}_I^{(m)} \\
&\epsilon_{IJK}\,\tilde{h}_{K,J} - \epsilon_o\frac{\partial\tilde{e}_I}{\partial t} = \tilde{j}_I^{(f)} + \tilde{j}_I^{(p)} \\
&\epsilon_o\,\tilde{e}_{K,K} = \tilde{\rho}^{(f)} + \tilde{\rho}^{(p)} \quad ; \quad \mu_o\tilde{h}_{K,K} = \tilde{\rho}^{(m)}
\end{aligned} \qquad (7.50)$$

where

$$\begin{aligned}
&\tilde{j}^{(p)} \equiv \frac{\partial\tilde{P}_I}{\partial t} + {}_oP_I\frac{\partial\tilde{u}_{K,K}}{\partial t} - {}_oP_K\frac{\partial\tilde{u}_{I,K}}{\partial t} \\
&\tilde{j}_I^{(m)} \equiv \frac{\partial\mu_o\tilde{m}_I}{\partial t} + \mu_o({}_o\mathcal{M}_I\frac{\partial\tilde{u}_{K,K}}{\partial t} - {}_o\mathcal{M}_K\frac{\partial\tilde{u}_{I,K}}{\partial t}) \\
&\tilde{\rho}^{(p)} \equiv -\tilde{P}_{K,K} \quad ; \quad \tilde{\rho}^{(m)} = -\mu_o\tilde{m}_{K,K}\,.
\end{aligned} \qquad (7.51)$$

From Eqs.(7.42-44), one obtains the equivalent form of (7.50) in terms of the potentials. From (7.42), there follows

$$\tilde{e}_I = -\tilde{\varphi}^{(e)}_{,I} - \frac{\partial \tilde{a}^{(e)}_I}{\partial t} - \frac{1}{\epsilon_0}\,\epsilon_{IKL}\,\tilde{a}^{(m)}_{L,K}$$
$$\tilde{h}_I = \frac{1}{\mu_0}\,\epsilon_{IKL}\,\tilde{a}^{(e)}_{L,K} - \tilde{\varphi}^{(m)}_{,I} - \frac{\partial \tilde{a}^{(m)}_I}{\partial t} \tag{7.52}$$

and the potentials satisfy

$$\bar{\Box}\,\tilde{\varphi}^{(e)} = -\frac{1}{\epsilon_0}\,(\tilde{\rho}^{(f)} + \tilde{\rho}^{(p)})\ ;\ \bar{\Box}\,\tilde{\varphi}^{(m)} = -\frac{1}{\mu_0}\,\tilde{\rho}^{(m)}$$
$$\bar{\Box}\,\tilde{a}^{(e)}_I = -\mu_0\,(\tilde{J}^{(f)}_I + \tilde{J}^{(p)}_I)\ ;\ \bar{\Box}\,\tilde{a}^{(m)}_I = -\epsilon_0\,\tilde{J}^{(m)} \tag{7.53}$$

and the Lorentz condition

$$\tilde{a}^{(e)}_{K,K} + \frac{1}{c^2}\frac{\partial \varphi^{(e)}}{\partial t} = 0 \quad ;\quad \tilde{a}^{(m)}_{K,K} + \frac{1}{c^2}\frac{\partial \varphi^{(m)}}{\partial t} = 0. \tag{7.54}$$

The boundary conditions and the Maxwell - stress tensor are the same as those in the alternative decomposition; however, the electromagnetic body force (7.47) becomes

$$\begin{aligned}\rho\tilde{f}^{(em)}_I &= {}_0\rho^{(f)}\tilde{e}_I + \tilde{\rho}^{(f)}\,{}_0E_I + \mu_0\,\epsilon_{IJK}\Big[J^{(f)}_J\tilde{h}_K + \tilde{J}^{(f)}_J\,{}_0H_K \\ &+ \frac{\partial \tilde{u}_{L,L}}{\partial t}\,({}_0P_J\,{}_0H_K - \epsilon_0\,{}_0M_J\,{}_0E_K)\Big] + \tilde{e}_{I,K}\,{}_0P_K + \mu_0\,\tilde{h}_{I,K}\,{}_0M_K \\ &+ \mu_0\,\epsilon_{IJK}\Big(\frac{\partial \tilde{P}_J}{\partial t}\,{}_0H_K - \epsilon_0\,\frac{\partial \tilde{m}_J}{\partial t}\,{}_0E_K\Big) - \tilde{u}_{L,N}\,\frac{\partial u_{N,L}}{\partial t}({}_0P_J\,{}_0H_K + \epsilon_0\,{}_0M_J\,{}_0E_K).\end{aligned} \tag{7.55}$$

From (7.49), one obtains the constitutive equations

$$\begin{aligned}\tilde{\eta} &= -\hat{\Sigma}^{(\theta)}\tilde{\theta} - \hat{\Sigma}^{(e\theta)}_{KL}\,\tilde{e}_{KL} - \frac{1}{\rho}\Big[\hat{\Sigma}^{(p\theta)}_K(\tilde{P}_K + \tilde{u}_{L,L}\,{}_0P_K + \tilde{u}_{L,K}\,{}_0P_L) \\ &+ \hat{\Sigma}^{(m\theta)}_K(\tilde{m}_K + \tilde{u}_{L,L}\,{}_0M_K + \tilde{u}_{L,K}\,{}_0M_L)\Big]\end{aligned}$$

$$\begin{aligned}\tilde{e}_K &= \hat{\Sigma}^{(p)}_{KL}\,\tilde{P}_L + \hat{\Sigma}^{(pm)}_{KL}\,\tilde{m}_L + \Big(\hat{\Sigma}^{(ep)}_{PQK} + \hat{\mathcal{J}}^{(p)}_{KQPM}\,{}_0P_M + \hat{\mathcal{J}}^{(pm)}_{KQPM}\,{}_0M_M \\ &+ \hat{\Sigma}^{(p\theta)}_Q\,\delta_{KP}\,{}_0\theta\Big)\tilde{u}_{P,Q} + \hat{\Sigma}^{(p\theta)}_K\,\tilde{\theta} - \epsilon_{KLN}\,\frac{\partial u_L}{\partial t}\Big(\hat{\Sigma}^{(pm)}_{MN}\,{}_0P_M \\ &+ \hat{\Sigma}^{(m)}_{NM}\,{}_0M_M + \hat{\Sigma}^{(m\theta)}_N\,{}_0\theta\Big)\end{aligned}$$

$$\mu_0 h_K = \hat{\Sigma}^{(pm)}_{LK} \tilde{P}_L + \hat{\Sigma}^{(m)}_{KL} \tilde{m}_L + (\hat{\Sigma}^{(em)}_{PQK} + \hat{\mathcal{J}}^{(pm)}_{QKPM} {}_\circ P_M + \hat{\mathcal{J}}^{(m)}_{KQPM} {}_\circ M_M$$

$$+ \hat{\Sigma}^{(m\theta)}_{Q} \delta_{KP} {}_\circ\theta) \tilde{u}_{P,Q} + \hat{\Sigma}^{(m\theta)}_{K} \tilde{\theta} + \frac{1}{c^2} \epsilon_{KLN} \frac{\partial \tilde{u}_L}{\partial t} (\hat{\Sigma}^{(p)}_{NM} {}_\circ P_M$$

$$+ \hat{\Sigma}^{(pm)}_{NM} {}_\circ M_M + \hat{\Sigma}^{(p\theta)}_{N} {}_\circ\theta) \qquad (7.56)$$

$$\tilde{t}_{KL} = (\hat{\underline{C}}^{(p)}_{KLM} + {}_\circ E_K \delta_{LM}) \tilde{P}_M + (\hat{\underline{C}}^{(m)}_{KLM} + \mu_0 {}_\circ H_K \delta_{LM}) \tilde{m}_M$$

$$+ [\hat{\underline{C}}^{(e)}_{KLPQ} + \hat{C}^{(v)}_{KLPQ} \frac{\partial}{\partial t} + \hat{\underline{\mathcal{J}}}^{(p)}_{KLQ} \delta_{NP} {}_\circ P_N + \hat{\underline{\mathcal{J}}}^{(m)}_{KLQ} \delta_{NP} {}_\circ M_N$$

$$+ (\hat{\underline{\mathcal{J}}}^{(\theta)}_{QL} \delta_{KP} - \hat{C}^{(\theta)}_{KL} \delta_{PQ}) {}_\circ\theta] \tilde{u}_{P,Q} + \hat{\underline{C}}^{(\theta)}_{KL} \tilde{\theta} + \hat{C}^{(g)}_{KLM} \tilde{\theta}_{,M}$$

$$+ \tilde{e}_K {}_\circ P_L + \mu_0 \tilde{h}_K {}_\circ M_L + \epsilon_{KMN} \frac{\partial u_M}{\partial t} (\mu_0 {}_\circ H_N {}_\circ P_L - \frac{1}{c^2} {}_\circ E_N {}_\circ M_L)$$

$$\tilde{J}_K = \hat{K}^{(p)}_{KL} \tilde{P}_L + \hat{K}^{(m)}_{KL} \tilde{m}_L + [\hat{K}^{(e)}_{KNM} + \hat{K}^{(v)}_{KNM} \frac{\partial}{\partial t} + (\hat{\underline{K}}^{(p)}_{ML} {}_\circ P_L$$

and $$+ \hat{\underline{K}}^{(m)}_{ML} {}_\circ M_L + \hat{\underline{K}}^{(\theta)} {}_\circ\theta) \delta_{KN}] \tilde{u}_{N,M}$$

$$\tilde{q}_K = \hat{G}^{(p)}_{KL} \tilde{P}_L + \hat{G}^{(m)}_{KL} \tilde{m}_L + [\hat{G}^{(e)}_{KNM} + \hat{G}^{(v)}_{KNM} \frac{\partial}{\partial t} + (\hat{\underline{G}}^{(p)}_{ML} {}_\circ P_L$$

$$+ \hat{\underline{G}}^{(m)}_{ML} {}_\circ M_L + \hat{\underline{G}}^{(\theta)}_{M} {}_\circ\theta) \delta_{KN}] \tilde{u}_{N,M} + \hat{G}^{(\theta)}_{K} \tilde{\theta} + \hat{G}^{(g)}_{KL} \tilde{\theta}_{,L} \quad .$$

For the materials with certain special symmetries, the constitutive equations (7.57) are simplified.

Considering isotropic materials, in particular, these equations are used partially in applications [13,26,34-39]. In the next chapter, the interactions between a magnetic field and viscoelastic solids will be discussed in a particular problem.

P A R T : I I I

A P P L I C A T I O N

CHAPTER 8

PLANE WAVES IN ELECTRICALLY CONDUCTING AND MAGNETIZABLE VISCOELASTIC ISOTROPIC SOLIDS SUBJECTED TO UNIFORM MAGNETIC FIELD

In the present chapter, the propagation of magneto-mechanical plane waves of small amplitudes in an initially isotropic, electrically conducting, soft ferromagnetic viscoelastic solid subjected to a uniform, primary magnetic field is studied both analytically and numerically. The secular equation for the plane waves propagating in an arbitrary direction is derived. The phase velocities and the attenuations per wavelength are obtained for given frequencies and an applied magnetic field in various directions. Furthermore, some interesting behaviors are numerically detected for certain frequencies and strong magnetic fields.

8.1. Résumé of the Governing Equations

The theories of magneto-elasticity and magneto-viscoelasticity are concerned with the interacting effects of an externally applied magnetic field on the deformation of elastic and/or viscoelastic solids and with the inverse effects. While the "electo-optical" and "magneto-optical" effects have played an essential part in the development of the electromagnetic theory of light in the beginning of the 20 th century, the subjects of electroelastic and magnetoelastic interactions have been rapidly developed in the last two decades [13,26,33-40,54,55].

In this section the governing equations of electrically conducting, magnetizable viscoelastic solids are presented. The medium is assumed to be magnetizable only, for which the magnetic hysteretic effect together with polarization charge and current are neglected. Furthermore, the material is assumed to be homogeneous and isotropic in its natural state. The starting point is the set of equations (7.50-56) which are going to be modified, assuming that the magnetic field is composed of a large static field plus a flactuating field, while the electric field and the displacement consist solely of small flactuating quantities. Thus, the equations present the excitation of visco-elastic waves in conducting, magnetizable solids arising from magneto-mechanical interactions.

If an electrically conducting viscoelastic solid is subjected to a mechanical load while immersed in a varying magnetic field, the laws of Cauchy and Maxwell will still determine the mechanical and the electromagnetic fields. In the present case, the mechanical traction and body forces are assumed to be absent while the electromagnetic traction and the ponderomotive Lorentz force are taken into account.

From Eq.$(7.50)_{2-5}$, with the assumptions stated above the Maxwell equations now become

$$\begin{aligned} &\epsilon_{ijk}\,\tilde{e}_{k,j} + \mu_0 \frac{\partial \tilde{h}_i}{\partial t} = -\tilde{j}_i^{(m)} \\ &\epsilon_{ijk}\,\tilde{h}_{k,j} - \epsilon_0 \frac{\partial \tilde{e}_i}{\partial t} = \tilde{j}_i' \\ &\epsilon_0\,\tilde{e}_{k,k} = 0 \quad ; \quad \mu_0 \tilde{h}_{k,k} = \tilde{\rho}^{(m)} \end{aligned} \tag{8.1}$$

where

$$\begin{aligned} &\tilde{j}_i^{(m)} = \frac{\partial \mu_0 \tilde{m}_i}{\partial t} + \mu_0\,{}_0M_i \frac{\partial \hat{u}_{k,k}}{\partial t} - \mu_0\,{}_0M_k \frac{\partial \hat{u}_{i,k}}{\partial t} \\ &\rho^{(m)} = -\mu_0\,\tilde{m}_{k,k} \end{aligned} \tag{8.2}$$

From Eq.$(7.50)_1$, the equation of motion now become

$$\rho\,\dot{v}_k = \tilde{t}_{lk,l} + \rho \tilde{f}_k^{(em)} \tag{8.3}$$

where

$$\rho \tilde{f}_i^{(em)} = \mu_0\,\epsilon_{ijk}\left({}_0J_j\,\tilde{h}_k + \tilde{j}_j\,{}_0H_k\right) + \mu_0\,\tilde{h}_{i,k}\,{}_0M_k \tag{8.4}$$

is the ponderomotive Lorentz force.

Substituting (6.89) into Eqs.(7.38,56) and neglecting the anisotropy due to the deformation and using the assumptions in this specific problem, one obtains

$$\mu_0\, {}_oH_k = \hat{c}_5\, {}_oM_k \quad ; \quad {}_ot_{kl} = \mu_0\, {}_oH_k\, {}_oM_l \quad ; \quad {}_oJ_k = 0 \tag{8.5}$$

and

$$\begin{aligned} \tilde{e}_k &= \hat{c}_4 \tilde{P}_k - \hat{c}_5 \epsilon_{kln} \frac{\partial \tilde{u}_l}{\partial t} {}_oM_n \quad ; \quad \mu_0 \tilde{h}_k = \hat{c}_5\, \tilde{m}_k \\ \tilde{t}_{kl} &= \mu_0\, {}_oH_k \tilde{m}_l + \mu_0 \tilde{h}_k\, {}_oM_l + (\hat{c}_1 + \hat{c}_1' \frac{\partial}{\partial t}) \tilde{\underline{e}}_{mm} \delta_{kl} \\ &\quad + 2\,(\hat{c}_2 + \hat{c}_2' \frac{\partial}{\partial t})\, \tilde{\underline{e}}_{kl} \\ \tilde{J}_k &= \hat{c}_8\, \tilde{P}_k \end{aligned} \tag{8.6}$$

respectively.

As it was obtained in Chapters 4,5, the electromagnetic fields influence the mechanical field through the ponderomotive Lorentz force and the actual stress tensor, while the mechanical field in turn affects the electromagnetic field by modifying Ohm's law of electric conduction and the Maxwell equations. The existence of a strain field may change the initial isotropy of the constitutive equations for electromagnetic fields. This effect, called dynamoptical effect which constitutes the basis for the photoelastic analysis of the medium, is neglected in the present study. This effect is studied in [31,32,53].

If one introduces new material constants as

$$\begin{aligned} \hat{\chi} &= \frac{\mu_0}{\hat{c}_5} \quad ; \quad \hat{\hat{\chi}} = \frac{1}{\hat{c}_4} \quad ; \quad \hat{\sigma} = \frac{\hat{c}_8}{\hat{c}_4} \\ \hat{\lambda} &= \hat{c}_1 \quad ; \quad \hat{\underline{\lambda}} = \hat{c}_1' \quad \hat{\mu} = \hat{c}_2 \quad ; \quad \hat{\underline{\mu}} = \hat{c}_2' \end{aligned} \tag{8.7}$$

the constitutive equations (8.5,6) take the form

$$\begin{aligned} {}_oM_k &= \hat{\chi}\, {}_oH_k \\ \tilde{P}_k &= \hat{\hat{\chi}}\, (\tilde{e}_k + \epsilon_{kln} \frac{\partial \tilde{u}_l}{\partial t} \mu_0\, {}_oH_k) \\ \tilde{J}_k &= \hat{\sigma} \quad (\tilde{e}_k + \epsilon_{kln} \frac{\partial \tilde{u}_l}{\partial t} \mu_0\, {}_oH_k) \\ \tilde{t}_{kl} &= \mu_0 ({}_oH_k \tilde{m}_l + \tilde{h}_k\, {}_oM_l) + (\hat{\lambda} + \hat{\underline{\lambda}} \frac{\partial}{\partial t}) \tilde{\underline{e}}_{mm} \delta_{kl} \\ &\quad + 2(\hat{\mu} + \hat{\underline{\mu}} \frac{\partial}{\partial t})\, \tilde{\underline{e}}_{kl} \quad . \end{aligned} \tag{8.8}$$

In $(8.8)_{1-3}$, $\hat{\chi}$, $\hat{\hat{\chi}}$ and $\hat{\sigma}$ are called the magnetic and electric susceptibilities and the electric conductivity. In $(8.8)_4$, $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\hat{\lambda}}$, $\hat{\hat{\mu}}$ are the elastic and viscous Lamé constants respectively. Eqs.(8.1,3) and $(8.8)_{1-4}$ form the basis of the magneto-viscoelasticity and are to be solved prescribed initial and boundary conditions.

We would like to emphasize the differences between the present treatment and the others [33-40,54,55]. First of all, it is usually considered that the electromagnetic body force is the Lorentz force, and the stress tensor is the Cauchy stress tensor in the equations of motion. However, Pao and Yeh [13] and Hutter and Pao [14] and Hutter [26] take into account the ponderomotive Lorentz force which is similar to ours in the Amperian formulation of the Maxwell's equations. Secondly, body is assumed to be elastic, i.e., mechanically nondissipative. Thirdly, Hutter [26] expresses the Maxwell's equations in terms of potentials in order to decrease the number of electromagnetic equations, but we directly deal with the electromagnetic fields, not the potentials. Also, our formulation is in terms of the Chu variables, but that of Hutter [26] is in terms of the Ampère variables and considers the waves of assigned wavelengths in elastic solids.

By inserting (8.8) into (8.1) and (8.3), with the aid of (8.2) and (8.4), one obtains

$$
\begin{aligned}
&\epsilon_{ijk}\,\tilde{e}_{k,j} + \mu_0(1+\hat{\chi})\frac{\partial \tilde{h}_i}{\partial t} + \hat{\chi}\left({}_0H_i\,\frac{\partial \tilde{u}_{k,k}}{\partial t} - {}_0H_k\,\frac{\partial \tilde{u}_{i,k}}{\partial t}\right) = 0 \\
&\epsilon_{ijk}\,\tilde{h}_{k,j} - \left(\hat{\sigma} + \epsilon_0\frac{\partial}{\partial t}\right)\tilde{e}_i + \mu_0\,\hat{\sigma}\,\epsilon_{ijk}\,{}_0H_j\,\frac{\partial \tilde{u}_k}{\partial t} = 0 \\
&-\mu_0\hat{\sigma}\,\epsilon_{ijk}\,{}_0H_j\,\tilde{e}_k + 2\mu_0\hat{\chi}\,\epsilon_{jk}\,\tilde{h}_{j,k} + \left(\hat{\lambda} + \hat{\hat{\lambda}}\frac{\partial}{\partial t}\right)\frac{\partial u_{k,ki}}{\partial t} \\
&\quad + \left(\hat{\mu} + \hat{\hat{\mu}}\frac{\partial}{\partial t}\right)\left(\tilde{u}_{k,ik} + \tilde{u}_{i,kk}\right) - \mu_0{}^2\hat{\sigma}\,\epsilon_{ijk}\,\epsilon_{kml}\,{}_0H_j\,{}_0H_l\,\frac{\partial \tilde{u}_m}{\partial t} \\
&\quad - \rho\,\frac{\partial^2 \tilde{u}_i}{\partial t^2} = 0 \quad .
\end{aligned}
\qquad (8.9)
$$

This coupled system of nine linear partial differential equations form the basis of the theory of electromagnetic and viscoelastic disturbances. This system is to be solved with the initial and boundary conditions which are to be modified for the assumptions stated above. These conditions are not written down here explicitly since wave propagation in an unbounded medium is considered in the remaining sections.

8.2. Plane Waves in an Unbounded Medium

There are many mechanical wave phenomena in solids that can be adequately described within pure mechanics, or at least within mechanics and thermodynamics. However, some of the most interesting and fruitful things about mechanical waves are the nonmechanical effects that accompany them.

Thus, the propagation of plane, steady state magneto-mechanical waves through a primary, uniform magnetic field are now investigated. Solutions to Eq.(8.9) will be sought which have the character of plane waves traveling in the positive $\underset{\sim}{x}$-direction. Since the solutions are being sought in a form of steady harmonic waves in the $\underset{\sim}{x}$-direction, $\tilde{\underset{\sim}{e}}$, $\tilde{\underset{\sim}{h}}$ and $\tilde{\underset{\sim}{u}}$ must all be functions of $\underset{\sim}{x}$ and t only. Thus one has

$$\begin{Bmatrix} \tilde{\underset{\sim}{e}} \\ \tilde{\underset{\sim}{h}} \\ \tilde{\underset{\sim}{u}} \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{e}^* \\ \underset{\sim}{h}^* \\ \underset{\sim}{u}^* \end{Bmatrix} \cdot \exp\left[i \left(\underset{\sim}{\mathbb{k}} \cdot \underset{\sim}{x} - \omega t \right) \right] \tag{8.10}$$

where $\underset{\sim}{e}^*$, $\underset{\sim}{h}^*$ and $\underset{\sim}{u}^*$ are the constant, complex amplitudes. In these expressions, which represent harmonic waves, $\underset{\sim}{\mathbb{k}}$ and ω are known as wave vector and angular frequency respectively. These two quantities are, in general, complex, so that the wavelength and the period are given by $2\pi / |Re\, \underset{\sim}{\mathbb{k}}|$ and $2\pi / (Re\, \omega)$ respectively. It is usual practice, however, to regard either $\underset{\sim}{\mathbb{k}}$ or ω as a real constant so that waves of given wavelength or of a specified frequency may be studied.

By inserting (8.10) into Eq.(8.9), one obtains finally

$$\epsilon_{ijk}\, \mathbb{k}_j\, e_k^* - \mu_0 (1+\hat{\chi})\, \omega\, h_i^* - i \mu_0 \hat{\chi} \omega \left({}_oH_i\, \mathbb{k}_k\, u_k^* - {}_oH_k\, \mathbb{k}_k\, u_i^* \right) = 0$$

$$(\hat{\sigma} - i \epsilon_0 \omega)\, e_i^* - \epsilon_{ijk}\, \mathbb{k}_j\, h_k^* + i \mu_0 \hat{\sigma}\, \omega\, \epsilon_{ijk}\, {}_oH_j\, u_k^* = 0 \tag{8.11}$$

$$\mu_0 \hat{\sigma} \epsilon_{ijk} {}_0H_k e_j^* + 2i\mu_0 \hat{\chi} k_j {}_0H_j h_i^* + [\rho\omega^2 - (\hat{\mu} - i\omega\hat{\underline{\mu}}) k_m k_m] u_i^*$$

$$- \{[\hat{\lambda} + \hat{\mu} - i\omega(\hat{\underline{\lambda}} + \hat{\underline{\mu}})] k_i k_m + i\mu_0 \hat{\sigma} \omega \epsilon_{ijk} \epsilon_{jmn} {}_0H_k {}_0H_n\} u_m^* = 0 \quad .$$

These nine equations have the following general form

$$\Lambda_{ik}(\underset{\sim}{k}, \omega, {}_0\underset{\sim}{H}, \hat{\chi}, \hat{\sigma}, \hat{\lambda}, \hat{\mu}, \hat{\underline{\lambda}}, \hat{\underline{\mu}}) \psi_k^* = 0 \quad {\scriptstyle (i,k=1,\dots,9)} \qquad (8.12)$$

where Λ_{ik} is the coefficient matrix of order 9x9 and ψ_k^* is a 9x1 column vector, $\underset{\sim}{\psi}^* = \| e_i^*, h_i^*, u_i^* \|^t$.

For a wave to exist, the amplitude $\underset{\sim}{\psi}^*$ must not be a null vector. Thus, one has the propagation condition

$$\det(\Lambda_{ik}) = 0 \quad . \qquad (8.13)$$

First we note that (8.13) may be simplified in the case when the planes of constant phase (wave front) are also planes of constant amplitude, i.e.,

$$\underset{\sim}{k} \cdot \underset{\sim}{x} = \text{constant} \quad . \qquad (8.14)$$

Now the planes of constant phase and the planes of constant amplitudes are, respectively,

$$\underset{\sim}{k}_r \cdot \underset{\sim}{x} = \text{constant} \quad ; \quad \underset{\sim}{k}_i \cdot \underset{\sim}{x} = \text{constant} \qquad (8.15)$$

since $\underset{\sim}{k} = \underset{\sim}{k}_r + i\underset{\sim}{k}_i$. When these two planes coincide one can write

$$\underset{\sim}{k} = k \underset{\sim}{n} \qquad (8.16)$$

where $\underset{\sim}{n}$ is a unit vector in the direction of the normal to the wave front.

Since one has a free choice of coordinate axes in an infinite medium, no loss of generality is involved in taking the x_1-direction to coincide with the direction of normal vector $\underset{\sim}{n}$, i.e., $n_i = \delta_{1i}$. We can then associate the displacement component $\tilde{u}_1$ with longitudinal (primary, P) waves and the transverse components $\tilde{u}_2$, $\tilde{u}_3$ with shear (secondary, S) waves. A primary, uniform magnetic field is assumed to exist in the form

$${}_0\underset{\sim}{H} = ({}_0H_1, {}_0H_2, 0)$$

see Fig.8.1. The choice of a zero x_3-component of the magnetic field is no restriction on the problem since the selected ${}_0\underset{\sim}{H}$ has components both parallel and perpendicular to the direction of wave motion, i.e., $k_i = k\, n_i = k\, \delta_{1i}$.

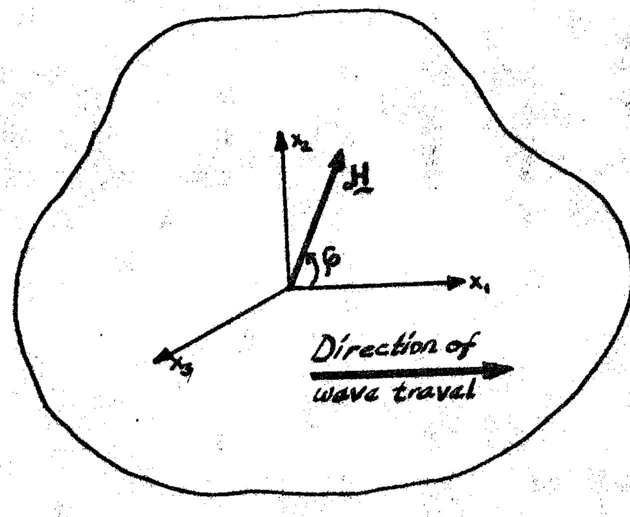


Fig.8.1 Primary Magnetic Field and Direction of Wave Travel.

The applied primary magnetic field may be expressed in the form

$$ {}_0H_i = {}_0H\,(\delta_{i1}\cos\varphi + \delta_{i2}\sin\varphi) \tag{8.17}$$

where φ is an angle between the magnetic field and the normal vector $\underset{\sim}{n}$ which is the direction of propagation.

As a result of (8.16) and (8.17), Eq.(8.11) become in matrix form

$$\begin{Vmatrix} 0 & 0 & 0 & a_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & a_{25} & 0 & a_{27} & a_{28} & 0 \\ 0 & a_{32} & 0 & 0 & 0 & a_{36} & 0 & 0 & a_{39} \\ a_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{49} \\ 0 & a_{52} & 0 & 0 & 0 & a_{56} & 0 & 0 & a_{59} \\ 0 & 0 & a_{63} & 0 & a_{65} & 0 & a_{67} & a_{68} & 0 \\ 0 & 0 & a_{73} & a_{74} & 0 & 0 & a_{77} & a_{78} & 0 \\ 0 & 0 & a_{83} & 0 & 0 & 0 & a_{87} & a_{88} & 0 \\ a_{91} & a_{92} & 0 & 0 & 0 & a_{96} & 0 & 0 & a_{99} \end{Vmatrix} \cdot \begin{Vmatrix} e_1^* \\ e_2^* \\ e_3^* \\ h_1^* \\ h_2^* \\ h_3^* \\ u_1^* \\ u_2^* \\ u_3^* \end{Vmatrix} = 0 \tag{8.18}$$

where

$$
\begin{aligned}
a_{14} &= -i a_{25} = -i a_{35} = \omega\,(1+\hat{\chi}) \\
a_{27} &= -\Bbbk\,\omega\,\hat{\chi}\,{}_0H \sin\varphi \\
a_{28} &= a_{39} = \Bbbk\,\omega\,\hat{\chi}\,{}_0H \cos\varphi \\
\mu_0 a_{23} &= -\mu_0 a_{32} = a_{56} = -a_{65} = i\,\Bbbk \\
a_{41} &= a_{52} = a_{63} = \hat{\sigma} - i\epsilon_0\,\omega \\
a_{49} &= -a_{67} = -i a_{73} = i\mu_0\omega\,\hat{\sigma}\,{}_0H \sin\varphi \\
a_{59} &= -a_{68} = -i\mu_0\,\omega\,\hat{\sigma}\,{}_0H \cos\varphi \\
a_{83} &= -a_{92} = \mu_0\,\hat{\sigma}\,{}_0H \cos\varphi \\
a_{74} &= a_{96} = 2i\mu_0\,\hat{\chi}\,\Bbbk\,{}_0H \cos\varphi \\
a_{78} &= a_{87} = -\frac{i}{2}\,\mu_0^2\,\omega\,\hat{\sigma}\,{}_0H^2 \sin 2\varphi \\
a_{77} &= \rho\omega^2 - [\hat{\lambda}+2\hat{\mu} - i\omega(\underline{\hat{\lambda}}+2\underline{\hat{\mu}})]\,\Bbbk^2 + i\mu_0^2\,\omega\,\hat{\sigma}\,{}_0H^2 \sin^2\varphi \\
a_{88} &= \rho\omega^2 - (\hat{\mu} - i\omega\underline{\hat{\mu}})\,\Bbbk^2 + i\mu_0^2\,\omega\,\hat{\sigma}\,{}_0H^2 \cos^2\varphi \\
a_{91} &= \mu_0\,\hat{\sigma}\,{}_0H \sin\varphi \\
a_{99} &= \rho\omega^2 - (\hat{\mu} - i\omega\underline{\hat{\mu}})\,\Bbbk^2 + i\mu_0^2\,\omega\,\hat{\sigma}\,{}_0H^2 \,.
\end{aligned}
\qquad (8.19)
$$

Eq.(8.18) can be rearranged so that the determinant of the coefficient matrix will be deduced easily. After making some lengthy elementary row operations, (8.18) is transformed into a diagonal form except the last row

$$
\left\|\begin{matrix}
\Lambda_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} & 0 & 0 & 0 & 0 & 0 \\
0 & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} & 0 & 0 & 0 & 0 & 0 \\
0 & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Lambda_{55} & \Lambda_{56} & \Lambda_{57} & 0 & 0 \\
0 & 0 & 0 & 0 & \Lambda_{65} & \Lambda_{66} & \Lambda_{67} & \Lambda_{68} & 0 \\
0 & 0 & 0 & 0 & 0 & \Lambda_{76} & \Lambda_{77} & \Lambda_{78} & 0 \\
0 & 0 & 0 & 0 & 0 & \Lambda_{86} & \Lambda_{87} & \Lambda_{88} & 0 \\
0 & \Lambda_{92} & \Lambda_{93} & \Lambda_{94} & 0 & 0 & 0 & 0 & \Lambda_{99}
\end{matrix}\right\| \cdot \left\|\begin{matrix} h_1^* \\ e_1^* \\ e_2^* \\ h_3^* \\ h_2^* \\ e_3^* \\ u_1^* \\ u_2^* \\ u_3^* \end{matrix}\right\| = 0 \qquad (8.20)
$$

where, the elements Λ_{ik} are given in terms of original ones a_{ik} as

$$
\begin{aligned}
&\Lambda_{11} = a_{14} \; ; \; \Lambda_{22} = -a_{39}\, a_{91} \; ; \; \Lambda_{23} = a_{39}\, a_{99} - a_{39}\, a_{92} \\
&\Lambda_{24} = a_{36}\, a_{99} - a_{39}\, a_{96} \; ; \; \Lambda_{32} = a_{41}\, a_{99} - a_{49}\, a_{91} \\
&\Lambda_{33} = \Lambda_{42} = -a_{49}\, a_{92} \; ; \; \Lambda_{34} = -a_{49}\, a_{96} \\
&\Lambda_{43} = a_{52}\, a_{99} - a_{59}\, a_{92} \; ; \; \Lambda_{44} = a_{56}\, a_{99} - a_{59}\, a_{96} \\
&\Lambda_{55} = a_{25}\, a_{88} \; ; \; \Lambda_{56} = a_{23}\, a_{88} - a_{28}\, a_{83} \\
&\Lambda_{57} = a_{27}\, a_{88} - a_{28}\, a_{87} \; ; \; \Lambda_{65} = a_{65} \; ; \; \Lambda_{66} = a_{63} \qquad (8.21) \\
&\Lambda_{67} = a_{67} \; ; \; \Lambda_{68} = a_{68} \; ; \; \Lambda_{76} = a_{73} \; ; \; \Lambda_{77} = a_{77} \\
&\Lambda_{78} = a_{78} \; ; \; \Lambda_{86} = a_{83} \; ; \; \Lambda_{87} = a_{87} \; ; \; \Lambda_{88} = a_{88} \\
&\Lambda_{92} = a_{91} \; ; \; \Lambda_{93} = a_{92} \; ; \; \Lambda_{94} = a_{96} \; ; \; \Lambda_{99} = a_{99} \, .
\end{aligned}
$$

For a possible wave motion, the propagation condition (8.13) is expected to yield a nontrivial solution for the amplitudes $\underset{\sim}{\varrho}^*$, $\underset{\sim}{h}^*$ and $\underset{\sim}{u}^*$ with a real nonvanishing speed of propagation. This leads to a polynomial equation with complex coeffients for the determinations of the phase velocity and attenuation constant, if they exist.

8.3. Dispersion Relations, Phase Velocities and Attenuations

It can be seen immediatly from the first row of (8.20) that the flactuating magnetic field $\underset{\sim}{\tilde{h}}$ has no component along the x_1-direction. The general propagation condition is

$$\Lambda_{99} \det \| \Lambda_{ij}^{(1)} \| \det \| \Lambda_{kl}^{(2)} \| = 0 \qquad (i,j=1,2,3;\ k,l=1,..,4) \quad (8.22)$$

which is the secular equation for the waves in an unbounded isotropic, conducting, magnetizable solid in the uniform magnetic field. In Eq.(8.22), $\det\|\Lambda_{ij}^{(1)}\|$ and $\det\|\Lambda_{kl}^{(2)}\|$ are given by

$$\begin{aligned} \det\| \Lambda_{ij}^{(1)} \| = {} & \Lambda_{22} (\Lambda_{33} \Lambda_{44} - \Lambda_{43} \Lambda_{34}) \\ & + \Lambda_{23} (\Lambda_{34} \Lambda_{42} - \Lambda_{32} \Lambda_{44}) + \Lambda_{24} (\Lambda_{43} \Lambda_{32} - \Lambda_{33} \Lambda_{42}) \end{aligned} \qquad (8.23)$$

and

$$\begin{aligned} \det\| \Lambda_{kl}^{(2)} \| = {} & \Lambda_{55} [\Lambda_{66} (\Lambda_{77} \Lambda_{88} - \Lambda_{87} \Lambda_{78}) + \Lambda_{67} (\Lambda_{78} \Lambda_{86} - \Lambda_{76} \Lambda_{88}) \\ & + \Lambda_{68} (\Lambda_{87} \Lambda_{76} - \Lambda_{77} \Lambda_{86})] \\ & + \Lambda_{65} [\Lambda_{56} (\Lambda_{78} \Lambda_{87} - \Lambda_{77} \Lambda_{88}) + \Lambda_{57} (\Lambda_{76} \Lambda_{88} - \Lambda_{78} \Lambda_{86})]. \end{aligned} \qquad (8.24)$$

For this degree of generality maintained up to now, the order of the polynomial equations in k is rather high, and in consequence all of its roots may not possibly be determined. In the remainder of this chapter, we shall confine our attention to the study of waves for the special direction of the applied primary magnetic field.

8.3.a) Propagation through conducting, magnetizable visoelastic solids when primary magnetic field is absent (${}_0\underset{\sim}{H} = 0$):

For the purpose of comparison, it is appropriate to consider the case of solids for which the applied primary magnetic field is zero, i.e., ${}_0H_1 = {}_0H_2 = 0$. Then some of the entries of the coefficient matrix (8.20) vanish and the others become simpler. Thus, from Eq.(8.19) one obtains

$$\begin{aligned} a_{27} &= a_{28} = a_{39} = a_{49} = a_{67} = a_{73} = a_{59} = a_{68} = a_{83} \\ &= a_{92} = a_{74} = a_{96} = a_{78} = a_{87} = a_{91} = 0 \\ a_{77} &= \rho\omega^2 - [\hat{\lambda} + 2\hat{\mu} - i\omega(\underline{\hat{\lambda}} + 2\underline{\hat{\mu}})]k^2 \\ a_{88} &= a_{99} = \rho\omega^2 - (\hat{\mu} - i\omega\underline{\hat{\mu}})k^2 \end{aligned} \tag{8.25}$$

which impllies that

$$\begin{aligned} \Lambda_{22} &= \Lambda_{33} = \Lambda_{42} = \Lambda_{34} = \Lambda_{57} = \Lambda_{67} = \Lambda_{68} \\ &= \Lambda_{76} = \Lambda_{78} = \Lambda_{86} = \Lambda_{87} = \Lambda_{92} = 0 \\ \Lambda_{23} &= a_{32}a_{99} \quad ; \quad \Lambda_{24} = a_{36}a_{99} \quad ; \quad \Lambda_{32} = a_{41}a_{99} \\ \Lambda_{43} &= a_{52}a_{99} \quad ; \quad \Lambda_{44} = a_{56}a_{99} \quad ; \quad \Lambda_{55} = a_{25}a_{88} \\ \Lambda_{56} &= a_{23}a_{88} \end{aligned} \tag{8.26}$$

and the remaining a_{ik} and Λ_{ik} are the same as in (8.19) and (8.21) respectively.

Upon substituting Eq.(8.26) into (8.22) and using (8.25), one obtains the dispersion relation which is equivalent to

$$\begin{aligned} (\hat{\sigma} - i\epsilon_0\omega)\left[V_0^2k^2 - \omega^2\left(1 + i\frac{\hat{\sigma}}{\omega\epsilon_0}\right)\right]^2 \left[\left(1 - \frac{i\omega}{\overline{\omega}_v}\right)V_s^2k^2 - \omega^2\right]^2 \cdot \\ \left[\left(1 - i\frac{\omega}{\omega_v}\right)V_p^2k^2 - \omega^2\right] = 0 \end{aligned} \tag{8.27}$$

where

$$V_0 = \frac{c}{\sqrt{1+\hat{\chi}}} \quad ; \quad V_s = \sqrt{\frac{\hat{\mu}}{\rho}} \quad ; \quad V_p = \sqrt{\frac{\hat{\lambda}+2\hat{\mu}}{\rho}} \tag{8.28}$$

$$\bar{\omega}_\nu = \frac{\hat{\mu}}{\hat{\underline{\mu}}} \quad \text{and} \quad \underline{\omega}_\nu = \frac{\hat{\lambda}+2\hat{\mu}}{\hat{\underline{\lambda}}+2\hat{\underline{\mu}}} .$$

In (8.28), V_0 , V_s and V_p are the speed of light in the medium of magnetic susceptibility $\hat{\chi}$, speeds of elastic S and P waves respectively. The new parameters $\bar{\omega}_\nu$ and $\underline{\omega}_\nu$ are in the dimension of frequency. Eq.(8.27) implies that, the relations between k and ω are

$$V_0^2 k^2 - \omega^2 (1+ i\nu_c) = 0 \quad ; \quad (1-i\bar{\omega}) V_s^2 k^2 - \omega^2 = 0 \tag{8.29}$$

$$(1-i\underline{\omega}) V_p^2 k^2 - \omega^2 = 0$$

where

$$\nu_c = \frac{1}{\omega} \left(\frac{\hat{\sigma}}{\epsilon_0}\right) \quad ; \quad \bar{\omega} = \frac{\omega}{\bar{\omega}_\nu} \quad ; \quad \underline{\omega} = \frac{\omega}{\underline{\omega}_\nu} . \tag{8.30}$$

The quantities in (8.30) are dimensionless. It should be noted that if the primary magnetic field is absent, the electromagnetic and mechanical waves propagate without coupling.

Supposing first that ω is a real constant, that is, a fixed frequency is assigned to the waves, from (8.29) it is seen that the medium is dispersive due to the electrical conductivity together with the viscosity of the medium*. Thus the phase velocity is not the same for each frequency of the wave. Consequently, different components of the wave travel with different speeds and tend to change phase with respect to one another. However, since the Maxwell's equations in vacuum are nondispersive, there is no difficulty in defining the velocity with which energy is transmitted through the medium by the wave motion. This velocity is simply equal to the phase velocity. When, however, the transmitting medium is dispersive, the definition of energy velocity requires special attention and is known to differ from phase velocity.

* It is also possible to regard k as a real constant, that is, (8.29) describes waves of an assigned wavelength.

To allow for the possibility of dispersion, either $\mathbb{k}$ or ω is viewed as the function of the other. Since the dispersive properties can not depend on whether the wave travels to the left or to the right in an unbounded medium, ω must be an even function of $\mathbb{k}$, $\omega(\mathbb{k})=\omega(-\mathbb{k})$. This implies that the secular equation is a polynomial in the even order of $\mathbb{k}$. For most frequencies (or wavelengths) $\mathbb{k}$ (or ω) is a smoothly varying function of ω (or $\mathbb{k}$). However, at certain frequencies (or wavelengths) there exist regions of so called "anomalous dispersion" where $\mathbb{k}$ (or ω) varies rapidly over a narrow interval of frequency (or wavelength). Supposing, for the present, $\mathbb{k}$ is complex, i.e., $\mathbb{k}=\mathbb{k}_r+i\,\mathbb{k}_i$ dissipative effects at assigned frequencies are permitted.

Through Eq.(8.29), one obtains the roots as

$$\mathbb{k}_{r_1}=\mp\frac{\omega}{\sqrt{2}\,V_o}\left(1\mp\sqrt{1+V_c^2}\right)^{1/2} \quad ; \quad \mathbb{k}_{i_1}=\mp\frac{\omega}{\sqrt{2}\,V_o}\left(\mp\sqrt{1+V_c^2}-1\right)^{1/2} \tag{8.31}$$

$$\mathbb{k}_{r_2}=\mp\frac{\omega}{\sqrt{2}\,V_s}\left(1\mp\sqrt{1+\bar{\omega}^2}\right)^{1/2} \quad ; \quad \mathbb{k}_{i_2}=\mp\frac{\omega}{\sqrt{2}\,V_s}\left(\mp\sqrt{1+\bar{\omega}^2}-1\right)^{1/2} \tag{8.32}$$

and

$$\mathbb{k}_{r_3}=\mp\frac{\omega}{\sqrt{2}\,V_p}\left(1\mp\sqrt{1+\underline{\omega}^2}\right)^{1/2} \quad ; \quad \mathbb{k}_{i_3}=\mp\frac{\omega}{\sqrt{2}\,V_p}\left(\mp\sqrt{1+\underline{\omega}^2}-1\right)^{1/2} \tag{8.33}$$

where $\mathbb{k}_{rn}$ ($\mathbb{k}_{in}$) stands for the real (imaginary) part of the n-th root. For $\mathbb{k}_r$ and $\mathbb{k}_i$ to be real, the positive sign in front of the square roots of (8.31-33) must be taken. Thus, the phase velocities and the attenuations per wavelength for the electromagnetic and viscoelastic waves are, respectively

$$\tilde{V}\equiv\frac{V}{V_o}=\mp\sqrt{\frac{2}{1+\xi_1}} \quad ; \quad \tilde{\alpha}=2\pi\sqrt{\frac{\xi_1-1}{\xi_1+1}} \tag{8.34}$$

$$\bar{V} \equiv \frac{V}{V_s} = \mp\sqrt{\frac{2}{1+\xi_2}} \quad ; \quad \bar{\alpha} = 2\pi\sqrt{\frac{\xi_2 - 1}{\xi_2 + 1}} \tag{8.35}$$

$$\underline{V} \equiv \frac{V}{V_p} = \mp\sqrt{\frac{2}{1+\xi_3}} \quad ; \quad \underline{\alpha} = 2\pi\sqrt{\frac{\xi_3 - 1}{\xi_3 + 1}} \tag{8.36}$$

where

$$\xi_1 = \sqrt{1+\gamma_c^2} \quad ; \quad \xi_2 = \sqrt{1+\bar{\omega}^2} \quad ; \quad \xi_3 = \sqrt{1+\underline{\omega}^2} \quad . \tag{8.37}$$

Here the phase velocity V is defined by $V = \frac{\omega}{Re\,k}$ and α is the attenuation per wavelength defined by $\alpha = (\Im m\, k)\lambda$ in which λ is the wavelength defined by $\lambda = 2\pi\, V/\omega$.

From Eqs.(8.34-36) it is seen that both of the electromagnetic and mechanical waves are dispersive and dissipative. The damping is coming from the electrical conductivity and the viscosity of the medium. Then the solutions for electromagnetic fields and displacement are

$$\begin{Bmatrix} \tilde{e}_2 \\ \tilde{e}_3 \\ \tilde{h}_2 \\ \tilde{h}_3 \end{Bmatrix} = \begin{Bmatrix} e_2^* \\ e_3^* \\ h_2^* \\ h_3^* \end{Bmatrix} \cdot \exp\left[-\frac{\omega x}{\sqrt{2}\,V_0}\sqrt{\xi_1 - 1} - i\omega\left(t - \frac{x}{\sqrt{2}\,V_0}\sqrt{1+\xi_1}\right)\right] \tag{8.38}$$

$$\tilde{u}_1 = u_1^* \cdot \exp\left[-\frac{\omega x}{\sqrt{2}\,V_p}\sqrt{\xi_3 - 1} - i\omega\left(t - \frac{x}{\sqrt{2}\,V_p}\sqrt{1+\xi_3}\right)\right]$$

$$\begin{Bmatrix} \tilde{u}_2 \\ \tilde{u}_3 \end{Bmatrix} = \begin{Bmatrix} u_2^* \\ u_3^* \end{Bmatrix} \cdot \exp\left[-\frac{\omega x}{\sqrt{2}\,V_s}\sqrt{\xi_2 - 1} - i\omega\left(t - \frac{x}{\sqrt{2}\,V_s}\sqrt{1+\xi_2}\right)\right] \tag{8.39}$$

which represent progressive electromagnetic, viscoelastic P and S waves travelling along the x_1-axis respectively. According to (8.38), the electromagnetic wave is transverse, i.e., it has no components along the x_1-axis, $\tilde{e}_1 = \tilde{h}_1 = 0$.

There is no difficulty in defining the group velocity V^g ($V^g \equiv d\omega/dk$) as long as the medium is purely dispersive, but if absorption also occurs, k becomes complex or imaginary and the group velocity ceases to have a clear physical meaning. So far if one assumes that the coefficient of absorption in the material is zero, then $V^g = V^e$ (velocity of the propagated energy).

We now determine the magnitude of the group velocity by means of (8.31-33). Using the definition of V^g in (8.31-33), one now obtains

$$\tilde{V}^g = \mp \frac{\sqrt{2(1+\xi_1)}}{1+\xi_1 - \nu_c^2/2\xi_1} \quad ; \quad \bar{V}^g = \mp \frac{\sqrt{2(1+\xi_2)}}{1+\xi_2 - \bar{\omega}^2/2\xi_2} \tag{8.40}$$

$$\underline{V}^g = \mp \frac{\sqrt{2(1+\xi_3)}}{1+\xi_3 - \underline{\omega}^2/2\xi_3} \, .$$

For the comparison of the phase velocities and the attenuations of the waves in an applied primary magnetic field, the phase velocities and the attenuations of the uncoupled waves, Eq.(8.34-36) are plotted as functions of frequency by parametrizing the conductivity, $\hat{\sigma}$ for the electromagnetic waves, $\bar{\omega}_\nu$ and $\underline{\omega}_\nu$ for the viscoelastic waves.

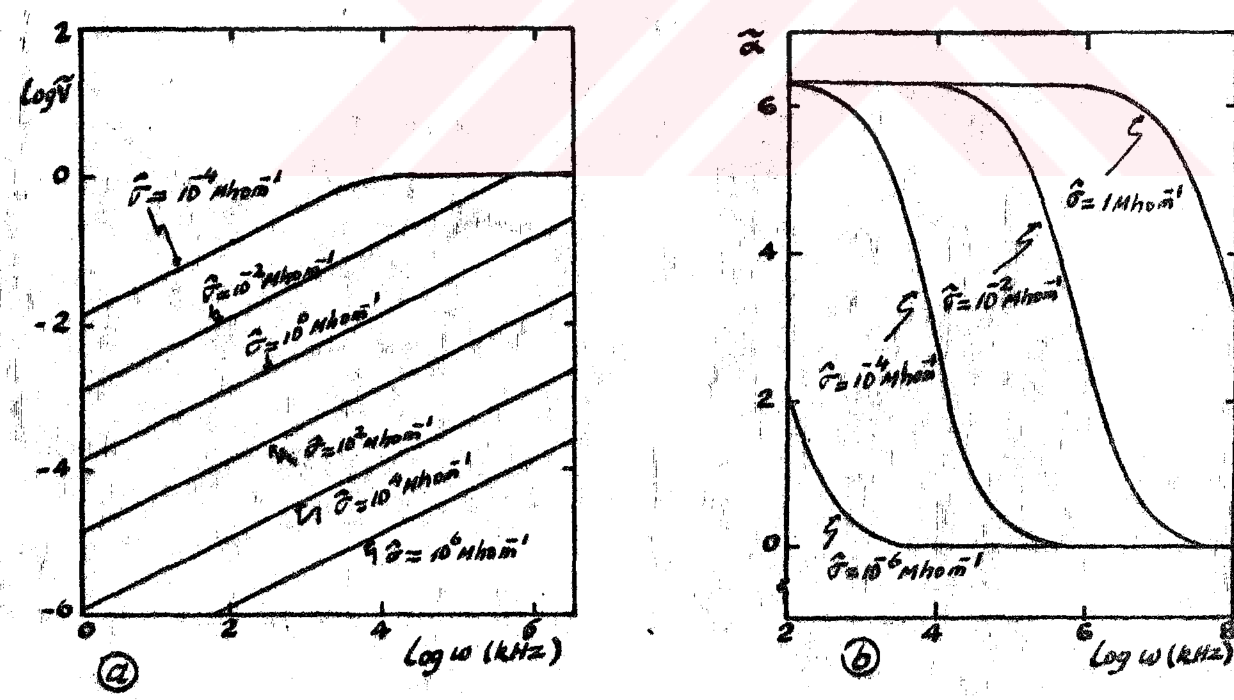


Fig.8.2 a) Phase Velocity $\tilde{V}$ and b) Attenuation $\tilde{\alpha}$ of Uncoupled Electromagnetic Wave as Function of Frequency ω, Parametrized for Conductivity $\hat{\sigma}$.

Fig. 8.2 gives plots of the nondimensional phase velocity

$\tilde{V}$ and attenuation $\tilde{\alpha}$ for the uncoupled electromagnetic waves for different electrical conductivities. The plots in Fig.8.3 are the phase velocity and the attenuation of the uncoupled electromagnetic waves as function of conductivity $\hat{\sigma}$ for certain frequencies.

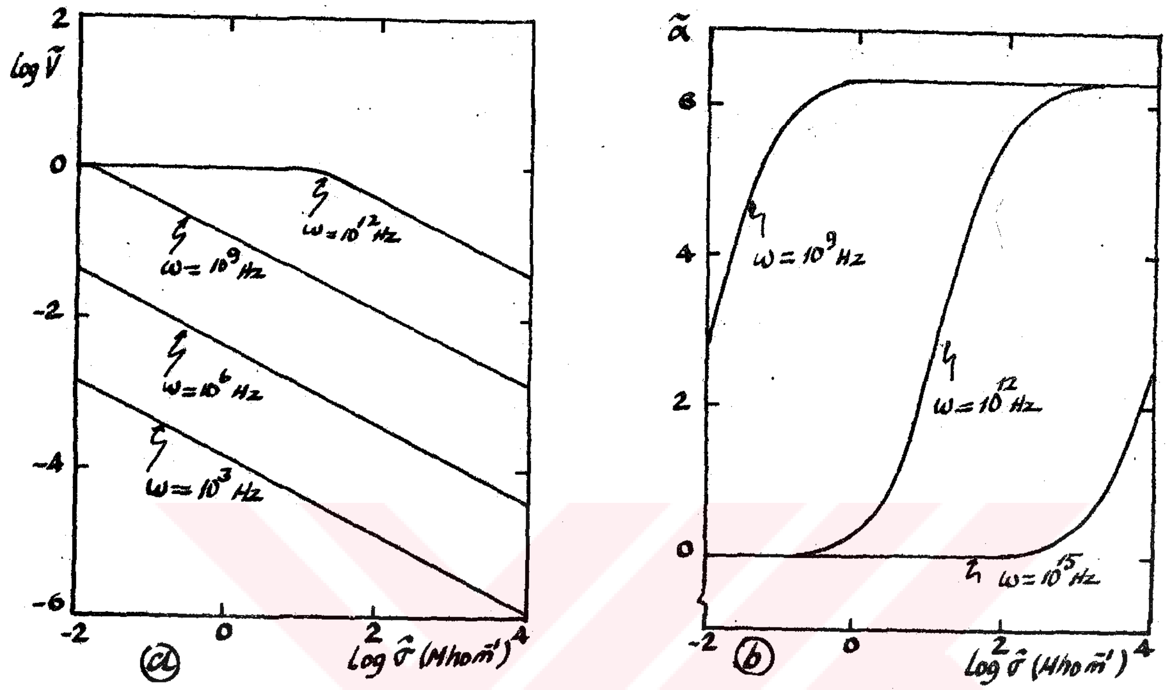


Fig.8.3 a) Phase Velocity $\tilde{V}$ and b) Attenuation $\tilde{\alpha}$ of Uncoupled Electromagnetic Wave as Function of Conductivity $\hat{\sigma}$ for Several Selected Values of Frequency ω .

It is seen from both these graphs that there are two limiting cases for the electromagnetic waves. One is the case of the nonconductor, i.e., $\hat{\sigma} \rightarrow 0$. For very small $\hat{\sigma}$, the electromagnetic wave is nondispersive and the attenuation per wavelength approaches zero beyond a certain frequency. It can also be seen immediately from $(8.40)_1$ that the group velocity is simply phase velocity. Another limiting case is that of superconductivity, i.e., $\hat{\sigma} \longrightarrow \infty$. For this case, the phase velocity and the group velocity approach zero very rapidly. Thus the waves immediately die out and the attenuation per wavelength approaches 2π .

Fig. 8.4 displays the phase velocity $\bar{V}$ and the attenuation $\bar{\alpha}$ of the uncoupled viscoelastic S wave as the function of frequency ω for different values of $\bar{\omega}_{\nu} = \frac{\hat{\mu}}{\bar{\mu}}$. Fig.8.5 gives the plots of $\bar{V}$ and $\bar{\alpha}$ as the function of $\bar{\omega}_{\nu}$ for several selected values of frequency ω . The graphs in Figs.

8.4 and 8.5 are also used as the plots of the phase velocity $\underline{V}$ and the attenuation $\underline{\alpha}$ of the uncoupled viscoelastic P wave as the function of ω (or $\underline{\omega}_\nu$) for several selected values of $\underline{\omega}_\nu$ (or ω).

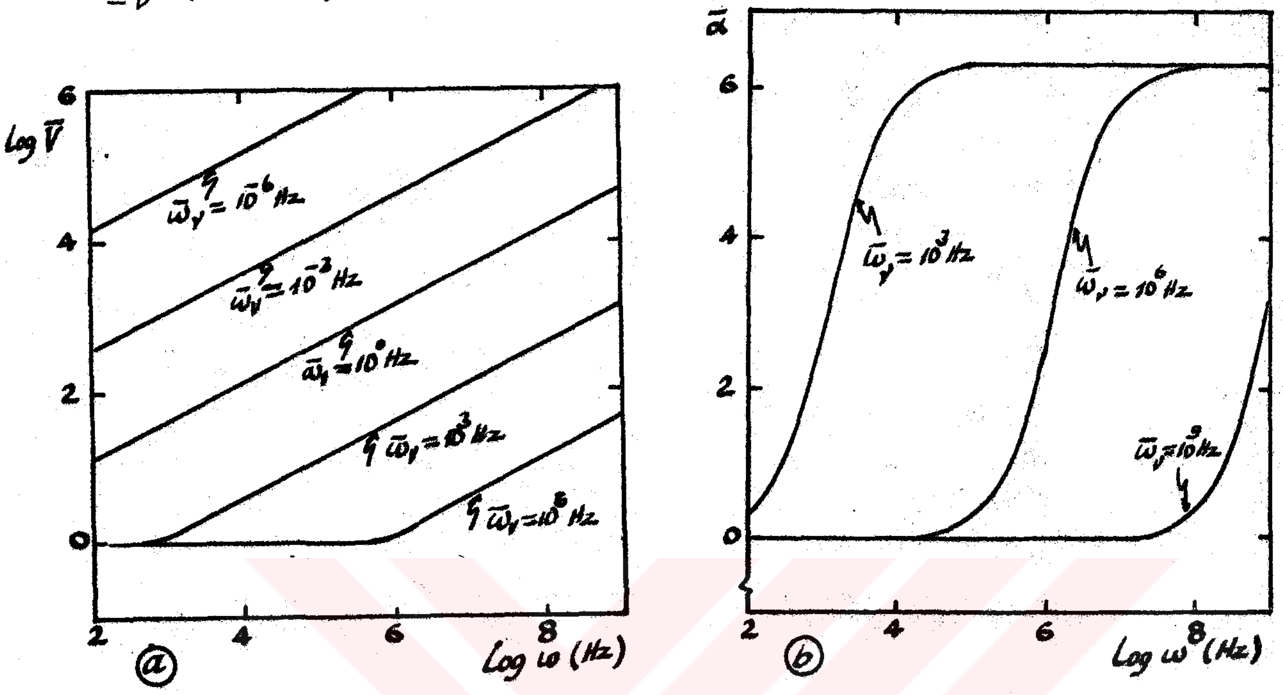


Fig.8.4 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Uncoupled Viscoelastic S Wave as Function of Frequency ω Parametrized for $\bar{\omega}_\nu$.

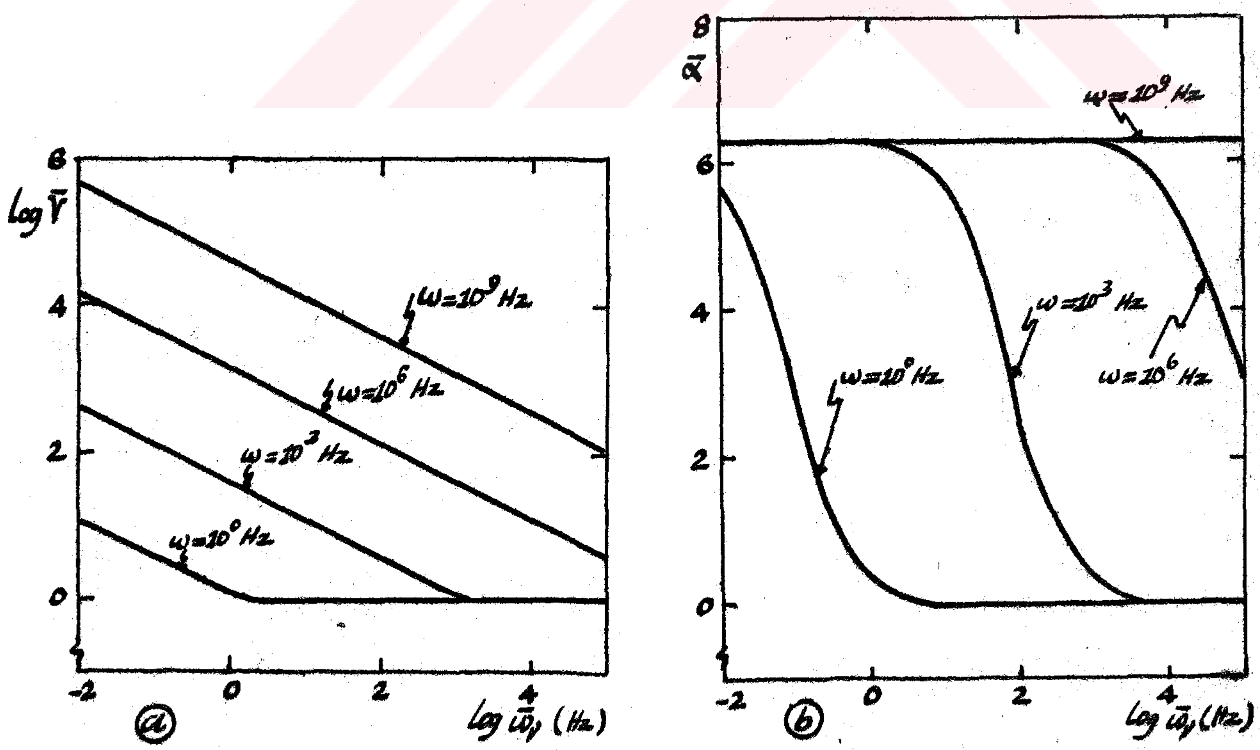


Fig.8.5 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Uncoupled Viscoelastic S Wave as Function of $\bar{\omega}_\nu$, for Several Selected Values of Frequency ω.

For the mechanical waves there are two limiting cases. One is the case of very small viscous Lamé constants, i.e., $\hat{\lambda} \to 0$ $\hat{\mu} \to 0$. For this case, S and P waves are both nondispersive and the group velocities are simply phase velocities. The attenuations approach zero. Another limiting case is the one of highly viscous media, i.e., $\hat{\lambda} \to \infty$, $\hat{\mu} \to \infty$. Then, it can be shown that $\bar{V}$ (or $\underline{V}$) is proportional to the square root of $\bar{\omega}$ (or ω). The attenuations per wavelength ~~for both waves approch~~ 2π and they are indicated in Figs.8.4 b and 8.5 b.

It is to be noted that the imaginary part of k appears as the coefficient measuring attenuation of the electromagnetic and mechanical waves in space. It is called the "spatial absorption coefficient". k is a commonly used coefficient in ordinary progressive wave propagation. However, it is equally possible to set

$$\omega = \omega_r + i\,\omega_i \tag{8.41}$$

into (8.29) and obtain the electromagnetic fields and the displacement in the form

$$\begin{Bmatrix} \tilde{\underset{\sim}{e}} \\ \tilde{\underset{\sim}{h}} \\ \tilde{\underset{\sim}{u}} \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{e}^* \\ \underset{\sim}{h}^* \\ \underset{\sim}{u}^* \end{Bmatrix} \cdot exp(-\omega_i t) \cdot exp\left[i(\omega_r t - kx)\right] \quad . \tag{8.42}$$

Here ω_i gives the coefficient which measures attenuation in time. It is then called the "temporal absorption coefficient". It applies appropriately to the temporal decay of a wave train or standing wave in a medium. When (8.29) is solved for complex ω at a given wavelength, the result obtained for the electromagnetic wave is the same as that given by Hutter ([26],p.1073) although a different formulation of Maxwell's equations is employed in there.

In concluding this section, it is said that if the applied primary magnetic field is absent, there are uncoupled modes of dispersive and dissipative electromagnetic and mechanical waves. The phase velocities and the attenuations of these modes are smoothly varying functions of frequency and the considered parameters, such as conductivity σ, and the viscous parameters $\bar{\omega}_\nu$ and w_ν. Thus, there is no anomalous dispersion over an interval of frequency or those of the considered parameters.

8.3.b) Propagation of waves under the applied longitudinal primary magnetic field (${}_0H_1 \neq 0$, ${}_0H_2 = 0$):

For the general form of the characteristic equation, $\det\|\Delta\| = 0$ is complicated and little analytic insight is gained by writing it down in full generality. Special cases, for example ${}_0H_2 = 0$ or ${}_0H_1 = 0$, can be treated analytically with reasonable effort.

Wave propagation along the applied magnetic field is obtained by substituting $\varphi = 0$, i.e., $\sin\varphi = 0$ in (8.19) and the resulting equations into (8.21). Thus one has

$$
\begin{aligned}
& a_{27} = a_{49} = a_{67} = a_{73} = a_{78} = a_{87} = a_{91} = 0 \\
& a_{28} = a_{39} = k\omega\hat{\chi}\,{}_0H \quad ; \quad a_{49} = -a_{68} = -\varepsilon\mu_0\omega\hat{\sigma}\,{}_0H \\
& a_{83} = -a_{92} = \mu_0\hat{\sigma}\,{}_0H \quad ; \quad a_{74} = a_{96} = 2i\mu_0\hat{\chi}\,k\,{}_0H \\
& a_{77} = \rho\omega^2 - [\hat{\lambda} + 2\hat{\mu} - i\omega(\underline{\hat{\lambda}} + 2\underline{\hat{\mu}})]k^2 \\
& a_{88} = a_{99} = \rho\omega^2 - (\hat{\mu} - i\omega\underline{\hat{\mu}})k^2 + i\mu_0^2\omega\hat{\sigma}\,{}_0H^2 \\
& \Lambda_{22} = \Lambda_{33} = \Lambda_{42} = \Lambda_{34} = \Lambda_{57} = \Lambda_{67} = \Lambda_{76} = \Lambda_{78} = \Lambda_{87} \\
& \qquad = \Lambda_{92} = 0 \qquad ; \quad \Lambda_{32} = a_{41}a_{99}
\end{aligned}
\tag{8.43}
$$

and for the remaining a_{ik} and Λ_{ik} , see Eqs.(8.19) and (8.21) respectively. The determinant (8.22) assumes the form

$$
\left(1 - i\frac{1}{V_c}\right)\Delta_{11}\Delta_{12}\Delta_{13}\Delta_{14} = 0 \tag{8.44}
$$

with

$$
\Delta_{11} = (1 - i\underline{\omega})V_p^2k^2 - \omega^2 \tag{8.45}
$$

$$
\Delta_{12} = (1 - i\bar{\omega})V_s^2k^2 - (1 + iV_cV_H)\omega^2 \tag{8.46}
$$

$$
\begin{aligned}
\Delta_{13} = {} & (1 - i\bar{\omega})k^4 - \left\{1 + (1 + iV_c)\left[(1 - i\bar{\omega})\left(\frac{V_s}{V_0}\right)^2 + 2\hat{\chi}^2V_H\right]\right. \\
& \left. - i(1 + \hat{\chi})V_cV_H\right\}\frac{\omega^2k^2}{V_s^2} + \left[1 + 2V_c^2V_H + i(1 - V_V)V_c\right]\frac{\omega^4}{V_s^2V_0^2}
\end{aligned}
\tag{8.47}
$$

$$\Delta_{14} = (1 - i^{\circ}\bar{\omega})k^4 - \left\{1 + \left[(1 - i^{\circ}\bar{\omega})(1 + i^{\circ}\nu_c) + i^{\circ}\bar{\nu}_H \nu_c\right]\left(\frac{V_s}{V_0}\right)^2\right\} \times$$

$$\times \frac{\omega^2 k^2}{V_s^2} + \left[1 + i^{\circ}(1 + \nu_H)\nu_c\right]\frac{\omega^4}{V_s^2 V_0^2} \qquad (8.48)$$

where new dimensionless quantities are introduced as

$$\nu_H = \frac{\mu_0\, {}_0H^2}{\rho c^2} \qquad \text{and} \qquad \bar{\nu}_H = \frac{\mu_0\, {}_0H^2}{\rho V_s^2} \ . \qquad (8.49)$$

Since $1 - i^{\circ}/\nu_c \neq 0$, two obvious solutions of (8.44) are $\Delta_{11} = 0$ and $\Delta_{12} = 0$. The first one represents the dispersive and dissipative uncoupled P wave and its behavior has been discussed in Section 8.3.a, i.e., $(8.29)_3$. This means that this mode of the mechanical wave propagates without being influenced by the applied magnetic field if the direction of the external magnetic field coincides with the direction of wave propagation. The second one is a dispersive and dissipative coupled mechanical wave which is now affected by the applied primary magnetic field. The phase velocity and the attenuation follow from (8.46)

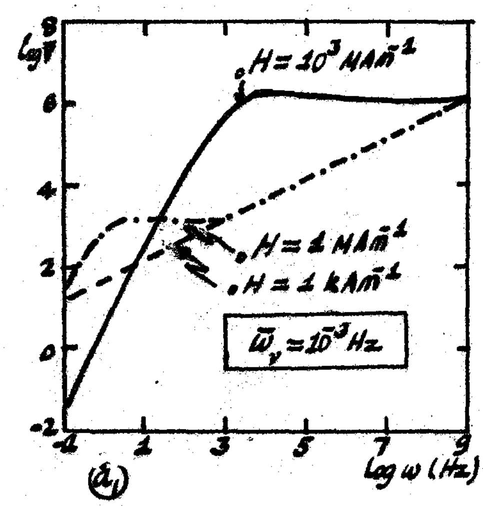
$$\bar{V} = \sqrt{\frac{2(1 + \bar{\omega}^2)}{\xi_4 - \xi_5}} \quad ; \qquad \alpha = 2\pi\sqrt{\frac{\xi_4 + \xi_5}{\xi_4 - \xi_5}} \qquad (8.50)$$

where

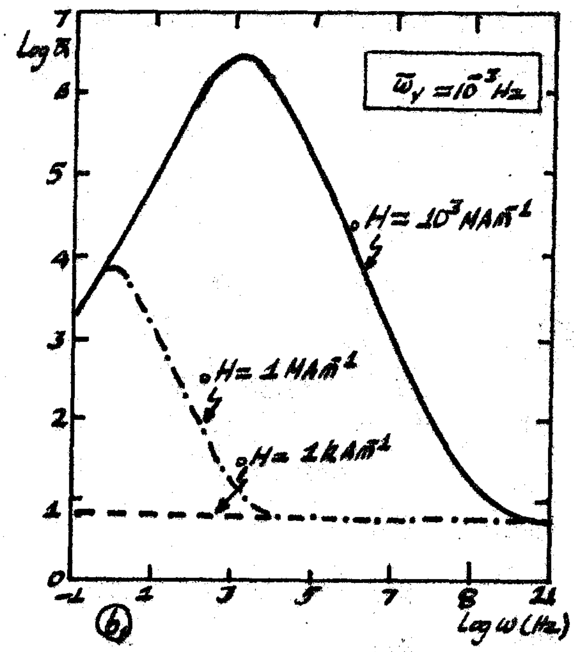
$$\xi_4 = \left[(1 + \bar{\omega}^2)(1 + \nu_c^2 \nu_H^2)\right]^{1/2} \quad ; \qquad \xi_5 = \bar{\omega}\, \nu_c \nu_H - 1 \ . \qquad (8.51)$$

As seen explicitly from the solution (8.50), the primary longitudinal magnetic field (PLMF) influences the phase velocity and the attenuation. If $\nu_H = 0$ (no magnetic field) or $\nu_c = 0$ (nonconductor) is substituted into (8.51), Eq.(8.50) reduces to (8.35).

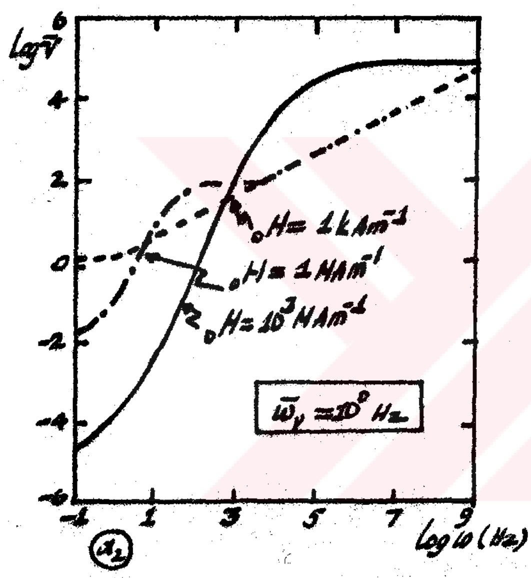
The numerical solution of (8.50) is carried out for a hypothetical test material, the electric conductivity of which is $\sigma = 5 \times 10^6\ Mho\ m^{-1}$. The phase velocity and the attenuation are plotted as functions of one variable, keeping others constant. Numerical calculations are performed for selected intervals of independent variable and parameters.



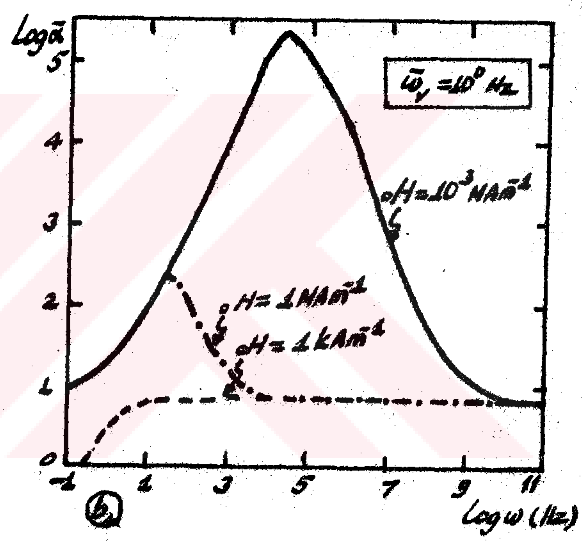
Log V̄
H = 10^3 MAm^-1
H = 1 MAm^-1
H = 1 kAm^-1
ω̄_v = 10^-3 Hz
Log ω (Hz)
a1



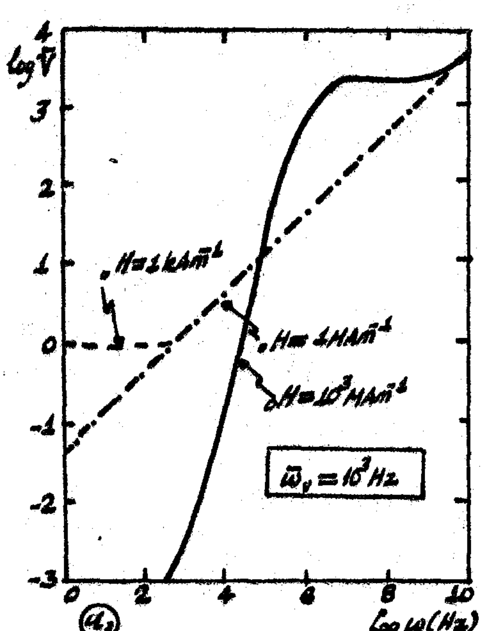
Log ᾱ
ω̄_y = 10^-3 Hz
H = 10^3 MAm^-1
H = 1 MAm^-1
H = 1 kAm^-1
Log ω (Hz)
b1



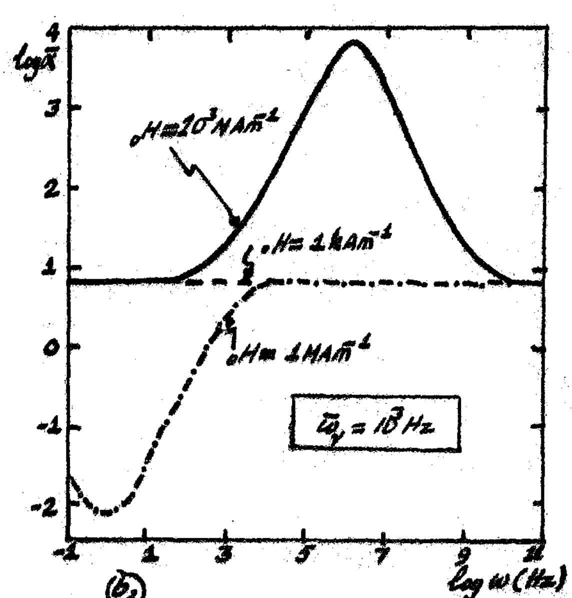
Log V̄
H = 1 kAm^-1
H = 1 MAm^-1
H = 10^3 MAm^-1
ω̄_y = 10^0 Hz
Log ω (Hz)
a2



Log ᾱ
ω̄_y = 10^0 Hz
H = 10^3 MAm^-1
H = 1 MAm^-1
H = 1 kAm^-1
Log ω (Hz)
b2



Log V̄
H = 1 kAm^-1
H = 1 MAm^-1
H = 10^3 MAm^-1
ω̄_y = 10^3 Hz
Log ω (Hz)
a3



Log ᾱ
H = 10^3 MAm^-1
H = 1 kAm^-1
H = 1 MAm^-1
ω̄_y = 10^3 Hz
Log ω (Hz)
b3

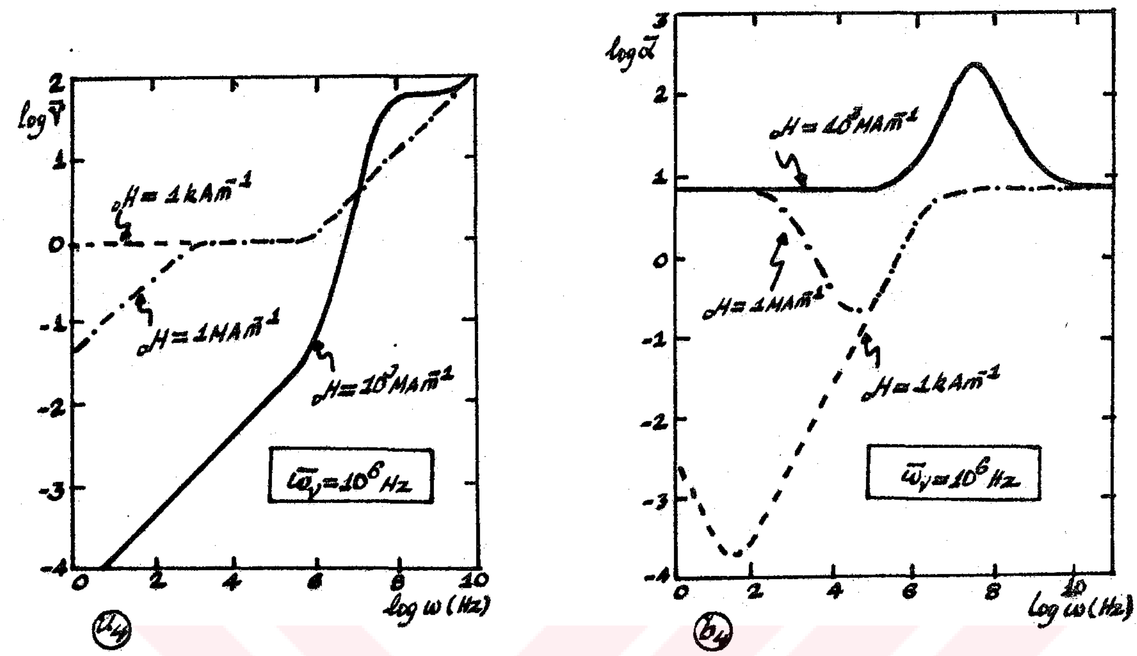


Fig.8.6 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Coupled Mechanical Wave as Function of Frequency ω for Different Selected Values of Primary Longitudinal Magnetic Field $_oH$ and Parameter $\bar{\omega}_\nu$.

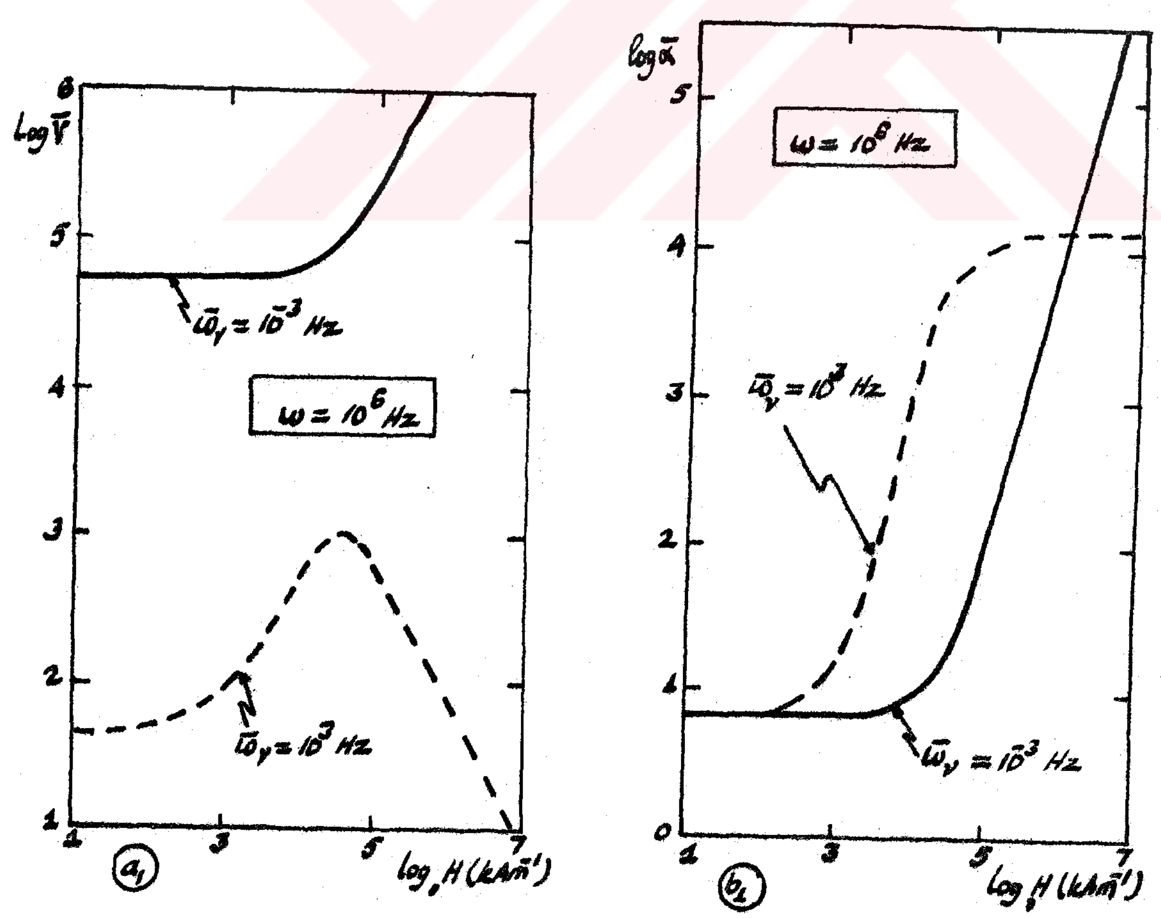
Fig.8.6 gives plots of the phase velocity $\bar{V}$, and the attenuation $\bar{\alpha}$ for the coupled mechanical wave at several selected values of the parameter $\bar{\omega}_\nu$ and the applied primary longitudinal magnetic field $_oH$.

In Fig.8.6$_1$, it is shown that $\log \bar{V}$ and $\log \bar{\alpha}$ are linear functions of $\log \omega$ for small values of $\bar{\omega}_\nu$ and the applied magnetic field in the considered range of frequency. In other words, the phase velocity $\bar{V}$ and the attenuation per wavelegth $\bar{\alpha}$ are parabolic functions of the frequency ω for highly viscous solids i.e., $\bar{\omega}_\nu \ll 1$. As long as the applied magnetic field is small enough (1kAm^{-1}) the phase velocity and the attenuation do not change with respect to the longitudinal magnetic field.However, when the primary magnetic field is of the order 1MAm^{-1} or greater, both $\log \bar{V}$ and $\log \bar{\alpha}$ are nonlinear functions of $\log \omega$ in a certain range of ω, but beyond certain values of ω, they are again linear functions of $\log \omega$. On the other hand, there is an anomalous dispersion at a certain small interval of the frequency of the propagated mechanical wave depending upon the value of the applied magnetic field. Observations show that there are significant differences between the plots in Figs. 8.4 and 8.6_1.

Fig.8.6_{2-4} displays that the variations of $\bar{V}$ and $\bar{\alpha}$ are similar with those of Fig.8.6_1, but the anomalous dispersion occurs at different intervals of frequency depending upon the values of the viscosity of materials. The peak of the anomalous dispertion occurs at a certain frequency and this frequency becomes higher when the solid becomes more elastic (i.e., $\bar{\omega}_\nu$ is very large) in the same magnetic field.

Fig.8.7 gives the magnetic field dependence of the phase velocity and the attenuation in a certain interval of ${}_oH$ i.e., $1\mathrm{kAm}^{-1} \leq {}_oH \leq 10^5\,\mathrm{MAm}^{-1}$ at several frequencies keeping $\bar{\omega}_\nu$ constant.

Fig.8.7_1 displays, however, that $\bar{V}$ and $\bar{\alpha}$ of an ultrasonic wave in a very high viscous solid (i.e., $\bar{\omega}_\nu << 1$) are not affected by the primary magnetic field of the order 1 MAm^{-1}. If the solid becomes more elastic, $\bar{V}$ and $\bar{\alpha}$ are affected by



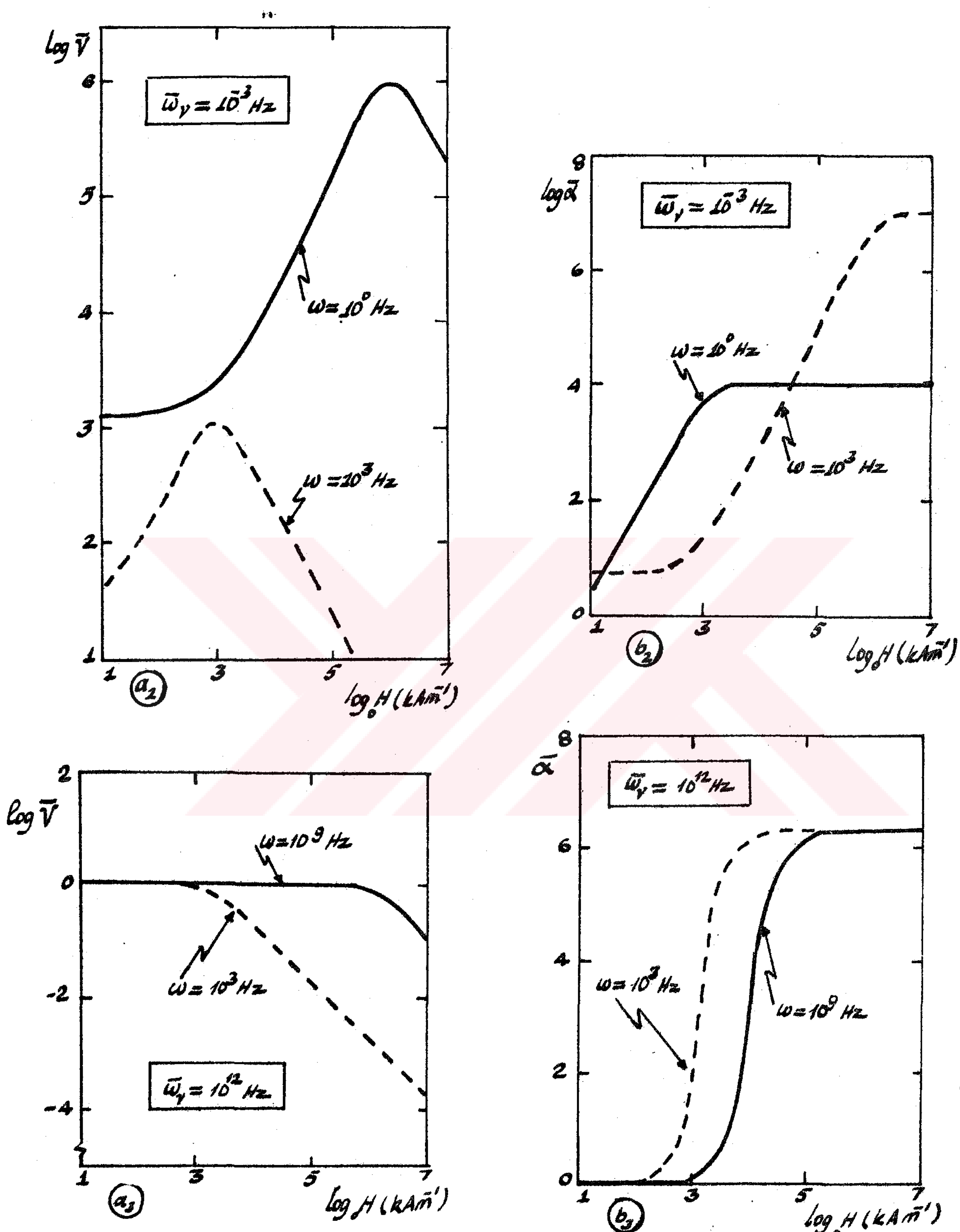
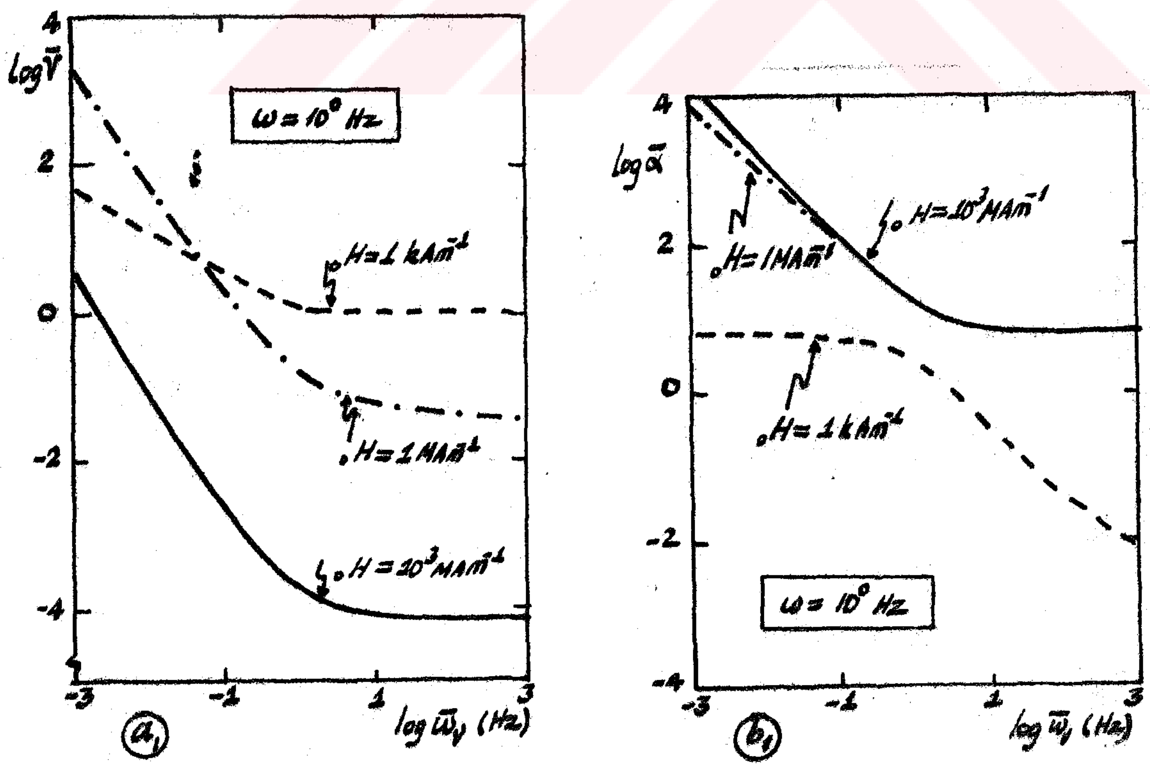


Fig.8.7 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Coupled Mechanical Wave as Function of Primary Longitudinal Magnetic Field $_oH$ for Certain Values of Frequency ω and Parameter $\bar{\omega}_\nu$.

the same magnetic field. In Fig.8.7_2, it is shown that, depending upon the frequency of the propagating mechanical wave, $\bar{V}$ and $\bar{\alpha}$ reach their maximum values at the specific applied magnetic field. Maximum values of $\bar{V}$ and $\bar{\alpha}$ associated with ${}_oH$ are not significant when the solids become very elastic. This is shown in Fig.8.7_3. Beyond a certain value of ${}_oH$, for example ${}_oH_c$, depending upon the frequency of the coupled mechanical wave, the phase velocity changes very rapidly when ${}_oH \geq {}_oH_c$. On the other hand, as it is illustrated in Fig.8.7_3, $\bar{V}$ is constant up to a certain value of ${}_oH$ and then $\log \bar{V}$ varies linearly with $\log {}_oH$ and becomes very small. Physically this means that when ${}_oH \rightarrow \infty$, the mechanical wave can not propagate. This curious behavior is only detected when a detailed numerical investigation is performed.

One can easily see the difference between the plots in Figs. 8.8 and 8.5, i.e., the difference between the coupled and uncoupled S waves. The effect of the applied longitudinal magnetic field tends to disappear when the frequency of the propagated wave becomes very high (very high ultrasonic wave, VHUW).



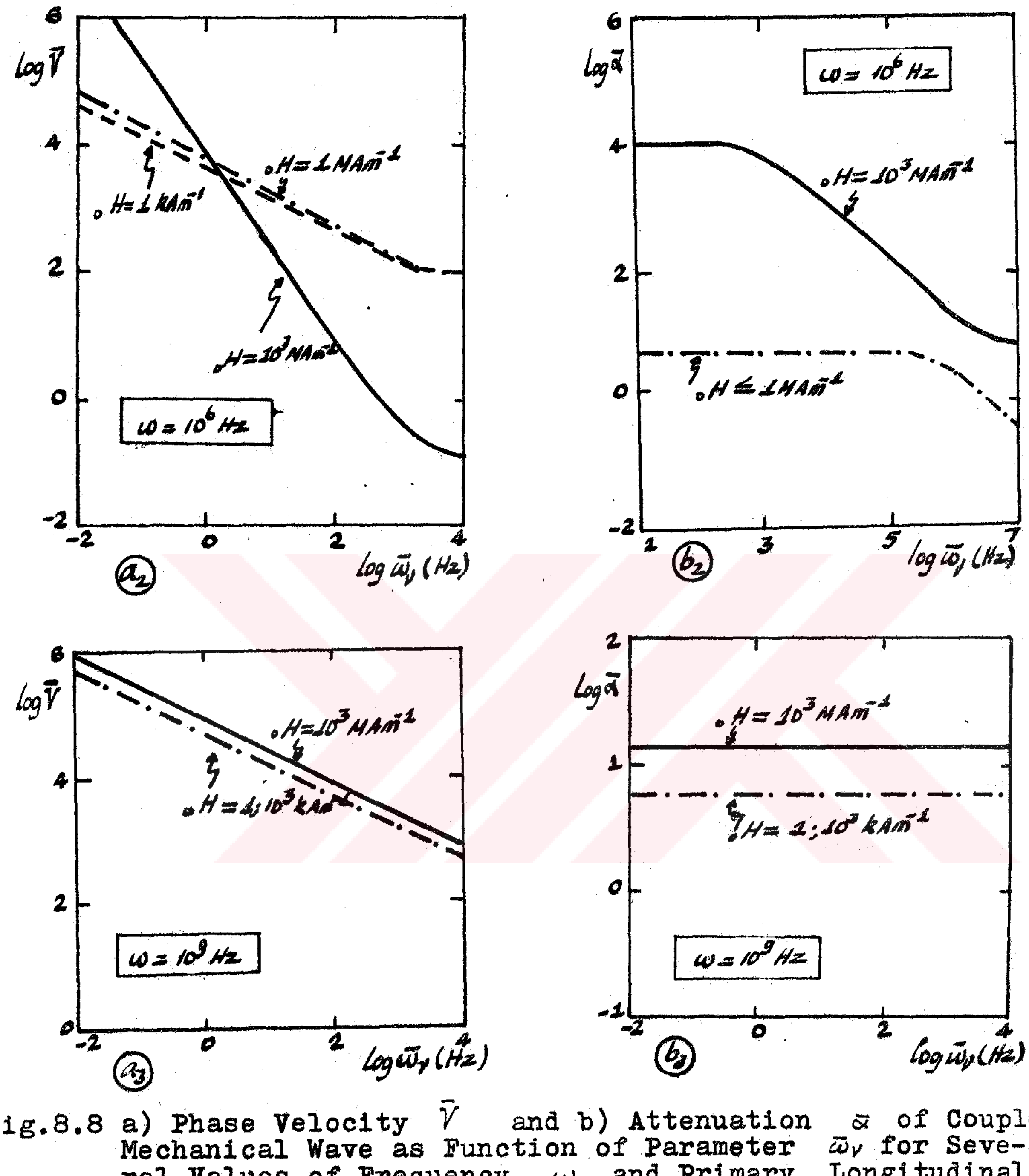


Fig.8.8 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Coupled Mechanical Wave as Function of Parameter $\bar{\omega}_\nu$ for Several Values of Frequency ω and Primary Longitudinal Magnetic Field $_oH$.

For the solutions of $\Delta_{13} = 0$ in (8.47) and $\Delta_{14} = 0$ in (8.48), it is appropriate to introduce some new dimensionless quantities such as

$$k^* = \frac{k}{k_0} \quad ; \quad \omega^* = \frac{\omega}{\omega_0} \quad ; \quad V_s^* = \frac{V_s}{c} \qquad (8.52)$$

$$V_0^* = \frac{V_0}{c} \quad ; \quad \mathbb{R} = \frac{k_0 c}{\omega_0}$$

where k_0 and ω_0 are, respectively, reference wave number and frequency. Then equating the polynomial expressions (8.47, 48) to zero, one obtains dimensionless equations in the form

$$(1-i\bar{\omega})k^{*4} - \left\{1+(1+i\nu_c)\left[(1-i\bar{\omega})\left(\frac{V_s^*}{V^*}\right)^2 + 2\hat{\chi}^2\nu_H - i(1+\hat{\chi})\nu_c\nu_H\right\} \times$$
$$\times \left(\frac{\omega^*}{\mathbb{R}V_s^*}\right)^2 k^{*2} + \left[1+2\nu_c^2\nu_H + i(1-\nu_H)\nu_c\right]\left(\frac{\omega^{*2}}{V_0^* V_s^* \mathbb{R}^2}\right)^2 = 0 \quad (8.53)$$

and

$$(1-i\bar{\omega})k^{*4} - \left\{1+\left[(1-i\bar{\omega})(1+i\nu_c) + i\bar{\nu}_H\nu_c\right]\left(\frac{V_s^*}{V_0^*}\right)^2\right\}\left(\frac{\omega^*}{\mathbb{R}V_s^*}\right)^2 k^{*2}$$
$$+ \left[1+i(1+\nu_H)\nu_c\right]\left(\frac{\omega^{*2}}{V_0^* V_s^* \mathbb{R}^2}\right)^2 = 0 \quad (8.54)$$

respectively. These two equations lead to couped modes of electromagnetic and mechanical waves.

In the absence of the primary magnetic field ($\nu_H = 0$, $\bar{\nu}_H = 0$) the solutions of Eqs.(8.53,54) reduce to (8.34,35) which are the uncoupled electromagnetic and mechanical waves discussed in Case (a).

If the material is assumed to be electrically nonconductive ($\nu_c = 0$), Eq.(8.53) becomes

$$(1-i\bar{\omega})k^{*4} - \left[1+(1-i\bar{\omega})\left(\frac{V_s^*}{V^*}\right)^2 + 2\hat{\chi}^2\nu_H\right]\left(\frac{\omega^*}{\mathbb{R}V_s^*}\right)^2 k^{*2}$$
$$+ \left[\frac{1}{V_0^* V_s^*}\left(\frac{\omega^*}{\mathbb{R}^*}\right)^2\right]^2 = 0 \quad (8.55)$$

which is the coupled nondispersive electromagnetic and dispersive mechanical waves affected by the primary, longitudinal magnetic field. However, (8.54) reduces to the uncoupled electromagnetic wave which propagates at a constant speed without an attenuation and to the uncoupled dispersive and dissipative S wave.

Eqs.(8.53,54) are the quadratic expressions in k^{*2} with complex coefficients. The solutions of (8.53) and (8.54) are, respectively,

$$\mathbb{k}^* = \mp \frac{\omega^*}{\mathbb{R} V_s^*} \left\{ \frac{(1+i\bar{\omega})\beta}{2(1+\bar{\omega}^2)} \left[1 \mp (1-4\xi_{CH})^{1/2} \right] \right\}^{1/2} \tag{8.56}$$

and

$$\mathbb{k}^* = \mp \frac{\omega^*}{\mathbb{R} V_s^*} \left\{ \frac{(1+i\bar{\omega})\bar{\beta}}{2(1+\bar{\omega}^2)} \left[1 \mp (1-4\bar{\xi}_{CH})^{1/2} \right] \right\}^{1/2} \tag{8.57}$$

where

$$\begin{aligned} \beta &= 1 + (1+i\nu_c)\left[(1-i\bar{\omega})\left(\frac{V_s^*}{V_o^*}\right)^2 + 2\hat{\chi}^2\nu_H\right] - i(1+\hat{\chi})\nu_c\,\nu_H \\ \bar{\beta} &= 1 + \left[(1-i\bar{\omega})(1+i\nu_c) + i\nu_c\,\bar{\nu}_H\right]\left(\frac{V_s^*}{V_o^*}\right)^2 \end{aligned} \tag{8.58}$$

$$\xi_{CH} = \frac{(1-i\bar{\omega})\,\delta}{\beta^2}\left(\frac{V_s^*}{V_o^*}\right)^2 \; ; \; \bar{\xi}_{CH} = \frac{(1-i\bar{\omega})\bar{\delta}}{\bar{\beta}^2}\left(\frac{V_s^*}{V_o^*}\right)^2 . \tag{8.59}$$

In Eq.(8.59), δ and $\bar{\delta}$ are defined as

$$\delta = 1 + 2\chi_c^2\nu_H + i(1-\nu_H)\nu_c \quad \text{and} \quad \bar{\delta} = 1 + i(1+\nu_H)\nu_c \tag{8.60}$$

respectively. The solution of (8.55) is obtained easily by substituting $\nu_c = 0$ in $(8.58)_1$ and $(8.60)_1$, and the resulting expressions into (8.56) which indicates that the electromagnetic and mechanical waves are still coupled.

Observation of Eqs.(8.45-48) shows that in the absence of the applied magnetic field, all the modes become uncoupled and they are the usual modes of electromagnetic and mechanical waves considered in Case (a).

Phase velocities and attenuations are obtained from the real and imaginary parts of $\mathbb{k}^*$ in (8.56,57). As seen from these equations, one can gain more insight from even an approximate solution of actual equations (8.56,57).

For the test material for which the material constants $\rho = 7.8\times10^3$ kgm^{-3}, $\hat{\mu} = 8.1\times10^{10}$ Nm^{-2}, $\hat{\lambda} = 1.12\times10^{11}$ Nm^{-2}, $\hat{\chi} = 10^4$, $\hat{\sigma} = 5\times10^6$ Mhom^{-1} are taken for the numerical solutions of (8.56, 57), and the ranges 10^0 Hz $\leq \omega \leq 10^{12}$ Hz, 10^{-3} Hz $\leq \bar{\omega}_\nu, \omega_\nu$ 10^9 Hz and 10^{-3} MAm$^{-1} \leq H_0 \leq 10^4$ MAm^{-1}, the orders of the quantities $4\,\xi_{CH}$ and $4\,\bar{\xi}_{CH}$ are between 10^{-20} - 10^{-4}. These are obviously small compared to 1. Therefore approximate solutions of (8.56,57) can be obtained by using the Binomial expansion retaining the first order terms. From the approximate solution of (8.56) the phase velocities and the attenuations are obtained as

$$V_1^* = \mathbb{R} V_0^* \left\{ \frac{2\frac{\beta_1}{\delta_1}\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]}{1+\beta_2\delta_2/\beta_1\delta_1+\sqrt{\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]\left[1+\left(\frac{\delta_2}{\delta_1}\right)^2\right]}} \right\}^{1/2} \tag{8.61}$$

$$\alpha_1 = 2\pi \left\{ \frac{\sqrt{\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]\left[1+\left(\frac{\delta_2}{\delta_1}\right)^2\right]} - \left(1+\frac{\beta_2\delta_2}{\beta_1\delta_1}\right)}{\sqrt{\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]\left[1+\left(\frac{\delta_2}{\delta_1}\right)^2\right]} + 1 + \frac{\beta_2\delta_2}{\beta_1\delta_1}} \right\}^{1/2}$$

and

$$V_2^* = \mathbb{R} V_s^* \left\{ \frac{2(1+\bar{\omega}^2)\frac{1}{\beta_1}}{1-\beta_2/\beta_1\,\bar{\omega}+\sqrt{(1+\bar{\omega}^2)\left[1+(\beta_2/\beta_1)^2\right]}} \right\}^{1/2} \tag{8.62}$$

$$\alpha_2 = 2\pi \left\{ \frac{\sqrt{(1+\bar{\omega}^2)\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]} - \left(1-\frac{\beta_2}{\beta_1}\bar{\omega}\right)}{\sqrt{(1+\bar{\omega}^2)\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]} + 1 - \frac{\beta_2}{\beta_1}\bar{\omega}} \right\}^{1/2}$$

where

$$\begin{aligned} \beta_1 &= 1 + 2\hat{x}^2\nu_H + (1+\bar{\omega}\,\nu_c)\left(\frac{V_s^*}{V_0^*}\right)^2 \\ \beta_2 &= 2\hat{x}\,\nu_c\,\nu_H + (\nu_c-\bar{\omega})\left(\frac{V_s^*}{V_0^*}\right)^2 \\ \delta_1 &= 1+2\nu_c^2\nu_H \quad ; \quad \delta_2 = (1-\nu_H)\,\nu_c \end{aligned} \tag{8.63}$$

Observation shows that the phase velocities and the attenuations in (8.61,62) are, respectively, associated with the predominantly electromagnetic and mechanical waves.

Similarly, from the approximate solution of (8.57) the phase velocities and the attenuations are obtained as

$$V_3^* = \mathbb{R} V_0^* \left\{ \frac{2\bar{\beta}_1\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\right]}{1+\frac{\bar{\beta}_2}{\bar{\beta}_1}\bar{\delta}_2+\sqrt{\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\right](1+\bar{\delta}_2^2)}} \right\}^{1/2} \tag{8.64}$$

$$
\alpha_3 = 2\pi \left\{ \frac{\sqrt{\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\left(1+\bar{\delta}_2^2\right)\right]} - \left(1+\frac{\bar{\beta}_2}{\bar{\beta}_1}\,\bar{\delta}_2\right)}{\sqrt{\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\left(1+\bar{\delta}_2^2\right)\right]} + 1+\frac{\bar{\beta}_2}{\bar{\beta}_1}\,\bar{\delta}_2} \right\}^{1/2}
$$

and

$$
V_4^* = R\,V_s^* \left\{ \frac{2/\bar{\beta}_1}{(1-\bar{\omega})\,\frac{\bar{\beta}_2}{\bar{\beta}_1} + \sqrt{(1+\bar{\omega}^2)\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\right]}} \right\}^{1/2}
$$

$$
\alpha_4 = 2\pi \left\{ \frac{\sqrt{(1+\bar{\omega}^2)\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\right]} - \left(1-\bar{\omega}\,\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)}{\sqrt{(1+\bar{\omega}^2)\left[1+\left(\frac{\bar{\beta}_2}{\bar{\beta}_1}\right)^2\right]} + 1-\bar{\omega}\,\frac{\bar{\beta}_2}{\bar{\beta}_1}} \right\}^{1/2} \qquad (8.65)
$$

where

$$
\bar{\beta}_1 = 1 + (1+\bar{\omega}\,\nu_c)\left(\frac{V_s^*}{V_0^*}\right)^2 \; ; \; \bar{\beta}_2 = \left[(1+\bar{\nu}_H)\nu_c - \bar{\omega}\right]\left(\frac{V_s^*}{V_0^*}\right)^2 \qquad (8.66)
$$

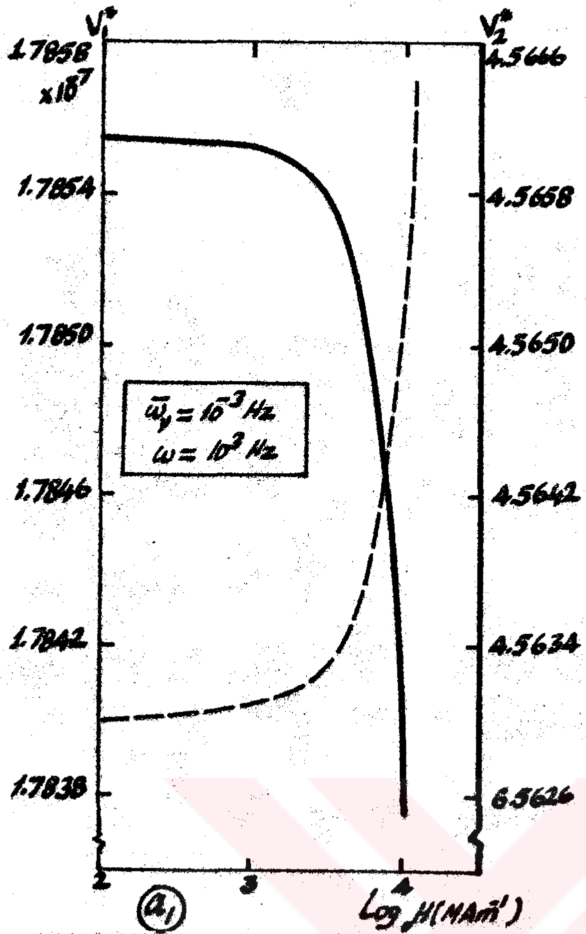
$$
\bar{\delta}_2 = (1+\nu_H)\,\nu_c
$$

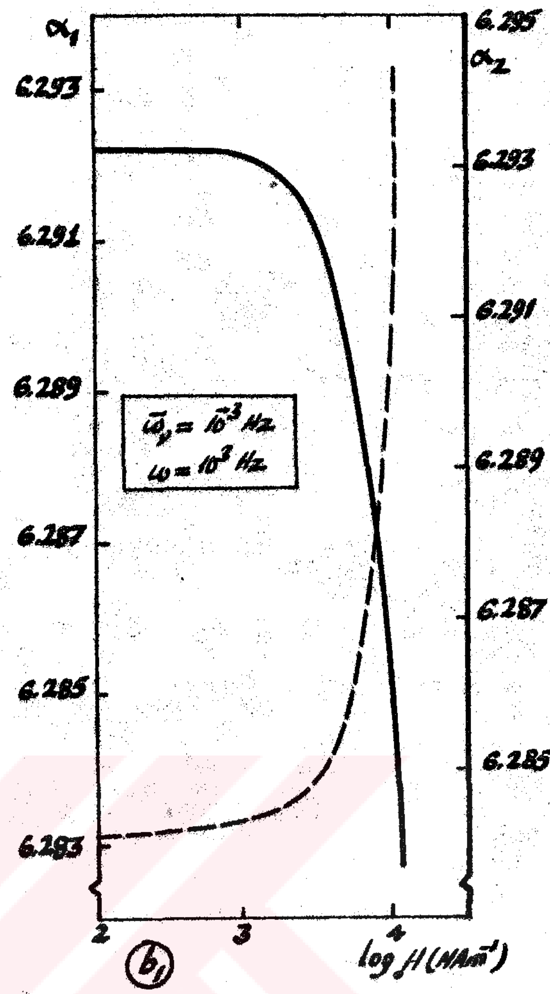
Observation shows that (8.64) and (8.65) are, respectively, the phase velocities and the attenuations of the predominantly electromagnetic and mechanical waves.

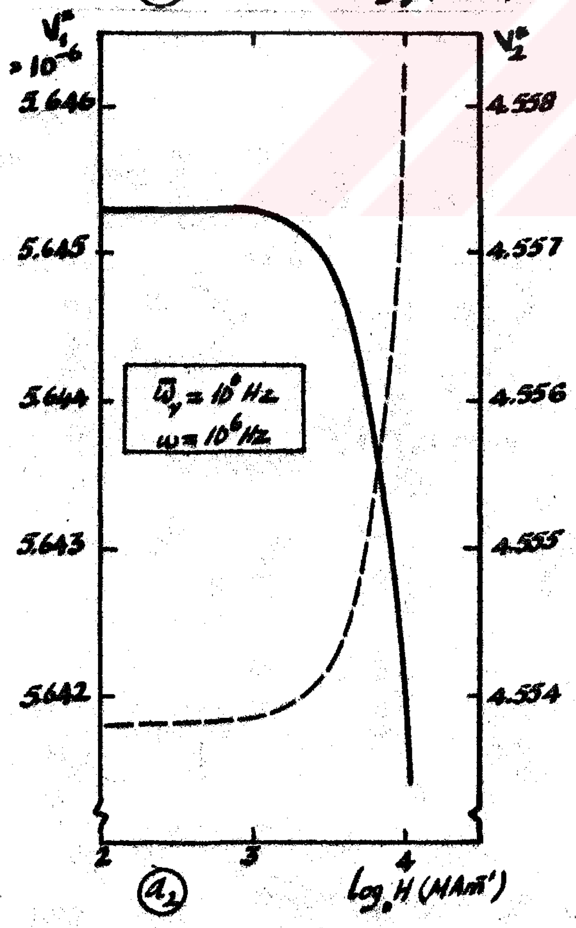
Further insight conserning the behavior of phase velocities and attenuations per wavelength with respect to the frequency and the applied magnetic field is gained for the coupled modes, (8.55-57), when numerical solutions are obtained. For the test material, the wave speeds V_p, V_s of uncoupled elastic waves and that of the electromagnetic wave V_0 in a nonconductor are obtained as follows:

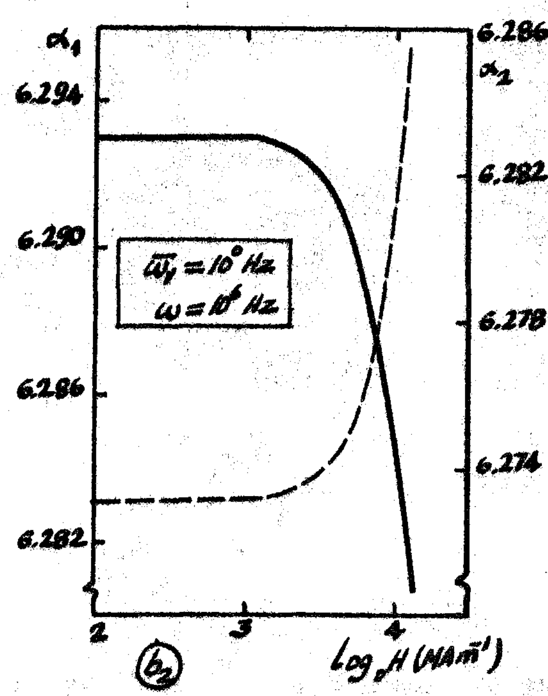
$$
V_p = 5.92691\times10^3\,m\,s^{-1} \; ; \; V_s = 3.22252\times10^3\,m\,s^{-1} \; ; \; V_0 = 2.99985\times10^8\,m\,s^{-1} \qquad (8.67)
$$

Again considering the same values for the constants of the test material used for the approximation of (8.57), one can numerically determine the phase velocities and the attenuations of the coupled modes. The graphs of these modes are shown in Figs. 8.9-11.


V_1^*
V_2^*
$\times 10^{7}$
1.7858
1.7854
1.7850
1.7846
1.7842
1.7838
4.5666
4.5658
4.5650
4.5642
4.5634
6.5626
$\bar{\omega}_y = 10^{-3}$ Hz
$\omega = 10^{3}$ Hz
2
3
4
$\log_e H$ (MAm^{-1})
a_1


α_1
α_2
6.293
6.291
6.289
6.287
6.285
6.283
6.295
6.293
6.291
6.289
6.287
6.285
$\bar{\omega}_y = 10^{-3}$ Hz
$\omega = 10^{3}$ Hz
2
3
4
$\log_e H$ (MAm^{-1})
b_1


V_1^*
$\times 10^{-6}$
V_2^*
5.646
5.645
5.644
5.643
5.642
4.558
4.557
4.556
4.555
4.554
$\bar{\omega}_y = 10^{6}$ Hz
$\omega = 10^{6}$ Hz
2
3
4
$\log_e H$ (MAm^{-1})
a_2


α_1
α_2
6.294
6.290
6.286
6.282
6.286
6.282
6.278
6.274
$\bar{\omega}_y = 10^{0}$ Hz
$\omega = 10^{6}$ Hz
2
3
4
$\log_e H$ (MAm^{-1})
b_2

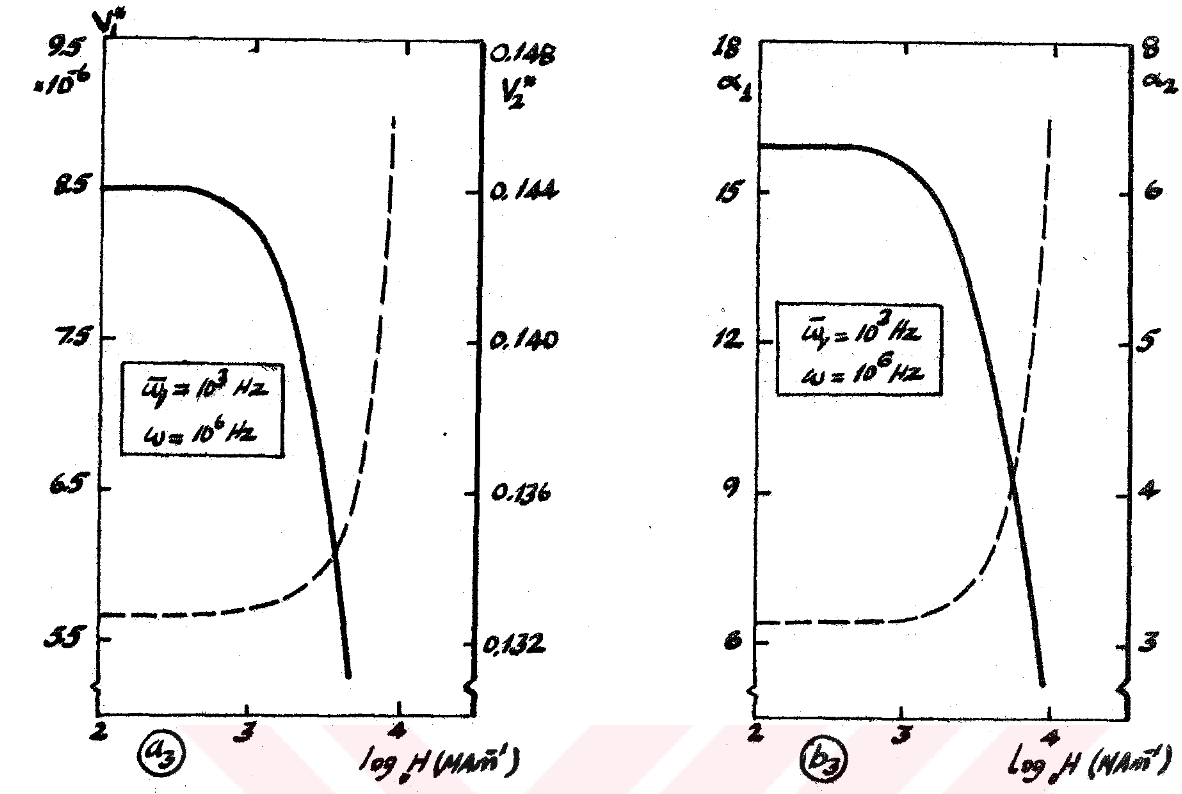


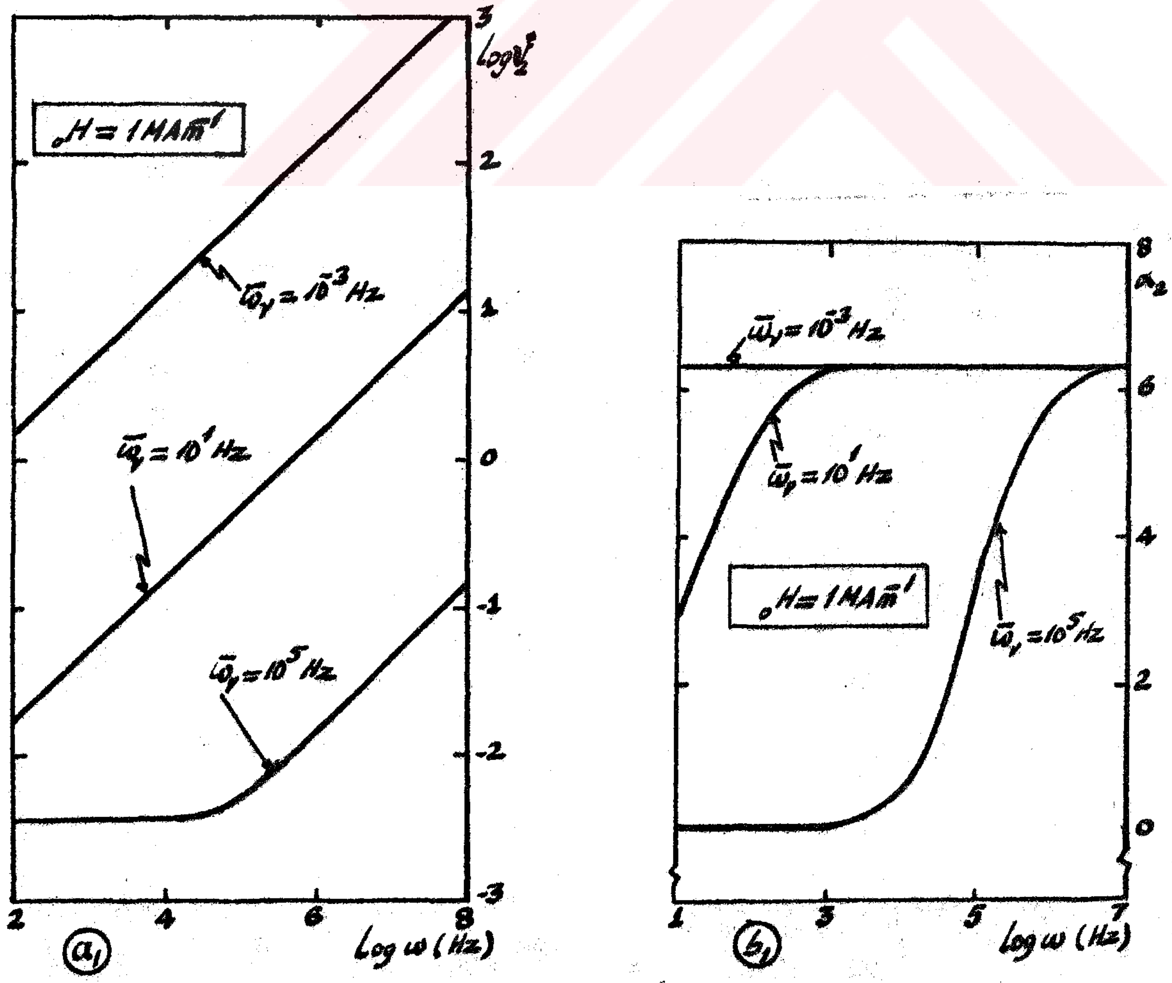
Fig.8.9 a) Phase Velocities V_1^* , V_2^* and b) Attenuations α_1 , α_2 of Coupled Modes of Electromagnetic and Mechanical Waves as Functions of Primary Longitudinal Magnetic Field ${}_0H$, for Certain Selected Values of Frequency ω and Parameter $\bar{\omega}_\gamma$.

Fig.8.9 gives the phase velocities and the attenuations of coupled electromagnetic and mechanical waves as the functions of the applied primary, longitudinal magnetic field for different values of frequency and the parameter $\bar{\omega}_\gamma$. The vertical axes at the left and right of each figure are, respectively, due to the predominantly electromagnetic and mechanical waves. For example, it is seen in Fig.8.9 a_1 that the phase velocity V_1^* of the predominantly electromagnetic wave reaches 1.7854×10^{-7} while the phase velocity V_2^* of the predominantly mechanical wave reaches 4.5650 at ${}_0H$ = 10 MAm^{-1} for $\omega = 10^{-3}$Hz and $\bar{\omega}_\gamma = 10^{-3}$ Hz.

Observation of Fig.8.9 indicates that the existing primary magnetic field is less than 10^2 MAm^{-1}, its influence on the phase velocities and the attenuations of the coupled modes is negligible. However, if the existing primary magnetic field is sufficiently strong, the effects on the propagated waves are significant. If the order of ${}_0H$ is beyond 10^3 MAm^{-1}, while V_1^* and

α_1 increase, V_2^* and α_2 decrease very rapidly with respect to the variation of ${}_oH$. Physically, this means that energy might be transfered from the mechanical wave to the electromagnetic wave beyond a certain value of the applied magnetic field. Therefore, in a large interval of the appropriate magnetic field, it is sufficient to consider the predominantly electromagnetic and mechanical waves propagating without the influence of the longitudinal magnetic field.

Fig.8.10 displays the phase velocity V_2^* and the attenuation α_2 of the predominantly mechanical wave as the function of frequency ω at different selected values of the primary magnetic field ${}_oH$ for parametrized $\bar{\omega}_\nu$. Observation of Fig. 8.10 and the data for the predominantly electromagnetic wave show that both waves are dispersive and dissipative. Another common characteristics of this case is that the effect of the primary magnetic field disappears if the wave is ultrasonic (US) or VHUS. Another significant feature of this result is that the dependence of V_1^* and V_2^* on the frequency at



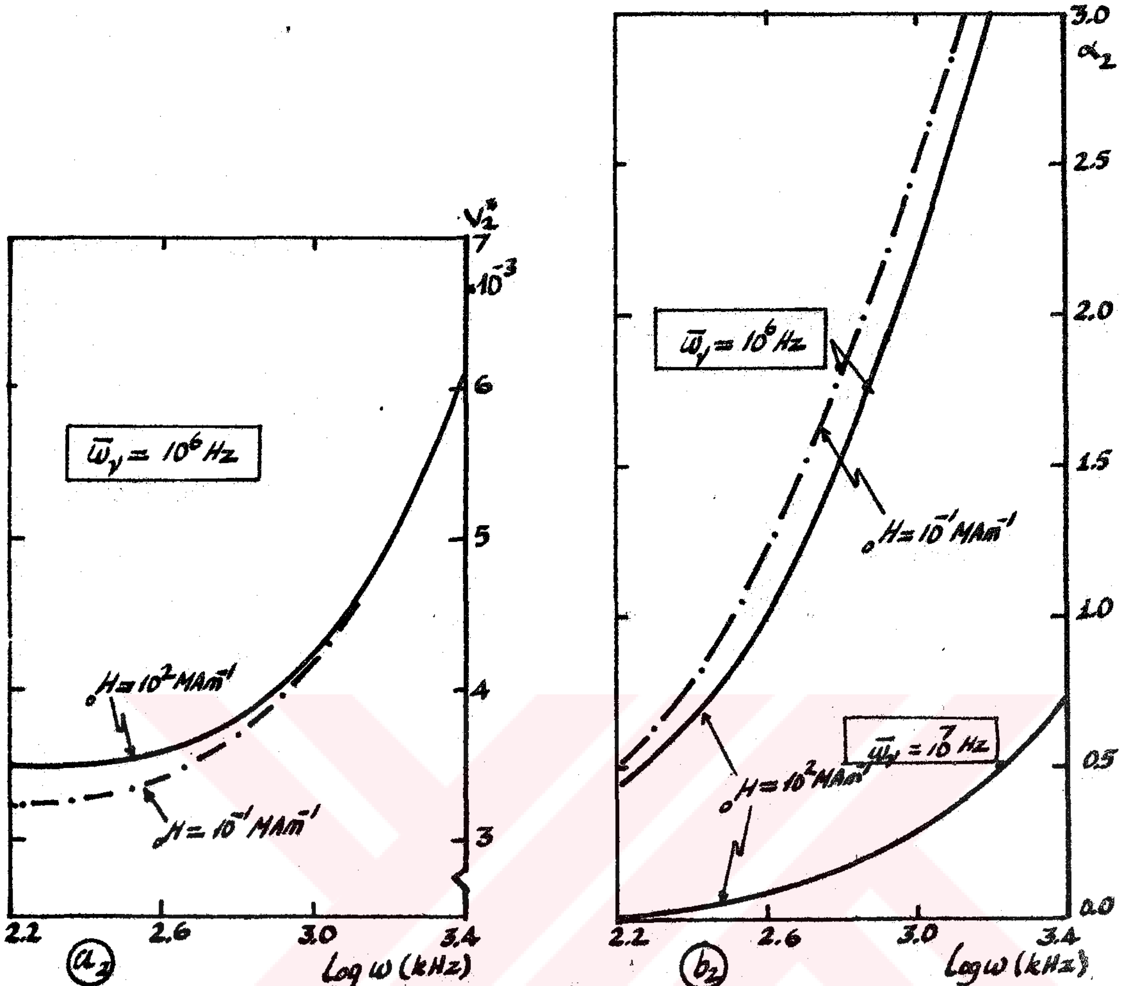


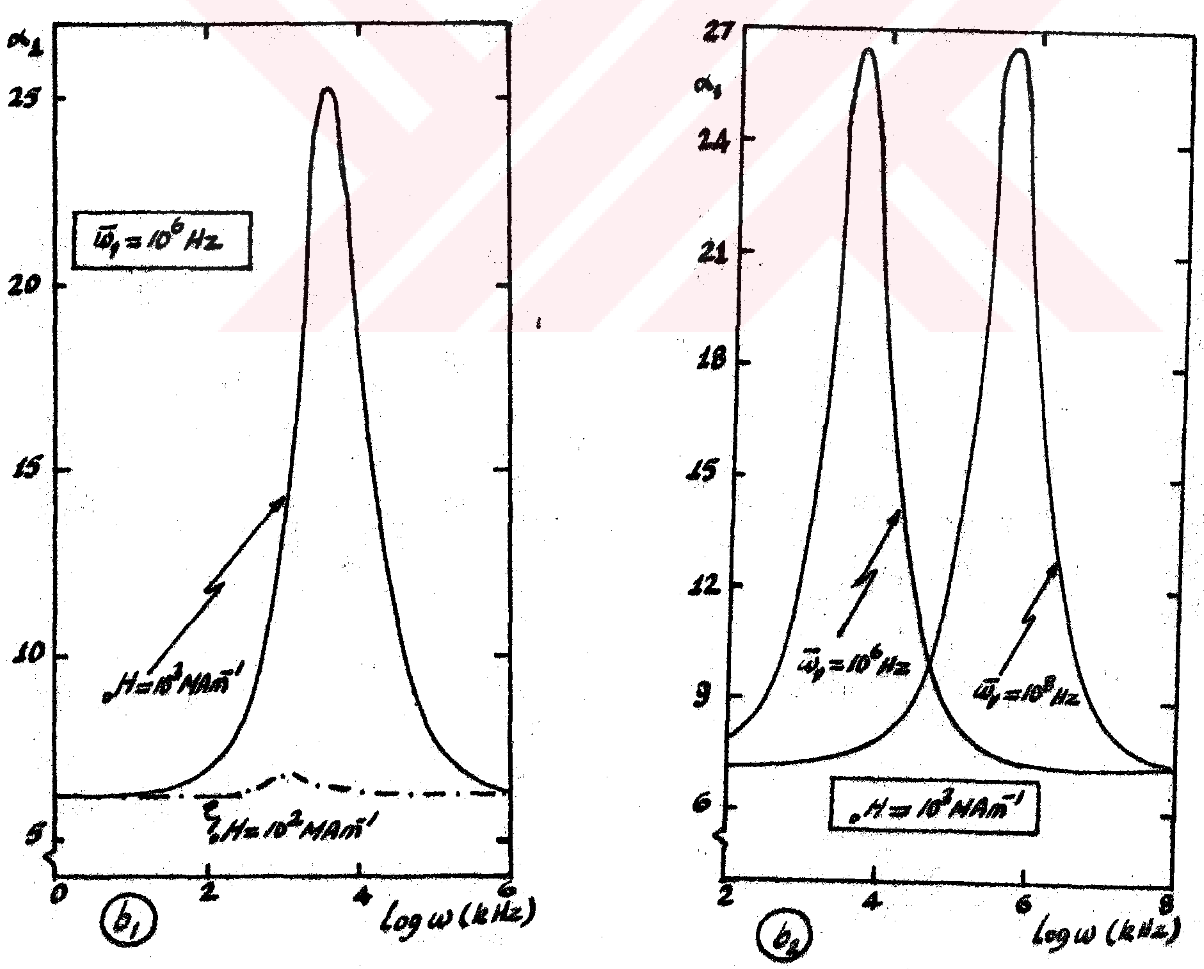
Fig.8.10 a) Phase Velocity V_2^* and b) Attenuation α_2 of Predominantly Mechanical Wave as Function of Frequency ω, for Selected Values of Primary Longitudinal Magnetic Field ${}_oH$ and Parameter $\bar{\omega}_y$.

specified ${}_oH$ is related with the parameter $\bar{\omega}_y$, i.e., the viscosity of the solids.

As was mentioned in Case (a), there is no anomalous dispersion of the electromagnetic wave. This is not so if there exists a uniform, longitudinal strong magnetic field. The attenuation of the predominantly electromagnetic wave in a strong magnetic field is seen in Fig.8.11. The attenuations are seen to be sufficiently large so that they may be detected experimentally if a strong enough magnetic field could be produced.

Figs.8.12-14 give the plots of the phase velocities $\bar{V}_1, \bar{V}_2$, and the attenuations α_1 , α_2 of the coupled modes of the waves given by (8.55).

The plots in Fig.8.12 indicate that $\bar{V}_1$ and α_1 are due to the predominantly nondispersive and nondissipative electromagnetic wave and $\bar{V}_2$ and α_2 are due to the predominantly mechanical wave. The predominantly mechanical wave is dispersive and dissipative and affected by the applied primary longitudinal magnetic field, while the predominantly electromagnetic wave is not, when the material is highly viscous, as it is observed in Fig.8.12$_1$. If the material becomes more elastic, the effect of the magnetic field disappears. Another common characteristic is that the effect of the strong magnetic field on the mechanical wave tends to disappear if the frequency of the propagated wave goes beyond a certain limit.



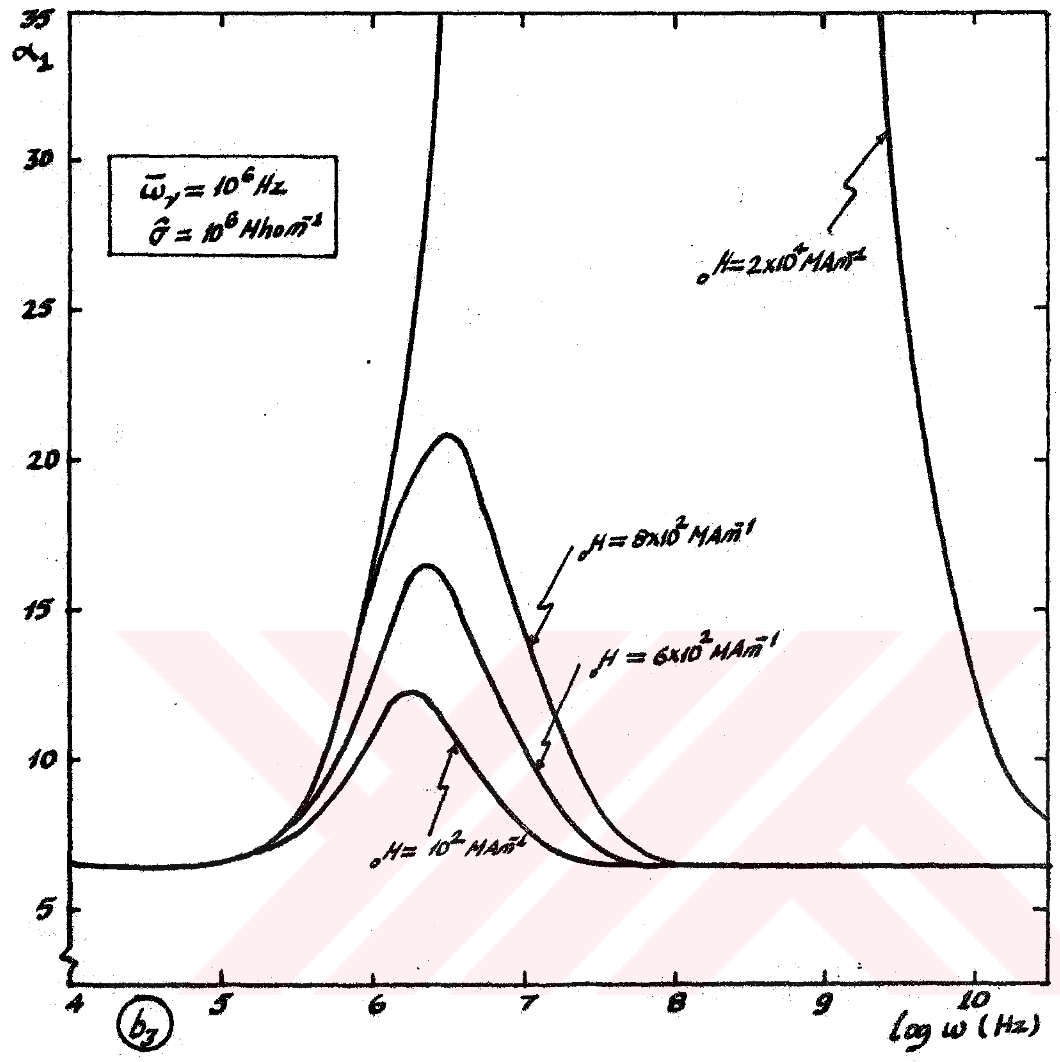


Fig.8.11 Attenuation $\alpha_{\perp}$ of Predominantly Electromagnetic Wave as Function of Frequency ω for Several Selected Values of Primary Longitudinal Magnetic Field ${}_{o}H$ and Parameter $\bar{\omega}_{\nu}$.

Fig.8.13 displays $\bar{V}$ and α of the coupled modes of electromagnetic and mechanical waves as functions of the applied longitudinal magnetic field at different values of the parameter $\bar{\omega}_{\nu}$ and the frequency ω. As seen in Fig.8.13, the phase velocities of both of the coupled waves are affected by the strong primary magnetic field, excep when the frequency of the propagated waves becomes very large. The dependence of the attenuations of the coupled modes is also seen in the same figure.

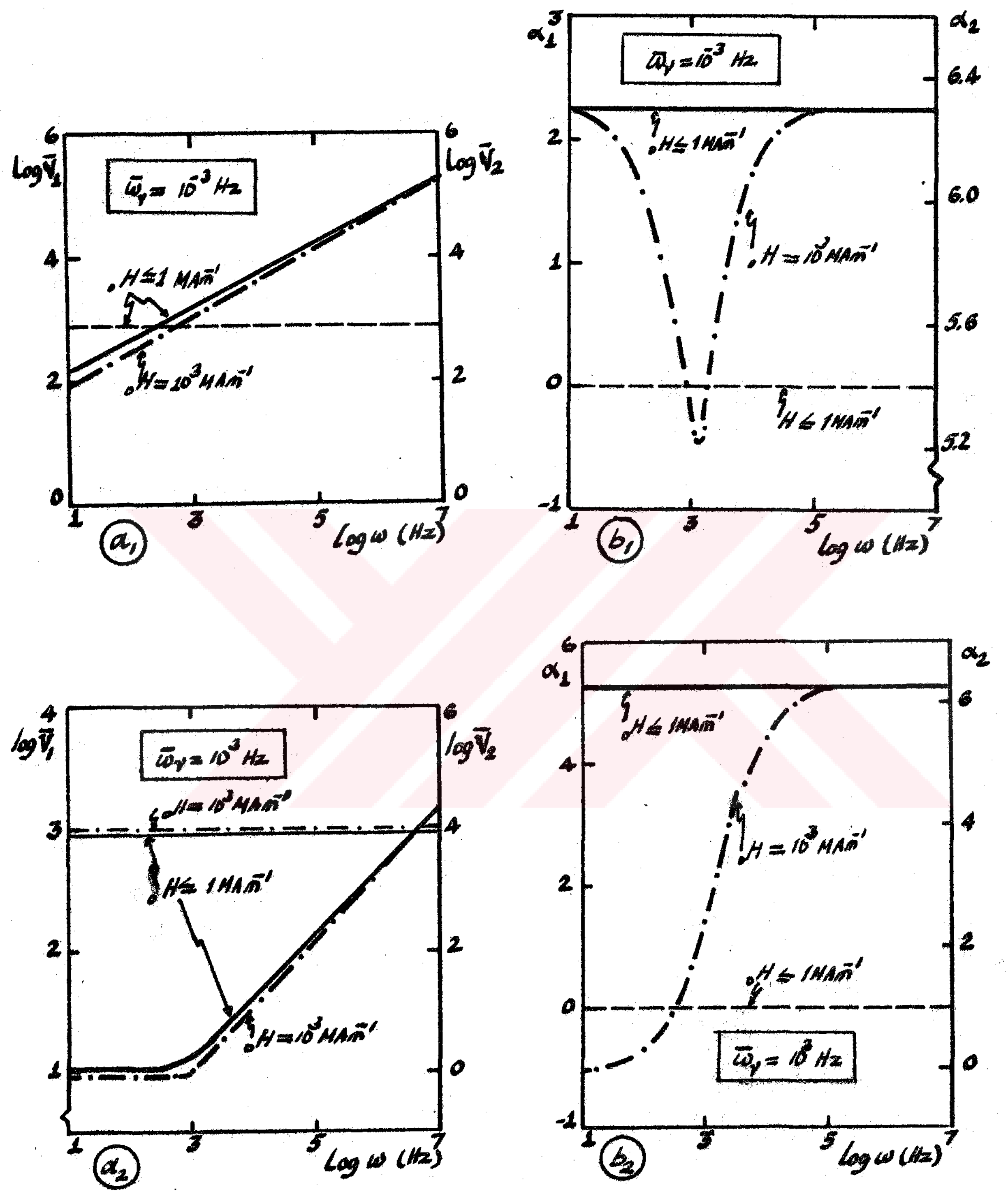


Fig.8.12 a) Phase Velocities $\bar{V}_1$, $\bar{V}_2$ and b) Attenuation α_1, α_2 of Coupled Modes of Predominantly Electromagnetic and Mechanical waves as Functions of Frequency ω for Selected Values of Primary Longitudinal Magnetic Field ${}_0H$ and Parameter $\bar{\omega}_y$.

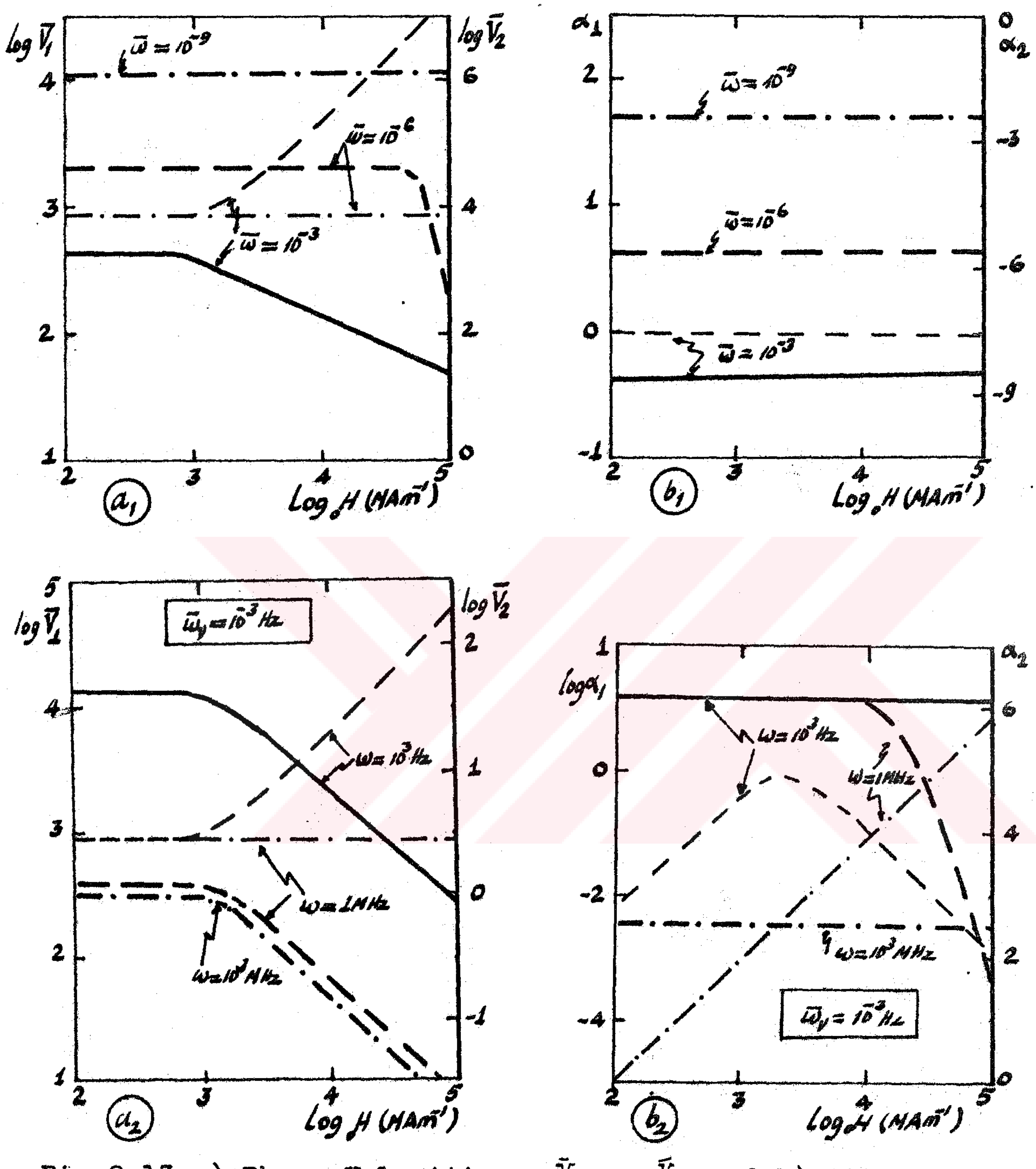


Fig.8.13 a) Phase Velocities $\bar{V}_1$, $\bar{V}_2$ and b) Attenuations α_1, α_2 of Coupled Modes of Predominantly Electromagnetic and Mechanical Waves as Functions of Primary Longitudinal Magnetic Field ${}_0H$, Parametrized for Frequency ω and Parameter $\bar{\omega}_y$.

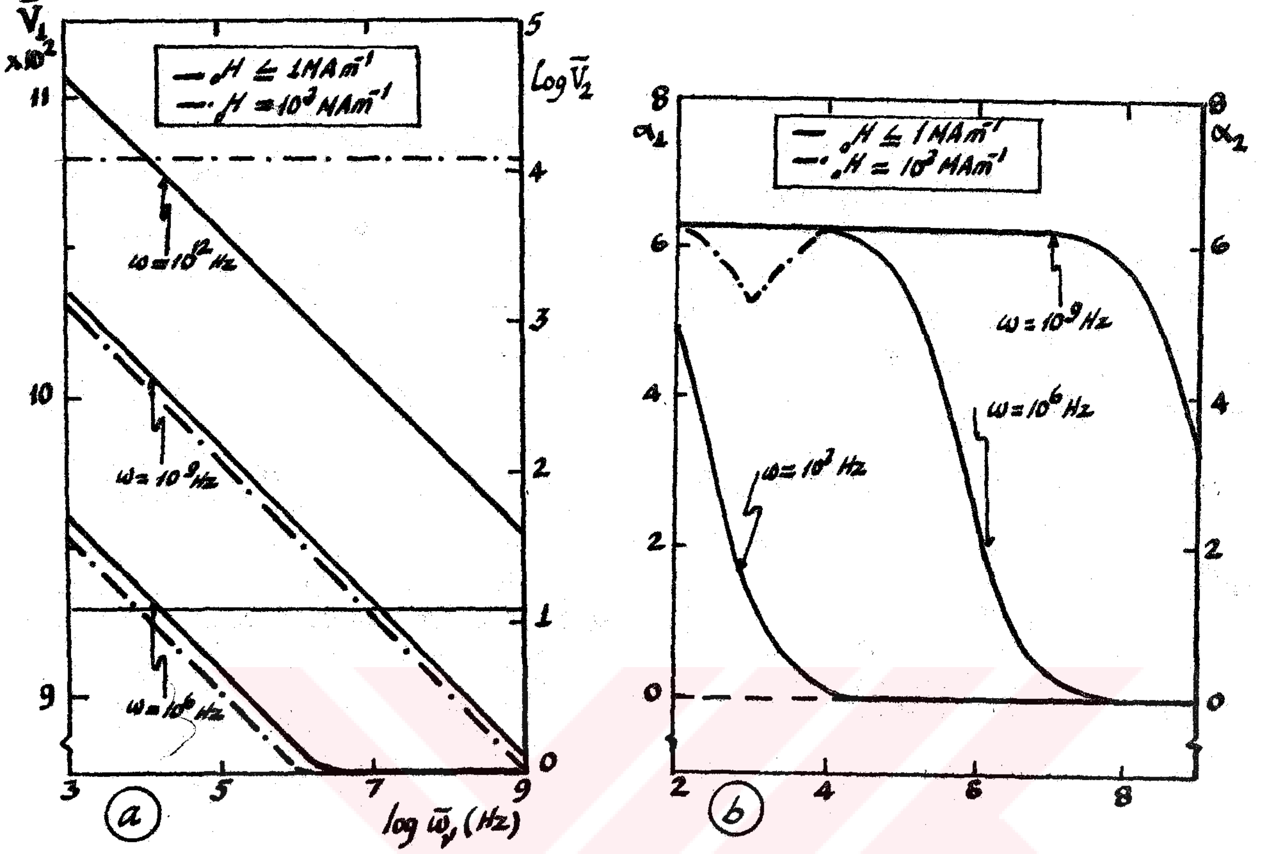


Fig.8.14 a) Phase Velocities $\bar{V}_1$, $\bar{V}_2$ and b) Attenuation α_1, α_2 of Coupled Modes of Predominantly Electromagnetic and Mechanical Waves as Functions of Viscosity Parameter $\bar{\omega}_\nu$ for Several Selected Values of Frequency ω and Primary Longitudinal Magnetic Field ${}_oH$.

Fig.8.14 depicits the behavior of the phase velocities and the attenuations of the coupled modes of the propagated waves with the parameter $\bar{\omega}_\nu$ (i.e., viscosity parameter) at several selected values of the frequency ω under the applied primary magnetic field ${}_oH$. The plot indicates that $\bar{V}_1$ does not vary with respect to the viscosity parameter $\bar{\omega}_\nu$ while $\bar{V}_2$ does. The phase velocity $\bar{V}_2$ of the propagated predominantly mechanical wave decreases very rapidly as the material becomes more elastic (less viscous) and reaches a constant value. On the other hand, the attenuation α_2 becomes zero as it is expected because the material is becoming more elastic. Thus, if the solid is electrically nonconductive and purely elastic (nonviscous), there is no decay of the amplitude of the propagated waves.

8.3.c) Propagation of waves under the applied transverse primary magnetic field (${}_oH_1 = 0$, ${}_oH_2 \neq 0$):

Substituting $\varphi = \pi/2$, i.e., $\cos\varphi = 0$ in (8.19) and the resulting equations into (8.21), one obtains

$$
\begin{aligned}
& a_{28} = a_{39} = a_{59} = a_{68} = a_{83} = a_{92} = a_{74} = a_{96} \\
& \quad = a_{78} = a_{87} = 0 \\
& a_{27} = -k\,\omega\,\hat{\chi}\,{}_oH \quad ; \quad a_{49} = -a_{67} = -i a_{73} = i\mu_o\,\omega\,\hat{\sigma}\,{}_oH \\
& a_{77} = \rho w^2 - [\hat{\lambda} + 2\hat{\mu} - i\omega\,(\hat{\underline{\lambda}} + 2\hat{\underline{\mu}})\,]k^2 + i\mu_o^2\omega\,\hat{\sigma}\,{}_oH^2 \\
& a_{88} = \rho w^2 - (\hat{\mu} - i\omega\hat{\underline{\mu}})\,k^2 \quad ; \quad a_{91} = \mu_o\hat{\sigma}\,{}_oH \\
& a_{99} = \rho w^2 - (\hat{\mu} - i\omega\hat{\underline{\mu}})\,k^2 + i\mu_o^2\omega\,\hat{\sigma}\,{}_oH^2
\end{aligned}
\tag{8.68}
$$

$$
\begin{aligned}
& \Lambda_{22} = \Lambda_{33} = \Lambda_{42} = \Lambda_{34} = \Lambda_{68} = \Lambda_{78} = \Lambda_{86} = \Lambda_{87} \\
& \quad = \Lambda_{93} = \Lambda_{94} = 0
\end{aligned}
$$

$$
\Lambda_{23} = a_{32}\,a_{99} \quad ; \quad \Lambda_{24} = a_{36}\,a_{99} \quad ; \quad \Lambda_{43} = a_{52}\,a_{99}
$$

$$
\Lambda_{44} = a_{56}\,a_{99} \quad ; \quad \Lambda_{56} = a_{23}\,a_{88} \quad ; \quad \Lambda_{57} = a_{27}\,a_{88}
$$

and the other coefficient remain the same. The determinant (8.22) assumes the form

$$
\Delta_{21}\,\Delta_{22}\,\Delta_{23}\,\Delta_{24}\,\Delta_{25} = 0 \tag{8.69}
$$

with

$$
\Delta_{21} = V_o^2 k^2 - (1 + i V_c)\,\omega^2 \tag{8.70}
$$

$$
\Delta_{22} = (1 - i\bar{\omega})\,V_t^2 k^2 - \omega^2 \tag{8.71}
$$

$$\Delta_{23} = (1-i\bar{\omega})V_s^2/k^2 - (1+iV_c\nu_H)\omega^2 \tag{8.72}$$

$$\Delta_{24} = (1-i\bar{\omega})\,(1+iV_c)V_s^2/k^2 - [1+i(1+\nu_H)V_c]\omega^2 \tag{8.73}$$

$$\Delta_{25} = (1-i\underline{\omega})k^4 - \left\{1 + \left[(1-i\underline{\omega})(1+iV_c) + iV_c\nu_H\right]\left(\frac{V_p}{V_s}\right)^2\right\} \times \frac{\omega^2/k^2}{V_p^2} + [1+i(1+\nu_H)V_c]\frac{\omega^4}{(V_sV_p)^2} \tag{8.74}$$

where

$$\nu_H = \frac{\mu_0\sigma H^2}{\rho V_p^2} \tag{8.75}$$

which is another dimensionless number. Three obvious solutions of Eqs.(8.70-72) are $\Delta_{21}=0$, $\Delta_{22}=0$ and $\Delta_{23}=0$. Now, $\Delta_{21}=0$ is identical to $(8.29)_1$ which is assumed with the purely electromagnetic wave considered in Case (a). Also $\Delta_{22}=0$ is the same as $(8.29)_2$ which is associated with the purely mechanical S wave. This means that the two modes of electromagnetic and mechanical waves propagate without being influenced by the magnetic field if the direction of the external magnetic field is transverse to the direction of wave propagation. Furthermore, $\Delta_{23}\equiv\Delta_{12}$, hence no extra discussion is needed.

The closed form solution of $\Delta_{24}=0$ is possible and it leads to

$$\bar{V} = \mp\left[\frac{2(1+\bar{\omega}^2)}{1+\sqrt{1+\xi_6}}\right]^{1/2} \tag{8.76}$$

$$\bar{\alpha} = 2\pi\left[\frac{\sqrt{1+\xi_6}-1}{\sqrt{1+\xi_6}+1}\right]^{1/2}$$

where

$$\xi_6 = \left[\frac{(1+\frac{1}{V_c^2})\bar{\omega} + \nu_H/V_c}{1+\frac{1}{V_c^2}+(V_c-\bar{\omega})\frac{\nu_H}{V_c}}\right]^{1/2} . \tag{8.77}$$

This is a coupled predominantly mechanical wave.

If either magnetic field or the electric conductivity is zero, $\Delta_{24}=0$ yields the same result as that of $\Delta_{22}=0$.

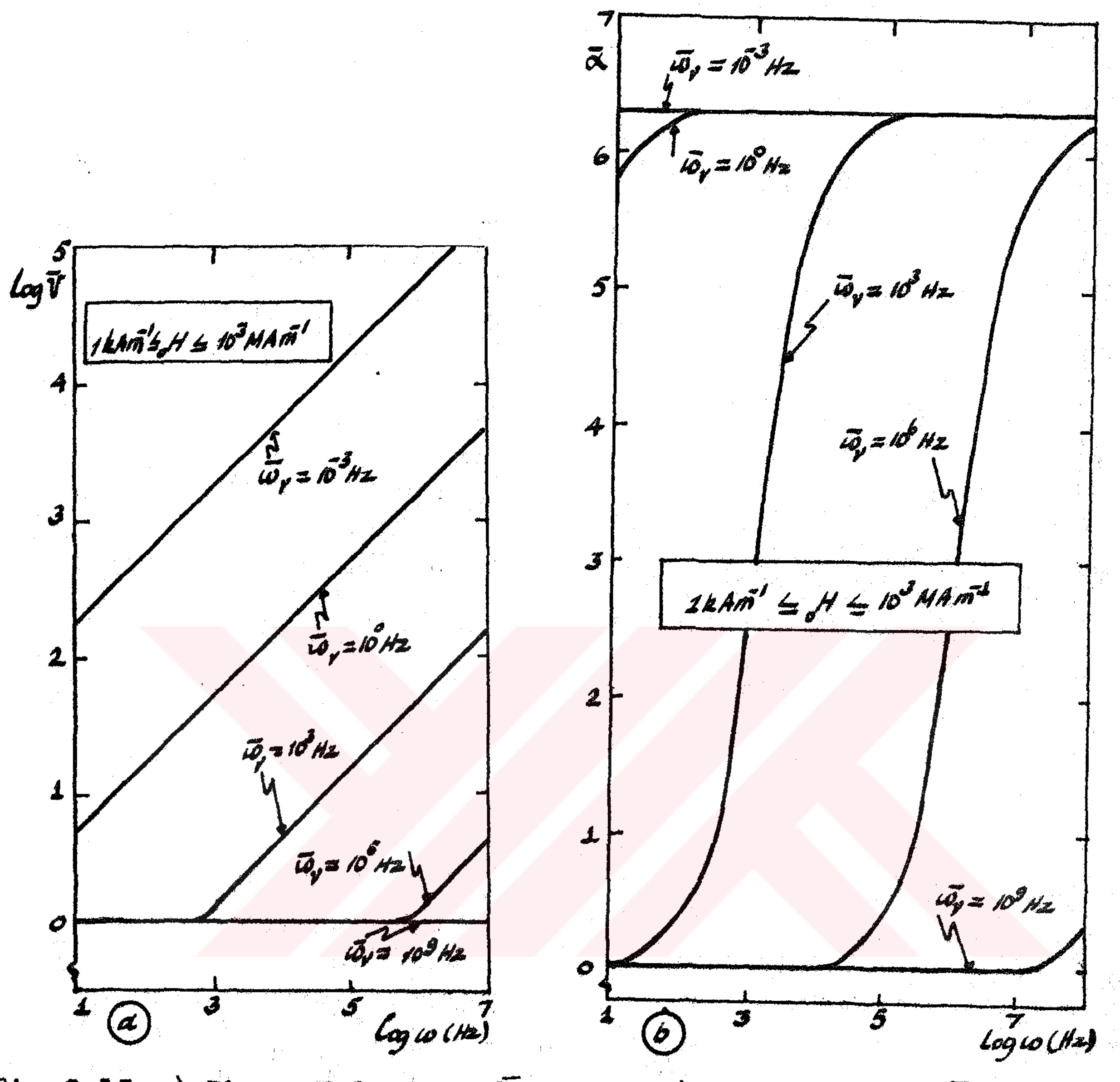


Fig.8.15 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Coupled Predominantly Mechanical Wave as Function of Frequency ω for Several Selected Values of Primary Transverse Magnetic Field ${}_oH$ and Parameter $\bar{\omega}_\gamma$.

For $\hat{\sigma} = 5x10^6$ Mho m^{-1}, graphs of Eq.(8.76) are plotted. Fig.8.15 shows the phase velocity $\bar{V}$ and the attenuation $\bar{\alpha}$ as the function of frequency ω at several selected values of the transverse magnetic field and the parameter $\bar{\omega}_\gamma$. Observation of the figure indicates that the applied primary magnetic field in the interval $1kAm^{-1} \leq {}_oH \leq 10^3\ MAm^{-1}$ has no effect on $\bar{V}$ and α. The comparison of the curves in Figs.8.4 and 8.15 indicates that the variation of $\bar{V}$ with respect to ω is the same, but the values of $\bar{V}$ at a specific value of ω in the two graphs are different because of electrical conductivity of the medium.

Fig.8.16 displays the phase velocity $\bar{V}$ and the attenuation $\bar{\alpha}$ as the function of the applied magnetic field ${}_oH$, for parametrized $\bar{\omega}$ ($\bar{\omega} \equiv \omega/\bar{\omega}_\nu$). Both $\bar{V}$ and $\bar{\alpha}$ do not depend upon the primary magnetic field ${}_oH$. Physically, the wave is not influenced by the existing primary, transverse magnetic field in the considered interval, but both $\bar{V}$ and $\bar{\alpha}$ vary with respect to $\bar{\omega}$.

Now considering (8.74) which is associated with the coupled modes of electromagnetic and mechanical waves, using the dimensionless quantities given by (8.52) and introducing a new dimensionless quantity

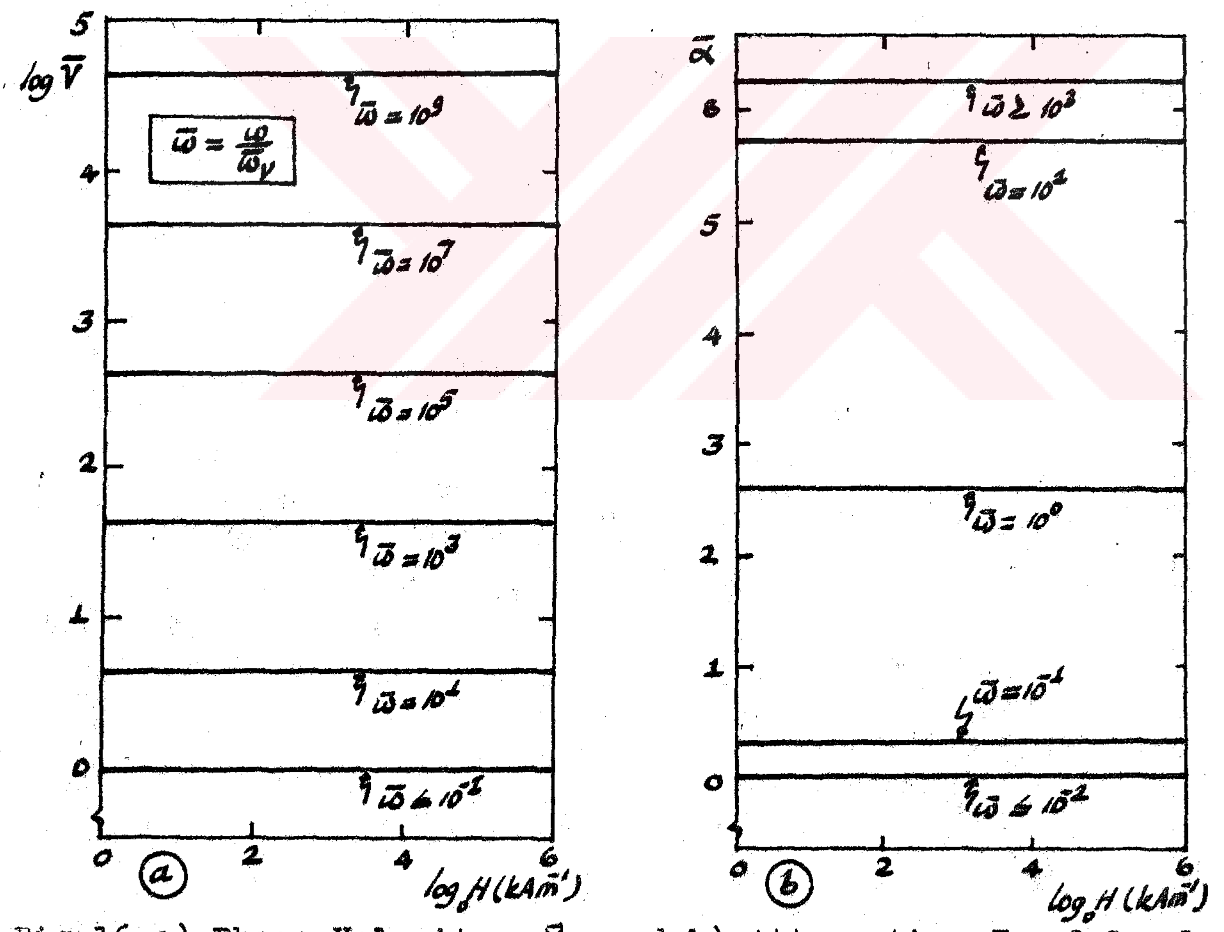


Fig.16 a) Phase Velocity $\bar{V}$ and b) Attenuation $\bar{\alpha}$ of Coupled Predominantly Mechanical Wave as Function of Primary Transverse Magnetic Field ${}_oH$ for Certain Values of Parameter $\bar{\omega}$.

$$V_p^* = V_p / c \tag{8.78}$$

into Eq.(8.74), one obtains a quadratic expression in k^{*2} with complex coefficients:

$$(1-i\underline{\omega})k^{*4} - \left\{1+\left[(1-i\underline{\omega})(1+i\nu_c)+i\nu_c\nu_H\right]\left(\frac{V_p^*}{V_0^*}\right)^2\right\}\left(\frac{\omega^*}{R V_p^*}\right)^2 k^{*2}$$
$$+\left[1+i(1+\nu_H)\nu_c\right]\left(\frac{\omega^{*2}}{V_0^* V_p^* R^2}\right)^2 = 0 \tag{8.79}$$

The solution of Eq.(8.79) is given by

$$k^* = \mp \frac{\omega^*}{R V_p^*}\left\{\frac{(1+i\underline{\omega})\beta}{2(1+\underline{\omega}^2)}\left[1 \mp (1-4\underline{\xi}_{cH})^{1/2}\right]\right\}^{1/2} \tag{8.80}$$

where

$$\underline{\beta} = 1+\left[(1-i\underline{\omega})(1+i\nu_c)+i\nu_c\nu_H\right]\left(\frac{V_p^*}{V_0^*}\right)^2$$

$$\underline{\xi}_{cH} = \frac{(1-i\underline{\omega})\underline{\delta}}{\underline{\beta}^2}\left(\frac{V_p^*}{V_0^*}\right)^2 \quad ; \quad \underline{\delta} = 1+i(1+\nu_H)\nu_c \; . \tag{8.81}$$

Thus, in the absence of the primary transverse magnetic field (i.e., $\nu_H=0$, $\nu_H=0$), (8.80) furnishes the waves of the two uncoupled modes given previosly by (8.34) and (8.36). If the material is nonconductive, then taking $\nu_c=0$ in (8.81), the coupling between two different modes of waves and the effect of the applied field disappear, which has not been observed in the case of (8.56).

As it was in Case (b), the phase velocities and the attenuations of coupled modes of propagated waves in (8.80) can not be deduced explicitly unless an approximate analytic treatment is employed. For the test material with the same material constants and the considered intervals of the variables in Case (b), the order of quantity $4\underline{\xi}_{cH}$ is between 10^{-20} - 10^{-4} which is obviosly very small compared to 1. In a similar manner, using the Binomial expansion, one obtains the phase velocities and the attenuations as follows:

$$V_1^* = \mathbb{R} V_0^* \left\{ \frac{2/\beta_1 \left[1 + \left(\frac{\beta_2}{\beta_1}\right)^2 \right]}{1 + \frac{\beta_2}{\beta_1}\delta_2 + \sqrt{\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right](1+\delta_2^2)}} \right\}^{1/2}$$

$$\alpha_1 = 2\pi \left\{ \frac{\sqrt{\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right](1+\delta_2^2)} - \left(1+\frac{\beta_2}{\beta_1}\delta_2\right)}{\sqrt{\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right](1+\delta_2^2)} + 1 + \frac{\beta_2}{\beta_1}\delta_2} \right\}^{1/2} \tag{8.82}$$

and

$$V_2^* = \mathbb{R} V_p^* \left\{ \frac{2/\beta_1}{(1-\underline{\omega})\frac{\beta_2}{\beta_1} + \sqrt{(1+\underline{\omega}^2)\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]}} \right\}^{1/2} \tag{8.83}$$

$$\alpha_2 = 2\pi \left\{ \frac{\sqrt{(1+\underline{\omega}^2)\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]} - (1-\underline{\omega})\frac{\beta_2}{\beta_1}}{\sqrt{(1+\underline{\omega}^2)\left[1+\left(\frac{\beta_2}{\beta_1}\right)^2\right]} + (1-\underline{\omega})\frac{\beta_2}{\beta_1}} \right\}^{1/2}$$

where

$$\beta_1 = 1 + (1+\underline{\omega}\,\nu_c)\left(\frac{V_p^*}{V_0^*}\right)^2 \; ; \; \beta_2 = \left[(1+\nu_H)\nu_c - \underline{\omega}\right]\left(\frac{V_p^*}{V_0^*}\right)^2 \tag{8.84}$$

$$\underline{\delta}_2 = (1+\nu_H)\,\nu_c$$

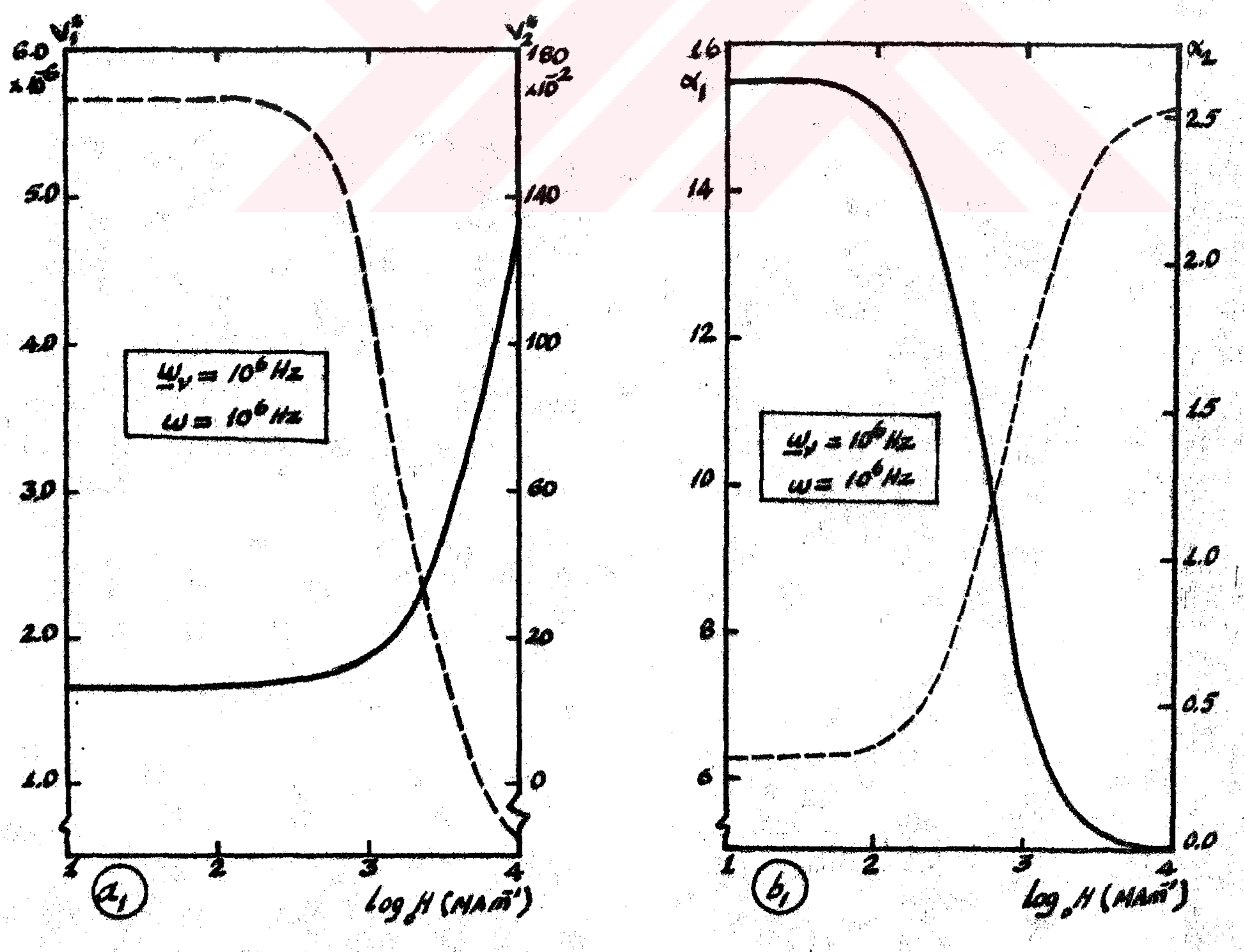
Observation of these equations shows that (8.82) and (8.83) are due to the phase velocities and the attenuations of predominantly electromagnetic and mechanical waves respectively.

Numerical solution of Eq.(8.80) is carried out for the same test material in certain ranges of the variables and the computed phase velocities and attenuations are plotted in Figs.8.17-19.

The transverse magnetic field dependence of phase velocities V^* and attenuations α are plotted for several selected values of frequency ω and the parameter $\underline{\omega}_p$ in Fig.8.17. As long as the applied magnetic field $_oH$ is less than 10^3 MAm^{-1},

V^* and α of both of the coupled modes are not changed with respect to ${}_oH$. However, for ${}_oH \geqslant 10^3$ MAm^{-1} they change very rapidly, such that V_1^* and α_1 of the predominantly electromagnetic wave increase while V_2^* and α_2 of the predominantly mechanical wave decrease. Another common characteristic of these curves is that the variations of V^* and α with respect to ${}_oH$ for different ω and ω_ν are the same, but their values at a specific ${}_oH$ are different. This is expected since both of the predominantly electromagnetic and mechanical waves are dispersive and dissipative. As was mentioned earlier in Case (a) α_2 tends to become small when ω_ν increases i.e., the viscous character diminishes.

Observation of Fig.8.18 leads one to conclude that both waves are dispersive and dissipative and the phase velocity of the predominantly mechanical wave is affected by the existing strong magnetic field. However, the influence of the transverse magnetic field, however strong it may be, on VHUS waves is not



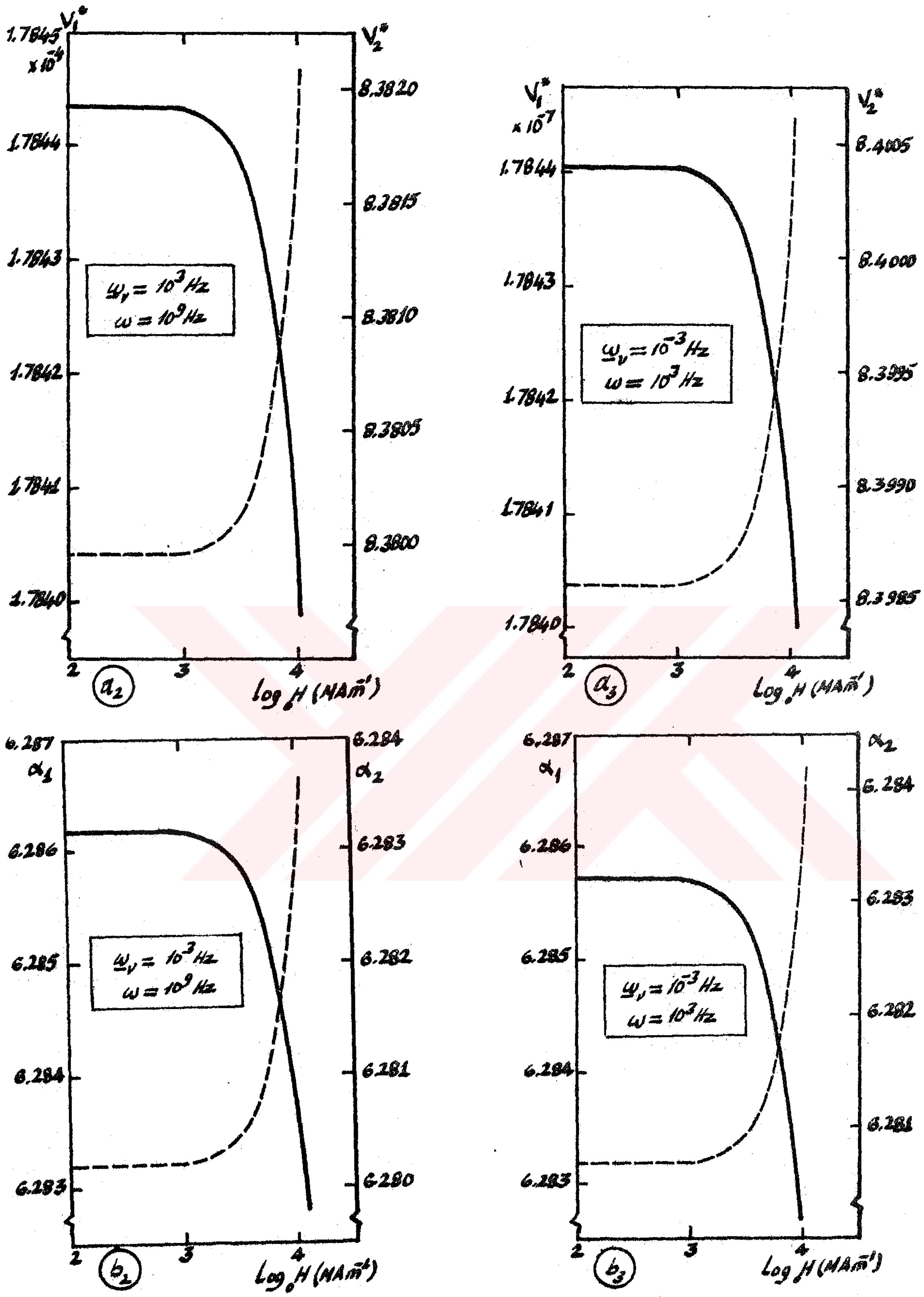


Fig.8.17 a) Phase Velocities V_1^*, V_2^* and b) Attenuation α_1, α_2 of Coupled Modes of Predominantly Electromagnetic and Mechanical Waves as Functions of Primary, Transverse Magnetic Field $_0H$ for Several Selected Values of Frequency ω and Parameter ω_ν.

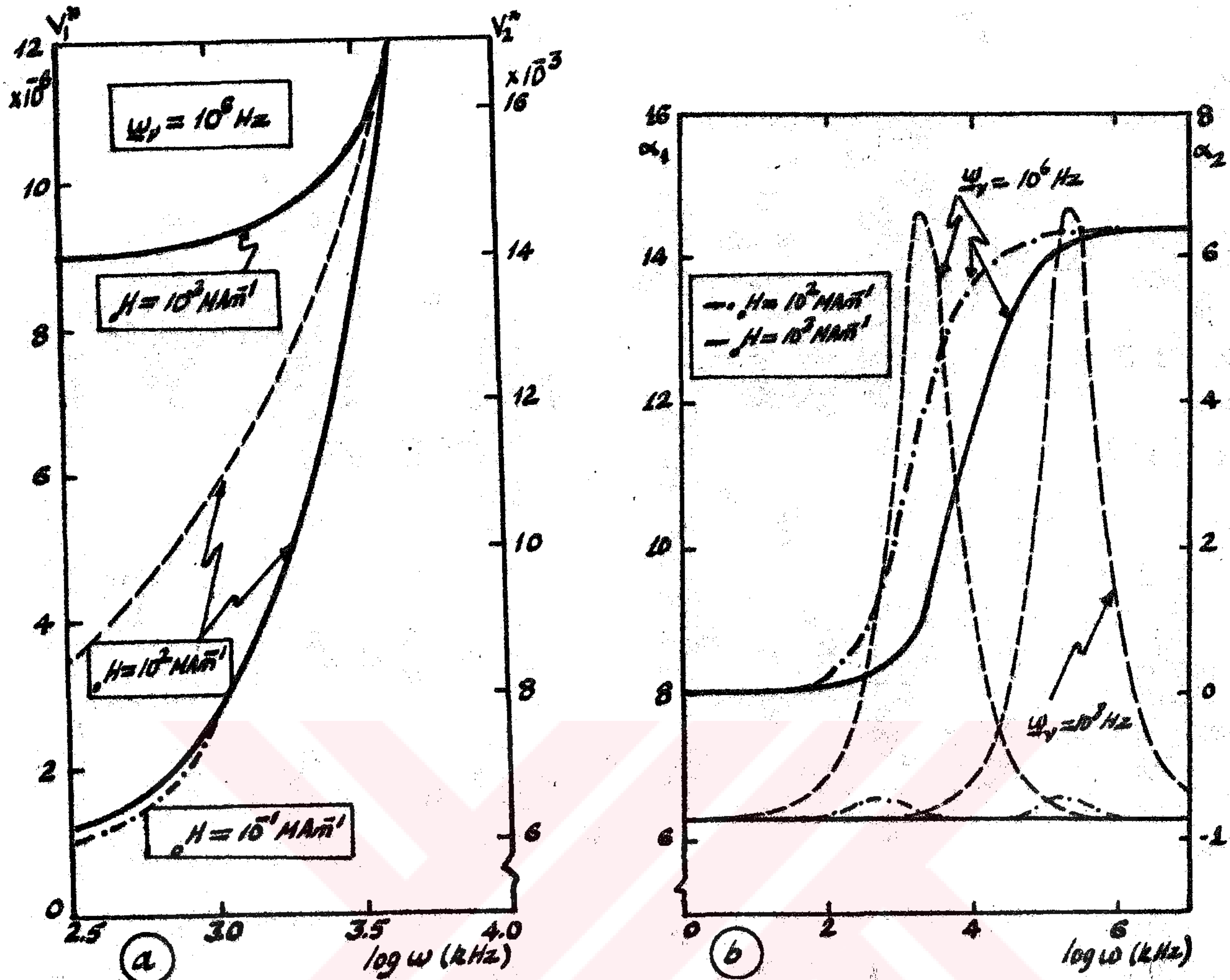


Fig.8.18 a) Phase Velocities V_1^*, V_2^* and b) Attenuations α_1, α_2 of Coupled Modes of Predominantly Electromagnetic and Mechanical Waves as Functions of Frequency ω for Several Values of Primary, Transverse Magnetic Field $_oH$ and Parameter ω_ν.

significiant. As seen in Fig.8.18 b, while there is no anomalous dispersion of the mechanical wave, there arises anomalous dispersion of the electromagnetic wave in the case of strong transverse magnetic fields. Another interesting result is that the interval of frequency of the anomalous dispersion depends upon the viscosity of the medium. Of course these results may provide some information about the mechanical and electromagnetic properties of the solids.

Fig.8.19 displays the attenuations of both waves as the function of frequency ω for specified values of $_oH$ and ω_ν. Observation of the curves in Fig.8.19 indicates that the dependence of the attenuation α on $\log \omega$ is nonlinear up to a certain frequency ω, but then the attenuation is constant for the considered magnetic field.

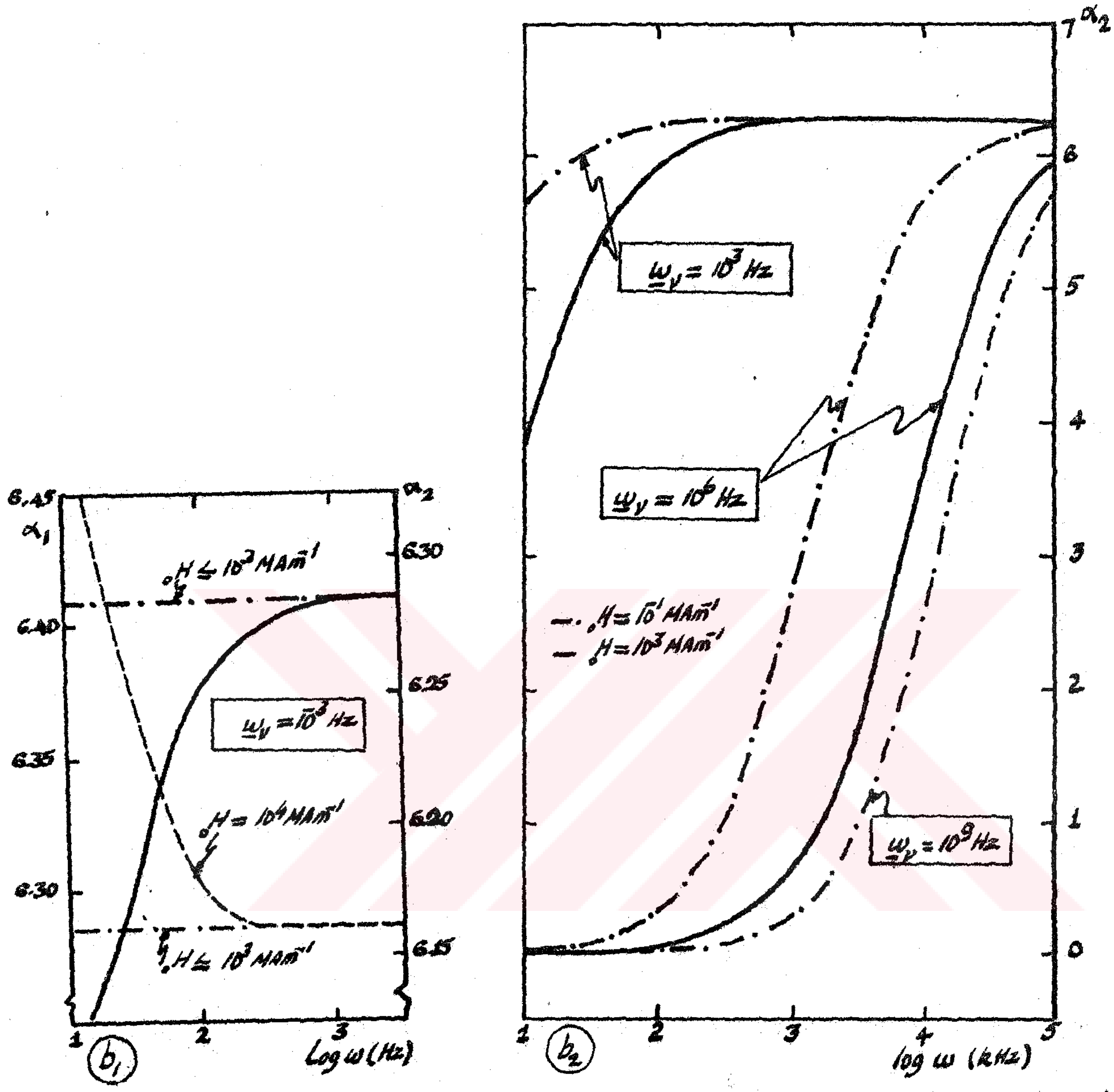


Fig.8.19 Attenuations α_1, α_2 of Coupled Modes of Predominantly Electromagnetic and Mechanical Waves as Function of Frequency ω for Several Selected Values of Primary, Transverse Magnetic Field $_oH$ and Parameter ω_ν.

CHAPTER 9

CONCLUSIONS

The primary objective of this dissertation has been to develop a macroscopic nonrelativistic theory of polarizable and magnetizable thermo-viscoelastic solids having thermal and electrical conductivity. The basic equations and the associated boundary conditions have been obtained by a systematic applications of the general balance laws of continuum physics.

When a fewer number of effects and special material symmetries are considered, the equations obtained in this thesis reduce to those given in the previous researches. The resulting equations are coupled and highly nonlinear; therefore, their solution is a very difficult task.

Even if the linear behavior of the material is considered in the derivation of the constitutive equations, the resulting governing equations still remain nonlinear.

The decomposition process presented uncouples the original governing equations as the equations associated with the rigid body and perturbation states. The equations in the rigid body state are in agreement with those of rigid body electrodynamics and the Euler's equations of rigid body motions. The set of equations in the perturbation state becomes linear whilethat of the rigid body state does not. However, when the body is constrained from the rigid body motions, the equations in the rigid body state now become linear.

The aim of our decomposition is to consider such circumtances where the dynamic infinitesimal deformation is superimposed on the prescribed rigid body motions. The decomposition process employed in [4,5,77] are not adequate to cover such a physical situation.

As an application, the propagation of magneto-mechanical waves in electrically conductive, magnetizable viscoelastic materials in an applied uniform magnetic field is discussed. Little physical insight is gained unless the numerical solutions of the dispersion relations are obtained. The plots of the numerical solutions reveal some interesting behavior of the phase velocities and the attenuations of certain modes.

It is observed that the waves, in general, coupled and they are dispersive and dissipative due to the conduction and the viscosity of the material.

Depending upon the direction of the applied magnetic field, several modes of the waves arise such as coupled mechanical and electromagnetic waves, and predominantly mechanical and electromagnetic waves. When the applied magnetic field is not present, the modes of the waves are uncoupled and both the phase velocities and the attenuations now vary smoothly with frequency. The effect of the primary magnetic field is negligable up to the order 1 MAm^{-1}, but when it is stronger, its effects become significant, depending upon the frequency of the propagating waves and the viscosity of the material.

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APPENDIX A

MAXWELL'S EQUATIONS IN FOUR-DIMENSIONAL SPACE

A point in a four-dimensional (Minkowski) space descrises an event with coordinates x_α ($\alpha = 1,2,3,4$), where the first three are the coordinates x_i in the inertial system, and $x_4 = ict$. This geometry is called the Minkowski geometry, generalizing the invariance of length of the Euclidean geometry to a four-dimensional space-time continuum [49].

A.1. Lorentz Group of Transformations

The group of orthogonal transformations of the four-dimensional space, leaving the four-dimensional distance invariant, is known as the group of Lorentz transformations, and the Maxwell's equations are invariant under such a group.

Lorentz transformations including proper and inproper Lorentz transformations plus translations are

$$x'_\alpha = \Lambda_{\alpha\beta}\, x_\beta + b_\alpha \tag{A.1}$$

where $\Lambda_{\alpha\beta}$ and b_α are two constants and $\underset{\sim}{\Lambda}$ satisfies

$$\Lambda_{\alpha\sigma}\,\Lambda_{\sigma\beta} = \delta_{\alpha\beta} \quad ; \quad \det \underset{\sim}{\Lambda} = \mp 1 \quad . \tag{A.2}$$

When two coordinate system coincide at time $t = 0$, the four-vector b_α vanishes and this transformation is called the "homogeneous" Lorentz transformation. If $\det \underset{\sim}{\Lambda} = +1$ the transformation is called "proper", otherwise (if $\det \underset{\sim}{\Lambda} = -1$), the transformation is "inproper".

The Lorentz group of transformations is used to transform the laws of physics from one inertial frame of reference to another. Hence, the matrix of the coordinate transformation in (A.1), in general, is given by

$$\| \underset{\sim}{\Lambda}(\underset{\sim}{v}) \| = \left\| \begin{array}{c|c} \underset{\sim}{I} + (\gamma-1)\dfrac{\underset{\sim}{v}\,\underset{\sim}{v}^t}{v^2} & i\gamma\dfrac{\underset{\sim}{v}}{c} \\ \hline -i\gamma\, \underset{\sim}{v}^t/c & \gamma \end{array} \right\| \tag{A.3}$$

where $$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \tag{A.4}$$

and $\underset{\sim}{v}^t$ denotes the transpose of the column vector $\underset{\sim}{v}$, $\underset{\sim}{v}$ being the velocity of primed coordinate system S' (rest frame of reference) with respect to the unprimed coordinate system S (laboratory frame of reference).

Introducing (A.3) into (A.1), one obtains (with $b_\alpha = 0$)

$$\underset{\sim}{x}' = \underset{\sim}{x} - \gamma \underset{\sim}{v} t + (\gamma - 1) \frac{\underset{\sim}{v} \cdot \underset{\sim}{x}}{v^2} \underset{\sim}{v}$$

and $$t' = \gamma \left(t - \frac{\underset{\sim}{v} \cdot \underset{\sim}{x}}{c^2} \right) \tag{A.5}$$.

The limit of (A.5) as $\left(\frac{v}{c}\right) \to 0$ (i.e., $\gamma \to 1$) gives

$$\underset{\sim}{x}' = \underset{\sim}{x} - \underset{\sim}{v} t \quad ; \quad t' = t \tag{A.6}$$

which is the homogeneous Galilean group of coordinate transformations between inertial systems. Therefore, the Galilean group is the limit of the Lorentz group when $\frac{v}{c}$ goes to zero. The transformation matrix corresponding to (A.6), i.e.,

$$\| \underset{\sim}{\Lambda}(\underset{\sim}{v}) \| = \left\| \begin{matrix} 1 & 0 & 0 & i v_1/c \\ 0 & 1 & 0 & i v_2/c \\ 0 & 0 & 1 & i v_3/c \\ 0 & 0 & 0 & 1 \end{matrix} \right\| \tag{A.7}$$

is not orthogonal.

The Maxwell equations are not invariant under the Galilean group of transformations whereas the governing equations of Newtonian mechanics are invariant under it. Thus, it is only for velocities small compared to the velocity of light ($v/c \ll 1$) that the Galilean group is a good approximation of the Lorentz group.

A.2. Four-dimensional Minkowski Formulation of Maxwell's Equations

Consider two antisymmetric field tensors $\underset{\sim}{\mathcal{F}}^{M}$, $\underset{\sim}{\mathcal{K}}^{M}$ and a four-vector $\underset{\sim}{\mathcal{J}}^{(4)}$ defined by

$$\|\underset{\sim}{\mathcal{T}}^M\| = \left\| \begin{matrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{matrix} \right\| \tag{A.8}$$

$$\|\underset{\sim}{\mathcal{K}}^M\| = \left\| \begin{matrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & -H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{matrix} \right\| \tag{A.9}$$

$$\underset{\sim}{\mathcal{J}}^{(f)} = (\underset{\sim}{J}^{(f)} , ic\rho^{(f)})^t \tag{A.10}$$

where superscript M denotes the quantities in the Minkowski formulation, and the fields $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{H}$ and $\underset{\sim}{D}$ are the quantities in the Minkowski formulation*.

Dual of the field tensors is defined by

$$dual\ \mathcal{T}^M_{\alpha\beta} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\sigma} \mathcal{T}^M_{\gamma\sigma} \tag{A.11}$$

where $\epsilon_{\alpha\beta\gamma\sigma}$ is a four-dim. alternating tensor being zero if any two indices are the same, $+1$ if the indices are an even permutation of 1234, and -1 if they are an odd permutation of 1234.

The Maxwell's equations, in four-dim. representation for any system of coordinates, are

$$\square .\ dual\ \underset{\sim}{\mathcal{T}}^M = 0 \quad ; \quad \square .\ \underset{\sim}{\mathcal{K}}^M = \underset{\sim}{\mathcal{J}}^{(f)} \tag{A.12}$$

where

$$\square \equiv (del, -\frac{i}{c}\frac{\partial}{\partial t}) \quad \text{or} \quad \frac{\partial}{\partial x_\alpha} \equiv (\frac{\partial}{\partial x_i}, -\frac{i}{c}\frac{\partial}{\partial t}) \tag{A.13}$$

is the four-dim. "del" operator. Abbreviating four-dim. divergence as Div , (A.12) is equivalent to

*Antisymmetric tensors in four-dim. space are equivalent to six vectors. Instead of (A.8) and (A.9) one might write $\underset{\sim}{\mathcal{T}}^M = (c\underset{\sim}{B} , -i\underset{\sim}{E})$ and $\underset{\sim}{\mathcal{K}}^M = (\underset{\sim}{H} , -ic\underset{\sim}{D})$ respectively.

$$Div\ dual\ \underset{\sim}{\mathcal{T}}^{M} = 0 \quad , \quad Div\ \underset{\sim}{\mathcal{K}}^{M} = \underset{\sim}{\mathcal{J}}^{(4)} \quad . \tag{A.14}$$

The conservation of the electric charge is given by

$$\Box \cdot \underset{\sim}{\mathcal{J}}^{(4)} = 0 \quad \text{or} \quad Div\ \mathcal{J}^{(4)} = 0 \quad . \tag{A.15}$$

The Maxwell equations in any inertial system S' moving with uniform velocity $\underset{\sim}{v}$ relative to the inertial system S are easily obtained by means of the tensor transformations. Thus, one has

$$\begin{aligned} \mathcal{T}'^{M}_{\alpha\beta} &= \Lambda_{\alpha\delta}\, \Lambda_{\beta\sigma}\, \mathcal{T}^{M}_{\delta\sigma} \\ \mathcal{K}'^{M}_{\alpha\beta} &= \Lambda_{\alpha\delta}\, \Lambda_{\beta\sigma}\, \mathcal{K}^{M}_{\delta\sigma} \\ \mathcal{J}'^{(4)}_{\alpha} &= \Lambda_{\alpha\beta}\, \mathcal{J}^{(4)}_{\beta} \end{aligned} \tag{A.16}$$

and the primed quantities satisfy the same equations, (A.12) or (A.14), provided that the operator $\Box$ is replaced by $\Box'$, i.e., the Maxwell equations are invariant under the Lorentz group of transformations.

One may introduce an auxilary polarization tensor as

$$\underset{\sim}{\mathbb{P}} \equiv \sqrt{\frac{\epsilon_0}{\mu_0}}\ \underset{\sim}{\mathcal{T}}^{M} - \underset{\sim}{\mathcal{K}}^{M} \tag{A.17}$$

so that the components of $\underset{\sim}{\mathbb{P}}$ are given by

$$\| \mathbb{P} \| = \left\| \begin{matrix} 0 & M_3 & -M_2 & icP_1 \\ -M_3 & 0 & M_1 & icP_2 \\ M_2 & -M_1 & 0 & icP_3 \\ -icP_1 & -icP_2 & -icP_3 & 0 \end{matrix} \right\| \quad . \tag{A.18}$$

$\underset{\sim}{\mathbb{P}}$ is an antisymmetric second order tensor.

The constitutive equations of free space are very simple

$$\mathcal{K}^{M}_{\alpha\beta} = \sqrt{\frac{\epsilon_0}{\mu_0}}\ \mathcal{T}^{M}_{\alpha\beta} \quad ; \quad \mathcal{J}^{(4)}_{\alpha} = 0 \tag{A.19}$$

where the free charge is assumed to be zero. It should be noted that (A.19) is invariant under a general Lorentz group of transformations which implies that the electromagnetic constitutive equations for free space are independent of the observer. This is not true for electromagnetic materials.

In a material medium the left hand side of $(A.19)_1$ will differ from the right hand side through a second rank tensor $\underset{\sim}{\mathbb{P}}$ defined in (A.17).

The constitutive equations of rigid bodies are assumed to be in the form

$$\mathbb{P}'_{\alpha\beta} = \hat{\mathbb{P}}'_{\alpha\beta}(\underset{\sim}{\mathcal{T}}'^{M}) \; ; \; \mathbb{J}'_{\alpha} = \hat{\mathbb{J}}'_{\alpha}(\underset{\sim}{\mathcal{T}}'^{M}) \tag{A.20}$$

where superscript "hat" over $\underset{\sim}{\mathbb{P}}'$ and $\underset{\sim}{\mathbb{J}}'$ denotes the function of the arguments in the rest frame of reference.

Restrictions on functions $\hat{\underset{\sim}{\mathbb{P}}}$ and $\hat{\underset{\sim}{\mathbb{J}}}$ can be imposed through symmetry considerations and through the energy principles. We now consider a special case of the constitutive equations (A.20), applicable to rigid, isothermal moving materials. In this case, $\hat{\underset{\sim}{\mathbb{P}}}'$ and $\hat{\underset{\sim}{\mathbb{J}}}'$ are assumed to be linear functions of the components $\underset{\sim}{\mathcal{T}}^{M}$. Hence, the constitutive equations of electrodynamics describing polarizable, magnetizable magneto-electric linear media reduce to

$$\mathbb{P}'_{\alpha\beta} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\; \hat{\xi}_{\alpha\beta\gamma\sigma}\mathcal{T}'^{M}_{\gamma\delta} \; ; \; \mathbb{J}'_{\alpha} = \frac{1}{2}\hat{\xi}_{\alpha\beta\gamma}\mathcal{T}'^{M}_{\beta\gamma} \tag{A.21}$$

where $\hat{\xi}_{\alpha\beta\gamma\sigma}$ is the generalized dimensionless susceptibility tensor and $\hat{\xi}_{\alpha\beta\gamma}$ is the generalized conductivity tensor [50, 51]. Since both $\underset{\sim}{\mathcal{T}}^{M}$ and $\underset{\sim}{\mathbb{P}}$ are antisymmetric, $(A.21)_1$ implies that

$$\hat{\xi}_{\alpha\beta\gamma\delta} = -\hat{\xi}_{\beta\alpha\gamma\delta} = \hat{\xi}_{\beta\alpha\delta\gamma} \,. \tag{A.22}$$

In addition to these intrinsic symmetries, one can expect the symmetry

$$\hat{\xi}_{\alpha\beta\gamma\delta} = \hat{\xi}_{\gamma\delta\alpha\beta} \tag{A.23}$$

which follows from the thermodynamic potential expression. Similarly, it is easily shown that

$$\hat{\xi}_{\alpha\beta\delta} = -\hat{\xi}_{\alpha\delta\beta} \qquad (\gamma \neq \beta) \tag{A.24}$$

A more conventional form of $(A.21)_1$ is obtained by introducing $(A.21)_1$ into $(A.19)_1$. That is

$$\mathcal{K}'^{M}_{\alpha\beta} = \sqrt{\frac{\epsilon_0}{\mu_0}} \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2} \hat{\xi}_{\alpha\beta\gamma\delta} \right) \mathcal{T}'^{M}_{\gamma\delta} \tag{A.25}$$

where $\delta_{\alpha\beta}$ is four-dim. Kroneker delta, being equal to 1 if $\alpha = \beta$, and zero if $\alpha \neq \beta$.

It is possible to introduce complex elements in $(A.21)_1$ in order to extend the conclusions to dispersive media. The elements of the generalized susceptibility tensor are then the function of frequency. The nondissipative behavior of the medium is expressed by the Hermitian symmetry of the matrix associated with the generalized susceptibility tensor, which means that

$$Re\, \hat{\xi}_{\alpha\beta\gamma\delta} = Re\, \hat{\xi}_{\gamma\delta\alpha\beta} \quad ; \quad Im\, \hat{\xi}_{\alpha\beta\gamma\delta} = - Im\, \hat{\xi}_{\gamma\delta\alpha\beta} \; . \tag{A.26}$$

It follows from the symmetry conditions (A.22,23) that the general susceptibility tensor can be represented by a six by six symmetric matrix which has 21 independent components [51]

$$\begin{array}{c|c|cccccc}
\mathcal{T}_{\alpha\beta}: & & 41 & 42 & 43 & 23 & 31 & 12 \\
\mathbb{P}_{\alpha\beta} & \hat{\xi}_{\alpha\beta\gamma\delta} & E_1 & E_2 & E_3 & cB_1 & cB_2 & cB_3 \\
\hline
14 & cP_1 & \hat{\chi}^{(p)}_{11} & \hat{\chi}^{(p)}_{12} & \hat{\chi}^{(p)}_{13} & \hat{\gamma}_{11} & \hat{\gamma}_{12} & \hat{\gamma}_{13} \\
24 & cP_2 & \hat{\chi}^{(p)}_{21} & \hat{\chi}^{(p)}_{22} & \hat{\chi}^{(p)}_{23} & \hat{\gamma}_{21} & \hat{\gamma}_{22} & \hat{\gamma}_{23} \\
34 & cP_3 & \hat{\chi}^{(p)}_{31} & \hat{\chi}^{(p)}_{32} & \hat{\chi}^{(p)}_{33} & \hat{\gamma}_{31} & \hat{\gamma}_{32} & \hat{\gamma}_{33} \\
23 & M_1 & \hat{\gamma}^{*}_{11} & \hat{\gamma}^{*}_{21} & \hat{\gamma}^{*}_{31} & \hat{\chi}^{(m)}_{11} & \hat{\chi}^{(m)}_{12} & \hat{\chi}^{(m)}_{13} \\
31 & M_2 & \hat{\gamma}^{*}_{12} & \hat{\gamma}^{*}_{22} & \hat{\gamma}^{*}_{32} & \hat{\chi}^{(m)}_{21} & \hat{\chi}^{(m)}_{22} & \hat{\chi}^{(m)}_{23} \\
12 & M_3 & \hat{\gamma}^{*}_{13} & \hat{\gamma}^{*}_{23} & \hat{\gamma}^{*}_{33} & \hat{\chi}^{(m)}_{31} & \hat{\chi}^{(m)}_{32} & \hat{\chi}^{(m)}_{33}
\end{array} \tag{A.27}$$

where the diagonal submatrices represent electric susceptibility $\hat{\chi}^{(p)}_{ij}$ and the inverse of the magnetic susceptibility $\hat{\chi}^{(m)}_{ij}$.

Superscript "star" over the elements of the submatrices denotes the complex conjugate of a particular element. Thus $\hat{\gamma}_{ij}$ and γ_{ji}^{*} represent the material property tensor of magneto-electric media. A material is called magneto-electric if it is polarized in a magnetic field and similarly magnetized in an electric field.

Introducing $\underset{\sim}{P} \equiv \underset{\sim}{D} - \epsilon_0 \underset{\sim}{E}$ and $\mu_0 \underset{\sim}{M} \equiv \underset{\sim}{B} - \mu_0 \underset{\sim}{H}$, one can write (A.27) explicitly in three-dim. space in the rest frame of references as

$$\begin{aligned} D_i' &= \epsilon_0 \left(\delta_{ij} + \hat{\chi}_{ij}^{(p)} \right) E_j' + \sqrt{\frac{\epsilon_0}{\mu_0}}\ \hat{\gamma}_{ij}\ B_j' \\ H_i' &= -\frac{1}{c}\ \hat{\gamma}_{ji}^{*}\ E_i' + \frac{1}{\mu_0} \left(\delta_{ij} - \frac{1}{\mu_0} \hat{\chi}_{ij}^{(m)} \right) B_j' \ . \end{aligned} \qquad (A.28)$$

The constitutive equations (A.28) become simpler for hemitropic (hemiheadral isotropic) and holotropic (holohedral isotropic)materials*. The constitutive equations are now

$$\begin{aligned} \underset{\sim}{D}'^{M} &= \epsilon_0 \left(1 + \hat{\chi}^{(p)} \right) \underset{\sim}{E}'^{M} + \sqrt{\frac{\epsilon_0}{\mu_0}}\ \hat{\gamma}\, \underset{\sim}{B}'^{M} \\ \underset{\sim}{H}'^{M} &= -\frac{1}{c}\ \hat{\gamma}^{*} \underset{\sim}{E}'^{M} + \frac{1}{\mu_0} \left(1 - \frac{1}{\mu_0}\ \hat{\chi}^{(m)} \right) \underset{\sim}{B}'^{M} \end{aligned} \qquad (A.29)$$

if the material is hemitropic, and

$$\underset{\sim}{D}'^{M} = \epsilon_0 \left(1 + \hat{\chi}^{(p)} \right) \underset{\sim}{E}'^{M} \quad ; \quad \underset{\sim}{H}'^{M} = \frac{1}{\mu_0} \left(1 - \frac{1}{\mu_0}\ \hat{\chi}^{(m)} \right) \underset{\sim}{B}'^{M} \qquad (A.30)$$

if the material is holotropic.

A similar type of reasoning can be applied to the components of $\hat{\xi}_{\alpha\beta\gamma}$ in $(A.21)_2$. For hemitropic material one has

$$\underset{\sim}{J}' = \hat{\sigma}^{(p)} \underset{\sim}{E}'^{M} + \hat{\sigma}^{(m)} \underset{\sim}{B}'^{M} \qquad (A.31)$$

and for holotropic material $\hat{\sigma}^{(m)}$ vanishes. The constants $\hat{\sigma}^{(p)}$ and $\hat{\sigma}^{(m)}$ can be called electric and magnetic conductivities respectively.

Magneto-electric materials for a stationary, isotropic,

*If the material is hemihedral isotropic, then the constitutive equations must be form-invariant under the proper orthogonal group of transformations. The constitutive equations for a holohedral isotropic material must be form-invariant under the central inversion transformation.

homogeneous linear medium were first proposed by Curié [52]. Later, the magneto-electric interactions are measured by Astrov [23], Al'shin and Astrov [24]. The electromagnetic wave propagation in an anisotropic rigid material is studied by Birss and Shrubsall [29] and Fuchs [30], and in a finitely deformed hemitropic magneto-electric material by Ersoy and Kıral [53].

From the transformation laws (A.16), one obtains the transformation of the electromagnetic quantities in the Minkowski formulation as

$$
\begin{aligned}
\underset{\sim}{E}' &= \gamma(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) + (1-\gamma)\frac{\underset{\sim}{E}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\
\underset{\sim}{B}' &= \gamma(\underset{\sim}{B} - \tfrac{1}{c^2}\underset{\sim}{v} \times \underset{\sim}{E}) + (1-\gamma)\frac{\underset{\sim}{B}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\
\underset{\sim}{D}' &= \gamma(\underset{\sim}{D} + \tfrac{1}{c^2}\underset{\sim}{v} \times \underset{\sim}{H}) + (1-\gamma)\frac{\underset{\sim}{D}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\
\underset{\sim}{H}' &= \gamma(\underset{\sim}{H} - \underset{\sim}{v} \times \underset{\sim}{D}) + (1-\gamma)\frac{\underset{\sim}{H}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\
\underset{\sim}{J}'^{(f)} &= \underset{\sim}{J}^{(f)} - \gamma\underset{\sim}{v}\rho^{(f)} + (\gamma-1)\frac{\underset{\sim}{J}^{(f)}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\
\rho'^{(f)} &= \gamma(\rho^{(f)} - \tfrac{1}{c^2}\underset{\sim}{J}^{(f)}\cdot\underset{\sim}{v}) \;.
\end{aligned}
\tag{A.32}
$$

The transformation of the auxilary polarization tensor (A.17) gives

$$
\begin{aligned}
\underset{\sim}{P}' &= \gamma(\underset{\sim}{P} - \tfrac{1}{c^2}\underset{\sim}{v} \times \underset{\sim}{M}) + (1-\gamma)\frac{\underset{\sim}{P}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \\
\underset{\sim}{M}' &= \gamma(\underset{\sim}{M} + \underset{\sim}{v} \times \underset{\sim}{P}) + (1-\gamma)\frac{\underset{\sim}{M}\cdot\underset{\sim}{v}}{v^2}\underset{\sim}{v} \;.
\end{aligned}
\tag{A.33}
$$

According to (A.33), the polarized (magnetized) material in the rest frame of reference appears to be both polarizable and magnetizable in the moving frame of reference.

If the velocity of the bulk material is very small compared to the velocity of light, then Eqs.(A.32) and (A.33) reduce to

$$
\begin{aligned}
\underset{\sim}{E}' &\cong \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} \quad ; \quad \underset{\sim}{B}' \cong \underset{\sim}{B} - \tfrac{1}{c^2}\underset{\sim}{v} \times \underset{\sim}{E} \\
\underset{\sim}{D}' &\cong \underset{\sim}{D} + \tfrac{1}{c^2}\underset{\sim}{v} \times \underset{\sim}{H} \quad ; \quad \underset{\sim}{H}' \cong \underset{\sim}{H} - \underset{\sim}{v} \times \underset{\sim}{D}
\end{aligned}
\tag{A.34}
$$

$$\underset{\sim}{J}^{(f)'} \cong \underset{\sim}{J}^{(f)} - \rho^{(f)} \underset{\sim}{v} \quad ; \quad \rho'^{(f)} \cong \rho^{(f)} - \frac{1}{c^2} \underset{\sim}{J}^{(f)} \cdot \underset{\sim}{v}$$

and

$$\underset{\sim}{P}' \cong \underset{\sim}{P} - \frac{1}{c^2} \underset{\sim}{v} \times \underset{\sim}{M} \quad ; \quad \underset{\sim}{M}' \cong \underset{\sim}{M} + \underset{\sim}{v} \times \underset{\sim}{P} \tag{A.35}$$

neglecting the terms containing $(v/c)^2$ with respect to the first order terms. If Eq.(A.34) is substituted into (A.28-31) the associated constitutive equations for the rigid body in the laboratory frame of reference are obtained.

The transformations of the gradient and the partial derivative with respect to time are obtained by means of (A.5) and the chain rule of differentiation. Thus $\frac{\partial}{\partial x'_\alpha} = \frac{\partial x_\beta}{\partial x'_\alpha} \frac{\partial}{\partial x_\beta}$ implies that

$$\frac{\partial}{\partial x'_i} = \frac{\partial}{\partial x_i} + (\gamma - 1) \frac{v_i v_j}{v^2} \frac{\partial}{\partial x_j} + \frac{\gamma}{c^2} v_i \frac{\partial}{\partial t} \tag{A.36}$$

$$\frac{\partial}{\partial t'} = \gamma \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) .$$

The material time derivative at the proper frame of reference is

$$\frac{d}{dt^\circ} \cong \frac{\partial}{\partial t^\circ} = \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \text{grad} + O\left(\frac{v^2}{c^2}\right) \cong \frac{d}{dt} . \tag{A.37}$$

Thus, the material derivative at the proper frame of reference is equal to that at the laboratory frame of reference if one neglects the terms containing $(v/c)^2$.

The four-dim. energy-momentum tensor in the Minkowski formulation of electrodynamics is

$$\| S_{\mu\gamma} \| = \left\| \begin{array}{c|c} S_{jk} & S_{j4} \\ \hline S_{4k} & S_{44} \end{array} \right\| \tag{A.38}$$

where

$$S_{jk} = \frac{1}{2} (D_m E_m + B_m H_m) \delta_{jk} - (D_j E_k + B_j H_k) \tag{A.39}$$

$$S_{4k} = ic \, (\underset{\sim}{D} \times \underset{\sim}{B})_k \quad ; \quad S_{j4} = \frac{i}{c} (\underset{\sim}{E} \times \underset{\sim}{H})_j$$

$$S_{44} = -\frac{1}{2} (D_m E_m + B_m H_m) .$$

The divergence of $S_{\mu\nu}$ gives a four-vector the space part of which is related to the electromagnetic body force,

$$f_\nu = (f_i, -icf_4) = \frac{\partial S_{\mu\nu}}{\partial x_\mu} \quad . \tag{A.40}$$

Thus, one has the body force $\underset{\sim}{f}^{(em)}$

$$\underset{\sim}{f}^{(em)} = -\underset{\sim}{f}_{space} = \hat{\sigma}\underset{\sim}{E} + \underset{\sim}{J}\times\underset{\sim}{B} - \frac{1}{2}\left(\nabla\underset{\sim}{B}\cdot\underset{\sim}{H} - \nabla\underset{\sim}{H}\cdot\underset{\sim}{B} + \nabla\underset{\sim}{D}\cdot\underset{\sim}{E} - \nabla\underset{\sim}{E}\cdot\underset{\sim}{D}\right) \tag{A.41}$$

and the energy supply $\rho r_E^M = icf_4 + f_j v_j$

$$\rho r_E^M = (\underset{\sim}{J} - \hat{\sigma}\underset{\sim}{v})\cdot(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}) + \frac{1}{2}\left(\underset{\sim}{H}\cdot\dot{\underset{\sim}{B}} - \underset{\sim}{B}\cdot\dot{\underset{\sim}{H}} + \underset{\sim}{D}\dot{\underset{\sim}{E}} - \underset{\sim}{E}\cdot\dot{\underset{\sim}{D}}\right). \tag{A.42}$$

The antisymmetric part of S_{jk} yields the body couple

$$\rho \underset{\sim}{l}^M = \underset{\sim}{D}\times\underset{\sim}{E} + \underset{\sim}{B}\times\underset{\sim}{H} \quad . \tag{A.43}$$

A.3. <u>Four-dimensional Chu Formulation of the Maxwell's Equations</u>

In four-dim. space one may define two antisymmetric second rank tensors and three four-vectors as

$$\|\underset{\sim}{T}^c\| = \begin{Vmatrix} 0 & H_3 & -H_2 & -ic\epsilon_0 E_1 \\ -H_3 & 0 & H_1 & -ic\epsilon_0 E_2 \\ H_2 & -H_1 & 0 & -ic\epsilon_0 E_3 \\ ic\epsilon_0 E_1 & ic\epsilon_0 E_2 & ic\epsilon_0 E_3 & 0 \end{Vmatrix}$$

$$\|\underset{\sim}{K}^M\| = \begin{Vmatrix} 0 & -E_3 & E_2 & -ic\mu_0 H_1 \\ E_3 & 0 & -E_1 & -ic\mu_0 H_2 \\ -E_2 & E_1 & 0 & -ic\mu_0 H_3 \\ ic\mu_0 H_1 & ic\mu_0 H_2 & ic\mu_0 H_3 & 0 \end{Vmatrix} \tag{A.44}$$

$$\underset{\sim}{J}^{(4)} = (\underset{\sim}{J}^{(4)}, ic\rho)^t$$

$$\underset{\sim}{P}^c = \frac{1}{\bar{\gamma}}\left(\underset{\sim}{P} + \frac{\bar{\underset{\sim}{v}}\,\underset{\sim}{P}\cdot\bar{\underset{\sim}{v}}}{c^2 - \bar{v}^2}\ ,\ ic\,\frac{\underset{\sim}{P}\cdot\bar{\underset{\sim}{v}}}{c^2 - \bar{v}^2}\right)^t \tag{A.45}$$

$$\underset{\sim}{M}^{c} = \frac{1}{\bar{\gamma}} \left(\underset{\sim}{M} + \frac{\bar{\underset{\sim}{V}}\, \underset{\sim}{M}\cdot\bar{\underset{\sim}{V}}}{c^2 - \bar{V}^2} \;,\; ic\, \frac{\underset{\sim}{M}\cdot\bar{\underset{\sim}{V}}}{c^2 - \bar{V}^2} \right)^t$$

where

$$\bar{\gamma} = \frac{1}{\sqrt{1-\bar{V}^2/c^2}} \tag{A.46}$$

and $\underset{\sim}{v}$ is the velocity of the material point with respect to S, superscript c denotes the quantities in the Chu formulation of electromagnetism [19]. Since both $\underset{\sim}{\mathcal{T}}^c$ and $\underset{\sim}{\mathcal{K}}^c$ contain $\underset{\sim}{E}$ and $\underset{\sim}{H}$, they are interdependent.

$$\text{dual}\ \underset{\sim}{\mathcal{T}}^c = ic\,\epsilon_0\, \underset{\sim}{\mathcal{K}} \quad ; \quad \text{dual}\ \underset{\sim}{\mathcal{K}}^c = -ic\mu_0\, \underset{\sim}{\mathcal{T}}^c . \tag{A.47}$$

The Maxwell s equations in four-dim. formulation are

$$\Box\cdot(\underset{\sim}{\mathcal{T}}^c + \bar{\underset{\sim}{\mathcal{T}}}^c) = \underset{\sim}{J}^{(4)} \quad ; \quad \Box\cdot(\underset{\sim}{\mathcal{K}}^c + \bar{\underset{\sim}{\mathcal{K}}}^c) = 0 \tag{A.48}$$

where

$$\bar{\mathcal{T}}^c_{\alpha\beta} = \bar{V}_\alpha \mathbb{P}^c_\beta - \bar{V}_\beta \mathbb{P}^c_\alpha \quad ; \quad \bar{\mathcal{K}}^c_{\alpha\beta} = \mu_0 (\bar{V}_\alpha M^c_\beta - \bar{V}_\beta M^c_\alpha) \tag{A.49}$$

In Eq.(A.49), $\bar{V}_\alpha$ is known as the four-velocity vector defined by

$$\bar{V}_\alpha = \bar{\gamma}\,(\underset{\sim}{\bar{v}}, ic) \quad ; \quad \bar{V}_\alpha \bar{V}_\alpha = -c^2 \quad . \tag{A.50}$$

By means of the transformation laws, the transformations of $\underset{\sim}{E}$, $\underset{\sim}{H}$, $\underset{\sim}{P}$ and $\underset{\sim}{M}$ (from an unprimed framed to a primed frame at velocity) are obtained. They are

$$\begin{aligned}
\underset{\sim}{E}' &= E_{\parallel} + \gamma\,(E_{\perp} + \underset{\sim}{v} \times \mu_0 \underset{\sim}{H}) \\
\underset{\sim}{H}' &= \underset{\sim}{H}_{\parallel} + \gamma\,(\underset{\sim}{H}_{\perp} - \underset{\sim}{v} \times \epsilon_0 \underset{\sim}{E}) \\
\underset{\sim}{P}' &= \underset{\sim}{P}_{\parallel} + \gamma \underset{\sim}{P}_{\perp} - \gamma\, \frac{\underset{\sim}{v} \times (\underset{\sim}{P} \times \underset{\sim}{v})}{c^2} \\
\mu_0 \underset{\sim}{M}' &= \mu_0 \underset{\sim}{M}_{\parallel} + \gamma \mu_0 \underset{\sim}{M}_{\perp} - \gamma\, \frac{\underset{\sim}{v} \times (\mu_0 \underset{\sim}{M} \times \underset{\sim}{v})}{c^2}
\end{aligned} \tag{A.51}$$

where $\underset{\sim}{v}$ is the velocity of the material in the unprimed frame and γ is given by (A.4). The subscripts $\parallel$ and $\perp$

in (A.51) denote the parallel and perpendicular components of the associated variables to the velocity $\underset{\sim}{v}$.

When the material velocity $\underset{\sim}{\bar{v}}$ is equal to the velocity of the primed frame $\underset{\sim}{v}$, and the terms which contain $(v/c)^2$ are neglected, the transformations (A.51) reduce to

$$\underset{\sim}{\mathcal{E}} \equiv \underset{\sim}{E}' \cong \underset{\sim}{E} + \underset{\sim}{v} \times \mu_0 \underset{\sim}{H} \ ; \ \underset{\sim}{\mathcal{H}} \equiv \underset{\sim}{H}' \cong \underset{\sim}{H} - \underset{\sim}{v} \times \epsilon_0 \underset{\sim}{E}$$

$$\underset{\sim}{\mathcal{P}} \equiv \underset{\sim}{P}' \cong \underset{\sim}{P} \ ; \ \underset{\sim}{\mathcal{M}} \equiv \underset{\sim}{M}' \cong \underset{\sim}{M} \ . \qquad (A.52)$$

Thus, in the nonrelativistic formulations of the Maxwell equations in terms of the Chu variables, the polarization and the magnetization at the rest and laboratory frames of reference are the same, however the electric and magnetic fields are not.

Using the constitutive equations in the laboratory frame of reference and the relations (3.39), the constitutive equations (A.29) are expressed in terms of the Chu variables.

APPENDIX B

DECOMPOSITIONS OF UNIT NORMAL AND TANGENT VECTORS

The surface of discontinuity is assumed to be the bounding of the deformable body. Our goal is to express the unknown unit normal and tangent vectors $\underset{\sim}{n}$ and $\underset{\sim}{k}$ in the present configuration $\mathcal{B}_t$ in terms of unit vectors $\underset{\sim}{N}$ and $\underset{\sim}{K}$ in the reference configuration $\mathcal{B}_R$. Let $S(X_K,t)=0$ and $s(x_k,t)=0$ be the same surface of the body in $\mathcal{B}_R$ and $\mathcal{B}_t$ respectively. It is assumed that $S(X_K,t)$ and $s(x_k,t)$ are both class C^1 in their arguments. By definition, one writes

$$N_K = S_{,K}\Big/\sqrt{S_{,L}S_{,L}} \quad ; \quad n_k = s_{,k}\Big/\sqrt{s_{,\ell}s_{,\ell}} \tag{B.1}$$

Since

$$s(x_k,t) = s[x_k(\underset{\sim}{X},t),t] = S(X_K,t) \tag{B.2}$$

one now expresses $(B.1)_2$ as

$$n_k = S_{,K}X_{K,k}\Big/(S_{,L}S_{,M}X_{L,\ell}X_{M,\ell})^{1/2} \tag{B.3}$$

Replacing $X_{K,k}$ by the displacement gradient $X_{K,k}=\delta_{Kk}-u_{K,k}$ in (B.3), one obtains

$$n_k = (\delta_{Kk}-u_{K,k})S_{,K}\Big/[(\delta_{L\ell}-u_{L,\ell})(\delta_{M\ell}-u_{M,\ell})S_{,L}S_{,M}]^{1/2} \tag{B.4}$$

If the linear terms in the displacement gradient are retained, and the definition of $\underset{\sim}{N}$ $(B.1)_1$ is used in the resulting equation, one can obtain

$$n_k \cong [(1+N_MN_L\,\tilde{e}_{ML})\delta_{Kk} - u_{K,L}\,\delta_{Lk}]N_K \tag{B.5}$$

where $\tilde{e}_{ML}$ is the infinitesimal strain tensor defined by (6.17). Therefore, for a prescribed $\underset{\sim}{N}$, the unit vector $\underset{\sim}{n}$ in $\mathcal{B}_t$ can be determined if the displacement gradient is known.

To find the transformation for the unit tangent vector, let λ be a parameter which characterize a space curve and t be the time. A material line of discontinuity is represented by

$$X_I = \hat{X}_I(\lambda, t) \quad ; \quad x_i = \hat{x}_i(\lambda, t) \tag{B.6}$$

in $\mathcal{B}_R$ and $\mathcal{B}_t$ respectively. In view of the mappings

$$x_i = \hat{x}_i(X_K, t) = \tilde{x}_i[X_K(\lambda, t), t] = \hat{x}_i(\lambda, t) \tag{B.7}$$

one now observes that the parametrization in the two representations of the curve are the same. One has

$$K_I = \frac{\partial X_I/\partial\lambda}{\sqrt{\frac{\partial X_L}{\partial\lambda}\frac{\partial X_L}{\partial\lambda}}} \quad ; \quad k_i = \frac{\partial x_i/\partial\lambda}{\sqrt{\frac{\partial x_\ell}{\partial\lambda}\frac{\partial x_\ell}{\partial\lambda}}} \ . \tag{B.8}$$

Using the chain rule of differentiation, $(B.8)_2$ is written as

$$k_i = x_{i,I}\frac{\partial X_I}{\partial\lambda} \Big/ \left(x_{\ell,L}\, x_{\ell,M} \frac{\partial X_L}{\partial\lambda}\frac{\partial X_M}{\partial\lambda} \right)^{1/2} . \tag{B.9}$$

Introducing the displacement field which is evaluated at the curve parametrized by λ

$$x_i = u_i(\lambda) + X_I \delta_{iI}$$

into (B.9), one obtains

$$k_i = (u_{i,I} + \delta_{iI})\frac{\partial X_I}{\partial\lambda} \Big/ \left[(\delta_{LM} + u_{L,M} + u_{M,L} + u_{\ell,L} u_{\ell,M}) \frac{\partial X_L}{\partial\lambda}\frac{\partial X_M}{\partial\lambda} \right]^{1/2}$$

$$\cong \frac{\partial X_I/\partial\lambda}{\sqrt{\frac{\partial X_K}{\partial\lambda}\frac{\partial X_K}{\partial\lambda}}} \; \frac{\delta_{iI} + u_{i,I}}{\sqrt{1 + \left[\frac{\partial X_L}{\partial\lambda}\frac{\partial X_M}{\partial\lambda}(u_{L,M} + u_{M,L})\right] \Big/ \left(\frac{\partial X_K}{\partial\lambda}\frac{\partial X_K}{\partial\lambda}\right)}} \tag{B.10}$$

If the linear terms are retained, one arrives at

$$k_i \cong [(1 - K_L K_M \tilde{e}_{ML})\delta_{iI} + u_{L,I}\delta_{iL}] K_I \tag{B.11}$$

where the definition of K_I $(B.8)_1$ is introduced. Thus $\underset{\sim}{k}$ is expressed in terms of the unit tangent vector $\underset{\sim}{K}$ and the displacement gradient.

VITA

The author, son of Galip and Döndü Ersoy, was born in Adana, Turkey in March 8,1945. After completing his primary and secondary education in Adana, he continued his education in the Department of Physics at M.E.T.U., and graduated in June 1968 with a B.S degree in Physics. He was also awarded a diploma by the Ankara Yüksek Öğretmen Okulu.

After a period of teaching physics at Konya Maarif Koleji and Selçuk Eğitim Enstitüsü, he joined the Department of Engineering Sciences at M.E.T.U. in December 1969 as an assistant and a doctoral student. He became an Instructor in July 1972. As a result of his studies he is the author of several research papers and one book.

He carried out his doctoral studies under a scholarship provided by TBTAK. He is married to Taliha and has one son born in Semptember 1969.

He is a citizen of the Republic of Turkey.