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RESEARCH

Optimal Control problems of NS-α and NS-ω turbulence models: analysis and numerical tests

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Abstract
In this study, optimal control problems for the Navier-Stokes-α (NS-α) model and the Navier-Stokes-ω (NS-ω) model are considered. Optimality conditions are derived, and semi-discrete a priori error estimates for all fluid variables are analyzed for both models. Numerical tests are performed to verify the accuracy of the theoretical findings and to demonstrate the effectiveness of optimal control. Given the proven utility of the NS-α and NS-ω models in fluid dynamics, this study addresses a significant gap by exploring the potential of optimal control to enhance the performance, efficiency, safety, and environmental impact of fluid systems.

Keywords Navier-Stokes equations · NS-α turbulence model · NS-ω turbulence model · finite element method · optimal control

Mathematics Subject Classification (2010) 65M60 · 76D05 · 49J20

1 Introduction

Consider the incompressible Navier-Stokes Equations (NSE) modeling conservation of linear momentum and the conservation of mass in a convex, polygonal domain Ω in $Q = [0, T] \times \Omega \subset \mathbb{R}^2$ by

$$
\begin{aligned}
y_t - \nu \Delta y + (y \cdot \nabla) y + \nabla p &= f \quad \text{in } Q, \\
\nabla \cdot y &= 0 \quad \text{in } Q.
\end{aligned}
\tag{1.1}
$$

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Here, y symbolizes velocity, $\nu > 0$ represents kinematic viscosity (inversely proportional to the Reynolds number $Re = \mathcal{O}(\nu^{-1})$), p denotes pressure, and f represents external forces.

This paper studies numerical analysis and testing of two optimal control methods of (1.1) including Navier Stokes-α (NS-α) and Navier Stokes-ω (NS-ω). Both models belong to the family of Large Eddy Simulation (LES) models and successively predict the larger scales of fluid flow on much coarse meshes.

The first (NS-α) model is also known as the viscous Camassa-Holm (CH) equation having the form

$$\begin{aligned} y_t - \nu\Delta y + (\nabla \times y) \times \overline{y} + \nabla P = f \quad &\text{in } Q, \\ \nabla \cdot \overline{y} = 0 \quad &\text{in } Q, \\ \overline{y} - \alpha^2 \Delta \overline{y} = y \quad &\text{in } Q, \end{aligned} \tag{1.2}$$

where, $\overline{y}$ is the averaged velocity (with filter radius $\alpha > 0$). In addition, the pressure includes some terms that arise in the derivation of this model and has the form $P = p + \overline{y} \cdot y$.

The second model, NS-ω, is a complement of NS-α which is obtained by averaging vorticity ω=$\nabla \times \overline{y}$ in (1.2) and given by

$$\begin{aligned} y_t - \nu\Delta y + (\nabla \times \overline{y}) \times y + \nabla P = f \quad &\text{in } Q, \\ \nabla \cdot y = 0 \quad &\text{in } Q, \\ \overline{y} - \alpha^2 \Delta \overline{y} = y \quad &\text{in } Q. \end{aligned} \tag{1.3}$$

The pioneering work of Chen et al. [1] develops the CH (1.2) by introducing a viscosity term as a closure approximation for the Reynolds-averaged equations of the incompressible NSE. In [2, 3], the analytical solutions obtained with the resulting viscous CH equation are compared with the experimental data, and the validity of this approach was tested for the mean turbulent channel flows and pipe flows. Meanwhile, NS-α models that include the effects of fluctuations on the mean motion have been considered in Chen et al. [4]. While the study [5] involves a review of NS-α model properties, including the filtration and re-derivation of fluid cycle velocity in Kelvin's circulation theorem, Connors [6] performs a complete analysis of convergence of finite element approximations of the NS-α regularization of the NSE. In addition, a numerical scheme based on the physics of (1.2), which preserves both energy and helicity, is also presented in Miles and Rebholz [7]. In a similar fashion for (1.3), as noted in Layton [8], the coefficient ω in (1.3) is effective in calculating the structure and the distribution of turbulence. The resulting algorithm is not only unconditionally stable but also conserves energy and helicity, [9, 10].

A mathematical theory for the continuous NS-ω model is presented in Layton et al. [10], and a high Reynolds fluid-fluid interaction problem is considered by using a new stable NS-ω turbulence model along with the partitioning method [11]. Furthermore, the NS-α and the NS-ω models behave differently in various turbulence situations, and each may be more suitable for specific scenarios due to their distinct properties. Therefore, the model choice depends on the application type and flow conditions. Authors refer to the comparative study of Layton et al. [9] for this purpose.

On the other hand, optimal control plays a vital role in many applications, such as designing more efficient fluid systems, reducing environmental impacts, and lowering costs [12–16]. In the turbulent case, optimal control strategies can be used to reduce or control the adverse effects caused by turbulence.

Optimal control of the NSE involves defining an objective function, simulating the solution of the equations, and determining a control strategy to optimize the objective function. The additional equation obtained for the optimal control problem of the NSE is given below. Optimality conditions for $\lambda \in L^2(0, T; L^2(\Omega))$ are as follows (see e.g., [17]):

$$\begin{aligned} -\lambda_t - \nu\Delta\lambda - (y\cdot\nabla)\lambda + (\nabla y)^T\lambda + \nabla r &= y_d - y && \text{in } Q,\\ \nabla\cdot\lambda &= 0 && \text{in } Q,\\ \lambda &= 0 && \text{on } [0,T]\times\partial\Omega,\\ \lambda(T,x) &= 0 && \text{in } \Omega, \end{aligned} \tag{1.4}$$

and

$$\big(\sigma u - \lambda, \hat{u} - u\big) \geq 0 \qquad \forall \hat{u} \in \mathbf{U^{ad}}, \tag{1.5}$$

where $\partial\Omega$ is the Lipschitz boundary of the domain Ω. $y_d : Q \mapsto \mathbb{R}^2$ is the desired state. Here, u is the control variable, and the admissible space of control constraints is denoted by $\mathbf{U^{ad}} = \{u \in L^2(0, T; L^2(\Omega)^2) : \ u_{a,i} \leq u_i \leq u_{b,i}, i = 1, 2, \ \text{a.e. on } Q\}$.

The usefulness of NS-α and NS-ω models has led to interest in their optimal control problems to improve fluid system performance, efficiency, safety, and environmental effects. The authors believe combining the NS-α and the NS-ω turbulence models with optimal control strategies is one of the initial steps in this direction. Therefore, the novelty of this paper lies in the numerical analysis of the optimal controls of these models and in demonstrating the accuracy of the results through numerical tests.

The outline of this paper is as follows. Differential filters and other notations are presented in Section 2. The optimality system for the examined model is given in Section 3. The adjoint and state equations' stability properties are explained in Section 4, and their error analyses are described in Section 5. Comparative numerical studies supporting the theoretical findings are presented in Section 6, and Section 7 concludes the paper.

2 Mathematical preliminaries

This section provides the necessary preliminary definitions and lemmas for the subsequent sections.

2.1 Problem statement and existence of solution

We consider the following objective functional to be minimized:

$$J(y,u) = \frac{1}{2}\int_Q |y((t,x) - y_d(t,x))|^2 dxdt + \frac{\sigma}{2}\int_Q |u(t,x)|^2 dxdt \quad (2.1)$$

subject to

for the NS-α model:

$$\begin{aligned} y_t - \nu\Delta y + ((\nabla\times y)\times \overline{y}) + \nabla p &= u && \text{in } Q,\\ y(0,x) &= y_0 && \text{in } \Omega,\\ \overline{y} - \alpha^2\Delta\overline{y} &= y && \text{in } Q,\\ \nabla\cdot\overline{y} &= 0 && \text{in } Q,\\ \overline{y} &= 0 && \text{on } (0,T)\times\partial\Omega,\\ u_{a,i} \le u_i &\le u_{b,i} \quad \text{a.e.} && \text{on } Q, \end{aligned} \quad (2.2)$$

for the NS-ω model:

$$\begin{aligned} y_t - \nu\Delta y + ((\nabla\times \overline{y})\times y) + \nabla p &= u && \text{in } Q,\\ y(0,x) &= y_0 && \text{in } \Omega,\\ \overline{y} - \alpha^2\Delta\overline{y} &= y && \text{in } Q,\\ \nabla\cdot\overline{y} &= 0 && \text{in } Q,\\ \overline{y} &= 0 && \text{on } (0,T)\times\partial\Omega,\\ u_{a,i} \le u_i &\le u_{b,i} \quad \text{a.e.} && \text{on } Q, \end{aligned} \quad (2.3)$$

where $\sigma > 0$ is the regularization parameter.

We note that the results on the existence, uniqueness, and regularity of solutions to (2.2) are discussed in [5, 18] along with a comparison for the NSE. The existence and regularity of a global attractor for (2.3) are presented in Layton [8]. Furthermore, the existence and uniqueness of the solutions of the NS-ω model are discussed in detail in Layton et al. [10].

2.2 Notations

This study will utilize Sobolev and Lebesgue spaces, following the notational standards established in Adams [19]. The symbols $\|\cdot\|_{L^p}$ and $\|\cdot\|_m$ represent the norms associated with the spaces $L^p(\Omega)$ and $H^m(\Omega)$, respectively, while $|\cdot|_m$ stands for the conventional semi-norm utilized in the Sobolev space of order m. Additionally, the closure of $C_0^\infty(\Omega)$ within the space $H^m(\Omega)$ is denoted by $H_0^m(\Omega)$. The dual norm of a function is given by

$$\|f\|_{-1} = \sup_{0\neq v\in H^1} \frac{(f,v)}{\|\nabla v\|}.$$

In particular, $(\cdot,\cdot)$ denotes the inner product and $\|\cdot\|$ stands for the norm in the space $L^2(\Omega)$. As usual, $\mathbf{Y} = \left(H_0^1(\Omega)\right)^2$, $M = L_0^2(\Omega)$ and $\mathbf{U} = (L^2(\Omega))^2$ stands for the velocity, the pressure, and the control spaces, respectively. Furthermore, the function space mapping from time interval $(0, T)$ to Q for $p \geq 1$ is given as

$$L^p(0, T; Q) = \left\{ z : [0, T] \to X \text{ measurable: } \int_0^T \|z(t)\|_Q^p \, dt < \infty \right\},$$

along with associated norm

$$\|z(t)\|_{L^p(0,T;Q)}^p = \begin{cases} \left(\int_0^T \|z(t)\|_Q^p \, dt \right)^{1/p}, & \text{if } \; 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t \in (0,T)} \|z(t)\|_Q, & \text{if } \; p = \infty. \end{cases}$$

Skew symmetric trilinear forms are considered for the arising non-linear terms to ensure numerical stability:

$$\begin{aligned} b(y, v, w) &= \frac{1}{2} \left(((y \cdot \nabla)v, w) - ((y \cdot \nabla)w, v) \right), \qquad (2.4) \\ b(y, v, w) &= ((\nabla v)^T w, y), \end{aligned}$$

for all $y, v, w \in \mathbf{Y}$.

The following vector identities will also be used. Let $a, b, c \in X : a(x), b(x), c(x) \in \mathbb{R}^3$ for all $x \in \Omega^d$. Then:

$$\begin{aligned} (\nabla \times a) \times b &= (b \cdot \nabla)a - \nabla(a \cdot b) + (\nabla b)^T a, \\ ((\nabla \times a) \times b, c) &= ((b \cdot \nabla)a, c) - ((c \cdot \nabla)a, b). \end{aligned}$$

2.3 Finite element discretization

To discretize the (2.1)-(2.3) using the finite element method, we employ a collection of triangulations denoted as $\zeta_h = \{K_j\}_{j=1}^M$. Suppose that Ω is partitioned into ζ_h in a quasi-uniform simplex fashion. For each triangle $K_j \in \zeta_h$, the discretization parameter h is given by $h = \max_{K_j \in \zeta_h} h_K$ where h_K is the diameter of K_j. The neighborhood of a node is denoted as $\varsigma_h(x) \in \Omega$ for each node x in the mesh ζ_h and is comprised of all cells K_j with $x \in \partial K_j$. Conforming finite element subspaces $\mathbf{Y}^h \subset \mathbf{Y}$ and $M^h \subset M$ are chosen for velocity and pressure variables fulfilling the discrete inf-sup condition:

$$\inf_{q^h \in M^h} \sup_{v^h \in \mathbf{Y}^h} \frac{(\nabla \cdot v^h, q^h)}{\|\nabla v^h\| \|q^h\|} \geq \beta > 0, \qquad (2.5)$$

where β is independent of the mesh size h. The Taylor-Hood and mini element pairs are known to satisfy (2.5), see, e.g., [20, 21]. Utilizing piecewise polynomials with degree $(k, k-1)$, the well-established approximation properties for the spaces $(\mathbf{Y}^h, M^h)$ can be expressed as follows:

$$\inf_{v^h \in \mathbf{Y}^h} \left(\|(y - v^h)\| + h\|\nabla(y - v^h)\| \right) \leq Ch^{k+1}\|y\|_{k+1} \quad y \in H^{k+1}(\Omega), \quad (2.6)$$
$$\inf_{q^h \in M^h} \|p - q^h\| \leq Ch^k \|p\|_k \quad p \in H^k(\Omega),$$

for $(v^h, q^h) \in (\mathbf{Y}^h, M^h)$. The discrete divergence-free subspace of $\mathbf{Y}^h$ is defined by

$$\mathbf{V}^h = \{v^h \in \mathbf{Y}^h : (\nabla \cdot v^h, q^h) = 0, \forall q^h \in M^h\}.$$

The equivalence between the weak formulations of the NSE in $\mathbf{Y}^h$ and $\mathbf{V}^h$ is established when the inf-sup condition is satisfied.

2.4 Properties of differential filter

Given that the NS-α and NS-ω models incorporate a differential filter, we now state the properties of the filtering process with some preliminary results used throughout the paper.

Definition 2.1 (Continuous differential filter) Let $y \in L^2(\Omega)$. Then the filter of y is the solution $\overline{y} \in \mathbf{Y}$ of

$$\alpha^2(\nabla\overline{y}, \nabla v) + (\overline{y}, v) = (y, v) \qquad \forall v \in \mathbf{Y}.$$

The concept of discrete filtering is defined similarly.

Definition 2.2 (Discrete differential filter) Let $y \in L^2(\Omega)$. Then the discrete filter of y is the solution $\overline{y}^h \in \mathbf{Y_h}$ of

$$\alpha^2(\nabla\overline{y}^h, \nabla v_h) + (\overline{y}^h, v_h) = (y, v_h) \qquad \forall v_h \in \mathbf{Y_h}.$$

Below, we present a series of lemmas that establish useful bounds upheld by the discrete filter.

Lemma 2.1 *(Stability of the discrete differential filter) Let $y \in \mathbf{Y}$, then it holds*

$$\|\overline{y}^h\| \leq \|y\|, \quad \|\nabla\overline{y}^h\| \leq \|\nabla y\|, \quad and \quad \|\nabla \times \overline{y}^h\| \leq \|\nabla y\|.$$

Proof The proof could be found, e.g., in John [20]. □

The continuous differential filter and the discrete differential filter differ in their respective formulations and the spaces in which they operate. The continuous filter is independent of any discretization. The discrete filter relies on a finite element mesh and works with discrete approximations of the continuous functions. The discrete filter introduces discretization errors and requires bounds or error estimates to measure its deviation from the continuous filter. Lemma 2.2 provides such error estimates. These differences highlight that the continuous filter is used for theoretical derivations, while the discrete filter is designed for practical numerical implementations.

Lemma 2.2 *(Error Estimate for the discrete differential filter) Let $y \in \mathbf{Y}$ with $\Delta y \in L^2(\Omega)$, then the error relation*

$$\|y - \overline{y}^h\|^2 + \alpha^2\|\nabla(y - \overline{y}^h)\|^2 \leq C \inf_{v_h \in \mathbf{y_h}} \{\|y - v_h\|^2 + \alpha^2\|\nabla(y - v_h)\|^2\} + C\alpha^4\|\Delta y\|^2,$$

is satisfied for constants C which is independent of α and h. Consequently, one has

$$\|\nabla(y - \overline{y}^h)\| \leq C\alpha^{-1}\left((\alpha + h)\|\nabla y\| + \alpha^2\|\Delta y\|\right),$$
$$\|y - \overline{y}^h\| \leq C\left((\alpha + h)\|\nabla y\| + \alpha^2\|\Delta y\|\right).$$

Proof See John [20] for the proof. □

3 Optimal control

By introducing the optimal control problems of NS-α and NS-ω turbulence models, this section focuses on the development of the optimality conditions for the NS-α and NS-ω models, including the state, auxiliary, and control variables and their role in minimizing the objective function. The optimality conditions of the recommended NS-α and NS-ω models are presented in the following theorem.

Theorem 3.1 *Let $(y, \bar{y}, u)$ be a solution of (2.2)-(2.1). Then, there exist Lagrange multipliers $(\lambda, \bar{\lambda})$ such that*

for NS-α:

$$-\lambda_t - \nu\Delta\lambda - (\overline{y}\cdot\nabla)\,\lambda - (\nabla\overline{y})^T\lambda + \nabla r = y_d - y + \overline{\lambda} \quad \text{in } Q,$$
$$\overline{\lambda} - \alpha^2\Delta\overline{\lambda} = (\lambda\cdot\nabla)y - (\nabla y)^T\lambda \text{ in } Q.$$

for NS-ω:

$$-\lambda_t - \nu\Delta\lambda - (\lambda\cdot\nabla)\,\overline{y} + (\nabla\overline{y})^T\lambda + \nabla r = y_d - y + \overline{\lambda} \quad \text{in } Q,$$
$$\overline{\lambda} - \alpha^2\Delta\overline{\lambda} = (y\cdot\nabla)\lambda + (\nabla y)^T\lambda \text{ in } Q.$$

and

$$\sigma u - \lambda = 0\ \forall\hat{u} \in \mathbf{U^{ad}},$$

where $\lambda \in L^2(0, T; \mathbf{Y})$.

Proof The proof is given in detail in the Appendix. □

The formulations characterized by the adjoint equations in their weak form are derived as follows:

for NS-α:

$$(-\lambda_t, w) + \nu(\nabla\lambda, \nabla w) + (\overline{y}^h \times (\nabla \times \lambda), w) - (R, \nabla \cdot w) = (y_d - y + \overline{\lambda}^h, w),$$
$$(\overline{\lambda}^h, w) + \alpha^2(\nabla\overline{\lambda}^h, \nabla w) = ((\nabla \times y) \times \lambda, w), \quad (3.1)$$

for NS-ω:

$$(-\lambda_t, w) + \nu(\nabla\lambda, \nabla w) + (\lambda \times (\nabla \times \overline{y}^h), w) - (R, \nabla \cdot w) = (y_d - y + \overline{\lambda}^h, w),$$
$$(\overline{\lambda}^h, w) + \alpha^2(\nabla\overline{\lambda}^h, \nabla w) = ((\nabla \times \lambda) \times y, w), \quad (3.2)$$

where $\nabla r = \nabla R + \nabla(\overline{y}^h \cdot \lambda)$.

Remark 3.1 It is worth noting that, to keep the presentation simple, the semi-discretizations of NS-α and NS-ω are considered. Optimal control problems of these models can be easily extended to the fully discrete case.

Now, we state the semi-discrete numerical schemes to be studied.

Algorithm 3.1 Find $y_h \in \mathbf{Y}_h$, $P_h \in M_h$, $u_h \in \mathbf{U_h^{ad}}$ satisfying:

$$\min J(y_h, u_h),$$

subject to: for all $(v_h, q_h) \in (\mathbf{Y_h}, M_h)$,

for the NS-α model:

$$(y_t, v_h) + \nu(\nabla y_h, \nabla v_h) + ((\nabla \times y_h) \times \overline{y_h}^h, v_h) - (P_h, \nabla \cdot v_h) = (u_h, v_h), \quad (3.3)$$
$$(\overline{y_h}^h, v_h) + \alpha^2(\nabla\overline{y_h}^h, \nabla v_h) = (y_h, v_h), \quad (3.4)$$

and for the NS-ω model:

$$(y_t, v_h) + \nu(\nabla y_h, \nabla v_h) + ((\nabla \times \overline{y_h}^h) \times y_h, v_h) - (P_h, \nabla \cdot v_h) = (u_h, v_h), \quad (3.5)$$
$$(\overline{y_h}^h, v_h) + \alpha^2(\nabla\overline{y_h}^h, \nabla v_h) = (y_h, v_h). \quad (3.6)$$

for all $(w_h, m_h) \in (\mathbf{Y_h}, M_h)$,

for the NS-α adjoint model:

$$(-\lambda_t, w_h) + \nu(\nabla\lambda_h, \nabla w_h) + (\overline{y_h}^h \times (\nabla \times \lambda_h), w_h) - (R_h, \nabla \cdot w_h) = (y_d - y_h + \overline{\lambda_h}^h, w_h), \quad (3.7)$$
$$\left(\sigma u_h - \lambda_h, \hat{u} - u_h\right) = 0 \quad (3.8)$$
$$(\overline{\lambda_h}^h, w_h) + \alpha^2(\nabla\overline{\lambda_h}^h, \nabla w_h) = ((\nabla \times y_h) \times \lambda_h, w_h) \quad (3.9)$$

and for the NS-ω adjoint model:

$$(-\lambda_t, w_h) + \nu(\nabla\lambda_h, \nabla w_h) + (\lambda_h \times (\nabla \times \overline{y_h}^h), w_h) - (R_h, \nabla \cdot w_h) = (y_d - y_h + \overline{\lambda_h}^h, w_h) \tag{3.10}$$

$$\big(\sigma u_h - \lambda_h, \hat{u} - u_h\big) = 0 \tag{3.11}$$

$$(\overline{\lambda_h}^h, w_h) + \alpha^2(\nabla\overline{\lambda_h}^h, \nabla w_h) = ((\nabla \times \lambda_h) \times y_h, w_h). \tag{3.12}$$

for all $\hat{u} \in \mathbf{U_{ad}}$.

There are two main computational approaches for solving optimal control problems constrained by partial differential equations: discretize-then-optimize and optimize-then-discretize. The discretize-then-optimize method transforms the problem into a nonlinear programming problem after discretization, while the optimize-then-discretize approach derives optimality conditions in functional spaces before discretization. This paper adopts the latter approach, leveraging the advantages of continuous optimization, such as the ability to make inferences near a point due to the smoothness of the function. For foundational knowledge on PDE-constrained optimal control, including the first and second-order optimality conditions, see references [15, 22–24].

4 Stability analysis

In this section, the stability analysis of the investigated problem is conducted by separately addressing the stability of the state, adjoint, and filter equations. This section derives bounds ensuring that the numerical methods remain stable under the defined assumptions and parameters, which is crucial for guaranteeing reliable solutions over time. To this end, we begin by establishing the stability of the state (3.3)-(3.5) and its corresponding filter (3.4)-(3.6). Subsequently, we proceed to outline the stability conditions pertaining to the adjoint (3.7)-(3.10) and its associated filter (3.9)-(3.12).

Lemma 4.1 *(Stability of state equations) For both NS-α and NS-ω, the equations of state satisfy the following inequalities:*

$$\|\overline{y_h(t)}^h\|^2 + \alpha^2\|\nabla\overline{y_h(t)}^h\|^2 + \nu\int_0^T \left(\|\nabla\overline{y_h(t)}^h\|^2 + \alpha^2\|\Delta\overline{y_h(t)}^h\|^2\right) dt \le C,$$

$$\|y_h(t)\|^2 + \nu\int_0^T \|\nabla y_h(t)\|^2 dt \le C.$$

where $C = C\,(y_h(0), f, \nu)$.

Proof The proof can be found in [6, 9]. □

The stability requirement for the filter related to the state equations corresponds to the statement in Lemma 2.1.

Lemma 4.2 *(Stability of the adjoint equations and its filters) The adjoint (3.7)-(3.10) and their filters (3.9)-(3.12) satisfy the following inequalities: for NS-alpha:*

$$\|\overline{\lambda}^h\| \leq \|\nabla \times y\|_\infty \|\lambda\|, , \tag{4.1}$$

$$\|\nabla\overline{\lambda}^h\| \leq \frac{\alpha^{-1}}{\sqrt{2}} \|\nabla \times y\|_\infty \|\lambda\|, \tag{4.2}$$

$$\|\lambda_h(t)\|^2 + \nu \int_t^T \|\nabla\lambda_h(s)\|^2 ds \leq S_{\lambda_\alpha}, \tag{4.3}$$

where

$$S_{\lambda_\alpha} = \Big(\|\lambda_h(T)\|^2 + C\nu^{-1} \int_t^T \|y_d - y_h\|^2 ds\Big) \exp\Big(C\nu^{-1} \int_t^T \|\nabla \times y_h\|_\infty^2 dt + C\nu^{-3} \int_t^T \|\nabla y_h\|^4 dt\Big),$$

for NS-omega:

$$\|\overline{\lambda}^h\| \leq \|\nabla \times \lambda\|_\infty \|y\|,$$

$$\|\nabla\overline{\lambda}^h\| \leq \frac{\alpha^{-1}}{\sqrt{2}} \|\nabla \times \lambda\|_\infty \|y\|,$$

$$\|\lambda_h(t)\|^2 + \nu \int_t^T \|\nabla\lambda_h(s)\|^2 ds \leq S_{\lambda_\omega}$$

where

$$S_{\lambda_\omega} = \Big(\|\lambda_h(T)\|^2 + C\nu^{-1} \int_t^T \|y_d - y_h\|^2 ds\Big) \exp\Big(C\nu^{-1} \int_t^T \|\nabla \times y_h\|_\infty^2 dt\Big),$$

and C is a constant that does not depend on mesh size h.

Proof The proof is given only for NS-α. Similar procedures can be followed for NS-ω.

To show the inequality in (4.1), $w_h = \overline{\lambda}^h$ is chosen as test function in (3.1). Then

$$\|\overline{\lambda}^h\|^2 + \alpha^2 \|\nabla\overline{\lambda}^h\|^2 = ((\nabla \times y) \times \lambda, \overline{\lambda}^h). \tag{4.4}$$

Applying skew-symmetric inequalities and Young's inequality for the term in the right-hand side of (4.4), one obtains

$$((\nabla \times y) \times \lambda, \overline{\lambda}^h) \leq \|\nabla \times y\|_\infty \|\lambda\| \|\overline{\lambda}^h\| \leq \frac{1}{2} \|\nabla \times y\|_\infty^2 \|\lambda\|^2 + \frac{1}{2} \|\overline{\lambda}^h\|^2.$$

If the resulting constraint is used in the equation, we get

$$\frac{1}{2} \|\overline{\lambda}^h\|^2 + \alpha^2 \|\nabla\overline{\lambda}^h\|^2 \leq \frac{1}{2} \|\nabla \times y\|_\infty^2 \|\lambda\|^2.$$

Since $2\alpha^2\|\nabla\overline{\lambda}^h\|^2 > 0$, the required statement (4.1) is obtained.
By using the discrete Laplacian definition in the second term of (3.1), we get

$$\alpha^2(\nabla\overline{\lambda}^h, \nabla w_h) = -\alpha^2(\Delta\overline{\lambda}^h, w_h).$$

To show the inequality in (4.2), let $w_h = \Delta\overline{\lambda}^h$ be a test function in (3.1). This results in

$$\|\nabla\overline{\lambda}^h\|^2 + \alpha^2\|\Delta\overline{\lambda}^h\|^2 = ((\nabla \times y) \times \lambda, \Delta\overline{\lambda}^h). \tag{4.5}$$

To bound the term in (4.5), one can use (4.6)

$$((\nabla \times y) \times \lambda, \Delta\overline{\lambda}^h) \leq \frac{\alpha^{-2}}{2}\|\nabla \times y\|_\infty^2\|\lambda\|^2 + \frac{\alpha^2}{2}\|\Delta\overline{\lambda}^h\|^2. \tag{4.6}$$

According to this, for (4.5) we get the following

$$\|\nabla\overline{\lambda}^h\|^2 + \frac{\alpha^2}{2}\|\Delta\overline{\lambda}^h\|^2 \leq \frac{\alpha^{-2}}{2}\|\nabla \times y\|_\infty^2\|\lambda\|^2. \tag{4.7}$$

Dropping $\frac{\alpha^2}{2}\|\Delta\overline{\lambda}^h\|^2 > 0$ on the left side (4.7) yields (4.2).
To show the inequality in (4.3), let $w_h = \lambda_h$ and $m_h = r_h$ in (3.7). Hence,

$$-\frac{1}{2}\frac{d}{dt}\|\lambda_h\|^2 + \nu\|\nabla\lambda_h\|^2 = (y_d - y_h, \lambda_h) + (\overline{\lambda_h}^h, \lambda_h) + ((\nabla \times \lambda_h) \times \overline{y_h}^h, \lambda_h). \tag{4.8}$$

The terms on the right-hand side of (4.8) must be bounded. Here, the first term is bounded as below by using the estimate for the duality pairing and Young's inequality:

$$(y_d - y_h, \lambda_h) \leq C\nu^{-1}\|y_d - y_h\|^2 + \frac{\nu}{6}\|\nabla\lambda_h\|^2.$$

If (4.1) is used for the second term (4.8), we get

$$\begin{aligned}(\overline{\lambda_h}^h, \lambda_h) &\leq C\nu^{-1}\|\overline{\lambda_h}^h\|^2 + \frac{\nu}{6}\|\nabla\lambda_h\|^2 \\ &\leq C\nu^{-1}\|\nabla \times y_h\|_\infty^2\|\lambda_h\|^2 + \frac{\nu}{6}\|\nabla\lambda_h\|^2.\end{aligned}$$

Applying skew symmetric inequalities, (2.1), and Young's inequality give

$$\begin{aligned}((\nabla \times \lambda_h) \times \overline{y_h}^h, \lambda_h) &\leq C\|\lambda_h\|^{1/2}\|\nabla\lambda_h\|^{3/2}\|\nabla\overline{y_h}^h\| \\ &\leq C\nu^{-3}\|\lambda_h\|^2\|\nabla y_h\|^4 + \frac{\nu}{6}\|\nabla\lambda_h\|^2.\end{aligned}$$

Arranging,

$$-\frac{1}{2}\frac{d}{dt}\|\lambda_h\|^2+\frac{\nu}{2}\|\nabla\lambda_h\|^2 \le C\left(\nu^{-1}\|\nabla\times y_h\|_\infty^2\|\lambda_h\|^2+\nu^{-1}\|y_d-y_h\|^2+\nu^{-3}\|\lambda_h\|^2\|\nabla y_h\|^4\right).$$

and integrating over $[t, T]$ gives

$$\|\lambda_h(t)\|^2+\nu\int_t^T\|\nabla\lambda_h\|^2dt \le \|\lambda_h(T)\|^2+C\Big(\nu^{-1}\int_t^T\|y_d-y_h\|^2dt+\nu^{-1}\int_t^T\|\nabla\times y_h\|_\infty^2\|\lambda_h\|^2dt$$
$$+\nu^{-3}\int_t^T\|\lambda_h\|^2\|\nabla y_h\|^4dt\Big).$$

Finally, (4.3) is obtained by applying Gronwall's inequality. □

5 Error analysis

This section establishes the convergence rates for the state, adjoint, and control variables, demonstrating how the numerical errors decrease with mesh refinement and validating the theoretical accuracy of the numerical schemes. First, we state the regularity assumptions for (1.1) and (1.4).

Assumption 5.1 Assume that

$$y, \lambda \in L^\infty(0, T; H^1)^2 \cap H^1(0, T; H^{k+1})^2 \cap H^3(0, T; L^2)^2 \cap H^2(0, T; H^1)^2;$$
$$p, r \in L^2(0, T; H^{s+1})^2 \cap H^2(0, T; L^2)^2;$$
$$f \in L^2(0, T; L^2)^2, \text{ and } u \in L^\infty(0, T; H^{k+1})^2.$$

The error estimations of the discrete differential filters of (3.1) and (3.2) are stated in the following lemma.

Lemma 5.1 *(Error Estimates for the discrete differential filters of adjoint) Let $\lambda \in \mathbf{Y}$ with $\Delta\lambda \in L^2(\Omega)$, the following estimations are hold:*
for NS-α:

$$\|\lambda-\overline{\lambda}^h\|^2+\alpha^2\|\nabla(\lambda-\overline{\lambda}^h)\|^2 \le K_{\lambda_\alpha},$$

where

$$K_{\lambda_\alpha}=C\Big(\inf_{v_h\in\mathbf{y_h}}\{\|\nabla(\lambda-v_h)\|^2+\alpha^2\|\nabla(\lambda-v_h)\|^2\}+\|\nabla\lambda\|^2\left(1+\alpha^2+\|\nabla\times y\|_\infty^2\right)\Big),$$

for NS-ω:

$$\|\lambda-\overline{\lambda}^h\|^2+\alpha^2\|\nabla(\lambda-\overline{\lambda}^h)\|^2 \le K_{\lambda_\omega},$$

where

$$K_{\lambda_\omega}=C\Big(\inf_{v_h\in\mathbf{Y_h}}\{\|\nabla(\lambda-v_h)\|^2+\alpha^2\|\nabla(\lambda-v_h)\|^2\}+(1+\alpha^2)\|\nabla\lambda\|^2+\|\nabla\times\lambda\|_\infty^2\|\nabla y\|^2\Big),$$

where C does not depend on α and h.

Proof The proof of lemma is presented for NS-α only. Similar procedures can be followed for NS-ω. According to the regularity assumption, one has

$$(\lambda, w_h)+\alpha^2(\nabla\lambda,\nabla w_h)=(\lambda, w_h)-\alpha^2(\Delta\lambda, w_h). \tag{5.1}$$

Subtracting (3.1) from (5.1) yields

$$(e, w_h)+\alpha^2(\nabla e,\nabla w_h)=(\lambda, w_h)-\alpha^2(\Delta\lambda, w_h)-((\nabla\times y)\times\lambda, w_h)\,, \tag{5.2}$$

where $e=\lambda-\overline{\lambda}^h$. By decomposing the error e in (5.2), $e=\lambda-\tilde{\lambda}-(\overline{\lambda}^h-\tilde{\lambda})=\eta-\phi_h$, where $\tilde{\lambda}$ is the best approximation and ϕ_h is chosen as the test function, we get:

$$\|\phi_h\|^2+\alpha^2\|\nabla\phi_h\|^2=-(\lambda,\phi_h)+\alpha^2(\Delta\lambda,\phi_h)+\alpha^2(\nabla\eta,\nabla\phi_h)+(\eta,\phi_h)+\big((\nabla\times y)\times\lambda,\phi_h\big)\,. \tag{5.3}$$

The terms in the right hand side of (5.3) are estimated as:

$$\begin{aligned}
(\lambda,\phi_h)&\le\frac{3}{2}\|\lambda\|^2+\frac{1}{6}\|\phi_h\|^2,\\
\alpha^2(\nabla\lambda,\nabla\phi_h)&\le\alpha^2\|\nabla\lambda\|^2+\frac{\alpha^2}{4}\|\nabla\phi_h\|^2,\\
\alpha^2(\nabla\eta,\nabla\phi_h)&\le\alpha^2\|\nabla\eta\|^2+\frac{\alpha^2}{4}\|\nabla\phi_h\|^2,\\
(\eta,\phi_h)&\le\frac{3}{2}\|\eta\|^2+\frac{1}{6}\|\phi_h\|^2,\\
((\nabla\times y)\times\lambda,\phi_h)&\le\frac{3}{2}\|\nabla\times y\|_\infty^2\|\lambda\|^2+\frac{1}{6}\|\phi_h\|^2.
\end{aligned}$$

By using above estimations, (5.3) can be written as

$$\frac{1}{2}\|\phi^h\|^2+\frac{\alpha^2}{2}\|\nabla\phi^h\|^2\le C\Big(\|\lambda\|^2+\alpha^2\|\nabla\lambda\|^2+\alpha^2\|\nabla\eta\|^2+\|\eta\|^2+\|\nabla\times y\|_\infty^2\|\lambda\|^2\Big),$$

which completes the proof. □

The auxiliary problem, which will be used in the proof of error analysis, is given by: for $u\in\mathbf{U^{ad}}$, find $y_h(u),\lambda_h(u)\in\mathbf{Y}$.

For the NS-α model:

$$\begin{aligned}(y_t(u), v_h) + \nu(\nabla y_h(u), \nabla v_h) + ((\nabla \times y_h(u)) \times \overline{y_h(u)}^h, v_h) - (P_h, \nabla \cdot v_h) = (u, v_h),\\ (\nabla \cdot \overline{y_h(u)}^h, q_h) = 0,\end{aligned} \tag{5.4}$$

and for the NS-ω model:

$$\begin{aligned}(y_t(u), v_h) + \nu(\nabla y_h(u), \nabla v_h) + ((\nabla \times \overline{y_h(u)}^h) \times y_h(u), v_h) - (P_h, \nabla \cdot v_h) = (u, v_h),\\ (\nabla \cdot y_h(u), q_h) = 0,\end{aligned}$$

for all $(v^h, q^h) \in (\mathbf{Y^h}, M^h)$. Similarly, the adjoint models in each case become the NS-α adjoint model:

$$\begin{aligned}&(-\lambda_t(u), w_h) + \nu(\nabla\lambda_h(u), \nabla w_h) + (\overline{y_h(u)}^h \times (\nabla \times \lambda_h(u)), w_h) - (R_h, \nabla \cdot w_h)\\ &= (y_d - y_h(u) + \overline{\lambda_h(u)}^h, w_h),\\ &(\overline{\lambda_h(u)}^h, w_h) + \alpha^2(\nabla\overline{\lambda_h(u)}^h, \nabla w_h) = ((\nabla \times y_h(u)) \times \lambda_h(u), w_h),\end{aligned} \tag{5.5}$$

and for the NS-ω adjoint case:

$$\begin{aligned}&(-\lambda_t(u), w_h) + \nu(\nabla\lambda_h(u), \nabla w_h) + (\lambda_h(u) \times (\nabla \times \overline{y_h(u)}^h), w_h) - (R_h, \nabla \cdot w_h)\\ &= (y_d - y_h(u) + \overline{\lambda_h(u)}^h, w_h),\\ &(\overline{\lambda_h(u)}^h, w_h) + \alpha^2(\nabla\overline{\lambda_h(u)}^h, \nabla w_h) = ((\nabla \times \lambda_h(u)) \times y_h(u), w_h),\end{aligned}$$

for all $(w^h, m^h) \in (\mathbf{Y^h}, M^h)$.

Remark 5.1 Since both models yield similar results, the NS-α model is preferred for simplicity in the proofs; similar results can also be obtained for the NS-ω model. We conduct the error analysis by evaluating discrepancies between the equations at each step: Step 1 (1.1) vs. (5.4), Step 2 (1.4) vs. (5.5), Step 3 (3.3) vs. (5.4), and Step 4 (3.7) vs. (5.5), respectively.

Lemma 5.2 *(Step 1) Let y and $y_h(u)$ be solutions of (1.1) and (5.4). Then, there exists a positive constant C such that the following bound holds for the error $y - y_h(u)$*

$$\|y - y_h(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla(y - y_h(u))\|^2_{L^2(0,T;L^2(\Omega))} \leq E_y,$$

where

$$\begin{aligned}E_y = \Big[&\|y - y_h(0)\|^2_{L^\infty(0,T;L^2(\Omega))} + C\inf\Big\{\int_0^T \Big(\nu^{-1}\|(y-\tilde{y})_t\|^2 + \nu\|\nabla(y-\tilde{y})\|^2 + \nu^{-1}\|p - P_h\|^2\\ &+\nu^{-1}\|\nabla(y-\tilde{y})\|^2\|\nabla y\|^2 + \nu^{-1}\|\nabla(y-\tilde{y})\|^2\|\nabla y_h(u)\|^2 + \nu^{-1}\alpha^4\|\nabla y(u)\|^2_\infty|y|^2_2\Big)dt\Big\}\Big]\\ &\exp\left(C\nu^{-3}\|\nabla y\|^4\right).\end{aligned}$$

Proof The proof of this lemma is provided in detail in the Appendix. □

Lemma 5.3 *(Step 2) Let λ and $\lambda_h(u)$ be solutions of (1.4) and (5.5), respectively. Then, we have*

$$\|\lambda - \lambda_h(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla(\lambda - \lambda_h(u))\|^2_{L^2(0,T;L^2(\Omega))} \leq E_\lambda,$$

where

$$E_\lambda = \|\lambda - \lambda_h(0)\|^2_{L^\infty(0,T;L^2(\Omega))} + C\Big[\inf_{\tilde{\lambda}\in Y_h}\int_0^T \Big\{\nu^{-1}\|\nabla y\|^2\|\nabla\lambda\|^2 + \nu^{-1}\|\nabla y_h(u)\|^2\|\nabla\lambda_h(u)\|^2$$
$$+\nu^{-1}\|r - R_h\|^2 + \nu\|\nabla(\lambda - \tilde{\lambda})\|^2 + \nu^{-1}\|(\lambda - \tilde{\lambda})_t\|^2 + \nu^{-1}E_y + \nu^{-1}\|\nabla\times y_h(u)\|^2_\infty\|\lambda_h(u)\|^2\Big\}dt\Big].$$

Proof The error equation is obtained by subtracting (1.4) from (5.5).

$$(e_t, w_h) - \nu(\nabla e, \nabla w_h) + b(y, \lambda, w_h) - b(w_h, y, \lambda) + (\overline{y_h(u)}^h \times (\nabla\times\lambda_h(u)), w_h)$$
$$+(r - R_h, \nabla\cdot w_h) = (y - y_h(u), w_h) + (\overline{\lambda_h(u)}^h, w_h), \tag{5.6}$$

where $e = \lambda - \lambda_h(u)$. By decomposing the error e in (5.6) as $e = \lambda - \tilde{\lambda} - (\lambda_h(u) - \tilde{\lambda}) = \eta - \phi_h$ with the best approximation of λ and choosing the test function $w_h = \phi_h$, the following equation is obtained:

$$-\frac{1}{2}\frac{d}{dt}\|\phi_h(t)\|^2 + \nu\|\nabla\phi_h\|^2$$
$$= -b(y, \lambda, \phi_h) + b(\phi_h, y, \lambda) - (\overline{y_h(u)}^h \times (\nabla\times\lambda_h(u)), \phi_h) - (r - R_h, \nabla\cdot\phi_h)$$
$$+\nu(\nabla\eta, \nabla\phi_h) - (\eta_t, \phi_h) + (y - y_h(u), \phi_h) + (\overline{\lambda_h(u)}^h, \phi_h). \tag{5.7}$$

The terms to the right of (5.7) are bounded as:

$$b(y, \phi_h, \lambda) \leq C\nu^{-1}\|\nabla y\|^2\|\nabla\lambda\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2,$$

$$b(\phi_h, y, \lambda) \leq C\nu^{-1}\|\nabla y\|^2\|\nabla\lambda\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2,$$

$$(\overline{y_h(u)}^h \times (\nabla\times\lambda_h(u)), \phi_h) \leq C\|\nabla y_h(u)\|\|\nabla\lambda_h(u)\|\|\nabla\phi_h\|$$
$$\leq C\nu^{-1}\|\nabla y_h(u)\|^2\|\nabla\lambda_h(u)\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2$$

$$(r - R_h, \nabla\cdot\phi_h) \leq C\nu^{-1}\|r - R_h\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2,$$

$$\nu(\nabla\eta, \nabla\phi_h) \leq C\nu\|\nabla\eta\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2,$$

$$(\eta_t, \phi_h) \le C\nu^{-1}\|\eta_t\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2,$$

$$(y - y_h(u), \phi_h) \le C\nu^{-1}\|y - y_h(u)\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2,$$

$$\begin{aligned}(\overline{\lambda_h(u)}^h, \phi_h) &\le C\nu^{-1}\|\overline{\lambda_h(u)}^h\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2 \\ &\le C\nu^{-1}\|\nabla \times y_h(u)\|_\infty^2\|\lambda_h(u)\|^2 + \frac{\nu}{16}\|\nabla\phi_h\|^2.\end{aligned}$$

Inserting all bounds in (5.7) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\phi_h(t)\|^2 + \frac{\nu}{2}\|\nabla\phi_h\|^2 \\ &\le C\Big(\nu^{-1}\|\nabla y\|^2\|\nabla\lambda\|^2 + \nu^{-1}\|\nabla y_h(u)\|^2\|\nabla\lambda_h(u)\|^2 + \nu^{-1}\|r - R_h\|^2 + \nu\|\nabla\eta\|^2 + \nu^{-1}\|\eta_t\|^2 \\ &\quad + \nu^{-1}\|y - y_h(u)\|^2 + \nu^{-1}\|\nabla \times y_h(u)\|_\infty^2\|\lambda_h(u)\|^2\Big).\end{aligned}$$

Application of Lemma 5.2 gives

$$\begin{aligned}&\frac{d}{dt}\|\phi_h(t)\|^2 + \nu\|\nabla\phi_h\|^2 \\ &\le C\Big(\nu^{-1}\|\nabla y\|^2\|\nabla\lambda\|^2 + \nu^{-1}\|\nabla y_h(u)\|^2\|\nabla\lambda_h(u)\|^2 + \nu^{-1}\|r - R_h\|^2 + \nu\|\nabla\eta\|^2 + \nu^{-1}\|\eta_t\|^2 \\ &\quad + \nu^{-1}E_y + \nu^{-1}\|\nabla \times y_h(u)\|_\infty^2\|\lambda_h(u)\|^2\Big).\end{aligned}$$

The result of Lemma 5.3 is obtained along with the approximation properties and the triangle inequality. □

Lemma 5.4 *(Step 3) Let y_h and $y_h(u)$ be the solutions of (3.3) and (5.4), respectively. Then, the following bound holds:*

$$\begin{aligned}&\|y_h - y_h(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla(y_h - y_h(u))\|^2_{L^2(0,T;L^2(\Omega))} \\ &\le C\left[\|y_0 - y_h(0)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu^{-1}\int_0^T \|u - u_h\|^2 dt\right]\exp(C\nu^{-3}(\|\nabla y_h(u)\|^4 + \|\nabla y_h\|^4).\end{aligned}$$

Proof If the auxiliary (5.4) is subtracted from the discrete (3.3) and $v_h = y_h - y_h(u)$ is taken one gets

$$\begin{aligned}&((y_h - y_h(u))_t, y_h - y_h(u)) + \nu\,(\nabla(y_h - y_h(u)), \nabla(y_h - y_h(u))) - (\overline{y_h}^h \times (\nabla \times y_h), y_h - y_h(u)) \\ &+ (\overline{y_h(u)}^h \times (\nabla \times y_h(u)), y_h - y_h(u)) = (u_h - u, y_h - y_h(u)).\end{aligned}$$

Arranging the nonlinear terms here, we get

$$
\begin{aligned}
&-(\overline{y_h}^h \times (\nabla \times y_h), y_h - y_h(u)) + (\overline{y_h(u)}^h \times (\nabla \times y_h(u)), y_h - y_h(u)) \\
&= -\Big(\overline{y_h}^h \times (\nabla \times (y_h - y_h(u))), y_h - y_h(u)\Big) - \Big((\overline{y_h}^h - \overline{y_h(u)}^h) \times (\nabla \times y_h(u)), y_h - y_h(u)\Big)
\end{aligned}
$$

The terms on the right-hand side of the last equations are estimated as follows:

$$
(u_h - u, y_h - y_h(u)) \leq C\nu^{-1}\|u - u_h\|^2 + \frac{\nu}{6}\|\nabla(y_h - y_h(u))\|^2,
$$

$$
\begin{aligned}
&\Big((\overline{y_h}^h - \overline{y_h(u)}^h) \times (\nabla \times y_h(u)), y_h - y_h(u)\Big) \\
&\leq C\|\nabla y_h(u)\|\|\nabla(\overline{y_h}^h - \overline{y_h(u)}^h)\|\|\nabla(y_h - y_h(u))\|^{1/2}\|y_h - y_h(u)\|^{1/2} \\
&\leq C\|\nabla y_h(u)\|\|\nabla(y_h - y_h(u))\|^{3/2}\|y_h - y_h(u)\|^{1/2} \\
&\leq C\nu^{-3}\|\nabla y_h(u)\|^4\|y_h - y_h(u)\|^2 + \frac{\nu}{6}\|\nabla(y_h - y_h(u))\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\Big(\overline{y_h}^h \times (\nabla \times (y_h - y_h(u))), y_h - y_h(u)\Big) &\leq \|\nabla\overline{y_h}^h\|\|\nabla(y_h - y_h(u))\|^{3/2}\|y_h - y_h(u)\|^{1/2} \\
&\leq C\nu^{-3}\|\nabla y_h\|^4\|y_h - y_h(u)\|^2 + \frac{\nu}{6}\|\nabla(y_h - y_h(u))\|^2.
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
&\frac{d}{dt}\|y_h - y_h(u)\|^2 + \nu\|\nabla(y_h - y_h(u))\|^2 \\
&\leq C\nu^{-1}\Big(\|u - u_h\|^2 + \nu^{-2}\|\nabla y_h(u)\|^4\|y_h - y_h(u)\|^2 + \nu^{-2}\|\nabla y_h\|^4\|y_h - y_h(u)\|^2\Big).
\end{aligned}
$$

and integrating over $[0, T]$ yields in

$$
\begin{aligned}
&\|y_h - y_h(u)\|^2 + \nu\int_0^T \|\nabla(y_h - y_h(u))\|^2 dt \\
&\leq \|y_0 - y_h(0)\|^2 + C\nu^{-1}\int_0^T \Big(\|u - u_h\|^2 + \nu^{-2}\|\nabla y_h(u)\|^4\|y_h - y_h(u)\|^2 + \nu^{-2}\|\nabla y_h\|^4\|y_h - y_h(u)\|^2\Big)dt.
\end{aligned}
$$

Finally, the desired result is obtained by using Gronwall's inequality. □

Lemma 5.5 *(Step 4) Let λ_h and $\lambda_h(u)$ be the solutions of (3.7) and (5.5), respectively. Then, the following a priori bound holds:*

$$
\begin{aligned}
&\|\lambda_h - \lambda_h(u)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\|\nabla(\lambda_h - \lambda_h(u))\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq C\Big[\|\lambda_0^h - \lambda_h(0)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu^{-1}\int_0^T \Big(\|\nabla \times \lambda_h\|^2_\infty\|y_h - y_h(u)\|^2 + \|y_h - y_h(u)\|^2 + K_\lambda\Big)dt\Big] \\
&\times \exp(C\nu^{-1} + C\nu^{-3}\|y_h(u)\|^4).
\end{aligned}
$$

Proof If the discrete (3.7) is subtracted from the auxiliary (5.5), with the choice of $v_h = \lambda_h - \lambda_h(u)$, one gets

$$((\lambda_h - \lambda_h(u))_t, \lambda_h - \lambda_h(u)) - \nu(\nabla(\lambda_h - \lambda_h(u)), \nabla(\lambda_h - \lambda_h(u))) + (\overline{y_h(u)}^h \times (\nabla \times \lambda_h(u)), -(\overline{y_h}^h \times (\nabla \times \lambda_h), \lambda_h - \lambda_h(u)) = (y_h - y_h(u), \lambda_h - \lambda_h(u)) - (\overline{\lambda_h}^h - \overline{\lambda_h(u)}^h, \lambda_h - \lambda_h(u)). \tag{5.8}$$

Arranging the nonlinear terms in (5.8) gives

$$\begin{aligned}&(\overline{y_h(u)}^h \times (\nabla \times \lambda_h(u)), \lambda_h - \lambda_h(u)) - (\overline{y_h}^h \times (\nabla \times \lambda_h), \lambda_h - \lambda_h(u)) \\ &= -\left((\overline{y_h}^h - \overline{y_h(u)}^h) \times (\nabla \times \lambda_h), \lambda_h - \lambda_h(u)\right) - \left(\overline{y_h(u)}^h \times (\nabla \times (\lambda_h - \lambda_h(u))), \lambda_h - \lambda_h(u)\right).\end{aligned}$$

Also,

$$\begin{aligned}&\left(\overline{\lambda_h}^h - \overline{\lambda_h(u)}^h, \lambda_h - \lambda_h(u)\right) \\ &= \left(\overline{\lambda_h}^h - \overline{\lambda_h(u)}^h + \lambda_h - \lambda_h + \lambda_h(u) - \lambda_h(u), \lambda_h - \lambda_h(u)\right) \\ &= (\lambda_h(u) - \overline{\lambda_h(u)}^h, \lambda_h - \lambda_h(u)) - (\lambda_h - \overline{\lambda_h}^h, \lambda_h - \lambda_h(u)) + (\lambda_h - \lambda_h(u), \lambda_h - \lambda_h(u)).\end{aligned} \tag{5.9}$$

The terms on the right of (5.9) are bounded as follows:

$$\begin{aligned}&\left((\overline{y_h}^h - \overline{y_h(u)}^h) \times (\nabla \times \lambda_h), \lambda_h - \lambda_h(u)\right) \\ &\leq C\nu^{-1}\|\nabla \times \lambda_h\|_\infty^2 \|\overline{y_h}^h - \overline{y_h(u)}^h\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2 \\ &\leq C\nu^{-1}\|\nabla \times \lambda_h\|_\infty^2 \|y_h - y_h(u)\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2,\end{aligned}$$

$$\left(\overline{y_h(u)}^h \times (\nabla \times (\lambda_h - \lambda_h(u))), \lambda_h - \lambda_h(u)\right) \leq C\nu^{-3}\|y_h(u)\|^4\|\lambda_h - \lambda_h(u)\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2$$

$$(y_h - y_h(u), \lambda_h - \lambda_h(u)) \leq C\nu^{-1}\|y_h - y_h(u)\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2,$$

$$(\lambda_h - \overline{\lambda_h}^h, \lambda_h - \lambda_h(u)) \leq C\nu^{-1}\|\lambda_h - \overline{\lambda_h}^h\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2,$$

$$(\lambda_h(u) - \overline{\lambda_h(u)}^h, \lambda_h - \lambda_h(u)) \leq C\nu^{-1}\|\lambda_h(u) - \overline{\lambda_h(u)}^h\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2,$$

$$(\lambda_h - \lambda_h(u), \lambda_h - \lambda_h(u)) \le C\nu^{-1}\|\lambda_h - \lambda_h(u)\|^2 + \frac{\nu}{12}\|\nabla(\lambda_h - \lambda_h(u))\|^2.$$

Using above estimations in (5.9) and Lemma 5.1 for filtered terms yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\lambda_h - \lambda_h(u)\|^2 + \frac{\nu}{2}\|\nabla(\lambda_h - \lambda_h(u))\|^2 \\
&\le C\nu^{-1}\Big(\|\nabla \times \lambda_h\|_\infty^2\|y_h - y_h(u)\|^2 + \nu^{-2}\|y_h(u)\|^4\|\lambda_h - \lambda_h(u)\|^2 + \|y_h - y_h(u)\|^2 \\
&+\|\lambda_h - \overline{\lambda_h}^h\|^2 + \|\lambda_h(u) - \overline{\lambda_h(u)}^h\|^2 + \|\lambda_h - \lambda_h(u)\|^2\Big) \\
&\le C\nu^{-1}\Big(\|\nabla \times \lambda_h\|_\infty^2\|y_h - y_h(u)\|^2 + \nu^{-2}\|y_h(u)\|^4\|\lambda_h - \lambda_h(u)\|^2 + \|y_h - y_h(u)\|^2 \\
&+K_{\lambda_\alpha} + \|\lambda_h - \lambda_h(u)\|^2\Big).
\end{aligned}$$

Integrating over $[0, T]$ gives

$$\begin{aligned}
&\|\lambda_h - \lambda_h(u)\|^2 + \nu\int_0^T \|\nabla(\lambda_h - \lambda_h(u))\|^2 dt \\
&\le C\nu^{-1}\int_0^T\Big(\|\nabla\times\lambda_h\|_\infty^2\|y_h - y_h(u)\|^2 + \nu^{-2}\|y_h(u)\|^4\|\lambda_h - \lambda_h(u)\|^2 + \|y_h - y_h(u)\|^2 \\
&+K_{\lambda_\alpha} + \|\lambda_h - \lambda_h(u)\|^2\Big)dt,
\end{aligned}$$

and by using Gronwall's inequality, we get the required result. □

Lemma 5.6 *Let (y, λ, u) and (y_h, λ_h, u_h) be the solutions of (1.1)-(1.4)-(1.5) and (3.3)-(3.7)-(3.1), respectively. Then, one has*

$$\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} \le C\left(\|\lambda - \lambda_h(u)\|^2_{L^2(0,T;L^2(\Omega))} + h^{2l+2}\right),$$

where l denotes the polynomial degree in the approximation properties of the control.

Proof The approach employed in Hacat et al. [25] applies equally to the NS-α and NS-ω models. □

We now state our main error estimation theorem.

Theorem 5.1 *Let (y, λ, u) be the solutions of (1.1)-(1.4)-(1.5) and (y^h, λ^h, u^h) be the solutions of (3.3)-(3.7)-(3.1), respectively. Then, the following estimate holds*

$$\|y - y_h\|^2 + \|\lambda - \lambda_h\|^2 + \|u - u_h\|^2 \le C\left(h^{2k} + (\alpha + h)\alpha^2 + h^{2l+2}\right).$$

Proof Through the utilization of the auxiliary problem, it is possible to formulate

$$\|y - y^h\|^2 \leq \|y - y^h(u)\|^2 + \|y^h(u) - y^h\|^2.$$

The results of the inequalities in Steps 1 and 3 lead to

$$\|y - y^h\|^2 \leq C\left(E_y + \|u - u^h\|^2\right).$$

Similarly, for adjoint error, we apply the triangle with the use of Step 2 and Step 4

$$\|\lambda - \lambda^h\|^2 \leq C\left(E_\lambda + \|u - u^h\|^2\right).$$

By combining the errors, the following result is obtained:

$$\|y - y^h\|^2 + \|\lambda - \lambda^h\|^2 + \|u - u^h\|^2 \leq C\left(E_y + E_\lambda + \|u - u^h\|^2\right).$$

Applying Lemma 5.6 and the approximation estimate (2.6) results in the statement of the main theorem. □

6 Numerical experiments

In this section, we conduct numerical experiments to demonstrate the effectiveness of the proposed method and validate some of the theoretical predictions outlined in the previous sections. For these tests, a public domain finite element software, *FreeFem++*, is used; see Hecht [26]. We consider the inf-sup stable finite element pair, Taylor-Hood finite elements, on a regular triangulation of the computational domains. Newton's method is considered to deal with non-linearities, and the second-order Crank-Nicolson method is chosen for temporal discretization. Algorithm 3.1, which consists of the optimality conditions of the discrete control-constrained problem, is solved by the primal-dual active set algorithm as a semi-smooth Newton step, as described in Bergounioux [27]. The algorithm stops at a feasible and optimal solution.

6.1 Convergence Study

The following manufactured solutions of Wachsmuth [14] are used:

$$y(t,x) = e^{-\nu t}\begin{bmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{bmatrix},$$

$$\lambda(t,x) = \left(-e^{\nu(\tau-T)} + e^{-\nu}\right)\begin{bmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{bmatrix},$$

Table 1 Errors and rates of convergence for NS-α

h	$\|y - y^h\|$	Rate	$\|\lambda - \lambda^h\|$	Rate
2^{-1}	0.0014818	–	1.38351e–7	–
2^{-2}	2.87621e–4	2.37	8.3041e–8	0.74
2^{-3}	3.25096e–5	3.15	9.25913e–9	3.17
2^{-4}	3.81827e–6	3.09	9.18238e–10	3.33
2^{-5}	5.13114e–7	2.92	1.2709e–10	2.86
2^{-6}	6.53542e–8	2.98	4.62331e–11	2.98

so that the right-hand side functions and the desired state are computed with respect to

$$\begin{aligned} y_d(t,x) &= y + \left(-\lambda_t - \nu\Delta\lambda - (y\cdot\nabla)\,\lambda + (\nabla y)^T\,\lambda\right), \\ y_0 &= y(0), \\ f &= y_t - \nu\Delta y + (y\cdot\nabla)\,y - u, \end{aligned}$$

in which $u = \min(b, \max(a, -\lambda))$ is used as the control variable and choose $a = 0$, and $b = 1$, specific to this test case. This form for u is chosen to ensure the control variable remains within the admissible bounds $a \le u \le b$. This formulation balances optimization effectiveness and adherence to problem constraints, ensuring a feasible and stable control process. As the Reynolds number, Re, increases, the flow's viscous effects decrease, and the flow exhibits more complex dynamics. Therefore, we choose $Re = 10^4$ to investigate the results in a more challenging scenario. A relatively small time step size, $\Delta t = 10^{-4}$, is selected to isolate spatial error and calculate the errors for different spatial mesh sizes. Table 1 presents the results obtained for the NS-α model, while Table 2 presents the results for the NS-ω model. Third-order convergence (for $k = 2$) is achieved for the state and adjoint variables, as predicted by the theory in Section 5. Furthermore, second-order convergence (for $l = 0$) is obtained for the control variable Hacat et al. [25].

Optimal control problems aim to optimize the NSE under a given performance criterion (cost function). In NSE optimal control problems, the cost function measures the performance of a given fluid dynamics problem in achieving specific objectives. This cost function usually reflects the effort to achieve a desired system behavior under particular control strategies. A good outcome usually refers to the case where the cost

Table 2 Errors and rates of convergence for NS-ω

h	$\|y - y^h\|$	Rate	$\|\lambda - \lambda^h\|$	Rate
2^{-1}	0.0014818	–	1.38352e–7	–
2^{-2}	2.87622e–4	2.37	8.30412e–8	0.74
2^{-3}	3.25097e–5	3.15	9.25943e–9	3.17
2^{-4}	3.81835e–6	3.09	9.18261e–10	3.33
2^{-5}	5.13147e–7	2.92	1.27023e–10	2.86
2^{-6}	6.5368e–8	2.98	4.59479e–11	2.98

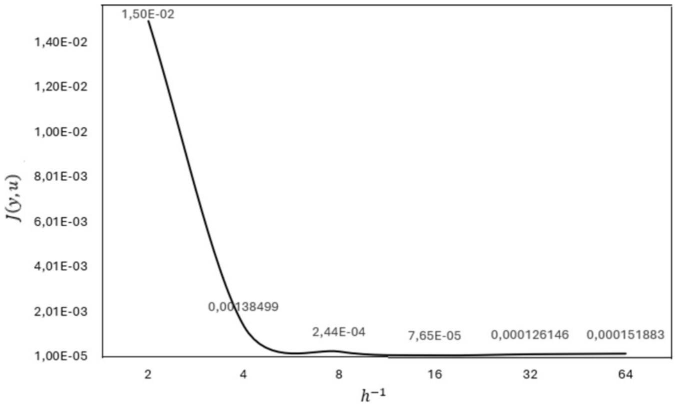

Fig. 1 Variation of cost function according to mesh size, for $\sigma = 10$

function is minimized. Figure 1 shows the behavior of the cost functional with respect to the mesh size h. As h becomes finer, the cost function tends to zero as expected. This shows the success of the optimal control strategy applied here.

6.2 Flow around a cylinder

A well-known benchmark problem for testing codes of flow problems, the two-dimensional flow around a cylinder test [28], is considered here to assess the effect of optimal control. A detailed presentation of this test problem without consideration of optimal control can be found in John [29]. Some numerical instabilities are expected in this flow setup for smaller viscosity values due to fluid interaction with obstacles.

We expect our optimal control strategy to regularize the flow characteristics according to y_d and to obtain a better approximation for the flow patterns. The computational domain is a 4×1 rectangle with a cylinder obstacle close to the inflow boundary. We study with different values of ν (1 and 2.10^{-3}) in the time interval $[0, 1]$ to better understand the effect of the optimal control for varying Re.

The boundary conditions are chosen to be no-slip on the horizontal walls of the domain and around the cylinder, while the vertical inlet and outlet boundary conditions are parabolic in the x-direction and are specified precisely as follows:

$$y = \begin{pmatrix} 4x_2(1 - x_2) \\ 0 \end{pmatrix}.$$

While the desired velocity is taken to be the solution of steady Stokes flow for the same domain, other variables are chosen as $a = 0$, $b = 0.5$, $\sigma = 10$, $\Delta t = 0.1$, $\alpha = 0.125$.

Controlled flows typically outperform uncontrolled solutions in all flow parameters and closely mimic y_d. It should be noted that, due to the nature of the problem, neither the controlled nor uncontrolled solutions can exactly match y_d. Figure 2 shows the speed contours for y_d, the controlled and uncontrolled cases. Since $Re = 1$ in this case, no significant difference could be observed between the controlled and uncontrolled

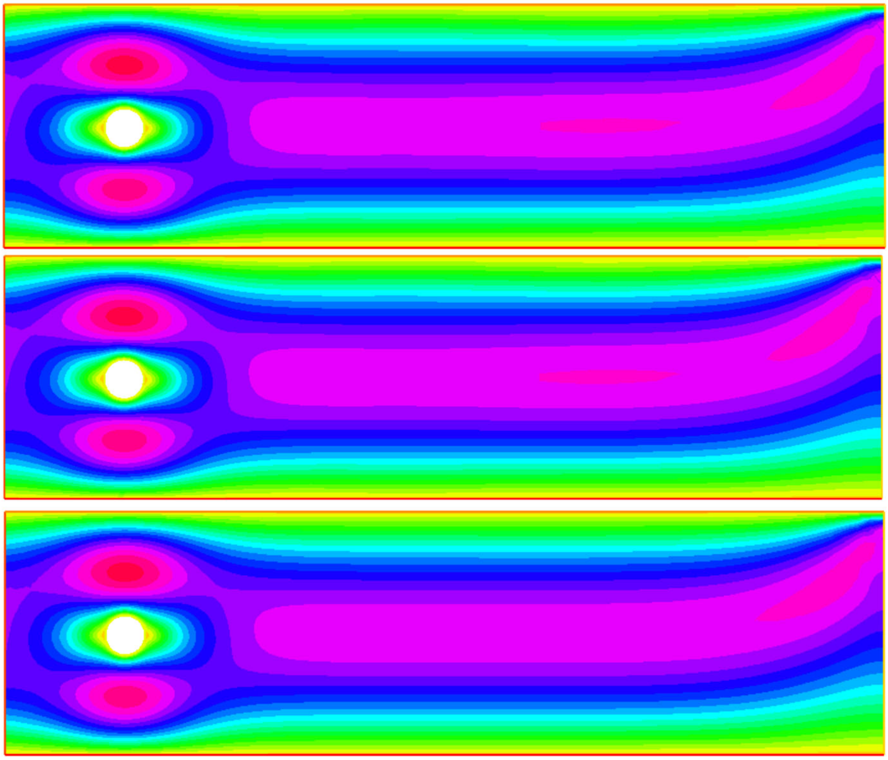

Fig. 2 Velocity profiles for uncontrolled, controlled, and desired states from top to bottom, respectively for $Re = 1$

states, even with y_d, which is expected due to the flow's higher viscosity and laminar behavior.

We expect the real difference between controlled and uncontrolled states for higher Reynolds numbers. Figure 3 presents the speed contours for the y_d, controlled and uncontrolled case for $Re = 2000$. As it could be easily deduced at a glance, the controlled flow is far more similar to y_d than the uncontrolled case. The fluctuations seen in the uncontrolled state are reduced, and the flow behavior is close to the desired state. The controlled results are closer to the desired state because the optimal control strategy minimizes the difference between the flow and desired states as defined by the objective function. While the controlled results show improvement, the observed deviation from the desired state reflects the difficulties of achieving precise control in complex turbulent flows, even with advanced optimization techniques. Thus, the effect of optimal control on the flow patterns is observed, which verifies the effectiveness of our schemes.

As a last effectiveness indicator, we show the energy dissipation rate, $(\nu\|\nabla y\|^2)$, statistics for the same flow, see Fig. 4. In particular, considering the turbulent nature of the flow, it is essential to control and optimize energy dissipation. In [30], energy dissipation rates were determined by establishing links between the reduced NS-α (rNS-α) model and some well-studied models. Tests have shown that this ratio exhibits

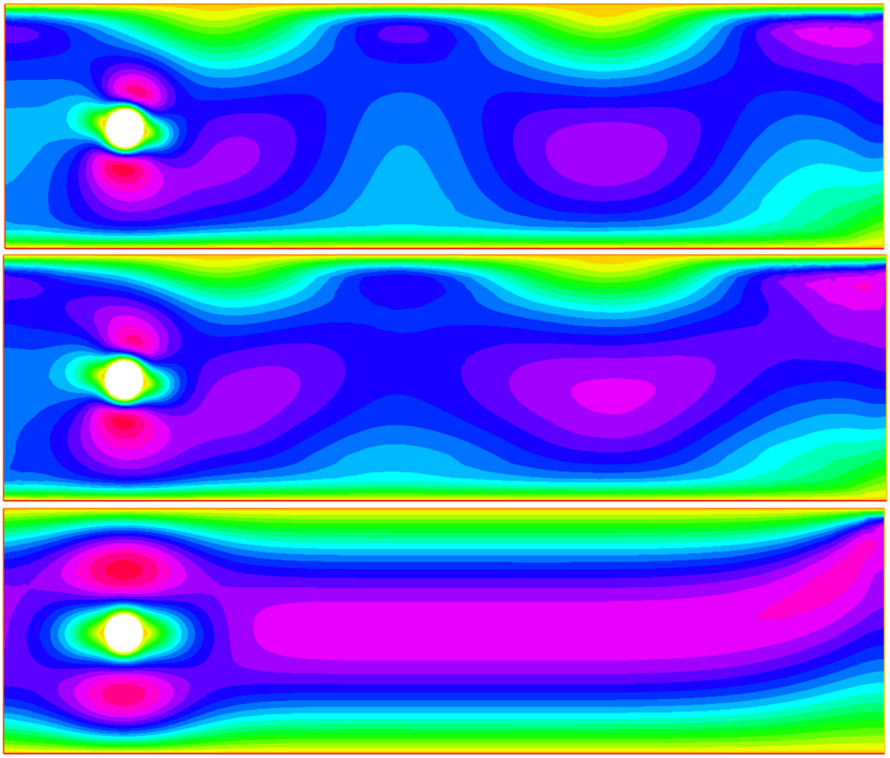

Fig. 3 Velocity profiles for uncontrolled, controlled, and desired states from top to bottom, respectively for $Re = 2000$

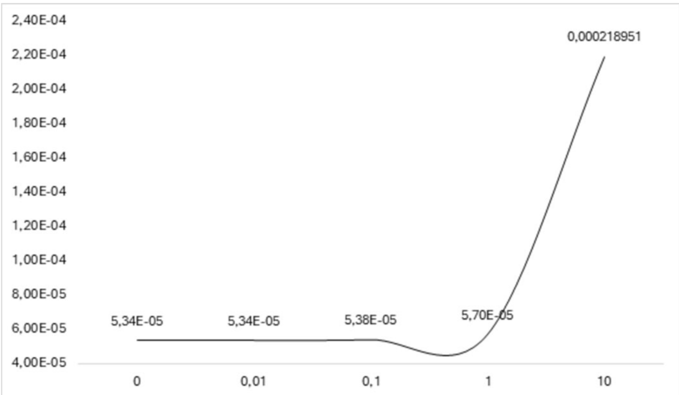

Fig. 4 Impact of the parameter σ on system performance

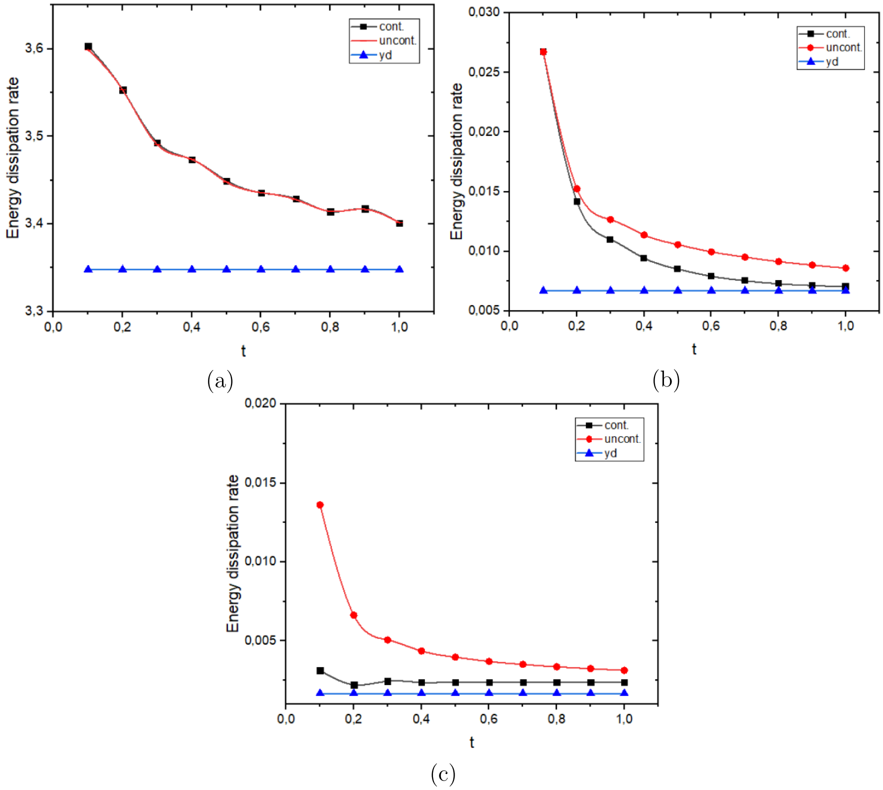

Fig. 5 Energy dissipation rates; a) $Re = 1$, b) $Re = 100$, c) $Re = 2000$

a steady and stable state in a turbulent flow. For our case, we aim to compare the controlled and uncontrolled case and their resemblance to the desired state.

As shown in Fig. 5, the optimal control strategy is found to optimize the energy dissipation more effectively as Re increases. We should also point out that since conservative Crank-Nicolson time stepping is considered here, numerical dissipation is generally low in magnitude. The optimal control method better adjusts the control parameters to achieve the objectives for optimizing the system's behavior. It also reduces the system's sensitivity to external influences and enables more precise management by adapting to changing requirements. This leads to more stable and desirable regulation of energy dissipation.

7 Conclusions

This study gave a detailed presentation of the optimal control problems for the NS-α and NS-ω turbulence models of NSE. The problem was discretized with the finite element method in space and analyzed regarding stability and convergence. The performances of the proposed schemes were assessed with quantitative and qualitative numerical tests.

Utilizing continuous data assimilation strategies with data from real-life applications in these models is the next step to explore.

A Proof of Lemma 5.2

Proof First, to obtain the required inequality, subtract (5.4) from the weak formulation of the rotation NSE. Then, for NS-α, one has

$$(e_t, v_h) + \nu(\nabla e, \nabla v_h) + ((\nabla \times y) \times y, v_h) - ((\nabla \times y_h(u)) \times \overline{y_h(u)}^h, v_h) - (p - P_h, \nabla \cdot v_h) = 0 \quad \text{(A.1)}$$

where $e = y - y_h(u)$. Decompose the error as $e = y - \tilde{y} - (y_h(u) - \tilde{y}) = \eta - \phi_h$ where $\tilde{y}$ is the best approximation of y. Choosing the test function $v_h = \phi_h$ in (A.1) results in

$$\begin{aligned}(\phi_t, \phi_h) + \nu(\nabla\phi_h, \nabla\phi_h) &= (\eta_t, \phi_h) + \nu(\nabla\eta, \nabla\phi_h) + ((\nabla \times y) \times y, \phi_h) \\ &\quad -((\nabla \times y_h(u)) \times \overline{y_h(u)}^h, \phi_h) - (p - P_h, \nabla \cdot \phi_h). \qquad \text{(A.2)}\end{aligned}$$

One can arrange the nonlinear terms as

$$\begin{aligned}&((\nabla \times y) \times y, \phi_h) - ((\nabla \times y_u(u)) \times \overline{y_h(u)}^h, \phi_h) \\ &= ((\nabla \times (y - y_h(u))) \times y, \phi_h) + ((\nabla \times y_h(u)) \times (y - \overline{y_h(u)}^h), \phi_h) \\ &= ((\nabla \times \eta) \times y, \phi_h) - ((\nabla \times \phi_h) \times y, \phi_h) + ((\nabla \times y_h(u)) \times (y - \overline{y_h(u)}^h), \phi_h).\end{aligned}$$

Then, (A.2) is written

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \nu\|\nabla\phi_h\|^2 &= (\eta_t, \phi_h) + \nu(\nabla\eta, \nabla\phi_h) - (p - P_h, \nabla \cdot \phi_h) \\ +((\nabla \times \eta) \times y, \phi_h) &- ((\nabla \times \phi_h) \times y, \phi_h) + ((\nabla \times y_h(u)) \times (y - \overline{y_h(u)}^h), \phi_h).\end{aligned}$$

The terms on the right-hand side are bounded in a standard way:

$$\begin{aligned}(\eta_t, \phi_h) &\le C\nu^{-1}\|\eta_t\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\ \nu(\nabla\eta, \nabla\phi_h) &\le C\nu\|\nabla\eta\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\ (p - P_h, \nabla \cdot \phi_h) &\le C\nu^{-1}\|p - P_h\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\ ((\nabla \times \eta) \times y, \phi_h) &\le C\nu^{-1}\|\nabla\eta\|^2\|\nabla y\|^2 + \frac{\nu}{14}\|\nabla\phi_h\|^2 \\ ((\nabla \times \phi_h) \times y, \phi_h) &\le C\nu^{-3}\|\phi_h\|^2\|\nabla y\|^4 + \frac{\nu}{14}\|\nabla\phi_h\|^2\end{aligned}$$

$$\begin{aligned}((\nabla \times y_h(u)) \times (y - \overline{y_h(u)}^h), \phi_h) &= ((\nabla \times y_h(u)) \times (y - \overline{y}^h), \phi_h) + ((\nabla \times y_h(u)) \times (\overline{y}^h - \overline{y_h(u)}^h), \phi_h) \\ &= ((\nabla \times y_h(u)) \times (y - \overline{y}^h), \phi_h) + ((\nabla \times y_h(u)) \times \overline{\eta}^h, \phi_h) \\ &= T_1 + T_2\end{aligned}$$

For T_1,

$$((\nabla \times y_h(u)) \times (y - \overline{y}^h), \phi_h) \leq \|\nabla \times y_h(u)\|_\infty \|y - \overline{y}^h\| \|\nabla \phi_h\|$$
$$\leq C\nu^{-1}\|\nabla \times y_h(u)\|_\infty^2 \|y - \overline{y}^h\|^2 + \frac{\nu}{14}\|\nabla \phi_h\|^2$$
$$\leq C\nu^{-1}\alpha^4\|\nabla \times y(u)\|_\infty^2 |y|_2^2 + \frac{\nu}{14}\|\nabla \phi_h\|^2.$$

For T_2,

$$((\nabla \times y_h(u)) \times \overline{\eta}^h, \phi_h) \leq C\|\nabla y_h(u)\| \|\nabla \overline{\eta}^h\| \|\nabla \phi_h\|$$
$$\leq C\nu^{-1}\|\nabla y_h(u)\|^2 \|\nabla \eta\|^2 + \frac{\nu}{14}\|\nabla \phi_h\|^2$$

Combining all these estimates gives us

$$\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \frac{\nu}{2}\|\nabla \phi_h\|^2$$
$$\leq C\Big(\nu^{-1}\|\eta_t\|^2 + \nu\|\nabla \eta\|^2 + \nu^{-1}\|p - P_h\|^2 + \nu^{-1}\|\nabla \eta\|^2\|\nabla y\|^2 + \nu^{-3}\|\phi_h\|^2\|\nabla y\|^4$$
$$+\nu^{-1}\alpha^4\|\nabla y(u)\|_\infty^2 |y|_2^2 + \nu^{-1}\|\nabla y_h(u)\|^2\|\nabla \eta\|^2\Big).$$

Integrating over $[0, T]$

$$\|\phi_h(T)\|^2 + \nu \int_0^T \|\nabla \phi_h\|^2 dt$$
$$\leq \|\phi_h(0)\|^2 + C\int_0^T \Big(\nu^{-1}\|(y - \tilde{y})_t\|^2 + \nu\|\nabla(y - \tilde{y})\|^2 + \nu^{-1}\|p - P_h\|^2 + \nu^{-1}\|\nabla(y - \tilde{y})\|^2\|\nabla y\|^2$$
$$+\nu^{-3}\|\phi_h\|^2\|\nabla y\|^4 + \nu^{-1}\alpha^4\|\nabla y(u)\|_\infty^2 |y|_2^2 + \nu^{-1}\|\nabla y_h(u)\|^2\|\nabla(y - \tilde{y})\|^2\Big).$$

and the application of Gronwall's inequality, the approximation properties, and the triangle inequality yields the required result. □

B Proof of Theorem 3.1

Proof For NS-α:

The Lagrange approach is used to obtain the continuous optimality system. First, define the Lagrangian function for the control problem by

$$\begin{aligned} L(y, \overline{y}, u, \lambda, \overline{\lambda}) &= J(y, u) + (y_t - \nu\Delta y + (\overline{y} \cdot \nabla)y + (\nabla\overline{y})^T y + \nabla p - u, \lambda) + (\overline{y} - y - \alpha^2\Delta\overline{y}, \overline{\lambda}) \\ &= \frac{1}{2}\int_Q (y - y_d)^2\, dxdt + \frac{\sigma}{2}\int_Q u^2 dxdt \\ &\quad + \int_Q (-\lambda_t \cdot y - \nu\Delta\lambda \cdot y + (\nabla p - u)\lambda)\, dxdt \\ &\quad + \int_Q (\overline{y} \cdot \nabla)y\lambda dxdt + \int_Q (\nabla\overline{y})^T y\lambda dxdt + \int_Q (\overline{y} \cdot \overline{\lambda} - \alpha^2\Delta\overline{y} \cdot \overline{\lambda} - y\overline{\lambda})dxdt \end{aligned} \quad (B.1)$$

Then, the first-order optimality conditions are obtained by letting the first-order partial derivatives of the Lagrangian function L be zero, similar to in [31–33]. Denoting $(\lambda, \bar{\lambda})$ adjoint parameters and Lagrange multipliers, respectively, optimality conditions are obtained by solving the system given below:

$$\begin{aligned}
\nabla_y L(y, \bar{y}, u, \lambda, \bar{\lambda}) &= 0 \\
\nabla_{\bar{y}} L(y, \bar{y}, u, \lambda, \bar{\lambda}) &= 0 \\
\nabla_u L(y, \bar{y}, u, \lambda, \bar{\lambda}) &= 0 \\
\nabla_\lambda L(y, \bar{y}, u, \lambda, \bar{\lambda}) &= 0 \\
\nabla_{\bar{\lambda}} L(y, \bar{y}, u, \lambda, \bar{\lambda}) &= 0.
\end{aligned}$$

These solutions are obtained from (B.1) as follows:

$$\begin{aligned}
\nabla_y L(\delta y) =& \int_Q ((y - y_d)\delta y) dxdt + \int_Q (-\lambda_t.\delta y - \nu\Delta\lambda.\delta y) dxdt + \nabla y(\delta y)\left(\int_Q (\bar{y}.\nabla) y\lambda dxdt\right) \\
&+ (\nabla y)^T(\delta y)\left(\int_Q (\nabla\bar{y})^T y\lambda dxdt\right) - \int_Q (\delta y)\bar{\lambda} dxdt \\
=& \int_Q (y - y_d - \lambda_t - \nu\Delta\lambda) dxdt - \int_Q (\bar{y}.\nabla)\lambda(\delta y) dxdt - \int_Q ((\nabla\bar{y})^T)\lambda(\delta y) dxdt - \int_Q (\delta y)\bar{\lambda} dxdt = 0,
\end{aligned}$$

and from here

$$-\lambda_t - \nu\Delta\lambda - (\bar{y}\cdot\nabla)\lambda - (\nabla\bar{y})^T\lambda = y_d - y + \bar{\lambda}.$$

$$\begin{aligned}
\nabla_{\bar{y}} L(\delta\bar{y}) =& \nabla_{\bar{y}} L(\delta\bar{y})\left(\int_Q (\bar{y}\cdot\nabla) y\lambda dxdt + \int_Q (\nabla\cdot\bar{y})^T y\lambda dxdt + \int_Q (\bar{y}\bar{\lambda} - \alpha^2\Delta\bar{\lambda}\cdot\bar{y} - y\cdot\bar{\lambda}) dxdt\right) \\
=& \nabla_{\bar{y}} L(\delta\bar{y})\left(\int_Q (\nabla y)^T\lambda\bar{y} dxdt - \int_Q (\lambda\cdot\nabla) y\bar{y}\lambda dxdt + \int_Q (\bar{\lambda} - \alpha^2\Delta\bar{\lambda})\delta\bar{y} dxdt\right) \\
=& \int_Q \left((\nabla y)^T\lambda\delta\bar{y} - (\lambda\cdot\nabla) y(\delta\bar{y}) + \bar{\lambda}\delta\bar{y} - \alpha^2\Delta\bar{\lambda}\delta\bar{y}\right) dxdt = 0,
\end{aligned}$$

and from here

$$\bar{\lambda} - \alpha^2\Delta\bar{\lambda} = -(\nabla y)^T\lambda + (\lambda\cdot\nabla) y.$$

$$\Delta_u L(\delta u) = \int_Q \sigma u\delta u dxdt - \int_Q \lambda\delta u dxdt = 0,$$

and from here

$$\sigma u - \lambda = 0.$$

Accordingly, Theorem 3.1 is obtained. □

Proof For NS-ω:

As in the NS-α case, the Lagrange approach is used to obtain the continuous optimality system. Lagrangian function for the control problem is defined by

$$\begin{aligned} L(y,\overline{y},u,\lambda,\overline{\lambda}) &= J(y,u) + (y_t - \nu\Delta y + (y\cdot\nabla)\overline{y} + (\nabla y)^T\overline{y} + \nabla p - u, \lambda) + (\overline{y} - y - \alpha^2\Delta\overline{y}, \overline{\lambda}) \\ &= \frac{1}{2}\int_Q (y - y_d)^2\,dxdt + \frac{\sigma}{2}\int_Q u^2 dxdt \\ &\quad + \int_Q (-\lambda_t\cdot y - \nu\Delta\lambda\cdot y + (\nabla p - u)\lambda)\,dxdt \\ &\quad + \int_Q (y\cdot\nabla)\overline{y}\lambda dxdt + \int_Q (\nabla y)^T\overline{y}\lambda dxdt + \int_Q (\overline{y}\cdot\overline{\lambda} - \alpha^2\Delta\overline{y}\cdot\overline{\lambda} - y\overline{\lambda})dxdt \end{aligned} \tag{B.2}$$

Then, optimality conditions are obtained by solving the system given below:

$$\begin{aligned} \nabla_y L(y,\bar{y},u,\lambda,\bar{\lambda}) &= 0 \\ \nabla_{\bar{y}} L(y,\bar{y},u,\lambda,\bar{\lambda}) &= 0 \\ \nabla_u L(y,\bar{y},u,\lambda,\bar{\lambda}) &= 0 \\ \nabla_\lambda L(y,\bar{y},u,\lambda,\bar{\lambda}) &= 0 \\ \nabla_{\bar{\lambda}} L(y,\bar{y},u,\lambda,\bar{\lambda}) &= 0. \end{aligned}$$

These solutions are obtained from (B.2) as follows:

$$\nabla_y L(\delta y) = \int_Q (y - y_d - \lambda_t - \nu\Delta\lambda)dxdt - \int_Q (\lambda.\nabla)\overline{y}(\delta y)dxdt + \int_Q (\nabla\overline{y})^T\lambda(\delta y)dxdt - \int_Q (\delta y)\bar{\lambda}dxdt = 0,$$

which results in

$$-\lambda_t - \nu\Delta\lambda + (\nabla\overline{y})^T\lambda - (\lambda\cdot\nabla)\overline{y} = y_d - y + \overline{\lambda}.$$

Similarly, the first order partial derivative of (B.2) in the direction of $\delta\overline{y}$ gives

$$\nabla_{\bar{y}} L(\delta\bar{y}) = \int_Q \left(-(y\cdot\nabla)\lambda\delta\bar{y} - (\nabla y)^T\lambda(\delta\bar{y}) + \bar{\lambda}\delta\bar{y} - \alpha^2\Delta\bar{\lambda}\delta\bar{y}\right)dxdt = 0,$$

which results in

$$\bar{\lambda} - \alpha^2\Delta\bar{\lambda} = (y\cdot\nabla)\lambda + (\nabla y)^T\lambda.$$

Finally,

$$\Delta_u L(\delta u) = \int_Q \sigma u\delta u dxdt - \int_Q \lambda\delta u dxdt = 0,$$

which results in

$$\sigma u - \lambda = 0.$$

Thus, the proof of Theorem 3.1 is completed. □

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval The authors declare that they followed all the rules of good scientific practice.

Competing interests The authors declare no competing interests.

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