MATHEMATICS Proceedings A 89 (4), December 15, 1986

Sections of $U(2n)/Sp(s)$ over S^{4n-1}

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Communicated by Prof. W.T. van Est at the meeting of March 24, 1986

ABSTRACT

We describe the sections of $U(2n)/Sp(s)$ over S^{4n-1} in terms of the sections of the symplectic Stiefel manifold $Sp(n)/Sp(s)$ and we express the orders of obstructions to sectioning $U(2n)/Sp(s)$ in terms of orders of obstructions to sectioning $Sp(n)/Sp(s)$. In certain cases we give the exact values of these orders.

1. INTRODUCTION

In [6] we have determined the integers n such that for a given s the fibration $U(n)/Sp(s)\rightarrow S^{2n-1}$ admits a cross section. According to these results, when n is even the existence of cross sections depends on the existence of cross sections of the symplectic Stiefel manifold $Sp(m)/Sp(s)$, where $m=n/2$. The purpose of the present paper is to shed some light onto the exact relationship between these two families of cross-sections.

In § 2 the relationship between the sections of $U(2n)/Sp(s)$ and the sections of $Sp(n)/Sp(s)$ is given. This relationship and the description of sections of $U(2n)/Sp(s)$ in terms of the sections of $Sp(n)/Sp(s)$ is expressed as our first main result in Theorem 1. These sections are especially important in the study of almost-quaternion substructures on the sphere and the results of Theorem 1 say something about the construction of these substructures. This is explained in § 3. To make the description of the relationship between sections of

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* This work is partially supported by the Scientific and Technical Research Council of Turkey.

$U(2n)/Sp(s)$ and $Sp(n)/Sp(s)$ complete, in § 4 we express the orders of obstructions to sectioning $U(2n)/Sp(s)$ in terms of the orders of obstructions to sectioning $Sp(n)/Sp(s)$. This is given as the second main result in Theorem 2. As a consequence of this and by the known results about the orders of obstructions to sectioning $Sp(n)/Sp(s)$, in certain cases we give the exact values of these orders (Theorem 3).

Throughout the paper the following conventions will be used.

Let F denote either $\mathbb{C}$ or $\mathbb{H}$ (quaternions). On the vector space F^n define the inner product $(x/y)=x_1\bar{y}_1+\ldots+x_n\bar{y}_n$ and the norm $\|x\|=(x/x)^{1/2}$ for $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in F^n$. The unitary group $U(n)$ and the symplectic group $Sp(n)$ are the norm preserving automorphisms of $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. Let e_j be the vector in $\mathbb{R}^n$ which has 1 in j^{th} entry and 0 elsewhere. Similarly we let a_j be the vector in $\mathbb{C}^n$ which has 1 in j^{th} entry and 0 elsewhere. If $s<n$ we shall consider $U(s)$ embedded in $U(n)$ by considering it to be the elements of $U(n)$ which leave $a_{s+1},\ldots,a_n$ fixed. The embedding $Sp(s)\subset Sp(n)$ is similarly defined. Every vector $(z_1,\ldots,z_n)$ in $\mathbb{C}^n$ will be considered as a vector $(y_1,x_1,\ldots,y_n,x_n)$ where $z_r=x_r+iy_r$, and every vector $(q_1,\ldots,q_n)$ in $\mathbb{H}^n$ will be considered as a vector $(w_1,z_1,\ldots,w_n,z_n)$ in $\mathbb{C}^{2n}$ where $q_r=z_r+jw_r$. The embedding $Sp(n)\subset U(2n)$ is defined in accordance with these conventions.

2. THE SECTIONS OF $U(2n)/Sp(s)$

To describe the sections of $U(2n)/Sp(s)$ and the relationship with the sections of $Sp(n)/Sp(s)$ let $1:Sp(n)/Sp(s)\to U(2n)/Sp(s)$ denote the inclusion, $h:U(2n)\to U(2n)/Sp(s)$ be the quotient map. Also, let $p:U(2n)\to S^{4n-1}$ be the projection which takes every $2n$-frame $(u_1,\ldots,u_{2n})$ in $\mathbb{C}^{2n}$ to u_{2n}. Denote the fibration $Sp(n)/Sp(s)\to S^{4n-1}$ by q and the fibration $U(2n)/Sp(s)\to S^{4n-1}$ by $\tilde{p}$. Throughout the paper let ι denote the positive generator of the group $\Pi_{4n-1}(S^{4n-1})\cong\mathbb{Z}$. Following [4], we call an element $\alpha\in\Pi_{4n-1}(Sp(n)/Sp(s))$ an m-section if $q(\alpha)=m\iota$, where m is an integer. Similarly we define an m-section of $U(2n)/Sp(s)$ to be an element $\beta\in\Pi_{4n-1}(U(2n)/Sp(s))$ such that $\tilde{p}\#(\beta)=m\iota$. By covering homotopy theorem every 1-section is represented by a cross section. Now, keeping in mind that cross-sections of $U(2n)/Sp(s)$ exist when $Sp(n)/Sp(s)$ has a cross section ([6]), we can state and prove the following theorem.

THEOREM 1. Let η_{s-1} denote the underlying real bundle of the quaternionic Hopf line bundle over $P_{s-1}(\mathbb{H}^n)$, and let c_s denote the order of $J(\eta_{s-1})$ in $J(P_{s-1}(\mathbb{H}^n))$. Assume $c_s|n$. In other words assume $Sp(n)/Sp(s)$ has a cross section. Then

(i) If m is any integer and θ is a $(2n-1)!m+1$ section of $Sp(n)/Sp(s)$, then $1_{\#}(\theta)-mh_{\#}(\alpha)$ is a 1-section of $\tilde{p}$ for a suitable generator α of $\Pi_{4n-1}(U(2n))\cong\mathbb{Z}$.

(ii) If n is even then any 1-section of $\tilde{p}$ is of the form $1_{\#}(\theta)-mh_{\#}(\alpha)$, where, θ, λ, α are as in part (i). Moreover, for every 1-section ψ of $U(2n)/Sp(s)$, 2ψ is of the form $1_{\#}(\varphi)$, where φ is a 2-section of $Sp(n)/Sp(s)$.

(iii) If n is odd every cross section of $U(2n)/Sp(s)$ is homotopic to a cross section of the form $1 \circ \sigma$ where σ is a cross section of $Sp(n)/Sp(s)$.

PROOF. (i) Consider the following exact sequence of the fibration $p: U(2n) \to \to S^{2n-1}$,

$$\to \Pi_{4n-1}(U_{2n-1}) \to \Pi_{4n-1}(U(2n)) \xrightarrow{p_\#} \Pi_{4n-1}(S^{4n-1}) \to^{\partial_p}$$

$\Pi_{4n-2}(U(2n-1)) \to \Pi_{4n-2}(U(2n))$.

We use the following results of [1], [2] and [5].

(2.1) $\Pi_{4n-1}(U(2n-1)) = 0$

(2.2) $\Pi_{4n-2}(U_{2n-1}) = 0$

(2.3) $\Pi_{4n-2}(U_{2n-1}) \cong \mathbb{Z}_{(2n-1)!}$

(2.4) $\Pi_{4n-1}(U(2n-1)) \cong \mathbb{Z}$.

By (2.2) ∂_p is an epimorphism. Therefore the generators of $\Pi_{4n-1}(S^{4n-1})$ are mapped onto the generators of $\Pi_{4n-2}(U(2n-1) \cong \mathbb{Z}_{(2n-1)!}$ (2.3). So, the image of $p_\#$ is generated by $[(2n-1)!]\iota$. Since by (2.1) p is injective, one of the generators of $\Pi_{4n-1}(U(2n)) \cong \mathbb{Z}$ goes to $[(2n-1)!]\iota$. Call it α. Now we have

$$\tilde{p}_\#(1_\#(\theta) - mh_\#(\alpha)) = q_\#(\theta) - mp_\#(\alpha) = [(2n-1)!m+1]\iota$$

$-m[(2n-1)!]\iota = \iota$. So, $1_\#(\theta) - mh_\#(\alpha)$ is a 1-section.

(ii) First, consider the following commutative diagram

$$\begin{tikzcd}
 & & & {} \\
 & & & \Pi_{4n-2}(Sp(n)) \arrow[u] \\
\cdots \arrow[r] & \Pi_{4n-1}(Sp(n)/Sp(s)) \arrow[r, "1_\#"] \arrow[dr, "q_\#"'] & \Pi_{4n-1}(U(2n)/Sp(s)) \arrow[r, "t_\#"] \arrow[d, "\tilde{p}_\#"] & \Pi_{4n-1}(U(2n)/Sp(n)) \arrow[r] \arrow[u] & \cdots \\
\cdots & & \Pi_{4n-1}(S^{4n-1}) \arrow[ll] & \Pi_{4n-1}(U(2n)) \arrow[l, "p_\#"'] \arrow[u, "r_\#"'] \arrow[ul, "h_\#"] & \cdots \arrow[l] \\
 & & & {} \arrow[u]
\end{tikzcd}$$

where the upper row is homotopy exact sequence of the projection

$$U(2n)/Sp(s) \to U(2n)/Sp(n).$$

Since $\Pi_{4n-2}(Sp(n)) = 0$ by [2], $r_\#$ is an epimorphism.

Now assume φ is a 1-section of $U(2n)/Sp(s)$. There exists an element $u \in \Pi_{4n-1}(U(2n))$ such that

$$r_\#(u) = t_\#(\varphi).$$

Then we have

$$t_{\#}(\varphi - h_{\#}(u)) = t_{\#}(\varphi) - r_{\#}(u) = 0.$$

So, by exactness of the upper horizontal row, there exists an element $\theta \in \Pi_{4n-1}(Sp(n)/Sp(s))$ such that

$$1_{\#}(\theta) = \varphi - h_{\#}(u).$$

Since α is a generator of $\Pi_{4n-1}(U(2n)) \cong \mathbb{Z}$ (2.4), there exist $m \in Z$ such that $u = -m\alpha$. Then we have

$$\varphi = 1_{\#}(\theta) - mh_{\#}(\alpha).$$

It remains to show that θ is $(2n-1)!m+1$ section of $Sp(n)/Sp(s)$. In fact

$$q_{\#}(\theta) = \tilde{p}_{\#}(1_{\#}(\theta)) = \tilde{p}_{\#}(\varphi + mh_{\#}(\alpha)) = \tilde{p}_{\#}(\varphi) + mp_{\#}(\alpha) =$$

$$= \imath + m[(2n-1)!]\imath = [(2n-1)!m+1]\imath.$$

This completes the proof of the first statement of (ii).

Next, let ψ be a 1-section of $U(2n)/Sp(s)$. Since $\Pi_{4n-1}(U(2n)/Sp(n)) \cong \mathbb{Z}_2$ for n even ([2]), we have

$$t_{\#}(2\psi) = 0.$$

Hence there exists an element $\varphi \in \Pi_{4n-1}(Sp(n)/Sp(s))$ such that

$$1_{\#}(\varphi) = 2\psi$$

by exactness of the upper horizontal row of the diagram above. Now, φ must be a 2-section of $Sp(n)/Sp(s)$ because

$$q_{\#}(\varphi) = \tilde{p}_{\#}(1_{\#}(\varphi)) = \tilde{p}(2\psi) = 2.$$

This completes the proof of part (ii).

(iii) Since $\Pi_{4n-1}(U(2n)/Sp(n)) \cong 0$ for n odd, ([2]), $1_{\#}: \Pi_{4n-1}(Sp(n)/Sp(s)) \rightarrow$ $\rightarrow \Pi_{4n-1}(U(2n)/Sp(n))$ is surjective. If β is a cross section $U(2n)/Sp(s)$ then there exists an element γ in $\Pi_{4n-1}(Sp(n)/Sp(s))$ with $1_{\#}(\gamma) = [\beta]$. Then

$$q_{\#}(\gamma) = \tilde{p}_{\#}(1_{\#}(\gamma)) = \tilde{p}_{\#}([\beta]) = \imath.$$

So, γ is a 1-section of $Sp(n)/Sp(s)$. By homotopy covering theorem γ is homotopic to a cross section σ of $Sp(n)/Sp(s)$. Therefore we have

$$\beta \simeq 1 \circ \gamma \simeq 1 \circ \sigma$$

as asserted.

REMARK. If n is even and σ is a cross section of $Sp(n)/Sp(s)$, then $1 \circ \sigma$ is a cross section but unlike the case n is odd not all cross sections are of this form. To see this it is sufficient to exhibit a 1-section φ of $U(2n)/Sp(s)$ such that $t_{\#}(\varphi) \neq 0$ in the diagram of the proof of part (ii). Since we have

$$\Pi_{4n-1}(U(2n)/Sp(n)) = \mathbb{Z}_2,\ \Pi_{4n-2}(Sp(n)) = 0,\ \Pi_{4n-1}U(2n) \cong \mathbb{Z},$$

$r_\#$ is onto and the generators of $\Pi_{4n-1}(U(2n))$ go to the generator of $\Pi_{4n-1}(U(2n)/Sp(n))$. Let α be the generator of $\Pi_{4n-1}(U(2n))$ such that $p_\#(\alpha) = [(2n-1)!]\iota$ (see the proof of part (i)). Choose any 1-section γ of $Sp(n)/Sp(s)$. Then $[(2n-1)!+1]\ \gamma$ is a $(2n-1)!+1$ section of $Sp(n)/Sp(s)$ and by (i) of Theorem 1 the element $\varphi = 1_\#([(2n-1)!+1)]\gamma) - h_\#(\alpha)$ is a 1-section of $U(2n)/Sp(s)$. However,

$$t_\#(\psi) = -t_\# h_\#(\alpha) = -r_\#(\alpha) = 1 \neq 0.$$

3. ALMOST QUATERNION s-SUBSTRUCTURES AND QUATERNIONIC $(n-s)$-FRAMES

As defined in [6] an almost-quaternion s-substructure on the canonical $\mathbb{C}^{m-1}$-bundle ξ_{m-1} over S^{2m-1} is a $4s$-dimensional subbundle η of the underlying real bundle $r\xi_{m-1}$ of ξ_{m-1} together with normalized almost-complex $2s$-substructure $G: E(\eta) \to E(\eta)$ defined on the total space of η such that $IG = -GI$ holds. Here I is the restriction of the complex structure of $\mathbb{R}^{2m}$ induced by the multiplication by the complex number i, to $E(\eta)$.

Each almost-quaternion s-substructure whose underlying bundle (considered as a complex bundle) has trivial orthogonal complement in ξ_{m-1} corresponds to a cross section of $U(m)/Sp(s)$. Let σ be a cross section of $U(m)/Sp(s)$, $x \in S^{2m-1}$, and let $\sigma(x)$ be represented by $L \in U(m)$. Then underlying real bundle η of the almost-quaternion s-substructure has fiber at x spanned by $L(e_1), \ldots, L(e_{4s})$. The structure map G on this fibre of η at x can be given by

$$G_x(L(e_\alpha)) = -L(e_{3\alpha}),\ G_x(L(e_{2\alpha})) = -L(e_{4\alpha})$$

$$G_x(L(e_{3\alpha})) = L(e_\alpha),\ G_x(L(e_{4\alpha})) = L(e_{2\alpha})\ \ \alpha = 1, 2, \ldots, s.$$

If L is another representative of $\sigma(x)$, $L' = AL$ where A is in $Sp(s)$, hence the definition of G is independent of the choice of the representative.

Now, let us restrict ourselves to the case m is even, and let $m = 2n$. If in the construction above we have $L \in Sp(n) \subset U(2n)$. Then G constructed is simply the quaternionic structure $J: \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ induced by the multiplication by the unit quaternion j.

Once a cross section σ of $Sp(n)/Sp(s)$ in S^{4n-1} is given we can construct the almost-quaternion s-substructure on ξ_{2n-1} corresponding to the cross section $1 \circ \sigma$ of Theorem 1, part (iii) as follows.

First we recall that a cross section σ of $Sp(n)/Sp(s)$ determines a $(4n-4s)$-frame of the form $(Ku_{s+1}, Ju_{s+1}, Iu_{s+1}, u_{s+1}, \ldots, Ku_n, Ju_n, Iu_n, u_n)$ (according to the conventions at the introduction), where $u_n = x$ and where the triple $I: \mathbb{R}^{4n} \to \mathbb{R}^{4n}$, $J: \mathbb{R}^{4n} \to \mathbb{R}^{4n}$, $K: \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ are the standart structure maps induced by the multiplication by the unit quaternions i, j, k respectively. The orthogonal complement of this frame is preserved under I, J, K. Thus the underlying real bundle η of the almost-quaternion s-substructure we want to construct has this orthogonal complement as fibre at x and G_x is the restriction of J to the fibre of η at x.

Theorem 1 (iii) says that for n odd, this type of substructures, up to homotopy, are the only ones.

4. ORDERS OF OBSTRUCTIONS

Consider the exact homotopy sequence

$$\ldots \to \Pi_{4n-1}(U(2n)/Sp(s)) \xrightarrow{\tilde{p}_{\#}} \Pi_{4n-1}(S^{4n-1}) \xrightarrow{\delta} \Pi_{4n-2}(U(2n-1)/Sp(s)) \to \ldots$$

of the fibration $U(2n)/Sp(s) \to S^{4n-1}$. Since the image of $\tilde{p}_{\#}$ is the kernel of δ, $U(2n)/Sp(s)$ has a cross section if and only if $\delta(\iota)=0$. So, we call $\delta(\iota)$ the obstruction to cross sectioning $U(2n)/Sp(s)$ over S^{4n-1}. Let $U\{2n,s\}$ denote the order of $\delta(\iota)$ if it is finite and 0 if it is infinite. The order of obstruction to cross sectioning symplectic Stiefel manifold is defined in [4] and is the subject of current research. Following the notation of James in [4] let us denote it by $X\{n,s\}$. In this section we shall express $U\{2n,s\}$ in terms of $X\{n,s\}$ and we shall give its values explicitely in cases the values of $X\{n,s\}$ are known.

THEOREM 2. For the values of $U\{2n,s\}$, the obstructions to cross sectioning $U(2n)/Sp(s)$, we have

$$U\{2n,s\} = \begin{cases} X\{n,s\} & \text{if } n \text{ is odd} \\ \text{g.c.d. } (X\{n,s\},(2n-1)!) & \text{if } n \text{ is even} \end{cases}$$

(Here g.c.d. stands for the greatest common divisor).

PROOF. First, let n be odd. In that case we have the commutative diagram.

$$\begin{tikzcd}
 & & 0 \\
\Pi_{4n-1}(Sp(n)/Sp(s)) \arrow[r] \arrow[dr, "q_{\#}"'] & \Pi_{4n-1}(U(2n)/Sp(s)) \arrow[r] \arrow[d, "\tilde{p}_{\#}"] & \Pi_{4n-1}(U(2n)/Sp(s)) \arrow[u, equal] \\
 & \Pi_{4n-1}(S^{4n-1}) \cong \mathbb{Z} &
\end{tikzcd}$$

Since $1_{\#}$ is onto, image of $q_{\#}$ and image of $\tilde{p}_{\#}$ are equal. Since $X\{n,s\}$, $U\{2n,s\}$ are the smallest integers such that $[X\{n,s\}]\iota$ and $[U\{2n,s\}]\iota$ are in the image of $q_{\#}$ and $p_{\#}$ respectively, they must be equal.

Assume n is even. We shall use the diagram in the proof of Theorem 1 (ii). We claim that Image $\tilde{p}_{\#}=$ Image $q_{\#}+$ Image $p_{\#}$. First let $a \in \operatorname{Im} q_{\#}$, $b \in \operatorname{Im} p_{\#}$. Then there exists $\sigma \in \Pi_{4n-1}(Sp(n)/Sp(s))$, $\alpha \in \Pi_{4n-1}(U(2n))$ such that $q_{\#}(\sigma)=a$, $p_{\#}(\alpha)=b$ it follows that $\tilde{p}_{\#}(1_{\#}(\sigma)+h_{\#}(\alpha)) = q_{\#}(\sigma)+p_{\#}(\alpha) = a+b$, so $a+b \in \operatorname{Im} \tilde{p}_{\#}$. Conversely, let $c \in \operatorname{Im} \tilde{p}_{\#}$ and $\tilde{p}(\varphi)=c$. Use a diagram chasing as in the proof of part (ii) of Theorem 1. Let $\upsilon \in \Pi_{4n-1}(U(2n))$ be such that $r_{\#}(\upsilon)=t_{\#}(\varphi)$. Then $t_{\#}(\varphi-h_{\#}(\upsilon))=0$. Hence there exists $\theta \in \Pi_{4n-1}(Sp(n)/Sp(s))$ with $1_{\#}(\theta)=\varphi-h_{\#}(\upsilon)$ by the exactness of the upper horizontal row in the diagram. So, $\varphi=1_{\#}(\theta)+h_{\#}(\upsilon)$. It follows that $c=\tilde{p}_{\#}(\varphi)=q_{\#}(\theta)+p_{\#}(\upsilon)$. So, $c \in \operatorname{Im} q_{\#}+\operatorname{Im} p_{\#}$.

Now, Im $q_{\#}$ is generated by $[X\{n,s\}]\iota$ and Im $p_{\#}$ is generated by $[(2n-1)!]\iota$ as is shown in the proof of Theorem 1 (i). It follows that

$$\operatorname{Im} \tilde{p}_{\#} = \{(\lambda X\{n,s\} + \mu[(2n-1)!])\iota|,\ \lambda, \mu \in \mathbb{Z}\}.$$

Therefore Im $\tilde{p}_{\#}$ is generated by $d\iota$, where d is the greatest common divisor of $X\{n,s\}$ and $(2n-1)!$. This completes the proof of Theorem 2.

Now, let c_s be as in Theorem 1. We have

LEMMA 1. If $\upsilon_p(n) \geq \upsilon_p(c_{s-1})$ then

$$\upsilon_p(X\{n,s\}) = \begin{cases} \upsilon_p(c_s) - \upsilon_p(n) & \text{if } \upsilon_p(n) \leq \upsilon_p(c_s) \\ 0 & \text{if } \upsilon_p(n) > \upsilon_p(c_s) \end{cases}$$

PROOF. The proof of this lemma is the same as the proof of Corollary 5.4 in [3]. Instead of the complex Hopf line bundle over $P_{s-1}(\mathbb{C})$, the quaternionic Hopf line bundle over $P_{s-1}(\mathbb{H})$ should be used.

THEOREM 3. If $\upsilon_p(n) \geq \upsilon_p(c_{s-1})$ then

$$U\{2n,s\} = \begin{cases} \upsilon_p(c_s) - \upsilon_p(n) & \text{if } \upsilon_p(n) \leq \upsilon_p(c_s),\ n \text{ odd} \\ \max\{\upsilon_p(c_s) - \upsilon_p(n), \upsilon_p((2n-1)!)\} & \text{if } \upsilon_p(n) \leq \upsilon_p(c_s),\ n \text{ even} \\ 0 & \text{if } \upsilon_p(n) > \upsilon_p(c_s) \end{cases}$$

PROOF. This follows from Theorem (2) and Lemma (1).

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