EXTENDING PERSISTENT HOMOLOGY THROUGH ZIGZAG MODULES

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ABSTRACT

EXTENDING PERSISTENT HOMOLOGY THROUGH ZIGZAG MODULES

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Zigzag persistence is a variant of the ordinary persistence in topological data analysis, which can be used to extract topological features from datasets. It was introduced by Carlsson and Silva [1] as a generalization of the original persistence algorithm for sequence of topological spaces, that are not related by only an increasing filtration, but also have inclusions and exclusions. The aim of this thesis is to present a survey of zigzag persistence by presenting its mathematical foundations and elaborating the key tools and algorithms used for its computation.

Keywords: Topological Data Analysis, Zigzag Module, Zigzag Persistence, The Diamond Principle

ÖZ

ZİGZAG MODÜLLERİYLE ISRARCI HOMOLOJİNİN GENİŞLETİLMESİ

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Zigzag ısrarcılığı, topolojik veri analizinde sıradan ısrarcılığın bir çeşididir ve veri kümelerinden topolojik özellikler çıkarmak için kullanılabilir. Carlsson ve Silva [1] tarafından yalnızca artan bir filtreleme ile ilişkili olmayan, aynı zamanda dahil olma ve dışlamaları da olan topolojik uzay dizileri için orijinal ısrarcılık algoritmasının bir genellemesi olarak tanıtılmıştır. Bu tezin amacı, matematiksel temellerini ve hesaplanması için kullanılan temel araçları ve algoritmaları detaylandırarak zikzak ısrarcılığın bir incelemesini sunmaktır.

Anahtar Kelimeler: Topolojik Veri Analizi, Zikzag modül, Zikzag Israrcılığı, Elmas Prensibi

To my family

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CHAPTER 1

INTRODUCTION

Over the past twenty years, Topological Data Analysis, also known as TDA, has become a rich framework for extracting significant topological features from point cloud data. The primary tool of Topological Data Analysis is the theory of persistent homology, which comes from the homology theory of Algebraic Topology in pure mathematics to capture the change of homological features throughout a filtration. It leverages tools such as filtrations and homology to extract meaningful topological features from data. A filtration is a nested sequence of topological spaces connected by inclusion maps, typically constructed from point cloud data. In the ordinary framework of persistent homology, the topological spaces extend monotonically, with all associated maps directed forward along the filtration. Applying the homology functor to a filtration induces a persistence module, a sequence of vector spaces connected by linear maps that reflect these inclusions.

However, ordinary persistence is limited for totally ordered filtrations, making it ineffective for a sequence of topological spaces with non-monotonic growth. To overcome this limitation, Gunnar Carlsson and Vin de Silva introduced the theory of Zigzag Persistence [1], which generalizes ordinary persistence by allowing both forward and backward morphisms in the sequence of topological spaces or vector spaces. In other words, Zigzag Persistence generalizes the ordinary framework of persistence homology by permitting forward and backward maps between spaces. Instead of requiring a nested sequence of topological spaces, zigzag persistence adjusts diagrams such as $\mathbb{X}_1 \to \mathbb{X}_2 \leftarrow \mathbb{X}_3 \to \mathbb{X}_4 \leftarrow \mathbb{X}_5$, determine persistence of a sequence of topological spaces where topological spaces have inclusions and exclusions, not only inclusions.

This thesis presents a survey of zigzag persistence. Based on the article "Zigzag Persistence" by Gunnar Carlsson and Vin de Silva, this thesis introduces the mathematical foundations of zigzag persistence and outlines the key tools and algorithms used for its computation.

This thesis is organized as follows: We begin with Chapter 2 as Background, which shows basic definitions and concepts of Filtration and Persistent Homology. Next, we describe Persistence Barcodes and Diagrams, which help as visual and algebraic summaries of topological features. Lastly, we handle the Stability of invariants, highlighting their robustness under perturbations. Chapter 3 introduces the theory of Zigzag Persistence. It begins with the mathematical theory of Zigzag Persistence and then describes the Right Filtration Operator and necessary definitions to understand and construct the decompositions of zigzag modules. This is followed by stating the theorem and algorithm for the Interval Decomposition Theorem. Lastly, we focus on the Diamond Principle, a foundational structural consequence of zigzag diagrams of vector spaces. In Chapter 4, we investigate the relationship between Zigzag Persistence and Real-Valued Functions. We start with level-set Zigzag Persistence, which captures topological changes in sublevel sets of topological spaces. Next, we introduce Pyramids, a diagram consisting of topological spaces or relative pairs derived from the slices of topological space $\mathbb{X}$. Lastly, we present the stability of level-set zigzag persistence.

CHAPTER 2

BACKGROUND

The background presented here is based on the book Introduction to Persistent Homology by Ziga Virk [8]. In this chapter, we present several methods for constructing complexes from a point cloud, a finite collection of points in a metric space. Such a point cloud may represent a sample from an underlying shape or a cluster of numerical data. We will define persistence homology, barcodes, and diagrams, and introduce definitions and theorems to understand the stability of persistence modules.

2.1 Filtration

Consider a real-valued function $f : \mathbb{T} \to \mathbb{R}$ defined on a topological space $\mathbb{T}$. For each real number a, define the sublevel set $\mathbb{T}_a = f^{-1}((-\infty, a])$. Clearly, these sublevel sets satisfy the inclusions

$$\mathbb{T}_a \subseteq \mathbb{T}_b \quad \text{whenever} \quad a \leq b.$$

Now, fix a sequence of real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$, typically chosen to be the critical values where the homology of the sublevel sets changes. Including the dummy value $a_0 = -\infty$, for which $\mathbb{T}_{a_0} = \emptyset$, we obtain a nested sequence of subspaces connected by inclusions, forming a *filtration* $\mathcal{F}_f$:

$$\mathcal{F}_f : \emptyset = \mathbb{T}_{a_0} \hookrightarrow \mathbb{T}_{a_1} \hookrightarrow \mathbb{T}_{a_2} \hookrightarrow \cdots \hookrightarrow \mathbb{T}_{a_n}.$$

The inclusions in the filtration induce linear maps between the singular homology groups of the corresponding subspaces. Specifically, if $\iota : \mathbb{T}_{a_i} \hookrightarrow \mathbb{T}_{a_j}$ for $i \leq j$

denotes the inclusion map, then the induced homomorphism

$$h_p^{i,j} = \iota_* : H_p(\mathbb{T}_{a_i}) \to H_p(\mathbb{T}_{a_j})$$

is defined for each $p \geq 0$ and $0 \leq i \leq j \leq n$.

Thus, we obtain a sequence of homomorphisms induced by inclusions, forming what is known as a *homology module*:

$$0 = H_p(\mathbb{T}_{a_0}) \to H_p(\mathbb{T}_{a_1}) \to H_p(\mathbb{T}_{a_2}) \to \cdots \to H_p(\mathbb{T}_{a_n}).$$

Definition 1 (Simplicial filtration). A *filtration* $\mathcal{F} = \mathcal{F}(K)$ of a simplicial complex K is a nested sequence of its subcomplexes,

$$\mathcal{F} : \emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$

which is also written with inclusion maps as

$$\mathcal{F} : \emptyset = K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_n = K.$$

$\mathcal{F}$ is called *simplex-wise* if either $K_i \setminus K_{i-1}$ is empty or a single simplex for every $i \in [1, n]$. Notice that the possibility of the difference being empty allows two consecutive complexes to be the same.

Naturally, every simplicial filtration gives rise to a sequence of homomorphisms $h_p^{i,j}$ induced by inclusions, again forming a homology module:

$$0 = H_p(K_0) \to \cdots \to H_p(K_i) \xrightarrow{h_p^{i,j}} H_p(K_j) \to \cdots \to H_p(K_n) = H_p(K)$$

Definition 2 (Abstract simplicial complex). Let V be a finite set. An *abstract simplicial complex* L on V is a family of non-empty subsets of V, such that if $\sigma \in L$ and $\tau \subseteq \sigma$ is non-empty, then $\tau \in L$.

Definition 3 (Rips complex). Let X be a metric space and let a sample $S \subset X$ be a finite subset. Choose a scale $r \geq 0$. The **Rips** complex $\mathrm{Rips}(S, r)$ is an abstract simplicial complex defined by the following rules:

1. The vertex set is S.

2. A subset $\sigma \subseteq S$ is a simplex if and only if $\mathrm{Diam}(\sigma) \leq r$.

Definition 4 (Rips filtration)**.** Let X be a metric space and let a sample $S \subset X$ be a finite subset. The **Rips filtration** on S is the collection of abstract simplicial complexes $\{\mathrm{Rips}(S, r)\}_{r \geq 0}$ along with inclusions

$$i_{r_1,r_2} : \mathrm{Rips}(S, r_1) \hookrightarrow \mathrm{Rips}(S, r_2) \quad \text{for all } r_1 \leq r_2.$$

Definition 5 (Čech complex)**.** Let X be a metric space and let a sample $S \subset X$ be a finite subset. Choose a scale $r \geq 0$. The Čech complex $\mathrm{Cech}(S, r)$ is an abstract simplicial complex defined by the following rules:

1. The vertex set is S.

2. A subset $\sigma \subseteq S$ is a simplex if and only if $\bigcap_{x \in \sigma} B(x, r) \neq \emptyset$.

Definition 6 (Čech filtration)**.** Let X be a metric space and let a sample $S \subset X$ be a finite subset. The **Čech filtration** of S is the collection of abstract simplicial complexes $\{\mathrm{Cech}(S, r)\}_{r \geq 0}$ along with inclusions

$$i_{r_1,r_2} : \mathrm{Cech}(S, r_1) \hookrightarrow \mathrm{Cech}(S, r_2) \quad \text{for all } r_1 \leq r_2.$$

Definition 7 (Nerve of a cover)**.** For $k \in \mathbb{N}$, let $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ be a collection of subsets of X. The *nerve* of $\mathcal{U}$ is the abstract simplicial complex $\mathcal{N}(\mathcal{U})$ defined by the following rules:

1. The vertex set is $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$, consisting of k elements.

2. A subset $\sigma \subseteq \mathcal{U}$ is a simplex if and only if $\bigcap_{U \in \sigma} U \neq \emptyset$.

Theorem 2.1.1 (Nerve theorem)**.** *Let $n \in \mathbb{N}$ and assume a collection*

$$\mathcal{U} = \{U_1, \ldots, U_k\}$$

consists of closed convex subsets of $\mathbb{R}^n$. Then

$$U_1 \cup U_2 \cup \cdots \cup U_k \simeq \mathcal{N}(\mathcal{U})$$

Definition 8 (ε-interleaving)**.** Choose $\varepsilon > 0$. Filtrations $\{A_r\}_{r\geq 0}$ and $\{B_r\}_{r\geq 0}$ (obtained by the Rips or the Čech construction) are ε-interleaved if there exist simplicial maps

$$\varphi_r : A_r \to B_{r+\varepsilon} \quad \text{and} \quad \psi_r : B_r \to A_{r+\varepsilon}$$

such that

$$\varphi_{r+\varepsilon} \circ \psi_r : B_r \to B_{r+2\varepsilon} \quad \text{and} \quad \psi_{r+\varepsilon} \circ \varphi_r : A_r \to A_{r+2\varepsilon}$$

are equal to the corresponding inclusions.

$$\begin{tikzcd}
\cdots \arrow[r] & A_r \arrow[r] \arrow[dr, "\varphi_r"'] & A_{r+\varepsilon} \arrow[r] \arrow[dr] & A_{r+2\varepsilon} \arrow[r] & \cdots \\
\cdots \arrow[r] & B_r \arrow[r] \arrow[ur, "\psi_r"] & B_{r+\varepsilon} \arrow[r] \arrow[ur] & B_{r+2\varepsilon} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 9 (Interleaving distance)**.** Given two filtrations, their *interleaving distance* is defined as the infimum of all values $\varepsilon > 0$, for which the filtrations are ε-interleaved.

Definition 10 (Vietoris complex)**.** For $k \in \mathbb{N}$, let $\mathcal{U} = \{U_1, U_2, \ldots, U_k\}$ be a collection of subsets of a finite space X, whose union is X. The *Vietoris complex* of $\mathcal{U}$ is the abstract simplicial complex $\mathcal{V}(\mathcal{U})$ defined by the following rules:

1. The vertex set is X.

2. A set $\sigma \subseteq X$ is a simplex if and only if there exists $U \in \mathcal{U}$ containing σ.

2.2 Persistent Homology

This section presents persistent homology by explaining persistent homology groups and persistent Betti numbers.

Definition 11 (Discrete filtration)**.** Let K be a simplicial complex. A (discrete) *filtration* of K is a sequence of subcomplexes

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K.$$

A filtration of a simplicial complex K in Definition 1 can be expressed as a sequence of natural inclusion maps denoted by

$$K_1 \xrightarrow{i_{1,2}} K_2 \xrightarrow{i_{2,3}} \cdots \xrightarrow{i_{m-1,m}} K_m = K.$$

Given a field $\mathbb{F}$ and $q \in \{0, 1, 2, \ldots\}$, we can apply homology $H_q(-;\mathbb{F})$ to obtain a sequence of homology groups connected by linear maps:

$$H_q(K_1;\mathbb{F}) \xrightarrow{(i_{1,2})_*} H_q(K_2;\mathbb{F}) \xrightarrow{(i_{2,3})_*} \cdots \xrightarrow{(i_{m-1,m})_*} H_q(K_m;\mathbb{F}) = H_q(K;\mathbb{F}).$$

Definition 12 (Persistent homology)**.** Assume K is a simplicial complex, $\mathbb{F}$ is a field, and $q \in \{0, 1, 2, \ldots\}$. Given a filtration

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K$$

of K, the corresponding q-dimensional *persistent homology groups* with coefficients in $\mathbb{F}$ are images of the maps

$$(i_{s,t})_* : H_q(K_s;\mathbb{F}) \to H_q(K_t;\mathbb{F})$$

for all $0 \leq s \leq t \leq m$. The corresponding ranks β^q_s are called *persistent Betti numbers* and are defined as

$$\beta^q_{s,t} = \operatorname{rank}(i_{s,t})_*.$$

2.3 Persistence Diagram and Barcodes

Definition 13 (Persistent Betti Number)**.** The persistent Betti number $\beta^q_{s,t}$ represents the dimension of the subspace of homology elements in K_t that have a representative in K_s. Equivalently, $\beta^q_{s,t}$ indicates the dimension of the collection of homology elements in K_s that are still non-trivial in K_t in the sense that

$$\beta^q_{s,t} = \dim H_q(K_s;\mathbb{F}) / \ker(i_{s,t})_*.$$

Definition 14 (Barcodes)**.** Barcodes give more specific information: a bar $[s, t)$ represents a homology element that is born precisely at s and terminates precisely at t.

Formally, the number of bars containing s and passing through t equals $\beta^q_{s,t}$.

Definition 15 (Homology Born at s)**.** Homology born at s is defined as

$$H_q(K_s; \mathbb{F})/(\mathrm{Im}(i_{s-1,s})_*).$$

Its dimension is $\beta_{s,s} - \beta_{s-1,s}$, representing the number of bars starting at s.

Definition 16 (Homology Terminating at t)**.** Homology terminating at t is defined as

$$\ker(i_{t-1,t})_*.$$

Its dimension is $\beta_{t-1,t-1} - \beta_{t-1,t}$, representing the number of bars terminating at t.

Definition 17 (Homology Born at s Still Alive at t)**.** The quantity $\beta_{s,t} - \beta_{s-1,t}$ represents the dimension of homology born at s which is still alive at t. It gives the number of bars starting at s and passing through t.

Definition 18 ($\mathbf{n}_{s,t}$ — Number of Bars Starting at s and Terminating at t)**.** The quantity $\mathbf{n}_{s,t}$ is defined as

$$\mathbf{n}_{s,t} = (\beta_{s,t-1} - \beta_{s-1,t-1}) - (\beta_{s,t} - \beta_{s-1,t}),$$

representing the number of bars starting at s and terminating at t.

Definition 19 ($\mathbf{n}_{s,\infty}$ — Number of Bars Starting at s and Persisting to Infinity)**.** We additionally define

$$\mathbf{n}_{s,\infty} = \beta_{s,m} - \beta_{s-1,m},$$

representing the number of bars starting at s that are still alive at the end of the filtration.

Definition 20 (Barcode Structure)**.** The q-dimensional *barcode* consists of intervals of the form

$i.$ $[s,t)$ for $1 \leq s < t \leq m$,

$ii.$ $[s,\infty)$ for $1 \leq s \leq m$.

Definition 21 (Multiplicity of barcode intervals)**.** A barcode can have multiple copies of each interval. Fix $1 \leq s < t \leq m$:

- The number of intervals of the form $[s,t)$ is denoted by $\mathbf{n}_{s,t}$.

- The number of intervals of the form $[s, \infty)$ is denoted by $\mathbf{n}_{s,\infty}$.

Definition 22 (Persistence diagrams)**.** Persistence diagrams are a method of visualizing persistent homology. Given a barcode, we can think of an interval $[s, t)$ as a point (s, t) in $\mathbb{R}^2$.

A point corresponding to (s, ∞) cannot be drawn directly, so we use a y-coordinate above m, usually $m + 1$, and represent (s, ∞) as $(s, m + 1)$. The persistence diagram is the collection of weighted points (s, t) in $\mathbb{R}^2$ with their multiplicities. It visualizes the lifetime of homological features.

Definition 23 (Multiplicity in persistence diagrams)**.** Each point (s, t) in a persistence diagram has an assigned multiplicity $\mathbf{n}_{s,t}$, representing the number of bars of the form $[s, t)$.

For points $(s, m + 1)$ representing bars $[s, \infty)$, the multiplicity is $\mathbf{n}_{s,\infty}$.

Lemma 2.3.1 (Fundamental lemma of persistent homology)**.**

$$\beta_{s,t} = \sum_{\substack{s' \leq s \\ t' > t}} \mathbf{n}_{s',t'}.$$

The formula in the lemma can be verified explicitly. However, the statement is apparent from the definitions:

- $\beta_{s,t}$ represents the homology born at s or before and terminating after t.
- $\mathbf{n}_{s,t}$ represents the homology born precisely at s and terminating precisely at t.

2.4 Stability

Definition 24 (Continuous ε-interleaving)**.** Choose $\varepsilon > 0$. Continuous filtrations $\{K_r\}_{r \geq 0}$ and $\{L_r\}_{r \geq 0}$ are ε-interleaved if there exist simplicial maps

$$\varphi_r : K_r \to L_{r+\varepsilon} \quad \text{and} \quad \psi_r : L_r \to K_{r+\varepsilon}$$

such that

$$\varphi_{r+\varepsilon} \circ \psi_r : L_r \to L_{r+2\varepsilon} \quad \text{and} \quad \psi_{r+\varepsilon} \circ \varphi_r : K_r \to K_{r+2\varepsilon}$$

are equal to the corresponding inclusions.

Proposition 2.4.1. Let K be a simplicial complex. Assume $f, g : K \to [0, \infty)$ are filtration functions. Then the sublevel filtrations of K corresponding to f and g are $\|f - g\|_\infty$-interleaved.

Definition 25 (Persistence module)**.** A *persistence module* is a collection of (finite-dimensional) vector spaces $\{V_r\}_{r\geq 0}$ along with linear maps

$$h_{r,q} : V_r \to V_q, \quad \forall r \leq q$$

satisfying

$$h_{r,q} = h_{r,s} \circ h_{s,q} \quad \text{and} \quad h_{r,r} = \mathrm{id}_{V_r} \quad \text{for all } r \leq q \leq s.$$

A scale $r \geq 0$ is said to be *regular* if there exists $\varepsilon > 0$ such that the maps $h_{p,q}$ are isomorphisms for all $p, q \in (r - \varepsilon, r + \varepsilon)$ or (in the case $r = 0$) for all $p, q \in [0, \varepsilon)$, i.e., the maps h are isomorphisms close to r.

A scale r is *critical* if it is not regular.

Definition 26 (Interval module)**.** Let $0 \leq p < q$. An *interval module* $\mathbb{F}_{p,q}$ corresponding to the pair (p, q) is a persistence module $\{V_r\}_{r\geq 0}$ defined as follows:

- $V_r = \mathbb{F}$ for $r \in [p, q)$ and $V_r = 0$ otherwise,
- Maps $h_{s,s'}$ are isomorphisms whenever possible.

Theorem 2.4.2 (Structure theorem for persistent homology)**.** *Each persistence module is isomorphic to a direct sum of interval modules. The decomposition is unique up to the permutation of the intervals.*

Definition 27 (ε-interleaving of persistence modules and interleaving distance)**.** Let $\varepsilon > 0$. Persistence modules $\{V_r\}_{r\geq 0}$ and $\{W_r\}_{r\geq 0}$ along with their respective linear maps $h_{r,q}$ and $h'_{r,q}$ are ε-interleaved if there exist linear maps

$$\varphi_r : V_r \to W_{r+\varepsilon} \quad \text{and} \quad \psi_r : W_r \to V_{r+\varepsilon}$$

such that

$$\varphi_{r+\varepsilon} \circ \psi_r : W_r \to W_{r+2\varepsilon} \quad \text{and} \quad \psi_{r+\varepsilon} \circ \varphi_r : V_r \to V_{r+2\varepsilon}$$

are equal to the respective structure maps $h'_{r,r+2\varepsilon}$ and $h_{r,r+2\varepsilon}$.

Given two persistence modules, their *interleaving distance* d_I is defined as the minimum of all values $\varepsilon > 0$ for which the modules are ε-interleaved.

Definition 28 (Hausdorff distance)**.** Let (X, d) be a metric space and assume $A, B \subset X$ are finite subsets. The *Hausdorff distance* $d_H(A, B)$ is defined as

$$d_H(A,B) = \max\left\{\max_{b\in B}\min_{a\in A} d(a,b),\ \max_{a\in A}\min_{b\in B} d(a,b)\right\}.$$

Definition 29 (Gromov–Hausdorff distance)**.** Suppose A, B are finite metric spaces. The *Gromov–Hausdorff distance* $d_{GH}(A, B)$ is defined as

$$d_{GH}(A,B) = \inf_{\mu,\nu}\left\{d_H(\mu(A),\nu(B))\right\},$$

where the infimum is over all isometric embeddings $\mu : A \to X$ and $\nu : B \to X$ into a metric space X.

Proposition 2.4.3. Let A, B be finite metric spaces with $\varepsilon = d_{GH}(A, B)$. Then for each $q \in \{0, 1, \dots\}$:

1. $\{H_q(\mathrm{Rips}(A, r))\}_{r\geq 0}$ and $\{H_q(\mathrm{Rips}(B, r))\}_{r\geq 0}$ are 2ε-interleaved.

2. $\{H_q(\mathrm{Cech}(A, r))\}_{r\geq 0}$ and $\{H_q(\mathrm{Cech}(B, r))\}_{r\geq 0}$ are ε-interleaved.

Definition 30 (Partial matching)**.** Let $v = (x, y) \in \mathbb{R}^2$ and define

$$\bar{v} = \left(\frac{x+y}{2}, \frac{x+y}{2}\right) \in \mathbb{R}^2.$$

A *partial matching* between persistence diagrams $\mathcal{A}$ and $\mathcal{B}$ is a bijective map

$$\varphi : A' \to B'$$

where $A' \subseteq \mathcal{A}$ and $B' \subseteq \mathcal{B}$.

Definition 31 (Matching distance)**.** The *matching distance* $d_M(\varphi)$ of a partial matching φ is defined as

$$d_M(\varphi) = \max\left\{\max_{v\in A'} d_\infty(v,\varphi(v)),\ \max_{v\in\mathcal{A}\setminus A'} d_\infty(v,\bar{v}),\ \max_{v\in\mathcal{B}\setminus B'} d_\infty(v,\bar{v})\right\}.$$

Definition 32 (Bottleneck distance)**.** The *bottleneck distance* between persistence diagrams $\mathcal{A}$ and $\mathcal{B}$ is the minimal matching distance between them, i.e.,

$$d_B(\mathcal{A}, \mathcal{B}) = \min_{\varphi \in \mu(\mathcal{A}, \mathcal{B})} d_M(\varphi).$$

Theorem 2.4.4 (Isometry theorem)**.** *The interleaving distance between persistence modules equals the bottleneck distance between the corresponding persistence diagrams.*

Theorem 2.4.5 (Stability theorem)**.** *Fix $q \in \{0, 1, 2, \ldots\}$. Assume persistence diagrams $\mathcal{A}$ and $\mathcal{B}$ represent q-dimensional persistent homologies of filtrations $\mathcal{V}$ and $\mathcal{W}$ obtained by one of the following procedures:*

1. *As the sublevel filtrations of filtration functions f and g satisfying $\|f - g\|_\infty \leq \varepsilon$*
2. *As the Rips filtrations of metric spaces X and Y satisfying $d_{GH}(X, Y) \leq \varepsilon/2$*
3. *As the Čech filtrations of metric spaces X and Y satisfying $d_{GH}(X, Y) \leq \varepsilon$*

Then

$$d_B(\mathcal{A}, \mathcal{B}) \leq \varepsilon.$$

CHAPTER 3

ZIGZAG PERSISTENCE

This chapter will describe the mathematical theory of zigzag modules or zigzag diagrams. We introduce the concept of zigzag persistence, an extension of persistent homology that accommodates both forward and backward maps in a sequence of vector spaces. We begin by defining zigzag diagrams of vector spaces and explaining how zigzag persistence is computed. The chapter also presents a key structural result known as the *Diamond Principle*, which provides a powerful tool for analyzing changes in persistence intervals.

The material presented here is based on the foundational article Zigzag Persistence by Carlsson and de Silva [1]. All constructions in this chapter are performed over a fixed field $\mathbb{F}$, and all vector spaces are assumed to be finite-dimensional unless otherwise specified.

3.1 Zigzag Diagrams of Vector Spaces

Definition 33. A zigzag diagram of vector spaces, or shortly a zigzag module $\mathbb{V}$ over a field $\mathbb{F}$ of length n is a sequence of vector spaces V_i's over $\mathbb{F}$ and linear maps between V_i's denoted by p_i's,

$$\mathbb{V} : V_1 \overset{p_1}{\longleftrightarrow} V_2 \overset{p_2}{\longleftrightarrow} \dots \overset{p_{n-2}}{\longleftrightarrow} V_{n-1} \overset{p_{n-1}}{\longleftrightarrow} V_n$$

where each linear map p_i represents either a forward map $\rightarrow$ denoted by f_i or a backward map $\leftarrow$ denoted by g_i. The sequence of linear maps f_i's and g_i's is called the type τ of the zigzag module $\mathbb{V}$.

For example, let $\mathbb{V}$ be a zigzag diagram of vector spaces $V_i's$ of length 5 with type

$\tau = fgfg$. So, $\mathbb{V}$ can be considered as a diagram like

$$\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3 \xrightarrow{f_3} V_4 \xleftarrow{g_4} V_5$$

Moreover, standard persistent modules, which we have discussed in the background chapter, are just zigzag modules with type $\tau = ff...ff$

$$V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-2}} V_{n-1} \xrightarrow{f_{n-1}} V_n$$

To understand zigzag modules more deeply, we need to consider their decomposition as "simpler" zigzag modules, which we are going to explain.

Definition 34. A submodule $\mathbb{W}$ of τ-module $\mathbb{V}$ is defined by subspaces $W_i \leq V_i$ such that $f_i(W_i) \subseteq W_{i+1}$ or $g_i(W_{i+1}) \subseteq W_i$ for all i.

As you can see, $\mathbb{W}$ is also a τ-module, and linear maps in $\mathbb{W}$ are just restrictions of f_i and g_i over W_i and W_{i+1}, respectively. If $\mathbb{W}$ is a submodule of $\mathbb{V}$, then we can write $\mathbb{W} \leq \mathbb{V}$.

Definition 35. A submodule $\mathbb{W}$ is called a summand of $\mathbb{V}$ if there exists a submodule $\mathbb{X} \leq \mathbb{V}$ that is a complement of $\mathbb{W}$ such that $V_i = W_i \oplus X_i$ for all i. Here, we say that $\mathbb{V}$ is the direct sum of $\mathbb{W}$ and $\mathbb{X}$; also, we write $\mathbb{V} = \mathbb{W} \oplus \mathbb{X}$.

Let us consider some examples of zigzag modules.

Example 3.1.1. Let

$$\mathbb{V} : \mathbb{F} \xrightarrow{id} \mathbb{F} \xleftarrow{\pi} \mathbb{F} \oplus \mathbb{F}$$

be a τ-module with type $\tau = fg$, where id is the identity homomorphism and $\pi : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. First consider the zigzag module

$$\mathbb{T} : \mathbb{F} \xrightarrow{id|_{\mathbb{F}}} 0 \xleftarrow{0} 0$$

which is not a submodule of $\mathbb{V}$ since $id|_{\mathbb{F}}(\mathbb{F}) = \mathbb{F} \not\subseteq 0$ even if $T_i \leq V_i$ for all i. Second,

$$\mathbb{S} : 0 \xrightarrow{id|_0} \mathbb{F} \xleftarrow{\pi|_{\mathbb{F}\oplus\mathbb{F}}} \mathbb{F} \oplus \mathbb{F}$$

is a submodule of $\mathbb{V}$ since $S_i \leq V_i$ such that $id|_0(0) \subseteq \mathbb{F}$ and $\pi|_{\mathbb{F}\oplus\mathbb{F}}(\mathbb{F} \oplus \mathbb{F}) = \mathbb{F} \subseteq \mathbb{F}$. However, $\mathbb{S}$ is not a summand of $\mathbb{V}$ because the complement of $\mathbb{S}$ is $\mathbb{T}$, and $\mathbb{T}$ is not a submodule of $\mathbb{V}$ by the first part. Lastly,

$$\mathbb{W} : \mathbb{F} \xrightarrow{id|_{\mathbb{F}}} \mathbb{F} \xleftarrow{\pi|_{\mathbb{F}\oplus 0}} \mathbb{F} \oplus 0$$

is a submodule of $\mathbb{V}$ since $W_i \leq V_i$ for all i such that $id|_{\mathbb{F}}(\mathbb{F}) \subseteq \mathbb{F}$ and $\pi|_{\mathbb{F}}(\mathbb{F}) \subseteq \mathbb{F}$. Moreover, $\mathbb{W}$ is also a submodule of $\mathbb{V}$ since $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$, where $\mathbb{U} : 0 \xrightarrow{id|_0} 0 \xleftarrow{\pi|_{0\oplus\mathbb{F}}} 0 \oplus \mathbb{F}$ is a submodule of $\mathbb{V}$.

Definition 36. A τ-module is decomposable if it can be written as a direct sum of non-zero submodules; otherwise, it is called indecomposable.

Let us recall that a Remak decomposition is a decomposition of an abelian group or a similar object into a finite direct sum of indecomposable objects.

Proposition 3.1.1. Any τ-module $\mathbb{V}$ has a Remak decomposition; in other words, we can write $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_N$ where the summands $\mathbb{W}_j$ are indecomposable.

Proof. Let $\mathbb{V} : V_1 \leftrightarrow V_2 \leftrightarrow \ldots \leftrightarrow V_{n-1} \leftrightarrow V_n$ be a τ-module. We will use induction on $\sum_i \dim(V_i)$. Assume that $\mathbb{V}$ is indecomposable, so then $\mathbb{V}$ has a Remak decomposition with one term. Assume that $\mathbb{V} : V_1 \leftrightarrow V_2 \leftrightarrow \ldots \leftrightarrow V_{n-1} \leftrightarrow V_n$ with $\sum_i \dim(V_i) \leq k$ for some integer k. Need to show that $\bar{\mathbb{V}} : \bar{V_1} \leftrightarrow \bar{V_2} \leftrightarrow \ldots \leftrightarrow \bar{V}_{n-1} \leftrightarrow \bar{V_n}$ with $\sum_i \dim(\bar{V_i}) = k+1$. If $\bar{\mathbb{V}}$ is indecomposable, clearly we are done. If $\bar{\mathbb{V}}$ is decomposable, then $\bar{\mathbb{V}}$ can be written as a direct sum of non-zero submodules such that

$$\bar{\mathbb{V}} = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \cdots \oplus \mathbb{X}_N.$$

Observe that for all $i \in 1, \ldots, N$, $\mathbb{X}_i$ has a total dimension less than $k+1$. By assumption, for all $i \in 1, \ldots, N$, $\mathbb{X}_i$ has Remak decomposition. So, $\bar{\mathbb{V}}$ has a Remak decomposition. Hence, any τ-module $\mathbb{V}$ has a Remak decomposition by induction. □

The following theorem allows us to think of these indecomposable summands as an invariant of a zigzag module $\mathbb{V}$ with respect to isomorphism.

Theorem 3.1.2 (Krull-Remak-Schmidt)**.** *Suppose τ-module $\mathbb{V}$ has Remak decompositions $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \ldots \oplus \mathbb{W}_M$ and $\mathbb{V} = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \ldots \oplus \mathbb{X}_N$. Then $M = N$ and there is a permutation σ of $\{1, 2, .., N\}$ such that $\mathbb{W}_i \cong \mathbb{X}_{\sigma(i)}$.*

Proof. All algebraic operations can be performed within $End(\mathbb{V})$.

Since all τ-modules have a finite total dimension, the ascending chain conditions and the descending chain conditions are automatic. Let $\mathbb{V}$ be the τ-module. Assume that

$$\mathbb{V} \cong \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_M \cong \mathbb{X}_1 \oplus \cdots \oplus \mathbb{X}_N$$

where the summands $\mathbb{W}_i$ and $\mathbb{X}_j$ are indecomposable τ-modules. We need to show that $M = N$ and there is some permutation σ such that $\mathbb{W}_i = \mathbb{X}_{\sigma(i)}$. Let w_i be the projection of $\mathbb{V}$ on $\mathbb{W}_i$, and x_j be the projection of $\mathbb{V}$ on $\mathbb{X}_j$, relative to the above direct sum decompositions. Define $v_j = w_1 x_j$ and $u_j = x_j w_1$. Since $\sum_j^N x_j = id_{\mathbb{V}}$,

$$\sum_{j=1}^{N} v_j u_j|_{\mathbb{W}_1} = \sum_{j=1}^{N} w_1 x_j x_j w_1|_{\mathbb{W}_1} = id_{\mathbb{W}_1}$$

Since $\mathbb{W}_1$ is indecomposable, $End(\mathbb{W}_1)$ is local by Proposition 7.4 in [7]. And therefore, $v_j u_j$ is an automorphism of $\mathbb{W}_1$. Then, renumbering $v_j u_j$, we may assume that that $v_1 u_1$ is an automorphism of $\mathbb{W}_1$. We claim that v_1 and u_1 induce isomorphism between $\mathbb{W}_1$ and $\mathbb{X}_1$. Thanks to Lemma 7.6 in [7], the claim was completed. Then, we have v_1 and u_1 induce isomorphism between $\mathbb{W}_1$ and $\mathbb{X}_1$. Now, we can say that

$$\mathbb{V} \cong \mathbb{X}_1 \oplus (\mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_M)$$

So, the projection map w_1 induces isomorphism from $\mathbb{X}_1$ to $\mathbb{W}_1$, and also since the kernel of w_1 is $\mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_M$, then $\mathbb{X}_1 \cap (\mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_M)$. Moreover, $\mathbb{X}_1 \equiv \mathbb{W}_1$ (mod $\mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_M$). So $\mathbb{V}$ is the direct sum of $\mathbb{X}_1$ and $\mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_M$, and also $\mathbb{V}$ is the direct sum as claimed. Therefore, we have

$$\mathbb{V}/\mathbb{X}_1 \cong \mathbb{X}_2 \oplus \cdots \oplus \mathbb{X}_N \cong \mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_M$$

So, we completed the proof by induction. □

Definition 37. Let τ be a type of length n and consider integers $1 \leq b \leq d \leq n$. The interval τ-module with birth time b and death time d is written $\mathbb{I}(b, d)$ and defined with spaces $I_i = \mathbb{F}$ if $b \leq i \leq d$, $I_i = 0$ otherwise, and with identity maps between adjacent copies of $\mathbb{F}$ and zero maps otherwise.

For example, let $\tau = fgfg$ and then $\mathbb{I}(3, 5)$ is interval τ-module

$$0 \xrightarrow{0} 0 \xleftarrow{0} \mathbb{F} \xrightarrow{id} \mathbb{F} \xleftarrow{id} \mathbb{F}$$

Theorem 3.1.3. *Intervals τ-modules are indecomposable.*

Proof. Assume to the contrary that the τ-module is decomposable. So, an interval τ-module can be written as

$$\mathbb{I}(b, d) = \mathbb{V} \oplus \mathbb{W}$$

This means that both $\mathbb{V}$ and $\mathbb{W}$ are summands of $\mathbb{I}(b, d)$, so they are complements of each other. By definition of interval modules, linear maps between vector spaces indexed from b to d are just identity maps. Since $\mathbb{V}, \mathbb{W} \subseteq \mathbb{I}(b, d)$, linear maps in $\mathbb{V}$ and $\mathbb{W}$ are just restrictions of identity maps. So, $\dim(V_i)$ and $\dim(W_i)$ can not decrease for $b \leq i \leq d$. Also, since $\mathbb{V}$ and $\mathbb{W}$ are complements of each other, $\dim(V_i)$ and $\dim(W_i)$ can not increase for $b \leq i \leq d$. So, $\dim(V_i)$ and $\dim(W_i)$ are constant for $b \leq i \leq d$, and so one of $\mathbb{V}$ and $\mathbb{W}$ must be zero space. Hence, $\mathbb{I}(b, d)$ is indecomposable.

□

Theorem 3.1.4 (Gabriel's Theorem)**.** *The indecomposable τ-modules are precisely the intervals $\mathbb{I}(b, d)$ where $1 \leq b \leq d \leq n$ = length(τ). Equivalently, every τ-module can be written as a direct sum of interval modules.*

Remark 3.1.1. This is a straightforward special case of Gabriel's theorem for the A_n type graphs. The original statement of Gabriel's Theorem is for Dynkin Graphs, which are A_n, D_n, and E type graphs. For a more accessible overview of Gabriel's theorem, see here [5].

Thanks to Gabriel's Theorem, we know that every zigzag module can be written as a direct sum of interval modules. Also, according to the Krull-Remak-Schmidt Principle, this direct sum decomposition is unique to the permutation.

3.2 Zigzag Persistence

In the previous section, we worked with the mathematical theory of zigzag diagrams and presented two fundamental results: the Krull–Remak–Schmidt Proposition and Gabriel's Theorem.

Gabriel's Theorem establishes that every zigzag module can be decomposed as a direct sum of interval modules. Furthermore, the Krull–Remak–Schmidt Principle guarantees that this decomposition is unique up to the permutation of the summands. We now turn to the definition of zigzag persistence and compare its properties with those of ordinary persistence. Following this, we introduce the right filtration operator, a key tool for analyzing and decomposing zigzag persistence. In particular, we will study how this operator facilitates the decomposition of filtered vector spaces associated with zigzag modules.

Definition 38. Let $\mathbb{V}$ be a zigzag module of arbitrary type. ***The zigzag persistence of*** $\mathbb{V}$ is defined to be the multiset $Pers(\mathbb{V}) = \{[b_j, d_j] \subseteq \{1, 2..., n\} : j = 1, 2..., n\}$ of integer interval modules derived from decomposition $\mathbb{V} = \mathbb{I}(b_1, d_1) \oplus \mathbb{I}(b_2, d_2) \oplus \cdots \oplus \mathbb{I}(b_n, d_n)$.

Definition 39. Let $\mathbb{V}$ be a zigzag module and $\mathbb{V}[p, q]$ denote the ***restriction*** of $\mathbb{V}$ to the index set $p \leq i \leq q$. A ***feature*** of $\mathbb{V}$ over the time interval $[p, q]$ is a summand of $\mathbb{V}[p, q]$ which is isomorphic to the interval module $\mathbb{I}(p, q)$.

Proposition 3.2.1. Let $\mathbb{V}$ be a persistence module of length n, and let $1 \leq p \leq q \leq n$. The following are equivalent:

(a) The composition map $V_p \to V_q$ is non-zero.

(b) There exist non-zero elements $x_i \in V_i$ for $p \leq i \leq q$, such that $x_{i+1} = f_i(x_i)$ for $p \leq i \leq q$.

(c) There exists a submodule of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$.

(d) There exists a summand of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$, i.e., a feature over $[p, q]$.

Proof. For (a) $\Rightarrow$ (b), Assume that the composition map $V_p \to V_q$ is non-zero. Let $x_p \in V_p$ such that it maps to a nonzero element in V_q. Let x_i be the image of x_p in V_i. So, x_i is non-zero. Also, there exists x_{i+1} an image of x_p such that $x_{i+1} = f_i(x_i)$ since the composition map is non-zero. Hence, there exist non-zero elements $x_i \in V_i$ for $p \leq i \leq q$, such that $x_{i+1} = f_i(x_i)$ for $p \leq i \leq q$.

For (b) ⇒ (c), assume that there exist nonzero elements $x_i \in V_i$ for $p \leq i \leq q$, such that $x_{i+1} = f_i(x_i)$ for $p \leq i \leq q$. Define $I_i = span(x_i)$ for $p \leq i \leq q$, and define $\mathbb{I}$ as a module with spaces I_i. Observe that $I_i \leq V_i$ for $p \leq i \leq q$ such that $f(I_i) \subseteq I_{i+1}$, $f(span(x_i)) \subseteq span(x_{i+1})$ since $x_{i+1} = f(x_i)$. So, $\mathbb{I}$ is a submodule of $\mathbb{V}[p, q]$.

For (c) ⇒ (a), assume that there exists a submodule of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$. By assumption, there is a submodule of $\mathbb{V}[p, q]$ which is isomorphic to $(I_p \to \cdots \to I_q)$. By the definition of interval modules, $I_i \cong \mathbb{F}$ for $p \leq i \leq q$, and all maps are identity maps. So, there exists a nonzero element x_p in $I_p \leq V_p$, which is mapped to the nonzero element x_q in $I_q \leq V_q$. Hence, the composition map $V_p \to V_q$ is non-zero.

For (d) ⇒ (c), this part is done by the definition of the summand of zigzag modules. Every summand of $\mathbb{V}[p, q]$ is also a submodule of $\mathbb{V}[p, q]$.

For (a) ⇒ (d), assume that the composition map $V_p \to V_q$ is nonzero. Consider an interval decomposition of $V[p, q]$.

$$\mathbb{V}[p, q] = \mathbb{I}(b_1, d_1) \oplus \cdots \oplus \mathbb{I}(b_N, d_N)$$

For each interval module $\mathbb{I}(b_j, d_j)$ for $p \leq j \leq q$, look at $I_p \to I_q$. If $b_j = p$ and $d_j = q$, the map $I_p \to I_q$ is nonzero in $\mathbb{I}(p, q)$. Otherwise, the map is a zero map. So, there exists at least one interval module that is isomorphic to $\mathbb{I}(p, q)$. □

In Proposition 3.2.1, we show some properties for ordinary persistence modules. But we know that ordinary persistence modules are just zigzag modules of type $\tau = f \ldots f$, i.e., all linear maps between vector spaces are forward. Here, we investigate whether these results hold for zigzag modules of arbitrary type. Let us check the results of the previous Proposition 3.2.1 for zigzag modules of arbitrary types. We need to consider three statements analogous to Proposition 3.2.1 from the perspective of zigzag persistence. Let $\mathbb{V}$ be a zigzag module of arbitrary type.

(a) There exists a nonzero element $x_i \in V_i$ for $p \leq i \leq q$, such that $x_{i+1} = f_i(x_i)$ or $x_i = g_i(x_{i+1})$ for $p \leq i \leq q$.

(b) There exists a submodule of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$.

(c) There exists a summand of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$, a feature over $[p, q]$.

We will show that (a) and (b) imply each other and that (c) implies both (a) and (b). Firstly, we start with (a) implying (b). Assume that there exist nonzero elements $x_i \in V_i$ for $p \leq i \leq q$, such that $x_{i+1} = f_i(x_i)$ or $x_i = g_i(x_{i+1})$ for $p \leq i \leq q$. We want to show that there exists a submodule of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$.

Define $I_i = \text{span}(x_i)$ for $p \leq i \leq q$, and also define a module whose spaces are the I_i. Clearly, $I_i \leq V_i$ for $p \leq i \leq q$. Let f be the forward map and g be the backward map. For the forward map f_i, we have $I_i \xrightarrow{f_i} I_{i+1}$. We need to show that $f_i(I_i) \subseteq I_{i+1}$. By assumption$f_i(x_i) = x_{i+1}$, so $f_i(\text{span}(x_i)) \subseteq \text{span}(x_{i+1})$. Therefore, we have $f_i(I_i) \subseteq I_{i+1}$. On the other hand, for the backward map g_i, we have $I_i \xleftarrow{g_i} I_{i+1}$. We need to show that $g_i(I_{i+1}) \subseteq I_i$. By assumption$g_i(x_{i+1}) = x_i$, so $g_i(\text{span}(x_{i+1})) \subseteq \text{span}(x_i)$. Therefore, we have $g_i(I_{i+1}) \subseteq I_i$.

For both the forward and backward maps, we have $I_i \leq V_i$ for $p \leq i \leq q$, such that $f_i(I_i) \subseteq I_{i+1}$ or $g_i(I_{i+1}) \subseteq I_i$ for all i. Hence, the module whose spaces are the I_i is a submodule of $\mathbb{V}$, and this submodule is isomorphic to $\mathbb{I}(p, q)$, since $I_i = \text{span}(x_i) = \mathbb{F}$.

Next, we show that (b) implies (a). Assume that there exists a submodule of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$. This means that there is a submodule $(\mathbb{F} \leftrightarrow \cdots \leftrightarrow \mathbb{F})$ over the interval $[p, q]$. Hence, there exist nonzero elements $x_i \in V_i$ for $p \leq i \leq q$, such that $x_{i+1} = f_i(x_i)$ or $x_i = g_i(x_{i+1})$ for $p \leq i \leq q$.

Let us show that (c) implies both (a) and (b). Assume that there exists a summand of $\mathbb{V}[p, q]$ isomorphic to $\mathbb{I}(p, q)$, i.e., a feature over $[p, q]$. By the definition of summands, we know that every summand is also a submodule. Thus, statement (b) is proven. Furthermore, since (b) implies (a), it follows that (c) also implies (a).

Remark: (a) and (b) do not imply (c) Example:

Definition 40. Let $\mathbb{V}$ be a zigzag module of length n, and let $K \subseteq \{1, \ldots, n\}$ be any subset. ***The restriction*** of Pers($\mathbb{V}$) to K is the multiset

$$\text{Pers}(\mathbb{V})|_K = \{I \cap K | I \in \text{Pers}(\mathbb{V}), I \cap K \neq \emptyset\}$$

Proposition 3.2.2. Let $\mathbb{V}$ be a zigzag module of length n and let $1 \leq p \leq q \leq n$. The following statements are equivalent.

1. There exists a summand of $V[p,q]$ isomorphic to $\mathbb{I}(p,q)$, i.e., a feature over $[p,q]$.
2. There exists a summand of $\mathbb{V}$ isomorphic to $\mathbb{I}(p',q')$ for some $[p',q'] \supseteq [p,q]$.

Indeed, there is a bijection between intervals $[p,q]$ in Pers($\mathbb{V}[p,q]$) and intervals $[p',q'] \supseteq [p,q]$ in Pers($\mathbb{V}$).

Proof. Let $\mathbb{V}$ be a zigzag module of length n, and let $1 \leq p \leq q \leq n$.

$$\mathbb{V} : V_1 \leftrightarrow \cdots \leftrightarrow V_p \leftrightarrow \cdots \leftrightarrow V_q \leftrightarrow \cdots \leftrightarrow V_n$$

We aim to establish a bijection between intervals $[p,q]$ in Pers($\mathbb{V}[p,q]$) and intervals $[p',q'] \supseteq [p,q]$ in Pers($\mathbb{V}$). Suppose that $\mathbb{V}$ has the interval decomposition:

$$\mathbb{V} = \mathbb{I}(b_1,d_1) \oplus \cdots \oplus \mathbb{I}(b_N,d_N)$$

where each $\mathbb{I}(b_i,d_i)$ is an interval module. By restriction of the interval $[p,q]$,

$$\mathbb{V}[p,q] : V_p \leftrightarrow \cdots \leftrightarrow V_q$$

So, the interval decomposition of $\mathbb{V}[p,q]$ is

$$\mathbb{V}[p,q] = \mathbb{I}(b_1,d_1)[p,q] \oplus \cdots \oplus \mathbb{I}(b_N,d_N)[p,q]$$

by restriction of the interval $[p,q]$. $\mathbb{I}(b_i,d_i)[p,q]$ denotes the restriction of $\mathbb{I}(b_i,d_i)$ for all i to the index set $p \leq i \leq q$. So, the interval $[b_i,d_i]$ restricts to $[p,q]$ if and only if $[b_i,d_i] \supseteq [p,q]$.

Thus, we established a bijection between intervals $[p,q]$ in Pers($\mathbb{V}[p,q]$) and intervals $[p',q'] \supseteq [p,q]$ in Pers($\mathbb{V}$). $\square$

Example 3.2.1. Let $\mathbb{V}$ be a zigzag module of type $\tau = fgf$ of length 4 as below:

$$\mathbb{V} = \mathbb{F} \xrightarrow{id} \mathbb{F} \xleftarrow{\pi} \mathbb{F}^2 \xrightarrow{id} \mathbb{F}^2$$

The zigzag module $\mathbb{V}$ has the interval decomposition:

$$\mathbb{V} = \mathbb{I}(1,3) \oplus \mathbb{I}(3,4)$$

where $\mathbb{I}(1,3) = (\mathbb{F} \to \mathbb{F} \leftarrow \mathbb{F} \to 0)$ and $\mathbb{I}(3,4) = (0 \to 0 \leftarrow \mathbb{F} \to \mathbb{F})$ are interval modules. By definition of Zigzag Persistence,Pers($\mathbb{V}$) $= \{[1,3],[3,4]\}$.

Consider $\mathbb{V}[2,3]$ as a restriction of $\mathbb{V}$ to the index set $[2,3]$; $\mathbb{V}[2,3] = \mathbb{F} \xleftarrow{\pi} \mathbb{F}^2$. Observe that $\mathbb{V}[2,3]$ has the interval decomposition as follows:

$$\mathbb{V}[2,3] = \mathbb{I}(2,3) \oplus \mathbb{I}(3,3) = \mathbb{I}(1,3)[2,3] \oplus \mathbb{I}(3,4)[2,3]$$

So, we have $\mathrm{Pers}(\mathbb{V}[2,3]) = \{[2,3],[3,3]\}$.

Thus, we can see the bijection between the interval $[2,3]$ in $\mathrm{Pers}(\mathbb{V}[2,3])$ and only the interval $[1,3] \supseteq [2,3]$ in $\mathrm{Pers}(\mathbb{V})$.

3.2.1 The Right Filtration Operator

We need to define the right filtration operator to understand and construct the decompositions of a τ-module $\mathbb{V}$.

Definition 41. The right filtration $\mathrm{R}(\mathbb{V})$ of τ-module $\mathbb{V}$ of length n takes the form $\mathrm{R}(\mathbb{V}) = (R_0, R_1, \ldots, R_n)$, where the R_i are subspaces of V_n satisfying the inclusion relation

$$0 = R_0 \leq R_1 \leq R_2 \ldots R_{n-1} \leq R_n = V_n$$

$\mathrm{R}(\mathbb{V})$ is defined recursively as follows :
Base step: If $\mathbb{V}$ has length 1, then $\mathrm{R}(\mathbb{V}) = (0, V_1)$. Recursive Step: Suppose that we have $\mathrm{R}(\mathbb{V}) = (R_0, R_1, \ldots R_n)$.

- If $\mathbb{V}^+$ is $\mathbb{V} \xrightarrow{f_n} V_{n+1}$, then $\mathrm{R}(\mathbb{V}^+) = (f_n(R_0), f_n(R_1), \ldots, f_n(R_n), V_{n+1})$
- If $\mathbb{V}^+$ is $\mathbb{V} \xleftarrow{g_n} V_{n+1}$, then $\mathrm{R}(\mathbb{V}^+) = (0, g_n^{-1}(R_0), g_n^{-1}(R_1), ..., g_n^{-1}(R_n))$

We observe that $f_n(R_0) = f_n(0) = 0$ and $g_n^{-1}(R_n) = g_n^{-1}(V_n) = V_{n+1}$, and also $\mathrm{R}(\mathbb{V}^+)$ is also filtration. It is easy to see that $\mathrm{R}(\mathbb{V})$ consists of all subspaces of V_n that are naturally defined by the linear maps p_i.
Also, we can say that, for each i, R_i/R_{i-1} carries information dating back to some earliest V_j in the sequence of vector spaces.

Definition 42. The birth time index $b(\tau) = (b_1, b_2, \ldots, b_n)$ is a vector of integers b_i that indicates the birth times associated with the subquotients R_i/R_{i-1} of the right filtration of τ-module. This is defined recursively as follows:

- Base Step: If τ is empty, then $b(\tau) = 1$. ($\mathbb{V}$ has length 1.)
- Recursive Step: Assume that we have $b(\tau) = (b_1, b_2, \ldots, b_n)$. If τ^+ is τf, then $b(\tau^+) = (b_1, b_2, \ldots, b_n, n+1)$. If τ^+ is τg, then $b(\tau^+) = (n+1, b_1, b_2, \ldots, b_n)$.

Example 3.2.2. Let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2$ be a zigzag module of length 2 of type $\tau = f$.

The module $\mathbb{V}$ has the right filtration

$$\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2) = (f_1(0), f_1(V_1), V_2) = (0, f_1(V_1), V_2)$$

as defined by the notion of right filtration. We can interpret $\mathrm{R}(\mathbb{V})$ as a filtration on V_2. By examining the subquotients of $\mathrm{R}(\mathbb{V})$, we observe that they encode information about previous V_j in the sequence of vector spaces. As in the example, $R_2/R_1 = f_1(V_1)/0$ corresponds to generators born at index 1 and persisting to index 2. Similarly, $R_2/R_1 = V_2/f_1(V_1)$ corresponds to generators born or appearing exclusively at index 2.
Let $\mathbb{V} : V_1 \xleftarrow{g_1} V_2$ be a zigzag module of length 2 of type $\tau = g$.

The module $\mathbb{V}$ has the right filtration

$$\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2) = (0, g_1^{-1}(0), g_1^{-1}(V_1) = (0, g_1^{-1}(0), V_2)$$

Examining the subquotients of $\mathrm{R}(\mathbb{V})$, we observe that $R_1/R_0 = g_1^{-1}(0)/0$ corresponds to generators that are at index 2 and destroy when returning to index 1. Similarly, $R_2/R_1 = V_2/g_1^{-1}(0)$ corresponds to generators that persist from index 2 to index 1.

Example 3.2.3. In this example, we will see the right filtrations of four types of zigzag modules with types $\tau = ff, \tau = fg, \tau = gf$ and $\tau = gg$.

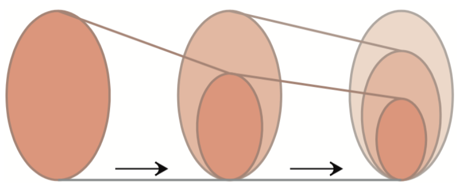

Figure 3.1: Illustration of the Right Filtration of the Zigzag Module $\mathbb{V}$ of the type $\tau = ff$ in [1]

Firstly, let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$ be a zigzag module of length 3 of type $\tau = ff$ as in the above figure. The module $\mathbb{V}$ has the right filtration $\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2, R_3)$

$$\mathrm{R}(\mathbb{V}) = (f_2f_1(0), f_2f_1(V_1), f_2(V_2), V_3) = (0, f_2f_1(V_1), f_2(V_2), V_3)$$

Examining the subquotients of $\mathrm{R}(\mathbb{V})$, we observe the following:

- $R_1/R_0 = f_2f_1(V_1)/0$ corresponds to generators that persist from index 1 to index 3.
- $R_2/R_1 = f_2(V_2)/f_2f_1(V_1)$ corresponds to generators that are born at index 2 and persist to index 3.
- $R_3/R_2 = V_3/f_2(V_2)$ corresponds to generators that are born and appear only at index 3.

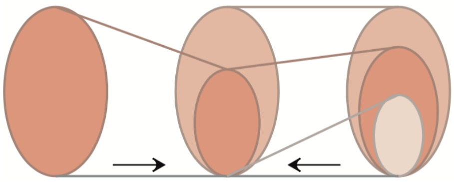

Figure 3.2: Illustration of the Right Filtration of the Zigzag Module $\mathbb{V}$ of the type$\tau = fg$ in [1]

Then, let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module of length 3 of type $\tau = fg$ as in the above figure. The module $\mathbb{V}$ has the right filtration $\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2, R_3)$

$$\mathrm{R}(\mathbb{V}) = (0, g_2^{-1}f_1(0), g_2^{-1}f_1(V_1), g_2^{-1}(V_2)) = (0, g_2^{-1}(0), g_2^{-1}f_1(V_1), V_3)$$

Examining the subquotients of $\mathrm{R}(\mathbb{V})$, we observe the following:

- $R_1/R_0 = g_2^{-1}(0)/0$ corresponds to generators that are at index 3 and destroy when returning to index 2.
- $R_2/R_1 = g_2^{-1}f_1(V_1)/g_2^{-1}(0)$ corresponds to generators that appear at index 1 and persist from index 1 to index 3.
- $R_3/R_2 = V_3/g_2^{-1}f_1(V_1)$ corresponds to generators that are born at index 2 and persist from index 2 to index 3.

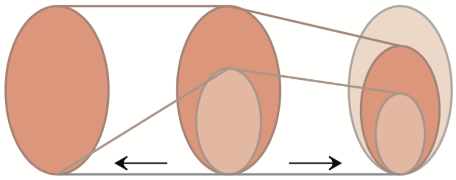

Figure 3.3: Illustration of the Right Filtration of the Zigzag Module $\mathbb{V}$ of the type $\tau = gf$ in [1]

Then, let $\mathbb{V} : V_1 \xleftarrow{g_1} V_2 \xrightarrow{f_2} V_3$ be a zigzag module of length 3 of type $\tau = gf$ as in the above figure. The module $\mathbb{V}$ has the right filtration $\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2, R_3)$

$$\mathrm{R}(\mathbb{V}) = (0, f_2(0), f_2g_1^{-1}(0), f_2g_1^{-1}(V_2), V_3) = (0, f_2g_1^{-1}(0), f_2(V_2), V_3)$$

Examining the subquotients of $\mathrm{R}(\mathbb{V})$, we observe the following:

- $R_1/R_0 = f_2g_1^{-1}(0)/0$ corresponds to generators that are born at index 2 and persist from index 2 to index 3.
- $R_2/R_1 = f_2(V_2)/f_2g_1^{-1}(0)$ corresponds to generators that appear at index 1 and persist from index 1 to index 3.
- $R_3/R_2 = V_3/f_2(V_2)$ corresponds to generators that are born and appear only at index 3.

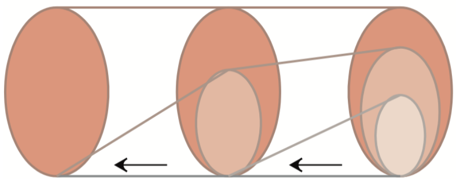

Figure 3.4: Illustration of the Right Filtration of the Zigzag Module $\mathbb{V}$ of the type $\tau = gg$ in [1]

Finally, let $\mathbb{V} : V_1 \xleftarrow{g_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module of length 3 of type $\tau = gg$ as in the above figure. The module $\mathbb{V}$ has the right filtration $\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2, R_3)$

$$\mathrm{R}(\mathbb{V}) = (0, g_2^{-1}(0), g_2^{-1}g_1^{-1}(0), g_2^{-1}g_1^{-1}(V_1)) = (0, g_2^{-1}(0), g_2^{-1}g_1^{-1}(0), V_3)$$

Examining the subquotients of $\mathrm{R}(\mathbb{V})$, we observe the following:

- $R_1/R_0 = g_2^{-1}(0)/0$ corresponds to generators that are born and appear only at index 3.

- $R_2/R_1 = g_2^{-1}g_1^{-1}(0)/g_2^{-1}(0)$ corresponds to generators that are born at index 2 and persist from index 2 to index 3.

- $R_3/R_2 = V_3/f_2(V_2)$ corresponds to generators that appear at index 1 and persist from index 1 to index 3.

Theorem 3.2.3. *Let τ be a type of length n, with $b(\tau) = (b_1, b_2, \ldots, b_n)$. For each $i = 1, 2, .., n$, we have isomorphism*

$$R(\mathbb{I}(b_i, n)) = \mathbb{J}(i, n)$$

where $\mathbb{J}(i, n) = (J_0, J_1, \ldots, J_n)$ is a filtration on k defined by $J_0 = J_1 = \cdots = J_{i-1} = 0$ and $J_i = J_{i+1} = \cdots = J_n = k$.

Proof. We will prove by induction on the length of τ.

- Base Step: Assume that τ is empty. By definition, $b(\tau) = (1)$. Then

  $$R(\mathbb{I}(1, 1)) = (0, k) = \mathbb{J}(1, 1)$$

  Assume that the result is true for τ of length n with $b(\tau) = (b_1, b_2, \ldots, b_n)$. Define τ^+ as a type of zigzag module of length $n+1$ obtained from τ by adding a forward map f or a backward map g to type τ. So, we have two choices for τ^+; $\tau^+ = \tau f$ or $\tau^+ = \tau g$.

- The first choice is $\tau^+ = \tau f$. Since our type has changed, the birth-time index has also changed. By the definition of the birth time index of τ^+, $b(\tau^+) = (b_1^+, b_2^+, ..., b_n^+, b_{n+1}^+) = (b_1, b_2, ..., b_n, n+1)$. For $1 \leq i \leq n$, the right filtration $R(\mathbb{I}_{\tau^+}(b_i, n+1)) = R(\mathbb{I}_\tau(b_i, n) \to k)$. By assumption, $R(\mathbb{I}_\tau(b_i, n)) = \mathbb{J}(i, n) = (J_0, J_1, \cdots, J_n)$. Then $R(\mathbb{I}_{\tau^+}(b_i, n+1) = (f(J_0), f(J_1), ..f(J_n), k)$ by definition of right filtration. Since f is a linear map in interval modules, then f is an identity map. So, $R(\mathbb{I}_{\tau^+}(b_i, n+1)) = (J_0, J_1, \ldots, J_n, k) = \mathbb{J}(i, n+1)$. We need to check $i = n+1$ case, $R(\mathbb{I}_{\tau^+}(n+1, n+1)) = R(0 \leftrightarrow 0 \cdots \leftrightarrow 0) \to k) = (0, 0, .., 0, k) = \mathbb{J}(n+1, n+1)$. So, it is satisfied.

- The second choice is $\tau^+ = \tau g$. Again, by the definition of birth time index of τ^+, we have $b(\tau^+) = (b_1^+, b_2^+, ..., b_n^+, b_{n+1}^+) = (n+1, b_1, b_2, ..., b_n)$. So, $b_i^+ = b_{i-1}$. For $2 \leq i \leq n+1$, the right filtration is $R(\mathbb{I}_{\tau^+}(b_i, n+1)) = R(\mathbb{I}_\tau(b_{i-1}, n) \leftarrow k)$. By assumption, $R(\mathbb{I}_\tau(b_{i-1}, n)) = \mathbb{J}(i-1, n) = (J_0, J_1, .., J_n)$. Then $R(\mathbb{I}_{\tau^+}(b_{i-1}, n+1) = (0, g^{-1}(J_0), .., g^{-1}(J_n)$ by the definition of right filtration. Since f is a linear map in interval modules, then f is an identity map. So, $R(\mathbb{I}_{\tau^+}(b_{i-1}, n+1)) = (0, J_0, .., J_n) = \mathbb{J}(i, n+1)$. We need to check $i = 1, b_1^+ = n+1$, $R(\mathbb{I}_{\tau^+}(n+1, n+1)) = ((0 \leftrightarrow 0 \cdots \leftrightarrow 0) \leftarrow k) = (0, g^{-1}(0), \ldots, g^{-1}(0), k) = (0, k, k \ldots, k) = \mathbb{J}(1, n+1)$. So, it is satisfied.

□

3.2.2 Decomposition of Filtered Vector Spaces

In the previous section, we defined the right filtration $\mathrm{R}(\mathbb{V})$ of the zigzag module $\mathbb{V}$ of length n. These filtrations belong to filtered vector spaces of depth n.

Definition 43. ***A filtered vector space of depth n*** is a sequence $\mathcal{R} = (R_0, R_1, \ldots, R_n)$ of vector spaces, where $0 = R_0 \leq R_1 \leq \cdots \leq R_n$. The class of such objects is denoted by Filt$_n$.

Example 3.2.4. Let $\mathbb{V} : \mathbb{F} \xrightarrow{id} \mathbb{F} \xleftarrow{\pi} \mathbb{F}^2$ be a zigzag module of length 3 with type $\tau = fg$. In this case, $f_1 = id$ and $g_2 = \pi$ Applying the right filtration operator to the zigzag module $\mathbb{V}$ by recursive definition of right filtration:

$$\mathrm{R}(\mathbb{V}) = (0, g_2^{-1}(0), g_2^{-1} f_1(V_1), V_3)$$

$$\mathrm{R}(\mathbb{V}) = (0, \pi^{-1}(0), \pi^{-1} id(\mathbb{F}), \mathbb{F} \oplus \mathbb{F}) = (0, \mathbb{F}, \mathbb{F} \oplus \mathbb{F}, \mathbb{F} \oplus \mathbb{F}).$$

Hence, $\mathrm{R}(\mathbb{V}) = (R_0, R_1, R_2, R_3) = (0, \mathbb{F}, \mathbb{F} \oplus \mathbb{F}, \mathbb{F} \oplus \mathbb{F}) \in$ Filt$_3$ such that $0 \leq \mathbb{F} \leq \mathbb{F} \oplus \mathbb{F} \leq \mathbb{F} \oplus \mathbb{F}$.

Definition 44. A filtered vector space $\mathcal{S} = (S_0, S_1, \ldots, S_n)$ is a ***subspace*** of $\mathcal{R}$ if $S_i \leq R_i$ for all i.

Example 3.2.5. For subspaces of a filtered vector space, consider the following example: Let $\mathbb{V} : \mathbb{F} \xrightarrow{id} \mathbb{F} \xleftarrow{\pi} \mathbb{F} \oplus \mathbb{F}$ be a zigzag module of length 3 with type $\tau = fg$ as in the previous example. Let

$$\mathcal{S} = (S_0, S_1, S_2, S_3) = (0, \mathbb{F}, \mathbb{F}, \mathbb{F}),$$

$$\mathcal{T} = (T_0, T_1, T_2, T_3) = (0, 0, \mathbb{F}, \mathbb{F} \oplus \mathbb{F})$$

be filtered vector spaces of length 3 with the same type τ as the type of $\mathbb{V}$. Since for all i, $S_i \leq R_i$ and $T_i \leq R_i$, $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathcal{R}(\mathbb{V})$.

Definition 45. A filtered vector space $\mathcal{S} = (S_0, S_1, \ldots, S_n)$ is a ***induced subspace*** of $\mathcal{R}$ if there exists a vector space $K \leq R_n$ such that $S_i = R_i \cap K$ for all i.

Example 3.2.6. In the previous example, we showed that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathrm{R}(\mathbb{V})$. $\mathcal{S} = (S_0, S_1, S_2, S_3) = (0, \mathbb{F}, \mathbb{F}, \mathbb{F})$ is induced subspace of $\mathrm{R}(\mathbb{V})$ since there exists a subspace K of R_3, $K = \mathbb{F} \leq \mathbb{F} \oplus \mathbb{F} = R_3$ such that $S_i = R_i \cap K$ for all i.

$$S_0 = 0 = R_0 \cap K = 0 \cap \mathbb{F} = 0$$

$$S_1 = \mathbb{F} = R_1 \cap K = \mathbb{F} \cap \mathbb{F} = \mathbb{F}$$

$$S_2 = \mathbb{F} = R_2 \cap K = \mathbb{F} \oplus \mathbb{F} \cap \mathbb{F} = \mathbb{F}$$

$$S_3 = \mathbb{F} = R_3 \cap K = \mathbb{F} \oplus \mathbb{F} \cap \mathbb{F} = \mathbb{F}.$$

However, $\mathcal{T} = (T_0, T_1, T_2, T_3) = (0, 0, \mathbb{F}, \mathbb{F}_2)$ is not induced subspace of $\mathrm{R}(\mathbb{V})$ since there is not a subspace K of R_3 such that $T_i = R_i \cap K$.

$$\text{If } K = 0, \text{then } S_3 = \mathbb{F}_2 \neq R_3 \cap K = \mathbb{F}_2 \cap 0 = 0$$

$$\text{If } K = \mathbb{F}, \text{then } S_3 = \mathbb{F}_2 \neq R_3 \cap K = \mathbb{F}_2 \cap \mathbb{F} = \mathbb{F}$$

$$\text{If } K = \mathbb{F}_2,\ \text{then } S_3 = \mathbb{F}_2 = R_3 \cap K = \mathbb{F}_2 \cap \mathbb{F}_2 = \mathbb{F}_2$$

$$\text{but, } S_2 = \mathbb{F} \neq R_2 \cap K = \mathbb{F}_2 \cap \mathbb{F}_2 = \mathbb{F}_2.$$

We can observe that by definition of the induced subspace of a filtered vector space $\mathcal{R}$, if $\mathcal{S}$ is an induced space, then there exists $K \leq R_n$, $S_i = R_i \cap K$ for all i. And, for n , $S_n = R_n \cap K$. So, $S_n = R_n \cap K = K$. Hence, $S_n = K$.

Definition 46. $\mathcal{R}$ is ***the direct sum*** of two subspaces $\mathcal{S}$ and $\mathcal{T}$, if $R_i = S_i \oplus T_i$. We can write $\mathcal{R} = \mathcal{S} \oplus \mathcal{T}$, the two subspaces $\mathcal{S}$ and $\mathcal{T}$ are said to be complementary summands.

Proposition 3.2.4. Every induced subspace of a filtered vector space has a complementary summand.

Proof. Let $\mathcal{R}$ be a filtered vector space, $\mathcal{R} = (R_0, R_1, \ldots, R_n)$ with $0 = R_0 \leq R_1 \leq \cdots \leq R_n$. Assume that $\mathcal{S} = (S_0, S_1, \ldots, S_n)$ is an induced subspace of $\mathcal{R}$ with $0 = S_0 \leq S_1 \leq \cdots \leq S_n$. Need to show that $\mathcal{S}$ has a complementary summand, i.e. , there exists a subspace $\mathcal{T} = (T_0, T_1, \ldots, T_n)$ of the filtered vector space $\mathcal{R}$ such that $\mathcal{S} \oplus \mathcal{T} = \mathcal{R}$ such that $S_i \oplus T_i = R_i$ for all i. We construct $\mathcal{T}$, inductively as follows: Since $R_0 = S_0 = 0$, we can take $T_0 = 0$. So, $R_0 = S_0 \oplus T_0$. Assume that we have T_k such that $R_k = S_k \oplus T_k$. This means that $S_k \cap T_k = 0$. So,

$$T_k \cap S_{k+1} \leq T_k \cap S_n \text{ since } S_{k+1} \leq S_n$$

$$T_k \cap S_{k+1} \leq (T_k \cap R_k) \cap S_n \text{ since } T_k = (T_k \cap R_k)$$

$$T_k \cap S_{k+1} \leq T_k \cap (R_k \cap S_n)$$

$$T_k \cap S_{k+1} \leq T_k \cap S_k = 0 \text{ since } R_k \cap S_n = S_k.$$

So, T_k and S_{k+1} are independent subspaces of R_{k+1}, and so T_k can be extended to a complement T_{k+1} of S_{k+1} in R_{k+1}. Hence, $R_{k+1} = S_{k+1} \oplus T_{k+1}$.

□

Corollary 3.2.5. The indecomposables in Filt_n are precisely the intervals $\mathbb{J}(i, n)$. Thus, every filtered vector space can be decomposed as a finite direct sum of intervals.

Proof. $\mathcal{S}$ is an induced subspace of $\mathcal{R}$ if there exists a subspace $K \leq R_n$ such that $S_i = R_i \cap K$ for all i. Also, by previous proposition 3.2.4, $\mathcal{R}$ has a nontrivial summand if and only if R_n has nontrivial vector subspaces. This means that $\dim(R_n) > 1$. But, for intervals $\mathcal{J}(i, n)$, $J_n = \mathbb{F}$, and so $\dim(J_n) = 1$. Hence, $\mathcal{J}(i, n)$ can not have a nontrivial summand. □

Definition 47. The ***dimension*** of $\mathcal{R} \in \text{Filt}_n$ is defined to be a vector of integers

$$\dim(\mathcal{R}) = (c_1, c_2, \ldots c_n),$$

where $c_i = \dim(R_i/R_{i-1})$ are dimensions of the subquotients of the filtration $\mathcal{R}$.

Proposition 3.2.6. Let $\mathcal{R}$ be a filtered vector space of depth n, with $\dim(\mathcal{R}) = (c_1, c_2, \dots, c_n)$. For any decomposition of $\mathcal{R}$ into intervals, the multiplicity of $\mathbb{J}(i, n)$ is c_i. Thus,

$$\mathcal{R} \cong \bigoplus_{1 \le i \le n} c_i \mathbb{J}(i, n)$$

.

Proof. $\mathcal{R} = (R_0, R_1, \dots, R_n)$ be a filtered vector space of depth n with $\dim(\mathcal{R}) = (c_1, c_2, \dots c_n)$. Assume that $\mathcal{R}$ has a decomposition into intervals $\mathcal{J}(i, n)$ as

$$R \cong \bigoplus_{1 \le i \le n} m_i \mathcal{J}(i, n)$$

for some $\mathcal{J}(i, n)$. Let m_i be a multiplicity of $\mathcal{J}(i, n)$. We need to show that$m_i = c_i$. For all $1 \le k \le n$, $\dim(R_k)$ is the sum of multiplicities of $\mathcal{J}(1, n)$,$\mathcal{J}(2, n)$, $\dots \mathcal{J}(k, n)$. So, $\dim(R_k) = m_1 + m_2 + \cdots + m_k$. Also, by definition of dimension $c_i' s$ of $\mathcal{R}$,

$$\dim(R_k) = \dim(R_1/R_0) + \dim(R_2/R_1) + \cdots + \dim(R_k/R_{k-1})$$

$$\dim(R_k) = c_1 + c_2 + \cdots + c_k.$$

So, by induction, for $k = 1, m_1 = c_1 = 1$. For $k = 2, m_1 + m_2 = c_1 + c_2$, so $m_2 = c_2$ since $m_1 = c_1$. This process continues, for k, $m_1 + m_2 + \cdots + m_k = c_1 + c_2 + \cdots + c_k$ and so $c_k = m_k$. Hence, $m_i = c_i$ for all i. □

Proposition 3.2.7. The right filtration operation respects the direct sum in the sense that

$$\mathrm{R}(\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N) = \mathrm{R}(\mathbb{V}_1) \oplus \cdots \oplus \mathrm{R}(\mathbb{V}_N)$$

for τ-modules $\mathbb{V}_1, \dots \mathbb{V}_N$.

Proof. Let $\mathbb{V}_1, \mathbb{V}_2, \dots \mathbb{V}_N$ be τ-modules such that $\mathbb{V}_i : V_{i,1} \leftrightarrow \cdots \leftrightarrow V_{i,n}$ and $\mathrm{R}(\mathbb{V}_i) = (R_{i,0}, R_{i,1}, \dots, R_{i,n})$ for $1 \le i \le N$. Define $\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N$ as τ-module such that $\mathbb{V} : V_1 \leftrightarrow \cdots \leftrightarrow V_n$ and $\mathrm{R}(\mathbb{V}) = (R_0, R_1, \dots, R_n)$. We will prove the equality by induction on type τ.

For the base of induction on τ, assume that type τ is empty. This means that $\mathbb{V}$ has length one, i.e, $\mathbb{V} : V_1$. So, $\mathrm{R}(\mathbb{V}) = (R_0, R_1) = (0, V_1)$.

$$\mathrm{R}(\mathbb{V}) = \mathrm{R}(\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N) = (0, V_1)$$

Since $\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N$, we have $V_1 = V_{1,1} \oplus \cdots \oplus V_{N,1}$. So,

$$\mathrm{R}(\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N) = (0, V_1) = (0, V_{1,1} \oplus \cdots \oplus V_{N,1})$$

$$\mathrm{R}(\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N) = (0, V_{1,1}) \oplus \cdots \oplus (0, V_{N,1})$$

$$\mathrm{R}(\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N) = \mathrm{R}(\mathbb{V}_1) \oplus \cdots \oplus \mathrm{R}(\mathbb{V}_N)$$

where right filtration of $\mathbb{V}_i$ of τ as assumption is $\mathrm{R}(\mathbb{V}_i) = (0, V_{i,1})$.
To proceed by induction on τ, assume that the statement holds for some τ. Then, $\mathrm{R}(\mathbb{V}) = R(\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N) = R(\mathbb{V}_1) \oplus \cdots \oplus R(\mathbb{V}_N)$ and so,

$$(R_0, R_1, \ldots, R_n) = (R_{1,0}, R_{1,1}, \ldots R_{1,n}) \oplus \cdots \oplus (R_{N,0}, R_{N,1}, \ldots R_{N,n}).$$

For $\tau^+ = \tau f$, $\mathbb{V}^+ = \mathbb{V} \xrightarrow{f} V_{n+1}$ and $\mathbb{V}_i^+ = \mathbb{V}_i \xrightarrow{f} V_{i,n+1}$ for $1 \leq i \leq N$.. Also, $\mathbb{V}^+ = \mathbb{V}_1^+ \oplus \cdots \oplus \mathbb{V}_N^+$. By the recursive definition of right filtration operator,

$$\mathrm{R}(\mathbb{V}^+) = (f(R_0), f(R_1), \ldots, f(R_n), V_{n+1}) \text{ and,}$$

$$\mathrm{R}(\mathbb{V}_i^+) = (f_i(R_{i,0}), f_i(R_{i,1}), \ldots f_i(R_{i,n}), V_{i,n+1}).$$

For some $0 \leq k \leq n$,

$$f(R_k) = f(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k})$$

$$f(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k}) = (f_1 \oplus \cdots \oplus f_N)(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k})$$

$$(f_1 \oplus \cdots \oplus f_N)(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k}) = f_1(R_{1,k}) \oplus f_2(R_{2,k}) \oplus \cdots \oplus f_N(R_{N,k})$$

from facts in linear algebra. So,

$$\mathrm{R}(\mathbb{V}^+) = (\ldots, f(R_k), \ldots) = (\ldots, f_1(R_{1,k}) \oplus f_2(R_{2,k}) \oplus \cdots \oplus f_N(R_{N,k}), \ldots)$$

$$\mathrm{R}(\mathbb{V}^+) = (f_1(R_{1,0}), \ldots, f_1(R_{1,n})) \oplus \cdots \oplus (f_N(R_{N,0}), \ldots, f_N(R_{N,n})).$$

And also, we have $V_{n+1} = V_{1,n+1} \oplus \cdots \oplus V_{i,n+1} \oplus \cdots \oplus V_{N,n+1}$. Hence,

$$\mathrm{R}(\mathbb{V}^+) = \mathrm{R}(\mathbb{V}_1^+ \oplus \cdots \oplus \mathbb{V}_N^+) = \mathrm{R}(\mathbb{V}_1^+) \oplus \cdots \oplus \mathrm{R}(\mathbb{V}_N^+)$$

For $\tau^- = \tau g$, $\mathbb{V}^- = \mathbb{V} \xleftarrow{g} V_{n+1}$ and $\mathbb{V}_i^- = \mathbb{V}_i \xleftarrow{g} V_{i,n+1}$ for $1 \leq i \leq N$.. Also, $\mathbb{V}^- = \mathbb{V}_1^- \oplus \cdots \oplus \mathbb{V}_N^-$. By the recursive definition of right filtration operator,

$$\mathrm{R}(\mathbb{V}^-) = (0, g^{-1}(R_0), g^{-1}(R_1), \ldots, g^{-1}(R_n)) \text{ and,}$$

$$\mathrm{R}(\mathbb{V}_i^-) = (0, g_i^{-1}(R_{i,0}), g_i^{-1}(R_{i,1}), \ldots g_i^{-1}(R_{i,n})).$$

For some $0 \leq k \leq n$,

$$g^{-1}(R_k) = g^{-1}(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k})$$

$$g^{-1}(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k}) = (g_1^{-1} \oplus \cdots \oplus g_N^{-1})(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k})$$

$$(g_1^{-1} \oplus \cdots \oplus g_N^{-1})(R_{1,k} \oplus R_{2,k} \oplus \cdots \oplus R_{N,k}) = g_1^{-1}(R_{1,k}) \oplus g_2^{-1}(R_{2,k}) \oplus \cdots \oplus g_N^{-1}(R_{N,k})$$

from facts in linear algebra. So,

$$\mathrm{R}(\mathbb{V}^-) = (\ldots, g_i^{-1}(R_k), \ldots) = (\ldots, g_1^{-1}(R_{1,k}) \oplus g_2^{-1}(R_{2,k}) \oplus \cdots \oplus g_N^{-1}(R_{N,k}), \ldots)$$

$$\mathrm{R}(\mathbb{V}^-) = (g_1^{-1}(R_{1,0}), \ldots, g_1^{-1}(R_{1,n})) \oplus \cdots \oplus (g_N^{-1}(R_{N,0}), \ldots, g_N^{-1}(R_{N,n})).$$

Hence,

$$\mathrm{R}(\mathbb{V}^-) = \mathrm{R}(\mathbb{V}_1^- \oplus \cdots \oplus \mathbb{V}_N^-) = \mathrm{R}(\mathbb{V}_1^-) \oplus \cdots \oplus \mathrm{R}(\mathbb{V}_N^-)$$

□

3.2.3 Streamlined Modules

In this subsection, we examine a special class of zigzag modules known as streamlined zigzag modules, defined by specific restrictions on the forward and backward maps, as described below.

Definition 48. A τ-module $\mathbb{V}$ is ***right-streamlined*** if each forward map $\xrightarrow{f_i}$ is injective and each backward map $\xleftarrow{g_i}$ is surjective.

Example 3.2.7. Let $\mathbb{V} : \mathbb{F} \xrightarrow{id} \mathbb{F} \xleftarrow{\pi} \mathbb{F} \oplus \mathbb{F}$ be a τ-module with type $\tau = fg$, where id is the identity homomorphism and $\pi : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. As you can see, $\mathbb{V}$ is a right-streamlined τ-module since id is injective and π is a surjective map.
Moreover, the interval modules $\mathbb{I}(b, n)$ of lenght n is right-streamlined where $1 \leq b \leq n$ since $(\mathbb{F} \xleftrightarrow{id} \mathbb{F})$ is always injective and surjective, and $(0 \xrightarrow{0} \mathbb{F})$ is injective, and also $(0 \xleftarrow{0} \mathbb{F})$ is surjective. However, the interval modules $\mathbb{I}(b, d)$ where $1 \leq b \leq d \leq n$ is not right streamlined since $(\mathbb{F} \xrightarrow{0} 0)$ is not injective and $(\mathbb{F} \xleftarrow{0} 0)$ is not surjective.

Proposition 3.2.8. A direct sum $\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N$ of τ-modules is right-streamlined if and only if each summand is right-streamlined.

Proof. Let $\mathbb{V}$ be a τ-module with direct sum decomposition $\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_N$ of τ-modules. Each $V_i \xrightarrow{f} V_{i+1}$ in $\mathbb{V}$ decomposes as $f = f_1 \oplus f_2 \oplus \cdots \oplus f_N$ where f_k is a forward map in $\mathbb{V}_k$ sor some $1 \leq k \leq N$. So, f is injective if and only if f_k is injective for all $1 \leq k \leq N$. Similary, each $V_j \xleftarrow{g} V_{j+1}$ in $\mathbb{V}$ decomposes as $g = g_1 \oplus g_2 \oplus \cdots \oplus g_N$ where g_k is a backward map in $\mathbb{V}_k$ sor some $1 \leq k \leq N$. So, g is surjective if and only if g_k is surjective for all $1 \leq k \leq N$. □

Theorem 3.2.9 (Decomposition Lemma)**.** *[1] Let $\mathbb{V}$ be a right-streamlined τ-module and let $\mathcal{R} = \mathrm{R}(\mathbb{V})$. For any decomposition $\mathcal{R} = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_N$, there exists an unique decomposition $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_N$ such that $\mathcal{S}_i = \mathrm{R}(\mathbb{W}_j)$ for some j.*

If the zigzag module is right-streamlined, it has a unique decomposition for two different decompositions of the right filtration of the zigzag module. See [1] for the proof of Theorem 3.2.9, as the Decomposition Lemma.

Theorem 3.2.10 (Interval decomposition for streamlined modules)**.** *Let $\mathbb{V}$ be a right-streamlined τ-module of length n, and write $\dim(\mathrm{R}(\mathbb{V})) = (c_1, c_2, \ldots, c_n)$, and $b(\tau) = (b_1, b_2, \ldots, b_n)$. Then there is an isomorphism of τ-modules*

$$\mathbb{V} \cong \bigoplus_{1 \leq i \leq n} c_i \mathbb{I}(b_i, n)$$

Proof. Let $\mathbb{V}$ be a right streamlined τ-module of length n with $\mathrm{R}(\mathbb{V})$ is the right filtration of $\mathbb{V}$. By Proposition 3.2.6, there is a decomposition of $\mathrm{R}(\mathbb{V})$ such that $\mathrm{R}(\mathbb{V}) = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_N$ where $\mathcal{J}_k$ is the interval filtered vector space for $1 \leq k \leq N$. In addition, in the decomposition, $\mathcal{J}(i, n)$ occurs with multiplicity c_i for $1 \leq i \leq n$. So, $N = c_1 + \cdots + c_n = \dim(\mathrm{R}(\mathbb{V}))$. By previous Proposition 3.2.9, the decomposition of $\mathrm{R}(\mathbb{V}) = \mathcal{J}_1 \oplus \cdots \oplus \mathcal{J}_N$ produces a decomposition of the right streamlined τ-module $\mathbb{V} = \mathbb{I}_1 \oplus \cdots \oplus \mathbb{I}_N$ where $\mathrm{R}(\mathbb{I}_k) = \mathcal{J}_k$ for all $1 \leq k \leq N$.
Since $\mathbb{V}$ is right streamlined, each $\mathbb{I}_k$ is also right streamlined. So, $\mathbb{I}_k$ is isomorphic to $\mathbb{I}(b, n)$ for some $b \in \mathbb{N}$ as in the example 3.2.7. We know that $\mathrm{R}(\mathbb{I}(b_i, n)) = \mathcal{J}(i, n)$ by the theorem 3.2.3. Therefore, for $\mathcal{J}(i, n)$ in the decoposition of $\mathrm{R}(\mathbb{V})$, we have an isomorhism $\mathbb{I}(b, n) = \mathbb{I}(b_i, n)$ for the birth time index $b(\tau) = (b_1, \ldots, b_n)$. Moreover, $\mathbb{I}(b_i, n)$ have multiplicity c_i since $\mathcal{J}(i, n)$ have multiplicity c_i in the decomposition of

filtered vector space $\mathrm{R}(\mathbb{V})$. Hence,

$$\mathbb{V} \cong \bigoplus_{1 \le i \le n} c_i \mathbb{I}(b_i, n)$$

$\square$

3.3 The Interval Decomposition Algorithm of Zigzag Modules

In this section, we will introduce the Interval Decomposition Theorem, a fundamental result for decomposing zigzag modules. We also provide an algorithm for computing this decomposition and illustrate its application with detailed examples.

3.3.1 The Interval Decomposition Theorem

Definition 49. Let $\mathbb{V} : V_1 \overset{p_1}{\longleftrightarrow} \dots \overset{p_{n-1}}{\longleftrightarrow} V_n$ be a τ-module with $\tau = p_1 \dots p_{n-1}$. The truncation of $\mathbb{V}$ of length k is denoted by $\mathbb{V}[k] : V_1 \overset{p_1}{\longleftrightarrow} \dots \overset{p_{k-1}}{\longleftrightarrow} V_k$ and type of truncation of $\mathbb{V}$ of length k is $\tau[k] = p_1 \dots p_{k-1}$.

Proposition 3.3.1. Let $\mathbb{V} : V_1 \overset{p_1}{\longleftrightarrow} \dots \overset{p_{n-1}}{\longleftrightarrow} V_n$ be an irreducible τ-module of length n. Then there exists a direct sum decomposition

$$\mathbb{V} = \mathbb{V}^1 \oplus \mathbb{V}^2 \oplus \dots \oplus \mathbb{V}^n$$

where each $\mathbb{V}^k$ is supported over the indices $\{1, \dots, k\}$ and is right streamlined over that range.

Proof. Let $\mathbb{V} : V_1 \overset{p_1}{\longleftrightarrow} \dots \overset{p_{n-1}}{\longleftrightarrow} V_n$ be a τ-module of length n. By induction on the length n of $\mathbb{V}$, we will prove this statement. For the base step, assume $n = 1$. So, we have $\mathbb{V} : V_1$ and we can set $\mathbb{V}^1 : V_1$. Since $\mathbb{V}^1$ is right streamlined, we are done. For the inductive step, assume that there exists a direct sum decomposition of

$$\mathbb{V}[k] \cong \mathbb{V}^1 \oplus \dots \oplus \mathbb{V}^{k-1} \oplus \mathbb{W}^k$$

where $\mathbb{V}^i$'s and $\mathbb{W}^k$ are right streamlined. We need to prove that $\mathbb{V}[k+1]$ has a direct sum decomposition as in the theorem. We can consider $\mathbb{V}[k+1]$ as follows:

$$\mathbb{V}[k+1] \cong (\mathbb{V}^1 \oplus \dots \oplus \mathbb{V}^{k-1} \oplus \mathbb{W}^k) \overset{p_k}{\longleftrightarrow} V_{k+1}$$

Since $\mathbb{V}^i$'s have no terms after index $(k-1)$, they cannot interact with the map p_k. So, we have

$$\mathbb{V}[k+1] \cong \mathbb{V}^1 \oplus \cdots \oplus \mathbb{V}^{k-1} \oplus (\mathbb{W}^k \overset{p_k}{\longleftrightarrow} V_{k+1})$$

If we can write $(\mathbb{W}^k \overset{p_k}{\longleftrightarrow} V_{k+1})$ as $\mathbb{V}^k \oplus \mathbb{W}^{k+1}$ where $\mathbb{V}^k$ has length k and both $\mathbb{V}^k$ and $\mathbb{W}^{k+1}$ are right streamlined modules. The last vector space in $\mathbb{W}^k$ is V_k. So, $R(\mathbb{W}^k)$ is a filtration on V_k. We have two cases, the map $p_k : \mathbb{W}^k \leftrightarrow V_{k+1}$ may be forward or backward.

Case f: Assume that we have $f_k : \mathbb{W}^k \to V_{k+1}$. This means that

$$\mathbb{W}^k : W_1^1 \leftrightarrow \ldots \leftrightarrow V_k \xrightarrow{f_k} V_{k+1}$$

Define $\mathcal{S} = R(\mathbb{W}^k) \cap \mathrm{Ker}(f_k)$. Obviously, $\mathcal{S}$ is an induced subspace of $R(\mathbb{W}^k)$. By Proposition 3.2.4, $\mathcal{S}$ has a complement in $R(\mathbb{W}^k)$ such that

$$R(\mathbb{W}^k) = \mathcal{S} \oplus \mathcal{T}.$$

By Proposition 3.2.9, there exists a direct sum decomposition

$$\mathbb{W}^k = \mathbb{V}^k \oplus \bar{\mathbb{W}}^k$$

such that $\mathcal{S} = R(\mathbb{V}^k)$ and $\mathcal{T} = R(\bar{\mathbb{W}}^k)$. So, this defines $\mathbb{V}^k$. Since $\mathcal{S} = R(\mathbb{W}^k) \cap \mathrm{Ker}(f_k)$, f_k is zero on the last term of $\mathbb{V}^k$. Moreover, we can set $\mathbb{W}^{k+1} = (\bar{\mathbb{W}}^k \xrightarrow{f_k} V_{k+1})$ as a right streamlined module because the kernel of f_k on $\bar{\mathbb{W}}^k$ is zero and so f_k is injective. i.e.

$$\mathbb{W}^{k+1} = (\bar{\mathbb{W}}^k \xrightarrow{f_k} V_{k+1}) \text{ where } \mathrm{Ker}(f_k)|_{\bar{\mathbb{W}}^k} = 0.$$

Case g: Assume that we have $g_k : \mathbb{W}^k \leftarrow V_{k+1}$. This means that

$$\mathbb{W}^k : W_1^1 \leftrightarrow \ldots \leftrightarrow V_k \xleftarrow{g_k} V_{k+1}$$

Define $\mathcal{S} = R(\mathbb{W}^k) \cap \mathrm{Im}(g_k)$. Clearly, $\mathcal{S}$ is an induced subspace of $R(\mathbb{W}^k)$. Similar to Case f, by Proposition 3.2.4, $\mathcal{S}$ has a complement in $R(\mathbb{W}^k)$ such that

$$R(\mathbb{W}^k) = \mathcal{S} \oplus \mathcal{T}.$$

By Proposition 3.2.9, there exists a direct sum decomposition

$$\mathbb{W}^k = \mathbb{V}^k \oplus \bar{\mathbb{W}}^k$$

such that $\mathcal{S} = R(\bar{\mathbb{W}}^k)$ and $\mathcal{T} = R(\mathbb{V}^k)$. So, this defines $\mathbb{V}^k$ and g_k is zero on the last term of $\mathbb{V}^k$ because $\mathcal{T} = R(\mathbb{V}^k)$ is defined as complementary of $\mathcal{S} = R(\mathbb{W}^k)\cap\mathrm{Im}(g_k)$. Also, we can set $\mathbb{W}^{k+1} = (\bar{\mathbb{W}}^k \xleftarrow{g_k} V_{k+1})$ g_k is surjective on the last term of V_{k+1} since $\mathcal{S} = R(\mathbb{W}^k) \cap \mathrm{Im}(g_k)$. So, $\mathbb{W}^{k+1}k$ is right streamlined.
Thus, we can define $\mathbb{V}^k$ as in the statement and $\mathbb{W}^{k+1}$ is right streamlined. So, we get

$$\mathbb{V}[n] \cong \mathbb{V}^1 \oplus \cdots \oplus \mathbb{V}^{n-1} \oplus \mathbb{W}^n$$

where $\mathbb{W}^n = \mathbb{V}^n$. □

Theorem 3.3.2. *Let $\mathbb{V}$ be a τ-module. For $1 \leq k \leq n$, define*

$$b(\tau[k]) = (b_1^k, b_2^k, \ldots, b_k^k)$$

Writing $\mathcal{R}_k = R(\mathbb{V}[k])$, define

$$(c_1^k, c_2^k, \ldots, c_k^k) = \begin{cases} \dim(\mathcal{R}_k \cap \mathit{Ker}(f_k)) \\ \dim(\mathcal{R}_k) - \dim(\mathcal{R}_k \cap \mathit{Im}(g_k)) \end{cases}$$

(whichever is applicable) when $k \neq n$, and

$$(c_1^n, c_2^n, \ldots, c_n^n) = \dim(\mathcal{R}_n).$$

Then

$$\mathbb{V} \cong \bigoplus_{1\leq i\leq k\leq n} c_i^k \mathbb{I}(b_i^k, k).$$

Proof. By using Proposition 3.3.3, we can decompose $\mathbb{V}$ as

$$\mathbb{V} \cong \mathbb{V}^1 \oplus \cdots \oplus \mathbb{V}^n$$

where each $\mathbb{V}^k$ is supported over the indices $\{1, \ldots, k\}$ and is right streamlined over that range.

Our aim is to calculate the decomposition of each $\mathbb{V}^k$ into intervals $\mathbb{I}(b, k)$. We have that

$$\mathbb{V}[k] \cong \mathbb{V}^k \oplus \mathbb{V}^{k+1}[k] \oplus \cdots \oplus \mathbb{V}^n[k]$$

So, we can define $\mathbb{W}^k = \mathbb{V}^{k+1} \oplus \cdots \oplus \mathbb{V}^n$. By using Proposition 3.2.7, we have

$$\mathcal{R}_k = \mathrm{R}(\mathbb{V}[k]) = \mathrm{R}(\mathbb{V}^k \oplus \mathbb{W}^k) = \mathrm{R}(\mathbb{V}^k) \oplus \mathrm{R}(\mathbb{W}^k)$$

This is a filtration on $V_k^k \oplus W_k^k$. Assume $k < n$. Observe that the module $\mathbb{W}^k$ is streamlined up to index $k+1$, whereas $\mathbb{V}^k$ vanishes at time $k+1$. The next map in the zigzag sequence takes one of the following forms:

$$V_k^k \oplus W_k^k \xrightarrow{f_k} W_{k+1}^k \quad \text{or} \quad V_k^k \oplus W_k^k \xleftarrow{g_k} W_{k+1}^k.$$

In the first case, it follows that $V_k^k = \ker(f_k)$ since f_k on W_k^k is injective, and thus

$$\mathsf{R}(\mathbb{V}^k) = \mathcal{R}_k \cap \ker(f_k).$$

In the second case, V_k^k is taken as a complement to $\operatorname{Im}(g_k)$ in V_k since g_k on W_{k+1}^k is surjective, implying that

$$\mathcal{R}_k = R(\mathbb{V}^k) \oplus (\mathcal{R}_k \cap \operatorname{Im}(g_k)).$$

Therefore, in either scenario, the dimension of $\mathsf{R}(\mathbb{V}^k)$ is given by:

$$\dim(\mathrm{R}(\mathbb{V}^k)) = \left\{ \begin{array}{l} \dim(\mathcal{R}_k \cap \ker(f_k)) \\ \dim(\mathcal{R}_k) - \dim(\mathcal{R}_k \cap \operatorname{Im}(g_k)) \end{array} \right\} = (c_1^k, \ldots, c_k^k),$$

Depending on which case applies. Finally, for $k = n$, we simply have:

$$\dim(\mathrm{R}(\mathbb{V}^n)) = \dim(\mathcal{R}_n) = (c_1^n, \ldots, c_n^n).$$

We know how to decompose a right streamlined zigzag module over the intervals by Theorem 3.2.10. By using Theorem 3.2.10, we have

$$\mathbb{V} = \bigoplus_{1 \le k \le n} \mathbb{V}^k \cong \bigoplus_{1 \le k \le n} \left\{ \bigoplus_{1 \le i \le k} c_i^k \, \mathbb{I}(b_i^k, k) \right\}$$

□

Remark 3.3.1. In Theorem 3.3.1, write

$$(r_1^k, r_2^k, \ldots, r_k^k) = \dim(\mathcal{R}_k)$$

for $1 \le k \le n$, and conventionally define $r_1^{n+1} = 0$ for all i. Then

$$c_i^k = \left\{ \begin{array}{ll} r_i^k - r_i^{k+1} & \text{when } \xrightarrow{f_k} \\ r_i^k - r_{i+1}^{k+1} & \text{when } \xleftarrow{g_k} \end{array} \right.$$

for all $1 \le i \le k \le n$.

Based on the method described, the steps of the algorithm are shown below:

Algorithm: [1]

Let $\mathbb{V} : V_1 \overset{p_1}{\longleftrightarrow} \ldots \overset{p_{n-1}}{\longleftrightarrow} V_n$ be a τ-module with $\tau = p_1 \ldots p_{n-1}$. We proceed step by step for $k = 1, 2, \ldots, n$, computing the filtration $\mathcal{R}_k = R(\mathbb{V}[k])$, the birth-time index $b(\tau[k])$, and the dimensions c_i^k at each stage.

- **Initialisation:** (for $k = 1$)

  (1) $\mathcal{R}_1 = (0, V_1)$

  (2) $b(\tau[1]) = (1)$.

- **Iterative Step:** (for $k = 1, 2, \ldots, n-1$)

  (3) Calculate R_{k+1} for $\mathcal{R}_k = (R_0^k, R_1^k, \ldots, R_k^k)$ using Definition 41.

$$\mathcal{R}_{k+1} = (R_0^{k+1}, \ldots, R_{k+1}^{k+1}) = \begin{cases} (f_k(R_0^k), \ldots, f_k(R_k^k), V_{k+1}) & \text{case } \xrightarrow{f_k} \\ (0, g_k(R_0^k), \ldots, g_k(R_k^k)) & \text{case } \xleftarrow{g_k} \end{cases}$$

  (4) Calculate $b(\tau[k+1])$ for $b(\tau[k]) = (b_1^k, \ldots, b_k^k)$ using Definition 42.

$$b(\tau[k+1]) = (b_1^{k+1}, b_2^{k+1}, \ldots, b_k^{k+1}) = \begin{cases} (b_1^k, \ldots, b_k^k, k+1) & \text{case } \xrightarrow{f_k} \\ (k+1, b_1^k, \ldots, b_k^k) & \text{case } \xleftarrow{g_k} \end{cases}$$

  (5) Calculate $(c_1^k, c_2^k, \ldots, c_k^k)$ using the formula in Theorem 3.3.2.

$$(c_1^k, c_2^k, \ldots, c_k^k) = \begin{cases} \dim(\mathcal{R}_k \cap \text{Ker}(f_k)) & \text{case } \xrightarrow{f_k} \\ \dim(\mathcal{R}_k) - \dim(\mathcal{R}_k \cap \text{Im}(g_k)) & \text{case } \xleftarrow{g_k} \end{cases}$$

  Alternatively, use the formula in Remark 3.3.1.

$$c_i^k = \begin{cases} r_i^k - r_i^{k+1} & \text{when } \xrightarrow{f_k} \\ r_i^k - r_{i+1}^{k+1} & \text{when } \xleftarrow{g_k} \end{cases}$$

- **Terminal Step:** (k=n)

  (6) Calculate $(c_1^n, c_2^n, \ldots, c_n^n) = \dim(R(\mathbb{V}))$

- **Print Results:**

  (7) For $1 \le i \le k \le n$, the interval module $\mathbb{I}(b_i^k, k)$ occurs with multiplicity c_i^k.

Let us illustrate the algorithm with an example.

Example 3.3.1. Let $\mathbb{V} : V_1 \xrightarrow{f} V_2$ be a zigzag module with type $\tau = f$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$ and $V_2 = \mathbb{F}$ and $f = \pi : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{\pi} \mathbb{F}$$

where $b(\tau) = (1, 2)$ and

$$R(\mathbb{V}) = (f(0), f(V_1), V_2))$$

$$R(\mathbb{V}) = (f(0), f(V_1), V_2)) = (f(0), f(\mathbb{F} \oplus \mathbb{F}), \mathbb{F}) = (0, \mathbb{F}, \mathbb{F}).$$

For the initial step in the above algorithm, we calculate the right filtration of $\mathbb{V}[1]$ and the birth time index of $\tau[1]$ where $\mathbb{V}[1] : V_1 \cong \mathbb{F} \oplus \mathbb{F}$.

$$(1) \quad \mathcal{R}_1 = (R_0^1, R_1^1) = R(\mathbb{V}[1]) = (0, V_1) = (0, \mathbb{F} \oplus \mathbb{F})$$

$$(2) \quad b(\tau[1]) = 1$$

Now, we proceed with the iterative step for $k = 1$. By using $\mathcal{R}_1$, we need to calculate $\mathcal{R}_2$ by the definition of the right filtration.

$$(3) \quad \mathcal{R}_2 = (R_0^2, R_1^2, R_2^2) = R(\mathbb{V}[2]) = (f(R_0^1), f(R_1^1), V_2)$$

$$\mathcal{R}_2 = (f(0), f(\mathbb{F} \oplus \mathbb{F}), \mathbb{F}) = (0, \mathbb{F}, \mathbb{F})$$

Then, calculate $b(\tau[2])$ by using $b(\tau[1])$ by the definition of the birth time index.

$$(4) \quad b(\tau[2]) = (b_1^2, b_2^2) = (b_1^1, 1 + 1) = (1, 2)$$

For $k = 1$, we need to compute c_1^1 as in the theorem .. . We know $\mathcal{R}_1 = (0, \mathbb{F} \oplus \mathbb{F})$ and $\mathcal{R}_2 = (0, \mathbb{F}, \mathbb{F})$.

$$\dim(\mathcal{R}_1) = (r_1^1) = (\dim(R_1^1/R_0^1)) = 2$$

$$\dim(\mathcal{R}_2) = (r_1^2, r_2^2) = (\dim(R_1^2/R_0^2), \dim(R_2^2/R_1^2))$$

$$\dim(\mathcal{R}_2) = (r_1^2, r_2^2) = (\dim(\mathbb{F}/0), \dim(\mathbb{F}/\mathbb{F})) = (1, 0)$$

So, by using $\dim(\mathcal{R}_1)$ and $\dim(\mathcal{R}_2)$ and the formula in the algorithm, we have

$$(5) \quad c_1^1 = r_1^1 - r_1^2 = 2 - 1 = 1 \quad \text{case } f$$

In the terminal step, for $k = 2$

$$(6) \quad (c_1^2, c_2^2) = \dim(R(\mathbb{V})) = (\mathbb{F}/0), \mathbb{F}/\mathbb{F}) = (1, 0)$$

So, we can calculate the interval modules $\mathbb{I}(b_i^k, k)$ that occur with multiplicity c_i^k for $1 \leq i \leq k \leq 2$. Hence, we have $\mathbb{I}(b_1^1, 1)$ with the multiplicity c_1^1 when $i = 1$ and $k = 1$,$\mathbb{I}(b_1^2, 2)$ with the multiplicity c_1^2 when $i = 1$ and $k = 2$, and $\mathbb{I}(b_2^2, 2)$ with the multiplicity c_2^2 when $i = 2$ and $k = 2$.

$$(7) \quad \mathbb{V} \cong c_1^1\mathbb{I}(b_1^1, 1) \oplus c_1^2\mathbb{I}(b_1^2, 2) \oplus c_2^2\mathbb{I}(b_2^2, 2)$$

where $b_1^1 = 1, b_1^2 = 1, b_2^2 = 2$ and $c_1^1 = 1, c_1^2 = 1, c_2^2 = 0$. Thus,

$$\mathbb{V} \cong \mathbb{I}(1, 1) \oplus \mathbb{I}(1, 2)$$

$$\mathbb{V} \cong (\mathbb{F} \to 0) \oplus (\mathbb{F} \to \mathbb{F})$$

Example 3.3.2. Let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$ be a zigzag module with type $\tau = f_1 f_2$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$, $V_2 = \mathbb{F}$ and $V_3 = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $f_1 : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component, and $f_2 : \mathbb{F} \to \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ is the inclusion on the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xrightarrow{f_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

where $b(\tau) = (1, 2, 3)$. We will follow the initial and iterative steps for $1 \leq k \leq 3$ as in the algorithm. For $k = 1$, we have only the space $\mathbb{V}[1] : V_1 \cong \mathbb{F} \oplus \mathbb{F}$. So, we can calculate $\mathcal{R}_1$ and $b(\tau[1])$ as follows:

$$\mathcal{R}_1 = (R_0^1, R_1^1) = (0, V_1) = (0, \mathbb{F} \oplus \mathbb{F})$$

$$b(\tau[1]) = (b_1^1) = (1)$$

Now, calculate $\mathcal{R}_2$ from $\mathcal{R}_1$ using the definition of the right filtration.

$$\mathcal{R}_2 = (R_0^2, R_1^2, R_2^2) = (f_1(0), f_1(\mathbb{F} \oplus \mathbb{F}), V_2)$$

$$\mathcal{R}_2 = (R_0^2, R_1^2, R_2^2) = (0, \mathbb{F}, \mathbb{F})$$

And, calculate $\mathcal{R}_3$ for $\mathcal{R}_2$ the same as above,

$$\mathcal{R}_3 = (R_0^3, R_1^3, R_2^3, R_3^3) = (f_2(0), f_2(\mathbb{F}), f_2(\mathbb{F}), V_3)$$

$$\mathcal{R}_3 = (R_0^3, R_1^3, R_2^3, R_3^3) = (0, \mathbb{F}, \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F})$$

Now, calculate $b(\tau[2])$ and $b(\tau[3])$ using the definition of the birth time index iteratively.

$$b(\tau[2]) = (b_1^2, b_2^2) = (1, 2)$$

$$b(\tau[3]) = (b_1^3, b_2^3, b_3^3) = (1, 2, 3)$$

At this point, we need to compute $\dim(\mathcal{R}_1)$ and $\dim(\mathcal{R}_2)$ to calculate c_i^k using the formula in the Proposition 3.3.2 for $1 \leq i \leq k \leq 2$.

$$\dim(\mathcal{R}_1) = (r_1^1) = \dim(\mathbb{F} \oplus \mathbb{F}/0) = (2)$$

$$\dim(\mathcal{R}_2) = (r_1^2, r_2^2) = (\dim(\mathbb{F}/0), \dim(\mathbb{F}/\mathbb{F})) = (1, 0)$$

At the terminal step, we determine $\dim(R_3) = \dim(R(\mathbb{V}))$ where

$$R(\mathbb{V}) = (0, \mathbb{F}, \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}).$$

So, we have

$$\dim(R_3) = \dim(R(\mathbb{V})) = (\mathbb{F}/0, \mathbb{F}/\mathbb{F}, {}^{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}}/_{\mathbb{F}})$$

$$\dim(R_3) = \dim(R(\mathbb{V})) = (r_1^3, r_2^3, r_3^3) = (c_1^3, c_2^3, c_3^3) = (1, 0, 2)$$

Since all maps in this example are forward, we apply the forward case of the algorithm. So, we have

$$c_1^1 = r_1^1 - r_1^2 = 2 - 1 = 1$$

$$c_1^2 = r_1^2 - r_1^3 = 1 - 1 = 0$$

$$c_2^2 = r_2^2 - r_2^3 = 0$$

Hence, by the Interval Decomposition Theorem 3.3.1, the zigzag module

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xrightarrow{f_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

decomposes as

$$\bigoplus_{1 \leq i \leq k \leq 3} c_i^k \mathbb{I}(b_i^k, k) \cong$$

$$c_1^1 \mathbb{I}(b_1^1, 1) \oplus c_1^2 \mathbb{I}(b_1^2, 2) \oplus c_2^2 \mathbb{I}(b_2^2, 2) \oplus c_1^3 \mathbb{I}(b_1^3, 3) \oplus c_2^3 \mathbb{I}(b_2^3, 3) \oplus c_3^3 \mathbb{I}(b_3^3, 3)$$

After determining c_i^k's, we have

$$\mathbb{V} \cong \mathbb{I}(1, 1) \oplus \mathbb{I}(1, 3) \oplus 2\mathbb{I}(3, 3)$$

Thus, we present the decomposition of the zigzag module $\mathbb{V}$ as follows:

$$\mathbb{V} = \begin{cases} (\mathbb{F} \to 0 \to 0) \\ \oplus \\ (\mathbb{F} \to \mathbb{F} \to \mathbb{F}) \\ \oplus \\ (0 \to 0 \to \mathbb{F}) \\ \oplus \\ (0 \to 0 \to \mathbb{F}) \end{cases}$$

Example 3.3.3. Let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module with type $\tau = f_1 g_2$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$, $V_2 = \mathbb{F}$ and $V_3 = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $f_1 : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component, and $g_2 : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

where $b(\tau) = (3, 1, 2)$. We will follow the initial and iterative steps for $1 \leq k \leq 3$ as in the algorithm. For $k = 1$, we have only the space $\mathbb{V}[1] : V_1 \cong \mathbb{F} \oplus \mathbb{F}$. So, we can calculate $\mathcal{R}_1$ and $b(\tau[1])$ as follows:

$$\mathcal{R}_1 = (R_0^1, R_1^1) = (0, V_1) = (0, \mathbb{F} \oplus \mathbb{F})$$

$$b(\tau[1]) = (b_1^1) = (1)$$

Now, calculate $\mathcal{R}_2$ from $\mathcal{R}_1$ using the definition of the right filtration.

$$\mathcal{R}_2 = (R_0^2, R_1^2, R_2^2) = (f_1(0), f_1(\mathbb{F} \oplus \mathbb{F}), V_2)$$

$$\mathcal{R}_2 = (R_0^2, R_1^2, R_2^2) = (0, \mathbb{F}, \mathbb{F})$$

And, calculate $\mathcal{R}_3$ for $\mathcal{R}_2$ the same as above,

$$\mathcal{R}_3 = (R_0^3, R_1^3, R_2^3, R_3^3) = (0, g_2^{-1}(0), g_2^{-1}(\mathbb{F}), g_2^{-1}(\mathbb{F}))$$

$$\mathcal{R}_3 = (R_0^3, R_1^3, R_2^3, R_3^3) = (0, \mathbb{F} \oplus \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F})$$

Now, calculate $b(\tau[2])$ and $b(\tau[3])$ using the definition of the birth time index iteratively.

$$b(\tau[2]) = (b_1^2, b_2^2) = (1, 2)$$

$$b(\tau[3]) = (b_1^3, b_2^3, b_3^3) = (3, 1, 2)$$

At this point, we need to compute $\dim(\mathcal{R}_1)$ and $\dim(\mathcal{R}_2)$ to calculate c_i^k using the formula in the Proposition 3.3.2 for $1 \leq i \leq k \leq 2$.

$$\dim(\mathcal{R}_1) = (r_1^1) = \dim(\mathbb{F} \oplus \mathbb{F}/0) = (2)$$

$$\dim(\mathcal{R}_2) = (r_1^2, r_2^2) = (\dim(\mathbb{F}/0), \dim(\mathbb{F}/\mathbb{F})) = (1, 0)$$

At the terminal step, we determine

$$\dim(R_3) = \dim(R(\mathbb{V})) \quad \text{where} \quad R(\mathbb{V}) = (0, \mathbb{F}, \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}).$$

So, we have

$$\dim(R_3) = \dim(R(\mathbb{V})) = ({}^{\mathbb{F} \oplus \mathbb{F}}/_{0}, {}^{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}}/_{\mathbb{F} \oplus \mathbb{F}}, {}^{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}}/_{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}})$$

$$\dim(R_3) = \dim(R(\mathbb{V})) = (r_1^3, r_2^3, r_3^3) = (c_1^3, c_2^3, c_3^3) = (2, 1, 0)$$

We now have the birth time indices

$$b(\tau[1]) = (b_1^1) = (1), \quad b(\tau[2]) = (b_1^2, b_2^2) = (1, 2), \quad \text{and}$$

$$b(\tau[3]) = (b_1^3, b_2^3, b_3^3) = (3, 1, 2)$$

Moreover, we have $\dim(\mathcal{R}_)$ for $1 \leq k \leq 3$ as follows:

$$\dim(\mathcal{R}_1) = (c_1^1) = (1), \quad \dim(\mathcal{R}_2) = (c_1^2, c_2^2) = (0, 0), \quad \text{and}$$

$$\dim(\mathcal{R}_3) = (c_1^3, c_2^3, c_3^3) = (2, 1, 0)$$

Hence, by the Interval Decomposition Theorem 3.3.1, the zigzag module $\mathbb{V}$ has decomposition as follows:

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \cong \bigoplus_{1 \leq i \leq k \leq 3} c_i^k \mathbb{I}(b_i^k, k)$$

where $\mathbb{I}(b_i^k, k)$ denotes the interval module associated with the birth-time index b_i^k. Hence, we obtain the decomposition

$$\mathbb{V} \cong \mathbb{I}(1, 1) \oplus \mathbb{I}(1, 3) \oplus 2\mathbb{I}(3, 3).$$

Thus, the zigzag module $\mathbb{V}$ can be represented as follows:

$$\mathbb{V} = \begin{cases} (\mathbb{F} \to 0 \to 0) \\ \oplus \\ (\mathbb{F} \to \mathbb{F} \to \mathbb{F}) \\ \oplus \\ (0 \to 0 \to \mathbb{F}) \\ \oplus \\ (0 \to 0 \to \mathbb{F}) \end{cases}$$

In this section, we outline an algorithm to solve the following problem. Let τ be a type of length n, and consider a τ-module V defined as follows: for each index i, let $V_i = \mathbb{F}^{a_i}$, where each $a_i \geq 0$. Depending on the type τ, the map f_i is represented by an $a_{i+1} \times a_i$ matrix M_i, or alternatively, the map g_i is given by an $a_i \times a_{i+1}$ matrix N_i. Given τ and the corresponding matrices M_i or N_i, the task is to compute the persistence $\mathrm{Pers}(V)$.

3.3.2 Concrete Vector Spaces

In this part, we present an algorithm [1] to solve the following problem. Let τ be a type of length n and $\mathbb{V}$ be a τ-module defined as follows :
Set $V_i = \mathbb{F}^{a_i}$ for each $1 \leq i \leq n$, where $a_i \geq 0$. Depending on the type τ, the forward map f_i given by an $a_{i+1} \times a_i$ matrix M_i, or the backward map g_i given by an $a_i \times a_{i+1}$ matrix N_i. Given the type τ and the matrices M_i and N_i, we want to compute $\mathrm{Pers}(\mathbb{V})$.

We proceed according to the Algorithm of Interval Decomposition Theorem $3.3.1$. The main computational task lies in step (3) in the algorithm, which involves determining the sequence of the right filtration $\mathcal{R}_k = \mathrm{R}(\mathbb{V}[k])$. The other steps follow the algorithm.

Our aim is to understand the maps M_i and N_i in an echelon form, which helps us identify persistence features in vector spaces.

We can change the bases of V_i, that is, a simple transformation of one vector by adding another multiplied by a scalar. Instead of explicitly changing bases of V_i's, we directly perform elementary row and column operations on M_i or N_i.

The elementary transformations $E_i(p, q, \lambda)$ change one basis vector β_q by adding λ times another basis vector β_p. This is a modification of $\mathcal{B}_i = (\beta_1, \ldots, \beta_{a_i})$ involving basis vectors β_p and β_q as follows:

$$\beta_p \leftarrow \beta_p$$

$$\beta_q \leftarrow \beta_q + \lambda \cdot \beta_q$$

If L is a matrix that represent a linear map $V_i \to W$ for some W, then we change the columns of L:

$$Column_p \leftarrow Column_p$$

$$Column_q \leftarrow Column_q + \lambda \cdot \text{Column}_q$$

If L is a matrix that represents a linear map $W \to V_i$ for some W, then we must apply the dual transformation to the rows of L:

$$Row_p \leftarrow Row_p - \lambda \cdot Row_p$$

$$Row_q \leftarrow Row_q$$

The filtration $R_k = R(V[k])$ on V_k is to be represented in the following way. We require the basis $B_k = (\beta_1, \ldots, \beta_{a_i})$ to be compatible with the filtration, in a sense that will be clarified shortly. Given such a basis, the filtration $R_k = (R_0, R_1, \ldots, R_k)$ is described by a non-decreasing function

$$\phi_k : \{1, 2, \ldots, a_i\} \to \{1, \ldots, k\}$$

such that for each $i = 1, \ldots, k$,

$$R_i = \text{Span}\{\beta_p | \phi_k(p) \leq i\}.$$

The dimension $r_i^k = \dim(R_i/R_{i-1})$ can be determined by the cardinality of $\phi_k^{-1}(i)$.

Now, assume that $\mathcal{B}_k$ and ϕ_k together represent the filtration $\mathcal{R}_k$, and the map ϕ_k identifies $\mathcal{R}_k$. Then, change $\mathcal{B}_{k+1}$ and determine ϕ_{k+1} to represent $\mathcal{R}_{k+1}$.There are two main cases based on whether we are working with M_k or N_k.

- *Case M*:

  Let M_k be a matrix representation of a linear map $V_k \to V_{k+1}$. Assume that $\mathcal{B}_k$ is compatible with a filtration $\mathcal{R}_k$ and that ϕ_k identifies $\mathcal{R}_k$. This gives a block structure

  $$M_k = \begin{bmatrix} K_1 & K_2 & \dots & K_k \end{bmatrix}$$

  where K_i gathers together the columns q with $\phi_k(q) = i$. Now, apply row operations to make the matrix M_k into row echelon form. Then, upgrade ϕ_{k+1} according to these pivots as follows:

  $$\phi_{k+1}(p) = \begin{cases} \phi_k(q) & \text{if row } p \text{ has a pivot in column } q \\ k+1 & \text{if row } p \text{ has no pivot} \end{cases}$$

- *Case N*:

  Let N_k be a matrix representation of a linear map $V_{k+1} \to V_k$. Assume that $\mathcal{B}_k$ is compatible with a filtration $\mathcal{R}_k$ and that ϕ_k identifies the filtration $\mathcal{R}_k$. This gives a block structure

  $$N_k = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_k \end{bmatrix}$$

  where L_i gathers together the rows q with $\phi_k(q) = i$. Applying column operations to make the matrix N_k into column echelon form. Then, upgrade ϕ_{k+1} according to these pivots as follows:

  $$\phi_{k+1}(p) = \begin{cases} \phi_k(q) + 1 & \text{if column } p \text{ has a pivot in row } q \\ 1 \quad \text{if column } p \text{ has no pivot} \end{cases}$$

Let us apply this to the examples discussed in the previous sections.

Example 3.3.4. Let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module with type $\tau = f_1 g_2$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$, $V_2 = \mathbb{F}$ and $V_3 = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $f_1 : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component, and $g_2 : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

where $b(\tau) = (3, 1, 2)$. Let the matrix representation of f_1 be a matrix M_1 , and let the matrix representation of g_2 be the matrix N_2 as follows:

$$M_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

We are given two matrices, each in echelon form. Also, each basis $\mathcal{B}_i$ is a standart basis of $V_i = \mathbb{F}^{a_i}$ for $1 \leq i \leq 3$.

$$\mathcal{B}_1 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \quad \text{and} \quad \mathcal{B}_2 = \{ \begin{bmatrix} 1 \end{bmatrix} \} \quad \text{and} \quad \mathcal{B}_3 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$$

Let us begin with $k = 1$. $\mathcal{B}_1$ is a basis for $V_1 = \mathbb{F} \oplus \mathbb{F}$. We have $\mathbb{V}[1] : V_1 \cong \mathbb{F} \oplus \mathbb{F}$. Clearly, we have

$$\mathrm{R}(\mathbb{V}[1]) = \mathcal{R}_1 = (0, \mathbb{F} \oplus \mathbb{F})$$

Define non-decreasing function

$$\phi_1 : \{1, 2\} \to \{1\}$$

such that $\mathcal{R}_1 = \mathrm{Span}\{\beta_p \mid \phi_1(p) \leq i\}$. Also, we have $\mathcal{R}_1 = \mathrm{Span}\{\beta_1, \beta_2\}$, and only possibilities of ϕ_1 are

$$\phi_1(1) = 1 \quad \text{and} \quad \phi_1(2) = 1$$

Now, we have $\mathcal{B}_1 = (\beta_1, \beta_2)$ is compatible with the filtration $\mathcal{R}_1$, and ϕ_1 represents $\mathcal{R}_1$.

Let us proceed to $k = 2$. We have

$$\mathbb{V}[2] : V_1 \xrightarrow{f_1} V_2 \cong \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F}$$

We know that M_1 is a matrix representation of f_1. There is no need to perform any transformation, as M_1 is already in echelon form. The block structure of M_1 with respect to K_i gathers the columns q with $\phi_1(q) = i$ (in our situation, $\phi_1(1) = 1$ and $\phi_1(2) = 1$) as follows:

$$M_1 = \begin{bmatrix} K_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let's define ϕ_2 by using ϕ_1 and M_1. ϕ_2 is a non-decreasing function as follows:

$$\phi_2 : \{1\} \to \{1, 2\}$$

such that

$$\phi_2(p) = \begin{cases} \phi_1(q) & \text{if row } p \text{ has a pivot in column } q \\ 2 & \text{if row } p \text{ has no pivot} \end{cases}$$

So, we have $\phi_2(1) = \phi_1(1) = 1$.

Now, for $k = 3$, we have

$$\mathbb{V} = \mathbb{V}[3] : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_2 \cong \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

We know that N_2 is the matrix representation of g_2. There is no need to perform any transformation, as N_2 is already in echelon form.The block structure of N_2 with respect to L_i gathers the rows q with $\phi_2(q) = i$ (in our situation, $\phi_2(1) = 1$) as follows:

$$N_2 = \begin{bmatrix} L_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Let's define ϕ_3 by using ϕ_2 and N_2. ϕ_3 is a non-decreasing function as follows:

$$\phi_3 : \{1, 2, 3\} \to \{1, 2, 3\}$$

such that

$$\phi_3(p) = \begin{cases} \phi_2(q) + 1 & \text{if column } p \text{ has a pivot in row } q \\ 1 & \text{if column } p \text{ has no pivot} \end{cases}$$

So,

$$\phi_3(1) = \phi_2(1) + 1 = 2$$

$$\phi_3(2) = 1 \quad \text{and} \quad \phi_3(3) = 1$$

Now, we define all ϕ_1,ϕ_2 and ϕ_3 that represent $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$, respectively. We know that the dimension $r_i^k = \dim({}^{R_i}/_{R_{i-1}})$ can be determined by the cardinality of $\phi_k^{-1}(i)$. So, $r_i^k = |\phi_k^{-1}(i)|$ for $1 \leq k \leq 3$ and $1 \leq i \leq k$. Hence, we will determine r_i^k's as follows:

$$\begin{aligned} &r_1^1 = |\phi_1^{-1}(1)| = 2 \\ &r_1^2 = |\phi_2^{-1}(1)| = 1 \quad , r_2^2 = |\phi_2^{-1}(2)| = 0 \\ &r_1^3 = |\phi_3^{-1}(1)| = 2 \quad , r_2^3 = |\phi_3^{-1}(2)| = 1 \quad r_3^3 = |\phi_3^{-1}(3)| = 0 \end{aligned}$$

As in the Algorithm, we can determine c_i^k's:

$$c_1^1 = r_1^1 - r_1^2 = 2 - 1 = 1$$

$$c_1^2 = r_1^2 - r_1^3 = 1 - 1 = 0 \quad \text{and} \quad c_2^2 = r_1^2 - r_2^3 = 1 - 1 = 0$$

$$c_1^3 = r_1^3 = 2 \quad \text{and} \quad c_2^3 = r_2^3 = 1 \quad \text{and} \quad c_3^3 = r_3^3 = 0$$

Also, we have the birth time indices

$$b(\tau[1]) = (b_1^1) = (1), \quad b(\tau[2]) = (b_1^2, b_2^2) = (1, 2),$$

$$b(\tau[3]) = (b_1^3, b_2^3, b_3^3) = (3, 1, 2)$$

Hence, by the Interval Decomposition Theorem 3.3.1, the zigzag module $\mathbb{V}$ has decomposition as follows:

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \cong \bigoplus_{1 \le i \le k \le 3} c_i^k \mathbb{I}(b_i^k, k)$$

where $\mathbb{I}(b_i^k, k)$ denotes the interval module associated with the birth-time index b_i^k. Hence, we obtain the decomposition

$$\mathbb{V} \cong \mathbb{I}(1, 1) \oplus \mathbb{I}(1, 3) \oplus 2\mathbb{I}(3, 3)$$

3.4 The Diamond Principle

In this section, we present a key structural result known as the *Diamond Principle*, which provides a powerful tool for analyzing changes in persistence intervals.

Definition 50. Let $\mathbb{V}$ be a zigzag module of length n. The ***left filtration*** of $\mathbb{V}$ is a filtration on V_1 of depth n, defined as

$$L(\mathbb{V}) = R(\bar{\mathbb{V}})$$

where $\bar{\mathbb{V}}$ is the reversal of $\mathbb{V}$; so $\bar{V}_i = V_{n+1-i}$ with maps $\bar{f}_i = g_{n-i}$ or $\bar{g}_i = f_{n-i}$
For any k, we have two natural filtrations on V_k as follows:

$$\mathcal{R}_k = (R_0, R_1, \ldots, \mathbb{R}_k) = R(\mathbb{V}[1, k])$$

$$\mathcal{L}_k = (L_0, L_1, \ldots, L_{n+1-k}) = L(\mathbb{V}[k, n])$$

where the right filtration over the indices $\{1, \ldots, k\}$ and the left filtration over the indices $\{k, \ldots, n\}$. We also have ***birth time*** and ***death time*** indices as follows:

$$b_k = (b_1, \ldots, b_k) = b(\tau[1, k])$$

$$d_k = (d_1, \ldots, d_{n+1-k}) = n + 1 - b(\bar{\tau}[k, n])$$

which indicate the birth and death times associated with the respective subquotients of $\mathcal{R}_k$ and $\mathcal{L}_k$.

Example 3.4.1. As in the previous example 3.3.3, let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module with type $\tau = f_1 g_2$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$, $V_2 = \mathbb{F}$ and $V_3 = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $f_1 : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component, and $g_2 : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

where $b(\tau) = (3, 1, 2)$. Let us choose $k = 2$. Then we will determine two natural filtrations on V_2, and also find the birth time and death time indices. Also, we can define $\bar{\mathbb{V}}$ as in the definition;

$$\bar{\mathbb{V}} : \bar{V_1} \xrightarrow{\bar{f_1}} \bar{V_2} \xleftarrow{\bar{g_2}} \bar{V_3} \cong V_3 \xrightarrow{g_2} V_2 \xleftarrow{f_1} V_1$$

$$\bar{\mathbb{V}} : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \xrightarrow{g_2} \mathbb{F} \xleftarrow{f_1} \mathbb{F} \oplus \mathbb{F}$$

where $\bar{\tau} = \bar{f_1}\bar{g_2} = g_2 f_1$.
For $\mathcal{R}_2 = R(\mathbb{V}[1, 2])$, we have $\mathbb{V}[1, 2] : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F}$, and so

$$\mathcal{R}_2 = R(\mathbb{V}[1, 2]) = (f_1(0), f_1(\mathbb{F} \oplus \mathbb{F}), \mathbb{F}) = (0, \mathbb{F}, \mathbb{F})$$

$$\mathcal{R}_2 = (R_0, R_1, R_2) = (0, \mathbb{F}, \mathbb{F})$$

For $\mathcal{L}_2 = L(\mathbb{V}[2, 3])$, we have $L(\mathbb{V}[2, 3]) = R(\bar{\mathbb{V}}[1, 2])$. So, $\bar{\mathbb{V}}[1, 2] : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \xrightarrow{\bar{f_1} = g_2} \mathbb{F}$. Then

$$\mathcal{L}_2 = L(\mathbb{V}[2, 3]) = (\bar{f_1}(0), \bar{f_1}(\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}), \mathbb{F})$$

$$\mathcal{L}_2 = (g_2(0), g_2(\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}), \mathbb{F}) = (0, \mathbb{F}, \mathbb{F})$$

Moreover, we can find birth and death time indices as follows:

$$\mathrm{b}_k = b(\tau[1, 2]) = (1, 2) = (b_2, b_2)$$

$$\mathrm{d}_k = n + 1 - b(\bar{\tau}[2, 3]) = (3, 2) = (d_1, d_2)$$

where $b(\bar{\tau}[2, 3]) = (1, 2)$.

Theorem 3.4.1 (Localization at index k [1]). *Let $\mathbb{V}$ be a zigzag module of length n and let $1 \leq k \leq n$. Let $\mathcal{R}_k, \mathcal{L}_k$ denote right and left filtration at k, and let $\mathrm{b}_k, \mathrm{d}_k$ denote the birth time and death time indices at k. Then, for all i, j in the range $1 \leq i \leq k$, $1 \leq j \leq n+1-k$, the multiplicity of $[b_i, d_j]$ in $\mathrm{Pers}(\mathbb{V})$ is equal to*

$$c_{ij} = \dim(R_i \cap L_j) - \dim(R_{i-1} \cap L_j) - \dim(R_i \cap L_{j-1}) + \dim(R_{i-1} \cap L_{j-1})$$

Equivalently,

$$c_{ij} = \dim\left({}^{(R_i \cap L_j)} \big/ {}_{(R_{i-1} \cap L_j) + (R_i \cap L_{j-1})} \right)$$

Example 3.4.2. As the previous example 3.4.1, let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module with type $\tau = f_1 g_2$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$, $V_2 = \mathbb{F}$, and $V_3 = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $f_1 : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component, and $g_2 : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

where $b(\tau) = (3, 1, 2)$. In the previous Example 3.4.1, we had computed both the right and left filtrations in k. Additionally, we determined the birth time and the death time indices for the zigzag module $\mathbb{V}$:

$$\mathcal{R}_2 = (R_0, R_1, R_2) = (0, \mathbb{F}, \mathbb{F})$$

$$\mathcal{L}_2 = (L_0, L_1, L_2) = (0, \mathbb{F}, \mathbb{F})$$

$$\mathrm{b}_k = (b_1, b_2) = (1, 2)$$

$$\mathrm{d}_k = (d_1, d_2) = (3, 2)$$

Moreover, from Example 3.3.3 we obtained the interval decomposition of the zigzag module $\mathbb{V}$:

$$\mathbb{V} \cong \mathbb{I}(1,1) \oplus \mathbb{I}(1,3) \oplus 2\mathbb{I}(3,3)$$

Hence, by Theorem 3.4.1.(Localization at index k), we can determine the multiplicities c_{ij} of the intervals $[b_i, d_j]$ in $\mathrm{Pers}(\mathbb{V})$ for all $1 \leq i \leq 2$ and $1 \leq j \leq 2$ as follows:

$$c_{11} = \dim(R_1 \cap L_1) - \dim(R_0 \cap L_1) - \dim(R_1 \cap L_0) + \dim(R_0 \cap L_0)$$

$$c_{11} = \dim(\mathbb{F} \cap \mathbb{F}) - \dim(0 \cap \mathbb{F}) - \dim(\mathbb{F} \cap 0) + \dim(0 \cap 0) = 1 - 0 - 0 - 0 = 1$$

$$c_{12} = \dim(\mathbb{F} \cap \mathbb{F}) - \dim(0 \cap \mathbb{F}) - \dim(\mathbb{F} \cap \mathbb{F}) + \dim(0 \cap \mathbb{F}) = 1 - 0 - 1 - 0 = 0$$

$$c_{21} = \dim(\mathbb{F} \cap \mathbb{F}) - \dim(\mathbb{F} \cap \mathbb{F}) - \dim(\mathbb{F} \cap 0) + \dim(\mathbb{F} \cap 0) = 1 - 1 - 0 - 0 = 0$$

$$c_{22} = \dim(\mathbb{F} \cap \mathbb{F}) - \dim(\mathbb{F} \cap \mathbb{F}) - \dim(\mathbb{F} \cap \mathbb{F}) + \dim(\mathbb{F} \cap \mathbb{F}) = 1 - 1 - 1 - 1 = 0$$

Therefore, by the theorem, the only nonzero multiplicity is c_{11} corresponding to the interval $[b_1, d_1]$. In $\mathrm{Pers}(\mathbb{V})$:

$$c_{11}[b_1, d_1] = 1 \cdot [1, 3]$$

$$c_{12}[b_1, d_2] = 0 \cdot [1, 2]$$

$$c_{21}[b_2, d_1] = 0 \cdot [2, 3]$$

$$c_{22}[b_2, d_2] = 0 \cdot [2, 2]$$

Thus, the results obtained here are consistent with those in Example $3.3.3$.

Example 3.4.3. Using the zigzag module from the previous example, we can also apply the theorem for $k = 3$ this time. Let $\mathbb{V} : V_1 \xrightarrow{f_1} V_2 \xleftarrow{g_2} V_3$ be a zigzag module with type $\tau = f_1 g_2$ where $V_1 = \mathbb{F} \oplus \mathbb{F}$, $V_2 = \mathbb{F}$, and $V_3 = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ and $f_1 : \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component, and $g_2 : \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$ is the projection onto the first component. So, we have

$$\mathbb{V} : \mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$$

where $b(\tau) = (3, 1, 2)$. Now, we will compute both the right and left filtration at $k = 3$, as well as determine the birth time and death time indices at $k = 3$.We have

$$\mathcal{R}_3 = \mathrm{R}(\mathbb{V}[1, 3]) = \mathrm{R}\left(\mathbb{F} \oplus \mathbb{F} \xrightarrow{f_1} \mathbb{F} \xleftarrow{g_2} \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}\right)$$

$$\mathcal{L}_3 = \mathrm{L}(\mathbb{V}[3, 3]) = \mathrm{L}\left(\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}\right)$$

Moreover, we compute

$$\mathcal{R}_3 = \left(0, g_2^{-1}(f_1(0)), g_2^{-1}(f_1(\mathbb{F} \oplus \mathbb{F})), g_2^{-1}(\mathbb{F})\right)$$

$$\mathcal{R}_3 = (R_0, R_1, R_2, R_3) = (0, \mathbb{F} \oplus \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F})$$

and

$$\mathcal{L}_3 = (L_0, L_1) = (0, \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F})$$

Also, we now determine the birth and death time indices at $k = 3$ as follows:

$$\mathrm{b}_3 = (b_1, b_2, b_3) = (3, 2, 1)$$

$$\mathrm{d}_3 = 3 + 1 - 1 = 3 = (d_1)$$

Hence, by Theorem 3.4.1.(Localization at index k), we can determine the multiplicities c_{ij} of the intervals$[b_i, d_j]$ in $\mathrm{Pers}(\mathbb{V})$ for all $1 \leq i \leq 3$ and $j = 1$ as follows:

$$c_{11} = \dim \left({}^{(R_1 \cap L_1)} / {}_{(R_0 \cap L_1) + (R_1 \cap L_0)} \right)$$

$$c_{11} = \dim \left({}^{(\mathbb{F}^2 \cap \mathbb{F}^3)} / {}_{(0 \cap \mathbb{F}^3) + (\mathbb{F}^2 \cap 0)} \right)$$

$$c_{11} = \dim \left({}^{\mathbb{F} \oplus \mathbb{F}} / {}_{0} \right) = 2$$

$$c_{21} = \dim \left({}^{(\mathbb{F}^3 \cap \mathbb{F}^3)} / {}_{(\mathbb{F}^2 \cap \mathbb{F}^3) + (\mathbb{F}^3 \cap 0)} \right)$$

$$c_{21} = \dim \left({}^{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}} / {}_{\mathbb{F} \oplus \mathbb{F}} \right) = 1$$

$$c_{31} = \dim \left({}^{(\mathbb{F}^3 \cap \mathbb{F}^3)} / {}_{(\mathbb{F}^3 \cap \mathbb{F}^3) + (\mathbb{F}^3 \cap 0)} \right)$$

$$c_{31} = \dim \left({}^{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}} / {}_{\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}} \right) = 0$$

Therefore, by the theorem, the nonzero multiplicity is c_{11} and c_{21} corresponds to the intervals $[b_1, d_1]$, and $[b_2, d_1]$, respectively. In $\mathrm{Pers}(\mathbb{V})$:

$$c_{11}[b_1, d_1] = 2 \cdot [3, 3]$$

$$c_{21}[b_2, d_1] = 1 \cdot [1, 3]$$

$$c_{22}[b_2, d_2] = 0 \cdot [2, 3]$$

Thus, the results found here correspond to those presented in Example 3.3.3.

3.4.1 The Diamond Principle

The Diamond Principle, introduced as a strong computational tool, provides a localization approach for computing zigzag barcodes. This principle is utilized to establish a method for comparing the zigzag barcodes of two naturally arising diagrams from a sequence of simplicial complexes. The article on Zigzag Persistent Homology and Real Valued Function [2] demonstrates that the Diamond Principle identifies isomorphisms among persistence barcodes associated with spaces equipped with real-valued functions, such as Morse functions. At the core of this framework lies the Diamond Principle, which connects the persistence intervals of two zigzag diagrams that differ by only a single local change. Consider the following diagram:

$$\begin{tikzcd}
 & & & & W_k & & & & \\
V_1 \arrow[r, leftrightarrow, "p_1"] & V_2 \arrow[r, leftrightarrow, "p_2"] & \cdots \arrow[r, leftrightarrow, "p_{k-1}"] & V_{k-1} \arrow[ur, "f_{k-1}"] & & V_{k+1} \arrow[ul, "g_k"'] \arrow[r, leftrightarrow, "p_{k+1}"] & \cdots \arrow[r, leftrightarrow, "p_{n-1}"] & V_n \\
 & & & & U_k \arrow[ul, "g_{k-1}"] \arrow[ur, "f_k"'] & & & &
\end{tikzcd}$$

Let $\mathbb{V}^+$ and $\mathbb{V}^-$ denote two zigzag modules displayed in the above diagram:

$$\mathbb{V}^+ : V_1 \overset{p_1}{\longleftrightarrow} \dots \overset{p_{k-2}}{\longleftrightarrow} V_{k-1} \xrightarrow{f_{k-1}} W_k \xleftarrow{g_k} V_{k+1} \overset{p_{k+1}}{\longleftrightarrow} \dots \overset{p_{n-1}}{\longleftrightarrow} V_n$$

$$\mathbb{V}^- : V_1 \overset{p_1}{\longleftrightarrow} \dots \overset{p_{k-2}}{\longleftrightarrow} V_{k-1} \xleftarrow{g_{k-1}} U_k \xrightarrow{f_k} V_{k+1} \overset{p_{k+1}}{\longleftrightarrow} \dots \overset{p_{n-1}}{\longleftrightarrow} V_n$$

We aim to compare $\mathrm{Pers}(\mathbb{V}^+)$ with $\mathrm{Pers}(\mathbb{V}^-)$, especially in relation to intervals that intersect $\{k+1, k, k-1\}$.

Definition 51. The diagram

$$\begin{tikzcd}
V_{k+1} \arrow[r, "g_k"] & W_k \\
U_k \arrow[u, "f_k"] \arrow[r, "g_{k-1}"'] & V_{k-1} \arrow[u, "f_{k-1}"']
\end{tikzcd}$$

is ***exact***, if $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$ in the following sequence

$$U_k \xrightarrow{D_1} V_{k+1} \oplus V_{k-1} \xrightarrow{D_2} W_k$$

where $D_1(u) = g_{k-1}(u) \oplus f_k(u)$ and $D_2(v \oplus v\prime) = f_{k-1}(v) - g_k(v\prime)$.

Theorem 3.4.2 (The Diamond Principle)**.** *Given $\mathbb{V}^+$ and $\mathbb{V}^-$, suppose that the middle diamond at indices $\{k+1, k, k-1\}$ is exact. Then, there is a bijection of the multisets $\mathrm{Pers}(\mathbb{V}^+)$ and $\mathrm{Pers}(\mathbb{V}^-)$ with the intervals matched according to the following rules:*

- *Intervals of type $[k, k]$ are unmatched.*
- *Type $[b, k]$ is matched with type $[b, k-1]$ and vice versa for $b \leq k-1$.*
- *Type $[k, d]$ is matched with type $[k+1, d]$ and vice versa for $d \geq k+1$.*
- *Type $[b, d]$ is matched with type $[b, d]$ in all other cases.*

Proof. Consider the right and left filtrations at V_{k+1} for the zigzag modules V^+ and V^-. Since $\mathbb{V}^+[k+1, n] = \mathbb{V}^-[k+1, n]$, it follows that $\mathcal{L}^+_{k+1} = \mathcal{L}^-_{k+1}$ and $\mathrm{d}^+_{k+1} = \mathrm{d}^-_{k+1}$. Additionally, by Proposition 3.4.4., we have

$$\mathcal{R}^+_{k+1} = \mathcal{R}^-_{k+1}$$

However, b^+_{k+1} agrees with b^-_{k+1} except that indices k and $k+1$ are interchanged as described in Proposition 3.4.3. as follows:

$$\mathrm{b}^+_{k+1} = (k+1, b_0, \ldots, b_k, k)$$

$$\mathrm{b}^-_{k+1} = (k, b_0, \ldots, b_k, k+1)$$

By applying Theorem 3.4.1 (Localization at index k), we conclude that the multiplicities of intervals $[b, d]$ with $b \leq k+1 \leq d$ are identical for both $\mathbb{V}^+$ and $\mathbb{V}^-$, except in cases where k or $k+1$ appears as the birth time. In such cases, the indices k and $k+1$ must be interchanged accordingly. Hence, this corresponds to the case where the intervals $[k, d]$ and $[k+1, d]$ are matched for all $d \geq k+1$.

Similarly, since $\mathbb{V}^+[k-1, n] = \mathbb{V}^-[k-1, n]$, it follows that $\mathcal{R}^+_{k-1} = \mathcal{R}^-_{k-1}$ and $\mathrm{b}^+_{k-1} = \mathrm{b}^-_{k-1}$. Additionally, by Proposition 3.4.4., we have

$$\mathcal{L}^+_{k-1} = \mathcal{L}^-_{k-1}$$

However, d^+_{k-1} agrees with d^-_{k-1} except that indices k and $k-1$ are interchanged as described in Proposition 3.4.3. as follows:

$$\mathrm{d}^+_{k-1} = (k-11, b_0, \ldots, b_k, k)$$

$$\mathrm{d}^-_{k-1} = (k, b_0, \dots, b_k, k-1)$$

By applying Theorem 3.4.1 (Localization at index k), we conclude that the multiplicities of intervals $[b, d]$ with $b \le k-1 \le d$ are identical for both $\mathbb{V}^+$ and $\mathbb{V}^-$, except in cases where k or $k-1$ appears as the death time. In such cases, the indices k and $k-1$ must be interchanged accordingly. Hence, this corresponds to the case where the intervals $[b, k]$ and $[b, k-1]$ are matched for all $b \ge k-1$.

Finally, we consider the intervals $[b, d]$ that do not intersect either $k-1$ or $k+1$; that is, the intervals satisfying either $b, d < k-1$ or $b, d > k+1$. The intervals with $b, d < k-1$, that is, those contained within $[1, k-2]$, are identical for the zigzag modules $\mathbb{V}^+$ and $\mathbb{V}^-$, since the two zigzag modules coincide up to the index $k-1$. Similarly, the intervals with $b, d > k+1$, that is, those contained within $[k+2, n]$, are identical for the zigzag modules $\mathbb{V}^+$ and $\mathbb{V}^-$, since the two zigzag modules coincide beyond the index $k-1$. □

Proposition 3.4.3. Let τ^+ and τ^- denote the types of zigzag modules $\mathbb{V}^+$ and $\mathbb{V}^-$, respectively. If we write

$$(b_1, b_2, \dots, b_{k-1}) = \mathrm{b}(\tau^+[1, k-1]) = \mathrm{b}(\tau^-[1, k-1])$$

for the birth time index up to time $k-1$, then

$$\mathrm{b}(\tau^+[1, k+1]) = (k+1, b_1, \dots, b_k, k)$$

$$\mathrm{b}(\tau^-[1, k+1]) = (k, b_1, \dots, b_k, k+1)$$

Similarly, if we write

$$(d_1, d_2, \dots, d_{n-k}) = \mathrm{d}(\tau^+[k+1, n]) = \mathrm{d}(\tau^-[k+1, n])$$

for the death time index from time $k+1$, then

$$\mathrm{d}(\tau^+[k-1, n]) = (k-1, b_1, \dots, b_k, k)$$

$$\mathrm{d}(\tau^-[k-1, n]) = (k, b_1, \dots, b_k, k-1)$$

Proof. Let $\tau_0 = \tau^+[1, k-1] = \tau^-[1, k-1]$ be the type of $\mathbb{V}[1, k-1]$, and let $\mathrm{b}(\tau_0) = (b_1, b_2, \dots, b_{k-1})$ be the corresponding birth time index. So, we have $\tau^+[1, k+1] = \tau_0 f g$ and $\tau^-[1, k+1] = \tau_0 g f$.By the definition of birth time index, it follows that

$$\mathrm{b}(\tau^+[1, k+1]) = (k+1, b_1, \dots, b_{k-1}, k)$$

$$\mathrm{b}(\tau^-[1,k+1]) = (k, b_1, \ldots, b_{k-1}, k+1)$$

Similarly, let $\tau_0\prime = \tau^+[k+1,n] = \tau^-[k+1,n]$,and let $\mathrm{d}(\tau_0\prime) = (d_1, d_2, \ldots, d_{n-k})$. So, we have $\tau^+[k+1,n] = fg\tau_0\prime$,and $\tau^-[k+1,n] = gf\tau_0\prime$. By the definition of the death time index, it follows that

$$\mathrm{d}(\tau^+[k-1,n]) = (k-1, d_1, \ldots, d_{n-k}, k)$$

$$\mathrm{d}(\tau^-[k-1,n]) = (k, d_1, \ldots, d_{n-k}, k-1)$$

□

Proposition 3.4.4. In the situation of Theorem 3.4.2. The Diamond Principle, the following filtrations are equal:

$$\mathrm{R}(\mathbb{V}^+[1,k+1]) = \mathrm{R}(\mathbb{V}^-[1,k+1])$$

$$\mathrm{L}(\mathbb{V}^+[k-1,n]) = \mathrm{L}(\mathbb{V}^-[k-1,n])$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & & & & W_k & & & \\
V_1 \arrow[r, leftrightarrow] & V_2 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & V_{k-1} \arrow[ur, "f_{k-1}"] & & V_{k+1} \arrow[ul, "g_k"'] \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & V_n \\
 & & & & U_k \arrow[ul, "g_{k-1}"] \arrow[ur, "f_k"'] & & &
\end{tikzcd}$$

Let $\mathbb{V}^+$ and $\mathbb{V}^-$ denote two zigzag modules displayed in the above diagram:

$$\mathbb{V}^+ : V_1 \overset{p_1}{\longleftrightarrow} \ldots \overset{p_{k-2}}{\longleftrightarrow} V_{k-1} \xrightarrow{f_{k-1}} W_k \xleftarrow{g_k} V_{k+1} \overset{p_{k+1}}{\longleftrightarrow} \ldots \overset{p_{n-1}}{\longleftrightarrow} V_n$$

$$\mathbb{V}^- : V_1 \overset{p_1}{\longleftrightarrow} \ldots \overset{p_{k-2}}{\longleftrightarrow} V_{k-1} \xleftarrow{g_{k-1}} U_k \xrightarrow{f_k} V_{k+1} \overset{p_{k+1}}{\longleftrightarrow} \ldots \overset{p_{n-1}}{\longleftrightarrow} V_n$$

Given $\mathbb{V}^+$ and $\mathbb{V}^-$, suppose that the middle diamond at indices $\{k+1, k, k-1\}$ is exact. Since above diamond is exact we have $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$ in the following sequence

$$U_k \xrightarrow{D_1} V_{k+1} \oplus V_{k-1} \xrightarrow{D_2} W_k$$

where $D_1(u) = g_{k-1}(u) \oplus f_k(u)$ and $D_2(v \oplus v\prime) = f_{k-1}(v) - g_k(v\prime)$. for $u \in U_k$, $v \in V_{k+1}$, and $v\prime \in V_{k-1}$.

We can write

$$\mathrm{R}(\mathbb{V}^+[1, k-1]) = \mathrm{R}(\mathbb{V}^-[1, k-1]) = (R_0, R_1, \dots, R_{k-1})$$

By definition of the right filtration, it follows that

$$\mathrm{R}(\mathbb{V}^+[1, k+1]) = \big(0, g_k^{-1} f_{k-1}(R_0), \dots, g_k^{-1} f_{k-1}(R_{k-1}), V_{k+1}\big)$$

and,

$$\mathrm{R}(\mathbb{V}^-[1, k+1]) = \big(0, f_k g_{k-1}^{-1}(R_0), \dots, f_k g_{k-1}^{-1}(R_{k-1}), V_{k+1}\big)$$

We know that $R_i \leq V_{k-1}$ for all $0 \leq i \leq k-1$. So, it is enough to show that

$$g_k^{-1} f_{k-1}(R_i) = f_k g_{k-1}^{-1}(R_i)$$

for any $R_i \leq V_{k-1}$. Let $x \in V_{k+1}$.

$$\begin{aligned} x \in f_k g_{k-1}^{-1}(R_i) \quad &\Longleftrightarrow \quad \exists y \in U_k, \exists z \in R_i \text{ such that } f_k(y) = x \text{ and } g_{k-1} = z \\ &\Longleftrightarrow \quad \exists y \in U_k, \exists z \in R_i \text{ such that } D_1(y) = z \oplus x \\ &\Longleftrightarrow \quad z \oplus x \in \mathrm{Im}(D_1) \end{aligned}$$

Conversely,

$$\begin{aligned} x \in g_k^{-1} f_{k-1}(R_i) \quad &\Longleftrightarrow \quad \exists z \in R_i \text{ such that } f_{k-1}(z) = g_k(x) \\ &\Longleftrightarrow \quad \exists z \in R_i \text{ such that } D_2(z \oplus x) = f_{k-1} - g_k(x) = 0 \\ &\Longleftrightarrow \quad \exists z \in R_i \text{ such that } z \oplus x \in \mathrm{Ker}(D_2) \end{aligned}$$

Since the middle diamond is exact, that is , $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$, it follows that

$$f_k g_{k-1}^{-1}(R_i) = g_k^{-1} f_{k-1}(R_i)$$

for any $R_i \leq V_{k-1}$.

For the second equality of the statement, we have $\mathrm{L}(\mathbb{V}^+[k-1, n]) = \mathrm{L}(\mathbb{V}^-[k-1, n])$.

We can write

$$(L_0, \dots, L_{n+1-k-1}) = (L_0, \dots, L_{n-k}) = \mathrm{L}(\mathbb{V}^+[k+1, n]) = \mathrm{L}(\mathbb{V}^-[k+1, n])$$

Then, by definition of left filtration

$$\mathrm{L}(\mathbb{V}^+[k-1, n]) = (0, f_{k-1}^{-1} g_k(L_0), \dots, f_{k-1}^{-1} g_k(L_{n-k}), V_{k-1})$$

$$\mathrm{L}(\mathbb{V}^-[k-1,n]) = (0, g_{k-1}f_k^{-1}(L_0), \ldots, g_{k-1}f_k^{-1}(L_{n-k}), V_{k-1})$$

We know that $L_i \leq V_{k+1}$ for all $0 \leq i \leq n-k$. So, it is enough to show that

$$f_{k-1}^{-1}g_k(L_i) = g_{k-1}f_k^{-1}(L_i)$$

for any subspace $L_i \leq V_{k+1}$. Let $x \in V_{k-1}$.

$$\begin{aligned} x \in g_{k-1}f_k^{-1}(L_i) &\iff \exists z \in L_i, \exists y \in U_k \quad \text{s.t. } g_{k-1}(y) = x \text{ and } f_k(y) = z \\ &\iff D_1(y) = g_{k-1}(y) \oplus f_k(y) = x \oplus z \text{ for some } z \in L_i \\ &\iff x \oplus z \in \mathrm{Im}(D_1) \end{aligned}$$

Conversely,

$$\begin{aligned} x \in f_{k-1}^{-1}g_k(L_i) &\iff f_{k-1}(x) = g_k(z) \text{ for some } z \in L_i \\ &\iff D_2(x \oplus z) = f_{k-1}(x) - g_k(z) = 0 \text{ for some } z \in L_i \\ &\iff x \oplus z \in \mathrm{Ker}(D_2) \text{ for some } z \in L_i \end{aligned}$$

Since the middle diamond is exact, that is, $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$, it follows that

$$g_{k-1}f_k^{-1}(L_i) = f_{k-1}^{-1}g_k(L_i)$$

for any $L_i \leq V_{k+1}$. Hence, we proved both equalities. $\square$

3.4.2 The Strong Diamond Principle

The Diamond Principle can be effectively applied to the diagram of topological spaces and continuous maps shown below.

$$\begin{tikzcd}
 & & & & A \cup B & & & & \\
X_1 \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & X_{k-2} \arrow[r, leftrightarrow] & A \arrow[ur] & & B \arrow[ul] \arrow[r, leftrightarrow] & X_{k+2} \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & X_n \\
 & & & & A \cap B \arrow[ul] \arrow[ur] & & & &
\end{tikzcd}$$

The four maps in the diamond are inclusions; the other maps $\leftrightarrow$ are arbitrary. Let $\mathbb{X}^+$ denote the upper zigzag diagram, and let $\mathbb{X}^-$ denote the lower zigzag diagram.

Theorem 3.4.5 (The Strong Diamond Principle)**.** *Given $\mathbb{X}^+$ and $\mathbb{X}^-$ as above. There is a complete bijection between the multiset* $\mathrm{Pers}(\mathrm{H}_*(\mathbb{X}^+))$ *and* $\mathrm{Pers}(\mathrm{H}_*(\mathbb{X}^-))$*. Intervals are matched according to the following rules:*

- $[k, k] \in \mathrm{Pers}(\mathrm{H}_{\mathrm{l}+1}(\mathbb{X}^{+}))$ *is matched with* $[k, k] \in \mathrm{Pers}(\mathrm{H}_{\mathrm{l}}(\mathbb{X}^{-}))$

In the other cases, the matching preserves homological dimensions:

- *Type* $[b, k]$ *is matched with type* $[b, k-1]$ *and vice versa for* $b \leq k-1$
- *Type* $[k, d]$ *is matched with type* $[k+1, d]$ *and vice versa for* $d \geq k+1$
- *Type* $[b, d]$ *is matched with type* $[b, d]$ *in all other cases.*

Proof. For any l, apply the homology functor H_l to the diagram. The middle diamond becomes as follows:

$$\begin{tikzcd} \mathrm{H}_l(A) \arrow[r] & \mathrm{H}_l(A \cup B) \\ \mathrm{H}_l(A \cap B) \arrow[u] \arrow[r] & \mathrm{H}_l(B) \arrow[u] \end{tikzcd}$$

The middle diamond is exact by the Mayer-Vietoris Theorem, where

$$\ldots \rightarrow \mathrm{H}_l(A \cap B) \xrightarrow{D_1} \mathrm{H}_l(A) \oplus \mathrm{H}_l(B) \xrightarrow{D_2} \mathrm{H}_l(A \cup B) \rightarrow \ldots$$

is an exact sequence. By the Diamond Principle Theorem $3.4.2.$, we have a partial bijection in $\mathrm{Pers}(\mathrm{H}_l(\mathbb{X}^{+}))$ and $\mathrm{Pers}(\mathrm{H}_l(\mathbb{X}^{-}))$ except for the intervals with type $[k, k]$.

Now, consider the connecting homomorphism in the same Mayer-Vietoris sequence:

$$\ldots \xrightarrow{D_2} \mathrm{H}_{l+1}(A \cup B) \xrightarrow{\partial} \mathrm{H}_l(A \cap B) \xrightarrow{D_1} \ldots$$

By exactness, ∂ induces an isomorphism as follows:

$$\mathrm{Coker}(D_2) = \mathrm{Ker}(D_1)$$

So, the $[k, k]$ summands of $\mathrm{Pers}(\mathrm{H}_{l+1}(\mathbb{X}^{+}))$ span

$$\mathrm{Coker}(D_2) = {\mathrm{H}_{l+1}(A \cup B)}/{\mathrm{Im}(D_2)}$$

In addition, the $[k, k]$ summands of $\mathrm{Pers}(\mathrm{H}_l(\mathbb{X}^{-}))$ span $\mathrm{Ker}(D_1)$.

Hence, by the exactness of the Mayer-Vietoris sequence, we can match the interval $[k, k] \in \mathrm{Pers}(\mathrm{H}_{\mathrm{l}+1}(\mathbb{X}^{+}))$ with $[k, k] \in \mathrm{Pers}(\mathrm{H}_{\mathrm{l}}(\mathbb{X}^{-}))$.

$\square$

CHAPTER 4

REAL-VALUED FUNCTIONS

The concept of zigzag modules is obtained by applying homology to a zigzag filtration and considering the induced linear maps, as discussed in Chapter 2. This chapter introduces zigzag persistence and outlines its computational framework based on the article "Zigzag Persistence and Real-Valued Functions" [2] by Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov.

A *zigzag diagram* of topological spaces is a sequence

$$\mathcal{X} : X_1 \leftrightarrow X_2 \leftrightarrow X_3 \leftrightarrow \cdots \leftrightarrow X_{n-1} \leftrightarrow X_n$$

where each X_i is a topological space and each $\leftrightarrow$ is a continuous map. We distinguish $X_i \to X_{i+1}$ are **forward** maps and $X_i \leftarrow X_{i+1}$ are **backward** maps.

Applying H_p, the homology functor with coefficients in a field $\mathbb{F}$, to the zigzag diagram yields a *zigzag diagram of vector spaces*, also called a *zigzag module*:

$$H_p(\mathcal{X}) : H_p(X_1) \leftrightarrow H_p(X_2) \leftrightarrow \cdots \leftrightarrow H_p(X_{n-1}) \leftrightarrow H_p(X_n).$$

The diamond principle describes the relationship between the persistent intervals of two zigzag diagrams that differ by a single local change in [2] as follows:

We consider the following zigzag diagrams:

$$\mathcal{X} : X_1 \leftrightarrow X_2 \leftrightarrow \cdots \leftrightarrow X_{k-1} \leftrightarrow U \leftrightarrow V \leftrightarrow X_{k+2} \leftrightarrow \cdots \leftrightarrow X_n$$

$$\mathcal{X}_U : X_1 \leftrightarrow \cdots \leftrightarrow X_{k-1} \leftrightarrow U \to U \cup V \leftarrow V \leftrightarrow X_{k+2} \leftrightarrow \cdots \leftrightarrow X_n$$

$$\mathcal{X}_\cap : X_1 \leftrightarrow \cdots \leftrightarrow X_{k-1} \leftrightarrow U \leftarrow U \cap V \to V \leftrightarrow X_{k+2} \leftrightarrow \cdots \leftrightarrow X_n$$

There exists a bijection:

$$\mathrm{Pers}(\mathcal{X}_\cap) \leftrightarrow \mathrm{Pers}(\mathcal{X}_U)$$

which transforms intervals according to the following rules:

- For $B \in \{X_1, X_2, \ldots, X_{k-2}, U\}$:

$$[B, U] \leftrightarrow [B, U \cup V]$$

$$[B, U \cap V] \leftrightarrow [B, U]$$

- For $D \in \{V, X_{k+2}, \ldots, X_n\}$:

$$[V, D] \leftrightarrow [U \cup V, D]$$

$$[U \cap V, D] \leftrightarrow [V, D]$$

- Otherwise:

$$[B, D] \leftrightarrow [B, D]$$

- Exceptional case: (Dimension shift +1 in homology)

$$[U \cap V, U \cap V] \in \mathrm{Pers}_p(\mathcal{X}_\cap) \leftrightarrow [U \cup V, U \cup V]^+ \in \mathrm{Pers}_p(\mathcal{X}_U)$$

  where the superscript $+$ indicates that in homology this interval appears in dimension $p + 1$.

4.1 Level Set Zigzag Persistence

Let X be a topological space, and let $f : X \to \mathbb{R}$ be a continuous function.

Definition 52. Let X be a topological space, and let $f : X \to \mathbb{R}$ be a continuous function. The *level set* at value $t \in \mathbb{R}$ is defined as

$$X_t = f^{-1}(t).$$

Definition 53. Let X be a topological space, and let $f : X \to \mathbb{R}$ be a continuous function. The *slices* corresponding to an interval $I \subset \mathbb{R}$ is defined as

$$X_I = f^{-1}(I).$$

Definition 54. We suppose that the pair (X, f), where X is a topological space and $f : X \to \mathbb{R}$ is continuous, is of *Morse type* if the following conditions hold:

- There exists a finite set of real numbers $a_1 < a_2 < \cdots < a_n$, called *critical values*, such that each open interval

$$I \in \{(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, \infty)\}$$

satisfies that the slice $X_I = f^{-1}(I)$ is homeomorphic to the product $Y \times I$, where f corresponds to the projection onto the second factor I.

- Each homeomorphism $Y \times I \to X_I$ extends to a continuous map

$$Y \times \overline{I} \to X_{\overline{I}},$$

where $\overline{I}$ denotes the closure of the interval $I \subset \mathbb{R}$.

Example 4.1.1 (Height Function on the Torus)**.** Let $X = T^2$, and $f : T^2 \to \mathbb{R}$ be the height function with critical values $a_1 < a_2 < a_3 < a_4$.

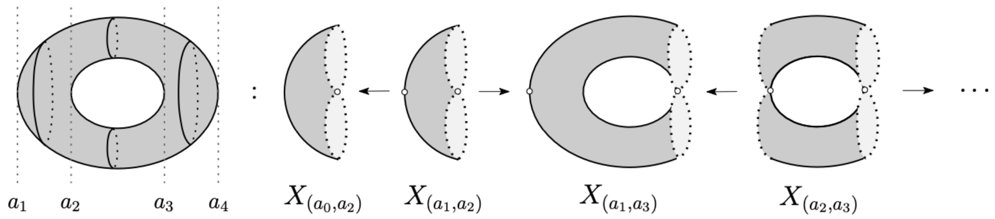

Figure 4.1: A torus with four critical values, where the height function gives the real-valued function in [6]

Consider intervals:

$$I_1 = (-\infty, a_1), \quad I_2 = (a_1, a_2), \quad I_3 = (a_2, a_3), \quad I_4 = (a_3, a_4), \quad I_5 = (a_4, \infty)$$

Slices:

- $X_{I_1} = \emptyset$
- $X_{I_2} \cong S^1 \times I_2$
- $X_{I_3} \cong (S^1 \sqcup S^1) \times I_3$

Now, let us analyze each slice:

- **Slice X_{I_1}:**
$$X_{I_1} = f^{-1}(I_1) = \emptyset.$$
We take $Y = \emptyset$, and thus:
$$X_{I_1} \cong \emptyset \times I_1.$$

- **Slice X_{I_2}:**
$$X_{I_2} = f^{-1}(I_2) \cong S^1 \times I_2.$$
Here, we take $Y = S^1$, so this slice has the structure of a cylindrical band wrapping around the torus.

- **Slice X_{I_3}:**
$$X_{I_3} = f^{-1}(I_3) = f^{-1}((a_2, a_3)) \cong (S^1 \sqcup S^1) \times I_3.$$
We take $Y = S^1 \sqcup S^1$, corresponding to two disjoint circles moving along the interval I_3.

Definition 55 (Level Set Zigzag Persistence)**.** Let (X, f) be a pair of Morse type. Choose a sequence of real numbers s_i such that

$$-\infty < s_0 < a_1 < s_1 < a_2 < \cdots < s_{n-1} < a_n < s_n < \infty,$$

where $a_1, a_2, \ldots, a_n$ are the critical values of f.

Using this sequence, we construct a zigzag diagram of topological spaces:

$$X : \quad X_0^0 \to X_0^1 \leftarrow X_1^1 \to X_1^2 \leftarrow \cdots \to X_{n-1}^n \leftarrow X_n^n,$$

where each space is defined by

$$X_i^j = X_{[s_i, s_j]} = f^{-1}([s_i, s_j]).$$

To highlight the role of critical values, we adopt the following labeling convention:

- Each X_i^i is associated with the critical value a_i.
- Each X_i^{i+1} corresponds to the regular interval containing it. For example, the sequence
$$X_0^0,\, X_1^1, \ldots, X_{n-1}^{n-1},\, X_n^n$$

corresponds to the intervals

$$(-\infty, a_1),\ (a_1, a_2),\ \ldots,\ (a_{n-1}, a_n),\ (a_n, \infty).$$

Zigzag persistence intervals of $\mathbb{X}$ are defined by taking the union of the labels of the consecutive modules $\mathbb{X}_i^i$ and $\mathbb{X}_{i-1}^i$ over which the interval is supported. As a result, each persistence interval is associated with an open, closed, or half-open interval on the real line. In practice, we translate between the $\mathbb{X}$ notation and critical value notation based on [2] as follows:

$$\begin{aligned}
\left[\mathbb{X}_{i-1}^i, \mathbb{X}_{j-1}^j\right] &\leftrightarrow [a_i, a_j] && \text{for } 1 \leq i \leq j \leq n,\\
\left[\mathbb{X}_{i-1}^i, \mathbb{X}_{j-1}^{j-1}\right] &\leftrightarrow [a_i, a_j) && \text{for } 1 \leq i < j \leq n+1,\\
\left[\mathbb{X}_{i}^i, \mathbb{X}_{j-1}^j\right] &\leftrightarrow (a_i, a_j] && \text{for } 0 \leq i < j \leq n,\\
\left[\mathbb{X}_{i}^i, \mathbb{X}_{j-1}^{j-1}\right] &\leftrightarrow (a_i, a_j) && \text{for } 0 \leq i < j \leq n+1.
\end{aligned}$$

Here, we set $a_0 = -\infty$ and $a_{n+1} = +\infty$ to account for unbounded intervals. Such infinite or semi-infinite intervals arise naturally unless $\mathbb{X}_0^0 = \mathbb{X}_n^n = \emptyset$, which holds when $\mathbb{X}$ is derived from a real-valued function defined on a compact topological space.

Definition 56 (Level Set Zigzag Persistence Diagram)**.** Each interval of any of the four types arising in level set zigzag persistence can be represented by a corresponding point $(a_i, a_j) \in \mathbb{R}^2$.

The collection of these points counted with multiplicity and labeled by type and homological dimension along with all points on the diagonal (i.e., $a_i = a_j$) in every dimension taken with infinite multiplicity, is called the *level set zigzag persistence diagram*, denoted by $\mathrm{DgmZZ}(f)$.

Remark 4.1.1 (Properties of Level Set Zigzag Persistence). Level set zigzag persistence exhibits two fundamental structural properties, which follow almost directly from the definitions:

- **Locality:** Let $I \subset \mathbb{R}$ be an interval. The zigzag diagram associated with the slice $\mathbb{X}_I = f^{-1}(I)$, equipped with the restricted function f_I, forms a subdiagram of the zigzag diagram for the original pair $(\mathbb{X}, f)$. As a result, there is a bijection between the level set zigzag intervals of $(\mathbb{X}, f)$ that intersect I, and those of $(\mathbb{X}_I, f_I)$.

- **Symmetry:** The level set zigzag persistence linked to $(\mathbb{X}, f)$ and $(\mathbb{X}, -f)$ are reflections of another. Specifically, their corresponding zigzag diagrams are isomorphic under reflection, and therefore, the level set zigzag intervals are in one-to-one correspondence via this symmetry.

4.2 Pyramids

Definition 57. Given a Morse-type pair (X, f), we construct a commutative diagram known as a *pyramid*. The nodes of this diagram consist of topological spaces or relative pairs derived from the slices of X. More precisely:

$$\begin{tikzcd}[column sep=small]
 & (X_0^n, X_0^i \cup X_j^n) \text{ with } i < j \arrow[dl] \arrow[dr] & \\
(X_0^j, X_0^i) \text{ with } i \leq j \arrow[dr] & & (X_i^n, X_j^n) \text{ with } i < j \arrow[dl] \\
 & X_j^i \text{ with } i \leq j &
\end{tikzcd}$$

The diagram is commutative and encodes how the various spaces and relative pairs relate via inclusion maps or projections between slices.

Example 4.2.1 (Case $n = 1$ Pyramid Diagram)**.** The pyramid diagram for the case $n = 1$ illustrates the simplest nontrivial structure derived from a Morse-type function with one critical value as follows:

$$\begin{tikzcd}[column sep=small]
 & & (X_0^1, X_0^0 \cup X_1^1) = (X_0^1, \partial X) & & \\
 & (X_0^1, X_0^0) \arrow[ur, dashed] & & (X_0^1, X_1^1) \arrow[ul, dashed] & \\
\emptyset \arrow[ur, dashed] & & X_0^1 \arrow[ul] \arrow[ur] & & \emptyset \arrow[ul, dashed] \\
 & X_0^0 \arrow[ul, dashed] \arrow[ur] & & X_1^1 \arrow[ul] \arrow[ur, dashed] &
\end{tikzcd}$$

Mayer–Vietoris pyramid for the levelset zigzag ($n = 3$) as below Figure 4.2:

$$
\begin{tikzcd}[column sep=small]
\emptyset & & (\mathbb{X}^3_0, {}^3_2\mathbb{X}) & & (\mathbb{X}^3_0, {}^2_1\mathbb{X}) & & (\mathbb{X}^3_0, {}^1_0\mathbb{X}) & & \emptyset \\
& (\mathbb{X}^3_0, \mathbb{X}^2_0) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, {}^3_1\mathbb{X}) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, {}^2_0\mathbb{X}) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, \mathbb{X}^3_1) \arrow[ul] \arrow[ur] & \\
\emptyset \arrow[ur] & & (\mathbb{X}^3_0, \mathbb{X}^1_0) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, {}^3_0\mathbb{X}) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, \mathbb{X}^3_2) \arrow[ul] \arrow[ur] & & \emptyset \arrow[ul] \\
& (\mathbb{X}^2_0, \mathbb{X}^1_0) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, \mathbb{X}^0_0) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_0, \mathbb{X}^3_3) \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_1, \mathbb{X}^3_2) \arrow[ul] \arrow[ur] & \\
\emptyset \arrow[ur] & & (\mathbb{X}^2_0, \mathbb{X}^0_0) \arrow[ul] \arrow[ur] & & \mathbb{X}^3_0 \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_1, \mathbb{X}^3_3) \arrow[ul] \arrow[ur] & & \emptyset \arrow[ul] \\
& (\mathbb{X}^1_0, \mathbb{X}^0_0) \arrow[ul] \arrow[ur] & & \mathbb{X}^2_0 \arrow[ul] \arrow[ur] & & \mathbb{X}^3_1 \arrow[ul] \arrow[ur] & & (\mathbb{X}^3_2, \mathbb{X}^3_3) \arrow[ul] \arrow[ur] & \\
\emptyset \arrow[ur] & & \mathbb{X}^1_0 \arrow[ul] \arrow[ur] & & \mathbb{X}^2_1 \arrow[ul] \arrow[ur] & & \mathbb{X}^3_2 \arrow[ul] \arrow[ur] & & \emptyset \arrow[ul] \\
& \mathbb{X}^0_0 \arrow[ul] \arrow[ur] & & \mathbb{X}^1_1 \arrow[ul] \arrow[ur] & & \mathbb{X}^2_2 \arrow[ul] \arrow[ur] & & \mathbb{X}^3_3 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 4.2: Pyramid for the levelset zigzag ($n = 3$) in [2]

Mayer-Vietoris pyramid for the levelset zigzag ($n = 3$). The two diagonals carry the extended persistence of f ($\nearrow$) and $-f$ ($\searrow$). Every diamond has the Mayer-Vietoris property. We use the shorthand ${}^{j}_{i}\mathbb{X} = \mathbb{X}^i_0 \cup \mathbb{X}^n_j$ to keep the diagram small.

Definition 58 (Monotone Zigzags)**.** The pyramid associated with a Morse-type pair (X, f) yields many zigzag diagrams by selecting paths through its commutative structure.

Among these, *monotone zigzags* are those paths that proceed strictly from west to east, without any backtracking. That is, the sequence of maps in a monotone zigzag always advances in one direction through the diagram.

These monotone zigzags have $2n+1$ nodes, excluding the initial and terminal $\emptyset$.

Theorem 4.2.1 ([2]). *Let (X,f) be of Morse type. Any two monotone zigzags $\mathcal{X}_1, \mathcal{X}_2$ in the pyramid diagram carry the same information in their persistent homology.*

Moreover, there exists an explicit bijection between

$$\mathrm{Pers}\big(H_*(\mathcal{X}_1)\big) \quad \textit{and} \quad \mathrm{Pers}\big(H_*(\mathcal{X}_2)\big),$$

which respects the homological dimension, except for possible shifts of degree ± 1.

Proof. We can transform $\mathcal{X}_1$ into $\mathcal{X}_2$ through a sequence of diamond moves. Each move makes a local adjustment to the diagram while preserving the global structure of the persistence intervals. The births and deaths of intervals are tracked as they move through these transformations.

In Theorem $3.4.5.$ of the previous chapter, the Strong Diamond Principle, we described how zigzag persistence barcodes, i.e., persistence intervals, match after applying the homology functor to the two zigzag diagrams of topological spaces that form a diamond.

Thus, we can identify the bijection between the persistence barcodes described in the Strong Diamond Principle (Theorem $3.4.5$) and the corresponding diamond regions in the pyramid structure shown in Figure 4.2. □

In principle, the rules outlined in the proof of the *Pyramid Theorem* can be used to determine the interval transformation law between any pair of monotone zigzags.

4.3 Stability

An important result in the study of persistent homology is the well-established stability of persistence diagrams based on the article "Stability of Persistence Diagrams" [3] by David Cohen-Steiner, Herbert Edelsbrunner and John Hare and stability of extended persistence diagrams based on the article "Extending Persistence Using Poincaré and Lefschetz Duality" [4] by David Cohen-Steiner, Herbert Edelsbrunner and John Harer. By leveraging the relationship between level-set zigzag persistence and extended persistence, the former's stability can be deduced from that of the latter.

Theorem 4.3.1 (Extended Persistence Stability Theorem [3], [4])**.** *Given two (Morse type) functions $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \mathbb{R}$, let*

$$\delta = \|f - g\|_\infty.$$

Then the bottleneck distance between their extended persistence diagrams is upper bounded by the L_∞ distance between them,

$$d_B(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) \leq \delta.$$

It is not difficult to observe that the transformation between extended and level set zigzag persistence preserves stability in terms of critical values.

Theorem 4.3.2 (Levelset Zigzag Stability Theorem [2])**.** *Given two (Morse type) functions $f : \mathbb{X} \to \mathbb{R}$ and $g : \mathbb{X} \to \mathbb{R}$ with $\delta = \|f - g\|_\infty$, let $\mathrm{DgmZZ}_p(f)$ and $\mathrm{DgmZZ}_p(g)$ be the p-dimensional persistence diagrams of the levelset zigzags of f and g. Then*

$$d_B(\mathrm{DgmZZ}_p(f), \mathrm{DgmZZ}_p(g)) \leq \delta.$$

Proof. As noted in [4], the Stability Theorem for extended persistence can be applied individually to each subdiagram:

$$\begin{aligned} d_B(\mathrm{Ord}_p(f), \mathrm{Ord}_p(g)) &\leq \delta, \\ d_B(\mathrm{Rel}_p(f), \mathrm{Rel}_p(g)) &\leq \delta, \\ d_B(\mathrm{Ext}_p(f), \mathrm{Ext}_p(g)) &\leq \delta, \end{aligned}$$

where d_B denotes the bottleneck distance.

This observation, together with the transformation rules summarized, implies that the intervals corresponding to ordinary and relative subdiagrams in level set zigzag persistence are stable.

While a potential instability may arise from points in the extended subdiagram switching from those below the diagonal to those above it (i.e., crossing the diagonal), such points are necessarily close to the diagonal. In that case, both the extended persistence diagram and the level set zigzag diagram enable these points to match with diagonal points, which preserves stability.

Therefore, the transformation from extended persistence to level set zigzag persistence maintains the stability of persistence diagrams.

□

CHAPTER 5

CONCLUSION

In this thesis, we have presented a survey on Zigzag Persistence, exploring the mathematical structure and computation of zigzag persistence modules. In Chapter 1, we showed a difference between ordinary persistence homology and zigzag persistence and motivated the introduction of zigzag modules and zigzag persistence. In Chapter 2, we provided some background information about filtration and persistent homology to introduce zigzag persistence.

Chapter 3 introduced the theory of Zigzag Persistence. In section 3.1, we formally described a zigzag diagram of vector spaces. In section 3.2, we examined three significant topics: the Right Filtration Operator, Decomposition of Filtered Vector Spaces, and Streamlined Modules. These contributed to the algorithm and theorem of the decomposition of Zigzag modules. Next, in section 3.3, we introduced the Interval Decomposition Theorem, which relates the decomposition of zigzag modules with type τ over the interval modules. We also examined the Diamond Principle and a stronger version of it, which are foundational structural consequences of zigzag diagrams of vector spaces.

Chapter 4 presented the relationship between Zigzag Persistence and Real-Valued Functions. We begin with level-set Zigzag Persistence, which captures topological changes in sublevel sets of topological spaces. Next, we introduce Pyramids, a diagram of topological spaces or relative pairs derived from the slices of topological space $\mathbb{X}$. Finally, we present the stability of level-set zigzag persistence.

There are huge number of applications using standard algorithms for computing persistence homology to real life problems. However, there are only a few applications of zigzag persistence like image processing and neural network analysis. For future work, since there are freely available algorithms, one can try to widen the applications of zigzag persistence. On the theoretic side, there is an algebraic stability for zigzag persistence modules, which is obtained by extending a zigzag persistence module to a two-dimensional persistent module which is block decomposable. One can explore on this more and obtain stability results for derived categories of zigzag or multidimensional persistence modules.

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