A DESCRIPTIVE SET-THEORETIC ANALYSIS OF PATH-CONNECTEDNESS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

YUSUF UYAR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

AUGUST 2025

Approval of the thesis:

A DESCRIPTIVE SET-THEORETIC ANALYSIS OF PATH-CONNECTEDNESS

submitted by **YUSUF UYAR** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Naci Emre Altun
Dean, Graduate School of **Natural and Applied Sciences** _____

Prof. Dr. Hasan Taşeli
Head of Department, **Mathematics** _____

Assist. Prof. Dr. Burak Kaya
Supervisor, **Mathematics, METU** _____

Examining Committee Members:

Prof. Dr. Yıldıray Ozan
Mathematics, METU _____

Assist. Prof. Dr. Burak Kaya
Mathematics, METU _____

Assist. Prof. Dr. Elif Medetoğulları
Mathematics and Science Education, TED University _____

Date:06.08.2025

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Surname: Yusuf Uyar

Signature :

ABSTRACT

A DESCRIPTIVE SET-THEORETIC ANALYSIS OF PATH-CONNECTEDNESS

Uyar, Yusuf
M.S., Department of Mathematics
Supervisor: Assist. Prof. Dr. Burak Kaya

August 2025, 57 pages

In this thesis, we analyze the complexity of path-connectedness and some other topological notions related to connectedness in $\mathbb{R}^2$ and $\mathbb{R}^3$ from the point of view of descriptive set theory. More specifically, following Debs, Saint Raymond and Becker, we survey the maximal descriptive complexity of the collection of subsets satisfying certain connectedness properties inside hyperspaces on Polish spaces and then, give examples of subspaces of $\mathbb{R}^n$ for $n = 2$ and $n = 3$ in which these complexity bounds are realized in various cases. In addition, we also examine the equivalence relation of path-connectedness on Polish subspaces of $\mathbb{R}^2$ using tools of Borel complexity theory. More specifically, we prove that the path-connectedness relation of a Polish subspace of $\mathbb{R}^2$ is an essentially countable Borel equivalence relation. We also show that the path-connectedness relation of the Knaster continuum is a non-smooth essentially hyperfinite Borel equivalence relation.

Keywords: Descriptive complexity, Path-connected, Connected, Borel reducible, Borel equivalence relations.

ÖZ

YOL BAĞLANTILILIĞIN BETİMSEL KÜME KURAMSAL ANALİZİ

Uyar, Yusuf
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Dr. Öğr. Üyesi. Burak Kaya

Ağustos 2025 , 57 sayfa

Bu tezde yol bağlantılılık ve bağlantılılık ile ilgili diğer bazı topolojik kavramları $\mathbb{R}^2$ ve $\mathbb{R}^3$'te betimsel küme kuramsal açıdan analiz ediyoruz. Daha spesifik olarak, Debs, Saint Raymond ve Becker'i takip ederek Leh uzaylarının hiper uzaylarında belirli bağlantılılık özelliklerini sağlayan alt küme topluluklarının maksimal betimsel karmaşıklığını inceleyip daha sonra bazı durumlarda bu karmaşıklık sınırlarının gerçeklendiği $n = 2$ ve $n = 3$ için $\mathbb{R}^n$'nin alt uzaylarının örneklerini veriyoruz. İlaveten, Borel karmaşıklık teorisinin yöntemlerini kullanarak $\mathbb{R}^2$'nin Leh alt uzaylarında yol bağlantılılık denklik bağıntısını da inceliyoruz. Daha spesifik olarak, $\mathbb{R}^2$'nin bir Leh alt uzayının yol bağlantılılık denklik bağıntısının özünde sayılabilir bir Borel denklik bağıntısı olduğunu kanıtlıyoruz. Knaster sürekliliğinin yol bağlantılılık denklik bağıntısının düzgün olmayan özünde hiper sonlu bir Borel denklik bağıntısı olduğunu da gösteriyoruz.

Anahtar Kelimeler: Betimsel karmaşıklık, yol bağlantılı, bağlantılı, Borel indirgenebilir, Borel denklik bağıntıları

To those who pursue goodness, beauty and truth

ACKNOWLEDGMENTS

Some parts of Chapter 5 of this thesis appeared in [24], which may or may not be included as a part of a future publication.

I'm deeply indebted to my family who believed in me and supported me throughout my whole life.

I would also like to thank to TÜBİTAK, the Scientific and Technological Research Council of Türkiye, for their financial support through 2211 - National Graduate Scholarship Programs (2210-E Program).

TABLE OF CONTENTS

ABSTRACT v

ÖZ vi

ACKNOWLEDGMENTS viii

TABLE OF CONTENTS ix

LIST OF TABLES xi

LIST OF FIGURES xii

CHAPTERS

1 INTRODUCTION 1

1.1 A Historical Overview 1

1.2 On Descriptive Complexity 2

1.3 Outline of the Thesis 4

2 PRELIMINARIES 7

2.1 Polish Spaces and Standard Borel Spaces 7

2.2 Hyperspaces on Polish Spaces 9

2.3 Trees 12

2.4 Borel and Projective Hierarchies 13

2.5 Borel Equivalence Relations 21

3 DESCRIPTIVE COMPLEXITY OF PATH-CONNECTEDNESS 25

4 DESCRIPTIVE COMPLEXITY OF CONNECTEDNESS 33

5 BOREL EQUIVALENCE RELATIONS OF PATH-CONNECTEDNESS . . 41

6 CONCLUSION 49

REFERENCES 53

APPENDICES 57

A COMPLEXITIES OF RELATIONS ON STANDARD BOREL SPACES 57

B COMPLEXITIES OF FUNCTIONS ON STANDARD BOREL SPACES 57

LIST OF TABLES

TABLES

Table 6.1 Upper bounds for $\mathcal{C}_{path}(X)$, $\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$ 49

LIST OF FIGURES

FIGURES

Figure 2.1 Vietoris topology $\subseteq$ Hausdorff metric topology 11

Figure 2.2 The Borel hierarchy 14

Figure 2.3 The projective hierarchy 17

Figure 3.1 Graphs of f and $f.s_\beta$ 27

Figure 3.2 $L_{\alpha,\beta}$ for $\alpha \in \Delta,\ \beta = (1, 1, 0, 0, 0...)$ 28

Figure 3.3 $L \cup K_T^0 \subseteq \mathbb{I}^2$ 30

Figure 3.4 $L \cup K_T^1$ 31

Figure 3.5 $L \cup K_T^2$ 31

Figure 5.1 A simple triod 42

Figure 5.2 Elements of $\mathcal{J}_n$ 44

Figure 5.3 The Knaster continuum (or the bucket handle) 48

CHAPTER 1

INTRODUCTION

1.1 A Historical Overview

The core of the descriptive set theory goes back to the work of Lebesgue, Baire, and Borel on definable subsets of $\mathbb{R}^n$ and definable functions of several real variables at the turn of the twentieth century. These French mathematicians were not comfortable with the idea of arbitrary function and arbitrary subset, such as subsets formed by Zermelo's choice function that "chooses" an element from each subset of a given set. They doubted the existence of functions or sets asserted without explicit definitions.

Baire introduced in his doctoral thesis the smallest collection of functions of several real variables containing continuous functions and closed under point-wise limits, which is now known as Baire functions. Lebesgue systematically worked in [15] what we can call the definability theory of continuum. He observed that Baire functions are exactly the ones that are "analytically representable" functions and that the inverse image of an open interval (an open rectangle in $\mathbb{R}^2$ etc.) under a Baire function coincides with the Borel measurable subsets of $\mathbb{R}^n$, the smallest collection of subsets containing open intervals and closed under complementation and countable union, which were introduced by Borel.

In the same article, Lebesgue used the argument that projection of a Borel subset of $\mathbb{R}^2$ onto $\mathbb{R}$ is still Borel measurable in one of his results. He assumed that this was a trivial argument, which was in fact incorrect. The error was identified by a young mathematics student Suslin who named projections of Borel sets as analytic, and he together with his professor Lusin founded basic properties of analytic sets. One of the most important results of Lusin was that Borel sets are those that are

analytic and whose complements are also analytic. Unfortunately, Suslin died in his mid-twenties. The investigation of analytic sets was then conducted by Lusin and his student Sierpinski. In 1925, they introduced a new class of definable sets called projective sets and they established fundamental properties of projective sets. It is the smallest collection of subsets obtained from Borel sets by applying successive projection and complementation operations.

In its early development, it was realized that the methods and techniques of the theory could be applied - sometimes simplifying the analysis - to all completely metrizable separable spaces, especially the Baire space $\mathbb{N}^{\mathbb{N}}$, which is the countable product space of the discrete topology on $\mathbb{N}$. If one has a nice way of coding through the infinite sequences of natural numbers $\mathbb{N}^{\mathbb{N}}$ or corresponding subspaces of it, then one can study the subject from the descriptive perspective. This is one of the reasons why descriptive set theory has applications in various other fields of mathematics today. We refer the reader to our references [8], [10], [18] and [20] for a detailed account of the history and development of descriptive set theory.

Since the techniques of descriptive set theory become available once there is a completely metrizable separable topology, numerous mathematicians studied applications of descriptive set theory to topology and analysis after the theory and the techniques have been well-established. In particular, logicians and topologists started to study the topological properties of Euclidean and other spaces through certain hyperspaces in the last decades.

1.2 On Descriptive Complexity

In usual mathematical practice, a topological space (X, τ) is typically defined by its open sets. So we may assume that open sets are our basic building blocks for definable subsets of X. We can then construct some other subsets of X using elements already defined in τ by applying the operations of countable union and complementation, which we take as our basic operations. The reason behind this choice is that, if we go beyond the countable iteration of union and complementation, we could exhaust all subsets of the space, and therefore get undesired pathological subsets.

What we consider is in fact simply the σ-algebra generated by open subsets, usually referred to as the Borel structure on the space, which is one of the central notions of measure theory.

Most of the sets that we explicitly define and use in practice can be obtained in this form. For example, consider any continuous function $f : [0,1] \to \mathbb{R}$. Then the set of differentiability points of f is

$$\bigcap_{\epsilon\in\mathbb{Q}^+}\bigcup_{\delta\in\mathbb{Q}^+}\bigcap_{m,n\in\mathbb{N}}\Big\{x : 0 < |q_n - x|, |q_m - x| < \delta \to \big|\tfrac{f(q_n)-f(x)}{q_n-x} - \tfrac{f(q_m)-f(x)}{q_m-x}\big| < \epsilon\Big\}$$

where we reserve the sequence $\{q_k\}_k$ as a fixed enumeration of rationals in $\mathbb{I} := [0,1]$ throughout the thesis. It is not that hard to check that the innermost set is a union of a closed subset of $\mathbb{R}$ with an open subset, and so $\{x \in [0,1] : f'(x) \text{ exists }\}$ can be obtained from open subsets of real numbers by applying finitely many allowed basic operations ([11, Page 70]).

It turns out that the Borel σ-algebra of a topological space can be stratified into a hierarchy. Given a topological space X, we denote $\mathbf{\Sigma}^0_1$ to be the open subsets of X. Then for any countable ordinal $\alpha > 0$, we set $\mathbf{\Pi}^0_\alpha$ as complements of sets in $\mathbf{\Sigma}^0_\alpha$ where $\mathbf{\Sigma}^0_\alpha$ is the collection of countable unions of elements in $\bigcup_{\xi<\alpha}\mathbf{\Pi}^0_\xi$. If X is metrizable, then collections $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$, which we call Borel pointclasses, lie properly in an increasing hierarchy as they satisfy $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_{\alpha+1}, \mathbf{\Pi}^0_{\alpha+1}$ for all $\alpha < \omega_1$ and exhaust all Borel subsets of X.

Although taking projection of a subset can be considered as a basic operation which "preserves definability", Borel sets are not closed under the usual projection operation as Lusin spotted in the work of Lebesgue. This motivates the definition of analytic sets, namely projections of Borel subsets of product spaces. Analytic sets are closed under countable unions and intersections but not under complementation operation. We then set $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ to be analytic and complements of analytic subsets of X respectively. Then we recursively define $\mathbf{\Pi}^1_n$ to be the complements of elements in $\mathbf{\Sigma}^1_n$ where $\mathbf{\Sigma}^1_n$ is the projections of $\mathbf{\Pi}^1_{n-1}$ subsets of product spaces of X, for any $n > 1$. These new pointclasses are called projective classes and it suffices to iterate them over natural numbers because they are clearly closed under taking projections and complementation (not class-by-class but as a whole).

The stratification of pointclasses can be visualized as

$$\begin{array}{cccccccccc} \mathbf{\Sigma}^0_1 & \mathbf{\Sigma}^0_2 & \mathbf{\Sigma}^0_3 & \dots & \mathbf{\Sigma}^0_\alpha & \dots & \mathbf{\Sigma}^1_1 & \mathbf{\Sigma}^1_2 & \dots & \mathbf{\Sigma}^1_n & \dots \\ \mathbf{\Pi}^0_1 & \mathbf{\Pi}^0_2 & \mathbf{\Pi}^0_3 & \dots & \mathbf{\Pi}^0_\alpha & \dots & \mathbf{\Pi}^1_1 & \mathbf{\Pi}^1_2 & \dots & \mathbf{\Pi}^1_n & \dots \end{array}$$

for $\alpha < \omega_1$ and $n \in \mathbb{N}$ where each class is a subset of the classes on the right of it (independent of the line on which they lie), as we shall discuss. We interpret that the complexity of sets in pointclasses increases as we go from left to right in the ordering above, as it requires "more steps" to be applied to construct them in that direction. We note that we will only consider these pointclasses (Borel and projective classes) throughout the thesis.

The definability theory is studied on completely metrizable separable topological spaces, which are called Polish spaces. This is because all Polish spaces and all Borel subsets of them share a common unique structure, as will become evident.

The tool that we are using to compare descriptive complexities of subsets is the method of reduction. Let A and B be subsets of Polish spaces X and Y respectively. If there is a continuous (respectively, Borel) map $f : X \to Y$ satisfying $f^{-1}(B) = A$, then we say that A is Wadge (respectively, Borel) reducible to B, written $A \leq_W B$ (respectively, $A \leq_B B$), which we interpret as the set B not to having less complexity than A. It can be easily checked that Borel and projective pointclasses are closed under continuous and Borel pre-images respectively. Therefore, the existence of an appropriate reduction from A to B implies that A cannot lie at higher levels of the hierarchies than B. We would like to remark that Wadge reductions can be used to analyze relative complexities in the Borel hierarchy, whereas, both Wadge and Borel reductions can be used to analyze relative complexities in the projective hierarchy.

1.3 Outline of the Thesis

In Chapter 2, we review the necessary background on descriptive set theory. We first introduce the basic properties of Polish spaces and discuss their relations with Borel subsets. Secondly, we put a Borel structure on $\mathcal{F}(X)$, the set of non-empty closed subsets of a Polish space X, induced from a Polish topology on it, and we

work with this same structure throughout the thesis. Thirdly, we give the definition of descriptive set-theoretic trees and briefly mention their relations with Polish spaces. Then we stratify Borel and projective classes, and present the notion of reducibility, which is our basic tool for complexity assignments. Lastly, we briefly introduce the theory of countable Borel equivalence relations and the Borel complexity theory to see how the reduction method works for equivalence relations on Polish spaces. In this chapter, we also state some lemmas that will simplify our work.

In Chapter 3, following the works of Debs, Saint Raymond and Becker, we review the descriptive complexity of $\mathcal{C}_{path}(X)$ for an arbitrary Polish space X as an upper bound where $\mathcal{C}_{path}(X) \subseteq \mathcal{F}(X)$ is the set of all closed path-connected subsets of X. We then show that the exact complexity of $\mathcal{C}_{path}(\mathbb{I}^3)$ is this upper bound, and therefore the bound cannot be lowered. We also give a lower bound for the complexity of $\mathcal{C}_{path}(\mathbb{I}^2)$, whose exact classification will be mentioned without a proof.

In Chapter 4, following the work of Debs and Saint Raymond, we determine the upper bounds for the complexities of $\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$ in $\mathcal{F}(X)$ for an arbitrary Polish space X where $\mathcal{C}(X)$ is the set of all connected subsets in $\mathcal{F}(X)$ and $\breve{\mathcal{C}}(X)$ is the set of all subsets in $\mathcal{F}(X)$ whose complement is connected. We give examples of Polish subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$ for which these bounds are realized. We mention how complexities change if we assume that X is compact.

In Chapter 5, we study the path-connectedness equivalence relation of a Polish subspaces of $\mathbb{R}^2$ using the tools of Borel complexity theory. In particular, we prove that such a Borel equivalence relation is Borel reducible to the universal countable Borel equivalence relation, which will be defined in Chapter 2. We also show that the path-connectedness relation of the Knaster continuum is a non-smooth essentially hyperfinite Borel equivalence relation.

In Chapter 6, we review the main results of the thesis and discuss some open problems for future research.

CHAPTER 2

PRELIMINARIES

In this chapter, our aim is to establish the necessary descriptive set-theoretic framework that we will use in descriptive complexity computations of path-connectedness and related notions in Polish spaces. Our main references for the subject are [9], [11], [22] and [23].

2.1 Polish Spaces and Standard Borel Spaces

A topological space (X, τ) is said to be completely metrizable if there is a complete metric on X that induces τ. Recall that a topological space which has a countable dense subset is called separable. For a metrizable space, being separable is equivalent to being second countable, i.e., having a countable basis. A topological space is called zero-dimensional if it has a basis whose elements are clopen.

Definition 2.1. *A **Polish space** is a topological space that is separable and completely metrizable.*

Examples of the most important Polish spaces are listed below.

- $\mathbb{R}^n$ with the Euclidean topology $(n \geq 1)$
- The Cantor space $2^{\mathbb{N}}$ where $2 = \{0, 1\}$ is endowed the discrete topology
- The Baire space $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ where $\mathbb{N}$ is endowed with the discrete topology
- The Hilbert cube $\mathbb{I}^{\mathbb{N}}$ where $\mathbb{I}$ is endowed with its Euclidean topology
- Any compact metrizable space

It is natural to ask then which subsets of a Polish space are Polish as a subspace. We need the following proposition to answer the question.

Proposition 2.1 ([22], Section 2.2)**.** *A subspace of a completely metrizable topological space is completely metrizable if and only if it is G_δ, i.e., a countable intersection of open subsets.*

One can use the equivalence of separability and second countability for metric spaces to show that any subspace of a separable metrizable space is actually separable. Then it follows from the proposition that the Polish subspaces of a Polish space are exactly G_δ subsets of it.

Polish spaces share common properties in terms of their mutual relationships. For example, any Polish space is a continuous injective image of a closed subset of the Baire space. Any zero-dimensional Polish space is homeomorphic to a Polish subspace of the Cantor space. The Baire space and the Cantor space can be embedded in each other, where the latter has the following characterization.

Proposition 2.2 ([22], Theorem 2.6.14)**.** *The Cantor space $2^{\mathbb{N}}$ is the unique zero-dimensional compact metrizable space without isolated points up to homeomorphism.*

This space will be used extensively in our discussions. It is simply the space of infinite binary sequences. The map

$$(\alpha_0, \alpha_1, \alpha_2, ...) \mapsto \sum_{i=0}^{\infty} \frac{2\alpha_i}{3^{i+1}}$$

is a canonical homeomorphism from the Cantor space to the set of numbers in the unit interval $\mathbb{I}$ whose ternary expansions contain only 0 and 2. The image of this homeomorphism gives the set

$$\left\{ \sum_{i=0}^{\infty} \frac{2\alpha_i}{3^i} \in \mathbb{I} : \alpha_i \in \{0, 1\} \text{ for all } i \in \mathbb{N} \right\},$$

which is known as the Cantor set. We will make use of this identification several times. For $\alpha \in 2^{\mathbb{N}}$, we let the boldface letter $\boldsymbol{\alpha}$ be the image of α under this homeomorphism.

If one considers Borel structures on Polish spaces, the relationship between them becomes much stronger. Given a topological space X, let $\mathcal{B}(X)$ be the $\sigma-$algebra

generated by open subsets of X, which is called the Borel $\sigma-$algebra on X. We call elements of $\mathcal{B}(X)$ Borel. A map $f : X \to Y$ between Polish spaces is called Borel if the inverse image of a Borel set is Borel. And two Borel subsets of Polish spaces are called Borel isomorphic if there is a Borel bijection between them whose inverse is also Borel. If X is a Polish space and $A \subseteq X$ is a Borel subset, then we observe that the Borel σ-algebra on A induced from the subspace topology is the same as the collection of Borel subsets of X that lie in A.

We note that the Borel isomorphism is a special case of measure isomorphism. A measurable space isomorphic to the Borel $\sigma-$algebra on a Polish space is of fundamental importance.

Definition 2.2. *A measurable space is called* ***standard Borel*** *if it is isomorphic to* $(X, \mathcal{B}(X))$ *for some Polish space* X.

It turns out that there is a unique uncountable standard Borel space up to measure isomorphism. In other words, we have the following theorem.

Theorem 2.3 (Borel Isomorphism Theorem)**.** *Any two uncountable standard Borel spaces are Borel isomorphic.*

Given a Polish space X and a Borel subset $B \subseteq X$, one can put a finer topology on X turning B into a clopen subset whereas the Borel σ-algebra on X remains the same. This induces a Polish topology on B as it is a G_δ subset of the new topology. So any Borel structure of a Borel subset of a Polish space is induced from a Polish topology on that subspace, and thus it is standard Borel. We treat Borel subsets, standard Borel spaces and Polish spaces as essentially the same and use these terms interchangeably if no confusion arises.

2.2 Hyperspaces on Polish Spaces

Since we investigate the complexity of some certain collections of subsets of a Polish space, we need to put a Polish topology (or at least a standard Borel structure) on subsets of the space.

Let X be a topological space, and let $\mathcal{K}(X)$ be the set of non-empty compact subsets of X. The topology generated by sets of the form

$$U^+ := \{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\} \text{ and } U^- := \{K \in \mathcal{K}(X) : K \subseteq U\}$$

where U varies over open subsets of X is called the **Vietoris topology** on X. One can easily check that sets of the form

$$[U_0 : U_1, U_2, ..., U_n] = U_0^- \cap U_1^+ \cap U_2^+ \cap ... \cap U_n^+$$

where U_i's vary over open subsets of X form a basis for $\mathcal{K}(X)$. What makes the Vietoris topology canonical is that many topological properties of X can be carried to $\mathcal{K}(X)$. That is, if X is separable, zero-dimensional, (completely) metrizable, compact metrizable, then so is $\mathcal{K}(X)$. In particular, $\mathcal{K}(X)$ is Polish whenever X is Polish.

We also have a canonical metric that gives this topology for metrizable spaces. Let (X, d) be a metrizable space. Then

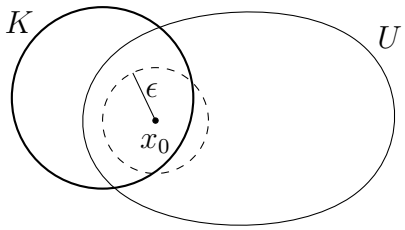
$$d_H(K, L) := \max\{\max_{x \in K} d(x, L), \max_{x \in L} d(x, K)\}$$

defines a metric, called the Hausdorff metric, on $\mathcal{K}(X)$. In fact, $d_H(K, L) < \epsilon$ is equivalent to have that for any point $x \in K$ there is $y \in L$ such that $d(x, y) < \epsilon$ and symmetrically for any point $x \in L$ there is $y \in K$ such that $d(x, y) < \epsilon$.

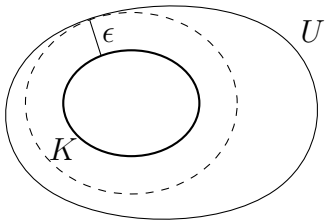
Proposition 2.4 ([22], Proposition 2.4.14)**.** *The Hausdorff metric induces the Vietoris topology.*

Proof. We shall first show that every subbasis element of the Vietoris topology is open in the topology of the Hausdorff metric. For this, let $U \subseteq X$ be open. Let us show that U^+ is open with respect to the metric topology. Let $K \in U^+$. Say $x_0 \in K \cap U$. Let $\epsilon > 0$ be such that $x_0 \in B_d(x_0, \epsilon) \subseteq U$. Then $\epsilon-$ball of the Hausdorff metric centered at K lies inside U^+. Because if a closed subset does not intersect U, then its distance to x_0, and hence to K, must be greater than or equal to ϵ. Thus, U^+ is open in the Hausdorff metric.

We also need to show that U^- is open. Let $K \subseteq U$ be compact, that is, $K \in U^-$. Then the distance of K to $X \setminus U$ is positive, say ϵ, by compactness of K. But then



(a) U^+ is open in the metric topology



(b) U^- is open in the metric topology

Figure 2.1: Vietoris topology $\subseteq$ Hausdorff metric topology

we obviously have $B_H(K, \epsilon) \subseteq U^-$. Using subbasis elements, we showed that the Hausdorff metric topology is finer than the Vietoris topology.

For the other direction, we need to show that for any Hausdorff ball $B_H(K, \epsilon)$ there is a Vietoris basis element $[U_0 : U_1, ..., U_n]$ such that $K \in [U_0 : U_1, ..., U_n] \subseteq B_H(K, \epsilon)$. Set $U_0 = \{x \in X : d(x, K) < \epsilon\}$ and let $U_i = B_d(x_i, \epsilon/2)$ where $\{x_1, ..., x_n\} \subseteq K$ is an $\epsilon/2$-net in K, that is, $K \subseteq \bigcup_{i \leq n} B_d(x_i, \epsilon/2)$. Let us show that $[U_0; U_1, ..., U_n]$ is the required basis element. We, by the definition of U_i's, have $K \in [U_0; U_1, ..., U_n]$. Let $L \in [U_0; U_1, ..., U_n]$. Then $L \subseteq U_0$, and so $\max_{x \in L} d(x, K) < \epsilon$. Let $x \in K$. Then $x \in U_{i_0} = B_d(x_{i_0}, \epsilon/2)$ for some $i_0 \in \{1, ..., n\}$. But since $L \cap U_{i_0} \neq \emptyset$, there is $y \in L \cap B_d(x_{i_0}, \epsilon/2)$. So $d(x, y) < \epsilon$. This shows that $d(x, L) < \epsilon$ for all $x \in K$. Thus, $\max_{x \in K} d(x, L) < \epsilon$. As a result, $d_H(K, L) < \epsilon$, showing that $L \in B_H(K, \epsilon)$. This completes the proof of the other direction. □

If X is compact metrizable, then the Borel σ-algebra on $\mathcal{K}(X)$ can be generated only by elements of the form U^+. This is because for any open subset U of a compact metrizable space X,

$$U^- = \bigcup_{n:\ U^c \subseteq U_n} (U_n^+)^c$$

where $\{U_n\}_n$ is a countable basis for X that is closed under finite unions (see [22, Page 96]).

We can generalize this idea to construct a standard Borel (hyper)space on non-empty closed subsets of a Polish space X. We define the Effros-Borel space on $\mathcal{F}(X)$, the

set of all non-empty closed subsets of X, to be the $\sigma-$algebra generated by

$$\{U^+ : U \subseteq X \text{ is open}\}$$

where U^+ is the set of all non-empty closed subsets intersecting U. The Effros-Borel structure on a Polish space is standard Borel. In other words, there is a Polish space on $\mathcal{F}(X)$ that generates the Effros-Borel structure. An example of such topology is the one induced from the injection of $\mathcal{F}(X)$ into the Cantor space by the following characteristic function $F \to (\chi_{U_0^+}(F), \chi_{U_1^+}(F), ...)$ where $\{U_n\}_n$ is a countable basis for X (see [23, Theorem 13.13]). However, it will be sufficient for us to consider only Borel subsets of $\mathcal{F}(X)$ instead of the topological structure on it.

If X is compact, we will use $\mathcal{K}(X)$ with the Vietoris topology on it rather than $\mathcal{F}(X)$ with the Effros-Borel structure, which are essentially the same.

Let X be Polish. Then there exists a Borel function $c : \mathcal{F}(X) \to X$ such that $c(F) \in F$ ([11, Theorem 12.13]). This function basically chooses an element from each non-empty closed subset of X. We call it a Borel selector function.

2.3 Trees

There is a tight relationship between Polish spaces and descriptive set-theoretic trees. We can see closed subsets of a Polish space $A^{\mathbb{N}}$ where A is a set endowed with the discrete topology as pruned trees on A. This identification is a coding tool for Polish spaces.

For any set A, we denote the set of finite sequences on A by $A^{<\mathbb{N}}$. If $s = (s_0 s_1 ... s_m)$ and $t = (t_0 t_1 ... t_n)$ are in $A^{<\mathbb{N}}$, we say that s is an initial segment of t whenever $m \leq n$ and $s_i = t_i$ for $0 \leq i \leq m$. For any $\mathbf{s} = (s_n)_n \in A^{\mathbb{N}}$ and $k \in \mathbb{N}^+$, $\mathbf{s}|k$ is defined to be the finite sequence $(s_0 s_1 ... s_{k-1})$ and $\mathbf{s}|0 := \emptyset$, the empty sequence.

Definition 2.3. *A **tree** T on a set A is a subset of $A^{<\mathbb{N}}$ that is closed under initial segments.*

An infinite sequence $\mathbf{x} \in A^{\mathbb{N}}$ is called a branch of a tree T on A if $\mathbf{x}|k \in T$ for all $k \in \mathbb{N}^+$. We denote $[T]$ as the set of all branches of T. A tree T is called pruned if

any element of the tree is a proper initial segment of some other element of it. A tree is well-founded if it does not have any branch. Otherwise, it is called ill-founded.

It is readily verified that sets of the form

$$N_s := \{\mathbf{x} \in A^{\mathbb{N}} : \mathbf{x}|k = s\}$$

where $s = (s_0 s_1 ... s_{k-1})$ give a basis for $A^{\mathbb{N}}$. For any tree T on a set A, $[T]$ is a closed subset of $A^{\mathbb{N}}$ where A is equipped with the discrete topology. To see this, assume that $\mathbf{x} \in A^{\mathbb{N}} \setminus [T]$. This means that there is $k \in \mathbb{N}^+$ such that $\mathbf{x}|k \notin T$. Then $\mathbf{x} \in N_{\mathbf{x}|k} \subseteq [T]^c$. So $[T]^c$ is open in $A^{\mathbb{N}}$. Conversely, any closed subset $F \subseteq A^{\mathbb{N}}$ can be viewed as the (pruned) tree $\{\mathbf{x}|k : \mathbf{x} \in F, k \in \mathbb{N}\}$ on A.

If T is a tree on the product set $A \times B$, we identify an element $(s_0, t_0)(s_1, t_1)...(s_k, t_k)$ of T with $(s_0 s_1 ... s_k, t_0 t_1 ... t_k) \in A^{<\mathbb{N}} \times B^{<\mathbb{N}}$ and we consider $[T]$ as a subspace of $A^{\mathbb{N}} \times B^{\mathbb{N}}$. For $\mathbf{x} \in A^{\mathbb{N}}$, we call the set

$$\{s \in B^{<\mathbb{N}} : \exists k \in \mathbb{N}\ (x|k, s) \in T\}$$

to be the $\mathbf{x}$ section of T and show it by $T(\mathbf{x})$.

2.4 Borel and Projective Hierarchies

In this subsection, our aim is to build the complexity hierarchy on subsets of Polish spaces and discuss their basic properties. Let X be a Polish space. In simple terms, all Borel subsets of X can be obtained from open sets by applying countable union and complementation operations in countably many steps. But the more steps it takes to construct a set, the more complex the set is. More precisely, we define $\mathbf{\Sigma}^0_1(X)$ and $\mathbf{\Pi}^0_1(X)$ as the collection of open subsets and closed subsets of X respectively. Then using transfinite recursion, we define

- $\mathbf{\Sigma}^0_\alpha(X) := \left\{ \bigcup_n A_n : A_n \in \mathbf{\Pi}^0_{\gamma_n}(X), \gamma_n < \alpha, n \in \mathbb{N} \right\}$
- $\mathbf{\Pi}^0_\alpha(X) := \{A^c : A \in \mathbf{\Sigma}^0_\alpha(X)\}$
- $\mathbf{\Delta}^0_\alpha(X) := \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$

for all $1 < \alpha < \omega_1$. We usually omit parentheses when the ambient space is clear from the context.

Since any open subset is equal to a countable union of closed sets in a metrizable space X, that is, $\mathbf{\Sigma}^0_1(X) \subseteq \mathbf{\Sigma}^0_2(X)$, one can use transfinite induction to show that $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{\Sigma}^0_{\alpha+1}(X)$ for any $\alpha < \omega_1$. Not only this, but we also have the following results by an application of transfinite induction:

Proposition 2.5 ([22], Proposition 3.6.1)**.** *For any ordinal $\alpha < \omega_1$,*

i. $\mathbf{\Pi}^0_\alpha(X), \mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{\Delta}^0_{\alpha+1}(X)$,

ii. $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X)$ *and* $\mathbf{\Delta}^0_\alpha(X)$ *are closed under finite unions and intersections and continuous pre-images,*

iii. $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$ *and* $\mathbf{\Delta}^0_\alpha(X)$ *are closed under countable unions, countable intersections and complements; respectively.*

Moreover, $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\alpha(X)$.

Using this proposition, we can picture Borel pointclasses as

$$\begin{array}{ccccccc} & \mathbf{\Sigma}^0_1 & & \mathbf{\Sigma}^0_2 & & \mathbf{\Sigma}^0_3 & \cdots & & \mathbf{\Sigma}^0_\alpha & \\ \mathbf{\Delta}^0_1 & & \mathbf{\Delta}^0_2 & & \mathbf{\Delta}^0_3 & \cdots & \mathbf{\Delta}^0_\alpha & & \alpha < \omega_1 \\ & \mathbf{\Pi}^0_1 & & \mathbf{\Pi}^0_2 & & \mathbf{\Pi}^0_3 & \cdots & & \mathbf{\Pi}^0_\alpha & \end{array}$$

Figure 2.2: The Borel hierarchy

where each class is a subset of the class on the right of it. A subtle diagonalization argument (see [11, 22A]) shows that if X is uncountable Polish, then $\mathbf{\Delta}^0_\alpha \subsetneq \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^0_\alpha$ for any $\alpha < \omega_1$, and thus any two distinct classes in the hierarchy are not equal. The classes are growing strictly towards the right.

Let us show as an example that the set of sequences on $\mathbb{N}$ that diverge to infinity is in

$\mathbf{\Pi}^0_3(\mathcal{N})$. We have

$$\begin{aligned}\Big\{(x_n)_n \in \mathcal{N} : \lim_n x_n = \infty\Big\} &= \{(x_n)_n : (\forall k)(\exists N)(\forall m)\ m > N \to x_m > k\} \\ &= \bigcap_k \{(x_n)_n : (\exists N)(\forall m)\ m > N \to x_m > k\} \\ &= \bigcap_k \bigcup_N \{(x_n)_n : (\forall m)\ m > N \to x_m > k\} \\ &= \bigcap_k \bigcup_N \bigcap_{m>N} \{(x_n)_n : x_m > k\}\end{aligned}$$

where all quantifiers range over $\mathbb{N}$. The inner-most set $\{(x_n)_n : x_m > k\}$ is clearly a closed subset of $\mathcal{N}$. So, the set $\Big\{(x_n)_n \in \mathcal{N} : \lim_n x_n = \infty\Big\}$ can be obtained by applying countable intersection, union, and intersection operations successively to closed subsets. This clearly shows that the set is in $\mathbf{\Pi}^0_3(\mathcal{N})$. It is worth noting that this observation does not prevent, in principle, this set from being in any lower classes like $\mathbf{\Pi}^0_2,\ \mathbf{\Sigma}^0_2,\ \mathbf{\Sigma}^0_1$. There can be simpler definitions for the set so that it has the less complexity.

One can see how the definition $(\forall k)(\exists N)(\forall m)\ m > N \to x_m > k$ gives rise to a $\mathbf{\Pi}^0_3$ set. Existential quantification over a countable set corresponds to taking countable union, whereas universal quantification over a countable set corresponds to taking countable intersection. This analogy shows the relationship between the definition of a subset and its class in the hierarchy. More precisely, if $\{x \in X : \varphi(n, x)\}$ is in $\mathbf{\Sigma}^0_{\alpha_n}(X)$ for all $n \in \mathbb{N}$, then $\{x \in X : \exists n \in \mathbb{N}\ \varphi(n, x)\}$ is in $\mathbf{\Sigma}^0_{\sup_n \alpha_n}(X)$ and $\{x \in X : \forall n \in \mathbb{N}\ \varphi(n, x)\}$ is in $\mathbf{\Pi}^0_{(\sup_n \alpha_n)+1}(X)$.

Thus, it usually suffices to check quantifiers and relations involved in the definition of a given subset of a Polish space to determine if it is Borel or not and its descriptive complexity within the Borel hierarchy. We will sometimes refer to Appendices A and B that include complexities of some relations and functions which are frequently used to define subsets of Polish spaces.

The collection of Borel sets are closed under countable union, countable intersection, and complementation operations. However, they are not closed under projections over Polish spaces. More specifically, the projection $\{x \in X : \exists y \in Y\ (x, y) \in B\}$ of a Borel subset $B \subseteq X \times Y$ of the product space of Polish spaces X and Y need

not be Borel in X. We will give examples of such sets in the thesis. In this regard, we define two operations on subsets of Polish spaces. For any subset $B \subseteq X \times Y$ we define

$$\exists^Y B = \{x \in X : \exists y \in Y \text{ such that } (x, y) \in B\}$$

and

$$\forall^Y B = \{x \in X : \forall y \in Y \text{ such that } (x, y) \in B\}$$

where the set $\exists^Y B$ is the usual projection of B over Y while we call the set $\forall^Y B$ to be the coprojection of the set B onto X over the space Y. In fact, $\forall^Y B = (\exists^Y B^c)^c$. We sometimes denote $\exists^Y B$ by $proj_X(B)$ for $B \subseteq X \times Y$.

Definition 2.4. *Let X be a Polish space and $A \subseteq X$. Then A is called **analytic** if it is a projection of a Borel subset over a Polish space. That is, if $A = \exists^Y B$ for some Borel subset $B \subseteq X \times Y$ and some Polish space Y.*

Analytic sets are proven to be closed under countable union, countable intersection and projections over Polish spaces. However, they are not closed under complementations. This leads us to define new complexity classes by applying projection and complementation operations starting from Borel sets in successive steps. We first set $\mathbf{\Sigma}^1_1(X)$ to be all analytic subsets of X and define $\mathbf{\Pi}^1_1(X)$ to be the set of complements of analytic sets, called coanalytic sets. Then we recursively define

$\mathbf{\Pi}^1_n(X) = \{A \subseteq X : A^c \in \mathbf{\Sigma}^1_n\}$ and

$\mathbf{\Sigma}^1_n(X) = \{A \subseteq X : A = \exists^Y B, B \in \mathbf{\Pi}^1_{n-1}(X \times Y), Y \text{ Polish }\}$ for $n > 1$,

$\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$ for any $n \geq 1$.

The classes defined above constitute a new hierarchy of complexity, called the projective hierarchy.

Any $\mathbf{\Sigma}^1_{n+1}$ class simply comprises the projections of $\mathbf{\Pi}^1_n$ sets, and $\mathbf{\Pi}^1_{n+1}$ contains complements of $\mathbf{\Sigma}^1_{n+1}$ sets. Equivalently, $\mathbf{\Pi}^1_{n+1}$ sets are coprojections of $\mathbf{\Sigma}^1_n$ sets. Indeed, the construction goes like

$$\mathcal{B}(X) \xrightarrow{\text{proj.}} \mathbf{\Sigma}^1_1(X) \xrightarrow{\text{comp.}} \mathbf{\Pi}^1_1(X) \xrightarrow{\text{proj.}} \mathbf{\Sigma}^1_2(X) \xrightarrow{\text{comp.}} \mathbf{\Pi}^1_2(X) \to \cdots$$

or

$$\mathcal{B}(X) \xrightarrow{\text{coproj.}} \mathbf{\Pi}^1_1(X) \xrightarrow{\text{comp.}} \mathbf{\Sigma}^1_1(X) \xrightarrow{\text{coproj.}} \mathbf{\Pi}^1_2(X) \xrightarrow{\text{comp.}} \mathbf{\Sigma}^1_2(X) \to \cdots .$$

Classes of the projective hierarchy have the following basic properties:

Proposition 2.6 ([22], Subsection 4.1)**.** *For any* $n \geq 1$*,*

i. $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$*,*

ii. $\mathbf{\Sigma}^1_n$ *is closed under Borel pre-images, countable intersections and unions, and projections over Polish spaces,*

iii. $\mathbf{\Pi}^1_n$ *is closed under Borel pre-images, countable intersections and unions, and coprojections over Polish spaces.*

iv. Moreover, $\mathcal{B}(X) = \mathbf{\Delta}^1_1$*.*

We can draw the following diagram

$$\begin{array}{ccccccccc} & \mathbf{\Sigma}^1_1 & & \mathbf{\Sigma}^1_2 & & \mathbf{\Sigma}^1_3 & \cdots & \mathbf{\Sigma}^1_n & \\ \mathcal{B}(X) = \mathbf{\Delta}^1_1 & & \mathbf{\Delta}^1_2 & & \mathbf{\Delta}^1_3 & \cdots & \mathbf{\Delta}^1_n & & n \in \mathbb{N}^+ \\ & \mathbf{\Pi}^1_1 & & \mathbf{\Pi}^1_2 & & \mathbf{\Pi}^1_3 & \cdots & \mathbf{\Pi}^1_n & \end{array}$$

Figure 2.3: The projective hierarchy

where each class is a subset of the class on the right of it. Similar to the Borel hierarchy, the projective hierarchy is strictly increasing towards the right for an uncountable Polish space. More precisely, $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n \subsetneq \mathbf{\Delta}^1_{n+1}$ for all $n \in \mathbb{N}^+$.

Analytic sets have several equivalent characterizations. For example, $A \subseteq X$ is analytic if and only if it is a projection of a closed subset of $X \times \mathcal{N}$. In fact, analytic sets could equally be characterized as projections of Borel subsets over a fixed uncountable Polish space such as $2^{\mathbb{N}}$ or $\mathcal{N}$. This is due to the Borel isomorphism theorem. More precisely, $A \subseteq X$ is analytic if and only if $A = proj_X(B)$ for some Borel $B \subseteq X \times 2^{\mathbb{N}}$. Obviously, the same holds for higher projective classes as well.

Let us give a generic example of a non-Borel $\mathbf{\Pi}^1_1$ set. Let **Tr** be the set of all trees on $\mathbb{N}$, identified as a subspace of $2^{\mathbb{N}^{<\mathbb{N}}}$. It is straightforward to check that **Tr** is closed in $2^{\mathbb{N}^{<\mathbb{N}}}$. Then let **WF** denote the set of well-founded trees in **Tr**. We have

$$\mathbf{WF}^c = \exists^{\mathcal{N}} \bigcap_{k\in\mathbb{N}} \{(T, \mathbf{x}) \in \mathbf{Tr} \times \mathcal{N} : \mathbf{x}|k \in T\}$$

where the inner set can be proven to be open with some effort. Then $\mathbf{WF}^c$ is a projection of a Borel set. So $\mathbf{WF}^c$ is $\mathbf{\Sigma}^1_1$, and therefore **WF** is $\mathbf{\Pi}^1_1$. That **WF** is non-Borel follows from Theorem 2.7.

Definition 2.5. *Let A, B be subsets of Polish spaces X, Y respectively. Then A is said to be **Wadge reducible** to B if there is a continuous function $f : X \to Y$ such that $f^{-1}(B) = A$. The function f is called a **continuous reduction** from A to B. In this case, we write $A \leq_W B$.*

Definition 2.6. *Let X be a Polish space, $A \subseteq X$ and Γ be a class of Polish space. We say that A is **$\mathbf{\Gamma}$-hard** if any Γ subset of a zero-dimensional Polish space is Wadge reducible to A. If we also have $A \in \Gamma$, then A is said to be **$\mathbf{\Gamma}$-complete**.*

If the function f in the first definition is Borel, we call it a Borel reduction, and write $A \leq_B B$. Likewise, if every Γ set is Borel reducible to A, then A is called Borel Γ-hard.

If A is a Γ-hard set and is Wadge reducible to B, then B is also Γ-hard. Because Wadge reducibility is a transitive relation.

Let f be a continuous reduction from A to B. Then $x \in A$ if and only if $f(x) \in B$ for all $x \in X$, which means that the membership relation on A can be checked by the membership relation on B through a continuous function. This suggests that B possibly contains "more information" compared to A. We also observe that the complexity class of A in the hierarchy cannot be higher than that of B. Likewise, a $\Gamma-$complete set can be considered as the most complex set in the class Γ.

Let Γ be any pointclass containing $\mathbf{\Pi}^0_2$. Since any zero-dimensional Polish space is homeomorphic to a G_δ subset of $2^{\mathbb{N}}$, any Γ subset of a zero-dimensional space can be seen as a Γ subset of $2^{\mathbb{N}}$ with some effort. Therefore, it is sufficient to reduce Γ subsets of $2^{\mathbb{N}}$ to show that a set is Γ-complete rather than considering an arbitrary zero-dimensional Polish space.

Theorem 2.7 ([11], Theorem 27.1)**.** ***WF is*** $\mathbf{\Pi}^1_1-$***complete.***

Proof. We showed earlier that **WF** is coanalytic. Now, let $A \subseteq 2^{\mathbb{N}}$ be any coanalytic set. Then there is a closed subset C of $2^{\mathbb{N}} \times \mathcal{N} \approx (2 \times \mathbb{N})^{\mathbb{N}}$ such that $A^c = \exists^{\mathcal{N}} C$. We know then that there is a (pruned) tree T on $2 \times \mathbb{N}$ such that $C = [T]$. The map $\mathbf{x} \mapsto T(\mathbf{x})$ defined from $2^{\mathbb{N}}$ to **Tr** is obviously continuous and we have

$$\mathbf{x} \in A \longleftrightarrow \mathbf{x} \notin \exists^{\mathcal{N}} C \longleftrightarrow \forall \mathbf{y} \in \mathcal{N},\ (\mathbf{x}, \mathbf{y}) \notin C = [T] \longleftrightarrow T(\mathbf{x}) \in \mathbf{WF}\ .$$

So, A is reducible to **WF**. This shows that $\mathbf{\Pi}^1_1$-hard. □

We now inductively define $\mathbf{\Pi}^1_n-$complete subset of $\mathcal{K}\left((2^{\mathbb{N}})^n\right)$ for all $n \geq 1$ that will be of constant use throughout the following chapters.

Notational Remark. Let X, Y be Polish spaces, $A \subseteq X \times Y$, and $x \in X$. We denote the x-section of the set A by $A(x)$. That is, $A(x) := \{y \in Y : (x, y) \in A\}$.

Let Δ be a homeomorph of the Cantor space and let $Q \subseteq \Delta$ be any countable dense subset. We recursively define

$$\begin{aligned}
\mathbb{P}_1 &= \mathcal{K}(Q) \\
\mathbb{P}_n &= \{K \in \mathcal{K}(\Delta^n) : \forall \alpha \in \Delta, K(\alpha) \in \mathbb{Q}_{n-1}\} \text{ for } n \geq 2 \\
\mathbb{Q}_n &= \mathcal{K}(\Delta^n) \setminus \mathbb{P}_n \text{ for } n \geq 1\ .
\end{aligned}$$

We will prove by induction on n that $\mathbb{P}_n$ is $\mathbf{\Pi}^1_n$-complete for all $n \geq 1$. The set $\mathcal{K}(Q)$ is known to be $\mathbf{\Pi}^1_1$-complete (see [11, Exercise 27.4]). So, we only prove the inductive step.

Theorem 2.8 ([5], Subsection 3.5)**.** $\mathbb{P}_n$ *is in* $\mathbf{\Pi}^1_n$ *for all* $n \geq 1$.

Proof. Assume that $\mathbb{P}_k$ is in $\mathbf{\Pi}^1_k$ where $k \in \mathbb{N}^+$. Using the fact that the intersection map is Borel, one can show that the set $\{(K, \alpha) \in \mathcal{K}(\Delta^{k+1}) \times \Delta : K(\alpha) \in \mathbb{Q}_k\}$ is a $\mathbf{\Sigma}^1_k$ set. Since $\mathbb{P}_{k+1}$ is the coprojection of this set, it is $\mathbf{\Pi}^1_{k+1}$. □

In what follows, we give an alternative proof, in which we construct continuous reductions, for completeness of $\mathbb{P}_n$ to the one presented in [5].

Theorem 2.9 ([5], Lemma 3.6)**.** $\mathbb{P}_n$ *is* $\mathbf{\Pi}^1_n$*-complete for all* $n \geq 1$.

Proof. We already showed that $\mathbb{P}_n \in \mathbf{\Pi}^1_n$ for all n. Assume that $\mathbb{P}_k$ is $\mathbf{\Pi}^1_k$-complete where $k \in \mathbb{N}^+$. Let $A \subseteq \Delta$ be any $\mathbf{\Pi}^1_{k+1}$ set. Then $A^c \subseteq \Delta$ is a $\mathbf{\Sigma}^1_{k+1}$ set, and so it is projection of a $\mathbf{\Pi}^1_k(\Delta^2)$ set, call B. Since $\mathbb{P}_k$ is $\mathbf{\Pi}^1_k$-complete, there is a continuous function $\varphi : \Delta^2 \to \mathcal{K}(\Delta^k)$ that reduces B to $\mathbb{P}_k$. Fix any homeomorphism $\iota : \Delta^2 \to \Delta$. Consider the continuous map $f : \Delta^2 \to \mathcal{K}(\Delta^{k+1})$ given by

$$(\alpha, \beta) \mapsto K_{\alpha,\beta} := \{\iota(\alpha, \beta)\} \times \varphi(\alpha, \beta) \ .$$

For any $\alpha \in \Delta$, the image $\{K_{\alpha,\beta} : \beta \in \Delta\}$ of the compact set $\{\alpha\} \times \Delta$ under f is compact. Since the union map $\bigcup : \mathcal{K}(\mathcal{K}(\Delta^{k+1})) \to \mathcal{K}(\Delta^{k+1})$ is continuous (see Appendix B), then so is the map $\psi : \Delta \to \mathcal{K}(\Delta^{k+1})$ given by $\psi(\alpha) = \bigcup_{\beta\in\Delta} K_{\alpha,\beta}$.

We have

$$\begin{aligned}
\alpha \in A &\longleftrightarrow \forall \beta \in \Delta,\ (\alpha, \beta) \notin B \\
&\longleftrightarrow \forall \beta \in \Delta,\ \varphi(\alpha, \beta) \notin \mathbb{P}_k \\
&\longleftrightarrow \bigcup_{\beta\in\Delta} K_{\alpha,\beta} \in \mathbb{P}_{k+1} \\
&\longleftrightarrow \psi(\alpha) \in \mathbb{P}_{k+1},
\end{aligned}$$

and so ψ reduces A to $\mathbb{P}_{k+1}$ continuously. □

Remark. The sets $\mathbb{P}_n$ and $\mathbb{Q}_n$ depend on the choices of Δ and Q. However, we will always specify them before each use.

Recall that a map $\psi : X \to \mathcal{K}(Y)$ where X is any topological space and Y is a metrizable space is called upper semi-continuous (u.s.c.) if for any open set $U \subseteq Y$, the set $\psi^{-1}(U^-)$ is open in X. The following technical lemma gives a way of constructing a Borel map from a given u.s.c. map, which is weaker than being Borel.

Lemma 2.10 ([5], Lemma 3.4)**.** *Let* X *be a Polish space,* Y *be a compact metrizable space, and* $\psi : X \to \mathcal{K}(Y)$ *be an u.s.c. map. Let* $Z \subseteq Y$ *be a* G_δ *subset so that* Z *is also Polish. Define* $\varphi : X \to \mathcal{F}(Z)$ *as* $\varphi(\alpha) = Z \cap \psi(\alpha)$*. If* $\varphi(\alpha)$ *is dense in* $\psi(\alpha)$ *for all* $\alpha \in X$*, then* φ *is a Borel map.*

Proof. It suffices to show that $\varphi^{-1}(U^+)$, U open in Z, is Borel in X for an arbitrary generator element U^+ of the Effros-Borel space on $\mathcal{F}(Z)$. Let V be open in Y such that $U = V \cap Z$. Say $V = \bigcup_n F_n$ where each F_n is closed in Y. And let $V_n = Y \setminus F_n$ for all n. If $\varphi(\alpha)$ is dense in $\psi(\alpha)$, then

$$\begin{aligned}\alpha \in \varphi^{-1}(U^+) \longleftrightarrow \varphi(\alpha) \cap U = \varphi(\alpha) \cap V \neq \emptyset &\longleftrightarrow \psi(\alpha) \cap V \neq \emptyset \\ &\longleftrightarrow \exists n,\ \psi(\alpha) \cap F_n \neq \emptyset \\ &\longleftrightarrow \exists n,\ \psi(\alpha) \nsubseteq V_n\ .\end{aligned}$$

Thus $\varphi^{-1}(U^+) = \bigcup_n (X \setminus \psi^{-1}(V_n^-))$ is a Borel set. □

We close this subsection with the following powerful theorem that follows from the proof of the result in [12] (see also [5, Theorem 2.1]).

Theorem 2.11. *Let X be a Polish space and $A \subseteq X$. If A is Borel $\mathbf{\Pi}^1_n$-hard, then it is $\mathbf{\Pi}^1_n$-hard.*

The theorem suggests that it suffices to find Borel reductions instead of continuous ones for completeness of projective classes, which is an easier task in general. Since $\mathbf{\Sigma}^1_n = \{A \subseteq X : A^c \in \mathbf{\Pi}^1_n\}$, it follows that the theorem holds for projective $\mathbf{\Sigma}$ classes as well.

2.5 Borel Equivalence Relations

One can use the idea of Borel reduction to compare Borel equivalence relations on Polish spaces as well. By a Borel equivalence relation E on a Polish space X what we mean is that E is a Borel subset of $X \times X$.

Definition 2.7. *Let E, F be Borel equivalence relations on Polish spaces X and Y respectively. Then E is **Borel reducible** to F if there exists a Borel function $f : X \to Y$ such that xEy if and only if $f(x)Ff(y)$ for all $x, y \in X$. In this case, f is called a **Borel reduction** from E to F, and we write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then we say that E and F are **Borel bireducible** with each other, written as $E \sim_B F$.*

If $E \leq_B F$, then we see F to be at least as complex as E because F contains all the information E has in the sense that the question of whether xEy or not can be decided inside F with the help of a Borel function.

Observe that, comparing this definition of Borel reducibility with that of Subsection 2.4, one may interpret $E \leq_B F$ in two different ways. If one wishes to compare the complexities of E and F merely as subsets of the ambient spaces $X \times X$ and $Y \times Y$, one would use the definition in Subsection 2.4. However, if one wishes to compare the complexities of E and F as equivalence relations, one would use the definition given above, which is more suitable for this purpose. Therefore, unless stated otherwise, while comparing the complexities of equivalence relations, one assumes that $E \leq_B F$ means the existence of a Borel reduction as defined above.

For Borel equivalence relations, we have the notion of universality analogous to the notion of completeness we had for Borel and projective classes. For a collection Γ of Borel equivalence relations on Polish spaces, we will call a Borel equivalence relation E to be universal Γ if any Γ equivalence relation is Borel reducible to E.

The simplest equivalence relation on a Polish space is the identity relation on the space, which is readily seen to be Borel. For any uncountable Polish spaces X, Y we have $id_X \sim_B id_Y$ as a consequence of the Borel isomorphism theorem. We then have the following hierarchy of the Borel reducibility

$$id_0 \leq_B id_1 \leq_B id_2 \leq_B ... \leq_B id_n \leq_B ... \leq_B id_{\mathbb{N}} \leq id_{\mathbb{R}} \tag{2.1}$$

where n denotes the discrete space $\{0, 1, ..., n-1\}$ of cardinality n.

If a Borel equivalence relation is Borel reducible to an identity relation on a Polish space, then it is no more complex than an identity relation.

Definition 2.8. *A Borel equivalence relation on a Polish space is called* ***smooth*** *if it is Borel reducible to the identity relation on a Polish space.*

If $E \subseteq X \times X$ has countably many equivalence classes, then it is clearly smooth and Borel bireducible with the corresponding identity relation. Moreover, all smooth relations are the ones listed in 2.1 up to Borel bireducibility due to the following dichotomy theorem of Silver.

Theorem 2.12 ([9], Theorem 5.7.1)**.** *Let E be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B id_{\mathbb{N}}$ or $id_{\mathbb{R}} \leq_B E$.*

The most important and basic non-smooth equivalence relation is E_0, the eventual equivalence, on $2^{\mathbb{N}}$ given by $\alpha E_0 \beta$ if and only if $\exists N \in \mathbb{N}, \forall n \geq N,\ \alpha(n) = \beta(n)$. Since it has uncountably many equivalence classes, $id_{\mathbb{R}} \leq_B E_0$ by the theorem above. It turns out that there is no equivalence relation in between. More precisely, we have the following dichotomy theorem.

Theorem 2.13 ([9], Theorem 5.7.2)**.** *Let E be a Borel equivalence relation on a standard Borel space. Then either $E \leq_B id_{\mathbb{R}}$ or $E_0 \leq_B E$.*

So E_0 is the simplest non-smooth relation and the following diagram represents the initial segment of Borel reducibility.

$$id_0 \leq_B id_1 \leq_B id_2 \leq_B ... \leq_B id_n \leq_B ... \leq_B id_{\mathbb{N}} \leq id_{\mathbb{R}} \leq_B E_0 \tag{2.2}$$

An equivalence relation is said to be countable (finite), if each equivalence class is countable (finite). We define the following notion for Borel equivalence relations to characterize E_0.

Definition 2.9. *Let E be a Borel equivalence relation on a Polish space X. Then E is called **hyperfinite** if $E = \bigcup_n E_n$ where each E_n is a finite Borel equivalence relation on X.*

A Borel equivalence relation is called essentially hyperfinite, if it is Borel reducible to a hyperfinite Borel equivalence relation.

For example, $E_0 = \bigcup_n F_n$ where F_n is the equivalence relation given by $\alpha F_n \beta$ if and only if $\alpha_i = \beta_i$ for all $i \geq n$, which is clearly finite.

E_0 turns out to be the universal hyperfinite Borel equivalence relation. We also observed that E_0 is reducible to any non-smooth Borel equivalence relation. Thus, E_0 is the unique hyperfinite non-smooth Borel equivalence relation up to Borel bireducibility. Moreover, countable Borel equivalence relations has an upper bound.

Theorem 2.14 ([9], Theorem 7.4.1)**.** *There is a universal countable Borel equivalence relation.*

It is clear that the universal countable Borel equivalence relation is unique up to Borel bireducibility. So we denote it by E_∞. As a fact, there are infinitely many equivalence relations between E_0 and E_∞. If E is reducible to a countable Borel equivalence relation, then it is below E_∞. That is, $E \leq_B E_\infty$. In this case, we say that E is essentially countable although it is not necessarily countable.

So we have the following diagram of Borel reducibility for essentially countable Borel equivalence relations.

$$\overbrace{\underbrace{id_0 \leq_B id_1 \leq_B ... \leq_B id_n \leq_B ... \leq_B id_{\mathbb{N}} \leq_B id_{\mathbb{R}}}_{\text{smooth}} \leq_B \underbrace{E_0 \leq_B ... \leq_B E_\infty}_{\text{non-smooth}}}^{\text{hyperfinite}}$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{essentially countable}}$$

Borel reducibility of equivalence relations is not a linear order, but just a pre-order for the linearity breaks at E_0. There are, for example, Borel equivalence relations that are not essentially countable and E_∞ is not reducible to them. Even non-smooth countable Borel equivalence relations are highly rich and far from being linear. It suffices, however, for us to consider only countable Borel equivalence relations given in the diagram above.

CHAPTER 3

DESCRIPTIVE COMPLEXITY OF PATH-CONNECTEDNESS

This chapter focuses on the descriptive complexity of path-connected subsets of a Polish space among all closed subsets. We denote the set of all path-connected non-empty closed subsets of a Polish space X by $\mathcal{C}_{path}(X)$. To be able to study paths in the space, we introduce the Polish space $\Pi(X)$ of all paths, i.e., continuous functions $\gamma : \mathbb{I} \to X$, where it is well-known that the space of all continuous functions from a compact metrizable space to a Polish space with respect to supremum metric induces a Polish topology. Our main references for the results in this chapter are [5] and [11].

The definition of $\mathcal{C}_{path}(X)$ in $\mathcal{F}(X)$ directly gives us an upper bound for its complexity. Any closed subset F of X is path-connected if and only if

$$\forall x, y \in F,\ \exists \gamma \in \Pi(X),\ \gamma(0) = x,\ \gamma(1) = y,\ \text{and}\ \forall k \in \mathbb{N}\ \gamma(q_k) \in F\ .$$

We shall show that this is a $\mathbf{\Pi}^1_2$ characterization.

Proposition 3.1 ([5], Proposition 5.1)**.** *Let X be a Polish space. Then $\mathcal{C}_{path}(X)$ is a $\mathbf{\Pi}^1_2$ subset of $\mathcal{F}(X)$.*

Proof. For each $n \in \mathbb{N}$, define

$$\mathcal{A}_n = \{(F, \gamma, x, y) : x, y \in F \to \gamma(0) = x\ \wedge\ \gamma(1) = y\ \wedge\ \gamma(q_n) \in F\}$$

as a subset of $\mathcal{F}(X) \times \Pi(X) \times X \times X$. We will show that $\mathcal{A}_n$ is Borel. Consider, for example, only the relation $\gamma(0) = x$. Tuples satisfying this relation can be obtained by taking the inverse image of the closed set

$$\{(F, t, x, y) \in \mathcal{F}(X) \times X^3 : t = x\}$$

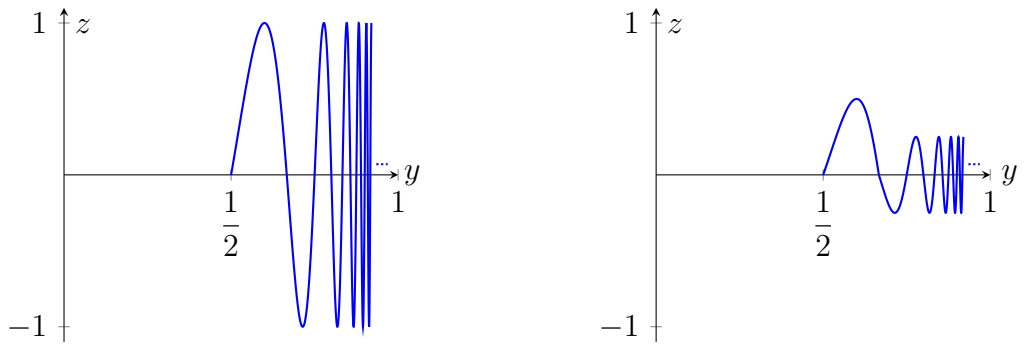
under the continuous function $(F, \gamma, x, y) \mapsto (F, \gamma(0), x, y)$. Thus, the relation "$\gamma(0) = x$" gives a closed subset and so a Borel subset. Similar to this observation, we can show that $\mathcal{A}_n$ is Borel because equality and membership relations are Borel (see Appendix A), γ is a continuous map and Borel sets are closed under continuous pre-images. Then, we set $\mathcal{A} = \bigcap_n \mathcal{A}_n$, which remains a Borel set. This is the set of all tuples (F, γ, x, y) for which γ is a path lying inside F from x to y if $x, y \in F$. Then $\exists^{\Pi(X)}\mathcal{A}$ is an analytic set and it consists of all triples (F, x, y) for which x is path-connected to y in F provided that $x, y \in F$. If this holds for any $x, y \in X$, then F is path-connected and vice versa. Thus, $\forall^{X^2}\exists^{\Pi(X)}\mathcal{A} = \mathcal{C}_{path}(X)$. Hence, $\mathcal{C}_{path}(X)$ is a coprojection of an analytic set, and so it is $\mathbf{\Pi}^1_2$. □

Let us show that this upper bound is realized for some Polish (in fact, compact) space showing that the bound is optimal. Due to Becker (and Ajtai, independently), $\mathcal{C}_{\text{path}}(\mathbb{I}^n)$ turns out to be $\mathbf{\Pi}^1_2-$complete for $n \geq 3$. It is enough to prove this for $n = 3$ because if $\mathcal{C}_{\text{path}}(\mathbb{I}^3)$ is $\mathbf{\Pi}^1_2$-complete, then we can embed $\mathbb{I}^3$ in $\mathbb{I}^n$ to establish the result for $n > 3$. Two proofs with similar techniques are included in [11, Theorem 37.11] and [5, Theorem 5.5] where it is proved for $n \geq 4$ in the latter. We improved the second proof using the ideas in the proof.

The well-known topologist's sine curve is not path-connected to the origin. But if we keep scaling the curve by smaller and smaller numbers as $x \to 0^+$ so that the corresponding function has a continuous extension at 0, then we will have a path to the origin. Building on this idea, we will relate eventually zero binary sequences with path-disconnected sets. First, let $\Delta = 2^{\mathbb{N}}$. Consider the function $z = f(y) = \sin\left(\frac{\pi}{1-y}\right)$ defined on $[1/2, 1)$ whose graph is presented in the Figure 3.1 a. For each $\beta \in \Delta$, consider the scaling function $s_\beta : [1/2, 1) \to \mathbb{R}$ whose graph is

$$\left\{(y, z) \in \mathbb{R}^2 : z = 2^{-\beta(0)-\beta(1)\ldots-\beta(k)}, y \in \left[1 - \frac{1}{k+2}, 1 - \frac{1}{k+3}\right), k \in \mathbb{N}\right\} .$$

Let $Q \subseteq \Delta$ be the set of eventually zero sequences, which is clearly a countable dense subset. If $\beta \in Q$, then the graph of the function $f.s_\beta$ will not be path-connected to the point $(1, 0)$ because the shrinking will stop at some point (see Figure 3.1 b). But if $\beta \notin Q$, then $f.s_\beta$ will have a continuous extension at $y = 1$, and so we will be able to "draw" a path from the graph of $f.s_\beta$ to the point $(1, 0)$.



(a) $z = f(y)$ (b) $z = f.s_\beta(y)$, $\beta = (1, 1, 0, 0, 0...)$

Figure 3.1: Graphs of f and $f.s_\beta$

We are now ready to prove our theorem.

Theorem 3.2 ([11], Theorem 37.11)**.** *$\mathcal{C}_{path}(\mathbb{I}^3)$ is a $\mathbf{\Pi}^1_2$-complete subset of $\mathcal{K}(\mathbb{I}^3)$.*

Proof. We shall reduce the complete $\mathbf{\Pi}^1_2$ set $\mathbb{P}_2$ to $\mathcal{C}_{\text{path}}(\mathbb{I}^2 \times \mathbb{J})$ where $\mathbb{J} := [-1, 1]$ by a continuous map. Let $\iota : \Delta^2 \to \Delta$ be a homeomorphism. For each $(\alpha, \beta) \in \Delta^2$, let $L_{\alpha,\beta}$ be the path-connected set of the line segment from $(0, 0, \boldsymbol{\alpha})$ to $(\boldsymbol{\iota}(\boldsymbol{\alpha}, \boldsymbol{\beta}), 1/2, 0)$ union the set

$$\left\{(\boldsymbol{\iota}(\boldsymbol{\alpha}, \boldsymbol{\beta}), y, z) \in \mathbb{I}^2 \times \mathbb{J} : z = f.s_\beta(y), y \in [1/2, 1)\right\}$$

and define L to be the plane segment $\mathbb{I} \times \{1\} \times \mathbb{J}$ union the set $\{(0, 0, \boldsymbol{\alpha}) : \alpha \in \Delta\}$. We refer the reader to the Figure 3.2 to visualize the set $L_{\alpha,\beta}$. We then define

$$\varphi(K) = L \cup \bigcup_{(\alpha,\beta)\in K} L_{\alpha,\beta}$$

for all $K \in \mathcal{K}(\Delta^2)$. The argument for well-definedness and continuity of φ follows as in the proof of Theorem 2.9 by the continuity of the map $(\alpha, \beta) \mapsto \varphi(\{(\alpha, \beta)\})$.

For $K \in \mathcal{K}(\Delta^2)$ and $\alpha \in proj_\Delta(K)$, the set

$$\varphi(\{\alpha\} \times K(\alpha)) \setminus L = \bigcup_{\beta\in K(\alpha)} L_{\alpha,\beta}$$

is path-connected as the point $(0, 0, \boldsymbol{\alpha})$ is common to $L_{\alpha,\beta}$ for all $\beta \in K(\alpha)$. And it can be easily seen that $\varphi(\{\alpha\} \times K(\alpha)) \setminus L$ and $\varphi(\{\alpha'\} \times K(\alpha')) \setminus L$ are not path-connected to each other for $\alpha \neq \alpha'$.

Assume that $K \in \mathbb{P}_2$. Then each section $K(\alpha)$ of K contains an element in $\Delta \setminus Q$. Thus,

$$\varphi(K) = \bigcup_{\alpha \in \Delta} \varphi(\{\alpha\} \times K(\alpha))$$

will be a path-connected subset of $\mathbb{I}^2 \times \mathbb{J}$. But if $K \notin \mathbb{P}_2$, then there will be a section $K(\alpha)$ that completely lies in Q, and therefore the point $(0, 0, \boldsymbol{\alpha})$ will not be path-connected to the plane segment at the level $y = 1$. We then get at least two distinct path components in $\varphi(K)$. Thus, φ is a continuous reduction from $\mathbb{P}_2$ to $\mathcal{C}_{path}(\mathbb{I}^2 \times \mathbb{J}) \approx \mathcal{C}_{path}(\mathbb{I}^3)$, showing that $\mathcal{C}_{path}(\mathbb{I}^3)$ is $\mathbf{\Pi}^1_2$-complete. □

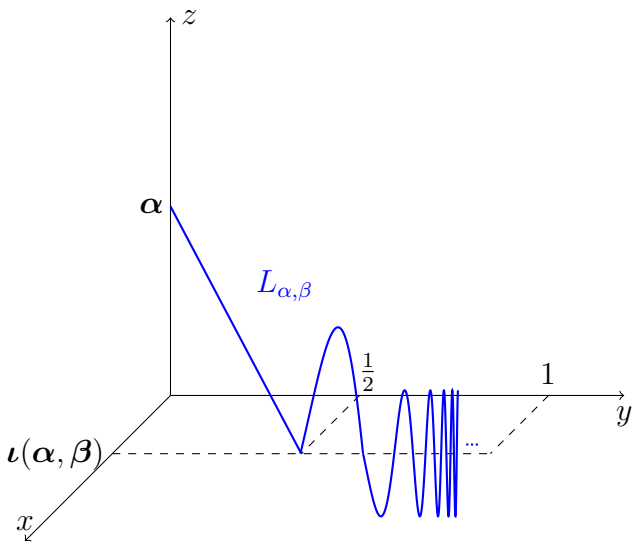


Figure 3.2: $L_{\alpha,\beta}$ for $\alpha \in \Delta,\ \beta = (1, 1, 0, 0, 0...)$

What about the complexity of $\mathcal{C}_{path}(\mathbb{I}^2)$? One might expect to have less complexity compared to $\mathcal{C}_{path}(\mathbb{I}^3)$. We shall prove that $\mathcal{C}_{path}(\mathbb{I}^2)$ is indeed not coanalytic, which is due to Becker.

That $\mathcal{C}_{path}(\mathbb{I}^2)$ is not analytic was proved by Just (unpublished). For the sketch of another proof given by Darji, see [16]. The exact classification of $\mathcal{C}_{path}(\mathbb{I}^2)$ was proven by Debs and Saint Raymond [3] to be a descriptive class obtained by applying certain union and intersection operations to $\mathbf{\Pi}^1_1$ sets.

Theorem 3.3 ([11], Page 257)**.** *$\mathcal{C}_{path}(\mathbb{I}^2)$ is $\mathbf{\Sigma}^1_1-$hard, and so it is not in $\mathbf{\Pi}^1_1(\mathcal{K}(\mathbb{I}^2))$.*

Proof. We will reduce the $\mathbf{\Sigma}^1_1-$complete subset $\mathbf{WF}^c \subseteq \mathbf{Tr}$ to $\mathcal{C}_{\text{path}}(\mathbb{I}^2)$ by a continuous map. For any tree T on $\mathbb{N}$, we will construct compact subsets $K^0_T \subseteq K^1_T \subseteq ... \subseteq \mathbb{I}^2$ so that $K_T := \bigcup_n K^n_T$ is also compact.

Let L be the union of line segments connecting points $(0,0) - (1,0) - (1,1)$.

Construction of K^0_T: Set K^0_T to be a "zig-zag" curve starting from the point $(0,1)$ contained in $\mathbb{I} \times [3/4, 1]$ as shown in the Figure 3.3. We label the local minimum points of the zig-zag curve by consecutive natural numbers.

Construction of K^1_T: For each natural number n, let ℓ_n be the vertical line segment starting from the point labeled by n in K^0_T to the level $y = \dfrac{1}{2}$. For each $(n) \in T$, draw ℓ_n and the same type of zig-zag curves placed inside $\mathbb{I} \times [3/8, 4/8]$ starting from the lower endpoint of ℓ_n whose x-coordinates do not exceed that of ℓ_{n+1}. Draw also the vertical line segment connecting the limit points of the zig-zag curve to the base line $y = 0$. For example, if $(0), (2), (4) \in T$, then K^1_T looks like Figure 3.4.

Construction of K^2_T: We label the local minimum points of the zig-zag curve corresponding to $(n) \in T$ by $n0, n1, n2, ...$ Then for each $(nm) \in T$, we draw the vertical line segment from the point labeled by (nm) up to the level $y = \dfrac{1}{4}$ and we call this line segment ℓ_{nm}. Then draw again the corresponding zig-zag curve from the lower endpoint of ℓ_{nm} and the corresponding vertical line segment to the base line $y = 0$. The illustration of K^2_T is provided in Figure 3.4 for $(01) \in T$.

The construction continues in this manner. We set K_T to be $L \cup \bigcup_n K^n_T$. It is routine to check that the map $T \mapsto K_T$ is continuous. The set K_T can have at most two path componants: the oscillating part and L. If T is ill-founded, say for example $0, 01, 012, 0123, ... \in T$, then the line segments $\ell_0, \ell_{01}, \ell_{012}, ...$ together with finitely many oscillations at each step will form a path from the oscillating part of K_T to the set L. Thus K_T will be path-connected. If, however, T is well-founded, then K_T is not path-connected, just like topologist's sine curve.

So, the map $T \to K_T$ is a continuous reduction from $\mathbf{WF}^c$ to $\mathcal{C}_{path}(\mathbb{I}^2)$. □

This proof (of the fact that $\mathcal{C}_{path}(\mathbb{I}^2)$ is $\mathbf{\Sigma}^1_1$-hard) can be lifted up to show that $\mathcal{C}_{path}(\mathbb{I}^3)$ is $\mathbf{\Pi}^1_2$-complete (see the proof of [11, Theorem 37.11]). In fact, Debs and Saint Raymond employed the same method in [5, Theorem 5.5] to show that $\mathcal{C}_{path}(\mathbb{I}^4)$ is $\mathbf{\Pi}^1_2$-complete using the proof that $\mathcal{C}_{path}(\mathbb{I}^3)$ is $\mathbf{\Sigma}^1_1$-hard. On the other hand, we proved that $\mathcal{C}_{path}(\mathbb{I}^3)$ is $\mathbf{\Pi}^1_2$-complete using a slightly different approach. Yet, we used the proof idea of Debs and Saint Raymond to relate eventually zero sequences with path-connectedness. In fact, we proved Theorem 2.9 by this lifting up method. This method was mentioned and used in [16, Theorem 2.17].

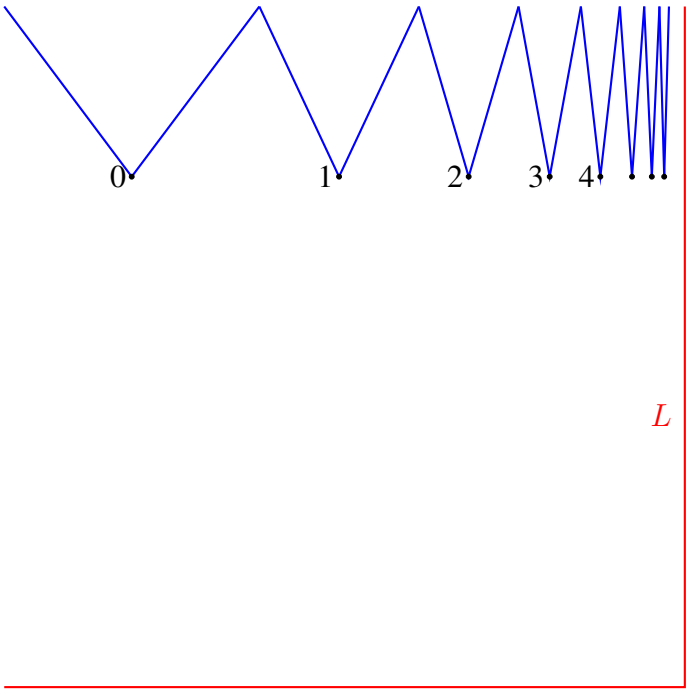


Figure 3.3: $L \cup K^0_T \subseteq \mathbb{I}^2$

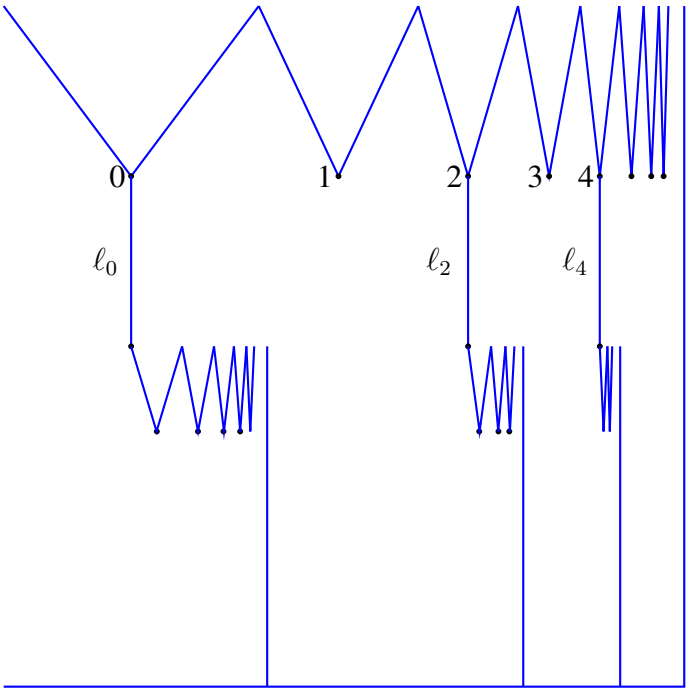


Figure 3.4: $L \cup K_T^1$

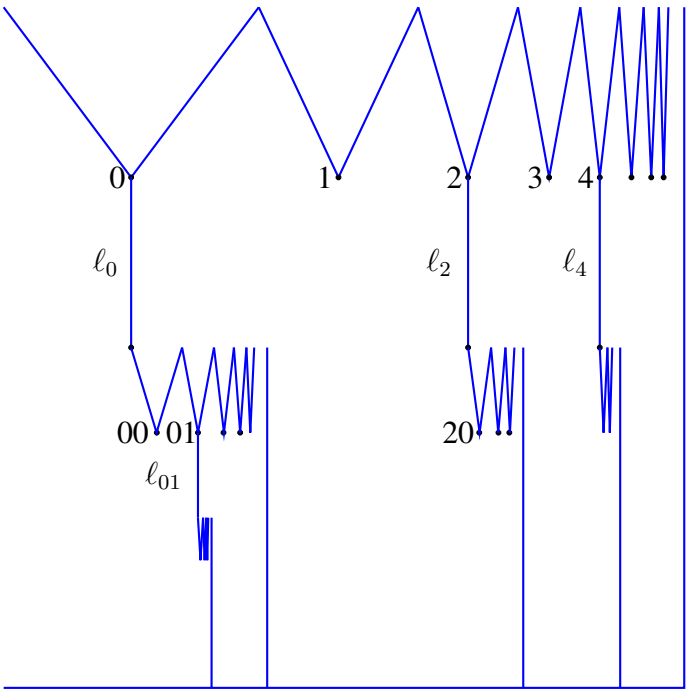


Figure 3.5: $L \cup K_T^2$

CHAPTER 4

DESCRIPTIVE COMPLEXITY OF CONNECTEDNESS

In this chapter, following the work of Debs and Saint Raymond [5], we shall investigate the descriptive complexity of the more general notion of conectedness. Let X be Polish. Define $\mathcal{C}(X)$ as the set of all closed connected subsets of X and $\breve{\mathcal{C}}(X)$ as the set of all closed subsets of X whose complement is connected. That is,

$$\mathcal{C}(X) = \{C \in \mathcal{F}(X) : \ C \text{ is connected}\}$$

and

$$\breve{\mathcal{C}}(X) = \{C \in \mathcal{F}(X) : \ X \setminus C \text{ is connected}\} \ .$$

We will first find the maximal possible complexity for these subsets in an arbitrary Polish space using their definitions. Since X is metrizable, a subset $A \subseteq X$ is disconnected if and only if there are disjoint open sets U, V such that $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$ and $A \subseteq U \cup V$. We need to reformulate this definition in terms of closed sets to be able to "talk" within $\mathcal{F}(X)$.

Let $F \subseteq X$ be closed. Then F is disconnected if and only if there are closed sets F_0, F_1 such that $F_0 \cap F_1 = \emptyset$, $F_0 \cap F \neq \emptyset$, $F_1 \cap F \neq \emptyset$ and $F = F_0 \cup F_1$. Here, F_0 and F_1 are witnessed by $F \cap U^c$ and $F \cap V^c$ in the previous definition.

Similarly, $X \setminus F$ is disconnected if and only if there are $F_0, F_1 \in \mathcal{F}(X)$ such that $F_0 \cup F \neq X$, $F_1 \cup F \neq X$, $F_0 \cup F_1 = X$ and $F_0 \cap F_1 \subseteq F$. Here, F_0 and F_1 are witnessed by U^c and V^c in the definition that uses open sets.

These definitions for $\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$ (actually for their complements) give upper bounds for their complexities as we shall prove below.

Theorem 4.1 ([5], Propositions 4.1 and 7.5.a)**.** *$\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$ are $\mathbf{\Pi}^1_2$ in $\mathcal{F}(X)$.*

Proof. Consider the set

$$\mathcal{A} = \{(F, F_0, F_1) : F_0 \cap F_1 = \emptyset,\ F_0 \cap F \neq \emptyset,\ F_1 \cap F \neq \emptyset \text{ and } F = F_0 \cup F_1\}$$

whose projection to the first component is equal to $\mathcal{F}(X) \setminus \mathcal{C}(X)$. The relation "$F = F_0 \cup F_1$" defines a Borel subset because the union map is Borel and the equality relation is Borel in $\mathcal{F}(X)$ (see Appendices A and B). On the other hand, "$F_0 \cap F \neq \emptyset$" defines an analytic relation because it can be obtained by taking the projection of

$$\{(F_0, F, x) \in \mathcal{F}(X) \times \mathcal{F}(X) \times X : x \in F_0,\ x \in F\}$$

which is a Borel set. It then easily follows that "$F_0 \cap F_1 = \emptyset$" is coanalytic. Therefore, $\mathcal{A}$ is an intersection of a Borel set with coanalytic and analytic sets. So, $\mathcal{A}$ is $\mathbf{\Sigma}^1_2$. But since $\mathcal{F}(X) \setminus \mathcal{C}(X) = \exists^{\mathcal{F}(X)\times\mathcal{F}(X)}\mathcal{A}$ and the projective sigma classes are closed under projection, $\mathcal{F}(X) \setminus \mathcal{C}(X)$ is still in $\mathbf{\Sigma}^1_2$. Thus, $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$.

Similarly, the equality and subset relations are Borel in $\mathcal{F}(X)$. Moreover, the relation "$F_0 \cap F_1 \subseteq F$" can be defined as $\forall x \in X\ (x \in F_0 \wedge x \in F_1 \to x \in F)$, which is a coanalytic definition. Hence the set

$$\mathcal{G} = \{(F, F_0, F_1) : F_0 \cup F \neq X,\ F_1 \cup F \neq X\ F_0 \cup F_1 = X,\ F_0 \cap F_1 \subseteq F\}$$

is coanalytic. Then $\mathcal{F}(X) \setminus \breve{\mathcal{C}}(X) = \exists^{\mathcal{F}(X)\times\mathcal{F}(X)}\mathcal{G}$ is $\mathbf{\Sigma}^1_2$, which implies that $\breve{\mathcal{C}}(X)$ is a $\mathbf{\Pi}^1_2$ subset of $\mathcal{F}(X)$. □

Remark. If X is additionally compact, then the intersection map turns out to be Borel, and so $\mathcal{A}$ defines a Borel subset. But then $\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$ become complements of projections of Borel sets. So they are $\mathbf{\Pi}^1_1$ in $\mathcal{K}(X)$ (see [5, Proposition 7.2]). In fact, we have more than that. Closed disconnected subsets of a compact space can be characterized in a much simpler way. If $\{U_n\}_n$ is a basis for X closed under finite union, then $F \in \mathcal{F}(X)$ is disconnected if and only if there are basic disjoint open sets U_n and U_m both intersecting F and that cover F. In other words,

$$\mathcal{F}(X) \setminus \mathcal{C}(X) = \bigcup_{n,m:\ U_n \cap U_m = \emptyset} U_n^+ \cap U_m^+ \cap (U_n \cup U_m)^-$$

and so it is open. Thus, $\mathcal{C}(X)$ is closed in $\mathcal{K}(X)$.

We will next show that these bounds for connectedness are optimal. What this means is that we will show that there exist

- a Polish space $X \subseteq \mathbb{R}^3$ for which $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$-complete,
- a Polish space $X \subseteq \mathbb{R}^2$ for which $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_2$-complete,
- a compact space $X \subseteq \mathbb{R}^2$ for which $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_1$-complete.

The proofs of these results are taken from [5]. For the first result, we provide the full proof with a slight modification of the original one whereas the last two results are proven with the help of some important lemmas whose proofs are skipped.

Theorem 4.2 ([5], Theorem 4.2)**.** *There is a Polish space $X \subseteq \mathbb{I}^3$ such that $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$-complete.*

Proof. Let Δ be a homeomorphic copy of the Cantor set such that $\{\frac{1}{4}, \frac{3}{4}\} \subseteq \Delta \subseteq \left[\frac{1}{4}, \frac{3}{4}\right] \subseteq \mathbb{R}$. Let Q be any countable dense subset of Δ, and set $P = \Delta \setminus Q$. Define

$$X = \{(x, y, z) \in \mathbb{I}^3 : x \notin \Delta \text{ or } (y \in P \text{ and } z = 0)\}$$

which is a G_δ subset of $\mathbb{I}^3$ because $\Delta \subseteq \mathbb{R}$ is a closed subset, $P \subseteq \mathbb{R}$ is a G_δ subset and "$z = 0$" defines a closed subset of $\mathbb{R}^3$. Thus, X is Polish by Theorem 2.1.

Let $\tilde{A}$ denote the set $A \times \{0\}$ for any $A \subseteq \mathbb{I}^2$. Define $\psi : \mathcal{K}(\Delta^2) \to \mathcal{K}(\mathbb{I}^3)$ as

$$\psi(K) = \{u = (x, y, z) \in \mathbb{I}^3 : z + d(x, \Delta) \geq \frac{1}{4} d(u, \tilde{K})\}$$

where d is the ℓ_1 metric on the corresponding space $\mathbb{I}^n$. That is,

$$d\left((x_1, ..., x_n), (x'_1, ..., x'_n)\right) = |x_1 - x'_1| + ... + |x_n - x'_n| \, .$$

It is clear that $\psi(K) \in \mathcal{K}(\mathbb{I}^3)$ because d is continuous in both uses and the inequality is not strict. In the rest of the proof, we shall be using the observation that $(x, y, 1) \in \psi(K)$ for any $(x, y) \in \mathbb{I}^2$ and any $K \in \mathcal{K}(\Delta^2)$ because the maximum possible value of $d(u, \tilde{K})$ is $1 + \frac{3}{4} + \frac{3}{4} = \frac{10}{4}$, and so $1 + d(x, \Delta) \geq \frac{1}{4} d(u, \tilde{K})$.

We observe that $\tilde{K} \subseteq \psi(K)$. Let us define $\varphi : \mathcal{K}(\Delta^2) \to \mathcal{F}(X)$ as

$$\varphi(K) = X \cap \psi(K) \, .$$

We claim then that φ is a Borel reduction from $\mathbb{P}_2$ to $\mathcal{C}(X)$.

Lemma 4.3. *The map φ is Borel.*

Proof. To prove that φ is Borel, we will use the technical Lemma 2.10. For the first assumption of the lemma, we will show that ψ is upper semi-continuous. Let $L \subseteq \mathbb{I}^3$ be closed. We will show that $\psi^{-1}(L^+)$ is closed where L^+ has the obvious meaning. Consider the set

$$\mathcal{A} = \{(K, u) \in \mathcal{K}(\Delta^2) \times L : u \in \psi(K)\} .$$

If $\{(K_n, u_n)\}_n$ is a sequence in $\mathcal{A}$ converging to $(K', u') \in \mathcal{K}(\Delta^2) \times \mathbb{I}^3$, then we have

$$z_n + d(x_n, \Delta) \geq \frac{1}{4} d(u_n, \tilde{K}_n) \text{ for all } n$$

and if take limits of both sides as $n \to \infty$, we get $z' + d(x', \Delta) \geq \frac{1}{4} d(u', \tilde{K'})$ since all functions involved are continuous. Hence $u' \in \psi(K')$. Moreover, $u_n \in L$ implies that $u' \in L$ as L is closed and $u_n \to u'$, showing that (K', u') is in $\mathcal{A}$. It follows that the set $\mathcal{A}$ is closed. But $\psi^{-1}(L^+)$ is equal to the projection of the closed set $\mathcal{A}$ onto the first component. It follows that $\psi^{-1}(L^+)$ is also closed as $\mathcal{A}$ is compact. But then $\psi^{-1}(U^-) = \psi^{-1}\left((\mathbb{I}^3 \setminus U)^+\right)^c$ is open for any open $U \subseteq \mathbb{I}^3$. Thus, ψ is upper semi-continuous.

The other assumption of the lemma was that $\varphi(K) = \psi(K) \cap X$ is dense in $\psi(K)$ for any $K \in \mathcal{K}(\Delta^2)$. Fix $K \in \mathcal{K}(\Delta^2)$ and $\epsilon > 0$. Then let $u_0 = (x_0, y_0, z_0) \in \psi(K)$ and consider the open rectangle

$$U = J^\epsilon_{x_0} \times J^\epsilon_{y_0} \times J^\epsilon_{z_0} \subseteq \mathbb{I}^3$$

where $J^\epsilon_t = (t - \epsilon/2, t + \epsilon/2) \cap \mathbb{I}$ for any $t \in \mathbb{I}$.

If $x_0 \notin \Delta$, then u_0 is in X, and so $U \cap \varphi(K)$ is non-empty. Similarly, if $z_0 = 1$, then $U \cap \varphi(K)$ is non-empty because $\mathbb{I} \times \mathbb{I} \times \{1\} \subseteq \psi(K)$ as we observed above.

So assume that $x_0 \in \Delta$ and $z_0 < 1$. Choose any $z \in J^\epsilon_{z_0}$ with $z > z_0$. Then choose

$x \in J^{\delta}_{x_0} \setminus \Delta$ where $\delta < \dfrac{z - z_0}{3}$. Set $u = (x, y_0, z)$, which is in $U \cap X$. Then

$$\begin{aligned} d(u, \tilde{K}) &\leq d(u, u_0) + d(u_0, \tilde{K}) \\ &\leq d(u, u_0) + 4z_0 \\ &= |x - x_0| + |z - z_0| + 4z_0 \\ &\leq \frac{z - z_0}{3} + z + 3z_0 \\ &= \frac{4}{3}z + \frac{8}{3}z_0 \\ &\leq 4z \\ &\leq 4z + 4d(x, \Delta) \end{aligned}$$

which implies that $u \in \psi(K)$. Thus $U \cap \varphi(K)$ is non-empty. This finishes the proof that $\varphi(K)$ is dense in $\psi(K)$. It follows from the Lemma 2.10 that φ is a Borel map. $\square$

Let us show that φ is a Borel reduction from $\mathbb{P}_2$ to $\mathcal{C}(X)$.

Let $K \notin \mathbb{P}_2$. Then there is $x_0 \in \Delta$ for which $K(x_0) \cap P$ is empty. It follows that $\varphi(K)(x_0) = \emptyset$ since otherwise $u' = (x_0, y, z) \in \varphi(K)$ would imply that $y \in P,\ z = 0$, and so $d(u', \tilde{K}) = 0 \Rightarrow (x_0, y) \in K$, which would give a contradiction. Hence

$$\{(x, y, z) \in X : x < x_0\} \text{ and } \{(x, y, z) \in X : x > x_0\}$$

clearly form a separation of $\varphi(K)$, and so $\varphi(K) \notin \mathcal{C}(X)$.

Now, let $K \in \mathbb{P}_2$. We shall show that $\varphi(K)$ is connected.

Before showing this, we observe that if $(x, y, z) \in \psi(K)$ where $x \notin \Delta$, then $(x, y, t) \in \varphi(K)$ for any $t \in [z, 1]$. This readily follows from the definition of $\psi(K)$. Now, let $\mathcal{J}$ be the set of connected components of $\mathbb{I} \setminus \Delta$ and let $\varphi_J(K) = \varphi(K) \cap (J \times \mathbb{I}^2)$ for all $J \in \mathcal{J}$. Then each $\varphi_J(K)$ is path-connected because any point in this set is connected by a vertical path to the level $z = 1$.

Assume towards a contradiction that $\varphi(K)$ is disconnected. Say V_0 and V_1 is a separation of $\varphi(K)$ with $(0, 0, 1) \in V_0$. Since each $\varphi_J(K)$ is path-connected,

$$\mathcal{J}_0 = \{J \in \mathcal{J} : \varphi_J(K) \subseteq V_0\} \text{ and } \mathcal{J}_1 = \{J \in \mathcal{J} : \varphi_J(K) \subseteq V_1\}$$

give a partition of the set $\mathcal{J}$.

Claim: If $u = (a, b, 0) \in \tilde{K}$ where $a \in \overline{\bigcup \mathcal{J}_0}$, then $u \in V_0$.
Proof: Let $x_k \in J_k$ and $J_k \in \mathcal{J}_0$ for each $k \in \mathbb{N}$ such that $x_k \to a$. Then set $z_k = |x_k - a|$ and $u_k = (x_k, b, z_k)$ for all $k \in \mathbb{N}$.

Since

$$d(u_k, \tilde{K}) \leq d(u_k, u) \leq |x_k - a| + z_k = 2z_k \leq 4z_k \leq 4\big(z_k + d(x_k, \Delta)\big),$$

we get $u_k \in \psi(K) \cap X = \varphi(K)$ for all $k \in \mathbb{N}$. So $u_k \in \varphi_{J_k}(K) \subseteq V_0$ for all $k \in \mathbb{N}$. This implies that $u \in V_0$ as $u_k \to u$ and V_0 is closed in $\varphi(K)$. The claim is now proved. Symmetrically, $u = (a, b, 0) \in \tilde{K}$ where $a \in \overline{\bigcup \mathcal{J}_1}$ implies $u \in V_1$.

Assume that $\mathcal{J}_1 \neq \emptyset$. Let $a = \inf\big(\bigcup \mathcal{J}_1\big)$. Since $[0, 1/4) \in \mathcal{J}_0$, $\inf J \in \Delta$ for any $J \in \mathcal{J}_1$. Hence, $a = \inf\big(\bigcup \mathcal{J}_1\big) \in \Delta$. Then $a \in \overline{\bigcup \mathcal{J}_0} \cap \overline{\bigcup \mathcal{J}_1}$. Since $K \in \mathbb{P}_2$, there is $b \in P$ such that $(a, b) \in K$, and so $(a, b, 0) \in \tilde{K}$. But since $a \in \overline{\bigcup \mathcal{J}_0} \cap \overline{\bigcup \mathcal{J}_1}$, we have $(a, b, 0) \in V_0 \cap V_1$ by the claim, which is a contradiction. Thus, $\mathcal{J} = \mathcal{J}_0$, and so $\varphi_J(K) \subseteq V_0$ for all $J \in \mathcal{J}$. Moreover, $\tilde{K} \cap \varphi(K) \subseteq V_0$ by the claim. Then it follows that

$$\varphi(K) = \big[\tilde{K} \cap \varphi(K)\big] \cup \bigcup_{J \in \mathcal{J}} \varphi_J(K) \subseteq V_0 \,.$$

This contradicts the assumption that V_0 and V_1 are a separation of $\varphi(K)$. We then conclude that $\varphi(K)$ is connected.

So, φ is a Borel reduction from $\mathbb{P}_2$ to $\mathcal{C}(X)$. It follows from Theorem 2.11 that $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$-complete. □

There is no known example of Polish $X \subseteq \mathbb{I}^2$ for which $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$-complete whereas in what follows, we will demonstrate an example of a Polish space $X \subseteq \mathbb{I}^2$ such that $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_2$-complete.

Lemma 4.4 ([5], Lemma 7.7)**.** *Let $\Delta \approx 2^{\mathbb{N}}$. Then for any $A \in \mathbf{\Sigma}^1_2(\Delta)$ there exists $G \in \mathbf{\Pi}^0_2(\Delta^2)$ and a continuous map $\alpha \mapsto K_\alpha$ from Δ to $\mathcal{K}(G)$ such that*

$$A = \{\alpha \in \Delta : \exists x \in \Delta,\ G(x) = K_\alpha(x)\} \,.$$

Theorem 4.5 ([5], Theorem 7.6)**.** *There exists a Polish space $X \subseteq \mathbb{I}^2$ for which $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_2$-complete.*

Proof. Let $\Delta \subseteq (0,1)$ be homeomorphic to $2^{\mathbb{N}}$ and choose any $\mathbf{\Sigma}^1_2$-complete set $A \subseteq \Delta$. Consider the set $G \subseteq \Delta^2$ and the map $\varphi : \Delta \to \mathcal{K}(G)$ associated to A as in the lemma above. Then define $X = [(\mathbb{I} \setminus \Delta) \times \mathbb{I}] \cup G$. Since $\mathbb{I} \setminus \Delta$ is open and G is a G_δ subset, X is G_δ, and so it is Polish. We will prove that the continuous map φ considered as a map from Δ to $\mathcal{F}(X)$ reduces the $\mathbf{\Pi}^1_2$-complete set A^c to $\breve{\mathcal{C}}(X)$.

If $\alpha \in A$, then $\exists x_0 \in \Delta$ such that $G(x_0) = K_\alpha(x_0)$. Thus $X \setminus K_\alpha = X \setminus \varphi(\alpha)$ can be separated by $\{(x,y) \in X : x < x_0\}$ and $\{(x,y) \in X : x > x_0\}$. So, $\varphi(\alpha) \notin \breve{\mathcal{C}}(X)$.

Assume now that $\alpha \notin A$. Assume towards a contradiction that V_0 and V_1 form a separation of $X \setminus \varphi(\alpha)$ with $(0,0) \in V_0$. Then set $\mathcal{J}$ to be the set of connected components of $\mathbb{I} \setminus \Delta$. Moreover, let

$$\mathcal{J}_0 = \{J \in \mathcal{J} : J \times \mathbb{I} \subseteq V_0\} \text{ and } \mathcal{J}_1 = \{J \in \mathcal{J} : J \times \mathbb{I} \subseteq V_1\} .$$

We can then apply exactly the same idea as in the proof of Theorem 4.2 to show that $\mathcal{J}_1$ is empty, and so $X \setminus \varphi(\alpha) \subseteq V_0$ leading to the same contradiction. We then conclude that $X \setminus \varphi(\alpha)$ is connected, and therefore $\varphi(\alpha) \in \breve{\mathcal{C}}(X)$.

So, φ is a continuous reduction from A^c to $\breve{\mathcal{C}}(X)$. This finishes the proof that $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_2$-complete. □

Lastly, we will prove the third result. Let $P = [0,1] \setminus \mathbb{Q}$. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \sin\left(\frac{1}{x - q_n}\right)$$

defined for all $x \in P$. We can write f as $g_n + h_n$ where

$$g_n(x) = \frac{1}{2^n} \cdot \sin\left(\frac{1}{x - q_n}\right) \text{ and } h_n(x) = \sum_{k \neq n} \frac{1}{2^k} \cdot \sin\left(\frac{1}{x - q_k}\right)$$

for any $n \in \mathbb{N}$. Being uniform limit of continuous functions at $x = q_n$, the function h_n is also continuous at q_n. On the other hand, we clearly have that

$$\left\{ \lim_{k\to\infty} g_n(x_k) : \lim_{k\to\infty} x_k = q_n \right\} = [-2^{-n}, 2^{-n}] .$$

But then

$$J_n := \left\{ \lim_{k\to\infty} f(x_k) : \lim_{k\to\infty} x_k = q_n \right\} = [h_n(q_n) - 2^{-n}, h_n(q_n) + 2^{-n}] .$$

So for each n, the set J_n of "cluster values" of f is a closed interval. Let X be the closure of the graph of the function f defined above. Since f is continuous, we have

$$X = Graph(f) \cup \bigcup_n J_n \, .$$

Moreover, any $H \subseteq X$ is connected if $H(x) \neq \emptyset$ for all $x \in \mathbb{I}$. We refer the reader to [4, Section 8] for the proof of this fact and for a detailed discussion of the function f.

Theorem 4.6 ([5], Theorem 7.3)**.** *There is a compact set $X \subseteq \mathbb{I}^2$ for which $\breve{\mathcal{C}}(X)$ is a $\mathbf{\Pi}^1_1$-complete.*

Proof. Although f cannot be extended continuously to $\mathbb{I}$, there is a subset Δ of $\mathbb{I}$ such that $0, 1 \in \Delta \approx 2^{\mathbb{N}}$, $\Delta \cap \mathbb{Q}$ is dense in Δ, and $f|_{\Delta \cap P}$ has a continuous extension to Δ (see [5, Lemma 7.4]). Set $Q = \mathbb{Q} \cap \Delta$ and let $g : \Delta \to \mathbb{J}$ be a continuous extension of f. Define $\varphi : \mathcal{K}(\Delta) \to \mathcal{K}(X)$ as $\varphi(K) = Graph(g|_K)$. Then φ is a continuous reduction from $\mathbb{P}_1$ to $\breve{\mathcal{C}}(X)$. It is easy to see that this map is (uniformly) continuous. If $K \notin \mathbb{P}_1$, then there is x_0 in $K \setminus Q$. But then since $x_0 \notin \mathbb{Q}$, x_0-section of X is a singleton. Thus, $\{(x, y) \in X : x < x_0\}$ and $\{(x, y) \in X : x > x_0\}$ clearly form a separation of $X \setminus \varphi(K)$. Hence $\varphi(K) \notin \breve{\mathcal{C}}(X)$. Suppose now that $K \in \mathbb{P}_1$. Then $K \subseteq \mathbb{Q}$. It follows that each section of $X \setminus \varphi(K)$ is non-empty, and so $X \setminus \varphi(K)$ is connected. As a result, φ is a continuous reduction from $\mathbb{P}_1$ to $\breve{\mathcal{C}}(X)$. Since X is compact, $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_1$ as we observed. Thus $\breve{\mathcal{C}}(X)$ is $\mathbf{\Pi}^1_1$-complete. □

We close this chapter by noting that it can be easily shown that $\mathcal{C}(\mathbb{I})$ is not open in $\mathcal{K}(\mathbb{I})$. So $\mathbf{\Pi}^0_1$ is also an optimal bound for the complexity of $\mathcal{C}(X)$ for a compact Polish space X.

CHAPTER 5

BOREL EQUIVALENCE RELATIONS OF PATH-CONNECTEDNESS

In this chapter, we analyze the possible complexities of path-connectedness equivalence relation on G_δ subsets of $\mathbb{R}^2$. For a Polish subspace $X \subseteq \mathbb{R}^n$, consider the path-connectedness equivalence relation $\approx_X$ on X given by $x \approx_X y$ if and only if there exists a continuous function γ from the unit interval $\mathbb{I}$ into X with $\gamma(0) = x$ and $\gamma(1) = y$ for all $x, y \in X$. Then the equivalence relation $\approx_X$ is an analytic subset of $X \times X$ as it is the projection of the closed subset

$$\{(x, y, \gamma) \in X \times X \times \Pi(X) : \gamma(0) = x \text{ and } \gamma(1) = y\}$$

where $\Pi(X)$ is defined as in Chapter 3. Becker and Pol showed that if $X \subseteq \mathbb{R}^2$ is a G_δ subset, then $\approx_X$ being Borel is equivalent to all path components of X being Borel ([2]). In the last year, Debs and Saint Raymond proved that $\approx_X$ is Borel for any G_δ subset of the real plane ([6]).

Knowing that $\approx_X$ is Borel, we will show that it is essentially countable and therefore bounded by E_∞ for any G_δ subspace $X \subseteq \mathbb{R}^2$. We will also prove that the non-smooth Borel equivalence relation E_0 is realized on $\approx_X$ for some compact $X \subseteq \mathbb{R}^2$.

Now, let $X \subseteq \mathbb{R}^2$ be G_δ. We know that the set $\mathcal{J}(X)$ of all paths is Borel in $\mathcal{K}(X)$ where a path means a homeomorphic image of $\mathbb{I}$ in X. Moreover, the map $e : \mathcal{J}(X) \to \mathcal{K}(X)$ that assigns each path to the set of its endpoints is also Borel. The proofs of these facts can be found in [2]. For any path J, we denote the set $J \setminus e(J)$ by $\mathring{J}$.

Remark. We abuse the definition of a path by defining it both as a continuous function $\gamma : \mathbb{I} \to X$ and as a homeomorphic image of $\mathbb{I}$ in X. The latter is known as

arc in the literature. However, it is well-known that in Hausdorff spaces, there is a continuous function $\gamma : \mathbb{I} \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ if and only if there is a homeomorphic image of $\mathbb{I}$ in X whose endpoints are x and y assuming that $x \neq y$. It will be clear from the context which definition we mean.

Theorem 5.1. $\approx_X \leq_B E_\infty$

Before proceeding with the proof, let us make a simple observation on the Borel equivalence relations to simplify the proof.

Lemma 5.2. *Let Z be a standard Borel space, and $B_1, B_2, ..., B_n$ be a Borel partition of Z. Let $\sim_i$ be a Borel equivalence relation on B_i for $i = 1, 2, ..., n$. If each of $\sim_i$ is essentially countable, then so is $\bigcup_{i=1}^{n} \sim_i$.*

The lemma helps us to consider different types of $\approx_X$-classes separately. Define

X_T : the set of points whose path component contains a simple triod, that is, there are paths I, J in the component such that $I \cap J$ is singleton and $e(I) \cap e(J) = \emptyset$,

X_S : the set of points whose path component is singleton, and

X_P : the set of points whose path component is an injective continuous image of a real interval or S^1.

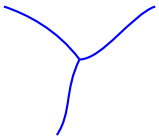


Figure 5.1: A simple triod

It is well-known that a path component of a separable metrizable space belongs to one of these three types (for example, see [2] and [7]). Therefore, $X = X_T \cup X_S \cup X_P$. Moreover, this is a Borel partition of X as shown by the following proposition.

Proposition 5.3 ([6], Proposition 10)**.** *X_T, X_S and X_P are Borel.*

Clearly, $\approx_X$ is a disjoint union of $\approx_{X_T}$, $\approx_{X_S}$ and $\approx_{X_P}$. We will show that all of these restrictions are essentially countable. Then Theorem 5.1 will follow by Lemma 5.2.

Theorem 5.4. *$\approx_{X_T}$, $\approx_{X_S}$ and $\approx_{X_P}$ are essentially countable.*

Proof. Moore's theorem ([17]) implies that X can have at most countably many triodic components. An elegant proof of this fact can be found in [19] by Pittman. Thus, having countably many Borel equivalence classes, $\approx_{X_T}$ is smooth and therefore essentially countable.

Since $\approx_{X_S}$ is the identity equivalence relation on a standard Borel space, it is smooth and so essentially countable.

Recall that a path component of X_P is a continuous injective image of a real interval or a circle. To show that $\approx_{X_P}$ is essentially countable, we will choose countably many paths from each path component of X_P.

Define a partition End and Int of X_P as the set of points which are endpoints of their path components and the set of points which are interior to some path in X_P respectively. We can apply a similar argument as in the proof of [6, Proposition 10] to the Borel set

$$\{(x, J) \in X \times \mathcal{J}(X) : x \in X_P, x \in J, x \notin e(J)\}$$

to show that its projection to the first component is Borel. But the projection of this set is the set of all points that is interior to some path in X_P, namely Int. Then both Int and End are Borel.

We need to construct codomain of our Borel reduction, which will be a Borel subset of $\mathcal{J}(X)$. Let $\{B_n\}_n$ be a basis of open discs for $\mathbb{R}^2$ and fix a Borel selector function $c : \mathcal{K}(X) \to X$. Then we define the set

$$\mathcal{J}_n = \{J \in \mathcal{J}(X) : J \subseteq \overline{B_n},\ J \cap \partial B_n \subseteq e(J) \subseteq \partial B_n \cup End,\ c(J) \in X_P\}$$

of all paths in X_P that lie in the closure of B_n, whose endpoints are on ∂B_n, or possibly on End, and do not intersect ∂B_n other than endpoints (see Figure 5.2). It is routine to check that $\mathcal{J}_n$ is Borel.

Define $\approx_{\mathcal{J}}$ to be the equivalence relation on $\mathcal{J} := \bigcup_n \mathcal{J}_n$ of being in the same path component.

Claim: $\approx_{\mathcal{J}}$ is a countable Borel equivalence relation.

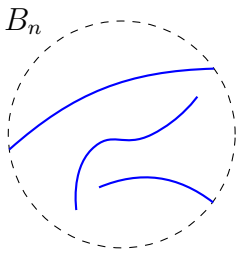


Figure 5.2: Elements of $\mathcal{J}_n$

Proof of Claim: The mapping $J \mapsto c(e(J))$ is a Borel reduction from $\approx_{\mathcal{J}}$ to $\approx_X$, which shows that $\approx_{\mathcal{J}}$ is Borel.

Let C be a path component in X_P. Say $\varphi : S \to C$ is a continuous injective parametrization of C where S is an interval or a circle. Let $n \in \mathbb{N}$ and let $I, J \in \mathcal{J}_n$ be in C. It is easily checked that I and J can intersect only at their endpoints, which implies that $\mathring{I}$ and $\mathring{J}$ are disjoint. Then $\{\varphi^{-1}(\mathring{J})\}_{J\subseteq C,\ J\in\mathcal{J}_n}$ is a disjoint family of open intervals in S, and therefore it is a countable family. This shows that $\approx_{\mathcal{J}}$ is an equivalence relation with countable classes and finishes the proof of the claim. □

To finish the proof of the theorem, observe that

$$\{(x, J) \in X_P \times \mathcal{J} : x \in J\}$$

is a Borel set with countable sections. Thus, by Lusin-Novikov theorem [11, Theorem 18.10], it has a Borel uniformization, that is, a Borel subset which is a function. The projection of this union to the first component is X_P. It is immediate from the construction then that this function is a Borel reduction from $\approx_{X_P}$ to $\approx_{\mathcal{J}}$. So $\approx_{X_P}$ is also essentially countable. □

This proves Theorem 5.1, which shows that if $\approx_X$ is non-smooth, then we necessarily have $E_0 \leq_B \approx_X \leq_B E_\infty$. But can $\approx_X$ ever be non-smooth? The answer turns out to be positive. Our next theorem will show that a non-smooth equivalence relation of path-connectedness can be realized for some compact subset of the plane. Let us introduce some auxiliary notations for the construction of the compact set and the reduction.

For each $\alpha \in 2^{\mathbb{N}}$, let $\overline{\alpha}$ denote the sequence obtained by taking the complement of

each entry, that is, $\overline{\alpha}_n = 1 - \alpha_n$. Consider the equivalence relation E_0^* on $2^{\mathbb{N}}$ given by

$$\alpha E_0^* \beta \longleftrightarrow (\alpha E_0 \beta \text{ or } \alpha E_0 \overline{\beta})$$

for any $\alpha, \beta \in 2^{\mathbb{N}}$. It is straightforward to check that E_0^* is a hyperfinite Borel equivalence relation. Moreover, the continuous map $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined as $f(\alpha) = (\alpha_0, 1, \alpha_1, 1, \alpha_2, 1, ...)$ for all $\alpha \in 2^{\mathbb{N}}$ is a Borel reduction from E_0 to E_0^*, showing that E_0^* is not smooth. Consequently, $E_0^* \sim_B E_0$.

We set $N_0 = 2^{\mathbb{N}}$ and for any $k \in \mathbb{N}^+$, we define $N_k \subseteq 2^{\mathbb{N}}$ as

$$N_k = \left\{\alpha \in 2^{\mathbb{N}} : \min\{i \in \mathbb{N} : \alpha_i = 1\} = k - 1\right\} .$$

For any sequence $\alpha \in N_k$ where $k > 0$, let $\widehat{\alpha}$ be the sequence $(0, 0, ..., 0, 1, \overline{\alpha}_k, \overline{\alpha}_{k+1}, \overline{\alpha}_{k+2}, ...)$, which is still in N_k.

We shall first define a sequence $(H_k)_k$ of compact subsets of $\mathbb{R}^2$ as follows. Let $x_0 = \frac{1}{2}$. For any $\alpha \in N_0$, let $\gamma^0_{\alpha,\overline{\alpha}}$ be the upper semicircle centered at $(x_0, 0)$ with radius $|x_0 - \boldsymbol{\alpha}|$ connecting $(\boldsymbol{\alpha}, 0)$ to $(\overline{\boldsymbol{\alpha}}, 0)$. Set

$$H_0 = \bigcup_{\alpha \in N_0} \gamma^0_{\alpha,\overline{\alpha}}$$

For each $k > 0$, let $x_k = \frac{\boldsymbol{\alpha}+\widehat{\boldsymbol{\alpha}}}{2}$ where α is some (equivalently, any) element of N_k. Then for each $\alpha \in N_k$, let $\gamma^k_{\alpha,\widehat{\alpha}}$ be the lower semicircle centered at $(x_k, 0)$ with radius $|x_k - \boldsymbol{\alpha}|$ connecting $(\boldsymbol{\alpha}, 0)$ to $(\widehat{\boldsymbol{\alpha}}, 0)$. Then set

$$H_k = \bigcup_{\alpha \in N_k} \gamma^k_{\alpha,\widehat{\alpha}}$$

Define $H := \bigcup_k H_k$. Indeed, the compact set H is shown in Figure 5.3. This set is a very well-known among continuum theorists as the Knaster continuum as an example of a compact connected metric space that cannot be written as a union of two proper compact connected subspaces.

Consider as an example $\alpha = (0, 1, 1, 0, ...)$ and $\beta = (0, 0, 0, 1, ...)$ where $\alpha F_4 \overline{\beta}$ and F_4 is defined as in the Subsection 2.5 so that $\alpha E_0^* \beta$. Then the diagram

$$\alpha = (0, 1, 1, 0, ...) \overset{\gamma^0_{\alpha,\overline{\alpha}}}{\longleftrightarrow} (1, 0, 0, 1, ...) \overset{\gamma^1_{\overline{\alpha},\widehat{\overline{\alpha}}}}{\longleftrightarrow} (1, 1, 1, 0, ...) \overset{\gamma^0_{\widehat{\overline{\alpha}},\overline{\widehat{\overline{\alpha}}}}}{\longleftrightarrow} (0, 0, 0, 1, ...) = \overline{\overline{\alpha}} = \beta$$

shows the path connecting $(\boldsymbol{\alpha}, 0)$ to $(\boldsymbol{\beta}, 0)$, which is $\gamma^0_{\alpha,\overline{\alpha}} \cup \gamma^1_{\overline{\alpha},\widehat{\overline{\alpha}}} \cup \gamma^0_{\widehat{\overline{\alpha}},\overline{\widehat{\overline{\alpha}}}}$.

Since E_0 ($\sim_B E_0^*$) is the unique hyperfinite non-smooth Borel equivalence relation up to Borel bireducibility, it follows from the next theorem that $\approx_H$ is non-smooth.

Theorem 5.5. *$\approx_H$ is Borel bireducible to E_0^*.*

Proof. To show that $E_0^* \leq_B \approx_H$, let $\psi : 2^{\mathbb{N}} \to H$ be defined as $\psi(\alpha) = (\boldsymbol{\alpha}, 0)$, which is clearly continuous. We claim that $\alpha E_0^* \beta$ if and only if $\psi(\alpha) \approx_H \psi(\beta)$.

Clearly, $\alpha E_0^* \beta$ if and only if $\alpha F_n \beta$ or $\alpha F_n \overline{\beta}$ for some $n \in \mathbb{N}$. We claim then that for any $n \in \mathbb{N}$ and for any $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha F_n \beta$ or $\alpha F_n \overline{\beta}$, there is a path in H connecting $\psi(\alpha)$ and $\psi(\beta)$. We shall prove this by induction on n.

For the base step $n = 0$, the claim trivially holds whenever $\alpha = \beta$ and, $\psi(\alpha)$ and $\psi(\beta)$ are connected by the path $\gamma^0_{\alpha,\overline{\alpha}}$ whenever $\overline{\alpha} = \beta$.

Let $n \in \mathbb{N}$ and assume that the claim holds for n. Let $\alpha, \beta \in 2^{\mathbb{N}}$ be such that $\alpha F_{n+1} \beta$ or $\alpha F_{n+1} \overline{\beta}$. Suppose that we are in the case that $\alpha F_{n+1} \beta$. If $\alpha_n = \beta_n$, then we have $\alpha F_n \beta$, and so we are done by the inductive hypothesis. So, assume $\alpha_n \neq \beta_n$. Without loss of generality, assume that $\alpha_n = 1$. We then have

$$\begin{aligned}
\psi(\alpha) &= \psi(\alpha_0, ..., \alpha_{n-1}, 1, \alpha_{n+1}, \alpha_{n+2}, ...) \\
&\approx_H \psi(0, ..., 0, 1, \alpha_{n+1}, \alpha_{n+2}, ...) \\
&\approx_H \psi(0, ..., 0, 1, \overline{\alpha}_{n+1}, \overline{\alpha}_{n+2}, ...) \\
&\approx_H \psi(1, ..., 1, 0, \alpha_{n+1}, \alpha_{n+2}, ...) \\
&\approx_H \psi(\beta_0, ..., \beta_{n-1}, 0, \alpha_{n+1}, \alpha_{n+2}, ...) = \psi(\beta)
\end{aligned}$$

where the first and the last equivalences follow from the inductive hypothesis whereas the second and the third ones are witnessed by some paths in H_{n+1} and H_0 respectively. The case for $\alpha F_{n+1} \overline{\beta}$, though slightly more involved, can be shown in a similar way. Thus, we have $\alpha E_0^* \beta$ implies $\psi(\alpha) \approx_H \psi(\beta)$.

For the other direction, assume that $\alpha, \beta \in 2^{\mathbb{N}}$ such that $\alpha \not E_0^* \beta$. Assume for the sake of contradiction that $\psi(\alpha) \approx_H \psi(\beta)$. Say $\gamma : \mathbb{I} \to H$ is a path from $\psi(\alpha)$ to $\psi(\beta)$. We will first prove that $\gamma(\mathbb{I})$ must intersect infinitely many H_k's. One can check that, by construction, for any $i \in \mathbb{N}$, the set $H_0 \cup ... \cup H_i$ is homeomorphic to

$\mathcal{C} \times \mathbb{I}$. It is therefore easy to see that $\psi(\alpha)$ and $\psi(\beta)$ are not path-connected inside $H_0 \cup ... \cup H_i$ for any $i \in \mathbb{N}$. Thus $\gamma(\mathbb{I})$ must intersect infinitely many H_k's. Now choose $e_n \in H_{k_n} \cap \gamma(\mathbb{I})$ for each $n \in \mathbb{N}$ from distinct H_{k_n}'s. Take $t_n \in \mathbb{I}$ such that $\gamma(t_n) = e_n \in H_{k_n}$ for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $(t_n)_n$ is an increasing (or decreasing) sequence. For if it is not the case, we can pass to an increasing (or decreasing) subsequence. For any $n \in \mathbb{N}$, $\gamma([t_n, t_{n+1}])$ must intersect the line $x = \frac{1}{2}$, call M. This is because $\gamma(t_n) \in H_{k_n}$ and $\gamma(t_{n+1}) \in H_{k_{n+1}}$ cannot be connected by a path inside $H \setminus M$. So for each $n \in \mathbb{N}$ let $\hat{t}_n \in (t_n, t_{n+1})$ such that $\gamma(\hat{t}_n) \in M$. Since $(t_n)_n$ and $(\hat{t}_n)_n$ converge to the same number and γ is a continuous function, we have

$$(0,0) = \lim_{n\to\infty} e_n = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \gamma(\hat{t}_n) \in M,$$

which is clearly not true. Thus, $\psi(\alpha) \not\approx_H \psi(\beta)$. This finishes the proof that $E_0^* \leq_B \approx_H$.

To show $\approx_H \leq_B \ E_0^*$, consider the map $\varphi : H \to 2^{\mathbb{N}}$ that sends each $e \in H$ to the sequence $\varphi(e)$ in $2^{\mathbb{N}}$ that maps under ψ to the closest point on the copy of the Cantor set on the x-axis, to which e is path-connected. Here we choose the left point in the case that there are two such points. It is straightforward to check that φ is a Borel reduction from $\approx_H$ to E_0^*. $\square$

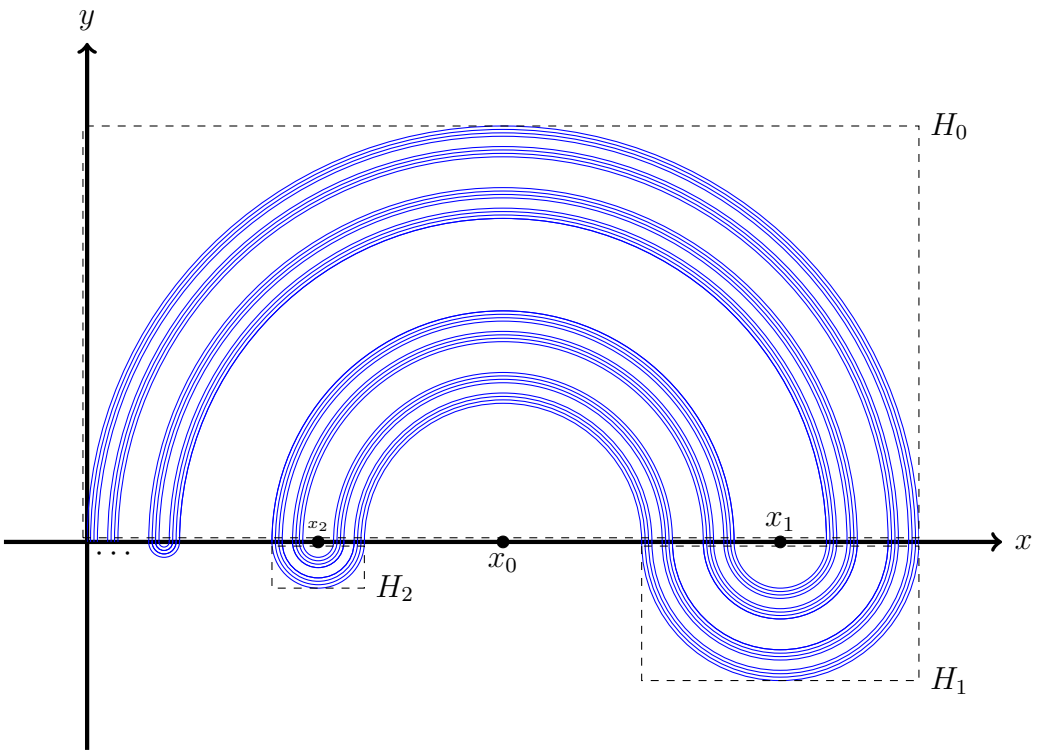


Figure 5.3: The Knaster continuum (or the bucket handle)

While the Borel equivalence relation $\approx_X$ is bounded for Polish $X \subseteq \mathbb{R}^2$, a compact subset of $\mathbb{R}^3$ with a non-Borel path-component was constructed in [13]. Not only this, but also an example of a compact subset of $\mathbb{R}^3$ none of whose path components is Borel was also given in [14] (see also [1]). The difference is notable.

CHAPTER 6

CONCLUSION

In this thesis, we surveyed the descriptive complexity of $\mathcal{C}_{path}(X)$, $\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$ in $\mathcal{F}(X)$ while we considered $\mathcal{K}(X)$ when X is compact following [5]. Together with the results in Chapters 3 and 4, we have the following table.

Table 6.1: Upper bounds for $\mathcal{C}_{path}(X)$, $\mathcal{C}(X)$ and $\breve{\mathcal{C}}(X)$

	X is Polish	X is compact
$\mathcal{C}_{path}(X)$	$\mathbf{\Pi}^1_2$	$\mathbf{\Pi}^1_2$
$\mathcal{C}(X)$	$\mathbf{\Pi}^1_2$	$\mathbf{\Pi}^0_1$
$\breve{\mathcal{C}}(X)$	$\mathbf{\Pi}^1_2$	$\mathbf{\Pi}^1_1$

These complexities are directly obtained by writing the definitions of the sets in an arbitrary Polish (and compact) space.

From the main results of the same chapters, we can conclude that all six bounds given in the Table 6.1 are the best possible choices. We showed, for example, that there is a Polish space $X \subseteq \mathbb{R}^3$ such that $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$-complete. By completeness, $\mathcal{C}(X)$ cannot be in the class $\mathbf{\Sigma}^1_2$ (and hence not in $\mathbf{\Delta}^1_1, \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1$ either). This suggests that it is never possible to define path-connected subsets of Polish spaces in a simpler way. To illustrate, one cannot characterize path-connectedness by a formula of the form

$$\forall y \in Y \; \varphi(y)$$

where Y is an uncountable Polish space and $\varphi(\cdot)$ defines a Borel relation. Therefore, the use of an existential quantifier over an uncountable Polish space is inevitable.

As we mentioned in the thesis, complete sets are the most complex sets in their

classes. It is interesting to note that connectedness and path-connectedness attain their most complex forms in two and three dimensional spaces.

It is probable that there is a Polish space $X \subseteq \mathbb{R}^2$ for which $\mathcal{C}(X)$ is $\mathbf{\Pi}^1_2$-complete. Yet, we only know an example in $\mathbb{R}^3$.

The difference between two and three dimensions is more apparent in path-connectedness. There can be two seemingly related reasons for this difference:

- If a path component in the plane is not uniquely path-connected, that is, if it contains a closed curve, then this component separates all path components into two distinct and unrelated path components to be examined. Moreover, any subset of the plane can contain at most countably many path components containing closed curve because of separability. This is clearly not the case in three dimension.

- The beautiful theorem of Moore shows that one cannot draw uncountably many mutually disjoint homeomorphic copies of the letter Y (or T,F,K etc.) in the plane. In $\mathbb{R}^3$, one can do this.

The second reason was mentioned in [3]. The difference between $\mathbb{R}^2$ and $\mathbb{R}^3$ is even sharper in terms of path-connectedness equivalence relation. There is an example of compact subset of $\mathbb{R}^3$ none of whose path components is Borel. Therefore, path-connectedness equivalence relation is not necessarily Borel in $\mathbb{R}^3$. On the other hand, the same relation is bounded by E_∞ if we consider Polish subspaces of $\mathbb{R}^2$. It is not known yet if this is the optimal upper bound or not.

In [1], Becker stated that there is no known example of a compact subset of the plane whose path-connectedness equivalence relation is non-smooth. There were also no works that addressed this question. After we solved the question by constructing such a compact set, Benjamin Vejnar communicated with us pointing out that exactly the same set, the Knaster continuum, is well-known as an example of indecomposable continua. He also informed us about the result of Solecki [21] that composant equivalence on indecomposable continua is Borel bireducible with either E_0 or E_1, which is an important Borel equivalence relation that is not essentially countable. So,

our result almost follow from Solecki's paper as in the case of the Knaster continuum composants coincide with path components.

Complexity classes of the set of simply connected subsets, the set of compact subsets with no holes, the set of uniquely path-connected subsets are also well studied (see [11] and [16]). It is possible to investigate other topological notions in terms of their descriptive complexities. For more open problems on the subject, we refer the reader to [5] and [16] in which some of the open problems discussed above are mentioned.

REFERENCES

[1] Howard Becker. The number of path-components of a compact subset of $\mathbf{R}^n$. In *Logic Colloquium '95 (Haifa)*, volume 11 of *Lecture Notes Logic*, pages 1–16. Springer, Berlin, 1998.

[2] Howard Becker and Roman Pol. Note on path-components in complete spaces. *Topology Appl.*, 114(1):107–114, 2001.

[3] Gabriel Debs and Jean Saint Raymond. The descriptive complexity of the set of arc-connected compact subsets of the plane. Preprint available at `https://www.karlin.mff.cuni.cz/~lhota/pres12/Debs%20-%20Saint%20Raymond%20Arc_Connect.pdf` Online; accessed July 2025.

[4] Gabriel Debs and Jean Saint Raymond. The descriptive complexity of the set of all closed zero-dimensional subsets of a Polish space. *Topology Appl.*, 249:43–66, 2018.

[5] Gabriel Debs and Jean Saint Raymond. The descriptive complexity of connectedness in Polish spaces. *Fundamenta Mathematicae*, 249(3):261–286, 2020.

[6] Gabriel Debs and Jean Saint Raymond. On the arc-wise connection relation in the plane. *Topology Appl.*, 353:Paper No. 108975, 6, 2024.

[7] G. R. Gordh, Jr. and Sam B. Nadler, Jr. Arc components of chainable Hausdorff continua. *General Topology and Appl.*, 3:63–76, 1973.

[8] Akihiro Kanamori. The emergence of descriptive set theory. In *From Dedekind to Gödel (Boston, MA, 1992)*, volume 251 of *Synthese Lib.*, pages 241–262. Kluwer Acad. Publ., Dordrecht, 1995.

[9] Vladimir Kanovei. *Borel equivalence relations*, volume 44 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. Structure and classification.

[10] V. G. Kanoveĭ. Development of descriptive set theory under the influence of N. N. Luzin's work. *Uspekhi Mat. Nauk*, 40(3(243)):117–155, 240, 1985.

[11] Alexander Kechris. *Classical descriptive set theory*, volume 156. Springer Science & Business Media, 2012.

[12] Alexander S. Kechris. On the concept of $\mathbf{\Pi}^1_1$-completeness. *Proc. Amer. Math. Soc.*, 125(6):1811–1814, 1997.

[13] K. Kunen and M. Starbird. Arc components in metric continua. *Topology Appl.*, 14(2):167–170, 1982.

[14] Attilio Le Donne. Arc components in metric continua. *Topology Appl.*, 22(1):83–84, 1986.

[15] Henri Lebesgue. Sur les fonctions représentables analytiquement. *Journal de mathematiques pures et appliquees*, 1:139–216, 1905.

[16] Alberto Marcone. Complexity of sets and binary relations in continuum theory: a survey. In *Set theory*, Trends Math., pages 121–147. Birkhäuser, Basel, 2006.

[17] RL Moore. Concerning triods in the plane and the junction points of plane continua. *Proceedings of the National Academy of Sciences*, 14(1):85–88, 1928.

[18] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[19] C. R. Pittman. An elementary proof of the triod theorem. *Proc. Amer. Math. Soc.*, 25:919, 1970.

[20] Glen Alan Schlee. On the development of descriptive set theory. thesis, University of North Texas, Denton, Texas, August 1988. Accessed June 9, 2025. Available via UNT Digital Library.

[21] Sławomir Solecki. The space of composants of an indecomposable continuum. *Adv. Math.*, 166(2):149–192, 2002.

[22] Sashi Mohan Srivastava. *A course on Borel sets*, volume 180. Springer Science & Business Media, 2008.

[23] Anush Tserunyan. Introduction to descriptive set theory. 2016. Available at https://www.math.mcgill.ca/atserunyan/Teaching_notes/dst_lectures.pdf Online; accessed July 2025.

[24] Yusuf Uyar. A nonsmooth path-connectedness relation in the real plane. *arXiv preprint arXiv:2501.07916*, 2025.

APPENDICES

We provide below some useful complexities of relations and continuous/Borel functions on standard Borel spaces. The results can be found in [11] and [22].

A COMPLEXITIES OF RELATIONS ON STANDARD BOREL SPACES

X	Relation	Space	Complexity
Polish	$x \in F$	$X \times \mathcal{F}(X)$	Borel
	$x \neq y$	$X \times X$	open
	$K \subseteq L$	$\mathcal{F}(X) \times \mathcal{F}(X)$	Borel
	$K = L$	$\mathcal{F}(X) \times \mathcal{F}(X)$	Borel
	$K \subseteq L$	$\mathcal{K}(X) \times \mathcal{K}(X)$	closed

B COMPLEXITIES OF FUNCTIONS ON STANDARD BOREL SPACES

X	Function	Domain and Codomain	Complexity
Polish	$(K, L) \mapsto K \cap L$	$\mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$	Borel
	$\{K_n\}_n \mapsto \bigcap_n K_n$	$\mathcal{K}(X)^{\mathbb{N}} \to \mathcal{K}(X)$	Borel
	$(K, L) \mapsto K \cup L$	$\mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$	Borel
metrizable	$K \mapsto \bigcup K$	$\mathcal{K}(\mathcal{K}(X)) \to \mathcal{K}(X)$	continuous
	$(K, L) \mapsto K \cup L$	$\mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$	continuous

Note. The map $K \to \bigcup K$ from $\mathcal{K}(\mathcal{K}(X))$ to $\mathcal{K}(X)$ is well-defined.