ANALYSIS AND DEVELOPMENT OF STATISTICAL PROPERTIES OF PERIODIC AUTOREGRESSIVE MOVING AVERAGE PROCESSES

A Ph.D. Thesis
Presented by
Abdullah Ahmed SMADI

to
the Graduate School of Natural and Applied Sciences
of Middle East Technical University
in Partial Fulfillment for the Degree of

DOCTOR OF PHILOSOPHY

in

STATISTICS

MIDDLE EAST TECHNICAL UNIVERSITY
ANKARA
January, 1994

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. İsmail TOSUN
Director

I certify that this thesis satisfies all the requirements of a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Yalçın TUNCER
Chairman of the Department

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy in Statistics.

Prof. Dr. Taylan ULA
Supervisor

Examining Committee in Charge:

Prof. Dr. Fuat ÖZKAN
Prof. Dr. Taylan ULA
Prof. Dr. Fetih YILDIRIM
Prof. Dr. İbrahim RAHİMOV
Prof. Dr. Ömer L. GEBİZLİOĞLU

ABSTRACT

ANALYSIS AND DEVELOPMENT OF STATISTICAL PROPERTIES OF PERIODIC AUTOREGRESSIVE MOVING AVERAGE PROCESSES

SMADI, Abdullah Ahmed
Ph.D. in Statistics
Supervisor: Prof. Dr. Taylan ULA
January, 1994, 124 pages

In this thesis, several features of the class of periodic autoregressive moving average (PARMA) processes are investigated. Firstly, the periodic stationarity and also the invertibility conditions of any PARMA process are reduced through its lumped-vector representation to an eigenvalue problem. It is also shown through a counterexample that for a PARMA process, stationarity of the aggregated process does not imply periodic stationarity.

For the identification of orders of PARMA processes, it is shown that this cannot be carried out by obtaining the orders of their marginal series. On the other hand, it is shown that the Box-Jenkins approach for identification of univariate ARMA processes can be generalized to univariate PARMA processes, following a seasonwise identification routine. For this, the seasonal autocorrelation function (ACF) and seasonal partial autocorrelation function (PACF), which play the same role as their ARMA counterparts, are employed. Approximated versions of the first and second order moments of the sample seasonal ACF are developed. Utilizing also some available results concerning both functions, confidence bands for the assessment of cut-off properties of these functions are developed. For the non-periodic case, these bands reduce to their well-known ARMA counterparts. The applicability of these bands are then illustrated through some simulations.

The last part of the study is devoted to estimation of PARMA processes. It is shown that, as in ARMA processes, the method of moment estimation in PARMA processes containing a moving average (MA) part is technically difficult and also does not give satisfactory results. On the other hand, it is shown that this method is straightforward and satisfactory for univariate or multivariate periodic autoregressive (PAR) processes. For PAR processes, the conditional least-squares (LS), conditional maximum likelihood (ML) and exact ML estimation methods are also studied for univariate and multivariate cases. It is shown that the first two methods give the same estimates of AR parameters for Gaussian processes. It is also shown that conditional LS estimates can be obtained in a season-wise manner, and regression methods can be employed directly both for univariate and multivariate cases. Detailed examples are given for some simple PAR processes. Estimates of error variances based on the same methods are also studied and compared. Simulation results indicate that conditional ML estimates are often superior in terms of MSE criterion. It is also shown that for PARMA processes contatining a MA part, the conditional LS and conditional ML estimates are difficult to obtain and also they are not equivalent. It is known that the exact likelihood function of any PARMA process is complicated. This is illustrated for two simple PAR and PMA processes.

Keywords: Periodically Correlated Process, PARMA Process, Periodic Stationarity, Lumped Process, Aggregation, Identification, Autocorrelation Function, Estimation, Method of Moments, Least Squares, Maximum Likelihood.

Science Code: 406.02.01

ÖZ

PERİYODİK OTOREGRESİF HAREKETLİ ORTALAMALAR SÜREÇLERİNİN İSTATİSTİKSEL ÖZELLİKLERİNİN ANALİZİ VE GELİŞTİRİLMESİ

SMADI, Abdullah Ahmed
Doktora Tezi, İstatistik Anabilim Dali
Tez Yöneticisi: Prof. Dr. Taylan ULA
Ocak, 1994, 124 Sayfa

Bu tezde, periyodik otoregresif hareketli ortalamalar (PARMA) süreçleri sınıfının çeşitli özellikleri araştırılmaktadır. İlk olarak, PARMA süreçlerinde durağanlılık ve çevrilebilirlik şartları, bileşik-vektör gösterimiyle özdeğer problemine indirgenmiştir. PARMA süreci için, agrege sürecin durağanlılığının periyodik durağanlılığı gerektirmediği de karşı bir örnekle gösterilmiştir.

PARMA süreçlerinin marjinal serilerinin derecelerinin elde edilmesi ile PARMA süreçlerinin derecelerinin teşhis edilemeyeceği gösterilmiştir. Diğer yandan, tek değişkenli ARMA süreçlerinin teşhisi için kullanılan Box-Jenkins yaklaşımının, mevsimsel teşhis rutiniyle, tek değişkenli PARMA süreçleri için genelleştirilebileceği gösterilmiştir. Bunun için, ARMA daki karşılıklarıyla aynı rolü oynayan mevsimsel otokorelasyon fonksiyonu (ACF) ve mevsimsel kısmi otokorelasyon fonksiyonu (PACF) kullanilmıştır. Örnek mevsimsel ACF'nin ilk ve ikinci derece momentlerinin yaklaşık formülleri geliştirilmiştir. Eldeki bazı sonuçlar da kullanılarak, bu fonksiyonların kesilme özelliklerinin tesbiti için güven sınırları geliştirilmiştir. Periyodik olmayan durumlar için, bu sınırlar bunların ARMA'daki bilinen karşılıklarına dönüşmektedir. Bu sınırların uygulanabilirliği bazı simülasyonlarla gösterilmiştir.

Çalışmanın son kısmı PARMA süreçlerinin tahminine ayrılmıştır. Hareketli ortalamalar (MA) içeren PARMA süreçlerinde, moment metodunun, ARMA süreçlerinde de olduğu gibi, teknik olarak zor olduğu ve tatmin edici sonuçlar vermediği gösterilmiştir. Diğer yandan, bu metodun tek ve çok değişkenli periyodik otoregresif (PAR) süreçleri için kolay ve tatmin edici olduğu gösterilmiştir. PAR süreçleri için, koşullu en küçük kare (LS), koşullu en çok olabilirlik (ML), ve kesin ML tahmin metotları da tek ve çok değişkenli durumlar için incelenmiştir. İlk iki metodun Gauss süreçlerinde AR parametreleri için aynı sonuçları verdiği gösterilmiştir. Koşullu LS tahminlerinin, mevsimsel biçimde elde edilebileceği ve regresyon metotlarının tek ve çok değişkenli durumlar için direk olarak kullanılabileceği de gösterilmiştir. Bazı basit PAR süreçleri için detaylı örnekler verilmiştir. Aynı metotlarla, hata varyansı tahminleri de incelenmiş ve karşılaştırılmıştır. Simülasyon neticeleri koşullu ML tahminlerinin ortalama kare yanılgı (MSE) kriteri yönünden daha üstün olduklarını göstermiştir. Hareketli ortalamalar içeren PARMA süreçleri için, koşullu LS ve koşullu ML tahminlerinin elde edilmelerinin zor oldukları ve eşdeğer de olmadıkları gösterilmiştir. Her PARMA süreci için kesin en çok olabilirlik fonksiyonunun karmaşık olduğu bilinmektedir. Bu, iki basit PAR ve PMA süreci için gösterilmiştir.

Anahtar Kelimeler: Periyodik Olarak Bağımlı Süreçler, PARMA Süreci, Periyodik Durağanlılık, Bileşik Süreç, Agregasyon, Teşhis, Otokorelasyon Fonksiyonu, Tahmin, Moment Metodu, En Küçük Kare, En Çok Olabilirlik.

Bilim Dalı Sayısal Kodu: 406.02.01

ACKNOWLEDGEMENTS

I am indebted to thank my Ph.D. supervisor, Prof. Dr. Taylan Ula, for suggesting the problem and for his continuous guidance throughout the course of this study.

I extend my appreciation to the examining committee and to the staff members at the Department of Statistics. The use of the computer facilities of the department is also appreciated.

I am also indebted to express my thanks to my parents, brothers, sisters, and my wife for their patience and support. Thanks are indeed to all my friends in Turkey and Jordan.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

ARMA	Autoregressive moving average
AR	Autoregressive
MA	Moving average
PARMA	Periodic autoregressive moving average
PAR	Periodic autoregressive
PMA	Periodic moving average
ω	Period
ν	Season
$p(\nu)$	Seasonal autoregressive order
$q(\nu)$	Seasonal moving average order
ϕ	Univariate autoregressive parameter
Φ	Multivariate autoregressive parameter
θ	Univariate moving average parameter
Θ	Multivariate moving average parameter
$\sigma_a^2(\nu)$	Univarite white noise variance
$\Sigma_a(\nu)$	Multivariate white noise varinace matrix
$\gamma_\ell(\nu)$	Univariate seasonal autocovariance function
$\Sigma_\ell(\nu)$	Multivariate seasonal autocovariance function
ACF	Autocorrelation function
PACF	Partial autocorrelation function
$\rho_\ell(\nu)$	Seasonal autocorrelation function
$r_\ell(\nu)$	Sample seasonal autocorrelation function
$\phi_{\ell\ell}(\nu)$	Seasonal partial autocorrelation function
$\hat{\phi}_{\ell\ell}(\nu)$	Sample seasonal partial autocorrelation function
LS	Least squares
ML	Maximum likelihood
RMSE	Root mean squared error

CHAPTER I

INTRODUCTION

1.1 Preliminaries

A time series $\{X_t;\ t \in \mathcal{T}\}$ can be defined as a collection of random variables which are ordered in time, where $\mathcal{T}$ denotes an index time points set.

In most statistical problems, we are concerned with estimating the properties of a population from a sample. In time series analysis, however, it is often impossible to have more than one observation at a given time, and the observed time series can be thought of as one realization of the parent time series. Thus, as there is a notional population, time series analysis is essentially concerned with evaluating the properties of the probability model which generated the observed time series. In the literature of time series analysis, modeling of stationary time series and transforming a non-stationary time series into a stationary one have taken a wide interest. An important class of models that are useful in fitting stationary time series is the widely known autoregressive moving average (ARMA) models (Box and Jenkins, 1976). In the next section, this class will be briefly discussed.

A time series is called second order stationary (or, covariance stationary) if its mean is constant and its second order moments are functions of time lag only. This type of stationarity is the most common version of stationarity and is often sufficient in practice. However, many seasonal time series cannot be filtered or standardized to achieve second order stationarity because the correlation structure of the time series may depend on the season. However, if it is assumed that the correlation structure depends on the season but not on the absolute time, then the time series is a periodically correlated process. A class of models useful in such situations is periodic autoregressive moving average (PARMA) models (Cleveland and Tiao, 1979; Tiao and Grupe, 1980), which are extensions of ARMA models that allow parameters to be periodic functions of time. This dissertation deals with the time domain analysis of PARMA processes. The definition and a review of literature for these models will be given in Section 1.3.

In the last section of this chapter, a brief summary of the succeeding chapters will be given.

1.2 ARMA Models

A time series $\{X_t\}$ is said to be a (mixed) autoregressive moving average process of order (p,q), denoted by ARMA(p,q), if it satisfies an equation of the form

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}, \qquad (1.1)$$

for all integers t. Here $\{a_t\}$ is a purely random (white noise) process. More precisely, $\{a_t\}$ is a sequence of uncorrelated and identically distributed random variables with zero mean and finite variance σ_a^2, which are usually assumed to be normally distributed, especially for inferential purposes. The real constants $\{\phi_1,\dots,\phi_p\}$ and $\{\theta_1,\dots,\theta_q\}$ are the autoregressive (AR) and moving average (MA) parameters, respectively. Furthermore, it is assumed in (1.1) without loss of generality that $E(X_t) = 0$, for all t. Nevertheless, if $E(X_t) = \mu \neq 0$, then we replace X_t by $X_t - \mu$.

Equation (1.1) may be written more compactly as

$$\Phi(B)X_t = \Theta(B)a_t, \qquad (1.2)$$

where $\Phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$, $\Theta(B) = 1 - \theta_1 B - \dots - \theta_q B^q$, in which B, the backwardshift operator, is defined as $B^j X_t = X_{t-j}$, j = 1, 2, ... Following (1.2), an ARMA(p,q) process is covariance stationary if and only if (iff) the roots of $\Phi(B) = 0$, with respect to B, lie outside the unit circle. It is also called invertible if it can be written as an infinite order AR process, which is satisfied iff the roots of $\Theta(B) = 0$ lie outside the unit circle (Box and Jenkins, 1976: 74).

An important extension of (1.1) is the multivariate ARMA(p,q) model. It is essentially obtained by replacing the scalar quantities in the univariate model by vector or matrix quantities. Employing the backwardshift operator, as in (1.2), an m-variate ARMA(p, q) model can be written as

$$\alpha(B)X_t = \beta(B)a_t. \tag{1.3}$$

Here, $X_t = (X_{t,1},\dots,X_{t,m})^T$, $\alpha(B) = I - \Phi_1 B - \dots - \Phi_p B^p$, $\beta(B) = I - \Theta_1 B - \dots - \Theta_q B^q$ and $\{\Phi_1,\dots,\Phi_p,\Theta_1,\dots,\Theta_q\}$ are m×m real matrices, where "T" stands for the transpose operator. The m-variate white noise process $\{a_t = (a_{t,1},\dots,a_{t,m})^T\}$ is a sequence of uncorrelated and identically distributed random vectors with $E(a_t) = 0$, $E(a_t a_t^T) = \Sigma_a$ and $E(a_t a_s^T) = 0$ for all $t \neq s$.

Following (1.3), an m-variate ARMA(p,q) process is covariance stationary iff all the roots of the determinantal polynomial $|\alpha(B)| = 0$ lie outside the unit circle, or equivalently iff all the roots λ of the determinantal equation

$$|\lambda^p I - \lambda^{p-1}\Phi_1 - \dots - \Phi_p| = 0 \tag{1.4}$$

are less than one in modulus (Fuller, 1976: 72), such that the modulus is defined for the complex number $c_1 + c_2 i$ as $(c_1^2 + c_2^2)^{1/2}$. A similar condition in terms of Θ's composes the invertibility conditions of the process.

Although the mixed ARMA model, including the multivariate version, involves only a finite number of parameters, it nevertheless possesses a remarkably wide range of applicability, and most stationary processes which arise in practice can usually be fitted by a model of this kind with suitably chosen values of p and q. Those models have been extensively studied in literature, and well-built methodologies for identification and estimation, especially for the univariate case, were developed and implemented in most statistical computer packages. For a more expository account, see, for example, Box and Jenkins (1976) for the univariate case, Hannan (1970) for the multivariate case, and Fuller (1976), Priestley (1981) and Wei (1990) for both.

1.3 PARMA Models

Time series which exhibit periodicity often arise in reality, for example, in economic and geophysical time series; see Bhuiya (1971), Salas (1974a, b) and Troutman (1978) for examples of the latter. A class of models which has been used widely in recent years for modeling univariate seasonal time series is that of multiplicative seasonal autoregressive integrated moving average (ARIMA) models (Box and Jenkins, 1976), which in turn is a nonstationary extension of ARMA models that can be transformed into stationary ARMA models through some appropriate differencing. A multiplicative seasonal ARIMA model, ARIMA$(p,d,q)\times(P,D,Q)_{\omega}$, is written as (Box and Jenkins, 1976: 305)

$$\phi_p(B)\Phi_P(B^{\omega})(1-B)^d(1-B^{\omega})^D X_t = \theta_q(B)\Theta_Q(B^{\omega})a_t,$$

where ω is the seasonal period, $\phi_p(B)$ and $\theta_q(B)$ are the regular autoregressive and moving average polynomials, and $\Phi_P(B^{\omega})$ and $\Theta_Q(B^{\omega})$ are the seasonal autoregressive and moving average polynomials, respectively, which are defined as $\Phi_P(B^{\omega}) = 1 - \Phi_1 B^{\omega} - \Phi_2 B^{2\omega} - \ldots - \Phi_P B^{P\omega}$ and $\Theta_Q(B^{\omega}) = 1 - \Theta_1 B^{\omega} - \Theta_2 B^{2\omega} - \ldots - \Theta_Q B^{Q\omega}$. Also here $(1-B)^d$ and $(1 - B^{\omega})^D$ are the regular and seasonal differencing operators, respectively. The quantities p, d and q are the ordinary autoregressive, differencing and moving average orders, and P, D and Q are their seasonal counterparts, respectively.

While ARIMA models have proved useful in practice, implicit in such models is the assumption of homogeneity, which means that the differenced series, $(1-B)^d(1-B^{\omega})^D X_t$, is stationary. However, in the analysis of series exhibiting a strong seasonal behavior, such a homogeneity assumption is sometimes clearly inappropriate (Tiao et al., 1975; Tiao et al., 1976). In such a situation, a possible substitute is the use of periodic time series models (Monin, 1963; Jones and Brelsford, 1967; Pagano, 1978; Cleveland and Tiao, 1979).

Prior to the discussion of the PARMA model, it is worth mentioning that the seasonal ARIMA model is completely different from the PARMA model in the following two senses: firstly, an appropriate differencing of the time series represented by a seasonal ARIMA model makes it stationary, while this is not true for a PARMA model where the differenced series is again of the PARMA type, and secondly, the seasonal ARIMA model does not allow for periodic parameters.

From a mathematical point of view, a function f(t) with domain $\mathcal{T}$ is said to be periodic if there exists a positive number ω, the period, such that, for all t, $t+\omega$ belongs to $\mathcal{T}$, and $f(t+\omega) = f(t)$ (Fuller, 1976: 12).

In time series terminology, an m-variate time series $\{X_t\}$, t belonging to the set of all integers, is said to be periodic stationary in the strict sense if the joint probability density function (pdf) of X_{t_1}, ..., X_{t_n} is the same as the joint pdf of $X_{t_1+k\omega}$, ..., $X_{t_n+k\omega}$ for any n time points $(t_1,\ldots,t_n)$, any integer k, and some positive integer ω, named as the period. In the wide sense, assuming that the second order moments of $\{X_t\}$ exist, $\{X_t\}$ is called a periodic covariance stationary process of period ω, if the elements of the mean vector and autocovariance matrix of X_t are finite and periodic with period ω, i.e. $E(X_t) = \mu_t = \mu_{t+k\omega}$ and $E([X_s - \mu_s][X_t - \mu_t]^T) = R(s,t) = R(s+k\omega,t+k\omega)$, for all integers s, t and k. In the remaining part of this thesis, the terms covariance stationarity and periodic covariance stationarity are abbreviated to stationarity and periodic stationarity, respectively.

Now we define the m-variate ω-period PARMA model of varying orders $(p(1),q(1);p(2),q(2);\ldots;p(\omega),q(\omega))$, denoted by $\text{PARMA}_\omega(p(\nu),\ q(\nu))$, for m = 1, 2, ..., ω = 2, 3, ..., ν = 1, ..., ω, and for all integers k as

$$X_{k\omega+\nu} = \Phi_1^{(\nu)} X_{k\omega+\nu-1} + \ldots + \Phi_{p(\nu)}^{(\nu)} X_{k\omega+\nu-p(\nu)} + a_{k\omega+\nu} - \Theta_1^{(\nu)} a_{k\omega+\nu-1} - \ldots - \Theta_{q(\nu)}^{(\nu)} a_{k\omega+\nu-q(\nu)} \qquad (1.5)$$

where $\{a_{k\omega+\nu}\}$ is an m-variate white noise process with $E(a_{k\omega+\nu}) = 0$, $E(a_{k\omega+\nu} a_{k\omega+\nu}^T) = \Sigma_a(\nu)$ and $E(a_t a_s^T) = 0$ for all $t \neq s$. It is also assumed in this model, for all k and ν, that $E(X_{k\omega+\nu}) = 0$. If, however, $E(X_{k\omega+\nu}) = \mu_\nu \neq 0$, then without loss of generality $X_{k\omega+\nu}$ is replaced by $X_{k\omega+\nu} - \mu_\nu$. The parameters, μ_ν, $\Phi_1^{(\nu)}$, ..., $\Phi_{p(\nu)}^{(\nu)}$, $\Theta_1^{(\nu)}$, ..., $\Theta_{q(\nu)}^{(\nu)}$ and $\Sigma_a(\nu)$, are also assumed to be periodic with period ω. For m = 1 in (1.5), we have the univariate case, and $\omega = 1$ gives the ARMA(p,q) model. If $p(1) = ... = p(\omega) = p$ and $q(1) = ... = q(\omega) = q$, the resulting PARMA model is said to have constant orders and, thence, will be denoted by $\text{PARMA}_\omega(p,q)$. If, in addition, $p(1) = ... = p(\omega) = 0$ $[q(1) = ... = q(\omega) = 0]$, then the resulting model is said to be a pure periodic autoregressive [moving average] model with period ω and denoted by $\text{PAR}_\omega(p(\nu))$ $[\text{PMA}_\omega(q(\nu))]$. Furthermore, if the white noise terms are independent and normally distributed, then the resulting PARMA process is a Gaussian PARMA process.

Note that in (1.5) the time index is written as $k\omega + \nu$ rather than t to show that it obeys modulo-ω arithmetic, so that for monthly data, say, k + 1 represents the year and ν represents the season, $\nu = 1, ..., 12$, so that, for example, the time point of 23 refers to year 2 and season 11 and that of 36 refers to the last season of year three. In the subscripts of X and a in (1.5), the terms $\nu - \ell$, $\ell = 1, 2, ...$, also represent different seasons which again must be between 1 and ω. Therefore, if $-\omega < \nu - \ell \leq 0$, say, then the subscript $k\omega + \nu - \ell$ can be rewritten as $(k-1)\omega + \omega + \nu - \ell$, so that this time point belongs to season $\omega + \nu - \ell$, which is between 1 and ω, but in the previous, say, year. This, for example, means that if $\nu - \ell = 0$, then this time point belongs to season ω, and $\nu - \ell = -1$ correponds to season $\omega - 1$, etc.Also, it is to be understood from (1.5) that for the k-th, say, year, the model is essentially represented by ω equations, such that ν-th equation corresponds to season ν.

Suppose that $\{X_{k\omega+\nu}\}$ is an m-variate periodic process. Gladyshev (1961) proved that this process is periodically stationary iff the $m\omega$-variate vector process

$$Y_k = (X_{k\omega+1}^T, ..., X_{k\omega+\omega}^T)^T \tag{1.6}$$

is stationary. Hereinafter, $\{Y_k\}$ is referred to as the lumped-vector process. Tiao and Grupe (1980) proved that the lumped-vector process corresponding to the univariate $PARMA_{\omega}(p(\nu),q(\nu))$ process follows an ω-variate $ARMA(p^*,q^*)$ model with p^* and q^* as given by

$$p^* = \max_{\nu} \{[(p(\nu)-\nu)/\omega] + 1\}$$
$$q^* = \max_{\nu} \{[(q(\nu)-\nu)/\omega] + 1\} \tag{1.7}$$

where [c] denotes the integral part of the real number c. This result is also valid for the m-variate PARMA process, in which case the lumped-vector process follows an $m\omega$-variate $ARMA(p^*,q^*)$ model. Vecchia (1985b) elaborated on the work of Tiao and Grupe (1980). He obtained the lumped-vector representation for the univariate varying orders PARMA model, and expressed its periodic stationarity conditions in terms of this representation. This result is generalized to the multivariate case in the next chapter, which is devoted to the discussion of periodic stationarity of univariate and multivariate PARMA processes. Obeysekera and Salas (1986) also obtained the periodic stationarity conditions of univariate $PAR_{\omega}(1)$ and $PARMA_{\omega}(1,1)$ processes from the stationarity conditions of the lumped-vector process. Latter on, their result was generalized to the multivariate case by Ula (1990).

An important implication of the periodic stationarity of an m-variate PARMA process, $\{X_{k\omega+\nu}\}$, is that it can be represented in the general linear form

$$X_{k\omega+\nu} = \sum_{j=0}^{\infty} \Psi_j^{(\nu)} a_{k\omega+\nu-j}, \tag{1.8}$$

where $\sum_{j=0}^{\infty} |\Psi_j^{(\nu)}| < \infty$, for all ν and k. The absolute summability of the m×m matrices $\{\Psi_j^{(\nu)}\}$ here is component-wise. This representation is valid in the mean square sense (Fuller, 1976: 72; Anderson and Vecchia, 1993). It is found helpful for forecasting PARMA processes (Ula, 1993), but may not provide much help for identification or estimation purposes.

Another approach for studying the PARMA model has been through its corresponding aggregated process. For an m-variate periodic process $\{X_{k\omega+\nu}\}$, we define the corresponding aggregated process, $\{W_k\}$, as

$$W_k = \sum_{\nu=1}^{\omega} X_{k\omega+\nu}, \tag{1.9}$$

for all integers k. It follows from Vecchia et al. (1983) that for univariate $PAR_\omega(1)$ and $PARMA_\omega(1,1)$ models, stationarity of the aggregated process implies periodic stationarity of the periodic process, and vice versa. Again, this result was generalized to the multivariate case by Ula (1990). He also proved that periodic stationarity of any periodic process implies stationarity of its corresponding aggregated process. Till then the reverse was proved to be true only for univariate and multivariate $PAR_\omega(1)$ and $PARMA_\omega(1,1)$ processes. However, in the next chapter, we show that the reverse is not always true. That is, stationarity of the aggregated process does not always imply periodic stationarity of the periodic process.

Although PARMA models are not as popular as ARMA or ARIMA models, they, nevertheless, proved to be useful in practice. For instance, they have found applications in modeling hydrological time series (Delleur et al., 1976; Salas et al., 1980; Vecchia, 1985b) and in signal processing (Sakai, 1982). Besides, the PARMA model provides a more natural mechanism to fit the periodic structure of a seasonal time series.

An inspection of the literature of PARMA models, especially the multivariate case, reveals the fact that it is much less than that of univariate and multivariate ARMA models. In an early paper, Jones and Brelsford (1967) considered moment estimation and prediction of periodic autoregressive models. Salas (1972) and Salas and Pegram (1979) obtained the moment equations for univariate and multivariate $PAR_\omega(p)$ models, respectively, and expressed covariance and correlation functions (matrices in the multivariate case) in terms of parameters of these models. Sakai (1982) studied partial autocorrelations of PAR processes. Pagano (1978) investigated statistical properties of moment estimators in univariate PAR models. For the same models, Troutman (1979) investigated their covariance properties, and represented a PAR model as an infinite order AR process. The results in the previous two papers were developed based on the lumped-vector representation. Salas et al. (1982) investigated correlation properties and moment estimation of univariate $PMA_\omega(1)$, $PARMA_\omega(1,1)$ and $PARMA_\omega(2,1)$ models. Vecchia (1985a, b) developed an algorithm for maximum likelihood estimation for univariate PARMA processes. Bartolini et al.

(1988) discussed moment estimation and aggregation in multivariate $PARMA_{\omega}(1,1)$ model. Vecchia and Ballerini (1991) proposed some statistical procedures to detect the existence of periodicity in the autocorrelation function of a general periodic process. Recently, Anderson and Vecchia (1993) developed some asymptotic results for the sample seasonal autocorrelation function of univariate PARMA processes. Ula (1993) investigated forecasting in multivariate PARMA models.

1.4 Aims and Scope of the Study

The aim of this study is to contribute to the theory and analysis of PARMA processes in various ways. Some contributions were already mentioned in the previous sections and more are mentioned below as we give a scope of the study. A detailed summary of contributions and results is given in Chapter V.

Since PARMA models are relatively new, many questions about them still have to be answered. An important problem is the determination of periodic stationarity conditions of PARMA processes, which is a prerequisite to their analysis. This is the subject of Chapter II. There, we obtain the lumped-vector representation of multivariate PARMA models by which we relate the parameters of the periodic process to the parameters of its corresponding lumped-vector process. Then, based on this representation, we simplify the periodic stationarity conditions to an eigenvalue problem. In the same chapter, as we have mentioned earlier, we show that stationarity of the aggregated process corresponding to a PARMA process does not imply that the later is periodic stationarity. The relations between periodic stationarity of PARMA processes and their covariance structure are also investigated.

In Chapter III, we discuss identification of PARMA models. It is shown, by analogy to ARMA models, that seasonal autocorrelation function (ACF) and partial autocorrelation function (PACF) play a primary role for the identification of PARMA models similar to the ACF and PACF in the context of ARMA models identification. In addition, by analogy to the cut-off properties of ACF and PACF in the case of pure MA and AR processes, respectively, similar cut-off properties of their seasonal counterparts are utilized. Approximate formulas for the first and second order moments of the sample seasonal ACF are developed, and similar available results for the sample seasonal PACF are utilized, and some asymptotic bands are developed for the assessment of such cut-off situations from the sample counterparts of these seasonal functions. Finally, the applicability of these results are investigated through simulation.

In Chapter IV, we consider estimation of PARMA models. It is shown there that in the context of PARMA models with a MA part, the method of moments is technically difficult and may give unsatisfactory results. Then this method together with conditional least squares (LS) method are studied for PAR models. Besides, we gain some insight into the likelihood function of PARMA model. Both the exact and conditional likelihood functions are obtained for $PAR_{\omega}(1)$ and $PMA_{2}(1)$ models. In view of these cases, the relations between the conditional maximum likelihood (ML) approach and the previous two approaches are also investigated. Furthermore, a comparison between these methods is carried out through simulation for univariate and bivariate PAR models. In this thesis, methods for solving non-linear equations or optimization techniques are not considered.

At the end, in Chapter V we summarize our findings and suggest some problems that deserve further research and investigation.

CHAPTER II

PERIODIC STATIONARITY OF PARMA PROCESSES

2.1 Introduction

In the previous chapter, we defined univariate and multivariate PARMA processes. We also pointed out that investigating periodic stationarity of these processes must precede their analysis.

In fact, the assumption of periodic stationarity imposes a set of conditions upon the autoregressive parameters. These conditions, however, are not easily stated in terms of the PARMA model as expressed by (1.5), and it is necessary to consider other approaches to obtain them. In Chapter I, we cited two approaches, namely the lumped-vector and the aggregated process approaches. In the following section, we will deduce periodic stationarity conditions for any m-variate PARMA process through the former approach and then simplify them to an eigenvalue problem. On the other hand, latter in this chapter, it will be shown that the aggregated process approach is not acceptable for this purpose.

Furthermore, we will consider the problem of imposing some constraints on the seasonal autocovariance matrices and the resulting constraints on the parameters of PAR processes in the sense of periodic stationarity. In this context, an available result of Troutman (1979) for the univariate $PAR_\omega(1)$ model is generalized to the multivariate case.

2.2 Lumped-Vector Representation of PARMA Processes

Consider the univariate $PARMA_\omega(p(\nu),q(\nu))$ model defined by (1.5) for m = 1. Then the ω-variate lumped-vector process $\{Y_k\}$ defined by (1.6) follows a ω-variate ARMA(p^*,q^*) model, written as

$$LY_k - \sum_{\ell=1}^{p^*} U_\ell Y_{k-\ell} = \Lambda\varepsilon_k - \sum_{\ell=1}^{q^*} V_\ell \varepsilon_{k-\ell}, \tag{2.1}$$

where p^* and q^* are as defined by (1.7), L is $\omega\times\omega$ with

$$(L)_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i < j \\ -\phi^{(i)}_{i-j} & , i > j \end{cases}$$

and U_ℓ is $\omega\times\omega$ with $(U_\ell)_{ij} = \phi^{(i)}_{\ell\omega+i-j}$ for i, j = 1, ..., ω, and Λ and V_ℓ have the same form as L and U_ℓ, respectively, but with θ replacing ϕ, and $\varepsilon_k = (a_{k\omega+1},\dots,a_{k\omega+\omega})^T$. The conventions $\phi^{(\nu)}_j = 0$, for all $j > p(\nu)$, and $\theta^{(\nu)}_j = 0$, for all $j > q(\nu)$, are assumed . The representation in (2.1) was given by Vecchia (1985b) and originally developed by Tiao and Grupe (1980).

The result above generalizes to the m-variate PARMA$_\omega$(p(ν),q(ν)) model such that $\{Y_k\}$ is an mω-variate ARMA(p^*,q^*) process written as (2.1) with p^* and q^* again as defined by (1.7). But L is $m\omega\times m\omega$ with

$$[L]_{ij} = \begin{cases} I_m & , i = j \\ 0_m & , i < j \\ -\Phi^{(i)}_{i-j} & , i > j \end{cases} \tag{2.2}$$

and U_ℓ is also $m\omega\times m\omega$ with

$$[U_\ell]_{ij} = \Phi^{(i)}_{\ell\omega+i-j}, \tag{2.3}$$

for i, j = 1, ..., ω, where $[A]_{ij}$ denotes the (i,j)th m×m sub-matrix of A, I_m is the m×m identity matrix, and 0_m is the m×m null matrix. Further, Λ and V_ℓ have the same form as L and U_ℓ, respectively, but in terms of Θ's, and the conventions $\Phi^{(\nu)}_j = 0_m$, for all $j > p(\nu)$ and $\Theta^{(\nu)}_j = 0_m$, for all $j > q(\nu)$, are also adopted. Also, $\varepsilon_k = (a^T_{k\omega+1},\dots,a^T_{k\omega+\omega})^T$.

We illustrate the lumped-vector representation of multivariate PARMA models through the following example.

EXAMPLE 2.1. Consider an m-variate $PARMA_3(2,0;2,1;1,2)$ model, which is defined, following (1.5), as

$$X_{3k+1} = \Phi_1^{(1)}X_{3(k-1)+3} + \Phi_2^{(1)}X_{3(k-1)+2} + a_{3k+1}$$

$$X_{3k+2} = \Phi_1^{(2)}X_{3k+1} + \Phi_2^{(2)}X_{3(k-1)+3} + a_{3k+2} - \Theta_1^{(2)}a_{3k+1}$$

$$X_{3k+3} = \Phi_1^{(3)}X_{3k+2} + a_{3k+3} - \Theta_1^{(3)}a_{3k+2} - \Theta_2^{(3)}a_{3k+1}$$

for all integers k.

It follows from (1.7) that $p^* = 1$ and $q^* = 0$, then, by (2.1), $Y_k = (X_{3k+1}^T, X_{3k+2}^T, X_{3k+3}^T)^T$ follows a 3m-variate AR(1) model, written as $LY_k - U_1Y_{k-1} = \Lambda\varepsilon_k$, where, by (2.2) and (2.3),

$$L = \begin{pmatrix} I_m & 0_m & 0_m \\ -\Phi_1^{(2)} & I_m & 0_m \\ 0_m & -\Phi_1^{(3)} & I_m \end{pmatrix}, \; U_1 = \begin{pmatrix} 0_m & \Phi_2^{(1)} & \Phi_1^{(1)} \\ 0_m & 0_m & \Phi_2^{(2)} \\ 0_m & 0_m & 0_m \end{pmatrix}, \; \Lambda = \begin{pmatrix} I_m & 0_m & 0_m \\ -\Theta_1^{(2)} & I_m & 0_m \\ -\Theta_2^{(3)} & -\Theta_1^{(3)} & I_m \end{pmatrix},$$

and $\varepsilon_k = (a_{3k+1}^T, a_{3k+2}^T, a_{3k+3}^T)^T$. This representation can also be verified from the three equations given at the beginning of the example. ■

Therefore, having any m-variate PARMA model in hand, it is not difficult to obtain its lumped-vector representation, (2.1). In the next section, this representation will be utilized to obtain the periodic stationarity conditions of any PARMA process.

2.3 Periodic Stationarity of Multivariate PARMA Processes As an Eigenvalue Problem

In this section and the next one, we refer to some results from matrix theory which we summarize in the next lemma. The proofs of these results can be found in most standard books on matrix theory as, for example, Graybill (1983).

LEMMA 2.1. Let A and B be any two m×m matrices, then
(i) if A is symmetric and for all non-zero m×1 vectors C, $C^TAC > 0$, then A

is positive definite (p.d.), and if, for all C, $C^TAC \geq 0$ with strict equality for at least one non-zero vector C, then A is positive semi-definite (p.s.d.).

(ii) If A is triangular, then its eigenvalues are its diagonal elements, and if these elements are all non-zero, then A is non-singular.

(iii) If A is p.d., then there exists a matrix B such that $A = B^2$, and B is also p.d.

(iv) If A is p.d. and B is non-singular, then both BAB^T and BAB^{-1} are p.d. and the eigenvalues of BAB^{-1} are the same as those of A.

The matrix L, defined by (2.2), is a lower triangular matrix and its diagonal elements are all equal to one. Thus, by part (ii) of Lemma 2.1, it is a non-singular matrix. Hence, multiplying (2.1) by L^{-1} we obtain

$$Y_k - \sum_{\ell=1}^{p^*} L^{-1}U_\ell Y_{k-\ell} = \varepsilon_k^* - \sum_{\ell=1}^{q^*} V_\ell^* \varepsilon_{k-\ell}^*, \qquad (2.4)$$

where $\varepsilon_k^* = L^{-1}\Lambda\varepsilon_k$ and $V_\ell^* = L^{-1}V_\ell\Lambda^{-1}L$.

Now, comparing (2.4) with (1.3), equation (1.4) implies that $\{Y_k\}$ is stationary iff all the roots of the determinantal equation

$$|\lambda^{p^*}I - \lambda^{p^*-1}L^{-1}U_1 - \ldots - L^{-1}U_{p^*}| = 0 \qquad (2.5)$$

are less than one in modulus. These conditions in turn are the periodic stationarity conditions of the periodic process $\{X_{k\omega+\nu}\}$ due to Gladyshev (1961), as mentioned in Chapter I. In particular, if $p^* = 1$, these conditions are functions of the eigenvalues of $L^{-1}U_1$, that is, these eigenvalues must be less than one in modulus.

If $p^* > 1$, the problem of checking whether a specific PARMA model is periodic stationary or not involves obtaining the roots of a complicated determinantal equation, (2.5). We can, however, overcome this problem by making use of the next lemma. It is noted that the statement of this lemma is found, for example, in Barone (1987), and proved for $p^* = 2$ (Fuller, 1976: 50), but no general proof was provided. Therefore, we give a general proof of it for sake of completeness.

LEMMA 2.2. The condition that all the roots of (1.4) are less than one in modulus is equivalent to the condition that all the eigenvalues of the matrix M defined by

$$M_{mp} = \left[\begin{array}{c|c} \Phi_1 & \\ \Phi_2 & \\ \vdots & I_{(p-1)m} \\ \Phi_{p-1} & \\ \hline \Phi_p & 0_{m\times(p-1)m} \end{array}\right]$$

are less than one in modulus.

PROOF. It is sufficient to show that

$$|\lambda I_{mp} - M| = |\lambda^p I_m - \lambda^{p-1}\Phi_1 - \dots - \lambda\Phi_{p-1} - \Phi_p|.$$

For this, note that the mp×mp matrix $\lambda I_{mp} - M$, which we denote by A, is given by

$$A = \begin{bmatrix} \lambda I_m - \Phi_1 & -I_m & 0_m & \dots & 0_m \\ -\Phi_2 & \lambda I_m & -I_m & \dots & 0_m \\ -\Phi_3 & 0_m & \lambda I_m & \dots & 0_m \\ \vdots & \vdots & \vdots & & \vdots \\ -\Phi_{p-1} & 0_m & 0_m & \dots & -I_m \\ -\Phi_p & 0_m & 0_m & \dots & \lambda I_m \end{bmatrix}.$$

If $\lambda = 0$, it can be easily shown that $|A| = (-1)^{2(p-1)m}|-\Phi_p| = |-\Phi_p|$ which, in this case, is the same as the right hand side of the above determinantal equation. For non-zero λ, we partition A as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right],$$

where $A_{11} = \lambda I_m - \Phi_1$, and the other submatrices are defined accordingly. Then

$$A_{22}^{-1} = \begin{bmatrix} (1/\lambda)I_m & (1/\lambda)^2 I_m & \cdots & (1/\lambda)^{p-1} I_m \\ 0_m & (1/\lambda)I_m & \cdots & (1/\lambda)^{p-2} I_m \\ 0_m & 0_m & \cdots & (1/\lambda)^{p-3} I_m \\ \vdots & \vdots & & \vdots \\ 0_m & 0_m & \cdots & (1/\lambda)I_m \end{bmatrix},$$

$A_{12}A_{22}^{-1}A_{21} = (1/\lambda)\Phi_2 + (1/\lambda)^2\Phi_3 + \ldots + (1/\lambda)^{p-1}\Phi_p$, and $|A_{22}| = \lambda^{(p-1)m}$. Then utilizing the fact that $|A| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{21}|$ leads to the desired result. ∎

Note that the above lemma expresses the stationarity conditions of m-variate AR(p) process as an eigenvalue problem. Thus comparing (2.4) with (1.3) and utilizing Lemma 2.2 result in the following proposition.

PROPOSITION 2.1. An m-variate $PARMA_{\omega}(p(\nu),q(\nu))$ process is periodic stationary iff all the eigenvalues of the matrix

$$R_{m\omega p^*} = \left[\begin{array}{c|c} L^{-1}U_1 & \\ L^{-1}U_2 & \\ \vdots & I_{(p^*-1)m\omega} \\ L^{-1}U_{p^*-1} & \\ \hline L^{-1}U_{p^*} & 0_{m\omega\times(p^*-1)m\omega} \end{array}\right] \qquad (2.6)$$

are less than one in modulus, where p^*, L and U_ℓ, $\ell = 1, \ldots, p^*$, are as defined by (1.7), (2.2) and (2.3), respectively.

The main conclusion that can be drawn from Proposition 2.1 is that periodic stationarity conditions of any PARMA model are expressible as an eigenvalue problem, which, for instance, is easy to solve via computer. In fact, this idea is implemented in the computer program in Appendix A which computes the eigenvalues of the matrix R in (2.6) for any input m-variate PARMA model.

As an application of Proposition 2.1, we consider the following example.

EXAMPLE 2.2. Consider an m-variate $\text{PARMA}_\omega(1,1)$ model, written as

$$X_{k\omega+\nu} = \Phi_1^{(\nu)} X_{k\omega+\nu-1} + a_{k\omega+\nu} - \Theta_1^{(\nu)} a_{k\omega+\nu-1}. \tag{2.7}$$

Then, by (1.7), $p^* = 1$ and $q^* = 1$, and following (2.2) and (2.3), we have

$$L = \begin{bmatrix} I_m & 0_m & \cdots & 0_m & 0_m \\ -\Phi_1^{(2)} & I_m & \cdots & 0_m & 0_m \\ 0_m & -\Phi_1^{(3)} & \cdots & 0_m & 0_m \\ \vdots & \vdots & & \vdots & \vdots \\ 0_m & 0_m & \cdots & I_m & 0_m \\ 0_m & 0_m & \cdots & -\Phi_1^{(\omega)} & I_m \end{bmatrix}, \quad U_1 = \left[\begin{array}{c|c} & \Phi_1^{(1)} \\ & 0_m \\ 0_{m\omega\times m(\omega-1)} & \vdots \\ & 0_m \end{array}\right].$$

It can be shown that

$$L^{-1} = \begin{bmatrix} I_m & 0_m & \cdots & 0_m & 0_m \\ \Phi_1^{(2)} & I_m & \cdots & 0_m & 0_m \\ \Phi_1^{(3)}\Phi_1^{(2)} & \Phi_1^{(3)} & \cdots & 0_m & 0_m \\ \vdots & \vdots & & \vdots & \vdots \\ \Phi_1^{(\omega)}\cdots\Phi_1^{(2)} & \Phi_1^{(\omega)}\cdots\Phi_1^{(3)} & \cdots & \Phi_1^{(\omega)} & I_m \end{bmatrix}.$$

Note that, in general, the elements of L^{-1} may be obtained recursively by utilizing $L^{-1}L = I$ and the fact that L is lower triangular.

It then follows that

$$L^{-1}U_1 = \left[\begin{array}{c|c} & \Phi_1^{(1)} \\ & \Phi_1^{(2)}\Phi_1^{(1)} \\ 0_{m\omega\times m(\omega-1)} & \vdots \\ & \Phi_1^{(\omega)}\Phi_1^{(\omega-1)}\cdots\Phi_1^{(1)} \end{array}\right]$$

which is an upper triangular partitioned form whose eigenvalues are those of $\Phi_1^{(\omega)}\Phi_1^{(\omega-1)}\cdots\Phi_1^{(1)}$ (Ula, 1990). Hence, by Proposition 2.1, an m-variate $\text{PARMA}_\omega(1,1)$ model is periodic stationary iff all the eigenvalues of the matrix $\Phi_1^{(\omega)}\cdots\Phi_1^{(1)}$ are less than one in modulus. This result was also obtained through an explicit derivation of the lumped-vector representation by Ula (1990). ■

Another important application of Proposition 2.1 is to investigate invertibility conditions of PARMA models. In fact, by analogy to ARMA models, invertibility conditions of any PARMA model are functions of its MA parameters only. Moreover, keeping in mind that in (2.1) Λ and V_ℓ have, in terms of Θ's, the same form as L and U_ℓ, respectively, an m-variate $PARMA_\omega(p(\nu),q(\nu))$ process is invertible iff all the eigenvalues of the matrix C are less than one in modulus, where C is the same as R, which is defined by (2.6), but with Λ and V_ℓ in place of L and U_ℓ, respectively. As an illustration, reconsider Example 2.2. Note that Λ and V_1 are identical with L and U_1, respectively, but with Θ in place of Φ. Thus, the m-variate $PARMA_\omega(1,1)$ process is invertible iff all the eigenvalues of $\Theta_1^{(\omega)}...\Theta_1^{(1)}$ are less than one in modulus. On this basis, the program listed in Appendix A can also be used to check whether a specific PARMA process is invertible or not.

2.4 Periodic Stationarity of PAR Processes and Their Autocovariance Function

We have seen in the previous section that periodic stationarity of any PARMA process imposes a set of constraints, summarized in Proposition 2.1, on its AR parameters only. This result, in fact, is in complete analogy with stationarity of ARMA processes. In the literature of ARMA processes, the relation between stationarity of these processes and their autocovariance (or autocorrelation) function is an important question. In this section, this problem is investigated in the context of PARMA models.

Let $\{X_{k\omega+\nu}\}$ be a periodic time series which follows a PARMA model and let

$$\Sigma_\ell(\nu) = \mathrm{Cov}(X_{k\omega+\nu}, X_{k\omega+\nu-\ell}) \qquad (2.8)$$

denote the autocovariance matrix for season ν at backward lag ℓ, for all $\nu = 1, ..., \omega$ and non-negative integers ℓ, which is periodic with period ω, that is, $\Sigma_\ell(\nu) = \Sigma_\ell(\nu + r\omega)$ for all integers r. Constraints on those autocovariance matrices are of particular interest. Troutman (1979) investigated such constraints for the univariate $PAR_\omega(1)$ model. In this case, the periodic stationarity condition is that $abs(\Pi_{\nu=1}^{\omega} \phi_1^{(\nu)}) < 1$, which is the univariate version of the result in Example 2.2. Letting $\gamma_0(\nu)$ and $\sigma_a^2(\nu)$ denote the variances of $X_{k\omega+\nu}$ and $a_{k\omega+\nu}$, respectively, he proved, under the assumption $\sigma_a^2(\nu) > 0$ for all ν, that imposing the condition $\gamma_0(1) = \ldots = \gamma_0(\omega) = 1$ results in additional constraints on the ϕ's, namely $abs(\phi_1^{(\nu)}) < 1$ for all ν.

Now we generalize Troutman's result to the multivariate case for equal, but not necessarily identity, autocovariance matrices. The following lemma is needed for this generalization.

LEMMA 2.3. Let A be any m×m square matrix having, in general, complex eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_m$, then

(i) If $0 \le \gamma_1^2 \le \gamma_2^2 \le \ldots \le \gamma_m^2$ are the eigenvalues of AA^T, then $\gamma_1^2 \le \lambda_i\bar{\lambda}_i \le \gamma_m^2$ for all i = 1, ..., m.

(ii) If the matrix $E = I - AA^T$ is p.d., then $mod(\lambda_i) < 1$ for all i.

(iii) If C is any p.d. matrix and the matrix $F = (C - ACA^T)$ is p.d., then $mod(\lambda_i) < 1$ for all i.

Here $\bar{\lambda}_i$ and $mod(\lambda_i)$ denote the complex conjugate and modulus of the complex number λ_i, respectively.

PROOF.

(i) For this, see (Householder, 1953: 146).

(ii) Let $0 < \delta_1 \le \delta_2 \le \ldots \le \delta_m$ be the eigenvalues of E. Since $|E - \delta_i I_m| = |AA^T - (1-\delta_i)I_m|$, we have, for all i = 1, ..., m, $\delta_i = 1 - \gamma_{m+1-i}^2$. Also, as AA^T is p.s.d., $\gamma_i^2 \ge 0$ for all i, and by assumption, $\delta_i > 0$. Thus, $0 \le \gamma_i^2 < 1$ for all i, which, by part (i) and utilizing the fact that $\lambda_i\bar{\lambda}_i = (mod(\lambda_i))^2$, gives the desired result.

(iii) Let $D = C^{1/2}$, which, by part (iii) of Lemma 2.1, is p.d. Then, by assumption, $F = D(I_m - (D^{-1}AD)(D^{-1}AD)^T)D$ is p.d. Also, let $W = D^{-1}AD$, and $R = I - WW^T$, then F = DRD. This implies that $R = D^{-1}F(D^{-1})^T$. Hence, by part (iv) of Lemma 2.1, R is p.d. Note that R satisfies the conditions of part (ii), so that, the eigenvalues of W are less than one in modulus.

But, since $W = D^{-1}AD$, then, by part (iv) of Lemma 2.1, the eigenvalues of A are the same as those of W and the result is proved. ■

In the following theorem, the result of Troutman (1979), previously explained, is generalized to the multivariate $PAR_\omega(1)$ model:

$$X_{k\omega+\nu} = \Phi_1^{(\nu)} X_{k\omega+\nu-1} + a_{k\omega+\nu}. \qquad (2.9)$$

PROPOSITION 2.2. If an m-variate $PAR_\omega(1)$ process is periodic stationary, and if $\Sigma_0(1) = \dots = \Sigma_0(\omega) = \Sigma_0$, and Σ_0 and $\Sigma_a(\nu)$, $\nu = 1, \dots, \omega$, are p.d., then, for all ν, the eigenvalues of $\Phi_1^{(\nu)}$ are less than one in modulus, where $\Sigma_0(\nu)$ is as defined by (2.8) and $\Sigma_a(\nu)$ is the variance-covariance matrix of $a_{k\omega+\nu}$.

PROOF. Making use of the periodic stationarity assumption, and post-multiplying (2.9) by $X^T_{k\omega+\nu}$ and $X^T_{k\omega+\nu-1}$, and then taking expectations, give, respectively, $\Sigma_0(\nu) = \Phi_1^{(\nu)}\Sigma_1^T(\nu) + \Sigma_a(\nu)$, and $\Sigma_1(\nu) = \Phi_1^{(\nu)}\Sigma_0(\nu-1)$. These equations imply that $\Sigma_a(\nu) = \Sigma_0(\nu) - \Phi_1^{(\nu)}\Sigma_0(\nu-1)(\Phi_1^{(\nu)})^T$, which, together with the assumptions of the theorem and part (iii) of Lemma 2.3 give the desired result. ■

We have seen in Example 2.2 that an m-variate $PAR_\omega(1)$ model is periodic stationary iff the eigenvalues of the matrix $\Phi_1^{(\omega)}\dots\Phi_1^{(1)}$ are all less than one in modulus. In Proposition 2.2, we impose further conditions on this model, which in turn impose more constraints on the Φ's, namely the eigenvalues of each $\Phi_1^{(\nu)}$ must be less than one in modulus.

It can be noted that if all the Φ's are diagonal (which includes the univariate case) or triangular (but all of same type, i.e. either lower or upper triangular), then these additional constraints imply the original condition that all the eigenvalues of $\Phi_1^{(\omega)}\dots\Phi_1^{(1)}$ are less than one in modulus. This is due to the fact that if all the Φ's are lower triangular, say, then their product is also lower triangular with diagonal elements (which are the same as eigenvalues) as the product of the corresponding diagonal elements (eigenvalues) of the individual Φ's. However, in the general case, this is not necessarily true. For this, consider a bivariate $PAR_2(1)$ model with

$$\Phi_1^{(1)} = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}, \quad \Phi_1^{(2)} = \begin{pmatrix} 0.5 & 0 \\ 1 & 0.5 \end{pmatrix}.$$

The eigenvalues of each $\Phi_1^{(1)}$ and $\Phi_1^{(2)}$, (0.5, 0.5), are less than one in modulus, whereas those of $\Phi_1^{(2)}\Phi_1^{(1)}$, (1.457, 0.043), are not.

In practice, the assumptions of Proposition 2.2 may apply, for instance, if we perform a seasonal standardization on the model, that is, if $E(X_{k\omega+\nu}) = \mu_\nu$ and $Var(X_{k\omega+\nu}) = \Sigma_0(\nu)$, then $X_{k\omega+\nu}$ is standardized as $[\Sigma_0(\nu)]^{-1/2}(X_{k\omega+\nu} - \mu_\nu)$, which has a zero mean vector and an identity variance-covariance matrix.

Now we turn to another related problem. It is known in the context of multivariate ARMA models that stationarity of any vector ARMA process implies positive definiteness of the matrix

$$\begin{bmatrix} \Gamma_0 & \Gamma_1^T & \cdots & \Gamma_{N-1}^T \\ \Gamma_1 & \Gamma_0 & \cdots & \Gamma_{N-2}^T \\ \vdots & \vdots & & \vdots \\ \Gamma_{N-1} & \Gamma_{N-2} & \cdots & \Gamma_0 \end{bmatrix},$$

where Γ_ℓ is the autocovariance matrix for lag ℓ, and N = 1, 2, ... (Fuller, 1976: 15). This result can be easily extended to PARMA processes by utilizing the lumped-vector approach, discussed in Section 2.2. More precisely, we have seen that corresponding to an m-variate periodic stationary PARMA process, $\{X_{k\omega+\nu}\}$, the lumped-vector process, $\{Y_k\}$, defined by (1.6), is a stationary $m\omega$-variate ARMA process. Thus, with $\Gamma_\ell = Cov(Y_k, Y_{k-\ell})$, the above matrix is p.d. for all N = 1, 2, ... This, for example, implies that $Var(Y_k) = \Gamma_0$, which is given by

$$\begin{bmatrix} \Sigma_0(1) & \Sigma_1^T(2) & \cdots & \Sigma_{\omega-1}^T(\omega) \\ \Sigma_1(2) & \Sigma_0(2) & \cdots & \Sigma_{\omega-2}^T(\omega) \\ \vdots & \vdots & & \vdots \\ \Sigma_{\omega-1}(\omega) & \Sigma_{\omega-2}(\omega) & \cdots & \Sigma_0(\omega) \end{bmatrix},$$

is p.d. This fact may be utilized for a crude check of periodic stationarity on an observed realization of a periodic time series in the sense that if the observed time series is periodic stationary, then the estimate of Γ_0 above for this realization should be p.d.

2.5 Stationarity of the Aggregated Process

In Chapter I, we defined the concept of aggregation of periodic processes. In fact, the essence of this concept comes from the requirement that the models specified at various levels of aggregation (daily, monthly, etc.) must be compatible with each other. Then the model selected for a given periodic time series (say, monthly) determines the model for the corresponding series of a higher level of aggregation (say, annual).

We have pointed out in Chapter I that Ula (1990) proved that periodic stationarity of any PARMA process implies stationarity of its corresponding aggregated process. The converse, however, was only shown to be true for $PAR_\omega(1)$ and $PARMA_\omega(1,1)$ processes by Vecchia et al. (1983) and Ula (1990), for univariate and multivariate cases, respectively. Ula (1990) posed the question whether this is true for all PARMA processes or not. In the following example, we answer this question, and show that it is not necessarily true for all PARMA processes.

EXAMPLE 2.3. Consider the following univariate $PAR_2(2)$ model

$$X_{2k+1} = 2X_{2(k-1)+2} + (1/2)X_{2(k-1)+1} + a_{2k+1},$$

$$X_{2k+2} = (-3/2)X_{2(k-1)+2} + a_{2k+2}.$$

The lumped-vector representation of this model is given by

$$\begin{pmatrix} X_{2k+1} \\ X_{2k+2} \end{pmatrix} = \begin{pmatrix} 1/2 & 2 \\ 0 & -3/2 \end{pmatrix} \begin{pmatrix} X_{2(k-1)+1} \\ X_{2(k-1)+2} \end{pmatrix} + \begin{pmatrix} a_{2k+1} \\ a_{2k+2} \end{pmatrix},$$

which is a bivariate AR(1) process with parameter matrix having eigenvalues (1/2, -3/2), which are not all less than one in modulus so that it is not stationary. Hence, our periodic process is not periodic stationary.

In contrast, defining the aggregated process W_k as $W_k = X_{2k+1} + X_{2k+2}$, it is easy to see that it follows

$$W_k = (1/2)W_{k-1} + (a_{2k+1} + a_{2k+2}),$$

which, by using the fact that $\{a_{2k+1}\}$ and $\{a_{2k+2}\}$ are two uncorrelated white noise processes, is a stationary univariate AR(1) process. ■

One conclusion that can be drawn from the above example is that although we may be interested in annual data, say, it is important to inspect the original seasonal data, say, monthly, if available, since a big loss of information is expected if the annual data are only utilized. A parallel comment was given by Vecchia et al. (1983). They pointed out for the univariate $PARMA_{\omega}(1,1)$ model that the estimates of the aggregated process parameters based on seasonal data will often be better than those based on the aggregated data only.

The achievement of compatibility between seasonal models in the case of non-stationarity is a question which we pose for investigation. However, as the aggregated process is not seasonal, standard transformations in the context of stationary processes, such as differencing, can be used to make it stationary. Moreover, although Example 2.3 shows that stationarity of the aggregated process does not necessarily imply that the corresponding periodic process is periodic stationary, more general conditions should be obtained for this relation to be true. This point is also suggested for future research.

CHAPTER III

IDENTIFICATION OF PARMA PROCESSES

3.1 Introduction

In time series analysis one of the most important steps is to identify a model based on an available realization. In the literature of PARMA models, however, no obvious methodologies for identification are available yet.

For ARMA models, the autocorrelation function (ACF) and partial autocorrelation function (PACF) play a primary role in their identification (Box and Jenkins, 1976). In this chapter, two analogous functions, named seasonal autocorrelation and seasonal partial autocorrelation functions, are utilized for the identification of the seasonal orders of PARMA models. Some properties of these functions are studied and then illustrated through a simulated example. In the next section, we report a result of Vecchia (1983) which discusses the relation between the orders of univariate PARMA model and the orders of its marginal series counterpart. Although this result may be used in a latter stage of identification, it is, however, proved through a counterexample that this result does not offer much help in the early stages of the identification of PARMA models. The identification of the period ω is not considered in this thesis, although it deserves investigation. In practice, however, the period is often determined according to the nature of problem.

3.2 Marginal Series of PARMA Processes

Let $\{X_{k\omega+\nu}\}$ be a periodic time series, with period ω. Then, for an arbitrary season ν, $\nu = 1, \ldots, \omega$, the time series $\{\ldots, X_{\nu-2\omega}, X_{\nu-\omega}, X_{\nu}, X_{\nu+\omega}, X_{\nu+2\omega}, \ldots\}$ is called the marginal series for season ν. Vecchia (1983) proved that if $\{X_{k\omega+\nu}\}$ is a univariate periodic stationary $\text{PARMA}_{\omega}(p(\nu),q(\nu))$ process, then its marginal series for season ν follows a stationary ARMA process. He also obtained upper bounds for the orders of these ARMA models. In particular, if $\{X_{k\omega+\nu}\}$ is a univariate $\text{PARMA}_{\omega}(p,q)$ process, then the marginal series for ν-th season follows an $\text{ARMA}(p'(\nu),q'(\nu))$ model such that $\max_{\nu}(p'(\nu)) \le p$ and $\max_{\nu}(q'(\nu)) \le p + [(q-1)/\omega]$, where $[c]$ stands for the integral part of the real number c.

One drawback of studying $\{X_{k\omega+\nu}\}$ in terms of its marginal series is that this approach does not take into account the inter-dependence of these series, which has already been expressed by the PARMA model. Besides, the following example shows that the relation between the orders of the marginal series and those of the periodic process is not one-to-one.

EXAMPLE 3.1. Consider the univariate $\text{PARMA}_2(1,1)$ model, with $\phi_1^{(\nu)} = \phi_\nu$ and $\theta_1^{(\nu)} = \theta_\nu$:

$$X_{2k+1} = \phi_1 X_{2(k-1)+2} + a_{2k+1} - \theta_1 a_{2(k-1)+2}$$

$$X_{2k+2} = \phi_2 X_{2k+1} + a_{2k+2} - \theta_2 a_{2k+1}.$$

It is easy to see that the two equations above give

$$X_{2k+1} = \phi_1\phi_2 X_{2(k-1)+1} + a_{2k+1} - (\theta_1-\phi_1)a_{2(k-1)+2} - \phi_1\theta_2 a_{2(k-1)+1},$$

$$X_{2k+2} = \phi_1\phi_2 X_{2(k-1)+2} + a_{2k+2} - (\theta_2-\phi_2)a_{2k+1} - \theta_1\phi_2 a_{2(k-1)+2}.$$

Rose (1977) proved that the sum of two uncorrelated MA(1) processes can uniquely be represented by an MA(1) process. Therefore, making use of this result and noting that, for all integers k, $\{a_{2k+1}\}$ and $\{a_{2k+2}\}$ are two uncorrelated white noise processes, the above two equations reveal that each of the two marginal series, namely $\{X_{2k+1}\}$ and $\{X_{2k+2}\}$, follows an ARMA(1,1) model.

In a similar manner, it is easy to show that the two marginal series corresponding to the $\text{PARMA}_2(1,2)$ process also follow an ARMA(1,1) model. ∎

In the previous example we proved that for two different PARMA processes with $\omega = 2$, the corresponding marginal series follow an ARMA(1, 1) model. In fact, for larger ω, it can be shown that more than two different PARMA processes will have marginal series of the same order. For instance, it can be shown that each of the marginal series of the processes $PMA_4(1)$, $PMA_4(2)$ and $PMA_4(3)$ will be a white noise process. The following example illustrates this fact through a simulated example.

EXAMPLE 3.2. Two different realizations, each of length 4×100, are simulated from the following two PMA_4 models:

(1) $PMA_4(1)$ model, with $\theta_1^{(1)} = 0.8$, $\theta_1^{(2)} = -0.6$, $\theta_1^{(3)} = 0.5$, $\theta_1^{(4)} = 0.9$, $\sigma_a^2(1) = 1$, $\sigma_a^2(2) = 4$, $\sigma_a^2(3) = 16$, $\sigma_a^2(4) = 5$,

(2) $PMA_4(2)$ model, with $\theta_1^{(1)} = 0.8$, $\theta_2^{(1)} = 0.9$, $\theta_1^{(2)} = -0.5$, $\theta_2^{(2)} = 0.7$, $\theta_1^{(3)} = 0.5$, $\theta_2^{(3)} = 0.2$, $\theta_1^{(4)} = -0.7$, $\theta_2^{(4)} = 0.6$, $\sigma_a^2(1) = 64$, $\sigma_a^2(2) = 4$, $\sigma_a^2(3) = 16$, $\sigma_a^2(4) = 1$,

such that the white noise processes $\{a_{k\omega+\nu}\}$ are assumed to be independently and normally distributed with mean zero. Then, using the program listed in Appendix B, each of the four marginal series is extracted from the simulated time series and its sample ACF and PACF are obtained. Tables 3.1 and 3.2 summarize these functions for different marginal series and for both models. For the simulation of uniform (0, 1) random numbers, the NAG subroutine G05CAF is utilized, based on which normal random deviates are simulated through the NAG subroutine G05EAF. The former subroutine is based on the multiplicative congruential algorithm $u_{i+1} = 13^{13} \times u_i \mod 2^{59}$. These subroutines are also used in the computer programs listed in Appendices C and D.

Table 3.1. Sample ACF for Marginal Series of Two PMA_4 Processes

lag	Seasons of Model(1) 1	2	3	4	Seasons of Model(2) 1	2	3	4
1	-.056	.069	.080	.055	.083	.122	.002	.024
2	-.002	.140	-.039	.057	.032	.009	-.003	-.027
3	-.148	.046	-.109	-.135	.040	.105	.060	-.019
4	.118	.125	-.176	-.065	.013	.085	.015	-.049
5	.003	-.062	-.019	-.009	.182	.076	.009	-.067
6	-.064	-.088	-.132	-.056	.050	.067	.030	-.043
7	-.108	-.011	.050	.000	.075	.151	.044	.062
8	.122	.022	.051	.084	-.129	-.087	-.064	-.006
9	.104	-.051	-.111	-.150	-.089	-.083	-.097	-.049
10	-.006	-.132	.001	.131	-.067	-.018	.037	.002
Q	8.116	7.521	8.668	7.931	8.095	8.159	2.198	1.716

Table 3.2. Sample PACF for Marginal Series of Two PMA_4 Processes

lag	Seasons of Model(1) 1	2	3	4	Seasons of Model(2) 1	2	3	4
1	-.056	.069	.080	.055	.083	.122	.002	.024
2	-.006	.136	-.046	.054	.025	-.005	-.003	-.028
3	-.149	.029	-.103	-.141	.036	.106	.060	-.018
4	.104	.104	-.164	-.054	.006	.061	.015	-.049
5	.012	-.088	-.003	.014	.180	.061	.010	-.066
6	-.085	-.116	-.162	-.069	.020	.044	.027	-.044
7	-.085	.011	.035	-.010	.063	.130	.043	.059
8	.107	.043	-.001	.093	-.161	-.140	-.066	-.016
9	.096	-.029	-.153	-.184	-.077	-.077	-.101	-.054
10	-.011	-.124	-.023	.141	-.096	-.046	.031	-.002

Following Box and Jenkins (1976), if a process is white noise, then for large N, where N denotes the length of the observed realization, each of the sample ACF and sample PACF is normally distributed with mean zero and variance 1/N for all lags. Therefore, for N = 100, it is seen that all the values in Tables 3.1 and 3.2 belong to the 95% interval, (-0.196, 0.196), which agrees with the discussion preceding this example. Another way of testing whether a process is white noise or not is through the Portmanteau test (Cryer, 1986: 153), for which the Box-Pierce test statistic (Q) is given in Table 3.1. If these Q values are compared with the upper 0.05 points of chi-square distribution with d.f. = 10 (equal to the maximum lag used), which is 18.3, it is seen that none of these values is significant, which assures our claim. ■

Hence, from the above discussion, it is apparent that the relations between the orders of any PARMA model and its corresponding marginal series orders is not one-to-one, so that the identification of the orders of PARMA models through those of its marginal series is not an accurate method. An alternative approach is proposed in the succeeding sections.

3.3 Seasonal Autocorrelation Function

3.3.1 Definition and Properties

Hereinafter the univariate $PARMA_{\omega}(p(\nu),q(\nu))$ model is mainly considered. Furthermore, the errors $\{a_{k\omega+\nu}\}$ are assumed to be independent. Besides, we will use $\gamma_{\ell}(\nu)$ instead of $\Sigma_{\ell}(\nu)$, which was defined by (2.8), to denote the univariate autocovariance function for season ν at backward lag ℓ, or simply the seasonal autocovariance function. Therefore, for univariate case (Vecchia, 1985b),

$$\gamma_{\ell}(\nu) = \mathrm{Cov}(X_{k\omega+\nu}, X_{k\omega+\nu-\ell}).$$

In Section 1.2 we have seen that if $\{X_t\}$ is a periodic stationary PARMA process with period ω, then the first two moments of this process are finite and periodic with period ω. Hence, we denote the periodic mean and variance for the ν-th season, $\nu = 1, \ldots, \omega$, by μ_{ν} and $\gamma_0(\nu)$, respectively. Then the ACF for season ν at backward lag ℓ (the seasonal ACF), denoted by $\rho_{\ell}(\nu)$, is defined as

$$\rho_{\ell}(\nu) = E\left[\left(\frac{X_{k\omega+\nu} - \mu_{\nu}}{[\gamma_0(\nu)]^{1/2}}\right)\left(\frac{X_{k\omega+\nu-\ell} - \mu_{\nu-\ell}}{[\gamma_0(\nu-\ell)]^{1/2}}\right)\right]$$

$$= E(Z_{k\omega+\nu}Z_{k\omega+\nu-\ell}) = \frac{\gamma_{\ell}(\nu)}{[\gamma_0(\nu)\gamma_0(\nu-\ell)]^{1/2}}, \qquad \ell \geq 0, \tag{3.1}$$

which is also the seasonal autocovariance and ACF of the seasonally standardized time series, $\{Z_{k\omega+\nu}\}$. The terms $\mu_{\nu-\ell}$ and $\gamma_0(\nu-\ell)$ are well defined by utilizing the facts, mentioned in the previous chapters, that μ_{ν} and $\gamma_{\ell}(\nu)$ are periodic (in ν) with period ω.

It is known that the ordinary ACF of a stationary time series, denoted by ρ_ℓ, is symmetric with respect to the time lag, that is, $\rho_\ell = \rho_{-\ell}$, for all integers ℓ. However, this property does not apply for $\rho_\ell(\nu)$, since the seasonal autocovariance function $\gamma_\ell(\nu)$ is not necessarily equal to $\gamma_{-\ell}(\nu)$, unless ℓ is a multiple of the period ω. This property can also be observed if we note that moving ℓ steps ahead from season ν and moving ℓ steps back from season ν do not imply that the resulting seasons are identical. This is the reason for restricting the definition of $\rho_\ell(\nu)$ in (3.1) for non-negative time lags only, although it is possible to waive this restriction by noting that

$$\rho_\ell(\nu) = E(Z_{k\omega+\nu}Z_{k\omega+\nu-\ell}) = E(Z_{k\omega+\nu-\ell}Z_{k\omega+\nu-\ell-(-\ell)}) = \rho_{-\ell}(\nu-\ell). \tag{3.2}$$

Another important property of the ordinary ACF of a stationary time series is that it vanishes as the time lag gets larger, or more precisely, $\lim_{\ell \to \infty} \rho_\ell = 0$. The following proposition, which is useful for the remaining part of this chapter, establishes a similar result for $\rho_\ell(\nu)$.

PROPOSITION 3.1. Let $\{X_{k\omega+\nu}\}$ be a univariate periodic stationary PARMA process. Then, for an arbitrary season ν, $\lim_{\ell \to \infty} \rho_\ell(\nu) = 0$.

PROOF. An important implication of the periodic stationarity of $\{X_{k\omega+\nu}\}$ is that for each season ν, the process can be represented in the general linear process form, (1.8), which is written for the univariate case as

$$X_{k\omega+\nu} = \sum_{j=0}^{\infty} \psi_j^{(\nu)} a_{k\omega+\nu-j}, \tag{3.3}$$

with $\sum_{j=0}^{\infty} |\psi_j^{(\nu)}| < \infty$. Then utilizing this result and (3.1), we have

$$\rho_\ell(\nu) = \frac{\mathrm{Cov}(X_{k\omega+\nu}, X_{k\omega+\nu-\ell})}{[\gamma_0(\nu)\gamma_0(\nu-\ell)]^{1/2}}$$

$$= c\ \mathrm{Cov}\left(\sum_{j=0}^{\infty} \psi_j^{(\nu)} a_{k\omega+\nu-j}, \sum_{n=0}^{\infty} \psi_n^{(\nu-\ell)} a_{k\omega+\nu-(\ell+n)}\right)$$

$$= c\ \mathrm{Cov}\left(\sum_{j=0}^{\infty} \psi_j^{(\nu)} a_{k\omega+\nu-j}, \sum_{i=\ell}^{\infty} \psi_{i-\ell}^{(\nu-\ell)} a_{k\omega+\nu-i}\right)$$

$$= c \sum_{j=\ell}^{\infty} \psi_j^{(\nu)} \psi_{j-\ell}^{(\nu-\ell)} \sigma_a^2(\nu-j),$$

so that

$$|\rho_\ell(\nu)| \le c\,\sigma^2 \sum_{j=\ell}^{\infty} |\psi_j^{(\nu)} \psi_{j-\ell}^{(\nu-\ell)}| \le c_1 \sum_{j=\ell}^{\infty} |\psi_j^{(\nu)} \psi_{j-\ell}^{(\nu-\ell)}|,$$

where $c = [\gamma_0(\nu)\gamma_0(\nu-\ell)]^{-1/2}$ and $\sigma^2 = \max_\nu[\sigma_a^2(\nu)] < \infty$, in which, for all ν, $\sigma_a^2(\nu) = \mathrm{Var}(a_{k\omega+\nu})$, and $c_1 = \sigma^2 \; [\gamma_0(\nu)]^{-1/2} \; \max_{\nu_1}[\gamma_0(\nu_1)]^{-1/2}$ which is finite due to the periodic stationarity assumption. Furthermore, $\sum_{j=\ell}^{\infty} \psi_j^{(\nu)} \psi_{j-\ell}^{(\nu-\ell)}$ is an absolutely convergent series since it is a convolution of two absolutely convergent series (Fuller, 1976: 28). Hence, $\sum_{j=\ell}^{\infty} |\psi_j^{(\nu)} \psi_{j-\ell}^{(\nu-\ell)}| \longrightarrow 0$ as $\ell \longrightarrow \infty$, by which and using the fact that $|\rho_\ell(\nu)| \ge 0$, the desired result is obtained. ■

The result in Proposition 3.1 is also valid for the multivariate case and the proof is much the same as the one above.

An important property of $\rho_\ell(\nu)$ is that if ν is an arbitrary season with $p(\nu) = 0$, then $\gamma_\ell(\nu) = 0$ for all $\ell > q(\nu)$, and, therefore,

$$\rho_\ell(\nu) = 0, \quad \ell > q(\nu). \tag{3.4}$$

This result can easily be verified for both univariate and multivariate cases by making use of (3.1) and the definition of PARMA model given by (1.5). It is known that the ordinary ACF possesses a similar cut-off property in ARMA models, namely it becomes zero for time lags larger than q if the underlying model is pure MA(q). On this basis we propose the sample seasonal ACF as a tool to check whether the ν-th season equation is pure MA or not. The next section is devoted to studying the sample seasonal ACF and the assessment of the cut-off property explained above.

3.3.2 Sample Seasonal Autocorrelation Function

Let $\{X_{k\omega+\nu}\}$ be a periodic stationary time series from which a realization of size $N\omega$ (say, N years), denoted by $\{X_1, X_2, \ldots, X_{N\omega}\}$, is realization of size $N\omega$ (say, N years), denoted by $\{X_1, X_2, \ldots, X_{N\omega}\}$, is observed. Then the sample counterpart of $\rho_\ell(\nu)$ is defined, by analogy to the ordinary sample ACF, as

$$r_\ell(\nu) = \frac{\hat{\gamma}_\ell(\nu)}{[\hat{\gamma}_0(\nu)\hat{\gamma}_0(\nu-\ell)]^{1/2}}, \qquad \ell \geq 0, \tag{3.5}$$

where $\hat{\gamma}_\ell(\nu)$ is the sample seasonal autocovariance function defined as

$$\hat{\gamma}_\ell(\nu) = \frac{1}{N}\sum_{k=0}^{N-1}(X_{k\omega+\nu} - \bar{X}_\nu)(X_{k\omega+\nu-\ell} - \bar{X}_{\nu-\ell}), \tag{3.6}$$

in which

$$\bar{X}_\nu = \frac{1}{N}\sum_{k=0}^{N-1} X_{k\omega+\nu}$$

is the sample mean for season ν. Note that $\bar{X}_\nu$ and $\hat{\gamma}_\ell(\nu)$ are also periodic with period ω, as μ_ν and $\gamma_\ell(\nu)$ are. In addition, the terms in (3.6) are set to zero whenever $k\omega + \nu - \ell < 1$. If we denote the number of these terms by $\varphi(\nu,\ell,\omega)$, then it can be shown that $\varphi(\nu,\ell,\omega) = [(\ell-\nu)/\omega] + 1$, where [c] stands for the integral part of c.

In (3.6), if $\{X_{k\omega+\nu}\}$ is Gaussian, i.e. if the white noise terms are independent and normal, Pagano (1978) proved that, for all ν and ℓ, $\hat{\gamma}_\ell(\nu)$ are consistent, converge almost surely and in mean square to $\gamma_\ell(\nu)$, and are asymptotically joint normal and unbiased. It is also known that $\bar{X}_\nu$ is an unbiased estimator of μ_ν and it can be shown that it is also consistent under the periodic stationarity assumption.

It can be seen that deriving the first and second order moments of $r_\ell(\nu)$, which involves the sample seasonal means, variances and autocovariances, is a formidable task. In the simpler case of ARMA models, for example, obtaining even the mean of the sample ACF is an extremely difficult job (see, Anderson, 1971). Therefore, we will try to obtain an approximate solution for the first and second order moments of $r_\ell(\nu)$ by pretending that the sample means and variances, $\bar{X}_\nu$ and $\hat{\gamma}_0(\nu)$, in $r_\ell(\nu)$, are equal to their population counterpart, μ_ν and $\gamma_0(\nu)$, respectively. This assumption will obviously be well justified for large samples, i.e.

for large N, due to the consistency property of these estimates mentioned earlier.

$$r_\ell(\nu) = \frac{1}{N} \sum_{k=\varphi(\nu,\ell,\omega)}^{N-1} Z_{k\omega+\nu} Z_{k\omega+\nu-\ell}, \qquad (3.7)$$

where $Z_{k\omega+\nu}$ are the sample values of the seasonally standardized time series, that is, $Z_{k\omega+\nu} = (X_{k\omega+\nu} - \mu_\nu) / [\gamma_0(\nu)]^{1/2}$ as defined in (3.1).

It follows from (3.2) and (3.7) that

$$E[r_\ell(\nu)] = \frac{N - \varphi(\nu,\ell,\omega)}{N} \rho_\ell(\nu).$$

In addition, as $\varphi(\nu,\ell,\omega)$ is a fixed quantity for fixed ν and ℓ, then this implies that $r_\ell(\nu)$ is asymptotically unbiased.

Next, in order to investigate the covariance of $r_{\ell_1}(\nu)$ and $r_{\ell_2}(\nu)$ for two different time lags ℓ_1 and ℓ_2, which we denote by $\alpha_{\ell_1,\ell_2}(\nu)$, we assume that $\{Z_{k\omega+\nu}\}$ has finite fourth order moments. Then, for $\ell_2 \geq \ell_1 \geq 0$,

$$\alpha_{\ell_1,\ell_2}(\nu) = E[r_{\ell_1}(\nu) r_{\ell_2}(\nu)] - E[r_{\ell_1}(\nu)]E[r_{\ell_2}(\nu)].$$

We now apply a standard result from quadravariate distributions (Priestley, 1981: 326) which states that (under the condition $E(Z_t) = 0$),

$$E(Z_t Z_{t+r} Z_s Z_{s+r+v}) = E(Z_t Z_{t+r})E(Z_s Z_{s+r+v}) + E(Z_t Z_s)E(Z_{t+r} Z_{s+r+v})$$
$$+ E(Z_t Z_{s+r+v})E(Z_{t+r} Z_s) + \kappa_4(s-t,r,v),$$

where $\kappa_4(s-t,r,v)$ is the fourth joint cumulant of the distribution of $[Z_t, Z_{t+r}, Z_s, Z_{s+r+v}]$. It then follows by some manipulations that

$$\alpha_{\ell_1,\ell_2}(\nu) = \frac{1}{N^2} \sum_{j=a}^{N-1} \sum_{k=b}^{N-1} \Big\{ \rho_{(j-k)\omega}(\nu) \rho_{(j-k)\omega+\ell_2-\ell_1}(\nu-\ell_1)$$
$$+ \rho_{(j-k)\omega+\ell_2}(\nu) \rho_{(j-k)\omega-\ell_1}(\nu-\ell_1) + \kappa_4[(k-j)\omega, -\ell_1, \ell_1-\ell_2] \Big\},$$

where $a = \varphi(\nu,\ell_1,\omega)$ and $b = \varphi(\nu,\ell_2,\omega)$. Pooling the two summations above, by making the transformation $m = j - k$, it follows that

$$\alpha_{\ell_1,\ell_2}(\nu) = \frac{1}{N}\sum_{m=a-(N-1)}^{N-1-b}\left(1 - \frac{\eta(m) + b}{N}\right)\left\{\rho_{m\omega}(\nu)\rho_{m\omega+\ell_2-\ell_1}(\nu-\ell_1) + \rho_{m\omega+\ell_2}(\nu)\rho_{m\omega-\ell_1}(\nu-\ell_1) + \kappa_4(-m\omega,-\ell_1,\ell_1-\ell_2)\right\}, \tag{3.8}$$

where

$$\eta(m) = \begin{cases} a-b-m, & a-(N-1) \le m < a-b \\ 0, & a-b \le m \le 0 \\ m, & m > 0. \end{cases}$$

Furthermore, it is known that if $\{Z_{k\omega+\nu}\}$ is a Gaussian process, then all the fourth joint cumulants, $\kappa_4(m_1,m_2,m_3)$, for all integers m_1, m_2 and m_3, are identically equal to zero (Anderson, 1971: 452). However, Bartlett (1946) proves a more general result. He proved that $\kappa_4(m_1,m_2,m_3)$ will be also zero if the error terms are independent (rather than being uncorrelated) and the process $\{Z_{k\omega+\nu}\}$ follows a general linear process form (Priestley, 1981: 332), (3.3). These conditions are satisfied here since we are assuming that $\{a_{k\omega+\nu}\}$ are independent and the process is periodic stationary. Thus, the term $\kappa_4(-m\omega,-\ell_1,\ell_1-\ell_2)$ will no longer appear in (3.8).

Setting $\ell_1 = \ell_2 = \ell$ in (3.8) gives

$$\mathrm{Var}[r_\ell(\nu)] = \frac{1}{N}\sum_{m=a-(N-1)}^{N-1-a}\left(1 - \frac{|m| + a}{N}\right)\left\{\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell) + \rho_{m\omega+\ell}(\nu)\rho_{m\omega-\ell}(\nu-\ell)\right\}.$$

Note that, in view of Proposition 3.1, for large m, the seasonal autocorrelations in the above summation are negligible. Besides, for small m and large N, the factor $\left(1 - \frac{|m| + a}{N}\right)$ is approximately one and the limits in the summation can be replaced by $-\infty$ and ∞. Then, for large N, the above expression becomes

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\sum_{m=-\infty}^{\infty}\left\{\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell) + \rho_{m\omega+\ell}(\nu)\rho_{m\omega-\ell}(\nu-\ell)\right\}, \tag{3.9}$$

which, by utilizing (3.2) and the fact that $\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell) + [\rho_{m\omega+\ell}(\nu) \times$

$\rho_{m\omega-\ell}(\nu-\ell)]$ is an even function of m, reduces to

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\left(1 + [\rho_\ell(\nu)]^2 + 2\sum_{m=1}^{\infty}\left\{\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell) + \rho_{m\omega+\ell}(\nu)\rho_{m\omega-\ell}(\nu-\ell)\right\}\right), \qquad (3.10)$$

where the symbol "$\cong$", wherever appears, means that the statement is true for large N.

The above equation and (3.8) also provide approximate expressions for $\mathrm{Var}[\hat{\gamma}_\ell(\nu)]$ and $\mathrm{Cov}[\hat{\gamma}_{\ell_1}(\nu),\hat{\gamma}_{\ell_2}(\nu)]$, respectively, but with γ in place of ρ in those expressions and such that $\hat{\gamma}_\ell(\nu)$ is as defined by (3.6) but with μ_ν replacing $\bar{X}_\nu$.

In the context of stationary processes, asymptotic formulas for the first and second order moments of sample ACF, r_ℓ, as defined by (3.5) with $\omega = \nu = 1$, have been derived by Bartlett (1946) under the assumption of white noise terms being independent. For instance, $\mathrm{Var}(r_\ell)$ is given by

$$\mathrm{Var}(r_\ell) \cong \frac{1}{N}\sum_{m=-\infty}^{\infty}\left\{\rho_m^2 + \rho_{m+\ell}\rho_{m-\ell} - 4\rho_\ell\rho_m\rho_{m-\ell} + 2\rho_m^2\rho_\ell^2\right\},$$

(Box and Jenkins, 1976: 35). Under the assumption of white noise terms being independent and normal, this result was then generalized to periodic stationary processes by Anderson and Vecchia (1993). Letting $R_\ell = [r_\ell(\omega), r_\ell(1), \ldots, r_\ell(\omega-1)]^T$, they have shown that the asymptotic variance-covariance matrix of R_ℓ is given by

$$W_\ell \cong \frac{1}{N}\sum_{m=-\infty}^{\infty}\left\{F_m\Pi^\ell F_m\Pi^{m-\ell} + F_{m+\ell}\Pi^\ell F_{m-\ell}\Pi^{m-\ell} - F_\ell(I + \Pi^\ell)F_mF_{m+\ell}\Pi^m - F_m\Pi^\ell F_{m-\ell}\Pi^{m-\ell}(I + \Pi^{-\ell})F_\ell + \frac{1}{2}F_\ell(I + \Pi^\ell)F_m^2\Pi^m(I + \Pi^{-\ell})F_\ell\right\},$$

where $F_m = \mathrm{diag}[\rho_m(\omega), \rho_m(1), \ldots, \rho_m(\omega-1)]$ and Π is an orthogonal $\omega\times\omega$ cyclic permutation matrix,

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

with $\Pi^{-\ell} \equiv (\Pi^T)^{\ell}$, Π^0 being the $\omega \times \omega$ identity matrix, and

$$\Pi^{\ell} \mathrm{diag}\{c_{\omega}, c_1, \dots, c_{\omega-1}\} \Pi^{-\ell} = \mathrm{diag}\{c_{\ell}, c_{\ell+1}, \dots, c_{\ell+\omega-1}\}$$

where $\{c_t\}$ is any sequence of constants satisfying $c_{k\omega+t} = c_t$ for all k. It was also shown that R_{ℓ} is asymptotically unbiased and jointly multivariate normal. Although the above formula is fairly complicated, it can be shown that the expression for $\mathrm{Var}[r_{\ell}(\nu)]$ in (3.9) is an approximate version of their corresponding formula, which differs by terms which include autocorrelations of third and higher orders, which, by Proposition 3.1, are usually negligible. For example, for a two-period case, it can be shown that their formulas for the first season, $\nu = 1$, reduce for odd lag ℓ to

$$\mathrm{Var}[r_{\ell}(1)] \cong \frac{1}{N} \sum_{m=-\infty}^{\infty} \Big\{ \rho_{2m}(1)\rho_{2m}(2) + \rho_{2m+\ell}(1)\rho_{2m-\ell}(2) - \rho_{\ell}(1)[\rho_m(2) \times \rho_{m+\ell}(1) + \rho_m(2+m)\rho_{m-\ell}(m)] + \rho_{\ell}^2(1)[\rho_{2m}^2(1) + \rho_{2m}^2(2)] \Big\}, \quad (3.11a)$$

and for even lag ℓ to

$$\mathrm{Var}[r_{\ell}(1)] \cong \frac{1}{N} \sum_{m=-\infty}^{\infty} \Big\{ \rho_{2m}^2(1) + \rho_{2m+\ell}(1)\rho_{2m-\ell}(1) - 2\rho_{\ell}(1)\rho_{2m}(1) \times [\rho_{2m+\ell}(1) + \rho_{2m-\ell}(2)] + 2\rho_{\ell}(1)\rho_{\ell}(2)\rho_{2m}^2(1) \Big\}. \quad (3.11b)$$

On the other hand, (3.9) reduces in this case to

$$\mathrm{Var}[r_{\ell}(1)] \cong \frac{1}{N} \sum_{m=-\infty}^{\infty} \Big\{ \rho_{2m}(1)\rho_{2m}(1-\ell) + \rho_{2m+\ell}(1)\rho_{2m-\ell}(1-\ell) \Big\},$$

where $1 - \ell = 1$ for even ℓ and $1 - \ell = 2$ for odd ℓ, which can easily be shown to differ from (3.11) in terms of third and higher order autocorrelations. More importantly, it can be shown that the formulas of Anderson and Vecchia (1993) and (3.9) are identical when $p(\nu) = 0$ and $\ell > q(\nu)$. Therefore, for the assessment of cut-off in seasonal autocorrelations, that is for checking whether season ν is a MA($q(\nu)$) or not, (3.10) is as applicable as the formulas of Anderson and Vecchia.

In addition, setting $\omega = \nu = 1$ in (3.9) gives

$$\mathrm{Var}(r_\ell) \cong \frac{1}{N}\sum_{m=-\infty}^{\infty}\left\{\rho_m^2 + \rho_{m+\ell}\rho_{m-\ell}\right\},$$

which differs from the corresponding formula of Bartlett (1946), given earlier in this section, by the term $-4\rho_\ell\rho_m\rho_{m-\ell} + 2\rho_m^2\rho_\ell^2$, which, for a MA(q) process, is equal to zero for $\ell > q$, since in this case $\rho_\ell = 0$. Furthermore, assuming that the time lag ℓ is large, a (large lag) formula for $\mathrm{Var}[r_\ell(\nu)]$ is obtained from (3.9) by setting $\rho_{m\omega+\ell}(\nu)$ to zero, by Proposition 3.1, and is given by

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\sum_{m=-\infty}^{\infty}\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell).$$

This formula reduces, by setting $\omega = \nu = 1$, to the corresponding formula in the context of stationary processes (Box and Jenkins, 1976: 35).

In the following proposition some properties of the variance of $r_\ell(\nu)$, which follow from (3.10), are summarized which are needed for the assessment of the cut-off property of the seasonal ACF for a season ν which follows a MA($q(\nu)$) process.

PROPOSITION 3.2. Let $\{X_{k\omega+\nu}\}$ be a periodic stationary PARMA$_\omega$($p(\nu),q(\nu)$) process. If ν is an arbitrary season with $p(\nu) = 0$, then, for positive integers ℓ, we have the following results for $\mathrm{Var}[r_\ell(\nu)]$:

(i) If $q(\nu) < w$, then

$$\mathrm{Var}[r_\ell(\nu)] \cong \begin{cases} \frac{1}{N}\left\{1+[\rho_\ell(\nu)]^2\right\}, & \ell \le q(\nu) \\ \frac{1}{N}, & \ell > q(\nu) \end{cases} \qquad (3.12)$$

(ii) If $k\omega \le q(\nu) < (k+1)w$, $k = 1, 2, \ldots$, then,

(a) for $1 \le \ell \le q(\nu) - k\omega$,

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\left(1 + [\rho_\ell(\nu)]^2 + 2\sum_{m=1}^{k}\left\{\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell) + \rho_{m\omega+\ell}(\nu)\rho_{m\omega-\ell}(\nu-\ell)\right\}\right),$$

(b) for $q(\nu) - (j+1)\omega < \ell \le q(\nu) - j\omega$, where $j = 1, \ldots, k - 1$,

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\left(1 + [\rho_\ell(\nu)]^2 + 2\left\{\sum_{m=1}^{k}\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell) \sum_{m=1}^{j}\rho_{m\omega+\ell}(\nu)\rho_{m\omega-\ell}(\nu-\ell)\right\}\right),$$

(c) for $q(\nu) - \omega < \ell \le q(\nu)$,

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\left(1 + [\rho_\ell(\nu)]^2 + 2\sum_{m=1}^{k}\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell)\right),$$

(d) and for $\ell > q(\nu)$,

$$\mathrm{Var}[r_\ell(\nu)] \cong \frac{1}{N}\left(1 + 2\sum_{m=1}^{k}\rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell)\right). \tag{3.13}$$

PROOF. Part (i) follows from (3.10) by utilizing (3.4). For part (ii), first notice that in parts (a) through (c), $\ell \le q(\nu)$, whereas in part (d), $\ell > q(\nu)$. This explains, due to (3.4), the presence of the term $[\rho_\ell(\nu)]^2$ in parts (a) through (c) but not in (d). Also, since $q(\nu) \ge k\omega$, $\rho_{m\omega}(\nu)$ is non-zero for $m \le k$, and this explains the presence of the term $\sum_{m=1}^{k} \rho_{m\omega}(\nu)\rho_{m\omega}(\nu-\ell)$ in all parts. For the terms including $\rho_{m\omega+\ell}(\nu)$, it can be easily seen that in part (a), $m\omega + \ell \le q(\nu)$ is true for $m = 1, \ldots, k$, in which case $\rho_{m\omega+\ell}(\nu)$ is non-zero. In part (c), since $q(\nu) - \omega < \ell \le q(\nu)$, no value of m satisfies $m\omega + \ell \le q(\nu)$. The same reasoning applies for part (d). In part (b), notice that $q(\nu) - (j+1)\omega < \ell \le q(\nu) - j\omega$, $j = 1, \ldots, k-1$, which means that $m\omega + \ell \le q(\nu)$ is valid only for $m = 1, \ldots, j$. We can denote j also as $j = [(q(\nu)-\ell)/\omega]$. ■

Setting $\omega = \nu = 1$, it can be easily seen from Proposition 3.2 that, for $\ell > q(\nu) = q$, case (i) reduces to the white noise process with

$Var(r_\ell) = 1/N$, $\ell \geq 1$, and case (ii) reduces to the MA(k) process, in which case (3.13) becomes

$$Var(r_\ell) \cong \frac{1}{N}\left(1 + 2\sum_{m=1}^{k}\rho_m^2\right), \quad \ell > k.$$

These are the well-known formulas for the identification of white noise and MA processes, respectively, in the context of stationary processes (Box and Jenkins, 1976: 35). However, note that for $\ell \leq q(\nu)$ the formulas for $Var(r_\ell)$ in case (ii) of Proposition 3.2 are rather approximated as they are based on (3.10).

For the assessment of the cut-off property of seasonal autocorrelations, we utilize the result of Anderson and Vecchia (1993) about $r_\ell(\nu)$ being asymptotically normal with mean $\rho_\ell(\nu)$. Then following the same methodology applied for ordinary sample autocorrelations in the context of stationary processes (Box and Jenkins, 1976), for a season ν which follows a MA($q(\nu)$) process, we start checking values of $q(\nu)$ successively, starting with $q(\nu) = 1$. Then for large N, as long as $q(\nu) < \omega$, (3.12) implies that $r_\ell(\nu)$, $\ell > q(\nu)$, is normally distributed with mean zero and variance 1/N, so that the 95% band $(-1.96/N^{1/2}, 1.96/N^{1/2})$ is applied to those autocorrelations. If $q(\nu) > \omega$, (3.13) should be utilized. More precisely, the theoretical seasonal autocorrelations in (3.13) are replaced with their sample estimates, and letting s_ν to denote the sample value of $(Var[r_\ell(\nu)])^{1/2}$, the 95% band is $(-1.96s_\nu, 1.96s_\nu)$ which should be applied for $r_\ell(\nu)$, $\ell > q(\nu)$. The accuracy of these bands are verified through simulation in Section 3.5.

3.4 Seasonal Partial Autocorrelation Function

Let $\{X_{k\omega+\nu}\}$ be a univariate periodic time series which follows a $PARMA_\omega(p(\nu),q(\nu))$ model. Also let ν be an arbitrary season such that $p(\nu) = p > 0$ and $q(\nu) = 0$, then the AR(p) equation for season ν is obtained, from (1.5), as

$$X_{k\omega+\nu} = \phi_1^{(\nu)}X_{k\omega+\nu-1} + \ldots + \phi_p^{(\nu)}X_{k\omega+\nu-p} + a_{k\omega+\nu}. \tag{3.14}$$

The above equation can be viewed as a regression model in which (the dependent variable) $X_{k\omega+\nu}$ is regressed on the regressors (independent variables) $X_{k\omega+\nu-1}, \ldots, X_{k\omega+\nu-p}$. A reasonable measure of the relationship between $X_{k\omega+\nu}$ and $X_{k\omega+\nu-p}$, keeping the other regressors fixed, is the partial correlation coefficient between $X_{k\omega+\nu}$ and $X_{k\omega+\nu-p}$, denoted by $\phi_{pp}(\nu)$, which is defined, for integers $p \geq 1$, as

$$\phi_{pp}(\nu) = \mathrm{Corr}(X_{k\omega+\nu}, X_{k\omega+\nu-p} | X_{k\omega+\nu-1}, \ldots, X_{k\omega+\nu-p+1}).$$

An alternative and more practical definition of $\phi_{pp}(\nu)$ is as follows (Cryer, 1986: 106-107): Let $\hat{X}_{k\omega+\nu}$ and $\hat{X}_{k\omega+\nu-p}$ be the best linear MSE predictors of $X_{k\omega+\nu}$ and $X_{k\omega+\nu-p}$ based on the set $\{X_{k\omega+\nu-1}, \ldots, X_{k\omega+\nu-p+1}\}$, respectively. Also, let $\hat{e}_{k\omega+\nu} = X_{k\omega+\nu} - \hat{X}_{k\omega+\nu}$ and $\hat{e}_{k\omega+\nu-p} = X_{k\omega+\nu-p} - \hat{X}_{k\omega+\nu-p}$ be the residuals. Then

$$\phi_{pp}(\nu) = \mathrm{Corr}(\hat{e}_{k\omega+\nu}, \hat{e}_{k\omega+\nu-p}), \tag{3.15}$$

which coincides with the previous definition if $\{X_{k\omega+\nu}\}$ is a Gaussian process. We adopt the second definition here.

The last parameter in (3.14), $\phi_p^{(\nu)}$, can be expressed in terms of the seasonal autocovariances by the method of moments (see Section 4.3). It can also be shown that $\phi_p^{(\nu)}$ can be related to the partial correlation $\phi_{pp}(\nu)$ as

$$\phi_p^{(\nu)} = \left(\frac{\mathrm{Var}(\hat{e}_{k\omega+\nu})}{\mathrm{Var}(\hat{e}_{k\omega+\nu-p})} \right)^{1/2} \phi_{pp}(\nu)$$

(see, for example, Ula, 1982). For example, if $p = 2$, (3.15) implies that $\phi_{22}(\nu) = \mathrm{Corr}(\hat{e}_{k\omega+\nu}, \hat{e}_{k\omega+\nu-2})$, where $\hat{e}_{k\omega+\nu} = X_{k\omega+\nu} - \hat{X}_{k\omega+\nu}$ and $\hat{e}_{k\omega+\nu-2} = X_{k\omega+\nu-2} - \hat{X}_{k\omega+\nu-2}$, $\hat{X}_{k\omega+\nu}$ and $\hat{X}_{k\omega+\nu-2}$ being the best linear MSE predictors of $X_{k\omega+\nu}$ and $X_{k\omega+\nu-2}$, respectively, in terms of $X_{k\omega+\nu-1}$. It can be easily shown that

$$\hat{X}_{k\omega+\nu} = \rho_1(\nu) \left(\frac{\gamma_0(\nu)}{\gamma_0(\nu-1)} \right)^{1/2} X_{k\omega+\nu-1}$$

and

$$\hat{X}_{k\omega+\nu-2} = \rho_1(\nu-1)\left(\frac{\gamma_0(\nu-2)}{\gamma_0(\nu-1)}\right)^{1/2} X_{k\omega+\nu-1}.$$

It then follows with some manipulations that $\text{Var}(\hat{e}_{k\omega+\nu}) = \gamma_0(\nu)[1 - \rho_1^2(\nu)]$ and $\text{Var}(\hat{e}_{k\omega+\nu-2}) = \gamma_0(\nu-2)[1 - \rho_1^2(\nu-1)]$. Then $\phi_{22}(\nu)$ is obtained as $\text{Cov}(\hat{e}_{k\omega+\nu}, \hat{e}_{k\omega+\nu-2})/[\text{Var}(\hat{e}_{k\omega+\nu})\text{Var}(\hat{e}_{k\omega+\nu-2})]^{1/2}$. Furthermore, if $p(\nu) = 2$, it can be shown that the moment equation for $\phi_2^{(\nu)}$ (Section 4.3) is equal to $[\text{Var}(\hat{e}_{k\omega+\nu})/\text{Var}(\hat{e}_{k\omega+\nu-2})]^{1/2}\phi_{22}(\nu)$.

As (3.15) is defined for all positive integers p, the seasonal PACF, $\phi_{\ell\ell}(\nu)$, can now be defined for any PARMA process for all lags $\ell = 1, 2, \ldots$, by replacing p in (3.15) by ℓ. The relation between $\phi_{\ell\ell}(\nu)$ and $\phi_\ell^{(\nu)}$ is again as given above, with p replaced by ℓ, where $\phi_\ell^{(\nu)}$ now is the last AR parameter in the ℓ-th order AR equation for season ν.

In the following proposition, the cut-off property of $\phi_{\ell\ell}(\nu)$ is established in the case season ν follows an AR(p(ν)) process.

PROPOSITION 3.3. Let $\{X_{k\omega+\nu}\}$ be a periodic stationary $\text{PARMA}_\omega(p(\nu),q(\nu))$ process. If ν is an arbitrary season with $q(\nu) = 0$, then $\phi_{\ell\ell}(\nu) = 0$ for all $\ell > p(\nu)$.

PROOF. For simplicity, let $p(\nu) = p$. Then, for $\ell > p$, (3.14) implies that the best linear MSE predictor of $X_{k\omega+\nu}$ in terms of $X_{k\omega+\nu-1}, \ldots, X_{k\omega+\nu-\ell}$ is $\hat{X}_{k\omega+\nu} = \phi_1^{(\nu)}X_{k\omega+\nu-1} + \ldots + \phi_p^{(\nu)}X_{k\omega+\nu-p}$, so that $\hat{e}_{k\omega+\nu} = a_{k\omega+\nu}$. Since $\hat{e}_{k\omega+\nu-\ell}$ is a linear combination of $(X_{k\omega+\nu-1},\ldots,X_{k\omega+\nu-\ell})$ and hence of $(a_{k\omega+\nu-1},\ldots,a_{k\omega+\nu-\ell})$, it is clear that it is not correlated with $\hat{e}_{k\omega+\nu}$. Thus, by (3.15), $\phi_{\ell\ell}(\nu) = 0$. For a similar argument in the context of ARMA models, see Cryer (1986: 107). ∎

In the context of ARMA models, the PACF , $\phi_{\ell\ell}$, $\ell = 1, 2, \ldots$, plays a primary role in the identification of pure AR processes. There, it is observed that for an AR(p) process, $\phi_{\ell\ell} = 0$ for all $\ell > p$. It can also be shown that $\phi_{\ell\ell}$ is the same as the last parameter in an AR(ℓ) model (Wei, 1990: 14). This also follows from the equation given above relating $\phi_{pp}(\nu)$ to $\phi_p^{(\nu)}$ (replacing p by ℓ and eliminating ν) by noting that the two residual variances in that equation are equal for a stationary process.

Therefore, $\phi_{\ell\ell}$ can be estimated as the estimate of an AR parameter, either by method of moments or more accurately by method of least squares. An iterative technique for the computation of the method of moment estimates of AR parameters was developed by Durbin (1960) (Box and Jenkins, 1976: 82). However, the above technique of estimation of PACF does not carry over to PARMA processes since the parameters (or their estimated values) do not behave as proper correlations. For example, for an arbitrary season ν, let $p(\nu) = 1$ and $q(\nu) = 0$. Then $X_{k\omega+\nu} = \phi_1^{(\nu)} X_{k\omega+\nu-1} + a_{k\omega+\nu}$, and it can be shown that $\phi_1^{(\nu)} = \gamma_1(\nu)/\gamma_0(\nu-1)$, where $\gamma_\ell(\nu)$ is the seasonal autocovariance function (Section 4.3). It is clear that $\phi_1^{(\nu)}$ does not necessarily belong to the interval (-1, 1) as a correlation should.

For the computation of the PACF $\phi_{\ell\ell}(\nu)$ through (3.15), we first report the following general result for the computation of partial correlations (Morrison, 1976: 94). Let $\{Y_1, Y_2, \ldots, Y_p\}$ be a set of p random variables, $p \geq 2$. Also, let Y_i and Y_j be any two variables from this set such that $i \neq j$, then we denote the partial correlation coefficient between Y_i and Y_j by $\rho_{ij.c}$, where $c = \{1, \ldots, p\} \setminus \{i,j\}$ (for any two sets A and B, $A \setminus B = A \cap B^c$). If $p = 2$, then the partial correlation coefficient between the two variables is taken by convention as their ordinary correlation coefficient. If $p > 2$, then all partial correlations can be obtained in terms of ordinary correlations iteratively through the following relation:

$$\rho_{ij.hc} = \frac{\rho_{ij.c} - \rho_{ih.c}\,\rho_{jh.c}}{\sqrt{(1-\rho^2_{ih.c})(1-\rho^2_{jh.c})}}, \tag{3.16}$$

$i, j, h = 1, \ldots, p$; $i \neq j \neq h$; and $c \subset \{1,\ldots,p\} \setminus \{i,j,h\}$. These are computed, firstly, by taking c to be the null set, so that

$$\rho_{ij.h} = \frac{\rho_{ij} - \rho_{ih}\,\rho_{jh}}{\sqrt{(1-\rho^2_{ih})(1-\rho^2_{jh})}}.$$

Then using these, the partial autocorrelations with c containing one element can be obtained by (3.16), and so on.

Now, by convention, $\phi_{11}(\nu) = \mathrm{Corr}(X_{k\omega+\nu}, X_{k\omega+\nu-1}) = \rho_1(\nu)$. Also, from (3.15) and (3.16), it is apparent that

$$\phi_{22}(\nu) = \text{Corr}(\hat{X}_{k\omega+\nu}, \hat{X}_{k\omega+\nu-2}) = \rho_{\nu,\nu-2.\nu-1}$$

$$= \frac{\rho_{\nu,\nu-2} - \rho_{\nu,\nu-1}\ \rho_{\nu-1,\nu-2}}{\sqrt{(1 - \rho^2_{\nu,\nu-1})(1 - \rho^2_{\nu-1,\nu-2})}},$$

where $\hat{X}_{k\omega+\nu}$ and $\hat{X}_{k\omega+\nu-2}$ are the best linear MSE predictors of $X_{k\omega+\nu}$ and $X_{k\omega+\nu-2}$, respectively, in terms of $X_{k\omega+\nu-1}$. Using $\rho_{\nu,\nu-\ell} = \text{Corr}(X_{k\omega+\nu}, X_{k\omega+\nu-\ell}) = \rho_\ell(\nu)$, $\ell = 1, 2$, we have

$$\phi_{22}(\nu) = \frac{\rho_2(\nu) - \rho_1(\nu)\rho_1(\nu-1)}{\sqrt{[1 - \rho_1^2(\nu)][1 - \rho_1^2(\nu-1)]}}. \tag{3.17}$$

This result can also be verified independently by obtaining the correlation of $\hat{X}_{k\omega+\nu}$ and $\hat{X}_{k\omega+\nu-2}$ directly from their equations given earlier in this section.

As an illustration of the cut-off property of the seasonal PACF, $\phi_{\ell\ell}(\nu)$, summarized in Proposition 3.3, let, for example, $p(\nu) = 1$. Then (3.14) is written as $X_{k\omega+\nu} = \phi_1^{(\nu)}X_{k\omega+\nu-1} + a_{k\omega+\nu}$. Multiplying this equation with $X_{k\omega+\nu-1}$ and $X_{k\omega+\nu-2}$ and then taking expectation give $\gamma_1(\nu) = \phi_1^{(\nu)}\gamma_0(\nu-1)$ and $\gamma_2(\nu) = \phi_1^{(\nu)}\gamma_1(\nu-1)$, respectively. Utilizing (3.1), it can then be easily shown that $\rho_2(\nu) = \rho_1(\nu)\rho_1(\nu-1)$. Hence, by (3.17), $\phi_{22}(\nu) = 0$. The same way can be used to show that $\phi_{\ell\ell}(\nu) = 0$ for all $\ell > 1$.

Although we could derive easily, through (3.16), explicit formulas for the first two seasonal partial autocorrelations, $\phi_{11}(\nu)$ and $\phi_{22}(\nu)$, this task for third or higher orders gets tedious. An alternative approach, which will be adopted here, is due to Sakai (1982). He proposed the following algorithm for calculating $\phi_{\ell\ell}(\nu)$ iteratively:

1) Initial Conditions
$\delta^2_\nu(0) = \tau^2_\nu(0) = \gamma_0(\nu)$, $\alpha_\nu(p,0) = 1$; $p = 0, 1, 2, \ldots, \nu = 1, \ldots, \omega$.

2) Order update from p to p+1 (p = 0, 1, 2, ...)
   a) Compute

$$\Delta_\nu(p) = \sum_{m=0}^{p} \gamma_{p+1-m}(\nu-m)\alpha_\nu(p,m)$$

$$\phi_{(p+1)(p+1)}(\nu) = \frac{\Delta_\nu(p)}{\delta_\nu(p)\tau_{\nu-1}(p)} \tag{3.18}$$

b) Update

$$\alpha_\nu(p+1,p+1) = -\Delta_\nu(p)/\tau^2_{\nu-1}(p), \quad \beta_\nu(p+1,p+1) = -\Delta_\nu(p)/\delta^2_\nu(p),$$

$$\delta^2_\nu(p+1) = \delta^2_\nu(p)[1 - \alpha_\nu(p+1,p+1)\beta_\nu(p+1,p+1)],$$

$$\tau^2_\nu(p+1) = \tau^2_{\nu-1}(p)[1 - \alpha_\nu(p+1,p+1)\beta_\nu(p+1,p+1)],$$

and for $i = 1, \ldots, p$,

$$\alpha_\nu(p+1,i) = \alpha_\nu(p,i) + \alpha_\nu(p+1,p+1)\beta_{\nu-1}(p,p+1-i),$$

$$\beta_\nu(p+1,i) = \beta_{\nu-1}(p,i) + \beta_\nu(p+1,p+1)\alpha_\nu(p,p+1-i),$$

where $\gamma_\ell(\nu)$ is again the seasonal autocovariance function and the subscript $\nu-1 = 0$ is always replaced by ω.

Now we will verify analytically that the first two seasonal partial autocorrelations obtained by the above algorithm are identical to those obtained on the basis of (3.16). For $\phi_{11}(\nu)$, following the above algorithm,

$$\phi_{11}(\nu) = \frac{\Delta_\nu(0)}{\delta_\nu(0)\tau_{\nu-1}(0)} = \frac{\gamma_1(\nu)}{[\gamma_0(\nu)\gamma_0(\nu-1)]^{1/2}} = \rho_1(\nu).$$

For $\phi_{22}(\nu)$, it can be shown that $\alpha_\nu(1,1) = -\gamma_1(\nu)/\gamma_0(\nu-1)$, $\beta_\nu(1,1) = -\gamma_1(\nu)/\gamma_0(\nu)$, $\delta^2_\nu(1) = \gamma_0(\nu)[1 - \rho^2_1(\nu)]$, $\tau^2_\nu(1) = \gamma_0(\nu-1)[1 - \rho^2_1(\nu)]$, and

$$\Delta_\nu(1) = \gamma_2(\nu) - \frac{\gamma_1(\nu-1)\gamma_1(\nu)}{\gamma_0(\nu-1)},$$

$$\phi_{22}(\nu) = \frac{\Delta_\nu(1)}{\delta_\nu(1)\tau_{\nu-1}(1)}.$$

Then it follows with simple manipulations that $\phi_{22}(\nu)$ is the same as that given by (3.17).

Now the sample seasonal PACF, $\hat{\phi}_{\ell\ell}(\nu)$, $\ell = 1, 2, \ldots$, can be obtained from (3.18) by replacing the theoretical seasonal autocovariances, $\gamma_{\ell}(\nu)$, by their corresponding sample estimates, $\hat{\gamma}_{\ell}(\nu)$, defined by (3.6).

Sakai (1982) also proved, under the assumption of white noise terms being independent and normal, that if a season ν follows an AR(p(ν)) process, then, for all ν and $\ell > p(\nu)$, the sample seasonal partial autocorrelations $\hat{\phi}_{\ell\ell}(\nu)$ are asymptotically independent (for each ν and ℓ), and normally distributed with zero mean and variance 1/N . Therefore, for large N, the 95% band $(-1.96/N^{1/2}, 1.96/ N^{1/2})$ should be applied for the sample seasonal partial autocorrelations for $\ell > p(\nu)$.

3.5 A Simulated Example

3.5.1 Simulation Results

To illustrate the results concerning the properties of the sample seasonal ACF and PACF, discussed in the previous sections, we consider the following $\text{PARMA}_4(2,2;0,1;3,0;0,4)$ model

$$X_{4k+1} = 0.8\, X_{4(k-1)+4} + 0.9\, X_{4(k-1)+3} + a_{4k+1} - 0.4\, a_{4(k-1)+4} - 0.9\, a_{4(k-1)+3}$$

$$X_{4k+2} = a_{4k+2} + 0.8\, a_{4k+1}$$

$$X_{4k+3} = 1.2\, X_{4k+2} - 0.7\, X_{4k+1} + 0.5\, X_{4(k-1)+4} + a_{4k+3}$$

$$X_{4k+4} = a_{4k+4} - 0.5\, a_{4k+3} - 0.7\, a_{4k+2} + 0.3\, a_{4k+1} - 1.1\, a_{4(k-1)+4}$$

First, we will show using the results of Chapter II that this process is periodic stationary and invertible. For this, note that following the lumped- vector representation as given by (2.1), $p^* = 1$, $q^* = 1$, and

$$L = \begin{bmatrix} 1. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. \\ 0.7 & -1.2 & 1. & 0. \\ 0. & 0. & 0. & 1. \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0. & 0. & 0.9 & 0.8 \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0.5 \\ 0. & 0. & 0. & 0. \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 1. & 0. & 0. & 0. \\ 0.8 & 1. & 0. & 0. \\ 0. & 0. & 1. & 0. \\ 0.3 & -.7 & -.5 & 1. \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0. & 0. & 0.9 & 0.4 \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1.1 \end{bmatrix}.$$

Then, following Proposition 2.1 and using the program listed in Appendix A, the eigenvalues of $L^{-1}U_1$ are (0, 0, -0.63, 0) and those of $\Lambda^{-1}V_1$ are (0, 0, 0.756, 0) which are all less than 1 in modulus. Thus the process above is periodic stationary and invertible.

Next, assuming that the white noise terms are independently and normally distributed with zero means and unit variances ($\sigma_a^2(\nu) = 1$, $\nu = 1, \ldots, 4$), 100 realizations each of length N (years), i.e. 4×N values, N = 30, 100, 500, are simulated from the above PARMA model. Then, for each realization, the sample seasonal autocorrelations, $r_\ell(\nu)$, and sample seasonal partial autocorrelations, $\hat{\phi}_{\ell\ell}(\nu)$, for $\nu = 1, 2, 3, 4$, and $\ell = 1, \ldots, 20$, are computed. Their averages over 100 realizations are also obtained. Besides, in each realization and for all seasons ν, the number of autocorrelations $r_\ell(\nu)$ going outside the corresponding 95% band is also obtained for $\ell > q(\nu)$. The same is done for $\hat{\phi}_{\ell\ell}(\nu)$ for $\ell > p(\nu)$. More precisely, note that the second season's equation is pure MA(1), so that, in view of (3.12), as $q = 1 < \omega = 4$, the relative frequency (over all realizations) of autocorrelations, $r_\ell(2)$, $\ell > q = 1$, going outside the 95% band $(-1.96/N^{1/2}, 1.96/N^{1/2})$ is obtained. This is also true for $\nu = 4$, in which case $q = 4 = \omega < 2\omega$, so that (3.13) applies with $k = 1$, and the relative frequency of $r_\ell(4)$, $\ell > q = 4$, going outside the corresponding 95% band is obtained. For $\nu = 3$, the season is pure AR(3), so that the relative frequency of $\hat{\phi}_{\ell\ell}(3)$, $\ell > p = 3$, going outside the 95% band $(-1.96/N^{1/2}, 1.96/N^{1/2})$ is obtained. This task is done by using the computer program listed in Appendix C, which is written for univariate $PARMA_\omega(p(\nu),q(\nu))$ models with $p(\nu) \le 2\omega$ and $q(\nu) \le 2\omega$. The results are summarized in Tables 3.3(a) through (c), in which the average values of autocorrelations and partial autocorrelations for the first 10 lags are presented only. The relative frequencies of autocorrelations and partial autocorrelations over all realizations going outside the 95% bands mentioned above are given in these tables in percentage, denoted as (fr%). We only include those corresponding to ν = 2 and 4 for the autocorrelations and ν = 3 for the partial autocorrelations since these are the pure MA and pure AR cases, respectively. For the other cases, relative frequencies are only presented in Table 3.3(a) and in parentheses for illustration. In those cases, the relative frequencies of autocorrelations and partial autocorrelations are computed assuming that the equation of season ν is pure MA or AR with orders p(ν) and q(ν), respectively.

To illustrate whether or not the variances $\sigma_a^2(\nu)$ affect the results, the same program (Appendix C) is executed for N = 100 and $\sigma_a^2(\nu)$ = 0.8, 0.5, 4.0, 0.2 for ν = 1, 2, 3, 4, respectively. The results for this case are summarized in Table 3.4(d).

Table 3.3. The Average Sample Seasonal ACF and PACF for $PARMA_4$(2,2; 0,1; 3,0; 0,4) Model

(a) $\sigma_a^2(\nu)$ = 1, N = 30

	Seasonal Autocorrelations				Seas. Partial Autocorr.			
	Season				Season			
lag	1	2	3	4	1	2	3	4
1	.342	.329	.552	-.436	.342	.329	.552	-.436
2	.278	-.000	-.154	-.193	-.508	.138	.434	-.058
3	.358	.012	.225	-.041	-.270	.198	-.514	.182
4	-.220	-.033	-.385	-.334	.318	.170	.123	.173
5	-.070	-.026	-.351	.002	-.271	-.152	.060	.373
6	-.348	.010	.095	.004	.179	.123	-.069	.118
7	-.333	-.013	-.092	-.005	.108	-.106	.036	-.052
8	.082	-.021	.197	-.021	-.024	-.028	-.003	.268
9	-.100	-.007	.192	.013	.118	.006	-.017	.109
10	.209	-.008	-.05	.006	.006	-.057	-.007	.026
fr%	(34.8)	3.1	(41.7)	2.7	(30.0)	(18.8)	1.7	(27.7)

Table 3.3 (cont'd)

(b) $\sigma_a^2(\nu) = 1$, N = 100

	Seasonal Autocorrelations				Seas. Partial Autocorr.			
	Season				Season			
lag	1	2	3	4	1	2	3	4
1	.361	.347	.573	-.451	.361	.347	.573	-.451
2	.315	-.001	-.179	-.193	-.577	.148	.496	-.089
3	.403	.000	.228	-.049	-.291	.281	-.590	.239
4	-.217	-.010	-.421	-.370	.350	.170	.050	.189
5	-.079	-.006	-.365	.010	-.384	-.201	.024	.450
6	-.397	.000	.103	.004	.183	.247	-.013	.136
7	-.347	-.004	-.121	-.005	.067	-.116	.021	-.068
8	.106	-.009	.256	.005	-.046	-.029	-.005	.376
9	-.103	-.001	.222	-.007	.145	.017	-.020	.175
10	.244	.007	-.063	-.001	.041	-.104	-.003	.046
fr%		4.2		3.0			4.4	

(c) $\sigma_a^2(\nu) = 1$, N = 500

	Seasonal Autocorrelations				Seas. Partial Autocorr.			
	Season				Season			
lag	1	2	3	4	1	2	3	4
1	.372	.353	.588	-.461	.372	.353	.588	-.461
2	.311	.001	-.184	-.205	-.587	.151	.519	-.092
3	.399	-.002	.223	-.041	-.285	.301	-.615	.245
4	-.212	-.002	-.423	-.355	.354	.161	.015	-.163
5	-.068	.003	-.359	.000	-.429	-.219	.004	.477
6	-.405	.002	.121	-.001	.143	.294	-.003	.132
7	-.345	.002	-.118	-.004	.045	-.109	.003	-.090
8	.111	-.002	.265	-.004	-.029	-.025	.000	.435
9	-.121	-.003	.228	-.000	.161	.015	-.013	.209
10	.258	.002	-.072	-.007	.073	-.119	-.002	.070
fr%		5.4		4.4			4.9	

Table 3.3 (cont'd)

(d) $\sigma_a^2(\nu)$ = 0.8, 0.5, 4.0, 0.2, N = 100.

	Seasonal Autocorrelations				Seas. Partial Autocorr.			
	Season				Season			
lag	1	2	3	4	1	2	3	4
1	.470	.404	.332	-.773	.470	.404	.332	-.773
2	-.218	-.014	-.096	-.118	-.262	.258	.269	-.233
3	.362	.013	.042	.069	-.424	.134	-.230	.135
4	-.090	.008	-.103	-.108	.415	.251	.016	-.045
5	.000	.000	-.193	-.006	-.413	-.285	.019	.374
6	-.158	-.008	.061	-.005	.086	.341	-.029	.206
7	-.277	.014	-.023	.001	.070	-.085	.006	-.278
8	.065	-.008	.061	-.007	-.041	-.030	.020	.447
9	-.070	-.016	.112	.003	.132	.039	-.000	.109
10	.114	-.011	-.033	.006	.035	-.089	-.003	.037
fr%		4.8		4.1			4.5	

3.5.2 Discussion

Generally speaking, the simulation results above are compatible with the theoretical results in the previous sections. In particular, the cut-off properties of the seasonal ACF are verified for seasons ν = 2 and 4, and those of the seasonal PACF for ν = 3. More precisely, the cut-off in seasonal autocorrelations is clear from the average values of the sample seasonal autocorrelations. For pure MA cases, ν = 2 and 4, for lags $\ell > q(2) = 1$ and $\ell > q(4) = 4$, respectively. Similar comment applies for the seasonal partial autocorrelations for the pure AR case, ν = 3, for lags $\ell > p(3) = 3$. These observations are apparent for all N = 30, 100, and 500, although they become more clear as N gets larger. This is also assured by the relative frequencies (fr%) corresponding to these cases which in turn agree with the proposed 95% bands. As N increases, the relative frequencies approach the theoretically expected 5% values. It should be remembered that the theoretical results we employ are large sample results.

It is seen that the comments above carry over to the case of varying (i.e. non-constant) white noise variances, given in Table 3.3(d).

On the other hand, making comments on the seasonal autocorrelations for ν = 1 and 3, and on the seasonal partial autocorrelations for ν = 1, 2 and 4 is not easy. It can be noted that in these cases no cut-off situations are apparent, as theoretically expected. In particular, for the mixed case, ν = 1, the relative frequencies presented in Table 3.3(a), 34.8% and 30%, are extremely high for both autocorrelations and partial autocorrelations which indicate that the equation of this season should be mixed, or possibly pure MA or AR, but of higher order. It is worth mentioning that, as in the context of ARMA models, the autocorrelations and partial autocorrelations do not offer much help for the identification of the orders of mixed processes.

CHAPTER IV

ESTIMATION IN PARMA PROCESSES

4.1 Introduction

The method of moments is one of the most common methods of estimation in statistical inference. It is also common in the context of time series analysis, as well. Although, in some situations, this method has some serious drawbacks, mainly in the estimation of the parameters in MA processes or ARMA processes in which an MA part is present, it is, however, quite satisfactory in the case of pure AR processes. For PARMA processes, the situation does not differ much. In the following section we will discuss moment estimation in a simple PARMA model, namely the univariate $PARMA_{\omega}(1,1)$ model, through which the difficulty and deficiency of the method of moments in the presence of a MA part is exposed. Pagano (1978) studied this method in univariate PAR processes and obtained some asymptotic properties of the estimates obtained by this method. Our main aim in this chapter is to compare this method for both univariate and multivariate PAR models with the conditional least squares (LS) method through some simulated examples.

In the context of AR models estimation it is known for small samples that the conditional LS method produces slightly better estimates than those obtained by the method of moments, and that the difference between these estimates is negligible for large samples (see, for example, Fuller, 1976: Ch. 8). In this chapter, this fact will be investigated for PAR models in the sense of bias and mean squared error (MSE) criteria through simulation.

4.2 Moment Estimation in $\text{PARMA}_\omega(1,1)$ Process

We know from (2.7), by setting m = 1, that the univariate $\text{PARMA}_\omega(1,1)$ model is written as

$$X_{k\omega+\nu} = \phi_1^{(\nu)} X_{k\omega+\nu-1} + a_{k\omega+\nu} - \theta_1^{(\nu)} a_{k\omega+\nu-1}, \tag{4.1}$$

for all integers k, where, for each fixed ν, $\nu = 1, \ldots, \omega$, $\{a_{k\omega+\nu}\}$ is a white noise process with zero mean and variance $\sigma_a^2(\nu)$.

Moment estimation of the seasonally standardized version of the univariate $\text{PARMA}_\omega(1,1)$ model has been investigated by Salas et al. (1982). Their result involves solving a set of ω simultaneous non-linear equations. The analogous result for the non-standardized model, (4.1), can be obtained by multiplying (4.1) by $X_{k\omega+\nu}$, $X_{k\omega+\nu-1}$ and $X_{k\omega+\nu-2}$, and then taking expectations and making some manipulations give, for each ν, the following set of equations

$$\gamma_0(\nu) = \phi_1^{(\nu)}\gamma_1(\nu) + \sigma_a^2(\nu) - \theta_1^{(\nu)}(\phi_1^{(\nu)} - \theta_1^{(\nu)})\sigma_a^2(\nu-1)$$

$$\gamma_1(\nu) = \phi_1^{(\nu)}\gamma_0(\nu-1) - \theta_1^{(\nu)}\sigma_a^2(\nu-1) \tag{4.2}$$

$$\gamma_2(\nu) = \phi_1^{(\nu)}\gamma_1(\nu-1). \tag{4.3}$$

These equations can also be found in Vecchia (1985a). Then, solving them for $\theta_1^{(\nu)}$ gives

$$\theta_1^{(\nu)} = \phi_1^{(\nu)} + \frac{\gamma_0(\nu) - \phi_1^{(\nu)}\,\gamma_1(\nu)}{\phi_1^{(\nu)}\,\gamma_0(\nu-1) - \gamma_1(\nu)} - \frac{\phi_1^{(\nu+1)}\,\gamma_0(\nu) - \gamma_1(\nu+1)}{\left[\phi_1^{(\nu)}\,\gamma_0(\nu-1) - \gamma_1(\nu)\right]\theta_1^{(\nu+1)}}. \tag{4.4}$$

It can be easily shown that the above equation reduces to equation (17) of Salas et al. (1982) by setting $\gamma_0(\nu) = 1$ and replacing $\gamma_1(\nu)$ by $\rho_1(\nu)$.

Now, replacing $\gamma_\ell(\nu)$ by their estimates $\hat{\gamma}_\ell(\nu)$, defined by (3.6), the moment estimates of $\phi_1^{(\nu)}$ can be obtained directly from (4.3) as $\hat{\phi}_1^{(\nu)} = \hat{\gamma}_2(\nu)/\hat{\gamma}_1(\nu-1)$. The moment estimates of $\theta_1^{(\nu)}$, $\hat{\theta}_1^{(\nu)}$, can then be obtained through (4.4) after replacing $\phi_1^{(\nu)}$ and $\gamma_\ell(\nu)$ by $\hat{\phi}_1^{(\nu)}$ and $\hat{\gamma}_\ell(\nu)$, respectively, and then solving the ω non-linear equations corresponding to all values of ν, $\nu = 1, \ldots, \omega$, simultaneously. Obviously, this is a difficult task. Finally, the moment estimates of $\sigma_a^2(\nu)$, $\hat{\sigma}_a^2(\nu)$, can be obtained from (4.2) by replacing $\phi_1^{(\nu)}$, $\theta_1^{(\nu)}$ and $\gamma_\ell(\nu)$ by their corresponding estimates.

It is known that moment estimation of the parameters of the ARMA(1,1) model produces poor estimates of the MA parameters. They even may not be real or satisfy invertibility conditions. Likewise, these drawbacks are expected to arise in the $\text{PARMA}_\omega(1,1)$ model, which reduces to an ARMA(1,1) model if $\omega = 1$. In fact, Vecchia (1985a) investigated moment estimation in $\text{PARMA}_\omega(1,1)$ model through simulation and reached such conclusions. In about 50% of the simulation runs, the moment estimates were not feasible. For feasible solutions, the moment estimates of AR parameters were usually good but those for MA parameters were usually not satisfactory.

It can also be shown that moment estimation in the pure $\text{PMA}_\omega(1)$ model has the same difficulties as in the $\text{PARMA}_\omega(1,1)$ model. This can be seen from (4.4) by setting $\phi_1^{(\nu)} = 0$, in which case we will again have a set of ω non-linear equations to be solved. For the m-variate $\text{PARMA}_\omega(1,1)$ model, an equation similar to (4.4) can be obtained in matrix terms. In that case, ω non-linear matrix equations are to be solved simultaneously, which is a formidable task.

Therefore, moment estimation in the presence of a MA part is not recommended for PARMA models due to its difficulties and deficiencies.

4.3 Moment Estimation in PAR Processes

We now consider the periodic stationary m-variate $\text{PAR}_\omega(p(\nu))$ model, which can be written, following (1.5) and setting all Θ's to zero, as

$$X_{k\omega+\nu} = \Phi_1^{(\nu)} X_{k\omega+\nu-1} + \ldots + \Phi_{p(\nu)}^{(\nu)} X_{k\omega+\nu-p(\nu)} + a_{k\omega+\nu}. \tag{4.5}$$

Now let ν be an arbitrary season and denote $p(\nu)$ by p for simplicity of presentation, then post-multiplying (4.5) by $X^T_{k\omega+\nu-1}$, ..., $X^T_{k\omega+\nu-p}$ and taking expectations, give, respectively,

$$\Sigma_1(\nu) = \Phi_1^{(\nu)}\Sigma_0(\nu-1) + \Phi_2^{(\nu)}\Sigma_1^T(\nu-1) + \ldots + \Phi_p^{(\nu)}\Sigma_{p-1}^T(\nu-1)$$

$$\Sigma_2(\nu) = \Phi_1^{(\nu)}\Sigma_1(\nu-1) + \Phi_2^{(\nu)}\Sigma_0(\nu-2) + \ldots + \Phi_p^{(\nu)}\Sigma_{p-2}^T(\nu-2)$$

$$\vdots$$

$$\Sigma_p(\nu) = \Phi_1^{(\nu)}\Sigma_{p-1}(\nu-1) + \Phi_2^{(\nu)}\Sigma_{p-2}(\nu-2) + \ldots + \Phi_p^{(\nu)}\Sigma_0(\nu-p)$$

where $\Sigma_\ell(\nu)$ are the seasonal autocovariance matrices as defined by (2.8). These equations are analogous to the multivarite Yule-Walker equations in the context of vector ARMA models (Fuller, 1976: 73), and therefore named as seasonal Yule-Walker equations (Pagano, 1978). Employing matrix notation, the above equations can be written as

$$\Sigma_\nu = \Phi_\nu\Gamma_\nu \quad \text{or} \quad \Phi_\nu = \Sigma_\nu\Gamma_\nu^{-1}, \tag{4.6}$$

where $\Phi_\nu = [\Phi_1^{(\nu)}, \ldots, \Phi_p^{(\nu)}]$, $\Sigma_\nu = [\Sigma_1(\nu), \ldots, \Sigma_p(\nu)]$, and

$$\Gamma_\nu = \begin{bmatrix} \Sigma_0(\nu-1) & \Sigma_1(\nu-1) & \ldots & \Sigma_{p-1}(\nu-1) \\ \Sigma_1^T(\nu-1) & \Sigma_0(\nu-2) & \ldots & \Sigma_{p-2}(\nu-2) \\ \vdots & \vdots & & \vdots \\ \Sigma_{p-1}^T(\nu-1) & \Sigma_{p-2}^T(\nu-2) & \ldots & \Sigma_0(\nu-p) \end{bmatrix}.$$

Thus, moment estimates, $\hat{\Phi}_1^{(\nu)}, \ldots, \hat{\Phi}_p^{(\nu)}$, of ν-th season AR parameters can be obtained through (4.6) by replacing the seasonal autocovariance matrices with their corresponding sample estimates which in turn are defined, in analogy with (3.6), as

$$\hat{\Sigma}_\ell(\nu) = \frac{1}{N}\sum_{k=0}^{N-1}(X_{k\omega+\nu} - \bar{X}_\nu)(X_{k\omega+\nu-\ell} - \bar{X}_{\nu-\ell})^T,$$

in which $\bar{X}_\nu = \frac{1}{N}\sum_{k=0}^{N-1} X_{k\omega+\nu}$ is the sample mean vector for season ν, and the terms are set to zero whenever $k\omega + \nu - \ell < 1$. The moment estimates, $\hat{\Sigma}_a(\nu)$, of the white noise variance-covariance matrices, $\Sigma_a(\nu)$, $\nu = 1, \ldots,$

ω, may be obtained through the following equation

$$\Sigma_a(\nu) = \Sigma_0(\nu) - \Phi_1^{(\nu)}\Sigma_1^T(\nu) - \dots - \Phi_p^{(\nu)}\Sigma_p^T(\nu) = \Sigma_0(\nu) - \Phi_\nu\Sigma_\nu^T$$

$$= \Sigma_0(\nu) - \Sigma_\nu\Gamma_\nu^{-1}\Sigma_\nu^T = \Sigma_0(\nu) - \Phi_\nu\Gamma_\nu\Phi_\nu^T, \quad (4.7)$$

which follows by post-multiplying (4.5) with $X_{k\omega+\nu}^T$ and then taking expectations.

The statistical properties of the moment estimators for univariate Gaussian PAR processes were investigated by Pagano (1978). He showed that these estimators are almost surely consistent and asymptotically efficient, and therefore asymptotically joint normal and unbiased. The moment estimators of ϕ's are also asymptotically independent.

4.4 Conditional Least Squares and Maximum Likelihood Estimation in PAR Processes

Consider the univariate $PAR_\omega(1)$ process, $X_{k\omega+\nu} = \phi_1^{(\nu)}X_{k\omega+\nu-1} + a_{k\omega+\nu}$. On the basis of an observed realization of size $N\omega$ (say, N years) from the time series, which we denote its seasonally mean-subtracted version by $X_1, \dots, X_{N\omega}$, we have the following relations

$$\left.\begin{aligned} a_1 &= X_1 - \phi_1^{(1)}X_0 \\ a_2 &= X_2 - \phi_1^{(2)}X_1 \\ &\vdots \\ a_{N\omega} &= X_{N\omega} - \phi_1^{(\omega)}X_{N\omega-1} \end{aligned}\right\} \quad (4.8)$$

Therefore, in the equations above, X_1, X_2 and $X_{N\omega}$, for example, represent in fact $X_1 - \mu_1$, $X_2 - \mu_2$ and $X_{N\omega} - \mu_\omega$, respectively. Then the conditional LS estimates of $\phi_1^{(\nu)}$, $\nu = 1, \dots, \omega$, are those which minimize

$$S^* = \sum_{j=2}^{N\omega} a_j^2 = \sum_{\nu=1}^{\omega}\sum_{k=\alpha_\nu}^{N-1} a_{k\omega+\nu}^2 = \sum_{\nu=1}^{\omega}\sum_{k=\alpha_\nu}^{N-1}(X_{k\omega+\nu} - \phi_1^{(\nu)}X_{k\omega+\nu-1})^2 = \sum_{\nu=1}^{\omega} S_\nu^*, \quad (4.9)$$

where $\alpha_1 = 1$, and $\alpha_\nu = 0$ for all other ν. The term conditional actually comes from the conditional likelihood function, as to be discussed later, and is related to the fact that in (4.8) it is not possible to obtain a_1 since an unknown observation, X_0, is involved. This in turn explains why j starts from 2 in the first formula above.

It can be seen from (4.9) that S^*, called the conditional sum of squares, is written as the sum of ω different sum of squares, S^*_ν, $\nu = 1, \ldots, \omega$, such that, for each ν, S^*_ν is the sum of squares of error terms which belong to season ν only. Thus, for an arbitrary ν, minimizing S^* with respect to $\phi_1^{(\nu)}$ reduces to minimizing S^*_ν only. Therefore, we will adopt the regression model and utilize it to obtain the conditional LS estimates of the parameters for each of the ω seasons, separately. More precisely, in minimizing S^*_ν with respect to $\phi_1^{(\nu)}$, $X_{k\omega+\nu}$ and $X_{k\omega+\nu-1}$, $k = \alpha_\nu, \ldots, N-1$, will be the values of the dependent and independent variables in regression, respectively, and the conditional LS estimate of $\phi_1^{(\nu)}$ will be obtained as the regression coefficient. This approach has already been adopted for estimation in both univariate and multivariate AR processes (see, for example, Priestley, 1981, and Wei, 1990).

Another important feature of the conditional LS estimates is that under the assumption that the white noise process is Gaussian, they are identical with the conditional maximum likelihood (ML) estimates. To see this, let us again consider the univariate $PAR_\omega(1)$ from which the realization, $X_1, \ldots, X_{N\omega}$ is observed. Assume that $a_{k\omega+\nu}$ are independently distributed as $N[0,\sigma_a^2(\nu)]$. Then it follows that the joint pdf of $a_2, \ldots, a_{N\omega}$ in (4.8) is

$$f(a_2,\ldots,a_{N\omega}) = (2\pi)^{-(N\omega-1)/2}\,[\sigma_a^2(1)]^{-(N-1)/2}\,[\prod_{2}^{\omega}\sigma_a^2(\nu)]^{-N/2}\exp\left\{-\frac{1}{2\sigma_a^2(1)}\right.$$

$$\left.\times\sum_{k=1}^{N-1}a_{k\omega+1}^2-\frac{1}{2\sigma_a^2(2)}\sum_{k=0}^{N-1}a_{k\omega+2}^2-\ldots-\frac{1}{2\sigma_a^2(\omega)}\sum_{k=0}^{N-1}a_{k\omega+\omega}^2\right\}$$

$$= (2\pi)^{-(N\omega-1)/2}\,[\sigma_a^2(1)]^{1/2}\,[\prod_{1}^{\omega}\sigma_a^2(\nu)]^{-N/2}\exp\left\{-\sum_{\nu=1}^{\omega}\frac{1}{2\sigma_a^2(\nu)}\sum_{k=\alpha_\nu}^{N-1}a_{k\omega+\nu}^2\right\},$$

where α_ν is as given in (4.9). Now, in (4.8), assuming that X_1 is fixed, the Jacobian of the transformation from $(a_2,\ldots,a_{N\omega})$ to $(X_2, \ldots, X_{N\omega})$ is one. Hence, the conditional likelihood function, conditional with respect to X_1, is given by

$$L^* = g^*(X_2, \ldots, X_{N\omega}|X_1) = (2\pi)^{-(N\omega-1)/2} [\sigma_a^2(1)]^{1/2} [\prod_1^\omega \sigma_a^2(\nu)]^{-N/2}$$

$$\times \exp\left\{-\sum_{\nu=1}^{\omega} \frac{1}{2\sigma_a^2(\nu)} \sum_{k=\alpha_\nu}^{N-1} (X_{k\omega+\nu} - \phi_1^{(\nu)} X_{k\omega+\nu-1})^2\right\}$$

$$= (2\pi)^{-(N\omega-1)/2} [\sigma_a^2(1)]^{1/2} [\prod_1^\omega \sigma_a^2(\nu)]^{-N/2} \exp\left\{-\frac{1}{2}\sum_{\nu=1}^{\omega} \frac{S_\nu^*}{\sigma_a^2(\nu)}\right\}, \qquad (4.10)$$

and its natural logarithm (ℓn) is

$$\ell n L^* = -\frac{N\omega-1}{2}\ell n(2\pi) + \frac{1}{2}\ell n[\sigma_a^2(1)] - \frac{N}{2}\sum_{\nu=1}^{\omega} \ell n[\sigma_a^2(\nu)] - \frac{1}{2}\sum_{\nu=1}^{\omega} \frac{S_\nu^*}{\sigma_a^2(\nu)}.$$

Then the conditional ML estimates of the parameters $\phi_1^{(\nu)}$ and $\sigma_a^2(\nu)$, $\nu = 1, \ldots, \omega$, can be obtained from $\ell n L^*$ by taking the partial derivatives with respect to the parameters and setting them to zero. For $\phi_1^{(\nu)}$, this process leads to minimizing S_ν^* with respect to $\phi_1^{(\nu)}$. Therefore, the conditional ML estimates of $\phi_1^{(\nu)}$ are the same as their conditional LS estimates. The conditional ML estimates of $\sigma_a^2(\nu)$, denoted by $\tilde{\sigma}_a^2(\nu)$, are obtained as $\tilde{\sigma}_a^2(1) = \tilde{S}_1^*/(N-1)$, and $\tilde{\sigma}_a^2(\nu) = \tilde{S}_\nu^*/N$, $\nu = 2, \ldots, \omega$, where $\tilde{S}_\nu^*$ is the estimated sum of squares for season ν. Note that since, in this case, $\alpha_1 = 1$ and $\alpha_\nu = 0$, $\nu = 2, \ldots, \omega$, then $\tilde{\sigma}_a^2(\nu) = \tilde{S}_\nu^*/(N-\alpha_\nu)$ for $\nu = 1, \ldots, \omega$, where $N - \alpha_\nu$ is the number of terms in $\tilde{S}_\nu^*$.

Although we are assuming that the time series $\{X_{k\omega+\nu}\}$ is seasonally mean-subtracted, it is worth mentioning that if estimation of the seasonal means, μ_ν, is of question, then, replacing $\{X_{k\omega+\nu}\}$ with $\{X_{k\omega+\nu} - \mu_\nu\}$ in (4.9), it can easily be seen that, under the assumption of independence and normality of the white noise terms, the conditional LS and conditional ML methods will again give the same estimates for μ_ν. However, in this case, the conditional approach will lose its simplicity since the conditional sum of squares, S_ν^*, will involve parameters other than those of the ν-th season, so that minimization can not now be performed in a season-wise manner. For example, for the univariate $PAR_2(1)$ model, written as

$$\begin{aligned} X_{2k+1} &= \phi_1^{(1)} X_{2(k-1)+2} + a_{2k+1} \\ X_{2k+2} &= \phi_1^{(2)} X_{2k+1} + a_{2k+2}, \end{aligned} \tag{4.11}$$

following (4.9), the conditional sum of squares is $S^* = \sum_{\nu=1}^{2} S_\nu^*$, such that

$$S_1^* = \sum_{k=1}^{N-1} [(X_{2k+1} - \mu_1) - \phi_1^{(1)}(X_{2(k-1)+2} - \mu_2)]^2$$

$$S_2^* = \sum_{k=0}^{N-1} [(X_{2k+2} - \mu_2) - \phi_1^{(2)}(X_{2k+1} - \mu_1)]^2.$$

Note that each of S_1^* and S_2^* is a function of μ_1 and μ_2, so that minimizing S^* with respect to μ_1, say, does not reduce to minimizing S_1^* only. This means that the conditional LS or ML estimates of μ_1 and μ_2 should be obtained by taking the partial derivatives of S^* with respect to μ_1 and μ_2, setting them to zero, and then solving the resulting two equations simultaneously for μ_1 and μ_2. This fact generalizes to $PAR_\omega(p(\nu))$ processes in which case ω simultaneous equations have to be solved.

The common way to overcome this problem is to estimate μ_ν by $\bar{X}_\nu$, the sample mean of season ν, which is also the moment estimate of μ_ν. This method is also adopted as an adequate approximation in the context of ML estimation of ARMA models (Box and Jenkins, 1976: 210). It should also be pointed out that the estimates of μ_ν affect the estimates of the other parameters. In using regression model for minimizing S_ν^*, the mean problem is usually handled by including a constant term in the regression equation of each season. For the univariate $PAR_2(1)$ model, for example, the constant terms corresponding to seasons $\nu = 1$ and 2 are, respectively, $\mu_1 - \phi_1^{(1)}\mu_2$ and $\mu_2 - \phi_1^{(2)}\mu_1$. It will be apparent from the next example that the estimation of these constants by regression is equivalent, for season 2, to estimating μ_ν by $\bar{X}_\nu$, but is slightly different for season 1. More generally, for any PAR model, it can be shown that, for seasons with $\alpha_\nu = 0$, these two methods are equivalent, but for seasons ν with $\alpha_\nu > 0$, there are slight differences. The variable α_ν was defined for the $PAR_\omega(1)$ model as $\alpha_1 = 1$ and $\alpha_\nu = 0$ for other ν, and it was mentioned that $N - \alpha_\nu$ is the number of terms in S^*_ν. For the more general case of (univariate or m-variate) $PAR_\omega(p(\nu))$ model, α_ν is given as

$$\alpha_\nu = \left[\frac{p(\nu) - \nu}{\omega} + 1\right], \qquad (4.12)$$

where [c] stands for the integral part of of the real number c, and $N - \alpha_\nu$ is the number of terms in the sum of squares of errors for season ν, S^*_ν, which is defined explicitly for the $PAR_\omega(p(\nu))$ model later in this section.

The next example illustrates the relations between conditional LS and ML estimates and moment estimates.

EXAMPLE 4.1. We again consider the univariate $PAR_2(1)$ model, (4.11). It can be easily seen that (4.6), with m = 1 and p = 1, implies that for ν = 2, for which $\alpha_2 = 0$, the moment estimate of $\phi_1^{(2)}$ is $\hat{\phi}_1^{(2)} = \hat{\gamma}_1(2) / \hat{\gamma}_0(1)$, which can be written as

$$\hat{\phi}_1^{(2)} = \frac{\sum_{k=0}^{N-1}(X_{2k+2} - \bar{X}_2)(X_{2k+1} - \bar{X}_1)}{\sum_{k=0}^{N-1}(X_{2k+1} - \bar{X}_1)^2},$$

by using (3.6) for $\hat{\gamma}_0(1)$ and $\hat{\gamma}_1(2)$. It can also be seen that this is identical with the conditional LS estimate of $\phi_1^{(2)}$ resulting from regressing X_{2k+2} on X_{2k+1} for k = 0, 1, ..., N - 1. Also, the LS estimate of the constant term in regression is $\bar{X}_2 - \hat{\phi}_1^{(2)}\bar{X}_1$. Therefore, by this constant, μ_ν, ν = 1, 2, in S^*_2 are being estimated by $\bar{X}_\nu$. The moment estimate of $\sigma_a^2(2)$ can be obtained through (4.7) as $\hat{\sigma}_a^2(2) = \hat{\gamma}_0(2) - \hat{\phi}_1^{(2)}\hat{\gamma}_1(2)$, which reduces to

$$\hat{\sigma}_a^2(2) = \frac{1}{N}\sum_{k=0}^{N-1}[(X_{2k+2} - \bar{X}_2) - \hat{\phi}_1^{(2)}(X_{2k+1} - \bar{X}_1)]^2,$$

which is identical with the conditional ML estimate, $\tilde{\sigma}_a^2(2) = \tilde{S}^*_2/N$, where

$\tilde{S}_2^*$ is the estimated S_2^* in which $\phi_1^{(2)}$ is being estimated by $\hat{\phi}_1^{(2)}$, and μ_ν, $\nu = 1, 2$, in S_2^* are being estimated by $\bar{X}_\nu$.

For $\nu = 1$, for which $\alpha_1 = 1$, the moment estimate of $\phi_1^{(1)}$ is $\hat{\phi}_1^{(1)} = \hat{\gamma}_1(1) / \hat{\gamma}_0(2)$, which can be written as

$$\hat{\phi}_1^{(1)} = \frac{\sum_{k=1}^{N-1}(X_{2k+1} - \bar{X}_1)(X_{2(k-1)+2} - \bar{X}_2)}{\sum_{k=0}^{N-1}(X_{2k+2} - \bar{X}_2)^2}.$$

On the other hand, the conditional LS estimate of $\phi_1^{(1)}$, $\tilde{\phi}_1^{(1)}$, resulting from regressing X_{2k+1} on $X_{2(k-1)+2}$ for $k = 1, \ldots, N - 1$, is given as

$$\tilde{\phi}_1^{(1)} = \frac{\sum_{k=1}^{N-1}(X_{2k+1} - \tilde{X}_1)(X_{2(k-1)+2} - \tilde{X}_2)}{\sum_{k=1}^{N-1}(X_{2(k-1)+2} - \tilde{X}_2)^2},$$

where $\tilde{X}_1 = \frac{1}{N-1} \sum_{k=1}^{N-1} X_{2k+1}$ and $\tilde{X}_2 = \frac{1}{N-1} \sum_{k=1}^{N-1} X_{2(k-1)+2}$. The adjusted sample means $\tilde{X}_1$ and $\tilde{X}_2$ differ from $\bar{X}_1$ and $\bar{X}_2$ by the exclusion of the observations X_1 and X_{2N}, respectively. Therefore, the moment estimate of $\phi_1^{(1)}$, $\hat{\phi}_1^{(1)}$, is not the same as its conditional LS counterpart, $\tilde{\phi}_1^{(1)}$. Aside from the difference in means, the denominator of $\hat{\phi}_1^{(1)}$ contains one additional term as compared to that of $\tilde{\phi}_1^{(1)}$. It should however be pointed out that, for large N, these differences can easily be shown to be negligible. Also the LS estimate of the constant term in regression will be $\tilde{X}_1 - \tilde{\phi}_1^{(1)}\tilde{X}_2$. Therefore, by this constant, μ_ν, $\nu = 1, 2$, in S_1^* are being estimated by $\tilde{X}_\nu$ rather than $\bar{X}_\nu$. The moment estimate of $\sigma_a^2(1)$ is $\hat{\sigma}_a^2(1) = \hat{\gamma}_0(1) - \hat{\phi}_1^{(1)}\hat{\gamma}_1(1)$, which reduces to

$$\hat{\sigma}_a^2(1) = \frac{1}{N}\left(\sum_{k=0}^{N-1}(X_{2k+1} - \bar{X}_1)^2 - [\hat{\phi}_1^{(1)}]^2 \sum_{k=0}^{N-1}(X_{2k+2} - \bar{X}_2)^2\right).$$

The conditional ML estimate of $\sigma_a^2(1)$ is $\tilde{\sigma}_a^2(1) = \tilde{S}_1^*/(N-1)$, which can be written as

$$\tilde{\sigma}_a^2(1) = \frac{1}{N-1}\sum_{k=1}^{N-1}[(X_{2k+1} - \tilde{X}_1) - \tilde{\phi}_1^{(1)}(X_{2(k-1)+2} - \tilde{X}_2)]^2.$$

$$= \frac{1}{N-1}\left[\sum_{k=1}^{N-1}(X_{2k+1} - \tilde{X}_1)^2 - [\tilde{\phi}_1^{(1)}]^2 \sum_{k=1}^{N-1}(X_{2(k-1)+2} - \tilde{X}_2)^2\right].$$

It may be seen that each of the two summations in $\hat{\sigma}_a^2(1)$ contains one additional term as compared to those in $\tilde{\sigma}_a^2(1)$. The two estimates also differ in terms of estimates of μ_ν and $\phi_1^{(1)}$, and also in terms of the divisor term. Therefore, the moment estimate of $\sigma_a^2(1)$ is not the same as its conditional ML counterpart. Again, the difference will be negligible for large samples. ■

More generally, for any PAR model, it can be shown that, for seasons with $\alpha_\nu = 0$, the conditional ML estimates of parameters are identical with the moment estimates, but for seasons with $\alpha_\nu > 0$, there are slight differences which are negligible for large samples.

Generalization of the above results to the univariate $PAR_\omega(p(\nu))$ process, which is defined by (4.5) with m = 1, is straightforward. More precisely, on the basis of an observed seasonally mean-subtracted realization, $X_1, \ldots, X_{N\omega}$, it can be shown, in analogy with (4.9), that the conditional sum of squares, S^*, is written as

$$S^* = \sum_{\nu=1}^{\omega} \sum_{k=\alpha_\nu}^{N-1} a_{k\omega+\nu}^2 = \sum_{\nu=1}^{\omega} \sum_{k=\alpha_\nu}^{N-1} (X_{k\omega+\nu} - \phi_1^{(\nu)} X_{k\omega+\nu-1} - \ldots - \phi_{p(\nu)}^{(\nu)} X_{k\omega+\nu-p(\nu)})^2$$

$$= \sum_{\nu=1}^{\omega} S_\nu^*,$$

where α_ν is as defined by (4.12). It should be noted that S_ν^*, the sum of squares for season ν, contains only $a_{k\omega+\nu}$ terms for season ν with $k = \alpha_\nu, \ldots, N-1$, which do not contain any unknown observations like X_0, X_{-1}, X_{-2}, ... For example, for a $PAR_2(2,4)$ process, for which $\alpha_1 = 1$ and $\alpha_2 = 2$ by (4.12), S_1^* contains $(a_3,a_5,\ldots)$ and S_2^* contains $(a_6,a_8,\ldots)$, because a_1, a_2 and a_4 involve unknown observations (X_0,X_{-1}), (X_0,X_{-1},X_{-2}) and (X_0), respectively. The conditional likelihood function $L^* = g^*(X_3,X_5,X_6,X_7, \ldots,X_{N\omega}|X_1,X_2,X_4)$ can be obtained from the pdf $f(a_3,a_5,a_6,a_7,\ldots,a_{N\omega})$ with a unit Jacobian. The exact likelihood function, L, can be obtained as $L = g(X_1,\ldots,X_{N\omega}) = L^* h(X_1,X_2,X_4)$, where $h(X_1,X_2,X_4)$ is the joint pdf of X_1,

X_2 and X_4. It can be shown, in the same manner as followed for the $PAR_\omega(1)$ model discussed previously, that the conditional LS and conditional ML estimates of the ϕ's are the same, both methods being based on minimizing S^* seasonwise with respect to these parameters. Multiple regression methods can be employed directly for this purpose. The conditional ML estimate of $\sigma_a^2(\nu)$, the white noise variance for season ν, can again be obtained as

$$\tilde{\sigma}_a^2(\nu) = \frac{\tilde{S}_\nu^*}{N-\alpha_\nu},$$

where $\tilde{S}_\nu^*$ is the estimated sum of squares for season ν, which is defined as S_ν^* above but with ϕ's replaced by their conditional ML estimates, and $N - \alpha_\nu$ is the number of terms in S_ν^*. The results in Example 4.1 also carry over to the univariate $PAR_\omega(p(\nu))$ model, as mentioned earlier, so that for seasons ν with $\alpha_\nu = 0$, the moment and conditional ML estimates of $\phi_1^{(\nu)}$, ..., $\phi_{p(\nu)}^{(\nu)}$, and $\sigma_a^2(\nu)$ are identical and μ_ν's in S_ν^* are estimated by the sample means $\bar{X}_\nu$ in regression. For seasons with $\alpha_\nu > 0$, the moment and conditional ML estimates are again slightly different, and the means are again estimated in a slightly different way, all these differences being negligible for large samples. The moment estimates of the parameters can be obtained from (4.6) and (4.7).

The least squares theory for regression (or more precisely general linear model) involves the crucial assumption that the independent variables are nonrandom variables. Therefore, the inferences there are conditional on the given values of the independent variables. However, most of the results in regression are valid for the case where the independent variables are random, provided that the joint distribution of dependent and independent variables satisfy certain conditions. A multivariate normal joint distribution is one such case. If the dependent and p independent variables have a multivariate normal joint distribution, and if N independent observation vectors are available from this joint distribution, then the regression coefficients estimated by LS are again (as in the nonrandom independent variables case) ML and also uniformly minimum variance unbiased (UMVU) estimates. The ML and UMVU estimates of the residual variance are again given as the sum of squares of estimated residuals divided by N and N - p - 1, respectively, where 1 accounts for the regression constant. For more details, see, for example, Ula (1983).

The independence and normality assumption on the white noise terms in PAR models imply joint multivariate normality of the process $\{X_{k\omega+\nu}\}$. However, any regression between these variables does not satisfy the crucial assumption in regression theory about the independence of observation vectors on these variables, as $\{X_{k\omega+\nu}\}$ is a correlated process. However, for the $PAR_{\omega}(p(\nu))$ process, $X_{k\omega+\nu} = \phi_1^{(\nu)}X_{k\omega+\nu-1} + \dots + \phi_{p(\nu)}^{(\nu)}X_{k\omega+\nu-p(\nu)} + a_{k\omega+\nu}$, the observations on the vector $\{X_{k\omega+\nu}, X_{k\omega+\nu-1}, \dots, X_{k\omega+\nu-p(\nu)}\}$ for $k = \alpha_{\nu}, \dots, N - 1$ for a given season will be at least $\omega - p(\nu)$ time units apart. Therefore, if ω is large and $p(\nu)$ is relatively small, the observation vectors will be very weakly correlated due to the decaying nature of the autocorrelation function. The PAR case has an advantage in this sense over the AR case, where no separation is available between the observation vectors. The different nature of time series and regression estimates can be clearly seen from the fact that conditional LS estimates for $\phi_1^{(\nu)}$ in $PAR_{\omega}(1)$ model, for example, are exact ML estimates under regression assumptions but they are only conditional ML estimates under PAR assumptions. Also the UMVU property, which is valid under regression assumptions, can not be attached to conditional LS estimates under PAR assumptions. As far as the white noise variance is concerned, a regression-type estimate (as we will call it) can still be defined for the univariate $PAR_{\omega}(p(\nu))$ model as

$$\approx{\sigma}_a^2(\nu) = \frac{\tilde{S}_{\nu}^*}{d.f.},$$

where the degrees of freedom $d.f. = N - \alpha_{\nu} - p(\nu) - 1$, in which $N - \alpha_{\nu}$ is the number of terms in $\tilde{S}_{\nu}^*$, $p(\nu)$ is the number of independent variables in regression, and 1 stands for the regression constant. This estimate corresponds to the UMVU estimate of the residual variance in regression. The ML estimate of the residual variance in regression is obtained by using $d.f. = N - \alpha_{\nu}$ in the above formula, which then is the same as the conditional ML estimate, $\tilde{\sigma}_a^2(\nu)$, of $\sigma_a^2(\nu)$. For the univariate $PAR_2(1)$ model, for example, the regression-type estimates of $\sigma_a^2(\nu)$ are $\tilde{\tilde{\sigma}}_a^2(1) = \tilde{S}_1^*/(N-3)$ and $\tilde{\tilde{\sigma}}_a^2(2) = \tilde{S}_1^*/(N-2)$, as $\alpha_1 = 1$, $\alpha_2 = 0$ and $p(1) = p(2) = 1$. For the same model, the conditional ML estimates of $\sigma_a^2(\nu)$ are $\tilde{\sigma}_a^2(1) = \tilde{S}_1^*/(N-1)$ and $\tilde{\sigma}_a^2(2) = \tilde{S}_2^*/N$. It had been shown in the context of AR models (Fuller, 1976: 337) that the regression-type estimator of the white noise variance is consistent. Also, in the same context, Jenkins and Watts (1968) suggest that the regression-type estimates are more appropriate.

A further generalization to the more general case of an m-variate $PAR_\omega(p(\nu))$ model can also be made. Here, the appropriate regression model to be adopted for conditional LS estimation is the multivariate multiple regression model, and the general form of the conditional sum of squares, in analogy with the univariate case, is given by

$$S^* = \sum_{\nu=1}^{\omega} \sum_{k=\alpha_\nu}^{N-1} a^T_{k\omega+\nu} a_{k\omega+\nu} = \sum_{\nu=1}^{\omega} S^*_\nu = \sum_{\nu=1}^{\omega} \sum_{j=1}^{m} S^*_{\nu,j},$$

where, by (4.5), $a_{k\omega+\nu} = X_{k\omega+\nu} - \Phi_1^{(\nu)} X_{k\omega+\nu-1} - \dots - \Phi_{p(\nu)}^{(\nu)} X_{k\omega+\nu-p(\nu)}$ is an m-variate error vector which can be written as $(a_{k\omega+\nu,1}, \dots, a_{k\omega+\nu,m})^T$, S^*_ν is again the sum of squares for season ν, $S^*_{\nu,j} = \sum_{k=\alpha_\nu}^{N-1} a^2_{k\omega+\nu,j}$ is the sum of squares for season ν for the j-th dimension, and α_ν is as defined by (4.12). It can be seen that minimizing S^* with respect to $\Phi_i^{(\nu)}$, $i = 1, \dots, p(\nu)$, reduces to minimizing S^*_ν separately, and it can also be seen that minimizing S^*_ν with respect to the elements in the j-th rows of $\Phi_i^{(\nu)}$ matrices reduces to minimizing $S^*_{\nu,j}$. Hence, the conditional LS estimates of the ϕ's in each dimension are those which minimize the sum of squares for that dimension. For the adoption of the multivariate regression model for minimization of S^*_ν, the m components of the m-dimensional vector $X_{k\omega+\nu}$, in (4.5), are treated as the set of dependent variables, while the $mp(\nu)$ components of $(X^T_{k\omega+\nu-1}, \dots, X^T_{k\omega+\nu-p(\nu)})^T$ stand for the independent variables. This is also in analogy with conditional LS estimation in vector AR processes (Fuller, 1976: 339). It can be shown that if the error vectors are independent and normally distributed, then the conditional ML estimates of $\Phi_1^{(\nu)}, \dots, \Phi_{p(\nu)}^{(\nu)}$ are again obtained by minimizing S^*_ν with respect to these parameters and therefore, they are again the same as the conditional LS estimates.

The conditional LS and conditional ML approaches for the m-variate case are clarified in the following example.

EXAMPLE 4.2. Consider the bivariate PAR$_\omega$(1) model $X_{k\omega+\nu} = \Phi_1^{(\nu)} X_{k\omega+\nu-1} + a_{k\omega+\nu}$, which can be expressed as

$$\begin{bmatrix} X_{k\omega+\nu,1} \\ X_{k\omega+\nu,2} \end{bmatrix} = \begin{bmatrix} \phi_{11}^{(\nu)} & \phi_{12}^{(\nu)} \\ \phi_{21}^{(\nu)} & \phi_{22}^{(\nu)} \end{bmatrix} \begin{bmatrix} X_{k\omega+\nu-1,1} \\ X_{k\omega+\nu-1,2} \end{bmatrix} + \begin{bmatrix} a_{k\omega+\nu,1} \\ a_{k\omega+\nu,2} \end{bmatrix}.$$

Then, viewing $X_{k\omega+\nu,1}$ and $X_{k\omega+\nu,2}$ as the dependent variables, and $X_{k\omega+\nu-1,1}$ and $X_{k\omega+\nu-1,2}$ as the independent variables, the following two regression equations are equivalent to the above matrix equation

$$X_{k\omega+\nu,1} = \phi_{11}^{(\nu)} X_{k\omega+\nu-1,1} + \phi_{12}^{(\nu)} X_{k\omega+\nu-1,2} + a_{k\omega+\nu,1}$$

$$X_{k\omega+\nu,2} = \phi_{21}^{(\nu)} X_{k\omega+\nu-1,1} + \phi_{22}^{(\nu)} X_{k\omega+\nu-1,2} + a_{k\omega+\nu,2}.$$

Based on this, the conditional sum of squares $S^* = S_1^* + S_2^*$, and $S_\nu^* = S_{\nu,1}^* + S_{\nu,2}^*$ with $S_{\nu,1}^* = \sum_{k=\alpha_\nu}^{N-1} a_{k\omega+\nu,1}^2$ and $S_{\nu,2}^* = \sum_{k=\alpha_\nu}^{N-1} a_{k\omega+\nu,2}^2$ for $\nu = 1, \ldots, \omega$, where, by (4.12), $\alpha_1 = 1$ and $\alpha_\nu = 0$ for other ν. Note, for example, that $S_{\nu,1}^*$ is a function of $\phi_{11}^{(\nu)}$ and $\phi_{12}^{(\nu)}$ only, and that the conditional LS estimates of them are those which minimize $S_{\nu,1}^*$. This explains the fact that the conditional LS estimates of the ϕ's in each dimension are those which minimize the sum of squares of that dimension.

For the conditional likelihood function of $(X_2,\ldots,X_{N\omega})$ given X_1, the system of equations (4.8) applies for the m-variate PAR$_\omega$(1) case with scalar quantities now being replaced by their vector or matrix counterparts. Assuming that the error vectors $\{a_{k\omega+\nu}\}$ are independent and multivariate normally distributed with zero mean vector and variance-covariance matrix $\Sigma_a(\nu)$, the joint pdf of $(a_2,\ldots,a_{N\omega})$ is

$$f(a_2,\ldots,a_{N\omega}) = \frac{\exp\left\{-\frac{1}{2} \sum_{\nu=1}^{\omega} \sum_{k=\alpha_\nu}^{N-1} a_{k\omega+\nu}^T \Sigma_a^{-1}(\nu)\, a_{k\omega+\nu}\right\}}{(2\pi)^{m(N\omega-1)/2}\, |\Sigma_a(1)|^{-1/2} \prod_1^{\omega} |\Sigma_a(\nu)|^{N/2}},$$

where $|.|$ stands for determinant. Now, conditioning on X_1, the Jacobian of the transformation from $(a_2,\ldots,a_{N\omega})$ to $(X_2,\ldots,X_{N\omega})$ is one, and therefore the conditional likelihood function $L^* = g^*(X_2,\ldots,X_{N\omega}|X_1)$ is the same as the one above but with $a_{k\omega+\nu}$ replaced by $X_{k\omega+\nu} - \Phi_1^{(\nu)}X_{k\omega+\nu-1}$. It can then be shown that

$$\ell nL^* = -\frac{m(N\omega-1)}{2}\ell n(2\pi) + \frac{1}{2}\ell n|\Sigma_a(1)| - \frac{N}{2}\sum_{\nu=1}^{\omega}\ell n|\Sigma_a(\nu)| - \frac{1}{2}\sum_{\nu=1}^{\omega}S_\nu^{**},$$

where

$$S_\nu^{**} = \sum_{k=\alpha_\nu}^{N-1}(X_{k\omega+\nu} - \Phi_1^{(\nu)}X_{k\omega+\nu-1})^T\,\Sigma_a^{-1}(\nu)\,(X_{k\omega+\nu} - \Phi_1^{(\nu)}X_{k\omega+\nu-1}).$$

Maximizing ℓnL^* with respect to $\Phi_1^{(\nu)}$ reduces to minimizing S_ν^{**}. It can also be shown that minimizing S_ν^{**} with respect to $\phi_{11}^{(\nu)}$ and $\phi_{12}^{(\nu)}$, for example for the bivariate case, reduces to minimizing $\sum_{k=\alpha_\nu}^{N-1}(X_{k\omega+\nu,1} - \phi_{11}^{(\nu)}X_{k\omega+\nu-1,1} - \phi_{12}^{(\nu)}X_{k\omega+\nu-1,2})^2$, which is the same as $S_{\nu,1}^*$ defined above (see Anderson, 1984: 287-291). Therefore, the conditional LS and conditional ML estimates of $\Phi_1^{(\nu)}$ are the same. It can also be verified that for seasons $\nu = 2, 3, \ldots, \omega$, for which $\alpha_\nu = 0$, the conditional ML estimate of $\Phi_1^{(\nu)}$ is the same as its moment estimate, which can be obtained from (4.6), but the two estimates are slightly different for season 1 for which $\alpha_1 = 1$. It should be noted that for the univariate case, S_ν^{**} reduces to $S_\nu^*/\sigma_a^2(\nu)$. ■

If we denote the conditional LS or ML estimates of $\Phi_1^{(\nu)}, \ldots, \Phi_{p(\nu)}^{(\nu)}$ by $\tilde{\Phi}_1^{(\nu)}, \ldots, \tilde{\Phi}_{p(\nu)}^{(\nu)}$, then it can be shown that the conditional ML and regression-type estimates of the white noise variance-covariance matrix for season ν, $\Sigma_a(\nu)$, which were given earlier for the univariate case, are expressed as

$$\tilde{\Sigma}_a(\nu) = \frac{\tilde{S}_\nu^{***}}{N-\alpha_\nu}, \tag{4.13}$$

following from Anderson (1984: 291), and

$$\tilde{\tilde{\Sigma}}_a(\nu) = \frac{\tilde{S}_\nu^{***}}{\text{d.f.}}, \tag{4.14}$$

respectively, where d.f. = N - α_ν - mp(ν) - 1, the m×m matrix $\tilde{S}_\nu^{***}$ is given by

$$\tilde{S}_\nu^{***} = \sum_{k=\alpha_\nu}^{N-1} \tilde{a}_{k\omega+\nu} \tilde{a}^T_{k\omega+\nu},$$

and, according to (4.5), $\tilde{a}_{k\omega+\nu} = X_{k\omega+\nu} - \tilde{\Phi}_1^{(\nu)} X_{k\omega+\nu-1} - \dots - \tilde{\Phi}_{p(\nu)}^{(\nu)} \times X_{k\omega+\nu-p(\nu)}$ is the estimated error vector. The d.f. in (4.14) is explained as the number of terms in $\tilde{S}_\nu^{***}$, N - α_ν, minus the number of independent variables in the multivariate regression equation including the constant term, mp(ν) + 1. For the bivariate PAR$_\omega$(1) model, for example, $\tilde{\Sigma}_a(1) = \tilde{S}_1^{***}/(N-1)$, $\tilde{\Sigma}_a(2) = \tilde{S}_2^{***}/N$, $\tilde{\tilde{\Sigma}}_a(1) = \tilde{S}_1^{***}/(N-4)$, and $\tilde{\tilde{\Sigma}}_a(2) = \tilde{S}_2^{***}/(N-3)$, as $\alpha_1 = 1$, $\alpha_2 = 0$, m = 2 and p(1) = p(2) = 1.

The regression-type estimate $\tilde{\tilde{\Sigma}}_a(\nu)$, (4.14), corresponds again to the UMVU estimate of the residual variance-covariance matrix in multivariate regression (Anderson, 1984: 291). The ML estimate of the residual variance-covariance matrix in multivariate regression is again the same as the conditional ML estimate $\tilde{\Sigma}_a(\nu)$ in (4.13) (Anderson, 1984: 291). It should be noted that the matrix $\tilde{S}_\nu^{***}$ reduces for the univariate case to $\tilde{S}_\nu^*$, the estimated version of S_ν^*. The relations between the conditional ML and moment estimates of $\Phi_1^{(\nu)}, \dots, \Phi_{p(\nu)}^{(\nu)}$ and $\Sigma_a(\nu)$ are still valid for the m-variate case. They are the same for seasons ν with $\alpha_\nu = 0$, but slightly different otherwise. The moment estimates of the parameters can be obtained from (4.6) and (4.7).

In Section (4.6), the relations between the methods of moments, conditional LS, and conditional ML for PAR models are verified for some simulated examples. In those examples, the regression-type estimate, (4.14), and the moment estimate, (4.6), for the estimation of the white noise variances are also considered and compared, and then for some univariate case a comparison is also carried out between these and conditional ML estimate (4.13).

So far we have discussed the conditional LS approach in PAR models and its relations with the method of moments and conditional ML estimation. Now we bring some insight into the exact ML estimation in PAR models through the univariate $PAR_\omega(1)$ model, $X_{k\omega+\nu} = \phi_1^{(\nu)} X_{k\omega+\nu-1} + a_{k\omega+\nu}$. The conditional likelihood function $L^* = g^*(X_2,\ldots, X_{N\omega}|X_1)$ for this case was already obtained earlier in this section. The exact (unconditional) likelihood function L can be obtained as $L = g(X_1, \ldots, X_{N\omega}) = L^* h(X_1)$, where $h(X_1)$ is the marginal pdf of X_1, which belongs to the first season.

It can be shown that the general linear process form, (3.3), of $X_{k\omega+1}$, following Ula (1993), is given by

$$
\begin{aligned}
X_{k\omega+1} &= a_{k\omega+1} + \phi_1^{(1)} a_{(k-1)\omega+\omega} + \phi_1^{(1)}\phi_1^{(\omega)} a_{(k-1)\omega+\omega-1} \\
&\quad + \phi_1^{(1)}\phi_1^{(\omega)}\phi_1^{(\omega-1)} a_{(k-1)\omega+\omega-2} + \ldots + (\prod_{\nu=1}^{\omega}\phi_1^{(\nu)})\, a_{(k-1)\omega+1} \\
&\quad + \phi_1^{(1)}(\prod_{\nu=1}^{\omega}\phi_1^{(\nu)}) a_{(k-2)\omega+\omega} + \ldots + (\prod_{\nu=1}^{\omega}\phi_1^{(\nu)})^2\, a_{(k-2)\omega+1} + \ldots \\
&= \left(\sum_{j=0}^{\infty} (\prod_{\ell=1}^{\omega}\phi_1^{(\ell)})^j\, a_{(k-j)\omega+1}\right) + \phi_1^{(1)} \sum_{\nu=2}^{\omega} \left\{ (\prod_{\ell=1}^{\omega-\nu} \phi_1^{(\omega+1-\ell)}) \right. \\
&\quad \left. \times \sum_{j=0}^{\infty} (\prod_{\ell=1}^{\omega}\phi_1^{(\ell)})^j\, a_{(k-1-j)\omega+\nu} \right\},
\end{aligned}
$$

where $\prod_{\ell=1}^{\omega-\nu}$ is taken as 1 for $\nu = \omega$. Then, it can easily be shown that

$$
Var(X_{k\omega+1}) = V_1 = \frac{1}{1-(\prod_{\nu=1}^{\omega}\phi_1^{(\nu)})^2} \left\{ \sigma_a^2(1) + (\phi_1^{(1)})^2 \sum_{\nu=2}^{\omega} \left(\prod_{\ell=1}^{\omega-\nu} \phi_1^{(\omega+1-\ell)} \right)^2 \sigma_a^2(\nu) \right\}.
$$

Therefore, $X_{k\omega+1}$ is $N(\mu_1, V_1)$, and the mean subtracted X_1 used in the likelihood function is $N(0, V_1)$.

Multiplying the conditional pdf $g^*(X_2,\ldots,X_{N\omega}|X_1)$, (4.10), with the marginal pdf of X_1 gives the exact likelihood function

$$L = (2\pi)^{-N\omega/2} \left[\prod_{1}^{\omega} \sigma_a^2(\nu)\right]^{-N/2} [V_1/\sigma_a^2(1)]^{-1/2} \exp\left\{-\frac{1}{2}\left(\frac{X_1^2}{V_1} + \sum_{\nu=1}^{\omega} \frac{S_\nu^*}{\sigma_a^2(\nu)}\right)\right\},$$

whose natural logarithm is

$$\ell n L = -\frac{N\omega}{2}\ell n(2\pi) - \frac{1}{2}\ell n\frac{V_1}{\sigma_a^2(1)} - \frac{N}{2}\left(\sum_{\nu=1}^{\omega} \ell n \sigma_a^2(\nu)\right) - \frac{X_1^2}{2V_1} - \frac{1}{2}\sum_{\nu=1}^{\omega} \frac{S_\nu^*}{\sigma_a^2(\nu)}$$

As V_1 is a highly non-linear function of all the ϕ's (it also involves all $\sigma_a^2(\nu)$), it is obvious that maximizing the above expression with respect to ϕ's involves solving ω non-linear equations simultaneously. Nevertheless, the desired accuracy in the estimates specifies what kind of simplification or approximation is allowed on the likelihood function. Clearly, the estimates based on the exact likelihood function as it is should be better than those resulting from any such approximation, in the sense that some information are lost by such approximation. The amount of this loss is an important question and should be investigated for PAR processes, although it is shown to be negligible in the context of AR(1) process (Cryer, 1986: 136). If the second term in $\ell n L$ which involves V_1 is neglected, which is usually dominated by the other terms, maximization of the log-likelihood function with respect to the ϕ's is equivalent to the minimization of $X_1^2/V_1 + \sum_{\nu=1}^{\omega} S_\nu^*/\sigma_a^2(\nu)$, which may be named as the unconditional sum of squares function in analogy with ARMA estimation. This function is again non-linear in the parameters and the estimates based on this have to be obtained through numerical iteration (for a similar discussion in the context of AR(1) model, see Cryer, 1986: 137). Existence and convergence of solutions in this case are additional problems which need investigation. A further approximation is to neglect the first term X_1^2/V_1 in the unconditional sum of squares function, which is usually dominated by the second term in this function. In this case, the estimation of the ϕ's becomes identical with the conditional ML estimation discussed earlier in this section. In this case, note also that the estimates of $\sigma_a^2(\nu)$ become $\tilde{S}_\nu^*/N$, for all ν, which differ from the conditional ML estimates, (4.13), only for the first season.

In view of the above discussion, it is foreseen that in higher order PAR processes, the terms corresponding to those involving V_1 in the likelihood function above will be much more complicated, and it may be very difficult, if not impossible, to obtain explicitly the exact likelihood function. For example, for the $PAR_2(2,4)$ model considered earlier, the variance-covariance matrix of (X_1,X_2,X_4) is needed, and obtaining of which is a difficult task. The additional complexity of the multivariate case is also obvious.

Finally, it is worth mentioning that in the context of ARMA model estimation, Box and Jenkins (1976) developed an iterative technique for the computation of unconditional sum of squares function, without having the need for its explicit expression, which is mainly based on an important concept, called backcasting. The essence of this concept is that a stationary ARMA process can be represented in two equivalent forms, namely the forward and backward forms. It then allows the unattainable initial error terms, like a_1 in (4.8) for the $PAR_\omega(1)$ model, to be obtained by back-forecasting (backcasting) using the backward form of the model. This technique, however, has no counterpart for PARMA models yet. The main problem here is that a backward form of a PARMA model can not be related to its forward form, (1.5), in the same manner as for ARMA models due to seasonality. We believe that this problem deserves to be considered for future research.

4.5 Conditional Least Squares and Maximum Likelihood Estimation in PMA Processes

The exact likelihood function of any PARMA model in which a MA part is present, as that of an ARMA model with a MA part, is very complicated. The conditional version of it is relatively less complicated but still difficult to maximize. Vecchia (1985a) followed a similar approach to that of Box and Jenkins (1976) and developed an algorithm for approximate conditional likelihood estimation for univariate PARMA models. If AR components are present, this algorithm assumes that some initial values of the realization are fixed. For pure PMA models, it obtains the exact ML estimates, but by setting the unattainable error terms (as a_0 in the next example) to their expected values which are zero. This algorithm, however, was tested only for univariate $PARMA_\omega(1,1)$ process, for which explicit formula of the conditional likelihood function is still obtainable, but is extremely difficult to apply for higher order PARMA models, as it is very difficult to obtain this function there.

In Section (4.2), we exposed the difficulty of the method of moments for estimation of a PARMA model with a MA part, and also mentioned its deficiencies as observed from some previous simulation studies. In this section, the exact likelihood function for a PARMA model with a MA part is investigated through the simple case of a $PMA_2(1)$ model along with its conditional version and its relations with the conditional LS approach.

The univariate $PMA_2(1)$ model can be obtained from (4.1) by taking $\omega = 2$ and setting $\phi_1^{(\nu)}$ to zero, and letting $\theta_1^{(\nu)} = \theta_\nu$ for $\nu = 1, 2$, as

$$X_{2k+1} = a_{2k+1} - \theta_1 a_{2(k-1)+2}$$

$$X_{2k+2} = a_{2k+2} - \theta_2 a_{2k+1},$$

where, for all integers k and $\nu = 1, 2$, $\{a_{2k+\nu}\}$ are assumed to be independently distributed as $N[0,\sigma_a^2(\nu)]$. Based on the realization $X_1, \ldots, X_{2N}$, we have the following equations

$$\begin{aligned} X_1 &= a_1 - \theta_1 a_0 \\ X_2 &= a_2 - \theta_2 a_1 \\ &\vdots \\ X_{2N} &= a_{2N} - \theta_2 a_{2N-1} \end{aligned} \tag{4.15}$$

Then, conditioning on a_0, the conditional sum of squares here becomes $S^* = \sum_{j=1}^{2N} a_j^2$. Note that, in this case, S^* is a highly non-linear function of the parameters. This can be seen by writing a_j in terms of X_j, $X_{j-1}, \ldots, X_1$ for $j = 1, \ldots, 2N$. For instance, setting $a_0 = 0$, zero being the expected value of a_0, $a_1 = X_1$, $a_2 = X_2 + \theta_2 X_1$, $a_3 = X_3 + \theta_1 X_2 +$

$\theta_1\theta_2 X_1$, $a_4 = X_4 + \theta_2 X_3 + \theta_1\theta_2 X_2 + \theta_1\theta_2^2 X_1$, etc. Also, minimizing this sum of squares with respect to, say, θ_1 does not only involve the error terms for the first season, but actually $(a_3, a_4, \ldots, a_{2N})$ are all involved. That is, although, in analogy with the conditional LS approach in PAR models, it is possible to write $S^* = S_1^* + S_2^*$, where $S_\nu^* = \sum_{k=0}^{N-1} a_{2k+\nu}^2$, this approach does not help much here since each of S_1^* and S_2^* is a function of both θ_1 and θ_2, so that $\partial S^*/\partial\theta_\nu = \partial S_1^*/\partial\theta_\nu + \partial S_2^*/\partial\theta_\nu \neq \partial S_\nu^*/\partial\theta_\nu$. Therefore, S^* can not be minimized in a seasonwise manner as in PAR models,, and also regression can not be employed due to non-linear nature of minimization.

To obtain the exact likelihood function, we follow Vecchia (1985a). Let $Y = (X_1, \ldots, X_{2N})^T$, $A = (a_1, \ldots, a_{2N})^T$ and $A^* = a_0$. Then the system of equations (4.15) are rewritten in matrix notation as $Y = SA - MA^*$, which in turn implies that $A = S^{-1}(Y + MA^*)$, where S and M are, respectively, 2N×2N and 2N×1 matrices defined as

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\theta_2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\theta_1 & 1 & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\theta_1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\theta_2 & 1 \end{bmatrix}$$

and $M = (\theta_1, 0, \ldots, 0)^T$.

The joint pdf of $a_1, \ldots, a_{2N}$ is given by

$$h(a_1, \ldots, a_{2N}) = (2\pi)^{-N}\,[\sigma_a^2(1)\sigma_a^2(2)]^{-N/2} \exp\left(\frac{-1}{2} A^T D^{-1} A\right),$$

where $D_{2N} = \mathrm{diag}[\sigma_a^2(1), \sigma_a^2(2), \sigma_a^2(1), \ldots, \sigma_a^2(2)]$, which in turn is the same as the conditional joint pdf of $a_1, \ldots, a_{2N}$ given A^*, due to independence of the error terms. The Jacobian of the transformation from A to Y, given A^*, is $|S^{-1}| = 1$. Then the conditional joint pdf of $X_1, \ldots, X_{2N}$ given A^* (or the conditional likelihood function) is given by

$$L^* = g^*(Y|A^*) = (2\pi)^{-N}\,[\sigma_a^2(1)\sigma_a^2(2)]^{-N/2}$$

$$\times \exp\left(\frac{-1}{2}(Y+MA^*)^T(S^T)^{-1}D^{-1}S^{-1}(Y+MA^*)\right). \tag{4.16}$$

Now, for the exact likelihood function, the marginal pdf of $A^* = a_0$ is needed. Note that a_0 belongs to the the second season, therefore, a_0 is $N[0,\sigma_a^2(2)]$. Hence, multiplying L^* by the pdf of a_0 gives

$$f(Y,A^*) = (2\pi)^{-(2N+1)/2}\,[\sigma_a^2(1)\sigma_a^2(2)]^{-N/2}\,[\sigma_a^2(2)]^{-1/2}$$

$$\times \exp\left(\frac{-1}{2}\left\{\frac{(A^*)^2}{\sigma_a^2(2)} + (Y+MA^*)^T(S^T)^{-1}D^{-1}S^{-1}(Y+MA^*)\right\}\right),$$

which may be rewritten as

$$f(Y,A^*) = (2\pi)^{-(2N+1)/2}\,|D_1|^{-1/2}\exp\left(\frac{-1}{2}(\Lambda Y-HA^*)^TD_1^{-1}(\Lambda Y-HA^*)\right),$$

where $(D_1)_{(2N+1)\times(2N+1)} = \text{diag}[\sigma_a^2(2),\ \sigma_a^2(1),\ \ldots,\ \sigma_a^2(2)]$, $\Lambda_{(2N+1)\times N} = \left[\frac{0}{S^{-1}}\right]$, and $H_{(2N+1)\times 1} = \left[\frac{1}{-S^{-1}M}\right]$. Let $\hat{A}^* = (H^TD_1^{-1}H)^{-1}H^TD_1^{-1}\Lambda Y$. Then it can be shown that the exponential term in $f(Y,A^*)$ above is written as

$$\exp\left(\frac{-1}{2}\left\{(\Lambda Y-H\hat{A}^*)^TD_1^{-1}(\Lambda Y-H\hat{A}^*) + (A^*-\hat{A}^*)^T(H^TD_1^{-1}H)(A^*-\hat{A}^*)\right\}\right),$$

which means that $f(Y,A^*) = f_1(Y)f_2(A^*|Y)$, and that $f_2(A^*|Y)$ is $N[\hat{A}^*, (H^TD_1^{-1}H)^{-1}]$. Thus, the exact likelihood function, $f_1(Y)$, is written as

$$L = (2\pi)^{-N}|D_1|^{-1/2}|H^TD_1H|^{-1/2}\exp\left(\frac{-1}{2}(\Lambda Y-H\hat{A}^*)^TD_1^{-1}(\Lambda Y-H\hat{A}^*)\right).$$

Furthermore, it can be shown that $(\Lambda Y-H\hat{A}^*)_{(2N+1)\times 1} = \left[\frac{-\hat{A}^*}{\hat{A}}\right]$ which implies that in the likelihood function above

$$\exp\left(\frac{-1}{2}(\Lambda Y-H\hat{A}^*)^TD_1^{-1}(\Lambda Y-H\hat{A}^*)\right) = \exp\left(\frac{-1}{2}\left\{\frac{(\hat{A}^*)^2}{\sigma_a^2(2)} + \hat{A}^TD^{-1}\hat{A}\right\}\right).$$

Besides, it can be shown that $H^TD_1H = 1/\sigma_a^2(2) + (S^{-1}M)^TD^{-1}(S^{-1}M)$ and that $S^{-1}M = (\theta_1,\ \theta_1\theta_2,\ \theta_1(\theta_1\theta_2),\ (\theta_1\theta_2)^2,\ \ldots,\ (\theta_1\theta_2)^N)^T$. Then it follows with simple manipulations that

$$H^T D_1 H = \frac{\theta_1^2}{\sigma_a^2(1)} \left(\frac{1 - (\theta_1\theta_2)^{2N}}{1 - (\theta_1\theta_2)^2} \right) + \frac{1}{\sigma_a^2(2)} \left(\frac{1 - (\theta_1\theta_2)^{2(N+1)}}{1 - (\theta_1\theta_2)^2} \right) = V_2.$$

Hence, the exact likelihood function L reduces to

$$L = f_1(Y) = (2\pi)^{-N} [\sigma_a^2(1)\sigma_a^2(2)]^{-N/2} [V_2\sigma_a^2(2)]^{-1/2}$$
$$\times \exp\left(\frac{-1}{2}\left\{ \frac{(\hat{A}^*)^2}{\sigma_a^2(2)} + \hat{A}^T D^{-1}\hat{A}\right\}\right).$$

We have seen earlier that the conditional sum of squares function for the $PMA_2(1)$ model is a non-linear function of the parameters, and minimizing this function requires solving a system of non-linear equations. The exact likelihood function above is clearly more complicated and its maximization is also more difficult. This fact carries over to the conditional likelihood function L^* given by (4.16). To see this, it can be shown that the natural logarithm of L^* reduces to

$$\ln L^* = -N\ln(2\pi) - \frac{N}{2}\{\ln\sigma_a^2(1) + \ln\sigma_a^2(2)\} - \frac{1}{2}\left(\frac{S_1^*}{\sigma_a^2(1)} + \frac{S_2^*}{\sigma_a^2(2)}\right).$$

Maximizing $\ln L^*$ with respect to θ_ν, $\nu = 1, 2$, gives $\sigma_a^{-2}(1)\partial S_1^*/\partial\theta_\nu + \sigma_a^{-2}(2) \times \partial S_2^*/\partial\theta_\nu = 0$. On the other hand, minimizing $S^* = S_1^* + S_2^*$ with respect to θ_ν, to obtain the conditional LS estimates, gives $\partial S_1^*/\partial\theta_\nu + \partial S_2^*/\partial\theta_\nu = 0$. Therefore, the conditional ML estimates of θ_ν are not the same as conditional LS estimates unless $\sigma_a^2(1) = \sigma_a^2(2)$. For PAR models, on the other hand, it was shown that the conditional ML estimates of ϕ's are the same as conditional LS estimates. Besides, maximizing $\ln L^*$ with respect to $\sigma_a^2(\nu)$ gives $\tilde{\sigma}_a^2(\nu) = \tilde{S}_\nu^*/N$ for $\nu = 1, 2$, where $\tilde{S}_\nu^*$ is the estimated sum of squares for season ν, whereas in the case of $PAR_2(1)$ model, the conditional ML estimates of $\sigma_a^2(\nu)$ were given as $\tilde{\sigma}_a^2(1) = \tilde{S}_1^*/(N-1)$ and $\tilde{\sigma}_a^2(2) = \tilde{S}_2^*/N$, $\tilde{\sigma}_a^2(1)$ being slightly different for the two models.

4.6 A Comparison of Estimation Methods for PAR Processes Through Simulated Examples

4.6.1 Simulation Results

In this section, we consider three different PAR models for simulations. We denote them, for simplicity, by Model (1), Model (2) and Model (3), and following (4.5), they are defined as follows:

(1) The univariate $PAR_4(1;3;1;2)$ model with $\phi_1^{(1)} = 0.9$, $\phi_1^{(2)} = 0.9$, $\phi_2^{(2)} = -0.8$, $\phi_3^{(2)} = 0.7$, $\phi_1^{(3)} = 1.2$, $\phi_1^{(4)} = -0.5$, $\phi_2^{(4)} = 0.6$,

(2) The univariate $PAR_4(1)$ model with $\phi_1^{(1)} = 1.4$, $\phi_1^{(2)} = -0.7$, $\phi_1^{(3)} = 1.1$, $\phi_1^{(4)} = -0.9$,

(3) The bivariate $PAR_2(1)$ model with

$$\Phi_1^{(1)} = \Phi_1 = \begin{bmatrix} 0.9 & -0.7 \\ 0 & 0.6 \end{bmatrix}, \quad \Phi_1^{(2)} = \Phi_2 = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{bmatrix}.$$

Also, in all cases, a zero mean process is assumed. That is, $\mu_\nu = 0$, for $\nu = 1, \ldots, \omega$.

It is readily shown, in view of Proposition 2.1, and by using the computer program listed in Appendix A, that the three processes defined by the models above are periodic stationary. More precisely, the lumped-vector process corresponding to each of the above models follows a 4-variate AR(1) model, and the eigenvalues of $L^{-1}U_1$ in Theorem 2.1 corresponding to Models (1), (2) and (3) are, in modulus, (0, 0, 0.84, 0), (0, 0, 0, 0.97) and (0, 0, 0.45, 0.36), respectively, which are all less than 1.

Our aim here is to compare estimation methods for PAR processes through simulation. We investigate the behavior of these methods, for various cases, such as the case of varying orders, as in Model (1), the case of fixed orders, as in Model (2), the case where the parameters are close to the boundary of the periodic stationarity region, as in Model

(2), and the bivariate case, as in Model (3). The cases of fixed and varying white noise variances are also considered.

The simulations are performed for the following cases:

(a) Model (1) with $\sigma_a^2(\nu) = 1$, $\nu = 1, \ldots, 4$,

(b) Model (1) with $\sigma_a^2(1) = 1$, $\sigma_a^2(2) = 4$, $\sigma_a^2(3) = 0.5$, $\sigma_a^2(4) = 2$,

(c) Model (2) with $\sigma_a^2(\nu) = 1$, $\nu = 1, \ldots, 4$,

(d) Model (3) with $\Sigma_a(1) = \Sigma_a(2) = I_2$, where I_2 denotes the 2×2 identity matrix.

In each case, n = 100 realizations each of length N (years), i.e, $N \times \omega$ values (or vectors), for N = 30, 100 and 300 are simulated assuming that the white noise terms are independently and normally distributed with zero means. Then, for each parameter, the average values (over n realizations) of its estimates and their corresponding root mean squared errors (RMSE) are obtained for different methods. We use the common definition of the RMSE, which is defined in terms of estimating a parameter ϕ by an estimate $\hat{\phi}$ as

$$\underset{\phi}{\text{RMSE}}(\hat{\phi}) = \left(\frac{1}{n} \sum_{i=1}^{n} (\hat{\phi}_i - \phi)^2 \right)^{1/2},$$

where $\hat{\phi}_i$ is the estimate of ϕ for i-th realization, $i = 1, \ldots, n$. The average estimate is then $(1/n)\sum_{i=1}^{n}(\hat{\phi}_i)$. The simulations are carried out using the programs listed in Appendix D.

The simulation results for cases (a), (b), (c) and (d) are summarized in Tables 4.1, 4.2, 4.3 and 4.4, respectively. In Table 4.5, case (a) is again considered for a comparison of white noise variance estimates. Tables 4.1- 4.4 give the average moment and conditional LS estimates for all parameters together with their RMSE. The conditional LS estimates of ϕ's were obtained by regression. As discussed previously, the conditional LS method does not provide estimates for white noise variances. However, we use a regression-type estimate instead, as given by

(4.14) for the multivariate case. In this section, we will refer to this estimate also as the conditional LS estimate.

Table 4.1. The Average Moment and Conditional LS Estimates and Their RMSE for Univariate $PAR_4(1;3;1;2)$ Model with $\phi_1^{(1)} = .9$, $\phi_1^{(2)} = .9$, $\phi_2^{(2)} = .8$, $\phi_3^{(2)} = .7$, $\phi_1^{(3)} = 1.2$, $\phi_1^{(4)} = -.5$, $\phi_2^{(4)} = .6$, $\sigma_a^2(\nu) = 1$, $\nu = 1, \ldots, 4$.

(a) N = 30

Parameters	Moment Estimates & (RMSE)	Cond. LS Estimates & (RMSE)
$\phi_1^{(1)}$	.894 (.187)	.927 (.183)
$\phi_1^{(2)}$ $\phi_2^{(2)}$ $\phi_3^{(2)}$	.873 .776 .645 (.289)(.393)(.086)	.896 .791 .679 (.226)(.307)(.064)
$\phi_1^{(3)}$	1.202 (.063)	1.202 (.063)
$\phi_1^{(4)}$ $\phi_2^{(4)}$	-.505 .554 (.208)(.251)	-.505 .554 (.208)(.251)
$\sigma_a^2(1)$ $\sigma_a^2(2)$	.985 1.416 (.277)(.669)	.995 1.024 (.291)(.277)
$\sigma_a^2(3)$ $\sigma_a^2(4)$	.947 .915 (.235)(.236)	1.015 1.016 (.246)(.246)

(b) N = 100

Parameters	Moment Estimates & (RMSE)	Cond. LS Estimates & (RMSE)
$\phi_1^{(1)}$	.896 (.090)	.904 (.090)
$\phi_1^{(2)}$ $\phi_2^{(2)}$ $\phi_3^{(2)}$	.930 .753 .682 (.112)(.142)(.033)	.913 .776 .690 (.100)(.119)(.031)
$\phi_1^{(3)}$	1.202 (.030)	1.202 (.030)
$\phi_1^{(4)}$ $\phi_2^{(4)}$	-.514 .609 (.099)(.120)	-.514 .609 (.099)(.120)
$\sigma_a^2(1)$ $\sigma_a^2(2)$	1.007 1.164 (.150)(.302)	1.012 .984 (.150)(.137)
$\sigma_a^2(3)$ $\sigma_a^2(4)$	.986 .970 (.142)(.140)	1.007 1.000 (.145)(.141)

Table 4.1 (cont'd)

(c) N = 300

Parameters	Moment Estimates & (RMSE)	Cond. LS Estimates & (RMSE)
$\phi_1^{(1)}$	.904 (.055)	.907 (.054)
$\phi_1^{(2)}$ $\phi_2^{(2)}$ $\phi_3^{(2)}$	.903 .794 .695 (.056)(.078)(.014)	.897 .804 .698 (.057)(.076)(.013)
$\phi_1^{(3)}$	1.199 (.015)	1.199 (.015)
$\phi_1^{(4)}$ $\phi_2^{(4)}$	-.504 .603 (.057)(.072)	-.504 .603 (.057)(.072)
$\sigma_a^2(1)$ $\sigma_a^2(2)$	.992 1.091 (.072)(.152)	.992 1.009 (.071)(.090)
$\sigma_a^2(3)$ $\sigma_a^2(4)$	.993 .995 (.074)(.073)	1.000 1.005 (.074)(.073)

Table 4.2. The Average Moment and Conditional LS Estimates and Their RMSE for Univariate $PAR_4(1;3;1;2)$ Model with $\phi_1^{(1)} = .9$, $\phi_1^{(2)} = .9$, $\phi_2^{(2)} = .8$, $\phi_3^{(2)} = .7$, $\phi_1^{(3)} = 1.2$, $\phi_1^{(4)} = -.5$, $\phi_2^{(4)} = .6$, $\sigma_a^2(1) = 1$, $\sigma_a^2(2) = 4$, $\sigma_a^2(3) = .5$, $\sigma_a^2(4) = 2$.

(a) N = 30

Parameters	Moment Estimates & (RMSE)	Cond. LS Estimates & (RMSE)
$\phi_1^{(1)}$	.896 (.138)	.926 (.134)
$\phi_1^{(2)}$ $\phi_2^{(2)}$ $\phi_3^{(2)}$	.925 .740 .630 (.430)(.512)(.109)	.843 .837 .655 (.365)(.417)(.086)
$\phi_1^{(3)}$	1.196 (.028)	1.196 (.028)
$\phi_1^{(4)}$ $\phi_2^{(4)}$	-.507 .568 (.364)(.441)	-.507 .568 (.364)(.441)
$\sigma_a^2(1)$ $\sigma_a^2(2)$	1.123 4.500 (.334)(1.531)	1.070 3.917 (.284)(1.139)
$\sigma_a^2(3)$ $\sigma_a^2(4)$	.459 1.748 (.125) (.535)	.491 1.942 (.127) (.528)

Table 4.2 (cont'd)

(b) N = 100

Parameters	Moment Estimates & (RMSE)	Cond. LS Estimates & (RMSE)
$\phi_1^{(1)}$	.895 (.065)	.903 (.064)
$\phi_1^{(2)}$ $\phi_2^{(2)}$ $\phi_3^{(2)}$	.950 .734 .677 (.226)(.264)(.043)	.908 .782 .684 (.201)(.230)(.038)
$\phi_1^{(3)}$	1.199 (.013)	1.199 (.013)
$\phi_1^{(4)}$ $\phi_2^{(4)}$	-.532 .632 (.184)(.219)	-.532 .632 (.184)(.219)
$\sigma_a^2(1)$ $\sigma_a^2(2)$	.972 4.317 (.140)(.830)	.958 3.964 (.146)(.561)
$\sigma_a^2(3)$ $\sigma_a^2(4)$	.478 1.933 (.067)(.266)	.488 1.992 (.066)(.266)

(c) N = 300

Parameters	Moment Estimates & (RMSE)	Cond. LS Estimates & (RMSE)
$\phi_1^{(1)}$	.896 (.043)	.900 (.044)
$\phi_1^{(2)}$ $\phi_2^{(2)}$ $\phi_3^{(2)}$	.919 .762 .692 (.138)(.152)(.019)	.906 .779 .695 (.132)(.144)(.018)
$\phi_1^{(3)}$	1.199 (.006)	1.199 (.006)
$\phi_1^{(4)}$ $\phi_2^{(4)}$	-.516 .618 (.102)(.121)	-.516 .618 (.102)(.121)
$\sigma_a^2(1)$ $\sigma_a^2(2)$	.994 4.155 (.079)(.404)	.989 4.016 (.081)(.322)
$\sigma_a^2(3)$ $\sigma_a^2(4)$	.495 1.975 (.048)(.158)	.498 1.995 (.048)(.158)

Table 4.3. The Average Moment and Conditional LS Estimates and Their RMSE for Univariate $PAR_4(1)$ Model with $\phi_1^{(1)} = 1.4$, $\phi_1^{(2)} = -.7$, $\phi_1^{(3)} = 1.1$, $\phi_1^{(4)} = -.9$, $\sigma_a^2(\nu) = 1$, $\nu = 1, \ldots, 4$.

Parameter	Moment Est. & (RMSE)			Cond. LS Est. & (RMSE)		
	N=30	N=100	N=300	N=30	N=100	N=300
$\phi_1^{(1)}$	1.263 (.170)	1.379 (.034)	1.390 (.015)	1.355 (.080)	1.396 (.019)	1.396 (.008)
$\phi_1^{(2)}$	-.674 (.049)	-.699 (.009)	-.696 (.006)	-.674 (.049)	-.699 (.009)	-.696 (.006)
$\phi_1^{(3)}$	1.057 (.087)	1.094 (.016)	1.097 (.009)	1.057 (.087)	1.094 (.016)	1.097 (.009)
$\phi_1^{(4)}$	-.859 (.073)	-.894 (.018)	-.896 (.008)	-.859 (.073)	-.894 (.018)	-.896 (.008)
$\sigma_a^2(1)$	4.573 (5.063)	3.312 (3.141)	1.675 (.976)	.950 (.263)	.999 (.135)	.996 (.073)
$\sigma_a^2(2)$	.935 (.257)	.972 (.137)	1.004 (.090)	1.002 (.267)	.992 (.137)	1.011 (.091)
$\sigma_a^2(3)$	.904 (.251)	.992 (.136)	.994 (.083)	.968 (.250)	1.012 (.140)	1.001 (.083)
$\sigma_a^2(4)$	.923 (.251)	1.004 (.134)	.976 (.073)	.989 (.257)	1.024 (.139)	.983 (.072)

Table 4.4. The Average Moment and Conditional LS Estimates and Their RMSE for Bivariate $PAR_2(1)$ Model with $\Phi_1 = \begin{bmatrix} .9 & -.7 \\ 0 & .6 \end{bmatrix}$, $\Phi_2 = \begin{bmatrix} .5 & .2 \\ 0 & .6 \end{bmatrix}$, $\Sigma_a(1) = \Sigma_a(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) N = 30

Par.	Moment Estimates		Moment Est. (RMSE)		Cond. LS Estimates		Cond. LS Est. (RMSE)	
Φ_1	.826 -.001	-.719 .521	(.167) (.178)	(.162) (.194)	.858 -.000	-.743 .537	(.159) (.183)	(.166) (.192)
Φ_2	.453 .035	.208 .603	(.128) (.122)	(.175) (.154)	.453 .035	.208 .603	(.128) (.122)	(.175) (.154)
$\Sigma_a(1)$	1.045	-.024 .946	(.277)	(.206) (.311)	1.034	.010 1.011	(.252)	(.215) (.321)
$\Sigma_a(2)$	.904	-.005 0.878	(.249)	(.164) (.250)	1.004	-.006 .975	(.256)	(.182) (.244)

Table 4.4 (cont'd)

(b) N = 100

Par.	Moment Estimates		Moment Est. (RMSE)		Cond. LS Estimates		Cond. LS Est. (RMSE)	
Φ_1	.898	-.697	(.069)	(.085)	.909	-.706	(.071)	(.085)
	.002	.564	(.082)	(.095)	.002	.570	(.083)	(.093)
Φ_2	.478	.183	(.061)	(.086)	.478	.183	(.061)	(.086)
	.005	.582	(.064)	(.080)	.005	.582	(.064)	(.080)
$\Sigma_a(1)$	1.040	-.020	(.152)	(.106)	1.028	-.006	(.143)	(.104)
		.988		(.153)		1.005		(.151)
$\Sigma_a(2)$	.966	-.002	(.149)	(.097)	.996	-.002	(.150)	(.100)
		.954		(.153)		.983		(.151)

(c) N = 300

Par.	Moment Estimates		Moment Est. (RMSE)		Cond. LS Estimates		Cond. LS Est. (RMSE)	
Φ_1	.894	-.691	(.045)	(.047)	.897	-.693	(.044)	(.047)
	-.004	.594	(.046)	(.046)	-.004	.596	(.046)	(.047)
Φ_2	.496	.202	(.032)	(.045)	.496	.202	(.032)	(.045)
	.004	.589	(.035)	(.049)	.004	.589	(.035)	(.049)
$\Sigma_a(1)$	1.010	.008	(.078)	(.058)	1.009	.012	(.079)	(.058)
		.990		(.084)		.996		(.084)
$\Sigma_a(2)$	.992	.003	(.079)	(.057)	1.002	.003	(.079)	(.057)
		.992		(.073)		1.002		(.073)

Table 4.5 The Average Moment, Conditional LS, and Conditional ML Estimates and Their RMSE for White Noise Variances for Case (a), $\sigma_a^2(\nu) = 1$, $\nu = 1, \ldots, 4$.

Est. Method	N	Parameter			
		$\sigma_a^2(1)$	$\sigma_a^2(2)$	$\sigma_a^2(3)$	$\sigma_a^2(4)$
Moments	30	.985 (.277)	1.416 (.669)	.947 (.235)	.915 (.236)
	100	1.007 (.150)	1.164 (.302)	.986 (.142)	.970 (.140)
	300	.992 (.072)	1.091 (.152)	.993 (.074)	.995 (.073)
Cond. LS	30	.995 (.291)	1.024 (.277)	1.015 (.246)	1.016 (.246)
	100	1.012 (.150)	.984 (.137)	1.007 (.145)	1.000 (.141)
	300	.992 (.071)	1.009 (.090)	1.000 (.074)	1.005 (.073)
Cond. ML	30	.926 (.280)	.883 (.265)	.947 (.235)	.915 (.236)
	100	.991 (.147)	.944 (.142)	.986 (.142)	.970 (.140)
	300	.985 (.072)	.995 (.088)	.993 (.074)	.995 (.073)

4.6.2 Discussion

The first conclusion that can be drawn from the tables above is that, as expected, all methods seem to be satisfactory in the sense that they produce estimates which are close to the actual values of the parameters. The differences between the estimates of different methods, if any, are generally small, and negligible for large N, for both univariate and bivariate cases.

We first summarize some theoretical results from Section 4.4 for the relationships of various estimates for PAR processes:

i) Conditional LS estimates for ϕ's are the same as conditional ML estimates.

ii) For seasons with $\alpha_\nu = 0$, the conditional LS estimates for ϕ's are the same as moment estimates.

iii) For seasons with $\alpha_\nu \neq 0$, the conditional LS estimates for ϕ's are not the same as moment estimates, but they are close for large samples.

iv) For seasons with $\alpha_\nu = 0$, the conditional ML estimates of white noise variances, $\sigma_a^2(\nu)$ or $\Sigma_a(\nu)$, are the same as moment estimates.

v) For seasons $\alpha_\nu \neq 0$, the conditional ML estimates of $\sigma_a^2(\nu)$ are not the same as moment estimates, but they are close for large samples.

vi) The conditional ML estimate of $\sigma_a^2(\nu)$ can be obtained from conditional LS estimate (meaning regression-type estimate here) as

$$\text{(conditional ML estimate)} = \frac{N - \alpha_\nu - mp(\nu) - 1}{N - \alpha_\nu}\text{(conditional LS estimate).}$$

This relation also applies to averages but not to RMSE.

Results (ii) through (vi) will be verified from the tables.

For the three models considered here, α_ν values, (4.12), are as follows:

Model (1): $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha_4 = 0$.
Model (2): $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$.
Model (3): $\alpha_1 = 1$, $\alpha_2 = 0$.

Although we assume a zero-mean process, we include a constant term in using regression for conditional LS estimation of ϕ's, because due to simulation, the sample means may not be exactly zero. For instance, the averages of the regression constants corresponding to each of the four seasons in the simulations summarized in Table 4.1(a) are -.001, 0.009, -.056, and .016, which are close to zero.

Result (ii) above is seen to be verified in Tables 4.1 and 4.2 for $\phi_1^{(3)}$, $\phi_1^{(4)}$ and $\phi_2^{(4)}$, in Table 4.3 for $\phi_1^{(2)}$, $\phi_1^{(3)}$ and $\phi_1^{(4)}$, and in Table 4.4 for Φ_2. Result (iii) is observed, for example, in Table 4.1c for $\phi_1^{(1)}$, $\phi_1^{(2)}$, $\phi_2^{(2)}$, $\phi_3^{(2)}$. The average conditional ML estimates of variances in Tables 4.1 - 4.4 can be obtained from the average conditional LS estimates by using the relation in result (vi). Then result (iv) can be verified, in terms of average values, in Tables 4.1 and 4.2 for $\sigma_a^2(3)$ and $\sigma_a^2(4)$, in Table 4.3 for $\sigma_a^2(2)$, $\sigma_a^2(3)$ and $\sigma_a^2(4)$, and in Table 4.4 for $\Sigma_a(2)$. Result (v) can also be observed in terms of average values, for example, in Table 4.1c for $\sigma_a^2(1)$ and $\sigma_a^2(2)$.

In Tables 4.1 - 4.4, the conditional LS estimates of ϕ's (which are the same as conditional ML estimates, by result (i)) can be compared with the moment estimates, for the cases where they are not identical (see result (ii)), in terms of RMSE and bias. A measure of bias for any parameter can be obtained as Bias = |(average estimate of parameter) - (actual value of parameter)|. It can be seen that, in terms of RMSE of $\hat{\phi}$'s, in 27 out of 39 cases, which we will simply denote as 27/39 (69%), conditional LS estimates are better, in 8/39 (21%) cases moment estimates are better, and in 4/39 (10%) cases they perform equally well. In terms of bias of $\hat{\phi}$'s, in 28/39 (72%) cases conditional LS estimates are better, in 7/39 (18%) cases moment estimates are better, and in the remaining 4/39 (10%) cases they perform equally well. Therefore, conditional LS estimates of ϕ's (which are identical with the conditional ML estimates) are superior to moment estimates (when the two are not identical) in terms of RMSE and bias.

In Tables 4.1 - 4.4, the conditional LS estimates of variances can be compared with the moment estimates. In terms of RMSE, in 21/54 (39%) cases conditional LS estimates are better, in 20/54 (37%) cases moment estimates are better, and in 13/54 (24%) cases they perform equally well. In terms of bias, in 41/54 (76%) cases conditional LS estimates are better, in 9/54 (17%) cases moment estimates are better, and in the remaining 4/54 (7%) cases they perform equally well. Therefore, for variances, conditional LS estimates are superior to moment estimates in terms of bias, but in terms of RMSE, they perform almost equally well. For seasons with $\alpha_\nu = 0$, the conditional ML estimates of variances are the same as moment estimates (result (iv)). For these seasons, in terms of RMSE of variances, in 14/30 (47%) cases conditional ML estimates (which are the same as moment estimates) are better, in 6/30 (20%) cases conditional LS estimates are better, and in 10/30 (33%) cases they perform equally well. In terms of bias, in 23/30 (77%) cases conditional LS estimates are better, in 4/30 (13%) cases conditional ML estimates are better, and in the remaining 3/30 (10%) cases they perform equally well. Therefore, for variances, conditional ML estimates are superior to conditional LS estimates in terms of RMSE, but in terms of bias conditional LS estimates are superior to conditional ML estimates. These observations also follow from Table 4.5 which give, for case (a), the conditional LS, conditional ML and moment estimates of variances. It may be noted that in this table, for $\sigma_a^2(3)$ and $\sigma_a^2(4)$, the conditional ML estimates are the same as moment estimates, which follow from result (iv). Result (v) can also be observed for $\sigma_a^2(1)$ and $\sigma_a^2(2)$ for N = 300. It can also be seen from this table that in terms of RMSE of variances, the conditional ML estimates are superior to conditional LS estimates and also to moment estimates (when they are not equal). In terms of bias, conditional LS estimates are superior to conditional ML and moment estimates. Since RMSE is a more reasonable measure of estimation accuracy as compared to bias, conditional ML estimates are recommended for estimation of white noise variances.

Therefore, among the method of moments, conditional LS and conditional ML, conditional ML estimation is recommended for all parameters of univariate and multivariate PAR models. Although this method gives satisfactory results, if, however, more accurate estimates are required, then exact maximum likelihood estimation should be considered.

Moreover, in the simulation results above, the bias and MSE criteria were considered for the comparison of differnt estimates. Other statistical properties of these estimates can also be investigated for such purpose.

CHAPTER V

SUMMARY AND CONCLUSIONS

Periodic autoregressive moving average (PARMA) processes are receiving considerable attention lately. The aim of this study is to contribute to the theory and analysis of these processes in various ways.

The first part of this study considers some items related to the periodic stationarity of PARMA processes, mainly through the lumped-vector representation of the process. It has been shown that a PARMA process is periodic stationary if and only if the corresponding lumped process is stationary. A compact lumped-vector representation previously developed for univariate PARMA processes is generalized to the multivariate case. Through this representation, it is shown that the periodic stationarity conditions (and also, analogously, the invertibility conditions) for any univariate or multivariate PARMA process can be reduced to an eigenvalue problem. In addition, in a previous work, it has been shown that the periodic stationarity of a PARMA process implies stationarity of the corresponding aggregated process, but the reverse was not proved or disproved, although was seen to be true for some special cases. It is shown through a counterexample here that the reverse is not always true.

For univariate periodic autoregressive (PAR) processes it was shown that if the periodic stationary process has unit variances for all seasons, then this imposes additional constraints on the autoregressive (AR) parameters. This result is generalized to the multivariate case in which all seasons have equal, but not necessarily identity, covariance matrices.

The relation between periodic stationarity of a PARMA process and positive definiteness of covariance matrices of this process is shown for the multivariate case to follow from the stationarity of the corresponding lumped process.

The identification of orders of PARMA processes, being a topic not completely resolved yet, receives considerable attention in this study. It is shown that the Box-Jenkins approach for identification of univariate ARMA processes can be generalized to univariate PARMA processes, following a seasonwise identification routine. For PARMA processes, the seasonal autocorrelation function (ACF) and seasonal partial autocorrelation function (PACF) play the same role as ACF and PACF in ARMA processes and they have analogous cut-off properties.

The seasonal ACF is employed for the identification of seasons following a pure moving average (MA) process, utilizing the cut-off property of the ACF of such seasons. For the assessment of this cut-off property from a given realization of the process, the sample seasonal ACF is utilized, and its first and second order moments are derived under certain approximations which are well justified for large samples. More refined asymptotic formulas recently developed for the second order moments of the sample seasonal ACF are difficult to work with. Nevertheless, under cut-off property, these formulas reduce to ours. The asymptotic normality of the sample seasonal ACF, which was recently proved, is used together with its asymptotic moments to obtain bands for the assessment of cut-off property. For the non-periodic case, i.e. taking period as one, these bands reduce to the well-known bands for ordinary MA processes.

On the other hand, the seasonal PACF is employed for the identification of seasons following a pure AR process, utilizing the cut-off property of the PACF of such seasons. It is shown that expressing partial autocorrelations as autoregressive parameters in ordinary AR processes does not carry over to periodic processes due to seasonally-varying variances. Therefore, the PACF is defined making use of the properties of partial autocorrelations in regression context. The well-known recursive formulas for partial autocorrelations in regression do not prove to be useful for the computation of PACF in periodic processes, for which we adapt a different algorithm. The sample seasonal PACF is utilized for the assessment of the cut-off property from a realization of the process. Available asymptotic results for the sample seasonal PACF are used to obtain bands for the assessment of cut-off property. These bands turn out to be the same as those for sample PACF in ordinary AR processes.

Simulation results with sample seasonal ACF and PACF bands agree well with the theoretical results. It is also shown that there is no one-to-one relationship between the orders of a PARMA process and the orders of its marginal series for each season. Therefore, a PARMA process cannot be identified from its marginal series.

The last part of the study is devoted to estimation of PARMA processes. It is shown through the univariate $PARMA_{\omega}(1,1)$ model that the method of moment estimation in PARMA processes in the presence of a MA component is technically difficult because a set of non-linear equations is to be solved simultaneously. Previous simulation studies have also shown that the moment estimates for this case are usually unsatisfactory or infeasible. Such observations are also valid for ordinary ARMA processes containing a MA component. On the other hand, it is shown that the moment estimation for univariate or multivariate PAR processes is straightforward and gives satisfactory results.

For PAR processes, the conditional least-squares (LS), conditional maximum likelihood (ML) and exact ML estimation methods are also studied for univariate and multivariate cases. Detailed examples are given using the univariate and multivariate $PAR_{\omega}(1)$ model. It is shown that conditional LS estimates of AR parameters can be obtained in a season-wise manner, and regression methods can be employed directly both for univariate and multivariate cases. The conditional LS method does not provide estimates for error variances; however, regression-type estimates can be used for these parameters, utilizing ML and uniformly minimum variance unbiased (UMVU) estimates in regression. It is shown that for a Gaussian process, the conditional ML estimates of AR parameters are the same as conditional LS estimates. The conditional LS or ML estimates of the AR parameters are also the same as the moment estimates, except for some initial seasons in which some observations are lost due to end effects. Even in that case, the estimates are not significantly different, being very close for large samples. The conditional ML estimates for error variances are the same as moment estimates, except again for some initial seasons. The conditional ML estimates for error variances are the same as regression-type estimates based on ML estimates in regression. The regression-type estimates based on UMVU estimates in regression differ from conditional ML estimates only in degrees of freedom of sum of squares, the difference being negligible for large samples.

The results of simulations, in which some Gaussian univariate and multivariate PAR processes are considered, are found compatible with the theoretical results above. More precisely, the relationships between the conditional ML estimates and moment estimates are verified, and in situations where these estimates are different, conditional ML estimates for AR parameters often dominate moment estimates in the sense of bias and mean squared error (MSE). For error variances, in terms of MSE, the conditional ML estimates are superior to regression type estimates and also to moment estimates (when they are not equal). In terms of bias, regression-type estimates dominate conditional ML and moment estimates. Although all methods considered here are satisfactory and produce estimates which are close to the actual values of the parameters, based on MSE criterion, the conditional ML estimation is recommended.

The exact ML estimation in PAR processes is not easy, firstly, because it is difficult to obtain the exact likelihood function, and, secondly, because numerical maximization algorithms are required for the resulting non-linear equations. These difficulties are illustrated in terms of the exact likelihood function of $PAR_{\omega}(1)$ model. For PARMA processes with a MA part, even the conditional LS and conditional ML estimates are difficult to obtain due to presence of simultaneous non-linear equations. These difficulties are shown in term of the conditional LS, conditional likelihood and exact likelihood functions of $PMA_{2}(1)$ model. In this case, the conditional LS estimates cannot be obtained in a season-wise manner and regression methods cannot be employed. It is also shown that, in this case, the conditional LS estimates are not the same as conditional ML estimates.

Computer programs for checking periodic stationarity (and also invertibility) of a PARMA process, and for identification and estimation results were developed and are provided in Appendices.

In view of this study, we summarize some problems concerning PARMA processes which deserve further investigation as follows:

1- Identification of Mixed PARMA Processes. Methods such as Akaike information criterion (Akaike, 1973; 1974) or the S and R array method (Gray et al., 1978), which are helpful in the identification of mixed ARMA models, should be investigated for PARMA models. Identification of multivariate PARMA processes is another important topic which was not investigated yet.

2- Estimation of Mixed PARMA or PMA Processes. Techniques such as backcasting should be investigated for ML estimation, and also some special optimization methods should be developed. Approaches like state space modeling and Kalman filtering (Wei, 1990: 384) should also be studied for PARMA models. This problem exists for the multivariate case as well.

3- Goodness-of-Fit Tests for PARMA Processes. A Portmanteau-type technique (Box and Jenkins, 1976: 290) should be investigated for this purpose.

4- Nonstationarity Transformations. It is known in the context of ARMA processes that techniques such as differencing reduce a non-stationary ARMA process to a stationary one. Such techniques for PARMA models, to achieve periodic stationarity, should be investigated.

5- Frequency-Domain Analysis. Almost all available results concerning PARMA processes are time-domain oriented. Hence, a parallel frequency-domain study of PARMA processes is to be investigated.

REFERENCES

Akaike, H., 1973. "Information Theory and an Extension of the Maximum Likelihood Principle", Proc. 2nd Int. Symp. on Information Theory, Eds. B. N. Petrov and F. Csaki, pp. 267-281, Akademia Kiado, Budapest.

Akaike, H., 1974. "A New Look at the Statistical Model Identification", IEEE Trans. Autom. Control, Vol. 19, pp. 716-723.

Anderson, P. L., and Vecchia, A. V., 1993. "Asymptotic Results for Periodic Autoregressive Moving- Average Processes", J. Time Series Analysis, Vol. 14, No. 1, pp. 1-18.

Anderson, T. W., 1971. The Statistical Analysis of Time Series, John Wiley, New York.

Anderson, T. W., 1984. An Introduction to Multivariate Statistical Analysis, 2nd Ed., John Wiley, New York.

Barone, P., "A Method for Generating Independent Realizations of a Multivariate Normal Stationary and Invertible ARMA(p,q) Process", J. Time Series Analysis, Vol. 8, No. 2, pp. 125-130.

Bartlett, M. S., 1946. "On the Theoretical Specification of Sampling Properties of Autocorrelated Time Series", J. Royal Stat. Soc., Series B, Vol. 8, pp. 27-41.

Bartolini, P., Salas, J. D., and Obeysekera, J. T. B., 1988. "Multivariate Periodic ARMA(1,1) Processes", Water Resour. Res., Vol. 24, No. 8, pp. 1237-1246.

Bhuiya, R. K., 1971. "Stochastic Analysis of Periodic Hydrologic Process", J. Hydraul. Div. Am. Soc. Civ. Eng., Vol. 97, No. HY7, pp. 949-962.

Box, G. E. P., and Jenkins, G. M., 1976. Time Series Analysis: Forecasting and Control, Revised Ed., Holden-Day, San Francisco.

Cleveland, W. P., and Tiao, G. C., 1979. "Modeling Seasonal Time Series", Economie Appliquée, Vol. 32, pp. 107-129.

Cryer, J. D., 1986. Time Series Analysis, PWS-KENT, Boston.

Delleur, J. W., Tao, P. C., and Kavvas, M. L., 1976. "An Evaluation of the Practicality and Complexity of Some Rainfall and Runoff Time Series Models", Water Resour. Res., Vol. 12, No. 5, pp. 953-970.

Durbin, J., 1960. "The Fitting of Time Series Models", Rev. Int. Inst. Statist., Vol. 28, pp. 233-244.

Fuller, W. A., 1976. Introduction to Statistical Time Series, John Wiley, New York.

Gladyshev, E. G., 1961. "Periodically Correlated Random Sequences", Soviet Math., Vol. 2, No. 2, pp. 385-388.

Gray, H. L., Kelley, G. D., and McIntire, D. D., 1978. "A New Approach to ARMA Modeling", Cumm. Stat., Series B, Vol. 7, No. 1, pp. 1-77.

Graybill, F. A., 1983. Matrices with Applications in Statistics, 2nd Ed., Wadsworth, Belmont.

Hannan, E. J., 1970. Multiple Time Series, John Wiley, New York.

Householder, A. S., 1953. Principles of Numerical Analysis, McGraw Hill, New York.

Jenkins, G. M., and Watts, D. G., 1968. Spectral Analysis and Its Applications, Holden-Day, San Francisco.

Jones, R. H., and Brelsford, W. M., 1967. "Time Series with Periodic Structure", Biometrika, Vol. 54, No. 3-4, pp. 403-408.

Monin, A. S., 1963. "Stationary and Periodic Time Series in the General Circulation of the Atmosphere", Proc. Symp. on Time Series Anal., Ed. M. Rosenblatt, pp. 144-151. John Wiley, New York.

Morrison, D. F., 1976. Multivariate Statistical Methods, 2nd Ed., McGraw-Hill, New York.

Obeysekera, J. T. B., and Salas, J. D., 1986. "Modeling of Aggregated Hydrologic Time Series", J. Hydrol., Vol. 86, pp. 197- 219.

Pagano, M., 1978. "On Periodic and Multiple Autoregressions", Annals Stat., Vol. 6, No. 6, pp. 1310-1317.

Priestley, M. B., 1981. Spectral Analysis and Time Series, Academic Press, London.

Rose, D. E., 1977. "Forecasting Aggregates of Independent ARIMA Processes", J. Econometrics, Vol. 5, pp. 323-345.

Sakai, H., 1982. "Circular Lattice Filtering Using Pagano's Method", IEEE Trans. Acoust. Sp. Signal Process., Vol. 30, No. 2, pp. 279- 287.

Salas, J. D., 1972. "Range Analysis for Storage Problems of Periodic Stochastic Processes", Hydrol. Paper, 57, Colo. State Univ., Fort Collins.

Salas, J. D., 1974a. "Range of Cumulative Sums, I. Exact and Approximate Expected Values", J. Hydrol., Vol. 23, pp. 39-66.

Salas, J. D., 1974b. "Range of Cumulative Sums, II. Application to Capacity of Reservoirs", J. Hydrol., Vol. 23, pp. 329-339.

Salas, J. D., Boes, D. C., and Smith, R. A., 1982. "Estimation of ARMA Models with Seasonal Parameters", Water Resour. Res., Vol. 18, No. 4, pp. 1006-1010.

Salas, J. D., Delleur, J. W., Yevjevich, V., and Lane, W. L., 1980. Applied Modeling of Hydrologic Time Series, Water Resources Publications, Littleton, Colo.

Salas, J. D., and Pegram, G. G. S., 1979. "A Seasonal Multivariate Multi-lag Auto-regressive Model in Hydrology", Proc. 3rd Int. Hydrol. Symp. on Theo. and Appl. Hydrology, Eds. H. J. Morel-Sertoux, et al., pp. 125-145, Water Resour. Publ., Littleton, Colo.

Tiao, G. C., Box, G. E. P., and Hamming, W. J., 1975. "Analysis of Los Angeles Photochemical Smog Data: A Statistical Overview", J. Air Pollution Control Assoc., Vol. 25, pp. 260-268.

Tiao, G. C., and Grupe, M. R., 1980. "Hidden Periodic Autoregressive-Moving Average Models in Time Series Data", Biometrika, Vol. 67, No. 2, pp. 365-373.

Tiao, G. C., Phadke, M. S., and Box, G. E. P., 1976. "Some Empirical Models for the Los Angeles Photochemical Smog Data", J. Air Pollution Control Assoc., Vol. 26, pp. 485-490.

Troutman, B. M., 1978. "Reservoir Storage with Dependent, Periodic Net Inputs", Water Resour. Res., Vol. 14, No. 3, pp. 395-401.

Troutman, B. M., 1979. "Some Results in Periodic Autoregression", Biometrika, Vol. 66, No. 2, pp. 219-228.

Ula, T. A., 1983. "Regression, Linear Model, and the Multivariate Normal Distribution", Proc. Statist. Seminars, METU Studies in Development, 1982 Special Issue, pp. 135-156.

Ula, T. A., 1990. "Periodic Covariance Stationarity of Multivariate Periodic Autoregressive Moving Average Processes", Water Resour. Res., Vol. 26, No. 5, pp. 855-861.

Ula, T. A., 1993. "Forecasting of Multivariate Periodic Autoregressive-Moving Average Processes", J. Time Series Anal., Vol. 14, No. 6, pp. 645-657.

Vecchia, A. V., 1983. "Aggregation and Estimation for Periodic Autoregressive-Moving Average Models", Unpublished Ph. D. Dissertation, Dept. of Stat., Colo. State Univ., Fort Collins, Colo.

Vecchia, A. V., 1985a. "Maximum Likelihood Estimation for Periodic Autoregressive Moving Average Models", Technometrics, Vol. 27, No. 4, pp. 375-384.

Vecchia, A. V., 1985b. "Periodic Autoregressive- Moving Average (PARMA) Modeling with Applications to Water Resources", Water Resour. Bull., Vol. 21, No. 5, pp. 721-730.

Vecchia, A. V., and Ballerini, R., 1991. "Testing for Periodic Autocorrelations in Seasonal Time Series Data", Biometrika, Vol. 78, No. 1, pp. 53-63.

Vecchia, A. V., Obeysekera, J. T., Salas, J. D., and Boes, D. C., 1983. "Aggregation and Estimation for Low-Order Periodic ARMA Models", Water Resour. Res., Vol. 19, No. 5, pp. 1297-1306.

Wei, W. W., 1990. Time Series Analysis: Univariate and Multivariate Methods, Addison-Wesley, Redwood City.

APPENDICES

APPENDIX A

COMPUTER PROGRAM FOR CHECKING PERIODIC STATIONARITY AND INVERTIBILITY OF PARMA PROCESSES

Description:

Computer programs in these appendices were all written in FORTRAN77 and developed using the main frame (CYBER 932/11) in the Department of Statistics, at Middle East Technical University.

This program performs a check on a given set of parameters (or their estimates based on an observed realization) of an m-variate $\text{PARMA}_{\omega}(p(\nu),\ q(\nu))$ process as to whether it is satisfying periodic covariance stationarity conditions or not. It is using the AR parameters only. The dimension of the vector time series (M) and the period (IW) are fixed beforehand in the first line in this program. It is designed for PARMA processes with lumped-vector process having an AR order of at most $p^* = 2$ (see (1.7)), p^* being denoted in this program as MAX1. The maximum value $p^* = 2$ is sufficient for common applications, and the program can easily be modified for larger values of p^*, if needed.

Also, due to the symmetry between the AR and MA parts in general PARMA models, this program can be used to check for the invertibility of PARMA processes in which case $q(\nu)$ and Θ's should be given in place of $p(\nu)$ and Φ's, respectively. Also, p^* in the above discussion will be replaced by q^* (see (1.7)).

INPUTS:

IP(IW): $p(\nu)$, $\nu = 1, \ldots, \omega$.

PHI(M*IW,3M*IW): This matrix can be written in terms of Φ matrices as follows:

$$\begin{bmatrix} \Phi_1^{(1)} & \Phi_2^{(1)} & \cdots & \Phi_{p(1)}^{(1)} & 0 & \cdots & 0 \\ \Phi_1^{(2)} & \Phi_2^{(2)} & \cdots & \Phi_{p(2)}^{(2)} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots & & \vdots \\ \Phi_1^{(\omega)} & \Phi_2^{(\omega)} & \cdots & \Phi_{p(\omega)}^{(\omega)} & 0 & \cdots & 0 \end{bmatrix}$$

OUTPUTS:

The outputs are L and U_ℓ matrices defined by (2.2) and (2.3), respectively. Also, it gives the eigenvalues of the matrix R defined by (2.6) and their modulus. For invertibility, the matrices L and U_ℓ will correspond to Λ and V_ℓ matrices in (2.1), respectively. The outputs are saved in file Tapel.

INPUT COMMANDS FOR A SAMPLE RUN:

Here we consider, as an example, the $PAR_2(1)$ model with $\phi_1^{(1)} = 0.8$, $\phi_1^{(2)} = 0.5$.

```
/ININ OSAMAO
REMARK: THIS PROGRAM IS DESIGNED FOR PARMA PROCESSES
FOR WHICH THE LUMPED PROCESS AR ORDER IS AT MOST 2
NOW INPUT (IP(V)), I.E, THE AR ORDERS
FOR THE V-TH SEASON, V=1,...,2
?1 1
MAX1=1
NOW, FOR SEASON 1 INPUT THE PHI MATRICES AS FOLLOWS
          1) INPUT THE FIRST ROW OF PHI1
          2) INPUT THE FIRST ROW OF PHI2, ETC...
          3) INPUT THE 2ND ROW OF PHI1, AND SO ON
?.8
NOW, FOR SEASON 2 INPUT THE PHI MATRICES AS FOLLOWS
          1) INPUT THE FIRST ROW OF PHI1
          2) INPUT THE FIRST ROW OF PHI2, ETC...
          3) INPUT THE 2ND ROW OF PHI1, AND SO ON
?.5
THE PARMA PROCESS YOU'VE DEFINED IS PCS.
FOR MORE DETAILS, LOOK INTO THE FILE TAPE1
/edif tapel            [outputs]

 THE L MATRIX...
 1. 0.
 -.5 1.
 THE U1 MATRIX...
 0. .8
```

```
0. 0.
EIGENVALUES
RR                  [real part of eigenvalue]
0. .4
RI                  [imaginary part of eigenvalue]
0. 0.
MODULUS OF THE EIGENVALUES
0. .4
HENCE,...THE PARMA PROCESS YOU'VE DEFINED IS PCS.
```

PROGRAM LISTING:

This program is stored in file OSAMAO.

```
      PARAMETER (M=1,IW=2,I1=M*IW,I2=2*I1)
      REAL          PHI(I1,3*I1),AL(I1,I1),AL1(I1,I1),U1(I1,I1),RI(I1),RR(I1)
      REAL       U2(I1,I1),AL2(I2,I2),RI1(I2),RR1(I2),U3(I1,I1)
      INTEGER    IP(IW),INT1(I1),INT2(I2)
      EXTERNAL F02AFF,F01AAF
      PRINT *,'REMARK: THIS PROGRAM IS DESIGNED FOR PARMA PROCESSES'
      PRINT *,'FOR WHICH THE LUMPED PROCESS IS AT MOST MULT. ARMA(2,2)'
      PRINT *,'NOW INPUT (IP(V)), I.E. THE AR ORDERS'
      PRINT *,'FOR THE V-TH SEASON, V=1,...,',IW
      READ   *,(IP(I),I=1,IW)
      MAX1=0
      DO 2 I=1,IW
      IF (IP(I).EQ.0) GOTO 2
      MOX1=(IP(I)-I)/IW+1
      IF (MOX1.GT.MAX1) MAX1=MOX1
2     CONTINUE
      PRINT *,'MAX1=',MAX1
      IF (MAX1.GT.0) GOTO 7
      PRINT *,'MAX1=0, I.E, THE PROCESS IS A PURE MA, HENCE PCS'
      STOP
7     IF (MAX1.LE.2) GOTO 10
      PRINT *,'MAX1>2, SORRY, THIS PROGRAM DOESN'T WORK HERE.'
      STOP
10    DO 1 K=1,IW
      IF (IP(K).LT.1) GOTO 1
      PRINT *,'NOW, FOR SEASON ',K,' INPUT THE PHI MATRICES AS FOLLOWS'
      PRINT *,'           1) INPUT THE FIRST ROW OF PHI1'
      PRINT *,'           2) INPUT THE FIRST ROW OF PHI2, ETC...'
      PRINT *,'           3) INPUT THE 2ND ROW OF PHI1, AND SO ON'
      DO 11 I=1,M
      DO 11 J=1,IP(K)
      READ   *,(PHI(M*(K-1)+I,L),L=M*(J-1)+1,M*J)
11    CONTINUE
1     CONTINUE
C                   **************
C        OBTAINING THE MATRICES L AND U1
C                   **************
      DO 3 I=1,I1
```

```
      DO  4  J=1,I1
      AL(I,J)=0E0
      U2(I,J)=0E0
      U1(I,J)=0E0
4     CONTINUE
      AL(I,I)=1E0
3     CONTINUE
      DO 5 I=2,IW
      DO  5  J=1,I-1
      DO 5 K=1,M
      DO 5 L=1,M
5         AL((I-1)*M+K,(I-J-1)*M+L)=-PHI((I-1)*M+K,(J-1)*M+L)
      WRITE(1,*)'THE  L  MATRIX...'
      DO  6  I=1,I1
      RI(I)=1E0
6         WRITE(1,*)(AL(I,J),J=1,I1)
      DO  8  I=1,IW
      DO  8  J=1,IW
      DO 8 K=1,M
      DO 8 L=1,M
8         U1((I-1)*M+K,(IW-J)*M+L)=PHI((I-1)*M+K,(I+J-2)*M+L)
      WRITE(1,*)'THE  U1  MATRIX...'
      DO 13 I=1,M*IW
13        WRITE(1,*)(U1(I,J),J=1,M*IW)
      IFAIL=1
      CALL     F01AAF(AL,I1,I1,AL1,I1,RR,IFAIL)
      CALL        MXMAB(I1,I1,I1,AL1,I1,U1,I1,U3,I1)
      IF (MAX1.EQ.2) GOTO 20
      IFAIL=0
      CALL    F02AFF(U3,I1,I1,RR,RI,INT1,IFAIL)
      WRITE(1,*)'EIGENVALUES'
      WRITE(1,*)'RR'
      WRITE(1,*)(RR(I),I=1,I1)
      WRITE(1,*)'RI'
      WRITE(1,*)(RI(I),I=1,I1)
      WRITE(1,*)'MODULUS OF THE EIGENVALUES'
      FLAG=0E0
      DO  16  I=1,I1
      RR(I)=SQRT(RR(I)**2+RI(I)**2)
      IF  (RR(I).GE.1E0)  FLAG=1E0
16    CONTINUE
      WRITE(1,*)(RR(I),I=1,I1)
      IF (FLAG.EQ.1E0) GOTO 18
      PRINT *,'THE PARMA PROCESS YOU'VE DEFINED IS PCS.'
      WRITE(1,*)'HENCE,...THE PARMA PROCESS YOU'VE DEFINED IS PCS.'
      GOTO 17
18    WRITE(1,*)'HENCE,...YOUR  PARMA  PROCESS  IS  NOT  PCS.'
      PRINT *,'YOUR PARMA PROCESS IS NOT PCS.'
17    PRINT *,'FOR MORE DETAILS, LOOK INTO THE FILE TAPE1'
      STOP
20    DO  21  I=1,IW
      DO  21  J=1,IW
      DO 21 K=1,M
      DO 21 L=1,M
21        U2((I-1)*M+K,(IW-J)*M+L)=PHI((I-1)*M+K,(I+J+IW-2)*M+L)
      WRITE(1,*)'THE  U2  MATRIX...'
```

```
      DO 22 I=1,I1
22       WRITE(1,*)(U2(I,J),J=1,I1)
      CALL     MXMAB(I1,I1,I1,AL1,I1,U2,I1,U1,I1)
      WRITE(1,*)'THE LINV*U2 MATRIX...'
      DO 23 I=1,I1
23       WRITE(1,*)(U1(I,J),J=1,I1)
      DO 24 I=1,I1
      DO 25 J=1,I1
      AL2(I,J)=U3(I,J)
25      AL2(I+I1,J)=U1(I,J)
24     AL2(I,I+I1)=1E0
      WRITE(1,*)'THE AL2 MATRIX...'
      DO 26 I=1,I2
26       WRITE(1,*)(AL2(I,J),J=1,I2)
      IFAIL=0
      CALL  F02AFF(AL2,I2,I2,RR1,RI1,INT2,IFAIL)
      WRITE(1,*)'EIGENVALUES'
      WRITE(1,*)'RR'
      WRITE(1,*)(RR1(I),I=1,I2)
      WRITE(1,*)'RI'
      WRITE(1,*)(RI1(I),I=1,I2)
      WRITE(1,*)'MODULUS OF THE EIGENVALUES'
      FLAG=0E0
      DO 27 I=1,I2
      RR1(I)=SQRT(RR1(I)**2+RI1(I)**2)
      IF (RR1(I).GE.1E0) FLAG=1E0
27   CONTINUE
      WRITE(1,*)(RR1(I),I=1,I2)
      IF (FLAG.EQ.1E0) GOTO 28
      PRINT *,'THE PARMA PROCESS YOU'VE DEFINED IS PCS.'
      WRITE(1,*)'HENCE,...THE PARMA PROCESS YOU'VE DEFINED IS PCS.'
      GOTO 29
28   WRITE(1,*)'HENCE,...YOUR PARMA PROCESS IS NOT PCS.'
      PRINT *,'YOUR PARMA PROCESS IS NOT PCS.'
29   PRINT *,'FOR MORE DETAILS, LOOK INTO THE FILE TAPE1'
      STOP
      END
```

APPENDIX B

COMPUTER PROGRAM FOR OBTAINING THE SAMPLE AUTOCORRELATION AND PARTIAL AUTOCORRELATION FUNCTIONS OF THE MARGINAL SERIES OF PARMA PROCESSES

Description:

This program computes the sample autocorrelation and partial autocorrelation functions (up to lag NOLAG) of the marginal series of simulated univariate $PARMA_{\omega}(p(\nu), q(\nu))$ processes based on a realization of length $N\omega$, for $p(\nu) \leq 2\omega$ and $q(\nu) \leq 2\omega$, and can be easily generalized for higher orders. The parameters IW, N and NOLAG are fixed beforehand in the first line in this program. The Box-Pierce statistic is also given for the autocorrelations, which is useful for identifying a white noise process. The NAG subroutine G13ABF is utilized to obtain these quantities.

In this and the succeeding programs, the error terms are assumed to be independently and normally distributed with mean zero, and the simulation of these terms are done by the Nag subroutine G05EAF. These programs also include an option to use actual data instead of simulated data, in which case the parameters in the first line of each program must be set accordingly. Also, the actual realization of the time series must be given in file "TAPE5" before running any of these programs. For an actual-data run, $p(\nu)$ and $q(\nu)$ are arbitrary.

INPUTS:

IP(IW): $p(\nu)$, $\nu = 1, \ldots, \omega$.

IQ(IW): $q(\nu)$, $\nu = 1, \ldots, \omega$.

PHI(IW,2*IW): As PHI matrix in the previous program with $m = 1$.

THITA(IW,2*IW): As PHI matrix but with θ's in place of ϕ's.

SIGMA(IW): The white noise variances, $\sigma_a^2(\nu)$, $\nu = 1, \ldots, \omega$.

ISEED: Seed point for random number generation; a positive integer.

OUTPUTS:

The ouputs are the sample ACF and PACF for all seasons of the given realization of PARMA process and the Box-Pierce statistic as well. The outputs are saved in file Tape166 (which is specified in the first line of the program as NOUT=166).

INPUT COMMANDS FOR A SAMPLE RUN:

Here we consider, as an example, the $PMA_2(1)$ model with $\theta_1^{(1)} = 0.8$, $\theta_1^{(2)} = 0.5$. $N = 30$, $\sigma_a^2(1) = \sigma_a^2(2) = 1$.

```
/ININ OSAMA1
INPUT IP(I), I=1,..., 2, I.E, THE AR ORDERS <=2*IW
?0 0
INPUT IQ(I), I=1,..., 2, I.E, THE MA ORDERS <=2*IW
?1 1
DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?
IF YES, TYPE "1"
?0
INPUT PHI"S FOR SEASON 1
?0
NOW, INPUT THITA"S FOR SEASON 1
?.8
INPUT PHI"S FOR SEASON 2
?0
NOW, INPUT THITA"S FOR SEASON 2
?.5
INPUT SIGMA(I), I=1,..., 2
?1 1
GIVE SEED...!
?100
FOR OUTPUTS, SEE TAPE166
/edif tape166      [outputs]

 ***********************************************
 ************
 SEASON 1
 ************
 LAG    AUTOCORR.
 *******************************
 1     .179200137863
 2     .1779862436263
 3     -.1855937913673
 4     .07190671454458
 5     -.0703076209482
```

```
********************************
BOX-PIERCE STAT. = 3.250517546349
********************************
********************************
LAG     PARTIAL AUTOCORR.
********************************
1     .179200137863
2     .1507133657195
3     -.2533980529208
4     .1343936020059
5     -.03505194214135
************************************************
************
SEASON 2
************
LAG     AUTOCORR.
********************************
1     .3691400857436
2     .06542144353839
3     -.1280877883058
4     .1144918706175
5     .03368578349005
********************************
BOX-PIERCE STAT. = 5.135819104121
********************************
********************************
LAG     PARTIAL AUTOCORR.
********************************
1     .3691400857436
2     -.08201926562076
3     -.1444666955753
4     .2607722773021
5     -.1245395870054
************************************************
```

PROGRAM LISTING:

This program is stored in file OSAMA1.

```
      PARAMETER (IW=2,N=30,NOLAG=5,NOUT=166)
      REAL PHI(IW,2*IW),SIGMA(IW),ZZ(10),ERROR(IW*(N+51))
      REAL  THITA(IW,2*IW),X(IW*(N+51)),Z(1),MU(IW),Y(N)
      REAL SIGG(IW),R(NOLAG),P(NOLAG),V(NOLAG),AR(NOLAG)
      INTEGER IP(IW),IQ(IW)
      EXTERNAL X04ABF,X02AJF,C05NBF
      CALL X04ABF(1,NOUT)
      PRINT *,'INPUT IP(I), I=1,..., ',IW,' I.E, THE AR ORDERS <=2*IW'
      READ  *,(IP(I),I=1,IW)
      PRINT *,'INPUT IQ(I), I=1,..., ',IW,' I.E, THE MA ORDERS <=2*IW'
      READ  *,(IQ(I),I=1,IW)
      PRINT *,'DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?'
      PRINT *,'IF YES, TYPE "1"'
```

```
      READ(*,*)IFL
      IF (IFL.EQ.1) GOTO 50
      DO 70 I=1,IW
      PRINT *,'INPUT PHI"S FOR SEASON ',I
      READ     *,(PHI(I,J),J=1,IP(I))
      PRINT *,'NOW, INPUT THITA"S FOR SEASON ',I
70      READ  *,(THITA(I,J),J=1,IQ(I))
      PRINT  *,'INPUT  SIGMA(I),  I=1,...,  ',IW
      READ  *,(SIGMA(I),I=1,IW)
      PRINT  *,'GIVE  SEED...!'
      READ(*,*)ISEED
      CALL G05CBF(ISEED)
      DO 17 I=1,IW
      VM=0E0
      CALL   G05EAF(VM,1,SIGMA(I),1,0.001,ZZ,10,IFAIL)
      DO 17 L=1,N+51
      IFAIL=0
      CALL G05EZF(Z,1,ZZ,10,IFAIL)
      ERROR((L-1)*IW+I)=Z(1)
17   CONTINUE
      DO 22 J=1,IW
22    X(J)=0E0
      DO 23 J=IW+1,IW*(N+51)
      SSU=ERROR(J)
      IV=J-(J/IW)*IW
      IF (IV.EQ.0) IV=IW
      DO 71 K=1,IP(IV)
71    SSU=SSU+X(J-K)*PHI(IV,K)
      DO 72 K=1,IQ(IV)
72   SSU=SSU-ERROR(J-K)*THITA(IV,K)
      X(J)=SSU
23   CONTINUE
      DO 35 J=1,N*IW
35     X(J)=X(51*IW+J)
      GOTO 51
50   PRINT *,'SINCE YOU ARE RUNNING THIS PROGRAM WITH ACTUAL DATA'
      PRINT *,'YOU SHOULD HAVE MODEIFIED THE PARAMETERS IN THE FIRST'
      PRINT *,'LINE OF THIS PROGRAM. ALSO, THE DATA, I.E. THE '
      PRINT *,'OBSERVED RELAIZATION OF THE TIME SERIES, MUST BE'
      PRINT *,'STORED IN FILE "TAPE5"'
      READ(5,*)(X(I),I=1,N*IW)
51    DO 100 I=1,IW
      DO 36 J=1,N
36      Y(J)=X((J-1)*IW+I)
      IFAIL=0
      CALL G13ABF(Y,N,NOLAG,XM,XV,R,STAT,IFAIL)
      WRITE(NOUT,*)'**********************************************'
      WRITE(NOUT,*)'************'
      WRITE(NOUT,*)'SEASON ',I
      WRITE(NOUT,*)'************'
      WRITE(NOUT,*)'LAG    AUTOCORR.'
      WRITE(NOUT,*)'******************************'
      DO 37 J=1,NOLAG
37     WRITE(NOUT,*)J,'       ',R(J)
      WRITE(NOUT,*)'******************************'
      WRITE(NOUT,*)'BOX-PIERCE STAT.TEST STAT. = ',STAT
```

```
      WRITE(NOUT,*)'********************************'
      IFAIL=0
      CALL G13ACF(R,NOLAG,NOLAG,P,V,AR,NVL,IFAIL)
      WRITE(NOUT,*)'********************************'
      WRITE(NOUT,*)'LAG    PARTIAL AUTOCORR.'
      WRITE(NOUT,*)'********************************'
      DO 38 J=1,NOLAG
38     WRITE(NOUT,*)J,'        ',P(J)
100   CONTINUE
      WRITE(NOUT,*)'************************************************'
      PRINT *,'FOR OUTPUTS, SEE TAPE',NOUT
      STOP
      END
```

APPENDIX C

COMPUTER PROGRAM FOR THE COMPUTATION OF THE SAMPLE SEASONAL AUTOCORRELATION AND PARTIAL AUTOCORRELATION FUNCTIONS OF PARMA PROCESSES

Description:

This program computes the average sample seasonal autocorrelation and partial autocorrelation functions (up to lag NOLAG) of simulated univariate PARMA processes over NREPT realizations each of length $N\omega$. It also gives the frequencies of autocorrelations and partial autocorrelations going outside the 95% bands developed in Chapter III. The parameters IW, N, NOLAG and NREPT are fixed beforehand in the first line of this program. This program can also be used for actual data, in which case NREPT = 1 (see the description of the program in Appendix B). This point carries over to the programs in the next appendix.

INPUTS:

The same as those of the previous program.

OUTPUTS:

The outputs are the average sample seasonal ACF and PACF, and frequencies for autocorrelations and partial autocorrelations falling beyond the 95% bands. The outputs are saved in file Tape100.

INPUT COMMANDS FOR A SAMPLE RUN:

Here we consider, as an example, the $PARMA_2(1,0;\ 0,1)$ model with $\phi_1^{(1)}$ =

0.8, $\theta_1^{(2)} = 0.5$. N = 30, NREPT = 2, NOLAG = 5, $\sigma_a^2(1) = \sigma_a^2(2) = 1$.

```
/ININ OSAMA2
INPUT IP(I), I=1,..., 2, I.E, THE AR ORDERS <=2*IW
?1 0
INPUT IQ(I), I=1,..., 2, I.E, THE MA ORDERS <=2*IW
?0 1
DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?
IF YES, TYPE "1"
?0
INPUT PHI"S FOR SEASON 1
?.8
NOW, INPUT THITA"S FOR SEASON 1
?0
INPUT PHI"S FOR SEASON 2
?0
NOW, INPUT THITA"S FOR SEASON 2
?.5
INPUT SIGMA(I), I=1,..., 2
?1 1
GIVE SEED...!
?209
FOR OUTPUTS, SEE TAPE100
/edif tape100      [outputs]

            SEASONAL AUTOCORRELATIONS
 *****************
   SEASON 1
 *****************
 LAG          AVERAGE
 1   .5317488596371
 2   -.01620124427954
 3   .1788676040135
 4   -.1458501267343
 5   .0168994940282
 TOTAL NO. OF AUTOCORR. COMPUTED AFTER LAG 0
  = 10
 NO. OF AUTOCORR. OUTSIDE THE BANDS, =2
 NO. OF AUTOCORR. OUTSIDE THE BANDS FOR 2
 ITERATIONS FOR SEASON 1 ARE AS FOLLOWS:
 1 1
 *****************
   SEASON 2
 *****************
 LAG          AVERAGE
 1   -.2077077712014
 2   -.06810569040779
 3   -.1187241427203
 4   .09762157960473
 5   -.07918171082883
 TOTAL NO. OF AUTOCORR. COMPUTED AFTER LAG 1
  = 8
 NO. OF AUTOCORR. OUTSIDE THE BANDS, =1
 NO. OF AUTOCORR. OUTSIDE THE BANDS FOR 2
 ITERATIONS FOR SEASON 1 ARE AS FOLLOWS:
```

```
1 0
    SEASONAL PARTIAL AUTOCORRELATIONS
******************
  SEASON 1
******************
LAG          AVERAGE
1   .5317488596371
2   -.1320125971226
3   -.1264080341135
4   .1019139702972
5   -.038491886237
TOTAL NO. OF AUTOCORR. COMPUTED AFTER LAG 1
 = 8
NO. OF AUTOCORR. OUTSIDE THE BANDS, =1
NO. OF AUTOCORR. OUSTIDE THE BANDS FOR 2
ITERATIONS FOR SEASON 1 ARE AS FOLLOWS:
1 0
******************
  SEASON 2
******************
LAG          AVERAGE
1   -.2077077712014
2   -.06170655635853
3   .1215119314638
4   -.2758046858665
5   -.05701102215099
TOTAL NO. OF AUTOCORR. COMPUTED AFTER LAG 0
 = 10
NO. OF AUTOCORR. OUTSIDE THE BANDS, =0
NO. OF AUTOCORR OUTSIDE THE BANDS FOR 2
ITERATIONS FOR SEASON 2 ARE AS FOLLOWS:
0 0
```

PROGRAM LISTING:

This program is stored in file OSAMA2.

```
      PARAMETER (IW=2,N=30,NOUT=100,NREPT=2,NOLAG=5)
      REAL  PHI(IW,2*IW),SIGMA(IW),R(10),ERROR(IW*(N+51))
      REAL THITA(IW,2*IW),X(IW*(N+51)),Z(1),MU(IW),ROW(IW,NOLAG)
      REAL SS1(IW,NOLAG),SIGG(IW),ASS(NOLAG,NOLAG),COV(IW,NOLAG)
      REAL SS2(IW,NOLAG),PAC(IW,NOLAG),DELTA(IW,NOLAG)
      REAL ALFA(IW,NOLAG,NOLAG),BETA(IW,NOLAG,NOLAG),SAG(IW,NOLAG)
      REAL TAO(IW,NOLAG)
      INTEGER  IP(IW),IQ(IW),ICC1(IW),ICON1(NREPT,IW)
      INTEGER ICC2(IW),ICON2(NREPT,IW)
      EXTERNAL X04ABF,X02AJF,C05NBF
      CALL X04ABF(1,NOUT)
      PRINT *,'INPUT IP(I), I=1,..., ',IW,' I.E, THE AR ORDERS <=2*IW'
      READ  *,(IP(I),I=1,IW)
      PRINT *,'INPUT IQ(I), I=1,..., ',IW,' I.E, THE MA ORDERS <=2*IW'
      READ  *,(IQ(I),I=1,IW)
```

```
      PRINT *,'DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?'
      PRINT *,'IF YES, TYPE "1"'
      READ(*,*)IFL
      IF (IFL.EQ.1) GOTO 55
      DO 70 I=1,IW
      PRINT *,'INPUT PHI"S FOR SEASON ',I
      READ     *,(PHI(I,J),J=1,IP(I))
      PRINT *,'NOW, INPUT THITA"S FOR SEASON ',I
70      READ *,(THITA(I,J),J=1,IQ(I))
      PRINT *,'INPUT  SIGMA(I),  I=1,...,  ',IW
      READ  *,(SIGMA(I),I=1,IW)
      PRINT *,'GIVE SEED...!'
      READ(*,*)ISEED
      CALL G05CBF(ISEED)
      DO 101 I=1,IW
      ICC1(I)=0E0
      ICC2(I)=0E0
      DO 101 J=1,NOLAG
      SS1(I,J)=0E0
101    SS2(I,J)=0E0
      CONS=1.96/N**0.5
55    DO 100 KK=1,NREPT
      IF (IFL.EQ.1) GOTO 50
      DO 17 I=1,IW
      VM=0E0
      CALL   G05EAF(VM,1,SIGMA(I),1,0.001,R,10,IFAIL)
      DO 17 L=1,N+51
      IFAIL=0
      CALL G05EZF(Z,1,R,10,IFAIL)
      ERROR((L-1)*IW+I)=Z(1)
17    CONTINUE
      DO 22 J=1,IW
22    X(J)=0E0
      DO 23 J=IW+1,IW*(N+51)
      SSU=ERROR(J)
      IV=J-(J/IW)*IW
      IF (IV.EQ.0) IV=IW
      DO 71 K=1,IP(IV)
71    SSU=SSU+X(J-K)*PHI(IV,K)
      DO 72 K=1,IQ(IV)
72    SSU=SSU-ERROR(J-K)*THITA(IV,K)
      X(J)=SSU
23    CONTINUE
      DO 35 J=1,N*IW
35     X(J)=X(51*IW+J)
      GOTO 51
50   PRINT *,'SINCE YOU ARE RUNNING THIS PROGRAM WITH ACTUAL DATA'
      PRINT *,'YOU SHOULD HAVE MODEIFIED THE PARAMETERS IN THE FIRST'
      PRINT *,'LINE OF THIS PROGRAM (NREPT=1). ALSO, THE DATA, I.E. '
      PRINT *,'THE OBSERVED RELAIZATION OF THE TIME SERIES, MUST BE'
      PRINT *,'STORED IN FILE "TAPE5"'
      READ(5,*)(X(I),I=1,N*IW)
51    DO 19 I=1,IW
      SS11=0E0
      SS22=0E0
      DO 18 J=1,N
```

```
      SS11=SS11+X(I+(J-1)*IW)
18      SS22=SS22+X(I+(J-1)*IW)**2
      MU(I)=SS11/N
      SIGG(I)=(SS22-N*(MU(I)**2))/(N-1)
19    CONTINUE
      DO 20 I=1,IW
      SIG=SIGG(I)**.5
      DO 20 J=1,N
      ERROR(I+(J-1)*IW)=(X(I+(J-1)*IW)-MU(I))/SIG
20        X(I+(J-1)*IW)=X(I+(J-1)*IW)-MU(I)
      DO 210 I=1,IW
      DO 21 L=1,NOLAG
      PAC(I,L)=0E0
      DELTA(I,L)=0E0
      TAO(I,L)=0E0
      SAG(I,L)=0E0
      DO 207 LL=1,NOLAG
      ALFA(I,L,LL)=0E0
207   BETA(I,L,LL)=0E0
      COV(I,L)=0E0
      ROW(I,L)=0E0
      DO 24 J=1,N-1
      IF (I+J*IW-L.LE.0) GOTO 24
      COV(I,L)=COV(I,L)+X(I+J*IW)*X(I+J*IW-L)
      ROW(I,L)=ROW(I,L)+ERROR(I+J*IW)*ERROR(I+J*IW-L)
24    CONTINUE
      COV(I,L)=COV(I,L)/N
      ROW(I,L)=ROW(I,L)/N
      SS1(I,L)=SS1(I,L)+ROW(I,L)
21    CONTINUE
      IF (IQ(I).LT.IW) GOTO 220
      IZ=IQ(I)/IW
      DO 230 L=IQ(I)+1,NOLAG
      CONS1=1E0
      IZZ=I-1
260   IF (IZZ.LE.0) IZZ=IZZ+IW
      IF (IZZ.LE.0) GOTO 260
      DO 240 JJ=1,IZ
240   CONS1=CONS1+2*ROW(I,JJ*IW)*ROW(IZZ,JJ*IW)
      CONS1=1.96*(CONS1/N)**0.5
      IF (ABS(ROW(I,L)).GE.CONS1) ICON1(KK,I)=ICON1(KK,I)+1
230   CONTINUE
      GOTO 250
220   DO 210 L=IQ(I)+1,NOLAG
      IF (ABS(ROW(I,L)).GE.CONS) ICON1(KK,I)=ICON1(KK,I)+1
210   CONTINUE
250   CONTINUE
C            **************************************************
C     NOW, CALCULATION OF PARTIAL AUTOCORRELATIONS
C            **************************************************
      DO 200 I=1,IW
      MAM=I-1
      IF (MAM.EQ.0) MAM=IW
      PAC(I,1)=COV(I,1)/(SIGG(I)*SIGG(MAM))**0.5
      SS2(I,1)=SS2(I,1)+PAC(I,1)
      ALFA(I,1,1)=-COV(I,1)/SIGG(MAM)
```

```
      BETA(I,1,1)=-COV(I,1)/SIGG(I)
      DELTA(I,1)=COV(I,2)+COV(MAM,1)*ALFA(I,1,1)
      SAG(I,1)=SIGG(I)*(1-ALFA(I,1,1)*BETA(I,1,1))
200       TAO(I,1)=SIGG(MAM)*(1-ALFA(I,1,1)*BETA(I,1,1))
      DO 201 L=1,NOLAG-1
      DO  201  I=1,IW
      DELTA(I,L)=COV(I,L+1)
      DO 203 M=1,L
      MM=I-M
204   IF (MM.LE.0) MM=MM+IW
      IF (MM.LE.0) GOTO 204
203   DELTA(I,L)=DELTA(I,L)+COV(MM,L+1-M)*ALFA(I,L,M)
      MAM=I-1
      IF (MAM.EQ.0) MAM=IW
      ALFA(I,L+1,L+1)=-DELTA(I,L)/TAO(MAM,L)
      BETA(I,L+1,L+1)=-DELTA(I,L)/SAG(I,L)
      SAG(I,L+1)=SAG(I,L)*(1-ALFA(I,L+1,L+1)*BETA(I,L+1,L+1))
      TAO(I,L+1)=TAO(MAM,L)*(1-ALFA(I,L+1,L+1)*BETA(I,L+1,L+1))
      DO 205 M=1,L
      ALFA(I,L+1,M)=ALFA(I,L,M)+ALFA(I,L+1,L+1)*BETA(MAM,L,L+1-M)
205    BETA(I,L+1,M)=BETA(MAM,L,M)+BETA(I,L+1,L+1)*ALFA(I,L,L+1-M)
      PAC(I,L+1)=-DELTA(I,L)/(SAG(I,L)*TAO(MAM,L))**0.5
201       SS2(I,L+1)=SS2(I,L+1)+PAC(I,L+1)
      DO  206  I=1,IW
      DO  206  L=IP(I)+1,NOLAG
      IF   (ABS(PAC(I,L)).GE.CONS)   ICON2(KK,I)=ICON2(KK,I)+1
206   CONTINUE
      DO  100  I=1,IW
      ICC1(I)=ICC1(I)+ICON1(KK,I)
      ICC2(I)=ICC2(I)+ICON2(KK,I)
100   CONTINUE
      DO  102  I=1,IW
      DO 102 J=1,NOLAG
      SS1(I,J)=SS1(I,J)/NREPT
102     SS2(I,J)=SS2(I,J)/NREPT
      WRITE(NOUT,*)'            SEASONAL  AUTOCORRELATIONS'
      DO  103  I=1,IW
      WRITE(NOUT,*)'*****************'
      WRITE(NOUT,*)'   SEASON  ',I
      WRITE(NOUT,*)'*****************'
      WRITE(NOUT,*)'LAG             AVERAGE   '
      DO 85 L=1,NOLAG
85     WRITE(NOUT,*)L,'       ',SS1(I,L)
      WRITE(NOUT,*)'TOTAL NO. OF AUTOCORR. COMPUTED AFTER LAG ',IQ(I)
      WRITE(NOUT,*)' = ',(NOLAG-IQ(I))*NREPT
      WRITE(NOUT,*)'NO. OF AUTOCORR. OUTSIDE THE BANDS, =',ICC1(I)
      WRITE(NOUT,*)'NO. OF AUTOCORR. OUTSIDE THE BANDS FOR ',NREPT
      WRITE(NOUT,*)'ITERATIONS FOR SEASON ',I,' ARE AS FOLLOWS:'
      WRITE(NOUT,*)(ICON1(JJ,I),JJ=1,NREPT)
103   CONTINUE
      WRITE(NOUT,*)'    SEASONAL PARTIAL  AUTOCORRELATIONS'
      DO  133  I=1,IW
      WRITE(NOUT,*)'*****************'
      WRITE(NOUT,*)'   SEASON  ',I
      WRITE(NOUT,*)'*****************'
      WRITE(NOUT,*)'LAG             AVERAGE   '
```

```
      DO 851 L=1,NOLAG
851    WRITE(NOUT,*)L,'      ',SS2(I,L)
      WRITE(NOUT,*)'TOTAL NO. OF AUTOCORR. COMPUTED AFTER LAG ',IP(I)
      WRITE(NOUT,*)' = ',(NOLAG-IP(I))*NREPT
      WRITE(NOUT,*)'NO. OF AUTOCORR. OUTSIDE THE BANDS, =',ICC2(I)
      WRITE(NOUT,*)'NO. OF AUTOCORR. OUTSIDE THE BANDS FOR ',NREPT
      WRITE(NOUT,*)'ITERATIONS FOR SEASON ',I,' ARE AS FOLLOWS:'
      WRITE(NOUT,*)(ICON2(JJ,I),JJ=1,NREPT)
133   CONTINUE
      PRINT *,'FOR OUTPUTS, SEE TAPE',NOUT
      STOP
      END
```

APPENDIX D

COMPUTER PROGRAMS FOR MOMENT AND CONDITIONAL LEAST-SQUARES ESTIMATION IN PAR PROCESSES

D.1 Univariate Case

Description:

This program computes the average moment and conditional least-squares (LS) estimates of the parameters of simulated $PAR_{\omega}(p(\nu))$ processes over NREPT realizations each of length $N\omega$. It is designed for $p(\nu) \leq \omega$, but can easily be generalized for higher orders. The conditional LS estimates are obtained by utilizing NAG subroutine G02CJF. The parameters IW, N and NREPT are fixed beforehand in the first line of this program. The estimates of the white noise variances in the conditional LS part are obtained in this program according to (4.14). This program can also be used for actual data (see the descriptions of the programs in Appendices B and C).

INPUTS:

The same as those of the previous program, but excluding IQ and THITA.

OUTPUTS:

The ouputs are the average moment and conditional LS estimates and their root mean squared error (RMSE). The outputs are saved in file Tape181.

INPUT COMMANDS FOR A SAMPLE RUN:

Here we consider, as an example, the $PAR_2(1)$ model with $\phi_1^{(1)} = 0.8$, $\phi_1^{(2)} =$

0.5. N = 30, NREPT = 100, $\sigma_a^2(1) = \sigma_a^2(2) = 1$.

```
/ININ OSAMA3
INPUT IP(I), I=1,..., 2, I.E, THE AR ORDERS <=IW
?1 1
DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?
IF YES, TYPE "1"
?0
INPUT PHI"S FOR SEASON 1
?.8
INPUT PHI"S FOR SEASON 2
?.5
INPUT SIGMA(I), I=1,..., 2
?1 1
GIVE SEED...!
?2203
FOR OUTPUTS, SEE TAPE181
/edif tape181     [outputs]
```

```
MOMENT ESTIMATES
PHI'S
.7214914765803
.4724707955424
THEIR RMSE
.1842630427989
.1435949086608
ERROR VARIANCE ESTIMATES
1.000873643797 .9577546466611
THEIR RMSE
.240318136315 .2547048038934
```

```
LS ESTIMATES
PHI'S
.7473786236009
.4724707955424
CONSTANT
-.01410239527737 .01747087316246
THEIR RMSE
.1739665984372
.1435949086608
ERROR VARIANCE ESTIMATES
1.001340916949 1.026165692851
THEIR RMSE
.2501273087882 .2703871964167
```

PROGRAM LISTING:

This program is stored in file OSAMA3.

```
      PARAMETER (IW=2,N=30,NOUT=181,NREPT=100)
```

```
      REAL   PHI(IW,IW),SIGMA(IW),R(10),ERROR(IW*(N+51)),A1(IW,IW)
      REAL    X(IW*(N+51)),Z(1),MU(IW),COV(IW,IW),PH1(IW,IW),PH2(IW,IW)
      REAL    SSM(IW,IW),SSL(IW,IW),CO(IW),A2(IW,IW),A3(N),A4(IW+1)
      REAL     A8(N),A9(N,IW+1),A10(N,IW+1),A11(IW+1,4),SAR2(IW)
      REAL SARV(IW),SAR1(IW),SARM1(IW),A0(IW)
      INTEGER   IP(IW),IPIV(IW+1),MIS(IW),MAS(IW)
      EXTERNAL X04ABF,X02AJF,C05NBF
      CALL X04ABF(1,NOUT)
      PRINT *,'INPUT  P(I),  I=1,...,',IW,'  I.E,  THE  AR  ORDERS  <=IW'
      READ   *,(IP(I),I=1,IW)
      DO 75 I=1,IW
      MIS(I)=0
      IF (IP(I)+1-I.GT.0) MIS(I)=1
      MAS(I)=IP(I)+MIS(I)+1
75    CONTINUE
      PRINT *,'DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?'
      PRINT *,'IF  YES,  TYPE  "1"'
      READ(*,*)IFL
      IF (IFL.EQ.1) GOTO 55
      DO 70 I=1,IW
      PRINT *,'INPUT PHI"S FOR SEASON ',I
70       READ  *,(PHI(I,J),J=1,IP(I))
      PRINT  *,'INPUT  SIGMA(I),  I=1,...,',IW
      READ  *,(SIGMA(I),I=1,IW)
      PRINT *,'GIVE  SEED...!'
      READ(*,*)ISEED
      CALL G05CBF(ISEED)
55    DO 100 KZ=1,NREPT
      IF (IFL.EQ.1) GOTO 50
      DO 17 I=1,IW
      VM=0E0
      CALL   G05EAF(VM,1,SIGMA(I),1,0.001,R,10,IFAIL)
      DO 17 L=1,N+51
      IFAIL=0
      CALL G05EZF(Z,1,R,10,IFAIL)
      ERROR((L-1)*IW+I)=Z(1)
17    CONTINUE
      DO 22 J=1,IW
22    X(J)=0E0
      DO 23 J=IW+1,IW*(N+51)
      SSU=ERROR(J)
      IV=J-(J/IW)*IW
      IF (IV.EQ.0) IV=IW
      DO 71 K=1,IP(IV)
71     SSU=SSU+X(J-K)*PHI(IV,K)
      X(J)=SSU
23    CONTINUE
      DO 35 J=1,N*IW
35     X(J)=X(51*IW+J)
      GOTO 51
50    PRINT *,'SINCE YOU ARE RUNNING THIS PROGRAM WITH ACTUAL DATA'
      PRINT *,'YOU SHOULD HAVE MODEIFIED THE PARAMETERS IN THE FIRST'
      PRINT *,'LINE OF THIS PROGRAM (NREPT=1). ALSO, THE DATA, I.E. '
      PRINT *,'THE OBSERVED RELAIZATION OF THE TIME SERIES, MUST BE'
      PRINT *,'STORED IN FILE "TAPE5"'
      READ(5,*)(X(I),I=1,N*IW)
```

```
C        ***********************************************
51       DO 19 I=1,IW
         SS1=0E0
         SS2=0E0
         DO 18 J=1,N
         SS1=SS1+X(I+(J-1)*IW)
18         SS2=SS2+X(I+(J-1)*IW)**2
         MU(I)=SS1/N
         CO(I)=(SS2-N*(MU(I)**2))/N
19     CONTINUE
         DO 20 I=1,IW
         DO 20 J=1,N
20          ERROR(I+(J-1)*IW)=X(I+(J-1)*IW)-MU(I)
         DO 21 I=1,IW
         DO 21 L=1,IW
         COV(I,L)=0E0
         DO 24 J=0,N-1
         IF (I+J*IW-L.LE.0) GOTO 24
         COV(I,L)=COV(I,L)+ERROR(I+J*IW)*ERROR(I+J*IW-L)
24     CONTINUE
         COV(I,L)=COV(I,L)/N
21     CONTINUE
C        ***********************************************
         DO 1000 I=1,IW
         DO 250 JJ=1,IW
         MU(JJ)=0E0
         DO 250 KK=1,IW
         A1(JJ,KK)=0E0
250     A2(JJ,KK)=0E0
         DO  2000  II=1,IP(I)
         MM=I-II
255    IF (MM.LE.0) MM=MM+IW
         IF (MM.LE.0) GOTO 255
         A1(II,II)=CO(MM)
         DO  2000  JJ=II+1,IP(I)
2000        A1(II,JJ)=COV(MM,JJ-II)
         IFAIL=0
         CALL  F01ABF(A1,IW,IP(I),A2,IW,MU,IFAIL)
         DO 350 II=1,IW
350      MU(II)=0E0
         DO  3000  II=1,IP(I)
         MU(II)=COV(I,II)
         DO  3000  JJ=II,IP(I)
3000         A2(II,JJ)=A2(JJ,II)
         IFAIL=0
         CALL F01CKF(MU,A2,MU,IW,1,IW,A3,N,3,IFAIL)
         DO  270  II=1,IP(I)
         PH1(I,II)=PH1(I,II)+MU(II)
270            SSM(I,II)=SSM(I,II)+(MU(II)-PHI(I,II))**2
         SSU=CO(I)
         DO  400  II=1,IP(I)
400       SSU=SSU-MU(II)*COV(I,II)
         SAR1(I)=SAR1(I)+SSU
         SARM1(I)=SARM1(I)+(SIGMA(I)-SSU)**2
C        ***********************************************
         DO 260  JJ=1,N-MIS(I)
```

```
      A8(JJ)=X(I+(JJ-1+MIS(I))*IW)
      A9(JJ,1)=1E0
      DO  260  II=1,IP(I)
260       A9(JJ,II+1)=X(I+(JJ-1+MIS(I))*IW-II)
      IFAIL=0
      CALL     G02CJF(A9,N,A8,N,N-MIS(I),IP(I)+1,1,A4,IW+1,SIGSQ,A10,N,
     *            IPIV,A11,A3,IFAIL)
      A0(I)=A0(I)+A4(1)
      DO  280  II=1,IP(I)
      A8(II)=0E0
      PH2(I,II)=PH2(I,II)+A4(II+1)
280         SSL(I,II)=SSL(I,II)+(A4(II+1)-PHI(I,II))**2
      SSU=0E0
      DO  381  JJ=1+MIS(I),N
      A8(JJ)=0E0
      DO  380  II=1,IP(I)
380       A8(JJ)=A8(JJ)+A4(II+1)*X(I+(JJ-1)*IW-II)
381     A8(JJ)=X(I+(JJ-1)*IW)-A8(JJ)-A4(1)
      DO  391  JJ=1+MIS(I),N
391   SSU=SSU+A8(JJ)**2
      SSU=SSU/(N-MAS(I))
      SAR2(I)=SAR2(I)+SSU
      SARV(I)=SARV(I)+(SIGMA(I)-SSU)**2
C      **********************************************
1000 CONTINUE
100  CONTINUE
      DO 290 I=1,IW
      A0(I)=A0(I)/NREPT
      SAR1(I)=SAR1(I)/NREPT
      SAR2(I)=SAR2(I)/NREPT
      SARM1(I)=SQRT(SARM1(I)/NREPT)
      SARV(I)=SQRT(SARV(I)/NREPT)
      DO  290  J=1,IP(I)
      PH1(I,J)=PH1(I,J)/NREPT
      PH2(I,J)=PH2(I,J)/NREPT
      SSM(I,J)=SQRT(SSM(I,J)/NREPT)
290    SSL(I,J)=SQRT(SSL(I,J)/NREPT)
      WRITE(NOUT,*)'____________________________________________________'
      WRITE(NOUT,*)'MOMENT ESTIMATES'
      WRITE(NOUT,*)'PHI'S'
      DO 800 I=1,IW
800      WRITE(NOUT,*)(PH1(I,J),J=1,IP(I))
      WRITE(NOUT,*)'THEIR RMSE'
      DO 801 I=1,IW
801      WRITE(NOUT,*)(SSM(I,J),J=1,IP(I))
      WRITE(NOUT,*)'ERROR VARIANCE ESTIMATES'
      WRITE(NOUT,*)(SAR1(I),I=1,IW)
      WRITE(NOUT,*)'THEIR RMSE'
      WRITE(NOUT,*)(SARM1(I),I=1,IW)
      WRITE(NOUT,*)'____________________________________________________'
      WRITE(NOUT,*)'LS ESTIMATES'
      WRITE(NOUT,*)'PHI'S'
      DO 802 I=1,IW
802      WRITE(NOUT,*)(PH2(I,J),J=1,IP(I))
      WRITE(NOUT,*)'CONSTANT'
       WRITE(NOUT,*)(A0(J),J=1,IW)
```

```
      WRITE(NOUT,*)'THEIR RMSE'
      DO 803 I=1,IW
803      WRITE(NOUT,*)(SSL(I,J),J=1,IP(I))
      WRITE(NOUT,*)'ERROR VARIANCE ESTIMATES'
      WRITE(NOUT,*)(SAR2(I),I=1,IW)
      WRITE(NOUT,*)'THEIR RMSE'
      WRITE(NOUT,*)(SARV(I),I=1,IW)
      WRITE(NOUT,*)'______________________________________________'
      PRINT *,'FOR OUTPUTS, SEE TAPE',NOUT
      STOP
      END
```

D.2 M-Variate Case

Description:

This program is the same as the previous program but written for M-variate (M>1) PAR processes. In this case, we are dealing with matrices rather than scalars and computations are more complicated. In addition to the other parameters, M is also fixed beforehand in the first line in this program.

INPUTS:

The same as those of the previous program but PHI matrix is now defined as in Appendix A.

OUTPUTS:

The ouputs are similar to those of the previous program. The outputs are saved in file Tape185.

INPUT COMMANDS FOR A SAMPLE RUN:

Here we consider, as an example, the bivariate $PAR_2(1)$ model considered in Section 4.6 and named Model (3), case (d). N = 30 and NREPT = 100. The outputs for this case are also summarized in Table 4.4(a).

```
/ININ OSAMA4
INPUT (IP(V)<=2), I.E, THE AR ORDERS
FOR THE V-TH SEASON, V=1,...,2
```

```
?1 1
DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?
IF YES, TYPE "1"
?0
NOW, FOR SEASON 1 INPUT THE PHI MATRICES AS FOLLOWS
         1) INPUT THE FIRST ROW OF PHI1
         2) INPUT THE FIRST ROW OF PHI2, ETC...
         3) INPUT THE 2ND ROW OF PHI1, AND SO ON
?.9 -.7
?0 .6
NOW, FOR SEASON 2 INPUT THE PHI MATRICES AS FOLLOWS
         1) INPUT THE FIRST ROW OF PHI1
         2) INPUT THE FIRST ROW OF PHI2, ETC...
         3) INPUT THE 2ND ROW OF PHI1, AND SO ON
?.5 .2
?0 .6
INPUT THE UPPER TRIANGLE OF THE ERROR COV. MATRIX
(ROW BY ROW) FOR SEASON 1
?1 0
?1
INPUT THE UPPER TRIANGLE OF THE ERROR COV. MATRIX
(ROW BY ROW) FOR SEASON 2
?1 0
?1
INPUT SEED POINT FOR RAN. VEC. GENERATION
?2605
FOR OUTPUTS, SEE TAPE185
/edif tape185     [outputs]

 ------------------------------------------------
 MOMENT ESTIMATES
 PHI'S
 SEASON 1
 .8266985161938 -.7196903151749
 -.001802415274898 .5214822721667
 SEASON 2
 .4532162716309 .208590727829
 .03574052257851 .6030688976322
 SIGMA'S
 SEASON 1
 1.045390422368 -.02431756270715
 -.02431756270715 .9469045275532
 SEASON 2
 .9042709791169 -.005832523436364
 -.005832523436364 .8781976289095
 MOMENT ESTIMATES RMSE
 PHI'S
 SEASON 1
 .1671098883147 .1626030695651
 .1784603854327 .1945296663978
 SEASON 2
 .1287638002777 .1750622450189
 .1223830692813 .1545066364721
 SIGMA'S
 SEASON 1
 .2779399233721 .2064018287171
```

.2064018287171 .3110346386067
SEASON 2
.2498065383558 .1641935598924
.1641935598924 .2509575959402

LS ESTIMATES
PHI'S
SEASON 1
.8588317404945 -.7433608800007
-.0007930156424487 .5375481279893
SEASON 2
.4532162716309 .208590727829
.03574052257851 .6030688976322
SIGMA'S
SEASON 1
1.034916784983 .01068118098242
.01068118098242 1.011597242355
SEASON 2
1.004745532352 -.006480581595956
-.006480581595956 .9757751432329
LS ESTIMATES RMSE
PHI'S
SEASON 1
.1591963230447 .1665281372361
.183974212163 .1927733436982
SEASON 2
.1287638002777 .1750622450189
.1223830692813 .154506636472
SIGMA'S
SEASON 1
.2527586546192 .2155687771436
.2155687771436 .3219507332793
SEASON 2
.2564176339459 .1824372887693
.1824372887693 .2449974571348

PROGRAM LISTING:

This program is stored in file OSAMA4.

```
      PARAMETER (M=2,IW=2,N=30,NOUT=185,NREPT=100)
      REAL PHI(M*IW,M*IW),SIG(IW,M,M),R(M+10),ERROR(M,IW*(N+51))
      REAL X(M,IW*(N+51)),Z(M),MU(IW,M),COV(M*IW,M*IW),SS1(M),VM(M)
      REAL C0(M*IW,M),A2(M*IW,M*IW),A3(M*IW),A4(M,M*IW),SS2(N,M)
      REAL  A5(M,M),A6(M,M),Z1(M),A7(1,M),A1(M*IW,M*IW),A8(N,M*IW+1)
      REAL SSM(IW,M,M*IW),SSMV(IW,M,M*IW),SSL(IW,M,M*IW),A9(N)
      REAL SSLV(IW,M,M*IW),A10(N,M*IW+1),SAR1(IW,M,M),SAR2(IW,M,M)
      REAL SARM(IW,M,M),SARV(IW,M,M),A11(M,M*IW),A12(M*IW+1,M)
      REAL B1(M*IW,M),B2(M*IW+1,4)
      INTEGER  IP(IW),IPIV(M*IW+1),MIS(IW),MAS(IW)
      EXTERNAL X04ABF
```

```
      CALL X04ABF(1,NOUT)
      PRINT *,'NOW INPUT (IP(V)<=',IW,'), I.E. THE AR ORDERS'
      PRINT *,'FOR THE V-TH SEASON, V=1,...,',IW
      READ  *,(IP(I),I=1,IW)
      DO 75 I=1,IW
      MIS(I)=0
      IF  (IP(I)+1-I.GT.0)   MIS(I)=1
      MAS(I)=M*IP(I)+MIS(I)+1
75    CONTINUE
      MAX1=0
      DO 2 I=1,IW
      IF (IP(I).EQ.0) GOTO 2
      MOX1=(IP(I)-I)/IW+1
      IF (MOX1.GT.MAX1) MAX1=MOX1
2     CONTINUE
      IF (MAX1.LE.1) GOTO 16
      PRINT *,'MAX1>1, SORRY, THIS PROGRAM DOESN'T WORK HERE.'
      STOP
C     REMARK: PHI MATRICES MUST SATISFY PCS CONDITIONS
16    PRINT *,'DO YOU WANT TO USE THIS PROGRAM FOR ACTUAL DATA ?'
      PRINT *,'IF YES, TYPE "1"'
      READ(*,*)IFL
      IF (IFL.EQ.1) GOTO 66
      DO 1 K=1,IW
      IF (IP(K).LT.1) GOTO 1
      PRINT *,'NOW, FOR SEASON ',K,' INPUT THE PHI MATRICES AS FOLLOWS'
      PRINT *,'     FIRST, INPUT THE FIRST ROW OF PHI1'
      PRINT *,'     THEN INPUT THE FIRST ROW OF PHI2, ETC...'
      PRINT *,'    AND THEN, INPUT THE 2ND ROW OF PHI1, AND SO ON'
      DO 11 I=1,M
      DO 11 J=1,IP(K)
      READ   *,(PHI(M*(K-1)+I,L),L=M*(J-1)+1,M*J)
11    CONTINUE
1     CONTINUE
      DO 170 I=1,IW
      IFAIL=0
      PRINT *,'INPUT THE UPPER TRIANGLE OF THE ERROR COV. MATRIX'
      PRINT *,'(ROW BY ROW) FOR SEASON ',I
      DO 18 K=1,M
18      READ(*,*)(SIG(I,K,J),J=K,M)
      DO 180 J=1,M
      DO 180 K=J,M
180     SIG(I,K,J)=SIG(I,J,K)
170   CONTINUE
      PRINT *,'INPUT SEED POINT FOR RAN. VEC. GENERATION'
      PRINT *,'ANY NON-NEGATIVE INTEGER'
      READ(*,*)ISEED
      CALL G05CBF(ISEED)
66    DO 100 KOZ=1,NREPT
      IF (IFL.EQ.1) GOTO 60
      DO 17 I=1,IW
      DO 19 II=1,M
      VM(II)=0E0
      DO 19 JJ=1,M
19        A5(II,JJ)=SIG(I,II,JJ)
      CALL G05EAF(VM,M,A5,M,0.001,R,M+10,IFAIL)
```

```
      DO 17 L=1,N+51
      IFAIL=0
      CALL G05EZF(Z,M,R,M+10,IFAIL)
      DO 21 J=1,M
21     ERROR(J,(L-1)*IW+I)=Z(J)
17   CONTINUE
      DO 22 I=1,M
      DO 22 J=1,IW
22     X(I,J)=0E0
      DO 23 J=IW+1,IW*(N+51)
      IV=J-(J/IW)*IW
      IF (IV.EQ.0) IV=IW
      DO 48 K=1,M
48    Z1(K)=ERROR(K,J)
      IF (IP(IV).LT.1) GOTO 51
      DO 26 I=1,IP(IV)
      DO 77 K=1,M
77     VM(K)=X(K,J-I)
      DO 27 K=1,M
      DO 27 L=1,M
27      A5(K,L)=PHI((IV-1)*M+K,(I-1)*M+L)
      CALL MXMAB(M,M,1,A5,M,VM,M,Z,M)
      DO 29 K=1,M
29     Z1(K)=Z1(K)+Z(K)
26   CONTINUE
51    DO 49 K=1,M
49     X(K,J)=Z1(K)
23   CONTINUE
      DO 35 J=1,N*IW
      DO 35 I=1,M
35      X(I,J)=X(I,51*IW+J)
      GOTO 61
60  PRINT *,'SINCE YOU ARE RUNNING THIS PROGRAM WITH ACTUAL DATA'
      PRINT *,'YOU SHOULD HAVE MODEIFIED THE PARAMETERS IN THE FIRST'
      PRINT *,'LINE OF THIS PROGRAM (NREPT=1). ALSO, THE DATA, I.E. '
      PRINT *,'THE OBSERVED RELAIZATION OF THE TIME SERIES, MUST BE'
      PRINT *,'STORED IN FILE "TAPE5"'
      DO 555 J=1,M
555     READ(5,*)(X(J,I),I=1,N*IW)
C      ****************************************
61    DO 999 JJ=1,M*IW
      DO 998 II=1,M
998    CO(JJ,II)=0E0
      DO 999 II=1,M*IW
999   COV(II,JJ)=0E0
      DO 190 I=1,IW
      DO 191 J=1,M
191   SS1(J)=0E0
      DO 192 K=1,M
      DO 185 J=1,N
185     SS1(K)=SS1(K)+X(K,I+(J-1)*IW)
192   MU(I,K)=SS1(K)/N
190   CONTINUE
      DO 20 I=1,IW
      DO 20 K=1,M
      DO 20 J=1,N
```

```
20        ERROR(K,I+(J-1)*IW)=X(K,I+(J-1)*IW)-MU(I,K)
      DO 207 I=1,IW
      DO 208 II=1,M
      DO 208 JJ=1,M
208    A5(II,JJ)=0E0
      DO 209 J=1,N
      DO 210 II=1,M
      SS1(II)=ERROR(II,I+(J-1)*IW)
210       A7(1,II)=ERROR(II,I+(J-1)*IW)
      CALL MXMAB(M,1,M,SS1,M,A7,1,A5,M)
      DO 211 II=1,M
      DO 211 JJ=1,M
211          CO((I-1)*M+II,JJ)=CO((I-1)*M+II,JJ)+A5(II,JJ)
209   CONTINUE
207   CONTINUE
      DO 212 I=1,M*IW
      DO 212 J=1,M
212     CO(I,J)=CO(I,J)/N
      DO 200 I=1,IW
      DO 200 L=1,IW
      DO 195 II=1,M
      DO 195 JJ=1,M
195    A5(II,JJ)=0E0
      DO 24 J=0,N-1
      IF (I+J*IW-L.LE.0) GOTO 24
      DO 196 II=1,M
      SS1(II)=ERROR(II,I+J*IW)
196     A7(1,II)=ERROR(II,I+J*IW-L)
      CALL MXMAB(M,1,M,SS1,M,A7,1,A5,M)
      DO 197 II=1,M
      DO 197 JJ=1,M
197          COV((I-1)*M+II,(L-1)*M+JJ)=COV((I-1)*M+II,(L-1)*M+JJ)+A5(II,JJ)
24   CONTINUE
200   CONTINUE
      DO 198 I=1,M*IW
      DO 198 J=1,M*IW
198    COV(I,J)=COV(I,J)/N
C      *****************************************
      DO 1000 I=1,IW
      DO 250 JJ=1,IW*M
      DO 250 KK=1,IW*M
      A1(JJ,KK)=0E0
250   A2(JJ,KK)=0E0
      DO 201 II=1,IP(I)
      MM=I-II
255  IF (MM.LE.0) MM=MM+IW
      IF (MM.LE.0) GOTO 255
      DO 201 IJ=1,M
      DO 201 IK=1,M
201        A1((II-1)*M+IJ,(II-1)*M+IK)=CO((MM-1)*M+IJ,IK)
      DO 2000 II=1,IP(I)
      DO 2000 JJ=II+1,IP(I)
      MM=I-II
2550 IF (MM.LE.0) MM=MM+IW
      IF (MM.LE.0) GOTO 2550
      DO 202 IJ=1,M
```

```
      DO 202 IK=1,M
202         A1((II-1)*M+IJ,(JJ-1)*M+IK)=COV((MM-1)*M+IJ,(JJ-II-1)*M+IK)
2000  CONTINUE
      DO  2001  IJ=1,M*IP(I)
      DO  2001  IK=1,IJ
2001     A1(IJ,IK)=A1(IK,IJ)
      IFAIL=0
      CALL  F01ABF(A1,M*IW,M*IP(I),A2,M*IW,A3,IFAIL)
      DO 350 II=1,M
      DO 350 JJ=1,M*IW
      A12(JJ+1,II)=0E0
350    A4(II,JJ)=0E0
      DO  3000  JJ=1,IP(I)
      DO 203 KK=1,M
      DO 203 JK=1,M
203     A4(KK,(JJ-1)*M+JK)=COV((I-1)*M+KK,(JJ-1)*M+JK)
3000  CONTINUE
      DO 204 KK=1,M*IP(I)
      DO 204 JK=KK,M*IP(I)
204   A2(KK,JK)=A2(JK,KK)
      IFAIL=0
      CALL MXMAB(M,M*IW,M*IW,A4,M,A2,M*IW,A11,M)
      DO 266 II=1,M
      DO 266 JJ=1,M*IW
      SSM(I,II,JJ)=SSM(I,II,JJ)+A11(II,JJ)
266           SSMV(I,II,JJ)=SSMV(I,II,JJ)+(A11(II,JJ)-PHI((I-1)*M+II,JJ))**2
      DO 560 IK=1,M
      DO 560 JK=1,M*IW
560    B1(JK,IK)=A4(IK,JK)
      CALL MXMAB(M,M*IW,M,A11,M,B1,M*IW,A6,M)
      DO 500 IK=1,M
      DO 500 JK=1,M
500     A6(IK,JK)=C0((I-1)*M+IK,JK)-A6(IK,JK)
      DO 540 IK=1,M
      DO 540 JK=1,M
      SAR1(I,IK,JK)=SAR1(I,IK,JK)+A6(IK,JK)
540      SARM(I,IK,JK)=SARM(I,IK,JK)+(SIG(I,IK,JK)-A6(IK,JK))**2
C      *****************************************
      DO  217  II=1,N-MIS(I)
      A8(II,1)=1E0
      DO  216  JJ=1,M
216       SS2(II,JJ)=X(JJ,I+(II-1+MIS(I))*IW)
      DO  217  JJ=1,M*IP(I)
      IZ=JJ-(JJ/M)*JJ
      IF  (IZ.EQ.0)  IZ=M
217       A8(II,JJ+1)=X(IZ,I+(II-1+MIS(I))*IW-(JJ+M-1)/M)
      IFAIL=0
      CALL   G02CJF(A8,N,SS2,N,N-MIS(I),M*IP(I)+1,M,A12,M*IW+1,VM,A10,N,
     *           IPIV,B2,A9,IFAIL)
      DO  218  II=1,M
      Z1(II)=A12(1,II)
      DO  218  JJ=1,M*IP(I)
      A2(II,JJ)=A12(JJ+1,II)
      SSL(I,II,JJ)=SSL(I,II,JJ)+A2(II,JJ)
218          SSLV(I,II,JJ)=SSLV(I,II,JJ)+(A2(II,JJ)-PHI((I-1)*M+II,JJ))**2
      DO 760 IK=1,M
```

```
        A7(1,IK)=0E0
        DO 761 JK=1,M
761     A6(IK,JK)=0E0
        DO 760 JK=1,N
760     SS2(JK,IK)=0E0
        DO  600  JJ=1+MIS(I),N
        DO  430  II=1,M
430     VM(II)=0E0
        DO   620   II=1,IP(I)
        DO 640 IK=1,M
        DO 660 JK=1,M
660       A5(IK,JK)=A2((II-1)*M+IK,JK)
640        Z(IK)=X(IK,I+(JJ-1)*IW-II)
        CALL MXMAB(M,M,1,A5,M,Z,M,SS1,M)
        DO 680 IK=1,M
680      VM(IK)=VM(IK)+SS1(IK)
620   CONTINUE
        DO 700 IK=1,M
        VM(IK)=X(IK,I+(JJ-1)*IW)-VM(IK)-Z1(IK)
700       SS2(JJ,IK)=SS2(JJ,IK)+VM(IK)
600   CONTINUE
        DO  604  JJ=1+MIS(I),N
        DO 605 IK=1,M
        VM(IK)=SS2(JJ,IK)
605      A7(1,IK)=VM(IK)
        CALL MXMAB(M,1,M,VM,M,A7,1,A5,M)
        DO 720 IK=1,M
        DO 720 JK=1,M
720      A6(IK,JK)=A6(IK,JK)+A5(IK,JK)
604   CONTINUE
        DO 740 IK=1,M
        DO 740 JK=1,M
        A6(IK,JK)=A6(IK,JK)/(N-MAS(I))
        SAR2(I,IK,JK)=SAR2(I,IK,JK)+A6(IK,JK)
740        SARV(I,IK,JK)=SARV(I,IK,JK)+(SIG(I,IK,JK)-A6(IK,JK))**2
C        ***************************************
1000  CONTINUE
100   CONTINUE
        DO 290 I=1,IW
        DO 290 J=1,M
        DO 291 K=1,M
        SAR1(I,J,K)=SAR1(I,J,K)/NREPT
        SAR2(I,J,K)=SAR2(I,J,K)/NREPT
        SARM(I,J,K)=SQRT(SARM(I,J,K)/NREPT)
291     SARV(I,J,K)=SQRT(SARV(I,J,K)/NREPT)
        DO  290  K=1,M*IP(I)
        SSM(I,J,K)=SSM(I,J,K)/NREPT
        SSL(I,J,K)=SSL(I,J,K)/NREPT
        SSMV(I,J,K)=SQRT(SSMV(I,J,K)/NREPT)
290      SSLV(I,J,K)=SQRT(SSLV(I,J,K)/NREPT)
        WRITE(NOUT,*)'--------------------------------------------------'
        WRITE(NOUT,*)'MOMENT ESTIMATES'
        WRITE(NOUT,*)'PHI'S'
        DO 800 I=1,IW
        WRITE(NOUT,*)'SEASON ',I
        DO 800 J=1,M
```

```
800       WRITE(NOUT,*)(SSM(I,J,K),K=1,IP(I)*M)
      WRITE(NOUT,*)'SIGMA'S'
      DO 810 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 810 J=1,M
810    WRITE(NOUT,*)(SAR1(I,J,K),K=1,M)
      WRITE(NOUT,*)'MOMENT ESTIMATES RMSE'
      WRITE(NOUT,*)'PHI'S'
      DO 801 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 801 J=1,M
801       WRITE(NOUT,*)(SSMV(I,J,K),K=1,IP(I)*M)
      WRITE(NOUT,*)'SIGMA'S'
      DO 811 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 811 J=1,M
811   WRITE(NOUT,*)(SARM(I,J,K),K=1,M)
      WRITE(NOUT,*)'-------------------------------------------------'
      WRITE(NOUT,*)'LS ESTIMATES'
      WRITE(NOUT,*)'PHI'S'
      DO 802 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 802 J=1,M
802       WRITE(NOUT,*)(SSL(I,J,K),K=1,IP(I)*M)
      WRITE(NOUT,*)'SIGMA'S'
      DO 820 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 820 J=1,M
820    WRITE(NOUT,*)(SAR2(I,J,K),K=1,M)
      WRITE(NOUT,*)'LS ESTIMATES RMSE'
      WRITE(NOUT,*)'PHI'S'
      DO 803 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 803 J=1,M
803       WRITE(NOUT,*)(SSLV(I,J,K),K=1,IP(I)*M)
      WRITE(NOUT,*)'SIGMA'S'
      DO 821 I=1,IW
      WRITE(NOUT,*)'SEASON ',I
      DO 821 J=1,M
821   WRITE(NOUT,*)(SARV(I,J,K),K=1,M)
      WRITE(NOUT,*)'-------------------------------------------------'
      PRINT *,'FOR OUTPUTS, SEE TAPE',NOUT
      STOP
      END
```

CURRICULUM VITAE

Abdullah A. Smadi was born in Nu'aimah, Jordan in 1965. He received his B.Sc. and M.Sc. degrees in Statistics from Yarmouk University, Irbid, Jordan in 1987 and 1989, respectively.

In 1990 he enrolled in the Ph.D. program of the Department of Statistics at Middle East Technical University, Ankara, Turkey. He worked as a research assistant in the same department from Spring 1990 to Spring 1991.