# NUCLEAR KÖTHE QUOTIENTS OF FRECHET SPACES

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ABSTRACT

# NUCLEAR KÖTHE QUOTIENTS OF FRECHET SPACES

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Let $T$ be an unbounded, continuous, linear operator from Fréchet space $E$ into Fréchet space $F$ and suppose $F$ satisfies the condition which is called (y). It is proved that $E$ and $F$ have a common quotient which is nuclear, has a basis and a continuous norm and it can be factored through $T$. By using this result, it is also proved that Fréchet spaces which have nuclear quotients with a basis and a continuous norm are those Fréchet spaces which do not satisfy the condition (b).

# ÖZET

## FRECHET UZAYLARININ NÜKLEER-KÖTHE BÖLÜM UZAYLARI

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T Fréchet uzayı E 'den Fréchet uzayı F 'ye sınırsız, sürekli, doğrusal dönüşüm olsun ve F uzayı da (y) olarak adlandırdığımız koşulu sağlasın. Bu durumda E ve F 'nin T tarafından faktöre edilen sürekli normlu ve bazlı nükleer ortak bölüm uzayının varlığı ispatlandı. Bu sonucun uygulamasıyla da sürekli normlu ve bazlı nükleer bölüm uzayı olan Fréchet uzaylarının (b) koşulunu sağlamayan Fréchet uzayları olduğu gösterilmektedir.

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# INTRODUCTION

A characterization of those Fréchet space which have nuclear Köthe quotients with a continuous norm was obtained by Bellenot and Dubinsky [2] for separable Fréchet spaces. More precisely, a separable Fréchet space has a nuclear Köthe quotient with a continuous norm if and only if it does not satisfy the condition (b), which will be defined subsequently. They also noted that the separability is not used their proof of one implication. Moreover T. Terzioğlu [13] proved that the equality L(E, F) = LB(E, F) holds when i) E satisfies (b), F has a basis and a continuous norm (which indicates that quotients with a basis and continuous norm of Fréchet space E which satisfies (b) are Banach spaces), ii) E satisfies (b), F" admits a continuous norm (which indicates that a Schwartz quotient of Fréchet space E which satisfies (b) must be $\omega$). This also shows that the non-existence of the condition (b) is essential so the condition (b) divides Fréchet spaces into two classes; one of them "Fréchet spaces do not have nuclear Köthe quotient with a continuous norm" and the other "Fréchet spaces may have nuclear Köthe quotients with a continuous norm".

Suppose a Fréchet space E does not satisfy (b). Then there exists an unbounded, continuous, linear operator from E into some nuclear Köthe space which has a continuous norm [13]. On the other hand, existence of an unbounded continuous linear operator between two Fréchet spaces tells us, under certain conditions, that they have common subspaces [12, 14]. It is natural to ask whether the same is true if "common subspace" is replaced by "common quotient". Positive answer to this question gives a complete characterization of Fréchet spaces which have a nuclear Köthe quotient with a continuous norm. Our main result (Theorem 2.13) gives a positive answer to this question.

In Chapter 1, we introduce some terminology and notation and state some preliminary results.

In Chapter 2, we are mainly interested in finding nuclear Köthe quotients of Fréchet spaces. We introduce a condition, which is called (y). If E satisfies (b) and F satisfies (y), then we have L(E, F) = LB(E, F). It is also showed that the same equality holds when E satisfies (b) and F is a $B_r$-complete Schwartz space with a continuous norm. Corollary 2.9 indicates that for a barrelled space with a basis the openness condition and the condition (b) are equivalent. We state and prove our main result (Theorem 2.13) and derive some of its consequences.

In Chapter 3, we give a characterization of those l.c.s's which satisfy (y) provided under their Mackey topology they are barrelled. We prove that if a l.c.s. E which does not satisfy (b) then E" does not satisfy the openness condition. This was obtained by Vogt [18] for Fréchet spaces. We generalizé the main theorem of the second chapter (Theorem 2.13). We can dualize some of our results and obtain the existence subspaces of df-spaces which are isomorphic to duals of nuclear Köthe spaces.

# CHAPTER 1

# PRELIMINARIES

In this chapter, our aim is to introduce some terminology and notation, to give some straightforward generalizations of some well-known concepts, such as unconditional bases, Köthe spaces and state some results which will be used subsequently.

We use many concepts and notation from the standard theory of locally convex spaces (l.c.s.). A l.c.s. is always *assumed to be Hausdorff*. For definitions and notations not explicitly explained here, we refer to the book [8] by H. Jarchow.

$U(E)$ denotes a neighborhood basis of a l.c.s. $E$ which we can assume to consist of absolutely convex and closed sets. $\hat{E}$ and $E'$ denote respectively the completion and the dual of $E$. The gauge of a barrel $U$ is denoted by $q_U$ and $E_U$ is the quotient space $E/q_U^{-1}(0)$. Let $U, V \in U(E)$ be such that $U \subset V$. Then we have a natural map $\rho_{UV}: E_U \to E_V$ which is called a *linking map.* $\beta_\tau$ is a bornology of $\tau$-bounded sets which we can assume to consist of closed and absolutely convex sets.

Let $A$ be a bounded and absolutely convex subset of a l.c.s. $E$. By $E[A]$ we mean the normed space $E[A] = \bigcup_{n=1}^{\infty} nA$ whose unit ball is $A$ and $\|\ \|_A$ denotes the norm of $E[A]$.

We will give a definition of a generalized unconditional basis.

*Definition 1.1.* A set $(x_\alpha)_{\alpha\varepsilon I}$ in a l.c.s. $E$ is called a *generalized unconditional basis* of $E$ if there is a set $(u_\alpha)_{\alpha\varepsilon I}$ in $E'$ such that $u_\alpha(x_\beta) = \delta_{\alpha\beta}$ for each $\alpha, \beta \varepsilon I$ and the net $x_F = \sum_{\alpha\varepsilon F} u_\alpha(x)x_\alpha$ which is indexed by finite subsets $F$ of $I$ and directed by inclusion converges to $x$, for each $x\varepsilon E$. The set $(u_\alpha)_{\alpha\varepsilon I}$ is called the associated set of coefficient functionals. Here the index set $I$ is arbitrary. Of course if $I = \mathbb{N}$, the set of positive integers, we have the usual notion of an unconditional basis.

It is easy to see that the set $(u_\alpha)_{\alpha\varepsilon I}$ is a generalized unconditional basis for $E'[\sigma(E', E)]$. We prove the following which will be used later.

*Lemma 1.2.* Let $(x_\alpha)_{\alpha\varepsilon I}$ be a generalized unconditional basis of a l.c.s. $E$ and $(u_\alpha)_{\alpha\varepsilon I}$ be the associated set of coefficient functionals. Then for each $\phi\varepsilon E'$ the net $(\phi_F)_{(\mathcal{F}, \subset)}$ $\sigma(E', E)$-converges to $\phi$ and it is $\sigma(E', E)$-bounded. Here $\mathcal{F} = [F | F\subset I$ and $F$ is finite$]$ and $\phi_F = \sum_{\alpha\varepsilon F} \phi(x_\alpha)u_\alpha$.

*Proof*: We have $\phi_F(x) = \phi(x_F)$ for every $x\varepsilon E$ and $\phi\varepsilon E'$. Hence $x = \lim x_F$ implies that $\phi = \sigma(E', E)\text{-}\lim \phi_F$. Suppose the net $(\phi_F)_{(\mathcal{F}, \subset)}$ is not $\sigma(E', E)$-bounded. Then we can find a sequence of finite subsets $F_n$ of $I$ and $x\varepsilon E$ such that $|\phi_{F_n}(x)| > n$ for each $n\varepsilon\mathbb{N}$. By the convergence of the net $(x_F)_{(\mathcal{F}, \subset)}$ we find a finite subset $G$ of $I$ such that for each $F\varepsilon\mathcal{F}$ with $G \subset F$ we have $|\phi(x_F) - \phi(x)| < 1$. So we must have $|\phi_{F_n \cup G}(x)| = |\phi(x_{F_n \cup G})| < C$

some $C > 0$. But we also have $\phi(x_{F_n \cup G}) = \phi(x_{F_n}) + \phi(x_{G \setminus F_n})$. Since $G$ is finite we can find an infinite subset $M$ of $\mathbb{N}$ such that $G \setminus F_n = G \setminus F_m$ for each $n, m \in M$. So we get $|\phi_{F_n}(x)| < C'$ some $C' > 0$, for each $n \in M$. This contradicts $|\phi_{F_n}(x)| > n$ for each $n \in \mathbb{N}$. So the net $(\phi_F)_{(F,\subset)}$ is bounded.

We generalize the usual concept of Köthe space in the following way.

*Definition 1.3.* Let $\Gamma$ and $I$ be sets such that there is a directed partial order on $I$. Then $A = [a_\beta^\alpha \in \mathbb{R} \mid \alpha \in I,\ \beta \in \Gamma]$ is called a ***generalized Köthe set*** if for each $\beta \in \Gamma$, there is $\alpha \in I$ such that $a_\beta^\alpha > 0$ and for each $\beta \in \Gamma$ for each $\alpha \leq \alpha' \in I$ we have $0 \leq a_\beta^\alpha \leq C\, a_\beta^{\alpha'}$ some $C > 0$. The vector space

$$\lambda^p(A) = [(\xi_\alpha) \in R^\Gamma \mid (\xi_\alpha\ a_\alpha^\beta) \in \ell_p(\Gamma) \quad \text{for each} \quad \beta \in I]$$

is called ***generalized Köthe space*** of order $p$, where $1 \leq p \leq \infty$. The topology on generalized Köthe space $\lambda^p(A)$ is defined by the family of seminorms $(q_\beta)_{\beta \in I}$, where $q_\beta(\xi_\alpha)_{\alpha \in \Gamma} = \| (\xi_\alpha\ a_\alpha^\beta)_{\alpha \in \Gamma} \|_{\ell_p(\Gamma)}$. Here $\| \ \|_{\ell_p(\Gamma)}$ denotes the usual norm of the Banach space $\ell_p(\Gamma)$. If $p=1$ we use $\lambda(A)$ instead of $\lambda^1(A)$.

Let $e_\alpha = (\delta_{\alpha\beta})_{\beta \in \Gamma}$. Then the set $(e_\alpha)_{\alpha \in \Gamma}$ is called the ***coordinate basis*** of $\lambda^p(A)$ which is a generalized unconditional basis for $1 \leq p < \infty$. In case $\Gamma = \mathbb{N}$ we have the usual notion of a Köthe space [8].

*Definition 1.4.* A l.c.s. $E$ is called *asymptotically normable* if there is $U_1 \varepsilon U(E)$ such that for each $V \varepsilon U(E)$ there is $W \varepsilon U(E)$ so that for every $\varepsilon > 0$ we can choose $M > 0$ so that

$$q_V(x) \leq M q_{U_1}(x) + \varepsilon q_W(x), \quad x \varepsilon E .$$

Clearly $q_{U_1}$ is a continuous norm on $E$, since the topology of $E$ is Hausdorff. Characterization of asymptotically normable Fréchet spaces and the implications between asymptotically normability and other related concepts can be found in [15]. In the last chapter, it will be proved that asymptotically normable l.c.s.'s are subspaces of some $\lambda^\infty(A)$ which admits a continuous norm.

It is easy to see that if a Schwartz space $E$ has a basis of neighborhoods such that completion of each linking map $\hat{\rho}_{UV}$ is one to one, then $E$ is asymptotically normable. Hence countably normed Fréchet Schwartz spaces are asymptotically normable.

A linear operator $T: E \to F$ is said to be bounded if $T(U)$ is bounded subset of the l.c.s. $F$ for some $U \varepsilon U(E)$. We denote the space of continuous, linear operators from $E$ into $F$ by $L(E, F)$ and space of bounded linear operators from $E$ into $F$ by $LB(E, F)$. Pairs of Fréchet space satisfying the relation $L(E, F) = LB(E, F)$ have been completely characterized by Vogt [17].

The following concept was introduced by Nachbin [11] and for Fréchet spaces by Bellenot and Dubinsky [2].

*Definition 1.5.* A l.c.s. $E$ satisfies the *openness condition* if for each $U \in \mathcal{U}(E)$ there is a $V \in \mathcal{U}(E)$ such that for each $W \in \mathcal{U}(E)$ we can find a constant $C > 0$ so that the inclusion

$$V \subset CW + q_U^{-1}(0)$$

holds [11]. A Fréchet space satisfying the openness condition is called a *quojection* [2].

If a complete l.c.s. satisfies the openness condition then it can be written as a projective limit of surjective maps between Banach spaces such that the canonical map from the projective limit into each Banach space is nearly open. It is easy to see that the openness condition is stable with respect to taking arbitrary quotients. A Schwartz space which satisfies the openness condition is a subspace of $\mathbb{R}^I$ [13]. Hence Schwartz quotients of l.c.s. which satisfies the openness condition are subspaces of $\mathbb{R}^I$.

Following was obtained by T. Terzioğlu [13].

*Proposition 1.6.* A l.c.s. $E$ satisfies the openness condition if and only if $L(E, F) = LB(E, F)$ for every l.c.s. $F$ which admits a continuous norm.

*Proof*: See [13].

Following condition on l.c.s. was introduced by T. Terzioğlu [13]. In case of Fréchet spaces, this condition is exactly the negation of the condition (*) introduced by Bellenot and Dubinsky [2] in examining nuclear Köthe quotient spaces of Fréchet spaces.

*Definition 1.7.* A l.c.s. $E$ satisfies the condition $(\underline{b})$ if for each $U \in \mathcal{U}(E)$ there is a $V \in \mathcal{U}(E)$ such that for every $W \in \mathcal{U}(E)$ we can find a constant $c > 0$ so that the following inclusion holds:

$$(b) \qquad W^0 \cap E'[U^0] \subset c\, V^0 .$$

It is easily seen that a l.c.s. satisfying the openness condition also satisfies (b). But the converse is not true [1]. However if a l.c.s. $E$ has a basis and satisfies (b) then it also satisfies the openness condition (Corollary 2.9). If a Fréchet space $E$ satisfies (b) then $E''$ is a quojection [9]. If $E''$ is a quojection then $E$ satisfies (b) as proved by Vogt in [18]. In the last chapter it is proved that the last implication is true for arbitrary l.c.s..

One of the results in [13] is that; If a Fréchet space $E$ does not satisfy (b) then there is an unbounded, continuous, linear operator from $E$ into some nuclear Köthe Fréchet space $\lambda(A)$ which admits a continuous norm. This result will be used for obtaining a quotient space which admits a continuous norm and has a basis of an arbitrary Fréchet space.

Suppose a Schwartz space $E$ satisfies (b). Let $U \in \mathcal{U}(E)$. Then we can find $V \in \mathcal{U}(E)$ as in (b) for this $U$. Since $E$ is a Schwartz space there exists $W \in \mathcal{U}(E)$ which is contained in $V$ and $V^0$ is compact in the Banach space $E'[W^0]$. But also there is a $c > 0$ such that $W^0 \cap E'[U^0] \subset c\, V^0$ holds. This implies the restriction of $\rho'_{WV}$ to $E'[U^0]$ is an isomorphism; but $\rho'_{WV}$ is also compact operator. Therefore $E'[U^0]$ is finite dimensional. So the topology on $E$ is the weak topology $\sigma(E, E')$. If a Fréchet Montel space $E$ satisfies

(b) then $E \cong E''$ is a quojection. Clearly Montel quojections are isomorphic to $\omega$. Hence $E \cong \omega$.

It is easy to see that the condition (b) is stable with respect to taking arbitrary quotients. Hence if a l.c.s. $E$ satisfies (b) then a Schwartz (Fréchet Montel) quotient spaces of $E$ is isomorphic to a subspace of $R^I$ (isomorphic to $\omega$).

Following result was obtained by T.Terzioğlu and M.Yurdakul in case $E$ and $F$ are Fréchet space [14]. The proof is almost same and therefore it is not given.

*Lemma 1.8.* Let $T: E \to F$ be a continuous, unbounded, linear operator. If $E$ is metrizable then there is a separable subspace $M$ of $E$ such that the restriction of $T$ to $M$ is still unbounded.

*Proof:* See [14].

The problem of finding a quotient space with a basis of an arbitrary Banach space was solved by Johnson and Rosenthal [7], for the separable case. Their result is that: If $(u_n)$ is a normalized $\sigma(E', E)$-null sequence in $E'$, dual of a separable Banach space $E$, then there exists a subsequence $(u_{n_k})$ of $(u_n)$ which is a $\sigma(E', E)$-basic sequence ($w^*$ basic sequence) in $E'$.

The following result due to Bellenot and Dubinsky [2] is a construction of a $\sigma(E', E)$-basic sequence in the dual of separable Fréchet space $E$. This was one of the main tools they used to prove that a Fréchet space which does not satisfy (b) and which is separable has a nuclear Köthe quotient which admits a continuous norm.

*Lemma 1.9.* Let $E$ be a separable Fréchet space which admits a continuous norm. Let $(d_n)$ be a dense sequence in $E$ and $E_0$ be the vector subspace it generates and suppose that there is a biorthogonal sequence $(x_n, u_n)$ satisfying

(a) $(x_n) \subset E_0$, $(u_n) \subset E'[U^0]$, where $U \in \mathcal{U}(E)$ and $q_U$ is a continuous norm on $E$.

(b) $u_m(d_n) = 0$ for $m > n$.

Then there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that the image of $(x_{n_k})$ is a basis in $E/(u_{n_k})^{\perp}$ and hence $(u_{n_k})$ is a $\sigma(E', E)$-basic sequence. That is for each $\phi \in \overline{sp(u_{n_k})}^{\sigma(E',E)}$ the sequence $\phi_k = \sum_{i=1}^{k} \phi(x_{n_i})u_{n_i}$ $\sigma(E', E)$-converges to $\phi$.

*Proof:* See [2].

# CHAPTER 2

# FRECHET SPACES CASE

In this chapter we are mainly interested in finding nuclear Köthe quotients of Fréchet spaces. We introduce a condition, which is called (y). If $E$ satisfies (b) and $F$ satisfies (y), then we have $L(E, F) = LB(E, F)$. We state and prove our main result (Theorem 2.13) and derive some of its consequences.

*Definition 2.1.* We say a l.c.s. $E$ satisfies the condition (y), if there exists a $\sigma(E', E)$-bounded subset $B_0$ of $E'$ such that the equality

$$\text{(y)} \qquad E' = \bigcup_{B \in \beta_{\sigma(E',E)}} \overline{E'[B_0] \cap B}^{\sigma(E', E)}$$

holds, where $\beta_{\sigma(E', E)}$ is the bornology of $\sigma(E', E)$ bounded subsets of $E'$.

We recall that the seminorm $q_U$ defined by a neighborhood $U$ is a norm if and only if $E'[U^0]$ is $\sigma(E', E)$-dense in $E'$. This means that every continuous linear functional on $E$ is the $\sigma(E', E)$-limit of a net $(u_\alpha)$ of linear functionals, such that each $u_\alpha$ satisfies

$$|u_\alpha(x)| \le c_\alpha\, q_u(x), \quad x \in E$$

for some $c_\alpha > 0$. If $E$ is barrelled and satisfies $(y)$, then certainly $E'[U^0]$ is $\sigma(E', E)$-dense in $E'$, where $U$ is the neighborhood $U = B_0^0$, $B_0$ as in $(y)$. However, in this case, $(y)$ implies that every $u \varepsilon E'$ is the $\sigma(E', E)$-limit of a $\sigma(E', E)$ bounded net $(u_\alpha)$ such that each $u_\alpha$ satisfies the above condition. Hence for a barrelled space the condition $(y)$ is stronger than the existence of a continuous norm. We shall see later that it is in fact strictly stronger. Even if a l.c.s. $E$ is not barrelled but satisfies $(y)$ then $E[\beta(E, E')]$ admits a continuous norm.

Now, we will prove that the condition $(y)$ is inherited by subspaces.

*Proposition 2.2.* If a l.c.s. $F$ satisfies $(y)$ then every subspace of $F$ also satisfies $(y)$.

*Proof:* Let $E$ be a subspace of $F$ and $i: E \to F$ be the inclusion map. For each $u \varepsilon E'$ there exists a $\sigma(F', F)$-bounded net $(v_\alpha)$ in $F'[B_0]$ such that an extension of $u$ on $F$ is the $\sigma(F', F)$-limit of $(v_\alpha)$, where $B_0$ is as in $(y)$. By the $\sigma(F', F)$-$\sigma(E', E)$ continuity of $i'$ we have the equality

$$E' = \bigcup_{B \varepsilon \beta_{\sigma(E', E)}} \overline{E'[i'(B_0)] \cap B}^{\,\sigma(E', E)}$$

and $i'(B_0)$ is a $\sigma(E', E)$-bounded subset of $E'$. These imply that $E$ satisfies $(y)$.

Next, we will specify some classes of l.c.s.'s satisfy the condition $(y)$.

*Proposition* 2.3. Each one of the following conditions implies that $E$ satisfies $(y)$.

i) $E[\sigma(E, E')]$ has a generalized unconditional basis $(x_\alpha)_{\alpha\varepsilon I}$ where the associated set of coefficient functionals $(u_\alpha)_{\alpha\varepsilon I}$ belongs to $E'[B_0]$ for some $B_0\varepsilon\beta_{\sigma(E', E)}$.

ii) $E[\sigma(E', E)]$ has a Schauder basis $(x_n)$ where the associated sequence of coefficient functionals $(u_n)$ belongs to $E'[B_0]$ for some $B_0\varepsilon\beta_{\sigma(E', E)}$.

iii) $E$ has a generalized unconditional basis and $E$ is a barrelled l.c.s. which admits a continuous norm.

iv) $E$ has a Schauder basis and $E$ is a barrelled l.c.s. which admits a continuous norm.

v) There exists $U_1\varepsilon U(E)$ such that for each $U\varepsilon U(E)$, there exist $V\varepsilon U(E)$ and $W\varepsilon U(E)$ such that $W \subset V \subset U$, $\hat{\rho}_{VU_1}$ is one to one and $\hat{\rho}_{WV}$ is weakly compact.

vi) There exists $U_1\varepsilon U(E)$, for each $V\varepsilon U(E)$ there exists $W\varepsilon U(E)$ such that $W \subset V$ and $\rho''_{WU_1}$ is one to one.

vii) $E$ is asymptotically normable.

*Proof*: i) Let $\phi\varepsilon E'$. Then the net $\phi_F = \sum_{\alpha\varepsilon F} \phi(x_\alpha)u_\alpha$, which is indexed by the finite subsets of $I$ and directed by inclusion, $\sigma(E', E)$-converges to $\phi$ and $\sigma(E', E)$-bounded (Lemma 1.2) and also belongs to $E'[B_0]$. This means that $E$ satisfies $(y)$.

ii) Let $\phi\varepsilon E'$. Then we have $\phi = \sigma(E', E)\text{-}\lim \phi_n$, where $\phi_n = \sum_{k=1}^{n} \phi(x_k)u_k$. Since $\phi_n$ $\sigma(E', E)$-convergent sequence it is $\sigma(E', E)$-bounded, which also belongs to $E'[B_0]$. So $E$ satisfies $(y)$.

iii) Let $q_u$ is a continuous norm on E. By barrelledness of E there exists $V\varepsilon U(E)$ such that

$$|u_\alpha(x)|\, q_u(x_\alpha) \leq q_v(x) \quad \text{for each} \quad x\varepsilon E.$$

Since $q_u$ is a norm on E, $q_u(x_\alpha) \neq 0$. This shows $u_\alpha \varepsilon E'[V^0]$. By (i) E satisfies (y).

iv) We find $V\varepsilon U(E)$, as in (iii), such that $u_n \varepsilon E'[V^0]$. By (ii) E satisfies (y).

v) Let $\phi\varepsilon E'$. Then $\phi\varepsilon E'[V^0]$ for some $V\varepsilon U(E)$, which we can assume to $\hat{\rho}_{vu_1}$ is one to one. Now we find W V so that $\hat{\rho}_{wv}$ is weakly compact. Then

$$\overline{\phi\varepsilon E'[U_1^0]}^{\,\sigma(E'_v,\, \hat{E}_v)} \subset \overline{\rho'_{wv}(E'[U_1^0])}^{\,\|\ \|_{W^0}} .$$

So $\phi = \lim \phi_n$ in the norm $\|\ \|_{W^0}$, where $\phi_n \varepsilon E'[U_1^0]$.

vi) Let $\phi\varepsilon E'$, find $V\varepsilon U(E)$ such that $\phi\varepsilon E'[V^0]$ and $\rho''_{vu_1}$ is one to one. Since $\rho''_{vu_1}$ is one to one, $E'[U_1^0]$ is $\sigma(E[V^0], E''_v)$-dense in $E'[V^0]$. Hence $\phi\varepsilon E'[V^0] = \overline{E'[U_1^0]}^{\,\|\ \|_{V^0}}$.

vii) Since E is asymptotically normable, there exists $U_1\varepsilon U(E)$, for each $V\varepsilon U(E)$, there exists $W\varepsilon U(E)$, for every $\varepsilon > 0$ there exists $M > 0$, such that following holds

$$q_v(x) \leq Mq_{u_1}(u) + \varepsilon q_w(x) \quad \text{for every} \quad x\varepsilon E.$$

This gives for each $V\varepsilon U(E)$, there exists $W\varepsilon U(E)$, for every $\varepsilon > 0$ following inclusion holds.

$$V^{0} \subset E'[U_1^0] + \varepsilon W^{0}$$

Let $\phi \varepsilon V^0$. By above inclusion, we can find a sequence $(\phi_n) \subset E'[U_1^0]$ such that $\phi - \phi_n \varepsilon \frac{1}{n} W^0$ for each $n \varepsilon N$. This shows the sequence $\phi_n$, which belongs to $E'[U_1^0]$, converges to $\phi$ and it is bounded.

*Remark:* If $E$ is a Fréchet space, this can also be shown by the following way. By [15] $E$ can be imbedded in $\ell_\infty(\Gamma) \otimes_\pi \lambda(A)$, where $\lambda(A)$ is a nuclear Köthe Fréchet space which admits a continuous norm. It is easy to show that $\ell_\infty(\Gamma) \otimes_\pi \lambda(A)$ satisfies $(y)$. By Proposition 2.2. $E$ satisfies $(y)$.

*Corollary 2.4.* Every Köthe space, which admits a continuous norm, satisfies $(y)$. Every countably normed Schwartz space satisfies $(y)$.

By a well-known result due to Pelczynski [10], if $E$ is a Fréchet space, which admits a continuous norm and has the bounded approximation property, then $E$ is a complemented subspace of a Fréchet space which has a basis and admits a continuous norm. Hence next result follows from Proposition 2.2 and 2.3.

*Corollary 2.5.* Every Fréchet space which has the bounded approximation property and admits a continuous norm, satisfies $(y)$.

Now, we consider the relation $L(E, F) = LB(E, F)$ for a pair of l.c.s.'s $E$ and $F$. Next two lemmas will be used in the following proposition.

*Lemma* 2.6. Let $T: E \to F$ be a continuous linear operator. If $E$ or $F$ is barrelled then for each $\sigma(F', F)$-bounded subset $B$ of $F'$ there exists $W \varepsilon U(E)$ such that the inclusion $T'(B) \subset W^0$ holds.

*Proof*: Let $B$ be a $\sigma(F', F)$-bounded subset of $F'$. By the weak continuity of $T'$, $T'(B)$ is a $\sigma(E', E)$ bounded subset of $E'$. If $E$ is barrelled, we can find $W \varepsilon U(E)$ such that $T'(B) \subset W^0$. If $F$ is barrelled then $B^0$ is a neighborhood in $F$. By the continuity of $T$, there exists $W \varepsilon U(E)$, such that $T(W) \subset B^0$ holds, and this gives $T'(B) \subset W^0$.

*Lemma* 2.7. Let $T: E \to F$ be a continuous linear operator. Then $T$ is bounded if and only if there exists $U \varepsilon U(E)$ such that the inclusion $T'(F') \subset E'[U^0]$ holds.

*Proof*: If $T$ is bounded, we find $U \varepsilon U(E)$ with $T(U)$ is bounded. By 8.3.2 in [8], $T(U)^0$ is a barrel in $F'$. Let $u \varepsilon F'$, we can find $c > 0$ such that $u \varepsilon c T(U)^0$. But this means $T'u \varepsilon c U^0$. Hence the inclusion $T'(F') \subset E'[U^0]$ holds. Conversely if the inclusion $T'(F') \subset E'[U^0]$ holds for some $U \varepsilon U(E)$, then for each $u \varepsilon F'$, we can find $c > 0$ such that $T'u \varepsilon c U^0$, equivalently $u \varepsilon c T(U)^0$. This shows that $T(U)^0$ is a barrel in $F'$. By 8.3.2 in [8], $T(U)^{00}$ is a bounded subset of $F$ which contains $T(U)$. Hence $T$ is bounded.

The following proposition shows $L(E, F) = LB(E, F)$ for a pair of l.c.s.'s one of which satisfies (b) and the other satisfies (b). A similar result was also obtained in [13]. In fact, the method used in the proof motivated by T. Terzioğlu.

*Proposition 2.8.* Let E satisfy (b) and F satisfy (y). If E or F is barrelled then $L(E, F) = LB(E, F)$.

*Proof*: Let $B_0$ be as in (y) for F. Let $T \varepsilon L(E, F)$. By the Lemma 2.6, we find $U \varepsilon U(E)$ with $T'(B_0) \subset U^0$ and $V \varepsilon U(E)$ as in the condition (b) for U. Let B be a $\sigma(F', F)$-bounded subset of $F'$. By the weak continuity of $T'$ we have the inclusion

$$T'(\overline{F'[B_0] \cap B}^{\sigma(F', F)}) \subset \overline{T'(F'[B_0] \cap B)}^{\sigma(E', E)}.$$

Since $T'(B_0) \subset U^0$, we get the inclusion

$$T'(\overline{F'[B_0] \cap B}^{\sigma(F', F)}) \subset \overline{E'[U^0] \cap T'(B)}^{\sigma(E', E)} .$$

Let $W \varepsilon U(E)$ be such that $T'(B) \subset W^0$, (Lemma 2.6) and $c > 0$ as in (b) for this W. We have

$$T'(\overline{F'[B_0] \cap B}^{\sigma(F', F)}) \subset \overline{E[U^0] \cap W^0}^{\sigma(E', E)} \subset cV^0 .$$

Since $F' = \bigcup_{B \varepsilon \beta_{\sigma(F',F)}} \overline{F'[B_0] \cap B}^{\sigma(F', F)}$, this last inclusion gives $T'(F') \subset E'[V^0]$ and so T is bounded by the Lemma 2.7.

Behrends, Dierolf and Harmand [1] constructed a proper Fréchet space which satisfies (b) and also admits a continuous norm. By Proposition 2.8 this Fréchet space does not satisfy (y). This example shows that the existence of a continuous norm is not sufficient for the condition (y).

*Corollary 2.9.* If $E$ satisfies (b) and $F$ is a barrelled quotient of it which has Schauder basis, then $F$ satisfies the openness condition.

*Proof:* Since the property (b) is stable with respect to taking quotients, $F$ satisfies (b) also. Let $(x_n)$ be a Schauder basis of $F$. For $U\varepsilon U(E)$, by barrelledness of $F$, we can find $V\varepsilon U(E)$ such that

$$|u_n(x)|\ q_u(x_n) \leq q_v(x), \quad x\varepsilon E$$

holds. This gives the following inclusion

$$q_v^{-1}(0) \subset \overline{sp[x_n|q_u(x_n) = 0]} \subset q_u^{-1}(0) \ .$$

Let $M = \overline{sp}[x_n|q_u(x_n) = 0]$. The image of $[x_n|q_u(x_n) \neq 0]$ is a Schauder basis in the quotient space $F/M$. By the above inclusion gauge at the image of $V$ is a continuous norm on $F/M$. By Proposition 2.8. the quotient map is bounded. This means the quotient space $F/M$ is normable. But above inclusion shows that $F/q_u^{-1}(0)$ is a quotient of $F/M$. So $F/q_u^{-1}(0)$ is also normable, which means that there is a $V\varepsilon U(E)$ such that for each $W\varepsilon U(E)$, there is $c > 0$, so that following holds.

$$V \subset q_u^{-1}(0) + cW \ .$$

So, $F$ satisfies the openness condition.

*Proposition 2.10.* If $E$ satisfies (b) and $F$ is a $B_r$-complete

Schwartz space which admits a continuous norm, then $L(E, F) = LB(E, F)$.

*Proof*: Let $T: E \to F$ be a continuous linear operator, and $q_{U_1}$ be a continuous norm on $F$. By the continuity of $T$, we find $U \varepsilon U(E)$ such that $T(U) \subset U_1$ and $V \varepsilon U(E)$ as the condition (b) for this $U$. Let $M = T'^{-1}(\overline{E'[U^0]}^{\| \|_{V^0}})$. Since $F'[U_1^0] \subset M$ and $F'[U_1^0]$ is $\sigma(F', F)$-dense in $F'$, $M$ is $\sigma(F', F)$-dense in $F'$. We will show that $M = F'$. Since $F$ is $B_r$ complete, it is sufficient to show that, $B \cap M$ is $\sigma(F', F)$-closed for every equicontinuous subset $B$ of $F'$. Let $B$ be a arbitrary equicontinuous subset of $F'$. Let $\phi_\alpha \varepsilon B \cap M$ and suppose $\phi_\alpha$ $\sigma(F', F)$-converges to $\phi$ in $B$. Since $F$ is a Schwartz space, we can find an equicontinuous subset $C$ of $F'$ such that the net $(\phi_\alpha)$ converges to $\phi$ in the normed space $F'[C]$. Since $C$ is equicontinuous, there is $W \varepsilon U(E)$ such that $T'(C) \subset W^0$. So the image $(T'\phi_\alpha)$ of the net $(\phi_\alpha)$ is converges to $T'\phi$ in the normed space $E'[W^0]$. Let $c > 0$ be as in (b) for this $W$. The inclusions $V^0 \subset W^0$, $E'[U^0] \cap W^0 \subset cV^0$ implies that the norm defined by $W^0$ and $V^0$ are equivalent on $E'[U^0]$. So $T'\phi \varepsilon \overline{E'[U^0]}^{\| \|_{W^0}}$ means $T'\phi \varepsilon \overline{E'[U^0]}^{\| \|_{V^0}}$. Hence $\phi \varepsilon M$. This shows $B \cap M$ is $\sigma(F', F)$-closed for an arbitrary equicontinuous subset $B$ of $F'$. So $M = F'$ and $T'(F') \subset E'[V^0]$. By the Lemma 2.7, $T$ is bounded.

*Remark:* We note that every Fréchet space is B-complete.

We need the following two rather technical lemmas, which will be used in the proof of our main result.

*Lemma 2.11.* Let $T: E \to F$ be a continuous, unbounded linear operator.

If $F$ satisfies (y) and $B_0$ as in (y), then for every $U \varepsilon U(E)$, for every $\sigma(F', F)$-bounded subset $B$ of $F'$, there exists a $\sigma(F', F)$ bounded subset $C$ of $F'$, which contains $B$ and $T'(F'[B_0] \cap C)$ is not absorbed by $U^0$.

*Proof*: Suppose the conclusion false for some $U \varepsilon U(E)$ and $B \varepsilon \beta_{\sigma(F',F)}$. Let $C \varepsilon \beta_{\sigma(F', F)}$, then the set $(B \cup C)^{00}$ belongs to $\beta_{\sigma(F', F)}$ and it contains $B$. Therefore $T'(F[B_0] \ (B \cup C)^{00})$ is absorbed by $U^0$. So for each $C \varepsilon \beta_{\sigma(F', F)}$ there exists $c > 0$ such that the inclusion $T'(F'[B_0] \cap C) \subset U^0$ holds. Weak continuity of $T'$ gives $T'(\overline{F'[B_0] \cap C}^{\sigma(F',F)}) \subset U^0$. But $F' = \bigcup_{C \varepsilon \beta_{\sigma(F',F)}} \overline{F'[B_0] \cap C}^{\sigma(F', F)}$ by (y). Hence $T'(F') \subset E'[U^0]$.

By the Lemma 2.7, this contradicts the unboundedness of $T$.

In the next result, we shall improve the previous lemma in the case the domain is a Fréchet space. This result should be compared with [14]; Lemma.

*Lemma 2.12*. Let $T: E \to F$ be a continuous, unbounded linear operator. If $F$ satisfies (y) and $B_0$ as in (y) for $F$ and $E$ is a Fréchet space, then there are $U_k \varepsilon U(E)$ which form a decreasing sequence of neighborhood basis of zero for $E$ and an increasing sequence of $\sigma(F', F)$-bounded subsets $B_k$ of $F'$, such that $T'(B_k) \subset U_k^0$ and $T'(F'[B_0] \cap B_{k+1})$ is not absorbed by $U_k^0$.

*Proof*: Let $W_n$ be a decreasing sequence of neighborhood basis for $E$. Take $B_1 = B_0$. By the continuity of $T'$ we find $W_{n_1}$ such that $T'(B_1) \subset W_{n_1}^0$ holds. By the Lemma 2.11 there exists $\sigma(F', F)$-bounded subset $B_2$ of $F'$ such that $B_1 \subset B_2$ and $T'(F'[B_0] \cap B_2)$ is not absorbed by $W^0_{n_1}$. We find $n_2$ such that, $T'(B_2) \subset W^0_{n_2}$ and again by the previous lemma there exists $B_3$ which contains $B_2$ and $T'(F'[B_0] \cap B_3)$ is not absorbed by $W^0_{n_2}$. Hence using the continuity and the Lemma 2.11 alternately, we choose $B_k \varepsilon \beta_{\sigma(F',F)}$ and $W_{n_k} \varepsilon U(E)$ with $n_{k+1} > n_k$, $T'(B_k) \subset W^0_{n_k}$ and $T'(F'[B_0] \cap B_{k+1})$ is not absorbed by $W^0_{n_k}$. We set $U_k = W_{n_k}$. So $U_k$ and $B_k$ are the desired sets.

If $F$ is also Fréchet space we can choose $B_k = V^0_k$ where $(V_k)$ forms a decreasing sequence of neighborhood basis for $F$.

With those preparations done, we now state and prove our main result.

*Theorem 2.13.* Let $E$ and $F$ be Fréchet spaces and suppose $F$ satisfies $(y)$. If $T: E \to F$ is a continuous unbounded, linear operator, then there exist a nuclear Köthe Fréchet space $\lambda(A)$ which admits a continuous norm and a continuous linear operator $S$ from $F$ onto $\lambda(A)$ such that $ST$ maps $E$ onto $\lambda(A)$.

*Proof*: By passing to the quotient $E/T^{-1}(0)$ we can assume $T$ is a one to one operator. Since $F$ is Fréchet space which satisfies $(y)$, $F$ admits a continuous norm. As $T$ is one to one, this means that $E$ also admits a continuous norm.

Now let $M$ be a separable subspace of $E$ such that the restriction of $T$ to $M$ is still unbounded (Lemma 1.8). If $i: M \to E$ is the imbedding and if we can find a continuous linear operator $S$

mapping $F$ onto $\lambda(A)$ so that $STi(M) = \lambda(A)$, then certainly $ST$ maps $E$ onto $\lambda(A)$. Therefore, it is sufficient to prove the theorem under the assumptions that $E$ is separable and admits a continuous norm.

Let $(d_n) \subset E$ be a dense sequence in $E$. We set $E_0 = sp[d_n]_{n=1}^{\infty}$. By Lemma 2.12, we select bases of neighborhoods $(U_n)$ and $(V_n)$ in $E$ and $F$ respectively such that $T'(V_n^0) \subset U_n^0$ and $T'(F'[V_1^0] \cap V_{n+1}^0)$ is not absorbed by $U_n^0$. Let $\| \ \|_n' = \| \ \|_{U_n^0}$ and $| \ |_n' = \| \ \|_{V_n^0}$ be the norms on $E'[U_n^0]$ and $F'[V_n^0]$, which are determined by the bounded sets $U_n^0$ and $V_n^0$ respectively.

Since $T'(F'[V_1^0] \cap V_{k+1}^0)$ is not absorbed by $U_k$, we have

$$\sup\left[\frac{\|T'u\|_k'}{|u|_{k+1}'} \;\middle|\; u \in F'[V_1^0]\right] = \infty$$

for each $k \in N$.

Now we will construct a sequence $(x_n)$ in $E_0$ and a sequence $(u_n)$ in $F'[V_1^0]$ so that the following are true:

(i) $u_j(Tx_i) = \delta_{ji}$

(ii) $u_j(Td_i) = 0$ for $i < j$

(iii) $\dfrac{\|T'u_n\|_k'}{|u_n|_{k+1}'} > 2^n$ for $k < n$.

Suppose that we have already found $x_1, \ldots, x_{m-1} \in E_0$ and $u_1, \ldots, u_{m-1} \in F'[V_1^0]$ so that (i), (ii) and (iii) hold for each $i, j, k, n < m$.

Let $G = [u \mid u \in F'[V_1^0],\ u(Tx_j) = 0,\ u(Td_j) = 0,\ 1 \leq j < m]$. $G$ is a $\sigma(F', F)$-closed, finite codimensional subspace of $F'[V_1^0]$. Hence it is closed in $F'[V_1^0]$ with respect to $|\ |_k'$ for each $k$. Let $L$ be an algebraic complement of $G$ in $F'[V_1^0]$. Since $G$ is $|\ |_k'$-closed, $L$ is a topological complement of $G$ in $F'[V_1^0]$ with respect to $|\ |_k'$ for each $k$. Since $T'L \subset E'[U_1^0]$ and $L$ is finite dimensional $T'(L \cap V_k^0)$ is absorbed by $U_i^0$ for each $i \in N$. Assume that $T'(G \cap V_{k+1}^0)$ is absorbed by $V_k^0$ for some $k$. Then $T'(G \cap V_{k+1}^0 + L \cap V_{k+1}^0)$ is absorbed by $U_k^0$. We have $cV_{k+1}^0 \cap F'[V_1^0] \subset G \cap V_{k+1}^0 + L \cap V_{k+1}^0$ for some $c > 0$. This last inclusion implies $T'(F'[V_1^0] \cap V_{k+1}^0)$ is absorbed by $U_k^0$ which is a contradiction. Therefore $T'(G \cap V_{k+1}^0)$ is not absorbed by $U_k^0$ for each $k$. Hence we have

$$\sup\left[\frac{\|T'u\|_k'}{|u|_{k+1}'} \mid u \in G\right] = \infty .$$

Now, we choose $v_m, \ldots, v_1$ in $G$ such that following hold.

$$|v_k|_{k+1}' < 2^{-k} \quad \text{and} \quad \|T'v_k\|_k' > (1+2^m)(1+|v_{k+1}|_k' + \ldots + |v_m|_k').$$

Set $u_m = v_1 + \ldots + v_m$. Then we have

$$\frac{|u_m|_{k+1}'}{\|T'u_m\|_k'} \leq \frac{|v_1|_{k+1}' + \ldots + |v_k|_{k+1}' + |v_{k+1}|_{k+1}' + \ldots + |v_m|_{k+1}'}{-(\|T'v_1\|_k' + \ldots + \|T'v_{k-1}\|_k') + \|T'v_k\|_k' - (\|T'v_{k+1}\|_k' + \ldots + \|T'v_m\|_k')}$$

$$\leq \frac{2^{-1} + \ldots + 2^{-k} + |v_{k+1}|_{k+1}' + \ldots + |v_m|_{k+1}'}{-(|v_1|_k' + \ldots + |v_{k-1}|_k') + (1+2^m)(1+|v_{k+1}|_k' + \ldots + |v_m|_k') - (|v_{k+1}|_k' + \ldots + |v_m|}$$

$$\leq \frac{1 + |v_{k+1}|'_{k+1} + \ldots + |v_m|'_{k+1}}{-1 + 1 + 2^m(1 + |v_{k+1}|'_k + \ldots + |v_m|'_k)}$$

$$\leq \frac{1 + |v_{k+1}|_k + \ldots + |v_m|'_{k+1}}{2^m(1 + |v_{k+1}|'_k + \ldots + |v_m|'_k)} \leq \frac{1}{2^m} .$$

So $|u_m|'_{k+1}(\|T'u_m\|'_k)^{-1} \leq 2^{-m}$ for each $k \leq m$ and $T'u_m \neq 0$. Since the set $[T'u_1, \ldots, T'u_m]$ is linearly independent and $E_0$ is dense in $E$ there exists $x_m \varepsilon E_0$ such that $T'u_m(x_m) = 1$ and $T'u_i(x_m) = 0$ for $i=1,\ldots,m-1$. The vectors $(x_i)_{i=1}^m$ and $(u_i)_{i=1}^m$ satisfy (i), (ii) and (iii). So the induction step is completed. The sequences $(x_n)$ and $(u_n)$ satisfy the hypothesis of the Lemma 1.9. There is a subsequence $(x_{n_k})$ of $(x_n)$ such that, the image of $(x_{n_k})$ under the natural quotient map from $E$ onto $E/[u_{n_k}]^{\perp}$ is a basis. By passing to a subsequence, we can assume that image of $(x_n)$ in $E/[u_n]^{\perp}$ is a basis.

Now we define $\lambda(A)$ and an operator $S: F \to \lambda(A)$ by setting $a_n^k = (|u_n|'_k)^{-1}$ and $Sx = (u_n(x))$. Since $T'(V_k^0) \subset U_k^0$ we have $|u_n|'_k(|u_n|'_{k+1})^{-1} \geq \|T'u_n\|'_k\ (|u_n|'_{k+1}) > 2^n$ for $k < n$ and

$$\sum_{n=1}^{\infty} a_n^k |u_n(x)| = \sum_{n=1}^{\infty} \frac{|u_n(x)|}{|u_n|'_k} \leq \sum_{n=1}^{\infty} \frac{|u_n|'_{k+1}}{|u_n|'_k} q_{U_{k+1}}(x) \leq q_{U_{k+1}}(x).$$

These show $\lambda(A)$ is nuclear, $S$ is well defined and continuous. Let $(e_n)$ and $(e_n^*)$ denote the coordinate bases of $\lambda(A)$ and $\lambda(A)'$. We have $STx_n = e_n$ and $T'S'e_n^* = T'u_n$. So $ST$ has dense range and

$$T'S'(\lambda(A)') \subset \overline{sp[T'u_n]}^{\sigma(E', E)} .$$

Since the image of $(x_n)$ in $E/[u_n]^{\perp}$ is a basis, the sequence $\phi_n = \sum_{i=1}^{n} \phi(x_i)T'u_i$ $\sigma(E', E)$-converges to $\phi$ for each $\phi \varepsilon \overline{sp[T'u_n]}^{\sigma(E',E)}$ Hence the sequence $(\phi(x_i)T'u_i)$ $\sigma(E', E)$-converges to zero and it is $\sigma(E', E)$-bounded. This means that $\phi(x_i)T'(u_i) \varepsilon U_k^0$ some $k$ so $|\phi(x_i)| \, \|T'u_i\|_k' \leqq 1$. But $|u_i|_{k+1}(\|T'u_i\|_k')^{-1} \leq 2^{-i}$ for each $i > k$, which gives $|\phi(x_i)| \, |u_i|_{k+1}' < c$ for some $c > 0$ and hence $\phi(x_i) = 0(a_n^{k+1})$. Let $\Theta = \sum_{i=1}^{\infty} \phi(x_i)e_i^* \varepsilon \lambda(A)'$. Then $T'S'\Theta = \phi$. This shows $T'S'(\lambda(A)')$ is $\sigma(E', E)$-closed in $E'$. By the closed range theorem (9.6.3, [8]), $ST(E)$ is closed in $\lambda(A)$, but it is also dense in $\lambda(A)$. Hence $ST$ maps $E$ onto $\lambda(A)$.

We will now state some immediate corollaries of our theorem. The next result implies that a subspace of a Fréchet space $F$ and $F$ have common nuclear Köthe quotient.

*Corollary 2.14*. Let $F$ be a Fréchet space which satisfies $(y)$. Let $E$ be a closed subspace of $F$. Then either $E$ is a Banach space or there is a nuclear Köthe space which admits a continuous norm and there is a quotient map $Q: F \to \lambda(A)$ such that $Q(E) = \lambda(A)$ also.

We can now generalize the theorem of Bellenot and Dubinsky [2] as a simple consequence of our theorem.

*Corollary 2.15*. Let $E$ be a Fréchet space. Then $E$ has nuclear Köthe quotient which admits a continuous norm if and only if $E$ does not satisfy the condition (b).

*Proof*: If $E$ satisfies (b) then by [13] $L(E, \lambda(A)) = LB(E, \lambda(A))$ for every Köthe space which admits a continuous norm.

Suppose $E$ does not satisfy (b) then by [13] there exists $T: E \to \lambda(A)$, $T$ is continuous, unbounded, linear operator and $\lambda(A)$ is a nuclear Köthe space which admits a continuous norm. By Theorem 2.13. there exists $S: \lambda(A) \to \lambda(B)$ such that $ST(E) = \lambda(B)$ and $\lambda(B)$ is nuclear and admits a continuous norm.

# CHAPTER 3

## GENERALIZATIONS

In this chapter we shall examine the condition (y) further, especially for spaces which are not necessarily metrizable. However if $E$ is a Fréchet space which satisfies (y), then we show that $E$ must be countably normed (Corollary 3.2). We generalize the main theorem of the previous chapter (Theorem 2.13) in Proposition 3.9. We can dualize some of our results and obtain the existence subspaces of df-spaces which are isomorphic to duals of nuclear Köthe spaces.

We give a characterization of those l.c.s.'s which satisfy (y) provided under their Mackey topologies they are barrelled.

*Proposition 3.1.* Let $E$ be a l.c.s. which satisfies (y) and suppose $\beta(E, E')$ is compatible with the duality $\langle E, E'\rangle$. Then there exist a generalized Köthe space $\lambda^{\infty}(A)$ which admits a continuous norm and a linear operator $T: E \to \lambda^{\infty}(A)$ such that $T$ is a $\sigma(E, E')$-$\sigma(\lambda^{\infty}(A), \lambda^{\infty}(A)')$ imbedding. If $E$ is barrelled, then $T$ is also continuous.

*Proof*: Let $B$ be as in (y). We define a Köthe space $\lambda^{\infty}(A)$ which admits a continuous norm and a linear operator $T: E \to \lambda^{\infty}(A)$ by setting $a_u^B = (\|u\|_B)^{-1}$ for each $u \in E'[B_0] - [0]$ and for each $B_0$ $B \in \beta_{\sigma(E', E)}$. We set $Tx = [\langle u, u(x)\rangle | u \in E'[B_0] - [0]]$. From

$$\sup[|u(x)|a_u^B|u\varepsilon E'[B_0]\sim[0]] = \sup[|u(x)|(\|u\|_B)^{-1}|u\varepsilon E'[B_0]\sim[0]] \leq q_{B^0}(x)$$

we see that $T$ is a well defined continuous map from $E[\beta(E, E')]$ into $\lambda^\infty(A)$. Since $\beta(E, E')$ is compatible with the duality $<E, E'>$, $T$ is $\sigma(E, E')$-$\sigma(\lambda^\infty(A), \lambda^\infty(A)')$ continuous. Since $E'[B_0]$ is $\sigma(E', E)$-dense in $E'$, $T$ is also one to one. Let $(x_\alpha)_{\alpha\varepsilon I}$ $E$ be a net which does not converge to zero in the $\sigma(E, E')$-topology. Let $(x_\alpha)_{\alpha\varepsilon J}$ be a subnet of $(x_\alpha)_{\alpha\varepsilon I}$ such that $\phi(x_\alpha) > 1$ for each $\alpha\varepsilon J$, for some $\phi\varepsilon E'$. Since $E$ satisfies $(y)$, there is a bounded net $(\phi_\beta)_{\beta\varepsilon L} \subset E'[B_0] \cap B$, $\phi_\beta \neq 0$ such that $\phi = \sigma(E', E)\text{-}\lim \phi_\beta$. Let $e_u^*\varepsilon\lambda^\infty(A)'$ such that $e_u^*[\xi_v|v\varepsilon E'[B_0]\sim[0]] = \xi_v$, for each $u\varepsilon E'[B_0]-[0]$. Since $(\phi_\beta)_{\beta\varepsilon L} \subset E'[B_0] \cap B$, the net $(e^*_{\phi_\beta})_{\beta\varepsilon L}$ is equicontinuous. Let $\Theta\varepsilon\lambda^\infty(A)'$ be the $\sigma(\lambda^\infty(A)', \lambda^\infty(A))$-limit of some convergent subnet of $(e^*_{\phi_\beta})_{\beta\varepsilon L}$. By the weak continuity of $T'$, we have $T'(\Theta) = \phi$. Hence $\Theta(Tx_\alpha) = \phi(x_\alpha) > 1$, for each $\alpha\varepsilon J$. This shows $T^{-1}$ is weakly continuous. So $T$ is a weak imbedding.

*Remark:* Let $(e^*_\alpha)_{\alpha\varepsilon\Gamma}$ be a subset of $\lambda^\infty(A)$ such that $e^*_\alpha(\xi_\beta)_{\beta\varepsilon\Gamma} = \xi_\beta$ for each $\alpha\varepsilon\Gamma$. If $q_U$ is a continuous norm on $\lambda^\infty(A)$ for some $U\varepsilon U(\lambda^\infty(A))$, then $(e^*_\alpha)_{\alpha\varepsilon\Gamma} \subset \lambda^\infty(A)'[U^0]$. By the Goldstein theorem we have
$V^0 = \overline{sp(e^*_\alpha)_{\alpha\varepsilon\Gamma}\ V^0}^{\sigma(\lambda^\infty(A)', \lambda^\infty(A))}$ for each $V\varepsilon U(\lambda^\infty(A))$. Hence $\lambda^\infty(A)$ satisfies $(y)$, if it admits a continuous norm.

*Corollary 3.2.* If $E$ is a Fréchet space which satisfies $(y)$, then $E$ is isomorphic to a subspace of some Fréchet Köthe space $\lambda^\infty(A)$ which admits a continuous norm.

*Proof:* If $E$ is a Fréchet space, then $\beta_{\sigma(E', E)}$ has a countable basis. Hence $\lambda^{\infty}(A)$ constructed in Proposition 3.1 has a countable basis of neighborhoods. It is also complete. So the map $T$ which is constructed in Proposition 3.1 becames a weak imbedding between two Fréchet spaces. By the closed range theorem (9.6.3 in [8]), $T$ is an imbedding.

*Remark:* If a Fréchet Köthe space $\lambda^{\infty}(A)$ admits a continuous norm, then it is countably normed. Hence every Fréchet space which satisfies $(y)$ is countably normed.

Next proposition indicates that the condition $(y)$, countably normability and asymptotically normability are equivalent in a Fréchet-Schwartz space.

*Proposition 3.3.* For a Fréchet-Schwartz space $E$ the following are equivalent.

(i) $E$ satisfies $(y)$,

(ii) $E$ is countably normed,

(iii) $E$ is asymptotically normable.

*Proof:* The remark above allows us to conclude that (i) implies (ii). By Proposition 3.1 in [15] we have that (ii) implies (iii). Proposition 2.2 gives that (iii) implies (i).

*Remark:* In [15] T. Terzioğlu and D. Vogt constructed a Fréchet-Montel space $\lambda(B)$ which admits a continuous norm and is not asymptotically normable. This $\lambda(B)$ is countably normed and also satisfies $(y)$. Therefore Schwartzness is essential in Proposition 3.3.

Our next result implies that $E$ satisfies (b) whenever $L(E, F) = LB(E, F)$ for every l.c.s. $F$ which satisfies (y).

*Proposition 3.4.* Let $E$ be a l.c.s. which does not satisfy (b). Then there exist generalized Köthe space $\lambda^{\infty}(A)$ which admits a continuous norm and a continuous linear operator $T: E''[\tau] \to \lambda^{\infty}(A)$ such that the restriction of $T$ to $E$ is unbounded, where $U(E''(\tau)) = [U^{00} | U \varepsilon U(E)]$.

*Proof:* Since $E$ does not satisfy (b), there is $U \varepsilon U(E)$ such that for each $V \varepsilon U(E)$, there exists $W \varepsilon U(E)$ so that $E'[U^0] \cap W^0$ is not absorbed by $V^0$. For our Köthe set, we take $a_u^V = (\|U\|_{V^0})^{-1}$ for each $u \varepsilon E'[U^0] - [0]$ and for each $V \varepsilon U(E)$ which is contained in $U$. We define $T: E''[\tau] \to \lambda^{\infty}(A)$ by setting $T(\phi) = [\langle u, \phi(u)\rangle | u \varepsilon E'[U^0] - [0]]$ for each $\phi \varepsilon E''(\tau)$. Clearly $T$ is well defined and continuous and $\lambda^{\infty}(A)$ admits a continuous norm. If $T(V)$ were to be bounded for some $V \varepsilon U(E)$, we would be able to find for each $W \varepsilon U(E)$ a constant $c_W > 0$ so that $\sup[|u(x)| \,|\, u \varepsilon E'[U^0] - [0]] < c_W$ for each $x \varepsilon V$. But this would give the following inclusion

$$E'[U^0] \cap W^0 \subset c_W V^0 .$$

This can't be true for each $W \varepsilon U(E)$.

*Corollary 3.5.* If $E$ is a l.c.s. which does not satisfy the condition (b), then $E''[\tau]$ does not satisfy the openness condition. If $E$ does not satisfy (b) and $E''[\tau]$ is barrelled, then $E''[\tau]$ does not satisfy (b).

*Proof:* By Proposition 3.4, we have $L(E''(\tau), \lambda^\infty(A)) \neq LB(E''(\tau), \lambda^\infty(A))$ for some Köthe space $\lambda^\infty(A)$, which admits a continuous norm. By a result of T. Terzioğlu [13], $E''(\tau)$ does not satisfy the openness condition. If $E''(\tau)$ is barrelled, then by Proposition 2.8, $E''(\tau)$ does not satisfy the condition (b).

Our next result shows that asymptotically normable l.c.s.'s are subspaces of generalized Köthe l.c.s.'s $\lambda^\infty(A)$ which admit a continuous norm.

*Proposition 3.6.* If $E$ is a asymptotically normable l.c.s., then $E''(\tau)$ can be imbedded in generalized Köthe space $\lambda^\infty(A)$ which admits a continuous norm.

*Proof:* Let $U_1 \varepsilon U(E)$ which satisfies the condition of asymptotically normability. We take Köthe set $A = [a_u^V | u \varepsilon E'[U_1^0] \sim [0], \ V \subset U_1, V \varepsilon U(E)]$ where $a_u^V = (\|u\|_{V^0})^{-1}$. Define $T: E''(\tau) \to \lambda^\infty(A)$ by setting $T(\phi) = [\langle u, \phi(u)\rangle | u \varepsilon E'[U_1^0] \sim [0]]$ for each $\phi \varepsilon E''(\tau)$. $T$ is well defined and continuous, $\lambda^\infty(A)$ admits a continuous norm. Let $(\phi_\alpha)_{\alpha\varepsilon I}$ be a net in $E''(\tau)$ which does not $\tau$-converge to zero. Then there exists $V \varepsilon U(E)$ such that $\phi_\alpha \notin V^{00}$ for each $\alpha \varepsilon J$, where $J$ is cofinal in $I$. Let $W \varepsilon U(E)$, such that $2W \subset V$ and $q_V(x) \leq M(\varepsilon) q_{U_1}(x) + \varepsilon q_W(x)$ holds for each $x \varepsilon E$ and each $\varepsilon > 0$. Polarization gives us

$$V^0 \subset E'[U_1^0] + \varepsilon V^0 \quad \text{for each} \quad \varepsilon > 0 .$$

This implies $V^0 \subset \overline{E'[U_1^0] \cap W^0}^{\| \|_{W^0}}$. So there is a net $(u_\alpha)_{\alpha\varepsilon J}$ in $E'[U_1^0] \cap W^0$ such that $u_\alpha(\phi_\alpha) > \frac{1}{2}$ for each $\alpha \varepsilon J$. We have

$$\sup[|u(\phi_\alpha)|a_u^W|u\varepsilon E'[U_1^0], u\neq 0] \geq \sup_{\beta\varepsilon J} |u_\beta(\phi_\alpha)|(\|u_\beta\|_{W^0})^{-1}$$

$$\geq \sup_{\beta\varepsilon J} |u_\beta(\phi_\alpha)|$$

$$\geq u_\alpha(\phi_\alpha) \geq \frac{1}{2}$$

for each $\alpha\varepsilon J$. So $(T\phi_\alpha)_{\alpha\varepsilon J}$ does not converge to zero in $\lambda^\infty(A)$. This shows $T$ is one to one and $T^{-1}$ is continuous. Hence $T$ is an imbedding.

*Corollary 3.7.* If $E$ is a asymptotically normable l.c.s. then $E''(\tau)$ satisfies $(y)$.

*Proof*: By the above proposition $E''(\tau)$ is a subspace of some $\lambda^\infty(A)$ which admits a continuous norm. By Proposition 2.2, $E''(\tau)$ satisfies $(y)$.

Now we consider unbounded, continuous, linear operators from a Fréchet space into a l.c.s. which satisfies $(y)$. We need the following lemma which is quite simple and its proof is straight forward.

*Lemma 3.8.* Let $T: E \to F$ and $S: F \to G$ be linear operators. If $T$ is continuous and $ST$ is an open mapping, then $S$ is also open mapping.

Next we generalize the main theorem of Chapter 2 in case the range space is not necessarily metrizable.

*Proposition* 3.9. Let $E$ be a Fréchet space and $F$ be l.c.s. which satisfies $(y)$. If $T: E \to F$ is a continuous, unbounded, linear operator, then there exist a nuclear Fréchet Köthe space $\lambda(A)$ which admits a continuous norm and a linear operator $S$ from $F$ onto $\lambda(A)$ such that $S$ is an open mapping, $ST$ is a continuous operator and $ST$ maps $E$ onto $\lambda(A)$. Further, if $F$ is $c_0$-barrelled, then $S$ is also continuous.

*Proof*: Since $E$ is a Fréchet space, $T: E \to F[\beta(F, F')]$ is continuous. Since $F$ satisfies $(y)$, $F[\beta(F, F')]$ admits a continuous norm. By passing to the quotient $E/T^{-1}(0)$, we can assume $E$ admits a continuous norm. By the Lemma 1.8 and a similar argument to which was used in the proof of Theorem 2.13, we can assume $E$ to be a separable Fréchet space. By the Lemma 2.12, we select decreasing base of neighborhoods $(U_n)$ in $E$ and an increasing sequence of $\sigma(F', F)$-bounded subsets $(B_n)$ of $F'$ such that $T'(B_n) \subset U_n^0$ and $T'(F'[B_0] \cap B_{n+1})$ is not absorbed by $U_n^0$, where $B_0$ is as in $(y)$ for $F$. As in the proof of Theorem 2.13, we can find a sequence $(x_n) \subset E$ and a sequence $(u_n) \subset F'[B_0]$ such that $(T'u_n)$ is a $\sigma(E', E)$-basic sequence, $T'u_n(x_m) = \delta_{nm}$ and $(\|u_n\|_{B_{k+1}} \|T'u_n\|_{U_k^0}^{-1}) \varepsilon \ell_1$. Now we define $\lambda(A)$ which is nuclear and admits a continuous norm and an operator $S: F \to \lambda(A)$ by setting $a_n^k = (\|u_n\|_{B_k})^{-1}$ and $Sx = (u_n(x))$ for each $x \varepsilon F$. Since $T'(B_k) \subset U_k^0$, $(\|u_n\|_{B_{k+1}} \|T'u_n\|_{U_k^0}^{-1}) \varepsilon \ell_1$, we have $(\|u_n\|_{B_{k+1}} \|u_n\|_{B_k}^{-1}) \varepsilon \ell_1$ and

$$\sum_{n=1}^{\infty} a_n^k |u_n(x)| = \sum_{n=1}^{\infty} \frac{u_n(x)}{\|u_n\|_{B_k}} \leq \sum_{n=1}^{\infty} \frac{\|u_n\|_{B_{k+1}}}{\|u_n\|_{B_k}} q_{B_{k+1}^0}(x) < \infty .$$

This shows $S$ is well defined. Let $S^*$ and $T^*$ denote the algebraic adjoint of $S$ and $T$ respectively. By the similar argument as in the proof of Theorem 2.13, we have $T^*S^*(\lambda(A)') = \overline{sp(T'u_n)}^{\sigma(E', E)}$. This shows $ST$ is a continuous operator (8.6.1 in [8]) and $ST$ has closed range (9.6.3 in [8]). But $ST$ is also dense in $\lambda(A)$. Hence we have $ST(E) = \lambda(A)$ and $ST$ is an open mapping. By the Lemma 3.8 $S$ is also open mapping. Now we suppose that $F$ is $c_0$-barrelled. Let $V_k = [(\xi_n)\in\lambda(A) | \Sigma(\xi_n)a_n^k \leq 1]$. $V_k^0$ is compact in the normed space $\lambda(A)'[V_{k+1}^0]$. By (9.4.2 in [8]) there is a $\| \ \|_{V_{k+1}^0}$-null sequence $(\phi_n) \subset sp(e_n^*)$ such that $V_k^0 \subset \overline{acx(\phi_n)}^{\| \ \|_{V_{k+1}^0}}$, where $(e_n)$ is the coordinate basis of $\lambda(A)$ and $(e_n^*)$ is the associated sequence of coefficient functionals of $(e_n)$. We have $S^*(sp(e_n^*) \cap V_i^0) \subset B_{i+1}^0$ for each $i$. Since $(\phi_n)$ is a $\| \ \|_{V_{k+1}^0}$-null sequence, $(S^*\phi_n)$ is a $\| \ \|_{B_{k+2}}$-null sequence in $F'$. By the continuity of $i: F'[B_k] \to F'[\sigma(F', F)]$, $(S^*\phi_n)$ is a $\sigma(F', F)$-null sequence. Since $F$ is $c_0$-barrelled, $(S'\phi_n)$ is an equicontinuous subset of $F'$. We have $S^*(V_k^0) \subset (S^*(\phi_n))^{00}$, which gives the continuity of $S$.

*Corollary 3.10.* Let $F$ be a proper Fréchet subspace of $c_0$-barrelled l.c.s. $E$, which satisfies $(y)$. Then $E$ and $F$ have common nuclear Köthe quotient which admits a continuous norm.

Let $F$ be a df-space ([8]; p.257) and $(A_n)$ a fundamental sequence of closed, bounded subsets of $F$ which is increasing. Let $T: E \to F$ be a continuous, linear operator and suppose there exists $k_0$ such that $T(E) \subset F[A_{k_0}]$. Then the neighborhood $A_{k_0}^0$ in $F'$ is mapped by $T'$, onto some bounded subset of $E'$. This means that the image of the barrel $(T'(A_{k_0}^0))^0$ is absorbed by the bounded set $A_{k_0}$.

Hence $T$ is bounded as a map from $E[\beta(E, E')]$ into $F$.

Suppose $T$ is a continuous, linear operator from l.c.s. $E$ into a df-space $F$ such that $T: E[\beta(E, E')] \to F$ is unbounded. By the above consideration, we can find a sequence $(x_n) \subset E$ such that $Tx_{n+1} \notin F[A_n]$. Since $F'[\beta(F', F)]$ is B-complete, $sp[Tx_n]_{n=1}^{\infty}$ is $\sigma(F'', F')$-closed in $F''$. Hence $sp[Tx_n]_{n=1}^{\infty}$ is $\sigma(F, F')$-closed in $F$. The inductive limit topology on $sp[Tx_n]$ is compatible with the relative $\sigma(F, F')$-topology. Clearly $S: sp[Tx_n] \to F$ is continuous when $sp[Tx_n]_{n=1}^{\infty}$ is equipped with the inductive limit topology where $S(Tx_n) = x_n$ for each $n$. Hence it is $\sigma(F, F')$-$\sigma(E, E')$ continuous ([8]; 8.6.3). This gives that the restriction of $T$ to $sp[x_n]_{n=1}^{\infty}$ is a weak isomorphism and $sp[x_n]_{n=1}^{\infty}$ is $\sigma(E, E')$-closed in $E$. We can improve this when $E'[\sigma(E', E)]$ satisfies $(y)$.

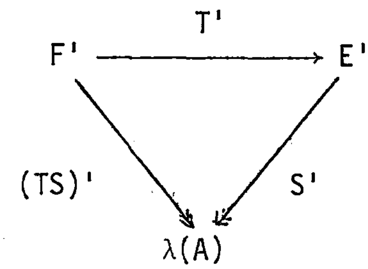
*Proposition 3.11*. Let $E$ be a locally complete l.c.s. such that $E'[\sigma(E', E)]$ satisfies $(y)$ and $F$ be a df-space. If $T: E \to F$ is a continuous linear operator which is unbounded as a map from $E[\beta(E, E')]$ into $F$, then there exist a nuclear Fréchet Köthe space $\lambda(A)$ which admits a continuous norm and continuous linear operator $S: \lambda(A)' \to F$ such that $S(\lambda(A)')$ is $\sigma(E, E')$-closed and the restriction of $T$ to $S(\lambda(A))'$ is an $\sigma(E, E')$-$\sigma(F, F')$ isomorphism. If the topology on $F$ is finer than $\tau_c(F, F')$, then the restriction of to $S(\lambda(A))'$ is an isomorphism.

*Proof*: By an easy modification of Lemma 2.12, we select $(B_n)$ and $(A_n)$ which are increasing sequences of bounded subsets of $E$ and

F respectively such that $(A_n)$ forms a fundamental sequence of bounded set, $T(B_n) \subset A_n$ and $T(E[B_0] \cap B_{n+1})$ is not absorbed by $A_n$, where $B_0$ is as in (y). Let $(x_n)$ be a sequence in $E[B_0]$ such that $(\|x_n\|_{B_{k+1}} \|Tx_n\|_{A_k}^{-1}) \in \ell_1$ and $\|x_n\|_{B_0} \neq 0$. Let $b_n^k = (\|x_n\|_{B_k})^{-1}$ and $R_1: \lambda(B)' \to E$, $R_1(\xi_n) = \sum_{n=1}^{\infty} \xi_n x_n$. By $(\|x_n\|_{B_{k+1}} \|x_n\|_{B_k}^{-1}) \in \ell_1$ and local completeness of E, $R_1$ is well defined. Since $\lambda(B)$ is a nuclear Fréchet space, $\lambda(B)'$ is ultra-bornological. So $R_1$ is continuous. It is easy see that $R_1'T': F' \to \lambda(B)$ is a continuous, unbounded linear operator. By the Theorem 2.13, there exist a nuclear Köthe Fréchet space $\lambda(A)$ which admits a continuous norm and a continuous operator $R_2': \lambda(B) \to \lambda(A)$ such that $R_2'R_1'T'(F') = \lambda(A)$. Let $S = R_1R_2$. $R_2: \lambda(A) \to \lambda(B)$ is continuous. So S is continuous. Since $(TS)'$ and $S'$ are open mappings, $TS = (TS)''$ and $S = S''$ are weak imbeddings. Suppose the topology on F is finer than $\tau_c(F, F')$. Let $U \in U(\lambda(A)')$, then $U^0$ is a compact subset of $\lambda(A)$. By ([8]; 9.4.5), there is a compact set K in F' such that $U^0 \subset S'T'(K)$. Since the topology on F is finer than $\tau_c(F, F')$, $K^0$ is a neighborhood in F. So $(TS)^{-1}$ is continuous and the restriction of T to $S(\lambda(A)')$ is an isomorphism.

*Corollary 3.12.* Let $T: E \to F$ be as in the above proposition. Then there exist nuclear Köthe Fréchet space $\lambda(C)$ which admits a continuous norm and an open mapping $R: F \to \lambda(C)'$ such that $RT(E) = \lambda(C)'$, R is $\sigma(F, F')$-$\sigma(\lambda(C), \lambda(C)')$-continuous and RT is an open mapping. If the topology on F is finer than $\tau_c(F, F')$ then R is continuous.

*Proof*: By the above proposition we have following diagram.




Here $S: \lambda(A)' \to E$ is continuous, $(TS)': F' \to \lambda(A)$ is a quotient map and $\lambda(A)$ is a nuclear Köthe Fréchet space which admits a continuous norm. The map $(TS)': F' \to \lambda(A)$ satisfies the hypothesis of theorem in [14]. So there exists a nuclear Köthe subspace $\lambda(C)$ of $F'$ such that the restriction of $(TS)'$ to $\lambda(C)$ is an isomorphism. We take $R'$ as the natural inclusion map from $\lambda(C)$ into $F'$. Since $S'T'R'$ is an imbedding, $RTS: \lambda(A)' \to \lambda(C)'$ is a quotient map. By the Lemma 3.8, $RT$ and $R$ are open mappings. Clearly $RT$ and $R$ are weakly continuous and $RT(E) = \lambda(C)'$. It is easy to see that $R$ is also continuous when the topology on $F$ is finer than $\tau_c(F, F')$.

Our next corollary asserts the existence of $\lambda(A)'$ subspaces and quotients of a df-space under certain conditions.

*Corollary 3.13.* Let $E$ be a locally complete df-space which is not strict L.B.-space. Then $E$ has a weakly closed copy of $\lambda(A)'$ such that $\sigma(\lambda(A)', \lambda(A))$, $\mu(\lambda(A)', \lambda(A))$ and $\beta(\lambda(A)', \lambda(A))$ are the relative topologies of $\sigma(E, E')$, $\mu(E, E')$ and $\beta(E, E')$ respectively and $E$ has a weak quotient $\lambda(B)'$ such that

$\sigma(\lambda(B)', \lambda(B))$, $\mu(\lambda(B)', \lambda(B))$ and $\beta(\lambda(B)', \lambda(B))$ are the quotient topologies of $\sigma(E, E')$, $\mu(E, E')$ and $\beta(E, E')$ respectively. Here $\lambda(A)$ and $\lambda(B)$ are nuclear Köthe Fréchet spaces which admit a continuous norm.

*Proof*: Let $(A_n)$ be a fundamental sequence of bounded subsets of E. Since E is not a strict L.B.-space, we can assume $E[A_1] \cap A_{n+1}$ is not absorbed by $A_n$ for each n. Let $(x_n)$ be a sequence in $E[A_1]$ such that $(\|x_n\|_{A_{k+1}} \|x_n\|_{A_k}^{-1}) \in \ell_1$ for each k. Now we define a nuclear Köthe Fréchet space $\lambda(C)$ which admits a continuous norm and an operator $T: \lambda(C)' \to E$ by setting $c_n^k = \|x_n\|_{A_k}^{-1}$ and $T(\xi_i) = \sum_{i=1}^{\infty} \xi_i x_i$. Since E is locally complete and $\lambda(C)'$ is ultrabornological, T is well defined and continuous. Clearly $T: \lambda(C)' \to E$ satisfies the hypothesis of Proposition 3.11 and Corollary 3.12 and the desired result follows from them.

*Corollary 3.14.* Every L.B.-space has a basic sequence.

*Proof*: Let E be a L.B.-space and $E = \text{ind } E_n$ regular representation of it. If E is a strict L.B.-space then by a result of Köthe [9] each $E_n$ is a subspace of E. So any basic sequence in $E_n$ is also basic sequence in E. If E is not a strict L.B.-space by the Corollary 3.12, E contains an isomorphic copy of $\lambda(A)'$, which has a basis.

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