ITERATED DEFECT CORRECTION METHODS FOR SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS

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ŞENNUR SOMALI (UZUNER)

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Approval of the Graduate School of Natural and Applied Sciences

Prof.Dr. Alpay ANKARA

Director

I certify that this thesis satisfies all the requirements of a thesis for the PhD degree in Mathematics.

Prof.Dr. Tosun TERZİOĞLU

Chairman of the Department

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, for a thesis towards the PhD degree in Mathematics.

Assoc.Prof.Dr. Bülent KARASÖZEN

Supervisor

Examining Committee in Charge:

Prof.Dr. Okay ÇELEBİ (Chairman)

Assoc.Prof.Dr. Mustafa AKGÜL

Assoc.Prof.Dr. Albert ERKİP

Assoc.Prof.Dr. Bülent KARASÖZEN

Asst.Prof.Dr. Tanıl ERGENÇ

ABSTRACT

ITERATED DEFECT CORRECTION METHODS FOR SEMI-EXPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS

Şennur Somalı (Uzuner)

PhD in Mathematics

Supervisor: Assoc. Prof. Dr. Bülent Karasözen

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The application of the iterated defect correction (IDeC) techniques to linear constant coefficient differential-algebraic equations (DAE's) of arbitrary index and nonlinear semi-explicit index one DAE's is analyzed. The convergence behavior of the defect corrections based on the linearly implicit Euler method is studied using the perturbed asymptotic expansions of the global error. Various numerical results are presented and discussed.

Key words: differential-algebraic equations, defect correction methods, linearly implicit Euler method, asymptotic expansions.

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ÖZET

YARI AÇIK DIFERANSİYEL-CEBİRSEL DENKLEMLER İÇİN YİNELEMELİ HATA DÜZELTME YÖNTEMLERİ

Şennur Somalı (Uzuner)
Doktora Tezi, Matematik Bölümü
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Bu çalışmada yinelemeli hata düzeltme yöntemlerinin sabit katsayılı isteksel indisli doğrusal ve bir indisli doğrusal olmayan yarı-açık difereansiyel-cebirsel denklemlere uygulanışı incelenmiştir. Doğrusal kapalı Euler yöntemine dayalı olan hata yinelemelerin yakınsama analizleri genel hatanın asimptotik açılımları kullanılarak yapılmıştır. Çeşitli sayısal sonuçlar verilmiş ve tartışlmıştır.

Anahtar Kelimeler: diferansiyel-cebirsel denklemler, hata düzeltme yöntemleri, doğrusal kapalı Euler yöntemi, asimptotik açılımlar.

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INTRODUCTION

The numerical and analytical treatment of differential-algebraic equations (DAE's) has been the subject of intensive research activity in the last few years (see monographs [1], [12], [14]). The application of ordinary differential equation (ODE) methods to these systems presents numerical difficulties. In the last twenty years several numerical methods were applied with some success to DAE's. Among them are the backward differentiation formulas (BDF) [9],[11], [19],[20], multistep methods [12]), one-leg methods ([12],[23], implicit Runge-Kutta methods [14], , [25], [27], Rosenbrock methods [26], [28] and extrapolation methods [4], [21]. The well known codes for DAE's are LSODI of Hindmarsh [18], DASSL of Petzold [24], LIMEX of Deufelhard et al. [5] and RADAU5 of Hairer and Wanner [16], [14].

In this thesis we will apply an efficient acceleration technique for stiff and non-stiff ODE's , namely the iterated defect correction (IDeC) method due to Frank et. al. [6] to DAE's and analyze its numerical behavior. We will outline briefly the DAE's considered here and give the formulation of the IDeC method for ODE's. DAE's are special implicit differential equations of the form

$$F(t, y(t), y'(t)) = 0 \tag{0.1}$$

with singular $F_{y'}$, where F and y are of the same dimension. Here and in the following we denote partial derivatives by subscripts, so that $F_{y'} = \partial F/\partial y'$. Equation (0.1) is also called a fully implicit DAE system. We are here especially interested in semi-explicit systems, differential equations with algebraic constraints of the form

$$y'(t) = f(t, y(t), z(t)) \qquad f : \mathbb{R}^{m+k+1} \to \mathbb{R}^m$$
$$0 = g(t, y(t), z(t)) \qquad g : \mathbb{R}^{m+k+1} \to \mathbb{R}^k \tag{0.2}$$

where y represents the differential variables and z the algebraic variables. The numerical methods devised for DAE's take into account the structure of the underlying DAE. We will give two commonly handled DAE types in the literature which are the subject of this work.

Linear constant coefficient DAE's are of the form

$$Ax'(t) + Bx(t) = f(t) \tag{0.3}$$

where A and B are $m \times m$ matrices and t is a real variable. This system is a special type of the fully implicit nonlinear DAE (0.1). It is known that if the matrix pencil $A + \lambda B$, with λ a complex variable, is a regular matrix pencil, then there exist matrices P and Q such that

$$PAQ = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \quad , \quad PBQ = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \tag{0.4}$$

where E is a nilpotent matrix of nilpotency index μ ($E^\mu = 0$ and $E^{\mu-1} \neq 0$) and I is the identity matrix. This transformation is known as the Kronecker canonical form (see [2, pp. 18]). The application of this transformation to (0.3) yields a linear constant coefficient DAE system of the form

$$y_1' + Cy_2 = f_1 \tag{0.5a}$$

$$Ey_2' + y_2 = f_2 \tag{0.5b}$$

The first equation is an explicit ODE system and does not interest us further. A nilpotent matrix of nilpotency index μ can always be transformed to a strictly upper or lower triangular matrix (see [2, pp.20, Theorem 2.3.5]). Therefore the second equation is a special case of the nonlinear semi-explicit DAE system (0.2).

The most characteristic value of a DAE system is its index. For linear constant coefficient DAE's it is defined as to the nilpotency index μ of the matrix E. It plays an important role in the analytical and numerical considerations. The numerical solution methods run into difficulties if the underlying DAE has a high index. The regularity of the matrix pencil $A + \lambda B$ is equivalent to solvability in the case of linear constant coefficient DAE's, which is not the case for linear DAE's with variable coefficients and nonlinear DAE's. The properties of the analytical solution of (0.3) are pointed out in [2, pp. 18, Theorem 2.3.1] and they will be discussed in sections I and III .

The nonlinear semi-explicit DAE systems of index 1 are an important subclass of (0.2) where f and g are sufficiently differentiable. We assume further that g_z has a bounded inverse in a neighborhood of the exact solution. Then the second equation of (0.2) can be transformed into $z = G(y)$ by the implicit function theorem, so that (0.2) becomes $y' = f(y, G(y))$. This proves the local solvability and regularity of a nonlinear system of index 1. We further assume that the initial values y_0 and z_0 are consistent with (0.2) , i.e. $g(y_0, z_0) = 0$. This type of DAE occurs very often in science and engineering.
There are two main definitions of the index for nonlinear DAE systems. One is due to Gear [10], [8] and it is called the differential index, di, of the system (0.1) ; it is the minimum integer such that the system of equations (0.1) and

$$\frac{d}{dt}F(t,y,y') = 0$$
$$\vdots$$
$$\frac{d^m}{dt^m}F(t,y,y') = 0$$

can be solved for $y' = y'(y)$. This means that the equation system above can be written as

$$
\begin{aligned}
F_{[0]}(t,y,y') &= 0 \\
F_{[1]}(t,y,y',y'') &= 0 \\
&\cdot \\
&\cdot \\
&\cdot \\
F_{[m]}(t,y,y',\ldots,y^{(m-1)},y^{(m)}) &= 0
\end{aligned}
\tag{0.6}
$$

where $F_{[j]}, j = 0,\ldots,m$ denote the total derivatives, for example $F_{[1]} = F_{y'}y'' + F_y y' + F_t$. We can write (0.6) as

$$\mathbf{F}_m(y,y,\mathbf{y}_m) = 0$$

where

$$\mathbf{y}_m = (y',\ldots,y^{(m)})^T$$

The index μ of (0.6) is the smallest number μ such that $\mathbf{F}_{\mu+1}$ uniquely determines the variable y' as a continuous function of y and t (see [1, pp. 33, Def. 2.5.1]). This definition can be satisfied in case of a semi-explicit DAE (0.2) in the following form :

If we differentiate the constraint equation (0.2b) with respect to t , we get

$$
\begin{aligned}
y' &= f(t,y,z) \\
g_y(t,y,z)y' + g_z(t,y,z)z' &= -g_t(t,y,z)
\end{aligned}
$$

If g_z is nonsingular, the system is an implicit ODE and we say that (0.2) has index 1. Otherwise we have to write the system by algebraic manipulation and coordinate change in the form of (0.2), and differentiate the constraint equation again. If an implicit ODE results, we say that the original problem has index 2. If the new system is not an implicit ODE , we repeat this process.

Hairer et. al. [14] gave a new definition of index as a measure of the sensitivity of the solutions to the perturbations in equation (0.1) , which is an important factor in the numerical solution of DAE's. Gear called this [7] the perturbation index

pi ; it is the smallest value of the integer m, such that the difference between the solution of (0.1) and the solution of the perturbed equation

$$F(t, \hat{y}, \hat{y}') = \delta(t)$$

can be bounded by an expression of the form

$$\|\hat{y}(t) - y(t)\| \leq C\|\hat{y}(0) - y(0)\| + \max_{0 \leq \xi \leq t} \|\delta(\xi)\| + \ldots + \max_{0 \leq \xi \leq t} \|\delta^{(m-1)}(\xi)\|.$$

For semi-explicit systems of index 1 these two definitions are equivalent i.e. $di = pi$, according to [14] and [7]. For general nonlinear DAE systems $di \leq pi \leq di + 1$ (see [7]).

IDeC methods originate from an idea of Zadunaisky [31] by estimation of the global discretization error of ODE's by means of Runge-Kutta (RK) methods. This idea has been modified by several authors and is applied to partial differential equations, to stiff and nonstiff ODE's (initial and boundary value problems). These lead to a class of fast converging numerical methods, known as IDeC methods.

We will now summarize the formulation of the IDeC method due to [6] for initial value problems (IVP) in the following form

$$\begin{aligned} y'(t) &= f(y(t)) \qquad\qquad t \in [0, T] \text{ and } y, y', f \in \mathbb{R}^m \\ y(0) &= y_0 \end{aligned} \tag{0.7}$$

with the exact solution $y(t)$. Subsequently this problem will be called the original problem (OP). This problem is first solved numerically by a method (basic method) on the grid

$$G = \{t_\nu : t_\nu = \nu \cdot h, \;\; \nu = 0, \ldots, m \cdot n, \;\; h := T/m \cdot n\}$$

where h is the constant step size in the interval $[0, T]$

The resulting numerical approximation η is denoted by $\eta^{[0]} := (\eta_0, \ldots \eta_{m \cdot n})$ (The meaning of m and n will be given in Chapter I). The global discretization error $\eta_\nu - y(t_\nu)$ can be estimated by computing the defect

$$d_h^{[0]} = P_h'^{[0]}(t) - f(P_h^{[0]}) \tag{0.8}$$

where P_h is computed by a piecewise interpolation of $\eta^{[0]}$ on the subintervals $[t^{i-1}, t^i], i = 1, \ldots, n$. Adding this defect to the right hand side of (0.7) an artificial IVP is obtain which is called the neighboring problem (NP) whose exact solution is $P_h^{[0]}$.

$$y'(t) = f(y(t)) + d_h^{[0]} \tag{0.9}$$

The NP may be solved by the same method as the OP or by another method. The numerical solution of (0.9) is denoted by $\pi^{[0]} := (\pi_0, \ldots, \ldots, \pi_{m \cdot n})$. Now the global discretization error $\pi_\nu^{[0]} - P_h^{[0]}(t_\nu)$ of the NP is available and can be used for the estimation of the global error of the OP :

$$\pi_\nu^{[0]} - P_h^{[0]}(t_\nu) \approx \eta_\nu^{[0]} - y(t_\nu)$$

Frank and Ueberhuber have shown in [6] that under suitable assumptions about the problem (0.7) , the methods for solving OP and NP approximately and the interpolation polynomial, $\pi^{[0]} - \eta^{[0]}$ give an error estimate of order $p+q$, where p and q are the orders of the methods for solving equations (0.7) and (0.9) numerically. In the identity

$$y(t_\nu) = \eta_\nu^{[0]} - (\eta_\nu^{[0]} - y(t_\nu))$$

the term $\eta_\nu^{[0]} - y(t_\nu)$ can be replaced by its estimate $\pi_\nu^{[0]} - P_h^{[0]}(t_\nu)$ which leads to an improved numerical approximation of order $p+q$. By subtracting the error estimate $\pi^{[0]} - \eta^{[0]}$ from the approximate solution $\eta^{[0]}$ we obtain an improved numerical approximation of order $p+q$.

$$\eta_\nu^{[1]} := \eta_\nu^{[0]} - (\pi_\nu^{[0]} - \eta_\nu^{[0]}) \tag{0.10}$$

Further defect correction steps can be constructed iteratively by interpolating $\eta^{[1]}$ by a new polynomial P_h and solving the new NP. In this way one obtains a sequence of approximations

$$\eta_\nu^{[j+1]} := \eta_\nu^{[0]} - (\pi_\nu^{[j]} - \eta_\nu^{[j]}) \qquad j = 0, 1, \ldots, j_{\max} \tag{0.11}$$

The number of maximum defect correction steps $j_{\max}$ is determined by the maximum attainable convergence order which in turn depends mainly on the degree of the interpolation polynomial and the convergence orders of the numerical methods for solving (0.7) and (0.9). We have chosen the linearly implicit Euler method as the basic method. It belongs to the class of semi-explicit methods. The major advantage of these methods is their low computational cost. They use a fixed Jocabian in each interval and only one Newton iteration is performed. This method is applied with success to stiff and nonstiff ODE's and is known in the literature as a very efficient acceleration technique. Like the extrapolation method, the error analysis and asymptotic order results are based on the asymptotic expansions of the global error in powers of the step size h. Applying the implicit Euler method to the explicit ODE (0.7) one obtains asymptotic expansions of the global error of the following form :

$$y(t) - y_h(t) = \sum_{i=1}^{N} e_i(t)h^i + E_h(t)h^{N+1} \tag{0.12}$$

where $y_h(t)$ is the implicit Euler approximation obtained with the step size h and y is the exact solution . For nonstiff ODE's and linear constant coefficient DAE's the asymptotic expansions can be generated in a simple manner with uniformly bounded remainder term $E_h(t)$. However, for stiff ODE's and nonlinear DAE's it is not possible to obtain the usual form of asymptotic expansions with a bounded remainder term (see [2, pp. 109]). Using the analogy between the stiff system

$$\begin{aligned} y'(t) &= f(t, y(t), z(t)) \qquad & y(0) &= y_0 \\ \epsilon z'(t) &= g(t, y(t), z(t)) \qquad & z(0) &= z_0 \end{aligned} \tag{0.13}$$

where $0 < \epsilon < 1$, and the corresponding semi-explicit DAE of index 1 obtained by setting ($\epsilon = 0$) the so called perturbed asymptotic expansions due to Hairer et. al are developed.

In Chapter I we consider a linear constant coefficient DAE of arbitrary index in semi-explicit form (0.5b). This system and the resulting NP's at defect correction steps are solved by the implicit Euler method. Using the Taylor expansions of the exact and approximate solutions asymptotic expansions of the global errors in unperturbed form are obtained for each component of the DAE system. This makes it possible to analyze the convergence behavior of each component of the

DAE system. It turns out that the maximum attainable convergenge order depends on the index of the underlying DAE and is limited by the degree of the interpolating polynomials.

In Chapter II a semi-explicit nonlinear DAE of index 1 of the form (0.2) is solved by the linearly implicit Euler method. The perturbed asymptotic expansions of the global error for constant step sizes are developed in the Appendix. Using these, convergenge results about approximate solutions at defect correction steps are obtained. The results are very similar to the nonstiff and stiff ODE's. The maximum attainable convergenge order is one less than the degree of the interpolating polynomial. On the other side both y differential and z algebraic components attain the same convergence order.

In Chapter III numerical results for two test examples are presented, one from the linear constant coefficient DAE and the other from the nonlinear index one DAE. These results confirm the order of convergence which are predicted theoretically in the chapter I and II. Using the step size and order control strategy for IDeC methods in [30] we have solved a pipeline problem as a technical application. The convergence behavior of the IDeC methods for the DAE's investigated here, is similar to the ODE's. Defect correction method belongs as extrapolation to the class of acceleration techniques. Extrapolation method is applied to various types of DAE's extensively in the last years. It is usual for these techniques to use a cheap method like the linearly implicit Euler method here. On the basis of numerical experiments and theoretical results obtained here, the IDeC techniques based on it seem to be efficient and promising for DAE's too and diverse further development.

Chapter 1

ITERATED DEFECT CORRECTION METHODS FOR CONSTANT COEFFICIENT LINEAR DAE'S

We consider the linear constant coefficient DAE (0.3) in Kronecker canonical form (0.4). The matrix E is a nilpotent block diagonal matrix $E = diag(E_1, E_2, \ldots, E_l)$ composed of elementary Jordan blocks of the form.

$$E_i = \begin{pmatrix} 0 & & & & \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & 0 \end{pmatrix}_{\mu \times \mu}$$

The system in (0.5a) and (0.5b) is called completely singular system, but if E contains only one Jordan block of the form above, then it will be called canonical (completely) singular system ([1, pp. 79]).

The behavior of the ODE methods are studied generally on the canonical singular subsystem (0.5b). We will apply the implicit Euler method as the basic method to this system and study its convergence behavior in defect correction steps.

The exact solution of the canonical subsystem of index μ

$$Ey' = y + g(t)$$

with $g(t) = (g_1(t), g_2(t), \ldots, g_\mu(t))$ and $y(t) = (y_1(t), y_2(t), \ldots, y_\mu(t))$ is given by

$$y_1(t) = -g_1(t)$$
$$y_2(t) = -g_2(t) - g_1'(t)$$
$$\vdots$$
$$y_\mu(t) = -g_\mu(t) + \sum_{i=1}^{\mu-1}(-1)^{\mu-i} g_i^{(\mu-i)}(t)$$

When the implicit Euler method is applied to this system the derivatives $g_i'(t)$ will be replaced by finite differences. The structure of the nilpotent matrix E makes the exact and approximate solutions of the ith equation depend only on the solutions of the first (i-1) equations. This situation is reflected to the behavior of the defect correction solutions. It turns out that the convergence orders of higher index components are lower then the lower index components. The maximum attainable order of convergence for the μ-index component is $m + 2 - \mu$, when a piecewise polynomial of degree $m \geq 2(\mu - 1)$ is used. The asymptotic expansions of the global errors are obtained using Taylor expansions and don't contain any perturbation terms.

1.1 Formulation of the IDeC method

In the following, we will describe briefly application of the IDeC method to linear constant coefficient DAE's in canonical singular form :

$$Ey' = y + g(t)$$
$$y(0) = y_0 \qquad (t, y) \in G \subset [0, T] \times \mathbb{R}^s \tag{1.1.1}$$

where $y(t)$ and $g(t)$ are vector-valued functions of dimension s and $g(t)$ is a sufficiently smooth function.

When the implicit Euler method is applied to the original problem (OP) (1.1.1) on the uniform grid $\{G := \{t_\nu = \nu \cdot h,\ \nu = 0, 1, \ldots, n \cdot m, \quad n, m \in N\}$ with the step size $h = T/m \cdot n$, the numerical solution obtained is denoted by $\eta := (\eta_0, \eta_1, \ldots, \eta_\nu, \ldots, \eta_{n \cdot m})$. The meaning of n and m will be given later.

$$E(\eta_{\nu+1} - \eta_\nu) = h\eta_{\nu+1} + hg(t_{\nu+1}) \tag{1.1.2}$$

$$\eta_0 = y_0$$

Interpolation of $\eta = (\eta_0, \ldots, \eta_{n \cdot m})^T$ by a vector-valued function $P_h^{[0]}(t)$ (i.e. $P_h(t_\nu) = \eta_\nu, \ \ \nu = 0, 1, \ldots, n \cdot m$) yields the defect

$$d_h^{[0]}(t) = EP_h^{[0]'}(t) - P_h^{[0]}(t) - g(t) \tag{1.1.3}$$

By adding the defect $d_h^{[0]}(t)$ to the right-hand side of the original problem (1.1.1), then the neighboring problem (NP) can be created analogously to the ODE's

$$Ey' = y + EP_h^{[0]'}(t) - P_h^{[0]}(t) \tag{1.1.4}$$

$$y(0) = y_0$$

The numerical approximation of (1.1.4) with the implicit Euler method on the same grid is denoted by $\pi := (\pi_0, \ldots, \pi_\nu, \ldots, \pi_{n \cdot m})$, where π_ν is an approximation to the exact solution $P_h^{[0]}(t)$ of (1.1.4) at t_ν.

$$E(\pi_{\nu+1}^{[0]} - \pi_\nu^{[0]}) = h\, \pi_{\nu+1}^{[0]} + hEP_h^{[0]'}(t_{\nu+1}) - hP_h^{[0]}(t_{\nu+1}) \tag{1.1.5}$$

The improved solution of (1.1.1) is given by

$$\eta_\nu^{[1]} = \eta_\nu - (\pi_\nu^{[0]} - P_h^{[0]}(t_\nu)) \tag{1.1.6}$$

the succeeding iterations can be constructed as follows; one can interpolate the values of $\eta^{[1]}$ by a new piecewise polynomial $P_h^{[1]}(t)$ and a new NP is constructed whose exact solution is $P_h^{[1]}(t)$, solving the new NP on the same grid G, it leads to the solution $\pi^{[1]}$ from which $\eta^{[2]}$ is computed, and so on. Continuing this we obtain the sequence of following approximations:

$$\eta_\nu^{[j]} = \eta_\nu^{[0]} - (\pi_\nu^{[j-1]} - P_h^{[j-1]}(t_\nu)) \quad j = 1, 2, \ldots, j_{\max} \tag{1.1.7}$$

where $\pi_\nu^{[j-1]}$ is the numerical solution of the NP, constructed at each defect correction step. Then $\eta_\nu^{[j]}$ denotes the corrected value of the approximation, where $\eta_\nu^{[0]} = \eta_\nu$.

The interpolant $P_h^{[j]}$ is a piecewise polynomial of fixed degree m. We use uniform grids, based on the stepsize $h = T/m \cdot n$ and consider the following subintervals of $[0,T] = \cup I_i$;

$$I_i = [t^{i-1}, t^i], \quad i = 1,2,\ldots,n \text{ with } t^i = i \cdot m \cdot h$$

Here n denotes the number of subintervals and m the degree of the interpolation polynomial in each subinterval. Our interpolating functions $P_h^{[j]}$ are defined piecewise as

$$P_h^{[j]}(t) = P_{i,h}^{[j]}(t), \quad t \in I_i, \ i = 1,2,\ldots,n$$

so that the interpolating polynomials $P_{i,h}^{[j]} : \mathbb{R} \to \mathbb{R}^s$

$$P_{i,h}^{[j]}(t_\nu) = \eta_\nu^{[j]}, \quad \nu = (i-1)\cdot m,\ (i-1)\cdot m + 1, \ldots, i \cdot m, \quad j = 0,1,\ldots,j_{\max} - 1$$

The defect $d_h^{[j]}(t)$ is also defined piecewise

$$d_h^{[j]}(t) := d_{i,h}^{[j]}(t) = EP_{i,h}^{[j]\prime}(t) - P_{i,h}^{[j]}(t) - g(t), \quad t \in (t^{i-1}, t^i), \ i = 1,2,\ldots,n$$

Every NP can be written then

$$Ey' = y + EP_{i,h}^{[j]\prime}(t) - P_{i,h}^{[j]}(t) \quad i = 1,2,\ldots,n, \quad j = 0,1,\ldots,j_{\max} - 1$$

$$y(t^{i-1}) = \begin{cases} y_0 & i = 1 \\ P_{i-1,h}^{[j]}(t^{i-1}) & i = 2,3,\ldots,n \end{cases} \tag{1.1.8}$$

The solution of the i-th part of (1.1.8) is $P_{i,h}^{[j]}(t)$. We consider (1.1.8) as an IVP defined in a piecewise fashion and having the solution $P_h^{[j]}$. This solution has jumps in its first derivative at the endpoints t^i of the subintervals I_i. This is not contradictory to the assumption that the original problem and NP are problems of the same kind (see [31]) since a smooth original problem is of course a special case of (1.1.8). With respect to the connection of the numerical values at the endpoints of the interpolation intervals, there exist two different connection strategies; local and global connection strategies. By the local connection strategy, all η-values and further corrections $\eta^{[j]}$ are computed in one interpolation interval, the numerical solution of the original problem starts with this corrected value at the beginning of the next interpolation interval. By the global connection strategy, firstly the original problem is solved on the whole integration interval $[0,T]$ and then the $\pi^{[j]}$ and $\eta^{[j]}$-values and so on are calculated. Our analysis is restricted to the global connection strategy.

1.2 Asymptotic Expansions of the Global Error of the Implicit Euler Method

In the following we will give the asymptotic expansions of the global error of the implicit Euler method in (1.1.2). The analytic solution of the DAE (1.1.1) is of the form

$$y(t) = -\sum_{i=0}^{\mu-1} E^i g^{(i)}(t)$$

where $g^{(i)}(t) = \frac{d^i}{dt^i} g(t)$ and μ is the nilpotency index of E. Solving (1.1.2) for $\eta_{\nu+1}$ and noting that

$$\left[E\left(\frac{\nabla}{h}\right) - I\right]^{-1} = -\sum_{i=0}^{\mu-1} E^i \left(\frac{\nabla}{h}\right)^i$$

we obtain

$$\eta_\nu = -\sum_{i=0}^{\nu} E^i \left(\frac{\nabla}{h}\right)^i g_\nu \qquad \text{for} \qquad \nu < \mu - 1 \tag{1.2.1a}$$

$$\eta_\nu = -\sum_{i=0}^{\mu-1} E^i \left(\frac{\nabla}{h}\right)^i g_\nu \qquad \text{for} \qquad \nu \geq \mu - 1 \tag{1.2.1b}$$

where

$$\nabla^i g_\nu = \sum_{k=0}^{i} (-1)^k \frac{i!}{k!(i-k)!} g_{\nu-k}$$

is the backward difference operator.

From definition of the global error $e_\nu = y(t_\nu) - \eta_\nu$, we write;

$$e_\nu = -\sum_{i=0}^{\nu} E^i g^{(i)}(t_\nu) + \sum_{i=0}^{\mu-1} E^i \left(\frac{\nabla}{h}\right)^i g_\nu \;\; \text{for} \;\; \nu = 1, 2, \ldots, nm,$$

For consistent initial conditions, i.e. $e_0 = 0$

$$y(0) = y_0 = -\sum_{i=0}^{\mu-1} E^i g^{(i)}(0)$$

Using the Taylor expansion,

$$\frac{1}{h^i}\nabla^i g_\nu = \frac{1}{h^i}\left\{\sum_{k=0}^{i}(-1)^k\frac{i!}{k!(i-k)!}\left[\sum_{l=0}^{m}(-k)^l\frac{h^l}{l!}g_\nu^{(l)} + \mathcal{O}(h^{m+1})\right]\right\}$$
$$= g_\nu^{(i)} + \sum_{k=0}^{i}\sum_{l=i+1}^{m}(-1)^k\frac{i!}{k!(i-k)!}\frac{h^{l-i}}{l!}(-k)^l g_\nu^{(l)} + \mathcal{O}(h^{m+1}) \quad (1.2.2)$$

and substituting (1.2.2) into the equation for the error e_ν, we get the asymptotic expansions corresponding to the two cases of nilpotency in the equations (1.2.1a) and (1.2.1b) ;

for $\nu < \mu - 1$

$$e_\nu = \sum_{i=1}^{\nu}\sum_{l=i+1}^{m} c_{l,i}E^i h^{l-i} g^{(l)}(t_\nu) - \sum_{i=\nu+1}^{\mu-1} E^i g^{(i)}(t_\nu) + E\mathcal{O}(h^{m+1})$$

and for $\nu \geq \mu - 1$

$$e_\nu = \sum_{i=1}^{\mu-1}\sum_{l=i+1}^{m} c_{l,i}E^i h^{l-i} g^{(l)}(t_\nu) + E\mathcal{O}(h^{m+1})$$

where

$$c_{l,i} = \sum_{k=0}^{i}(-1)^{k+1}\frac{i!}{k!(i-k)!}\frac{(-k)^l}{l!}$$

Since $g(t) = Ey' - y$, the error formula can be rewritten as

$$e_\nu = -E^{\nu+1}y^{\nu+1}(t_\nu) + \sum_{i=1}^{\mu}\sum_{l=i+1}^{m} a_{l,i}E^i h^{l-i} y^{(l)}(t_\nu) + E\mathcal{O}(h^{m+1}) \text{ for } \nu < \mu - 1 \quad (1.2.3)$$

and

$$e_\nu = \sum_{i=1}^{\mu-1}\sum_{l=i+1}^{m} a_{l,i}E^i h^{l-i} y^{(l)}(t_\nu) + E\mathcal{O}(h^{m+1}) \text{ for } \nu \geq \mu - 1 \quad (1.2.4)$$

where $a_{l,i} = c_{l-1,i-1} - c_{l,i}$.

We see that the global error behaves like $\mathcal{O}(h)$ after $\mu - 1$ steps and $\mathcal{O}(1)$ for $1 \leq \nu < \mu - 1$.

1.3 Error Analysis for Index μ-Systems

Theorem 1.1 : Let $g(t)$ be sufficiently smooth. For an index $\mu \geq 2$ linear DAE system, based on the implicit Euler method, one obtains the following orders of convergence for defect correction step j with an interpolating polynomial of degree $m \geq 2(\mu-1)$.

$$\eta^{[j]}_{\mu,\nu} - y_\mu(t_\nu) = \begin{cases} \mathcal{O}(h^{j+1}) & 0 \leq j < \mu - 1 \\ \\ \mathcal{O}(h^{m+2-\mu}) & j \geq \mu - 1 \end{cases} \tag{1.3.1}$$

where $\nu = m, \ldots, nm$, and the subscript μ denotes the μth component of the vectors η_ν and y respectively.

Proof : Similar to the (OP), we have for the (NP), asymptotic expansions of the global error in the following form,

$$\pi^{[j-1]}_\nu - P^{[j-1]}_h(t_\nu) = \sum_{i=1}^{\mu-1} \sum_{l=i+1}^{m} a_{l,i} E^i h^{l-i} P^{[j-1]}_h(t_\nu) + E\mathcal{O}(h^{m+1}) \tag{1.3.2}$$

where $P^{[j-1]}_h(t)$ is defined as in Section 1.1.

Subtracting (1.3.2) from the global error of the (OP) (1.2.4) and (1.2.5) , we obtain

$$\begin{aligned} \eta^{[j]}_\nu - y(t_\nu) &= [\eta_\nu - y(t_\nu)] - [\pi^{[j-1]}_\nu - P^{[j-1]}_h(t_\nu)] \\ &= \sum_{i=1}^{\mu-1} \sum_{l=i+1}^{m} a_{l,i} E^i h^{l-i} [y^{(l)}(t_\nu) - P^{[j-1](l)}_h(t_\nu)] + E\mathcal{O}(h^{m+1}) \end{aligned}$$

Written in the components of index μ system with

$$\eta^{[j]}_\nu = (\eta_{1,\nu}, \eta_{2,\nu}, \ldots, \eta_{\mu,\nu})^T, \quad y(t_\nu) = (y_1(t_\nu), y_2(t_\nu), \ldots, y_\mu(t_\nu))^T,$$

$$P^{[j]}_h(t_\nu) = (P_{1,h}(t_\nu), P_{2,h}(t_\nu), \ldots, P_{\mu,h}(t_\nu))^T,$$

and

$$E = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 1 & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \cdot \\ \vdots & & & & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{\mu\times\mu} \qquad E^i y = \begin{bmatrix} 0 \\ \vdots \\ y_1 \\ \vdots \\ y_{\mu-i} \end{bmatrix}_{\mu\times 1}$$

we have;

$$\eta^{[j]}_{\mu,\nu} - y_\mu(t_\nu) = \sum_{i=1}^{\mu-1} \sum_{l=i+1}^{m} a_{l,i} h^{l-i} \left[y^{(l)}_{\mu-i}(t_\nu) - P^{[j-1](l)}_{\mu-i,h}(t_\nu)\right] + \mathcal{O}(h^{m+1}) \tag{1.3.3}$$

i) For $j = 1$, using the Lemma 1.2 and (1.3.8) in Lemma 1.1;

$$\begin{aligned}
\left\|\eta^{[1]}_{\mu,\nu} - y_\mu(t_\nu)\right\| \le & \sum_{i=1}^{\mu-2}\sum_{l=i+1}^{m} |a_{l,i}| h^{l-i} \left\|y^{(l)}_{\mu-i}(t_\nu) - P^{[0](l)}_{\mu-i,h}(t_\nu)\right\| \\
& + \sum_{l=\mu}^{m} |a_{l,\mu-1}| h^{l-i} \left\|y^{(l)}_{1}(t_\nu) - P^{[0](l)}_{1,h}(t_\nu)\right\| \\
& + const.h^{m+1} \\
\left\|\eta^{[1]}_{\mu,\nu} - y_\mu(t_\nu)\right\| \le & \sum_{i=1}^{\mu-2}\sum_{l=i+1}^{m} const.h^{l-i+1} + \sum_{l=\mu}^{m} const.h^{m+2-\mu} + const.h^{m+1} \\
& \le const.h^2
\end{aligned}$$

where $m + 2 - \mu \ge 2$ and *const.* denotes from here on a constant which depends on the derivatives of the unknown solutions and it is independent of h and ν.

ii) For $2 \le j < \mu - 1$,

$$\begin{aligned}
\left\|\eta^{[j]}_{\mu,\nu} - y_\mu(t_\nu)\right\| \le & \sum_{i=1}^{\mu-2} \left\{ \sum_{l=i+1}^{m-\mu+i-j+1} |a_{l,i}| h^{l-i} \left\|y^{(l)}_{\mu-i}(t_\nu) - P^{[j-1](l)}_{\mu-i,h}(t_\nu)\right\| \right. \\
& + \sum_{l=m-\mu+i-j+2}^{m-\mu+i} |a_{l,i}| h^{l-i} \left\|y^{(l)}_{\mu-i}(t_\nu) - P^{[j-1](l)}_{\mu-i,h}(t_\nu)\right\| \\
& \left. + \sum_{l=m-\mu+i+1}^{m} |a_{l,i}| h^{l-i} \left\|y^{(l)}_{\mu-i}(t_\nu) - P^{[j-1](l)}_{\mu-i}(t_\nu)\right\| \right\} + const.h^{m+2-\mu}
\end{aligned}$$

Again using (1.3.10.b) in Lemma 1.2;

$$\begin{aligned}
\left\|\eta^{[j]}_{\mu,\nu} - y_\mu(t_\nu)\right\| \le & \sum_{i=1}^{\mu-2} \left\{ \sum_{l=i+1}^{m-\mu-j+1+i} const.h^{l-i+j} + \sum_{l=m-\mu-j+i+2}^{m-\mu+i} const.h^{m+2-\mu} \right. \\
& \left. + \sum_{l=m-\mu+1+i}^{m} const.h^{l+1-i} \right\} + const.h^{m+2-\mu} \\
& \le const.h^{j+1} + const.h^{m+2-\mu} \le const.h^{j+1}
\end{aligned}$$

where $m + 2 - \mu \ge j + 1, \;\; j_{\max} = \mu - 2$.

iii) For $j \ge \mu - 1$, using (1.3.10b) and (1.3.10c) in Lemma 1.2,

$$\begin{aligned}
\left\|\eta^{[j]}_{\mu,\nu} - y_\mu(t_\nu)\right\| \le & \sum_{i=1}^{\mu-2} \left\{ \sum_{l=i+1}^{m-\mu+1+i} const.h^{m+2-\mu} + \sum_{l=m+2-\mu+i}^{m} const.h^{l+1-i} \right\} \\
& + const.h^{m+2-\mu} \le const.h^{m+2-\mu}
\end{aligned}$$

To increase the order of convergence after $(\mu - 1)$ steps the following relation must be satisfied;

$$m + 2 - \mu \ge j + 1 \qquad \text{for} \qquad j = \mu - 1$$

and this gives the degree of the interpolation polynomial (i.e. $m \geq 2(\mu - 1)$)

Lemma 1.1 : For linear constant coefficient DAE's of index two, one gets the following orders;

$$\frac{d^k}{dt^k} P^{[0]}_{2,h}(t) - \frac{d^k}{dt^k} y_2(t) = \mathcal{O}(h) \quad k = 0, 1, \ldots, m \tag{1.3.4}$$

$$\frac{d^k}{dt^k} P^{[j]}_{2,h}(t) - \frac{d^k}{dt^k} y_2(t) = \begin{cases} \mathcal{O}(h^{m-k}) & k = 0, 1, \ldots, m-1 \\ \\ \mathcal{O}(h) & k = m \quad j \geq 1 \end{cases} \tag{1.3.5}$$

for $t \in [t^1, T] = I'$.

Proof : The error of the second component has the asymptotic expansions given in (1.2.3);

$$\eta_{2,\nu} - y_2(t_\nu) = \sum_{l=2}^{m} a_{l,1} h^{l-1} y_1^{(l)}(t_\nu) + R_{2,\nu} \quad \text{for} \quad \nu \geq \mu - 1$$

where $R_{2,\nu} = \mathcal{O}(h^{m+1})$.

For the proof of this Lemma, we follow similar consideration as in [6]. Let $R^{[0]}_{2,i,h}(t)$ be a mth degree polynomial which interpolates the $R_{\nu,2}$ values in the subinterval I_i,

$$R^{[0]}_{2,i,h}(t_\nu) = R_{2,\nu} \quad \text{for} \quad \nu = (i-1) \cdot m, \ldots, i \cdot m, \quad i = 2, 3, \ldots, n$$

and the function $R^{[0]}_{2,h}$ is defined as

$$R^{[0]}_{2,h}(t) = R^{[0]}_{2,i,h}(t), \quad t \in I_i, \quad i = 2, 3, \ldots, n$$

Let us define the auxiliary function

$$\psi^{[0]}_{2,h}(t) = y_2(t) + \sum_{l=2}^{m} a_{l,1} h^{l-1} y_1^{(l)}(t) + R^{[0]}_{2,h}(t)$$

From $R_{2,\nu} = \mathcal{O}(h^{m+1})$ and considering the following form of the interpolation

$$R^{[0]}_{2,i,h}(t) = \mathcal{L}_{(i-1)m}(t) R_{2,(i-1)m} + \cdots + \mathcal{L}_{i \cdot m}(t) R_{2,im} \quad i = 2, 3, \ldots, m$$

where

$$\frac{d^k}{dt^k} \mathcal{L}_{i \cdot m}(t) = \mathcal{O}(h^{-k}) \quad k = 0, 1, \ldots, m$$

we get;

$$\frac{d^k}{dt^k} R^{[0]}_{2,i,h}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0, 1, \ldots, m+1, \quad \text{and} \quad t \in (t^{i-1}, t^i)$$

it gives

$$\max_{t \in I'} \left\| \frac{d^k}{dt^k} R^{[0]}_{2,h}(t) \right\| \leq const. h^{m+1-k} \quad k = 0, 1, \ldots, m+1 \tag{1.3.6}$$

As a consequence, using (1.3.5), for $t \in I$ we have

$$\left\| \frac{d^k}{dt^k} \psi_{2,h}(t) - \frac{d^k}{dt^k} y_2(t) \right\| \leq \sum_{l=2}^{m} |a_{l,1}| h^{l-1} |y_1^{(l+k)}(t)| + \left\| \frac{d^k}{dt^k} R^{[0]}_{2,h}(t) \right\|$$

$$\leq \begin{cases} const. h & k = 0, 1, \ldots, m \\ \\ const. & k = m+1 \end{cases} \tag{1.3.7}$$

At the point t^i, where the kth derivative does not exist, (1.3.5) and (1.3.6) hold for the left and right kth derivative. Obviously $\psi^{[0]}_{2,h}(t_\nu) = \eta_{2,\nu}$ holds and we know that $P^{[0]}_{2,h}(t)$ interpolates $\eta_{2,\nu}$, that is $P^{[0]}_{2,h}(t)$ interpolates $\psi^{[0]}_{2,h}(t)$ at the points t_ν. Therefore we are able to apply a well known result in [17].

$$\max_{t \in I'} \left\| \frac{d^k}{dt^k} P^{[0]}_{2,h}(t) - \frac{d^k}{dt^k} \psi^{[0]}_{2,h}(t) \right\| \leq const. h^{m+1-k} \quad k = 0, 1, \ldots, m+1 \tag{1.3.8}$$

and together with (1.3.6),

$$\left\| \frac{d^k}{dt^k} P^{[0]}_{2,h}(t) - \frac{d^k}{dt^k} y_2(t) \right\| \leq \left\| \frac{d^k}{dt^k} P^{[0]}_{2,h}(t) - \frac{d^k}{dt^k} \psi^{[0]}_{2,h}(t) \right\| + \left\| \frac{d^k}{dt^k} \psi^{[0]}_{2,h}(t) - \frac{d^k}{dt^k} y_2(t) \right\|$$
$$\leq const. h^{m+1-k} + const. h \leq const. h \text{ for } k = 0, 1, \ldots, m$$

For the jth step of IDeC, all considerations are completely the same, let us define the auxiliary function $\psi^{[j]}_{2,h}(t)$ as

$$\psi^{[i]}_{2,h}(t) - y_2(t) = \sum_{l=2}^{m} a_{l,1} h^{l-1} \left(y_1^{(l)}(t) - P^{[j-1](l)}_{1,h}(t) \right) + R^{[j]}_{2,h}(t)$$

where $R^{[j]}_{2,h}(t) = R^{[j]}_{2,i,h}(t)$ for $t \in I_i$, $\quad i = 2, \ldots, n$, and $R^{[j]}_{2,i,h}(t)$ is a polynomial of degree m which interpolates the values $R_{2,\nu} - R^{[j]}_{2,\nu} = \mathcal{O}(h^{m+1})$. Obviously;

$$\frac{d^k}{dt^k} R^{[j]}_{2,h}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0, 1, \ldots, m+1$$

From the difference equation (1.1.2) and (1.1.4), it can be easily observed that, $\eta_{1,\nu}^{[j]} = y_1(t_\nu)$ then the interpolating polynomials $P_{1,h}^{[j]}(t)$ interpolates the exact values of $y_1(t_\nu)$. Therefore we can apply again (1.3.7) for $t \in (t^{i-1}, t^i)$

$$\frac{d^k}{dt^k} P_{1,i,h}^{[j]}(t) - \frac{d^k}{dt^k} y_1(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0, 1, \ldots, m+1 \tag{1.3.9}$$

and it gives

$$\max_{t \in I'} \left\| \frac{d^k}{dt^k} P_{1,h}(t) - \frac{d^k}{dt^k} y_1(t) \right\| \leq const.h^{m+1-k} \quad k = 0, 1, \ldots, m+1$$

Using the above relation, we get;

$$\left\| \frac{d^k}{dt^k} \psi_{2,h}^{[j]}(t) - \frac{d^k}{dt^k} y_2(t) \right\| \leq \sum_{l=2}^{m} |a_{l,1}| h^{l-1} \left\| y_1^{(l+k)}(t) - P_{1,h}^{[j-1](l+k)}(t) \right\| + \left\| R_{2,h}^{[j](k)}(t) \right\|$$

$$\leq \begin{cases} const.h^{m-k} & k = 0, 1, \ldots, m-1 \\ \\ const.h & k = m \end{cases} \tag{1.3.10}$$

Since $\psi_{2,h}^{[j]}(t_\nu) = P_{2,h}^{[j]}(t_\nu) = \eta_{2,\nu}^{[j]}$, from (1.3.7) for $t \in I'$

$$\left\| \frac{d^k}{dt^k} P_{2,h}^{[j]}(t) - \frac{d^k}{dt^k} \psi_{2,h}^{[j]}(t) \right\| \leq const.h^{m+1-k} \quad k = 0, 1, \ldots, m+1$$

then it gives (1.3.4b) together with (1.3.9)

Lemma 1.2 : For linear constant coefficient DAE's of index $\mu \geq 2$ one gets;

$$\frac{d^k}{dt^k} P_{\mu,h}^{[0]}(t) - \frac{d^k}{dt^k} y_\mu(t) = \mathcal{O}(h) \quad k = 0, 1, \ldots, m \tag{1.3.11}$$

for $1 \leq j < \mu - 1$

$$\frac{d^k}{dt^k} P_{\mu,h}^{[j]}(t) - \frac{d^k}{dt^k} y_\mu(t) = \begin{cases} \mathcal{O}(h^{j+1}) & 0 \leq k \leq m - \mu - j \\ \mathcal{O}(h^{m+2-\mu-k}) & m - j - \mu + 1 \leq k \leq m - \mu \\ \mathcal{O}(h) & m - \mu + 1 \leq k \leq m \end{cases} \tag{1.3.12}$$

for $j \geq \mu - 1$

$$\frac{d^k}{dt^k} P_{\mu,h}^{[j]}(t) - \frac{d^k}{dt^k} y_\mu(t) = \begin{cases} \mathcal{O}(h^{m+2-\mu-k}) & O \leq k \leq m - \mu + 1 \\ \mathcal{O}(h) & m - \mu + 2 \leq k \leq m \end{cases} \tag{1.3.13}$$

where $t \in [t^1, T] = I'$.

Proof : We will use induction. For $\mu = 2$, it is proved in Lemma 1.1 Suppose that they are true for $\bar{\mu} \leq \mu - 1$. Now we will prove for $\bar{\mu} = \mu$ using the same technique as Lemma 1.1 . Let us define the auxiliary function using the asymptotic expansion of the μth component.

$$\psi_{\mu,h}^{[0]}(t) = y_\mu(t) + \sum_{i=1}^{\mu-1} \sum_{l=i+1}^{m} a_{l,i} h^{l-i} y_{\mu-i}^{(l)}(t) + R_{\mu,h}^{[0]}(t) \tag{1.3.14}$$

where the mth degree polynomial $R_{\mu,h}^{[0]}(t)$ is defined as

$$R_{\mu,h}^{[0]}(t_\nu) = R_{\mu,\nu}^{[0]} = \mathcal{O}(h^{m+1}) \quad \nu = m, \ldots, nm \text{ so } \nu \geq \mu - 1$$

and

$$\frac{d^k}{dt^k} R_{\mu,h}^{[0]}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0, 1, \ldots, m \tag{1.3.15}$$

Using the (1.3.12) and (1.3.11), we have

$$\left\| \frac{d^k}{dt^k} \psi_{\mu,h}^{[0]}(t) - \frac{d^k}{dt^k} y_\mu(t) \right\| \leq \begin{cases} const.h & k = 0, 1, \ldots, m \\ \\ const. & k = m+1 \end{cases} \tag{1.3.16}$$

Using the same idea of the proof of Lemma 1.1 we get $P_{\mu,h}^{[0]}(t_\nu) = \psi_{\mu,h}^{[0]}(t_\nu) = \eta_{\mu,\nu}^{[0]}$

$$\left\| \frac{d^k}{dt^k} P_{\mu,h}^{[0]}(t) - \frac{d^k}{dt^k} \psi_{\mu,h}^{[0]}(t) \right\| \leq const.h^{m+1-k} \quad k = 0, 1, \ldots, m+1$$

for $t \in I'$.

This gives our assertion (1.3.10a) together with (1.3.13). For the jth step, the auxiliary function is defined as;

$$\psi_{\mu,h}^{[j]}(t) = y_\mu(t) + \sum_{i=1}^{\mu-1} \sum_{l=i+1}^{m} a_{l,i} h^{l-i} \left[y_{\mu-i}^{(l)}(t) - P_{\mu-i,h}^{[j-1](l)}(t) \right] + R_{\mu,h}^{[j]}(t)$$

where

$$R_{\mu,h}^{[j]}(t_\nu) = R_{\mu,\nu}^{[j]} \text{ and } \frac{d^k}{dt^k} R_{\mu,h}^{[j]}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0, 1, \ldots, m+1$$

Since $\psi_{\mu,h}^{[j]}(t_\nu) = \eta_{\mu,\nu}^{[j]} = P_{\mu,h}^{[j]}(t_\nu)$, we have

$$\max_{t \in I'} \left\| \frac{d^k}{dt^k} P_{\mu,h}^{[j]}(t) - \frac{d^k}{dt^k} \psi_{\mu,h}^{[j]}(t) \right\| \leq const.h^{m+1-k} \quad k = 0, 1, \ldots, m+1 \tag{1.3.17}$$

i) For $j=1$

$$\left\|\frac{d^k}{dt^k}\psi^{[1]}_{\mu,h}(t)-\frac{d^k}{dt^k}y_\mu(t)\right\| \le \sum_{i=1}^{\mu-2}\sum_{l=i+1}^{m}|a_{l,i}|h^{l-i}\left\|y^{(l+k)}_{\mu-i}(t)-P^{[0](l+k)}_{\mu-i,h}(t)\right\|$$

$$+\sum_{l=\mu}^{m}|a_{l,\mu-1}|h^{l+1-\mu}\left\|y^{(l+k)}_{1}(t)-P^{[0](l+k)}_{1,h}(t)\right\|+\mathcal{O}(h^{m+1-k})$$

$$\le \sum_{i=1}^{\mu-2} const.h\left\|y^{k+l-i}_{\mu-i}(t)-P^{[0](k+1+i)}_{\mu-i,h}(t)\right\|+\sum_{l=\mu}^{m} const.h^{m+2-\mu-k}+const.h^{m+1-k}$$

$$\le \begin{cases} const.h^2 & 0\le k\le m-\mu+1 \\ const.h & m-\mu+2\le k\le m \end{cases}$$

$$+\begin{cases} const.h^{m+2-\mu-k} & 0\le k\le m-\mu \\ const.h & m+1-\mu\le k\le m \end{cases}$$

$$\le \begin{cases} const.h^2 & 0\le k\le m-\mu \\ const.h & m-\mu+1\le k\le m \end{cases} \quad \text{for} \quad \mu\ge 2$$

ii) For $2\le j<\mu-1$, and $t\in[t^1,T]$

$$\left\|\frac{d^k}{dt^k}\psi^{[j]}_{\mu,h}(t)-\frac{d^k}{dt^k}y_\mu(t)\right\| \le \sum_{i=1}^{\mu-2} const.h\left\|y^{(i+1+k)}_{\mu-i}(t)-P^{[j-1](i+1+k)}_{\mu-i,h}(t)\right\|$$

$$+\sum_{l=i+1}^{m} const.h^{l+1-\mu}\left\|y^{(l+k)}_{1}(t)-P^{(l+k)[j-1](l+k)}_{1,h}(t)\right\|+\mathcal{O}(h^{m+1-k})$$

by substituting $j-1,\mu-i,i+1+k$ for j,μ,k in the relation (1.3.10b) we have;

$$y^{(i+1+k)}_{\mu-i}(t)-P^{[j-1](i+1+k)}_{\mu-i,h}(t)=\begin{cases} \mathcal{O}(h^j) & 0\le k\le m-\mu-j \\ \mathcal{O}(h^{m+1-\mu-k}) & m-\mu-j+1\le k\le m-\mu \\ \mathcal{O}(h) & m-\mu+1\le k\le m \end{cases}$$

then;

$$\left\|\frac{d^k}{dt^k}\psi^{[j]}_{\mu,h}(t)-\frac{d^k}{dt^k}y_\mu(t)\right\| \le \begin{cases} const.h^{j+1} & 0\le k\le m-\mu-j \\ const.h^{m+2-\mu-k} & m+1-j-\mu\le k\le m-\mu \\ const.h^2 & m-\mu+1\le k\le m \end{cases}$$

$$+\begin{cases} const.h^{m+2-\mu-k} & 0\le k\le m-\mu \\ const.h & m-\mu+1\le k\le m \end{cases}$$

$$\leq \begin{cases} const.h^{j+1} & 0 \leq k \leq m-\mu-j \\ const.h^{m+2-\mu-k} & m+1-j-\mu \leq k \leq m-\mu \\ const.h & m-\mu+1 \leq k \leq m \end{cases}$$

iii) For $j \geq \mu-1$, by substituting $\mu-i, i+1+k$ instead of μ, k in the relation (1.3.10c)

$$\frac{d^k}{dt^k} P_{\mu-i}^{[j-1]}(t) - \frac{d^k}{dt^k} y_{\mu-i}(t) = \begin{cases} \mathcal{O}(h^{m+1-\mu-k}) & 0 \leq k \leq m-\mu \\ \mathcal{O}(h) & m-\mu+1 \leq k \leq m \end{cases}$$

then;

$$\left\| \frac{d^k}{dt^k} \psi_{\mu,h}^{[j]}(t) - \frac{d^k}{dt^k} y_\mu(t) \right\| \leq \begin{cases} const.h^{m+2-\mu-k} & 0 \leq k \leq m-\mu \\ const.h^2 & m-\mu+1 \leq k \leq m \end{cases}$$

$$+ \begin{cases} const.h^{m+2-\mu-k} & 0 \leq k \leq m-\mu \\ const.h & m-\mu+1 \leq k \leq m \end{cases}$$

$$\leq \begin{cases} const.h^{m+2-\mu-k} & 0 \leq k \leq m-\mu+1 \\ const.h & m-\mu+2 \leq k \leq m \end{cases}$$

The relations (1.3.15), (1.3.16), (1.3.17) give the our assertion (1.3.10a, b, c) together with (1.3.14).

Chapter 2

IDeC METHODS FOR NONLINEAR INDEX ONE PROBLEMS

The linearly implicit Euler or semi-implicit Euler discretization of an ordinary differential equation

$$y' = f(y)$$

corresponds to one Newton iteration for the nonlinear system which arises in the implicit Euler discretization

$$(I - h \cdot f'(y_0))(y_{n+1} - y_n) = hf(y_n)$$

Applying this method to the system (0.11) and putting $\epsilon = 0$ one gets the formulation of the linearly-implicit Euler method to the nonlinear semi-explicit index one DAE system (0.2) :

$$\begin{bmatrix} I - hA_1 & -hA_2 \\ -hA_3 & -hA_4 \end{bmatrix} \begin{bmatrix} y_{n+1} - y_n \\ z_{n+1} - z_n \end{bmatrix} = h \begin{bmatrix} f(y_n, z_n, t_{n+1}) \\ g(y_n, z_n, t_{n+1}) \end{bmatrix}$$

where

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix}_{(y_0, z_0)}$$

In the following the linearly implicit Euler method will be formulated for defect correction steps. The convergence analysis is done using the perturbed asymptotic expansions given in ([4]) for the global connection strategy and constant steps sizes. The perturbed asymptotic expansions of the Neighboring Problems in the defect correction steps are developed in the Appendix.

The main result of this chapter is that at each defect correction step the order of convergence of the differential and algebraic components of (0.2) is increased by one. The maximal attainable convergence order is limited by the degree m of the interpolation polynomial.

2.1 Formulation of the IDeC method for nonlinear semi-explicit index one systems

In this section, we will describe briefly how defect correction can be applied to solve the non-autonomous index one problem

$$\begin{aligned} y' &= f(y,z,t) \qquad y(0) = y_0 \\ 0 &= g(y,z,t) \qquad z(0) = z_0 \qquad t \in [0,T] \end{aligned} \tag{2.1.1}$$

where f and g are vector-valued functions of dimension s_1, s_2 and f, g are sufficiently smooth. The initial conditions are consistent i.e. $g(y_0, z_0, t_0) = 0$ and $g_z^{-1}(y,z)$ exists

$$\|g_z^{-1}\| \leq \ const. \quad \text{for } t \in [0,T].$$

This system can be interpreted as a limiting case of singularly perturbed systems considered in [13]. By the linearly implicit Euler method the Jacobian is evaluated at the initial values and held constant in the whole integration interval. On the uniform grid $G := \{t_\nu = \nu h, \ \ \nu = 0, 1, \ldots, n \cdot m, \quad n, m \in N\}$ with the stepsize $h = T/m \cdot n$, the linearly implicit Euler method applied to the Original Problem (OP) (2.1.1) reads

$$\begin{bmatrix} I - hA_1 & -hA_2 \\ -hA_3 & -hA_4 \end{bmatrix} \begin{bmatrix} \eta_{\nu+1} - \eta_\nu \\ \xi_{\nu+1} - \xi_\nu \end{bmatrix} = h \begin{bmatrix} f(\eta_\nu, \xi_\nu, t_{\nu+1}) \\ g(\eta_\nu, \xi_\nu, t_{\nu+1}) \end{bmatrix} \tag{2.1.2}$$

where

$$A_1 = \frac{\partial f}{\partial y}(y_0, z_0, 0) \;\; A_2 = \frac{\partial f}{\partial z}(y_0, z_0, 0) \;\; A_3 = \frac{\partial g}{\partial y}(y_0, z_0, 0) \;\; A_4 = \frac{\partial g}{\partial z}(y_0, z_0, 0) \tag{2.1.3}$$

We get the following numerical solution;

$$\eta := (\eta_0, \ldots, \eta_\nu, \ldots, \eta_{n \cdot m}) \quad \xi := (\xi_0, \ldots, \xi_\nu, \ldots, \xi_{n \cdot m}) \tag{2.1.4}$$

Interpolation of η and ξ by a suitable vector-valued interpolation functions $P_h(t)$ and $Q_h(t)$ (i.e. $P_h(t_\nu) = \eta_\nu$ and $Q_h(t) = \xi_\nu, \;\; \nu = 0, 1, \ldots, n \cdot m$) yields the defect

$$\begin{aligned} d_{1,h} &= P_h' - f(P_h, Q_h, t) \\ d_{2,h} &= -g(P_h, Q_h, t) \end{aligned} \tag{2.1.5}$$

By adding the defect $d_{1,h}(t), \;\; d_{2,h}(t)$ to the right hand side of the (OP) (2.1.1), we construct the Neighboring Problem (NP) whose exact solutions are $P_h(t)$ and $Q_h(t)$

$$\begin{aligned} y' &= f(y, z, t) + d_{1,h}(t) \qquad y(0) = y_0 \\ 0 &= g(y, z, t) + d_{2,h}(t) \qquad z(0) = z_0 \end{aligned} \tag{2.1.6}$$

Solving the (NP) (2.1.6) with the Implicit Euler Method on the same grid leads to the following numerical solution

$$\begin{aligned} \pi &:= (\pi_0, \pi_1, \ldots, \pi_\nu, \ldots, \pi_{n \cdot m}) \\ \omega &:= (\omega_0, \omega_1, \ldots, \omega_\nu, \ldots, \omega_{n \cdot m}) \end{aligned}$$

where π_ν and ω_ν are the approximation to the exact solutions $P_h(t)$ and $Q_h(t)$ of (2.1.6) at the point t_ν.

$$\begin{bmatrix} I - hA_1 & -hA_2 \\ -hA_3 & -hA_4 \end{bmatrix} \begin{bmatrix} \pi_{\nu+1} - \pi_\nu \\ \omega_{\nu+1} - \omega_\nu \end{bmatrix}$$

$$= h \begin{bmatrix} f(\pi_\nu, \omega_\nu, t_{\nu+1}) + P'_h(t_{\nu+1}) - f(P_h(t_{\nu+1}), Q_h(t_{\nu+1}), t_{\nu+1}) \\ g(\pi_\nu, \omega_\nu, t_{\nu+1}) - g(P_h(t_{\nu+1}), Q_h(t_{\nu+1}), t_{\nu+1}) \end{bmatrix} \tag{2.1.7}$$

Then we compute an improved solution $\eta_\nu^{[1]}$ and $\xi_\nu^{[1]}$ of (OP) which is given by

$$\eta_\nu^{[1]} = \eta_\nu - (\pi_\nu - P_h(t_\nu))$$
$$\xi_\nu^{[1]} = \xi_\nu - (\omega_\nu - Q_h(t_\nu)) \quad (2.1.8)$$

and so on.

2.2 Asymptotic Behavior of the Linearly Implicit Euler Method and IDeC - Methods

In this section, we analyze the asymptotic behavior of the sub-class of IDeC-methods for the non-linear index one problem. Our analysis is restricted to the global connection strategy. At the beginning, we will transform the (OP) and (NP) into semi-explicit form in order to apply the theory of asymptotic expansions of the linearly implicit Euler method developed in [4] and [13].

We introduce the additional variable $y_{s_{1+1}} = t$, therefore the semi-explicit form for the (OP) is in autonomous form,

$$\begin{aligned} y' &= f(y,z) \qquad y(0) = y_0 \\ 0 &= g(y,z) \qquad z(0) = z_0 \quad t \in [0,T] \end{aligned} \quad (2.2.1)$$

where $y_{s_1+1} \in \mathbb{R}^{s_1+1}$, $f \in \mathbb{R}^{s_1+1}$ and $f_{s_1+1}(y,z) = 1 \quad y_{s_1+1}(0) = 0$. We consider the following subintervals of $[0,T]$.

$I_i = [t^{i-1}, t^i]$, $i = 1,2,\ldots,n$ with $t^i = i \cdot m \cdot h$ and the interpolating polynomials

$$P_i^{[j]} : \mathbb{R} \rightarrow \mathbb{R}^{s_1+1} \qquad Q_i^{[j]} : \mathbb{R} \rightarrow \mathbb{R}^{s_2}$$

$$P_i^{[j]}(t_\nu) = \eta_\nu^{[j]} \quad Q_i^{[j]}(t_\nu) = \xi_\nu^{[j]} \text{ for } \nu = (i-1)\cdot m, \ldots, i\cdot m, \quad j = 0,1,\ldots,j_{\max}-1 \quad (2.2.2)$$

The interpolating functions $P_h^{[j]}(t)$ and $Q_h^{[j]}(t)$ are now defined as

$$P_h^{[j]}(t) = P_i^{[j]}(t) \quad Q_h^{[j]}(t) = Q_i^{[j]}(t), \quad t \in I_i \quad i = 1,2,\ldots,n \quad (2.2.3)$$

According to the new additional variable, the (NP) is defined piecewise as an initial value problem in the interpolating intervals ;

$$
\begin{aligned}
y' &= F_i^{[j]}(y,z) \\
0 &= G_i^{[j]}(y,z) \quad i = 1,2,\ldots,n \quad j = 0,1,\ldots,j_{\max}-1
\end{aligned} \tag{2.2.4}
$$

$$
y(t^{i-1}) = \begin{cases} y_0 & i = 1 \\ P_{i-1}^{[j]}(t^{i-1}) & i = 2,3,\ldots,n \end{cases} \tag{2.2.5a}
$$

$$
z(t^{i-1}) = \begin{cases} z_0 & i = 1 \\ Q_{i-1}^{[j]}(t^{i-1}) & i = 2,\ldots,n \end{cases} \tag{2.2.5b}
$$

with the exact solutions $P_i^{[j]}(t)$ and $Q_i^{[j]}(t)$ where $F_{i,s_1+1}^{[j]}(y,z) = 1, \quad y \in \mathbb{R}^{s_1+1}$ and $P_h^{[j]}(t) \in \mathbb{R}^{s_1+1}$ and the $(s_1+1)-th$ component of the solution $(P_i^{[j]}(t))_{s_1+1} = t$,

$$(d_{1,i}^{[j]}(y_{s_1+1}))_{s_1+1} = 0$$

After the transformation of (NP) into semi explicit form, the perturbation $d_{1,i}^{[j]}(t)$ and $d_{2,i}^{[j]}(t)$ are functions of y_{s_1+1} only;

$$
\begin{aligned}
d_{1,i}^{[j]}(t) &= d_{1,i}^{[j]}(y_{s_1+1}) = P_i^{[j]'}(y_{s_1+1}) - f(P_i^{[j]}(y_{s_1+1}), Q_i^{[j]}(y_{s_1+1}), y_{s_1+1}) \\
d_{2,i}^{[j]}(t) &= d_{2,i}^{[j]}(y_{s_1+1}) = -g(P_i^{[j]}(y_{s_1+1}), Q_i^{[j]}(y_{s_1+1}), y_{s_1+1})
\end{aligned}
$$

In our later discussions, we will consider only the systems (2.2.1) and (2.2.4). In according to the semi-explicit form the asymptotic expansions of the global errors of the methods

$$
B \begin{bmatrix} y_{\nu+1} - y_\nu \\ z_{\nu+1} - z_\nu \end{bmatrix} = h \begin{bmatrix} f(y_\nu, z_\nu) \\ g(y_\nu, z_\nu) \end{bmatrix} \tag{2.2.6}
$$

and

$$
B \begin{bmatrix} \pi_{\nu+1}^{[j]} - \pi_\nu^{[j]} \\ \omega_{\nu+1}^{[j]} - \omega_\nu^{[j]} \end{bmatrix} = h \begin{bmatrix} F_i^{[j]}(\pi_\nu^{[j]}, \omega_\nu^{[j]}) \\ G_i^{[j]}(\pi_\nu^{[j]}, \omega_\nu^{[j]}) \end{bmatrix} \tag{2.2.7}
$$

are investigated in the Appendix.

These solutions have jumps in the first derivatives at the endpoints of the interpolation intervals as in case of linear problems in Chapter 1.

Theorem 2.1 : Let f and g be sufficiently smooth. For an IDeC method based on Linearly Implicit Euler Method of order 1 and on piecewise interpolation with polynomial of degree m, for the index one problem, we have;

$$\eta_\nu^{[j]} - y(t_\nu) = \mathcal{O}(h^{j+1})$$
$$\xi_\nu^{[j]} - z(t_\nu) = \mathcal{O}(h^{j+1}) \qquad j = 0, 1, \ldots, m-1 \tag{2.2.8}$$

In this situation further steps of the iteration do not increase the asymptotic order of the approximation.

Proof: From the iteration of IDeC methods; we write

$$\eta_\nu^{[j]} - y(t_\nu) = (\eta_\nu^{[0]} - y(t_\nu)) - (\pi_\nu^{[j-1]} - P_h^{[j-1]}(t_\nu)) \tag{2.2.9a}$$
$$\xi_\nu^{[j]} - z(t_\nu) = (\xi_\nu^{[0]} - z(t_\nu)) - (\omega_\nu^{[j-1]} - Q_h^{[j-1]}(t_\nu)) \tag{2.2.9b}$$

Using the asymptotic error formula (A.1,2) and (A.19a,b), by subtraction we get

$$\eta_\nu^{[j]} - y(t_\nu) = \sum_{l=1}^{m} h^l (a_l(t_\nu) - a_{l,h}^{[j-1]}(t_\nu))$$
$$+ \sum_{l=2}^{m} h^l (\alpha_\nu^l - \alpha_{\nu,h}^{[j-1]l}) + (R_\nu - R_{\nu,h}^{[j-1]}) \tag{2.2.10a}$$
$$\xi_\nu^{[j]} - z(t_\nu) = \sum_{l=1}^{m} h^l (b_l(t_\nu) - b_{l,h}^{[j-1]}(t_\nu))$$
$$+ \sum_{l=2}^{m} h^l (\beta_\nu^l - \beta_{\nu,h}^{[j-1]l}) + (\tilde{R}_\nu - \tilde{R}_{\nu,h}^{[j-1]}) \tag{2.2.10b}$$

where

$$\|R_\nu - R_{\nu,h}^{[j-1]}\| \leq const.\ \ h^{m+1} \quad \nu = 0, 1, 2, \ldots, n \cdot m \tag{2.2.11a}$$
$$\|\tilde{R}_\nu - \tilde{R}_{\nu,h}^{[j-1]}\| \leq const.\ \ h^{m+1} \quad j = 1, 2, \ldots, j_{\max} + 1 \tag{2.2.11b}$$

and *const.* denotes a constant which depends on the (OP) and (NP) respectively, but is independent of h and ν. Then (2.2.9a) would lead to

$$\|\eta_\nu^{[j]} - y(t_\nu)\| \leq \|\eta_\nu^{[0]} - y(t_\nu)\| + \|\pi_\nu^{[j-1]} - P_h^{[j-1]}(t_\nu)\|$$
$$\leq \sum_{l=1}^{m} h^l \|a_l(t_\nu) - a_{l,h}^{[j-1]}(t_\nu)\| + \sum_{l=2}^{m} h^l \|\alpha_\nu^l - \alpha_{\nu,h}^{[j-1]l}\| + \|R_\nu - R_{\nu,h}^{[j-1]}\|$$

From Lemma 2.9;

$$\|a_l(t_\nu) - a_{l,h}^{[j-1]}(t_\nu)\| \le \max_{t\in[0,T]} \|a_l(t) - a_{l,h}^{[j-1]}(t)\| \le \ const.\ h^j \quad \text{for} \quad l = 1,2,\ldots,m-j$$
$$\|a_l(t_\nu) - a_{l,h}^{[j-1]}(t_\nu)\| \le \max_{t\in[0,T]} \|a_l(t) - a_{l,h}^{[j-1]}(t)\| \le \ const.\ h^{m-l} \quad \text{for} \quad l = m-j+1,\ldots,m$$
$$\|\alpha_\nu^l - \alpha_{\nu,h}^{[j-1]l}\| \le \ const.\ \ h^j \qquad l = 1,2,\ldots,m-j$$
$$\|\alpha_\nu^l - \alpha_{\nu,h}^{[j-1]l}\| \le \ const.\ \ h^{m-l} \qquad l = m-j+1,\ldots,m$$

We can separate the summation terms in two parts, and substitute the above relations.

$$\|\eta_\nu^{[j]} - y(t_\nu)\| \le \sum_{l=1}^{m-j} \ const.\ \ h^{j+l} + \sum_{l=m-j+1}^{m} \ const.\ \ h^m + \mathcal{O}(h^{m+1})$$
$$\le const.\ h^{j+1} + const.\ h^{j+2} + \ldots + const.\ h^m + \mathcal{O}(h^{m+1})$$

and in a similar way, using Lemma 2.9 we obtain

$$\|\xi_\nu^{[j]} - z(t_\nu)\| \le \sum_{l=1}^{m} \|b_l(t_\nu) - b_{l,h}^{[j-1]}(t_\nu)\| h^l + \sum_{l=2}^{m} \|\beta_\nu^l - \beta_{\nu,h}^{[j-1]l}\| h^l + \|\tilde{R}_\nu - \tilde{R}_{\nu,h}^{[j-1]}\|$$
$$\le const.\ \ h^{j+1} + const.\ \ h^{j+2} + \ldots + const.\ \ h^m + \mathcal{O}(h^{m+1})$$

It is easily seen that the maximum attainable correction step j satisfies the relation $j_{\max} + 1 = m$ for $j = 0,1,\ldots,j_{\max}$, this would imply that

$$\|\eta_\nu^{[j]} - y(t_\nu)\| \le \ const.\ \ h^{j+1} \text{ and } \|\xi_\nu^{[j]} - z(t_\nu)\| \le \ const.\ \ h^{j+1}, \quad j = 0,1,\ldots,m-1$$

Lemma 2.1 For non-linear index-one problems (2.2.1)

$$\max_{t\in[0,T]} \|\frac{d^k}{dt^k} P_h^{[0]}(t) - \frac{d^k}{dt^k} y(t)\| \le \ \ const.\ h \ \text{ for } \ k = 0,1,\ldots,m \quad (2.2.12a)$$
$$\max_{t\in[0,T]} \|\frac{d^k}{dt^k} P_h^{[0]}(t) - \frac{d^k}{dt^k} y(t)\| \le \ \ const. \quad \text{for} \ \ k = m+1 \quad (2.2.12b)$$

$$\max_{t\in[0,T]} \|\frac{d^k}{dt^k} Q_h^{[0]}(t) - \frac{d^k}{dt^k} z(t)\| \le \ \ const.\ \ h \ \text{ for } \ k = 0,1,\ldots,m \quad (2.2.13a)$$
$$\max_{t\in[0,T]} \|\frac{d^k}{dt^k} Q_h^{[0]}(t) - \frac{d^k}{dt^k} z(t)\| \le \ \ const. \quad \text{for} \ \ k = m+1 \quad (2.2.13b)$$

where $P_h^{[0]}(t)$ and $Q_h^{[0]}(t)$ are the m-th degree polynomials which satisfy the properties (2.2.2) and (2.2.3).

Proof: We first define the functions

$$R_h^{[0]}(t) : [0,T] \to \mathbb{R}^{s_1+1} \text{ with } R_h^{[0]}(t) = R_i^{[0]}(t), \quad i = 1,2,\ldots,n \tag{2.2.14}$$

where $R_i^{[0]}(t)$ is a polynomial of degree m which interpolates the R_ν values in the equation (A.1)

$$R_i^{[0]}(t_\nu) = R_\nu \quad \nu = (i-1)\cdot m, \ldots, i\cdot m \quad i = 1,2,\ldots,n$$

From (A.18), we can write

$$\alpha_\nu^l = N_l(\alpha_0^2, \ldots, \alpha_0^l,\ \beta_0^2, \ldots, \beta_0^l,\ y(t_\nu - \nu\, h), y(t_\nu - (\nu-1)h)$$
$$,\ \ldots, y(t_\nu - h),\ z(t_\nu - \nu\, h), \ldots, z(t_\nu - h))$$

where $h = \frac{t_\nu - t_0}{\nu}$. Let $h(t) = \frac{t - t_0}{\nu}$. A new continuous function $\alpha^l(t)$ can now be defined such that

$$\alpha^l(t_\nu) = \alpha_\nu^l \tag{2.2.15}$$

Now, we define the auxiliary function

$$\psi_h^{[0]}(t) = y(t) + \sum_{l=1}^{m} h^l a_l(t) + \sum_{l=2}^{m} h^l \alpha^l(t) + R_h^{[0]}(t) \tag{2.2.16}$$

with

$$\psi_h^{[0]} = \psi_i^{[0]} \quad , i = 1,2,\ldots,n \tag{2.2.17}$$

From $R_\nu = \mathcal{O}(h^{m+1})$ it follows after some simple considerations ,

$$\frac{d^k}{dt^k} R_i^{[0]}(t) = \mathcal{O}(h^{m+1-k}) \ \ k = 0,1,\ldots,m+1 \ \text{ for all } \ i \in N \ \text{ and } \ t \in (t^{i-1}, t^i) \tag{2.2.18}$$

$$\frac{d^k}{dt^k} R_h^{[0]}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0,1,\ldots,m+1 \tag{2.2.19}$$

As a consequence, we get

$$\frac{d^k}{dt^k}\psi_h^{[0]}(t) - \frac{d^k}{dt^k}y(t) = \sum_{l=1}^{m} h^l \frac{d^k}{dt^k} a_l(t) + \sum_{l=2}^{m} h^l \frac{d^k}{dt^k}\alpha^l(t) + \frac{d^k}{dt^k} R_h^{[0]}(t)$$
$$= \mathcal{O}(h) \quad \text{for} \quad k = 0,1,\ldots,m \tag{2.2.20}$$

and

$$\frac{d^{m+1}}{dt^{m+1}}\psi_h^{[0]}(t) - \frac{d^{m+1}}{dt^{m+1}}y(t) = \mathcal{O}(1)$$

Remark: At the points t^i, where the k^{th} derivative does not exist, (2.2.19) and (2.2.20) hold for the left and right k^{th} derivative. Obviously $\psi_h^{[0]}(t_\nu) = \eta_\nu^{[0]}$, it means $P_h^{[0]}(t)$ interpolates $\psi_h^{[0]}(t)$ at the points t_ν

$$P_h^{[0]}(t_\nu) = \psi_h^{[0]}(t_\nu) = \eta_\nu^{[0]} \tag{2.2.21}$$

therefore we are able to apply a result due to [17]

$$\frac{d^k}{dt^k}P_i^{[0]}(t) - \frac{d^k}{dt^k}\psi_i^{[0]}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0,1,\ldots,m+1, \text{ for } \forall i \text{ and } t \in (t^{i-1}, t^i) \tag{2.2.22}$$

which together with (2.2.20) implies

$$\begin{aligned}\frac{d^k}{dt^k}P_h^{[0]}(t) - \frac{d^k}{dt^k}y(t) &= \left(\frac{d^k}{dt^k}P_h^{[0]}(t) - \frac{d^k}{dt^k}\psi_h^{[0]}(t)\right) + \left(\frac{d^k}{dt^k}\psi_h^{[0]}(t) - \frac{d^k}{dt^k}y(t)\right)\\ &= \mathcal{O}(h), \quad k = 0,1,\ldots,m\end{aligned}$$

and

$$\frac{d^{m+1}}{dt^{m+1}}P_h^{[0]}(t) - \frac{d^{m+1}}{dt^{m+1}}y(t) = O(1) \quad \forall t \in [0,T] \tag{2.2.23}$$

these give our assertion (2.2.12a) and (2.2.12b). The inequalities (2.2.13a) and (2.2.13b) can be proved similarly.

Lemma 2.2: Let f and g be sufficiently smooth on $[0,T]$ and K_s, $K_{s,i}^{[0]}$ be matrices whose entries are functions of y, z and $P_i^{[0]}(t)$, $Q_i^{[0]}(t)$ respectively, then

$$\|K_s(y,z) - K_{s,i}^{[0]}(P_i^{[0]}, Q_i^{[0]})\| \leq \text{ const. } h \tag{2.2.24}$$

$$\|K_s(y,z)y^{(l)}(t) - K_{s,i}^{[0]}(P_i^{[0]}, Q_i^{[0]})P_i^{[0](l)}(t)\| \leq \text{ const. } h \tag{2.2.25}$$

$$\|K_s(y,z)z^{(l)}(t) - K_{s,i}^{[0]}(P_i^{[0]}, Q_i^{[0]})Q_i^{[0](l)}(t)\| \leq \text{ const. } h \tag{2.2.26}$$

where $\|\cdot\|$ corresponds to the induced matrix norm of the maximum norm for vectors, for all s and $t \in I_i$, $l = 1,2,\ldots,m$.

Proof: Since f and g are sufficiently smooth , then the entries of $K_s(y,z)$ and $K_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]})$ are sufficiently smooth and bounded. Let $c_s(y,z)$ be any entry of $K_s(y,z)$ and $c_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]})$ be the corresponding entry of $K_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]})$. Then

$$\begin{aligned} c_s(y,z) - c_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]}) &= \frac{\partial c_s}{\partial y}(\lambda y + (1-\lambda)P_i^{[0]}, \lambda z + (1-\lambda)Q_i^{[0]}) \cdot (y - P_i^{[0]}) \\ &+ \frac{\partial c_s}{\partial z}(\lambda y + (1-\lambda)P_i^{[0]}, \lambda z + (1-\lambda)Q_i^{[0]}) \cdot (z - Q_i^{[0]}) \\ &, \text{ with } 0 < \lambda < 1 \end{aligned}$$

Using the inequality (2.2.12a), (2.2.13a) and the boundedness of c_y and c_z, we get

$$\|c_s(y,z) - c_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]})\| \leq const.\ h \quad \forall s \ , \ t \in I_i,$$

Now, the following result can be obtained

$$\|K_s(y,z) - K_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]})\| \leq \ const.\ h \quad \forall s, \ \forall i \text{ and } t \in I_i,$$

$$\begin{aligned} \|K_s(y,z)y^{(l)}(t) - K_{s,i}^{[0]}(P_i^{[0]},Q_i^{[0]})P_i^{[0](l)}(t)\| &= \|(K_s - K_{s,i}^{[0]})y^{(l)}(t) + K_{s,i}^{[0]}(y^{(l)}(t) - P_i^{[0](l)}(t))\| \\ &\leq \|K_s - K_{s,i}\| \ \|y^{(l)}(t)\| + \|K_{s,i}^{[0]}\| \ \|y^{(l)}(t) - P_i^{[0](l)}(t)\| \\ &\leq \ const.\ h + \ const.\ h \ , \ l = 1,2,\ldots,m \text{ and } \forall s \end{aligned}$$

The inequality (2.2.26) can be proved similarly. We will use the following lemmas taken from [6] in order to prove Lemma 2.5, 2.6, 2.9.

Lemma 2.3 Let $f,g : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be arbitrary functions defined on a region $[a,b] \times G$ with $G \subset R^n$ and $v,\omega : R^1 \to R^n$ be continuous functions on $[a,b]$, where $(t,v(t),t,\omega(t)) \in ([a,b]) \times G)$ for $t \in [a,b]$ with $v(t)$ being the solution of $v' = f(t,v)$, $v(a) = v_a$ and $\omega(t)$ is the solution of $\omega' = g(t,\omega)$, $\omega(a) = \omega_a$. Assume further that $f,g \in c^0([a,b] \times G)$ and that f is bounded and

$$\|f(t,y_1) - f(t,y_2)\| \leq L\|y_1 - y_2\| \qquad (2.2.27)$$

$$\|f(t,y) - g(t,y)\| \leq \delta \ \text{ for } \ (t,y) \in [a,b] \times G \qquad (2.2.28)$$

Under these assumptions

$$\|\nu(t) - \omega(t)\| \leq \|\nu_a - \omega_a\|e^{L(t-a)} + \frac{\delta}{k}(e^{L(t-a)} - 1) \ \text{ for } \ t \in [a,b] \qquad (2.2.29)$$

It is not possible to apply Lemma 2.3 immediately to the variational equations (A.21) since the solutions of the variational equations of the NP's have jumps in their first derivative at the points t^i. Therefore we need a slight modification of Lemma 2.3. whose proff can be found in [6].

Lemma 2.4: Replace the second IVP of Lemma 2.3 by a piecewise problem defined on the partition of $[a,b]$ in subintervals $I_i = [t^{i-1}, t^i]$:

$$\omega'_i = g_i(t, \omega_i), \;\; t \in I_i, \;\; \omega_i(t^{i-1}) = \begin{cases} \omega_a & i = 1 \\ \omega_{i-1}(t^{i-1}) & i \geq 2 \end{cases}$$

g_i is defined on $I_i \times G$ and ω_i is defined on I_i. If instead of (2.2.28) the following inequality is satisfied:

$$\|f(t,y) - g_i(t,y)\| \leq \delta \;\text{ for }\; (t,y) \in I_i \times G, \;\; i = 1, 2, \ldots, n \tag{2.2.30}$$

where δ is independent of i, then (2.2.29) is again valid.

Proof: (See [6]).

The variational equations of the NP's (A.21) are defined in a piecewise fashion. For $h \to 0$ the number of the points t^i tends to infinity. Since Lemma 2.4 holds for an arbitrary number of intervals I_i, it may be applied immediately to the variational equations for $a_l(t)$ and $a_{l,h}^{[j]}(t)$, $j = 0, 1, \ldots, j_{\max}$. **Lemma 2.5:** Let $a_1(t)$ and $a_{1,h}^{[0]}(t) = a_{1,i}^{[0]}(t)$ for $i = 1, 2, \ldots, n$ satisfy the differential equations (A.13) and (A.21) with $l = 1$, with the initial conditions (A.15a) and (A.23a) then

$$\max_{t \in [0,T]} \|a_1(t) - a_{1,h}^{[0]}(t)\| \leq \; const. \; h \tag{2.2.31}$$

Proof:

$$a'_1(t) = \varphi_1(t, a_1) \qquad\qquad t \in [0, T]$$

$$a_{1,i}^{[0]\prime}(t) = \varphi_{1,i}(t, a_{1,i}^{[0]}(t))$$

$$a_1(0)=0 \quad a_{1,i}^{[0]}(t^{i-1})=\begin{cases}0 & i=1\\ a_{1,i-1}(t^{i-1}) & i=2,\ldots,n\end{cases}$$

For the arbitrary $a_1(t)$ and $\bar{a}_1(t)\in G$, $\varphi_1(t,a_1)-\varphi_1(t,\bar{a}_1)=K_1(y,z)(a_1(t)-\bar{a}_1(t))$ and $\|K_1(y,z)\|$ is bounded therefore

$$\|\varphi_1(t,a_1)-\varphi_1(t,\bar{a}_1)\|\leq \ const. \ \|a_1-\bar{a}_1\| \tag{2.2.32}$$

and therefore φ_1 is the Lipschitz function. Now we will show that

$$\|\varphi_1(t,a_1)-\varphi_{1,i}^{[0]}(t,a_1)\|\leq \ const. \ \ h \text{ for } (t,a_1)\in(I_i\times G) \tag{2.2.33}$$

where the constant is independent of i. Another form of (2.2.33) is

$$\begin{aligned}&\|(K_1-K_{1,i}^{[0]})a_1(t)+(K_2y'(t)-K_{2,i}^{[0]}P_i^{[0]\prime}(t))\\ &(K_3z'(t)-K_{3,i}^{[0]}Q_i^{[0]\prime}(t))+(\frac{1}{2}y''(t)-\frac{1}{2}P_i^{[0]\prime\prime}(t))\|\\ &\leq\|K_1-K_{1,i}^{[0]}\|\ \|a_1(t)\|\\ &+\|K_2y'-K_{2,i}^{[0]}P_i^{[0]\prime}\|+\|K_3z'-K_{3,i}^{[0]}Q_i^{[0]\prime}\|+\frac{1}{2}\|y''(t)-P_i^{[0]\prime\prime}(t)\|\end{aligned} \tag{2.2.34}$$

it follows immediately from Lemma 2.1 and Lemma 2.2 that

$$\|\varphi_1(t,a_1)-\varphi_{1,i}^{[0]}(t,a_1)\|\leq \ const. \ \ h\|a_1(t)\|+ \ \ const.\ h$$

where the constants are independent of h and i. Since all solutions $a_1(t)$ are in a certain finite region, $\|a_1(t)\|\leq S$ then we get the inequality (2.2.30) the assumptions of Lemma 2.4 holds, then

$$\max_{t\in[0,T]}\|a_1(t)-a_{1,h}^{[0]}(t)\|\leq \ const. \ \ h$$

Lemma 2.6: Let $a_l(t)$ and $a_{l,i}^{[0]}(t)$ satisfy the variational equations (A.13) and (A.21) with their initial conditions, then

$$\max_{t\in[0,T]}\|a_l(t)-a_{l,h}^{[0]}(t)\|\leq \ \ const. \ \ h \quad l=1,2,\ldots,m-1 \tag{2.2.35a}$$

$$\max_{t\in[0,T]}\|a_m(t)-a_{m,h}^{[0]}(t)\|\leq \ const. \tag{2.2.35b}$$

Proof: When $l = 1$, the inequality (2.2.35a) is proved in Lemma 2.5 .Suppose the inequality is true for $1 \le l < m-1$. Let $l = m-1$, then the differential equations is of the form.

$$a'_{m-1}(t) = \varphi_{m-1}(t, a_{m-1}(t))$$
$$a^{[0]'}_{m-1,i}(t) = \varphi^{[0]}_{m-1,i}(t, a^{[0]}_{m-1,i}(t))$$

where $\|\varphi_{m-1}(t, a_{m-1}) - \varphi_{m-1}(t, \bar{a}_{m-1})\| \le const.\ \|a_{m-1} - \bar{a}_{m-1}\|$. From the equality of φ_{m-1} and $\varphi^{[0]}_{m-1,i}$, we write;

$$\|\varphi_{m-1}(t, a_{m-1}) - \varphi^{[0]}_{m-1,i}(t, a_{m-1})\| \le \|K_1 - K^{[0]}_{1,i}\| \cdot \|a_{m-1}(t)\| +$$
$$\frac{1}{(m-1)!}\|K_2 y^{(m-1)}(t) - K^{[0]}_{2,i} P^{[0](m-1)}_i(t)\| + \frac{1}{(m-1)!}\|K_3 z^{(m-1)}(t) - K^{[0]}_{3,i} Q^{[0](m-1)}_i(t)\|$$
$$+\frac{1}{m!}\|y^{(m)}(t) - P^{[0](m)}_i(t)\| + \|E_{m-1} - E^{[0]}_{m-1,i}\|$$

Since the smoothness of E_{m-1}, and $E_{m-1,i}$ using the mean value theorem

$$\|E_{m-1} - E^{[0]}_{m-1,i}\| \le \sum_{k=1}^{m-2} \|\frac{\partial E_{m-1}}{\partial a_k}\|\ \|a_k - a^{[0]}_{k,i}\|$$
$$+ \|\frac{\partial E_{m-1}}{\partial y}\|\ \|y - P^{[0]}_i\| + \|\frac{\partial E_{m-1}}{\partial z}\|\ \|z - Q^{[0]}_i\| \le\ const.\ \ h$$

Using the Lemma 2.1, 2.2, 2.5 we have

$$\|\varphi_{m-1}(t, a_{m-1}) - \varphi^{[0]}_{m-1,i}(t, a_{m-1})\| \le\ const.\ \ h$$

Again the assumptions of Lemma 2.4 holds, then

$$\max_{t\in[0,T]} \|a_{m-1}(t) - a^{[0]}_{m-1,h}(t)\| \le\ const.\ \ h$$

Let $l = m$, in the same way, we have

$$\|\varphi_m(t, a_m) - \varphi^{[0]}_{m,i}(t, a_m)\| \le\ const.\ \ h + \|y^{(m+1)}(t) - P^{[0](m+1)}_i(t)\|$$

Using the inequality (2.2.12b) in Lemma 2.1, we conclude that

$$\|\varphi_m(t, a_m) - \varphi^{[0]}_{m,i}(t, a_m)\| \le\ const.\qquad \forall t$$

therefore

$$\max_{t\in[0,T]} \|a_m(t) - a^{[0]}_{m,h}(t)\| \leq \ const.$$

Lemma 2.7 Let $b_l(t)$ and $b^{[0]}_{l,i}(t)$ satisfy the variational equation (A.14) and (A.22) with the initial conditions (A.15b), (A.23b) then

$$\max_{t\in[0,T]} \|b_l(t) - b^{[0]}_{l,h}(t)\| \leq \ const. \ \ h \ \ , \ \ l = 1,2,\ldots,m-1 \qquad (2.2.36a)$$

$$\max_{t\in[0,T]} \|b_m(t) - b^{[0]}_{m,h}(t)\| \leq \ const. \qquad (2.2.36b)$$

Proof: From the equation (A.14) and (A.22)

$$\|b_l(t) - b^{[0]}_{l,i}(t)\| \leq \|K_4 a_l(t) - K^{[0]}_{4,i} a^{[0]}_{l,i}(t)\| + \|K_5 y^{(l)}(t) - K^{[0]}_{5,i} P^{[0](l)}_i(t)\| + \|K_6 z^{(l)}(t) - K^{[0]}_{6,i} Q^{[0](l)}_i\| + \|\tilde{E}_l - \tilde{E}^{[0]}_{l,i}\| \qquad (2.2.37)$$

Since

$$\begin{aligned}\|K_4 a_l(t) - K^{[0]}_{4,i} a^{[0]}_{l,i}(t)\| &= \|(K_4 - K^{[0]}_{4,i}) a_l(t) + K^{[0]}_{4,i}(a_l - a^{[0]}_{l,i})\| \\ &\leq \|K_4(y,z) - K^{[0]}_{4,i}(P^{[0]}_i, Q^{[0]}_i)\| \ \|a_l(t)\| \\ &+ \|K^{[0]}_{4,i}\| \ \|a_l(t) - a^{[0]}_{l,i}(t)\| \qquad (2.2.38)\end{aligned}$$

from the inequalities (2.2.24) in Lemma 2.2, and the inequalities (2.2.35a) and (2.2.35b) in Lemma 2.6 we have

$$\max_{t\in I_i} \|K_4 a_l(t) - K^{[0]}_{4,i} a^{[0]}_{l,i}(t)\| \leq const. \ h \quad \text{for } l = 1,2,\ldots,m-1 \qquad (2.2.39a)$$

$$\max_{t\in I_i} \|K_4 a_m(t) - K^{[0]}_{4,i} a^{[0]}_{m,i}(t)\| \leq const. \qquad (2.2.39b)$$

and using the smoothness property of $\tilde{E}_l$ and $\tilde{E}^{[0]}_{l,i}$ we conclude that

$$\|\tilde{E}_l - \tilde{E}^{[0]}_{l,i}\| \leq \sum_{k=1}^{l-1} \left\| \frac{\partial \tilde{E}_{k+1}}{\partial a_k} \right\| \ \|a_k - a^{[0]}_{k,i}\| + \left\| \frac{\partial \tilde{E}_l}{\partial y} \right\| \ \|y(t) - P^{[0]}_i(t)\| + \left\| \frac{\partial \tilde{E}_l}{\partial z} \right\| \ \|z(t) - Q^{[0]}_i(t)\|$$

then for $l = 1,2,\ldots,m$,

$$\max_{t\in I_i} \|\tilde{E}_l - \tilde{E}_{l_i}^{[0]}\| \leq \; const. \;\; h \tag{2.2.40}$$

therefore (2.2.37) gives together with the inequalities (2.2.39a) and (2.2.39b), (2.2.40) and the inequalities (2.2.25), (2.2.26) in Lemma 2.2,

$$\max_{t\in I_i} \|b_l(t) - b_{l,i}^{[0]}(t)\| \leq \; const. \;\; h$$

$$\max_{t\in I_i} \|b_m(t) - b_{m,i}^{[0]}(t)\| \leq \; const. \;\; \text{and } i = 1,2,\ldots,n$$

Since $b_{l,h}^{[0]}(t) = b_{l,i}^{[0]}(t) \quad t \in I_i \quad i = 1,2,\ldots,n$, it leads to our assertion.

Lemma 2.8 Let the sequences $\{\alpha_\nu^l\}$, $\{\alpha_{\nu,h}^{[0]l}\}$, $\{\beta_\nu\}$, $\{\beta_{\nu,h}^{[0]l}\}$ satisfy the equations (A.18) and (A.24) then

$$\|\alpha_\nu^l - \alpha_{\nu,h}^{[0]l}\| \leq \; const. \;\; h \qquad l = 2,\ldots,m-1$$

$$\|\alpha_\nu^m - \alpha_{\nu,h}^{[0]m}\| \leq \; const. \text{ for } \; \forall\nu \in N \tag{2.2.41}$$

and

$$\|\beta_\nu^l - \beta_{\nu,h}^{[0]l}\| \leq \; const. \; h \quad l = 2,\ldots,m-1$$

$$\|\beta_\nu^m - \beta_{\nu,h}^{[0]m}\| \leq \; const. \qquad \text{for } \; \forall\nu \in N \tag{2.2.42}$$

Proof: We know that $a_l(0) = \alpha_0^l$, $b_l(0) = \beta_0^l$, $a_{l,h}^{[0]}(0) = \alpha_{0,h}^{[0]l}$ and $b_{l,h}^{[0]}(0) = \beta_{0,h}^{[0]l}$. From the inequalities (2.2.35a,b) and (2.2.36a,b) in Lemma 2.6, 2.7 respectively.

$$\|a_l(0) - a_{l,h}^{[0]}(0)\| = \|\alpha_0^l - \alpha_{0,h}^{[0]l}\| \leq \; const. \;\; h \quad l = 1,2,\ldots m-1$$

$$\|a_m(0) - a_{m,h}^{[0]}(0)\| = \|\alpha_0^m - \alpha_{0,h}^{[0]m}\| \leq \; const. \tag{2.2.43}$$

and

$$\|b_l(0) - b_{l,h}^{[0]}(0)\| = \|\beta_0^l - \beta_{0,h}^{[0]l}\| \leq \; const. \;\; h \quad l = 1,2,\ldots m-1$$

$$\|b_m(0) - b_{m,h}^{[0]}(0)\| = \|\beta_0^m - \beta_{0,h}^{[0]m}\| \leq \; const. \tag{2.2.44}$$

From the equation (A.18), we get

$$\|\alpha_\nu^l - \alpha_{\nu,h}^{[0]l}\| \le \sum_{k=1}^{l} \left\| \frac{\partial N_k}{\partial \alpha_0^k} \right\| \|\alpha_0^l - \alpha_{0,h}^{[0]l}\| + \sum_{i=0}^{\nu-1} \left\| \frac{\partial N_l}{\partial y(t_i)} \right\| \|y(t_i) - P_h^{[0]}(t_i)\|$$
$$+ \sum_{i=0}^{\nu-1} \frac{\partial N_l}{\partial z(t_i)} \|z(t_i) - Q_h^{[0]}(t_i)\| + \sum_{k=1}^{l} \left\| \frac{\partial N_k}{\partial \beta_0^k} \right\| \|\beta_0^l - \beta_{0,h}^{[0]l}\|, \text{ for all } \nu \in N$$

using the inequalities (2.2.43) and (2.2.44) we conclude the inequalities (2.2.41). The inequalities (2.2.42) can be proved similarly.

Lemma 2.9 For the same problem of Lemma 2.1, with $j = 0, 1, \ldots, m-1$

$$\max_{t\in[0,T]} \left\| \frac{d^k}{dt^k} P_h^{[j]} - \frac{d^k}{dt^k} y(t) \right\| \le \ const. \ \ h^{j+1} \quad k = 0, 1, \ldots, m-j \qquad (2.2.45a)$$

$$\max_{t\in[0,T]} \left\| \frac{d^k}{dt^k} P_h^{[j]}(t) - \frac{d^k}{dt^k} y(t) \right\| \le \ const. \ \ h^{m-k+1} \quad k = m-j+1, \ldots, m+1 \qquad (2.2.45b)$$

$$\max_{t\in[0,T]} \left\| \frac{d^k}{dt^k} Q_h^{[j]}(t) - \frac{d^k}{dt^k} z(t) \right\| \le \ const. \ \ h^{j+1} \quad k = 0, 1, \ldots, m-j \qquad (2.2.46a)$$

$$\max_{t\in[0,T]} \left\| \frac{d^k}{dt^k} Q_h^{[j]}(t) - \frac{d^k}{dt^k} z(t) \right\| \le \ const. \ \ h^{m-k+1} \quad k = m-j+1, \ldots, m+1 \qquad (2.2.46b)$$

and

$$\max_{t\in[0,T]} \|a_l(t) - a_{l,h}^{[j]}(t)\| \le \ const. \ \ h^{j+1} \quad l = 1, 2, \ldots, m-j-1 \ (2.2.47a)$$

$$\max_{t\in[0,T]} \|a_l(t) - a_{l,h}^{[j]}(t)\| \le \ const. \ \ h^{m-l} \quad l = m-j, \ldots, m \qquad (2.2.47b)$$

$$\max_{t\in[0,T]} \|b_l(t) - b_{l,h}^{[j]}(t)\| \le \ const. \ \ h^{j+1} \quad l = 1, 2, \ldots, m-j-1 \ (2.2.48a)$$

$$\max_{t\in[0,T]} \|b_l(t) - b_{l,h}^{[j]}(t)\| \le const. \ \ h^{m-l} \quad l = m-j, \ldots, m \qquad (2.2.48b)$$

and for all $\nu \in N, \ \ \nu = 0, 1, \ldots, n \cdot m$

$$\|\alpha_\nu^l - \alpha_{\nu,h}^{[j]l}\| \le const. \ \ h^{j+1} \quad l = 2, \ldots, m-j-1 \qquad (2.2.49a)$$

$$\|\alpha_\nu^l - \alpha_{\nu,h}^{[j]l}\| \le const. \ \ h^{m-l} \quad l = m-j, \ldots, m \qquad (2.2.49b)$$

$$\|\beta_\nu^l - \beta_{\nu,h}^{[j]l}\| \leq const. \;\; h^{j+1} \quad l = 2, \ldots, m-j-1 \qquad (2.2.50a)$$

$$\|\beta_\nu^l - \beta_{\nu,h}^{[j]l}\| \leq const. \;\; h^{m-l} \quad l = m-j, \ldots, m \qquad (2.2.50b)$$

where $P_h^{[j]}(t)$ and $Q_h^{[j]}(t)$ are the piecewise polynomial of fixed degree m. $a_l(t)$, $a_{l,i}^{[j]}(t)$, $b_l(t)$, $b_{l,i}^{[j]}(t)$, α_ν^l, $\alpha_{\nu,h}^{[j]l}$, β_ν^l, $\beta_{\nu,h}^{[j]l}$ satisfy the equations, (A.13), (A.21), (A.14), (A.22), (A.18), (A.24) with their initial conditions.

Proof: We will use mathematical induction; when $j = 0$, all inequalities in Lemma 2.9 are proved in Lemma 2.1, 2.6, 2.7, 2.8 respectively. Suppose these inequalities are true for $0 \leq j < m-1$. Let $j = m-1$; in this step, all considerations are completely the same. In order to prove the first inequalities (2.2.45a) and (2.2.45b), we will consider the new auxiliary function as the following;

$$\psi_h^{[m-1]}(t) = y(t) + \sum_{l=1}^{m} h^l (a_l(t) - a_{l,h}^{[m-2]}(t)) + \sum_{l=2}^{m} h^l (\alpha^l(t) - \alpha_h^{[m-2]l}(t)) + R_h^{[m-1]}(t)$$

where

$$R_h^{[m-1]}(t) = R_i^{[m-1]}(t), \quad t \in I_i \quad i = 1, 2, \ldots, n,$$

$\alpha_h^{[m-2]l}(t)$ are defined in a similar manner as discussed in proof of Lemma 2.1, and $R_i^{[m-1]}(t)$ is a polynomial of degree m which interpolates the values

$$R_\nu - R_{\nu,h}^{[m-2]} = \mathcal{O}(h^{m+1}) \;\text{ for }\; \nu = (i-1)m, \ldots, i \cdot m$$

and

$$\alpha_h^{[m-2]l}(t_\nu) = N_l^{[m-2]}(\alpha_0^{[m-2]l}, \ldots, \alpha_\nu^{[m-2]l}, \beta_0^{[m-2]l}, \ldots, \beta_\nu^{[m-2]l}$$
$$, \; P_h^{[m-2]}(t_\nu - \nu\, h), \ldots, P_h^{[m-2]}(t_\nu - h), Q_h^{[m-2]}(t_\nu - \nu\, h), \ldots, Q_h^{[m-2]}(t_\nu - h))$$

After the same consideration in Lemma 2.1, we get

$$\frac{d^k}{dt^k} R_h^{[m-1]}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0, 1, \ldots, m+1$$

According to our assumptions for $0 \leq j < m-1$, from (2.2.47a and b) we write that;

$$a_1(t) - a_{1,h}^{[m-2]}(t) = \mathcal{O}(h^{m-1})$$

$$a_l(t) - a_{l,h}^{[m-2]}(t) = \mathcal{O}(h^{m-l}) \qquad l = 2,3,\ldots,m$$
$$\alpha_\nu^l - \alpha_\nu^{[m-2]l} = \mathcal{O}(h^{m-l}) \qquad l = 2,3,\ldots,m$$

Differentiation with respect to the t of the variational equations (A.13) and (A.21), and similar considerations as in the proof of Lemma 2.1 leads to

$$\frac{d^k}{dt^k}a_l(t) - \frac{d^k}{dt^k}a_{l,h}^{[m-2]}(t) = \mathcal{O}(h^{m-l-k+1}) \quad k = 0,1,\ldots,m+1$$

Since the difference of $\frac{d^k}{dt^k}\alpha^l(t) - \frac{d^k}{dt^k}\alpha_h^{[m-2]l}(t)$ depend on the difference $\frac{d^k}{dt^k}P_h^{[m-2]}(t) - \frac{d^k}{dt^k}y(t)$ and $\frac{d^k}{dt^k}Q_h^{[m-2]}(t) - \frac{d^k}{dt^k}z(t)$,
we obtain that

$$\frac{d^k}{dt^k}\alpha^l(t) - \frac{d^k}{dt^k}\alpha_h^{[m-2]l}(t) = \mathcal{O}(h^{m-1}) \quad k = 0,1,2$$
$$\frac{d^k}{dt^k}\alpha^l(t) - \frac{d^k}{dt^k}\alpha_h^{[m-2]l}(t) = \mathcal{O}(h^{m-k+1}) \quad k = 3,\ldots,m+1$$

As a consequence we get

$$\psi_h^{[m-1]}(t) - y(t) = h(a_1 - a_{1,h}^{[m-2]}) + \sum_{l=2}^{m} h^l(a_l - a_{l,h}^{[m-2]}) + \sum_{l=2}^{m} h^l(\alpha^l(t) - \alpha_h^{[m-2]l}(t)) + R_h^{[m-1]}(t)$$

$$\begin{aligned}\frac{d^k}{dt^k}\psi_h^{[m-1]}(t) - \frac{d^k}{dt^k}y(t) &= \sum_{l=1}^{m}\left(\frac{d^k}{dt^k}a_l(t) - \frac{d^k}{dt^k}a_{l,h}^{[m-2]}(t)\right)h^l \\ &+ \sum_{l=2}^{m} h^l\left(\frac{d^k}{dt^k}\alpha^l(t) - \frac{d^k}{dt^k}\alpha_h^{[m-2]l}(t)\right) + \frac{d^k}{dt^k}R_h^{[m-1]}(t) \\ &= \mathcal{O}(h^{m+1-k}),\ k = 1,2,\ldots,m+1 \qquad (2.2.51)\end{aligned}$$

Obviously $\psi_h^{[m-1]}(t_\nu) = \eta_\nu^{[m-1]}$ holds, i.e. $P_h^{[m-1]}(t_\nu) = \psi_h^{[m-1]}(t_\nu) = \eta_\nu^{[m-1]}$ then $P_h^{[m-1]}(t)$ interpolates $\psi_h^{[m-1]}(t)$ at the points t_ν, therefore from [17]

$$\frac{d^k}{dt^k}P_h^{[m-1]}(t) - \frac{d^k}{dt^k}\psi_h^{[m-1]}(t) = \mathcal{O}(h^{m+1-k}) \quad k = 0,1,\ldots,m+1 \qquad (2.2.52)$$

which together with (2.2.51) as Lemma 2.1, we conclude that, $t \in [0,T]$

$$\frac{d^k}{dt^k}P_h^{[m-1]}(t) - \frac{d^k}{dt^k}y(t) = \mathcal{O}(h^m) \text{ for } k = 0,1$$
$$\frac{d^k}{dt^k}P_h^{[m-1]}(t) - \frac{d^k}{dt^k}y(t) = \mathcal{O}(h^{m+1-k}) \text{ for } k = 2,3,\ldots,m+1$$

For the function $z(t)$, define the auxiliary function $\Phi_h^{[m-1]}(t)$ as follows,

$$\Phi_h^{[m-1]}(t) = z(t) + \sum_{l=1}^{m} h^l(b_l(t) - b_{l,h}^{[m-2]}(t)) + \sum_{l=2}^{m} h^l(\beta^l(t) - \beta^{[m-2]l}(t)) + \tilde{R}_h^{[m-1]}(t)$$

According to above definition and the same consideration as y, we get the our assertion for $z(t)$.

Now we will show the inequality (2.2.47a,b) for $j = m-1$, using induction for $l = 1, 2, \ldots, m$ i.e.

$$\max_{t\in[0,T]} \|a_l(t) - a_{l,h}^{[m-1]}(t)\| \leq \; const. \;\; h^{m-l} \quad l = 1, 2, \ldots, m \tag{2.2.53}$$

when $l = 1$, in order to prove, we will use the same process of Lemma 2.5. The assumptions of Lemma 2.5 are valid for $a_l(t), a_{l,h}^{[m-1]}(t)$, φ_l and $\varphi_{l,i}^{[m-1]}(t)$, then we can write;

$$\begin{aligned}\|\varphi_1(t,a_1) - \varphi_{1,i}^{[m-1]}(t,a_1)\| \leq & \|K_1 - K_{1,i}^{[m-1]}\| \; \|a_1(t)\| + \|K_2 y'(t) - K_{2,i}^{[m-1]} P_i^{[m-1]'}(t)\| \\ & + \|K_3 z'(t) - K_{3,i}^{[m-1]} Q_i^{[m-1]'}(t)\| + \frac{1}{2}\|y''(t) - P_i^{[m-1]''}(t)\| \\ \leq & \; const. \;\; h^{m-1}\end{aligned}$$

From the mean value theorem;

$$\|K_s - K_{s,i}^{[m-1]}\| \leq \; const. \;\; \|y - P_i^{[m-1]}\| + \; const. \;\; \|z - Q_i^{[m-1]}\| \leq \; const. \;\; h^m$$

and in the similar way

$$\|K_s y' - K_{s,i}^{[m-1]} P_i^{[m-1]'}\| \leq \; const. \;\; h^m \;\; \text{and} \;\; \|K_s z' - K_{s,i}^{[m-1]} Q_i^{[m-1]'}\| \leq \; const. \;\; h^m$$

with

$$\|y''(t) - P_i^{[m-1]''}(t)\| \leq \; const. \;\; h^{m-1}$$

where $K_{s,i}^{[m-1]}$ is a function of $P_i^{[m-1]}$ and $Q_i^{[m-1]}$ as Lemma 2.2, then from Lemma 2.4 we conclude that

$$\max_{t\in[0,T]} \|a_1(t) - a_{1,h}^{[m-1]}(t)\| \leq \; const. \;\; h^{m-1}$$

let the inequality (2.2.53) be true for $l < m$, when $l = m$, in the same way we obtain

$$\begin{aligned}\|\varphi_m(t,a_m) - \varphi_{m,i}^{[m-1]}(t,a_m)\| \leq & \|K_1 - K_{1,i}^{[m-1]}\| \; \|a_m(t)\| + \|K_2 y^{(m)}(t) - K_{2,i}^{[m-1]}(t) P_i^{(m)}\| \\ & + \|K_3 z^{(m)}(t) - K_{3,i}^{[m-1]} Q_i^{[m-1](m)}(t)\| \\ & + \|y^{(m+1)}(t) - P_i^{[m-1](m+1)}(t)\| \\ & + \|E_m - E_{m,i}^{[m-1]}\| \leq \; const.\end{aligned}$$

Since $\|y^{(m+1)} - P_i^{[m-1](m+1)}\| \leq$ *const.* then it gives our assertion (2.2.53). Since $b_l(t)$ and $b_{l,h}^{[j]}(t)$ depends on $a_l(t)$ and $a_{l,h}^{[j]}(t)$, using the same consideration in Lemma 2.7, the inequalities (2.2.48a and b) can be proved. We can write from the inequality (2.2.47a,b) and (2.2.48a,b)

$$\|a_l(0) - a_{l,h}^{[j]}(0)\| = \|\alpha_0^l - \alpha_{0,h}^{[j]l}\|$$
$$\|b_l(0) - b_{l,h}^{[j]}(0)\| = \|\beta_0^l - \beta_{0,h}^{[j]l}\|$$

As the same consideration the proof of Lemma 2.8, we obtain the inequalities (2.2.49a,b) and (2.2.50a,b).

Chapter 3

NUMERICAL RESULTS

The IDeC methods discussed in the previous chapters are applied to a linear constant coefficient index four and nonlinear index one artificial test problems. The numerical results confirm the predicted orders of convergence given in Theorems 1.1 and 2.1. Using the step size control strategy for IDeC methods in ([30]) a pipeline model is integrated as a technical application of nonlinear index one DAE system. All computations are performed on a IBM PC PS 2/60 in FORTRAN 77 in DOUBLE PRECISION.

Before we present the numerical results, we want to discuss the extension of IDeC method to the index one problems of the form

$$By' = f(y), \quad y(0) = y_0 \tag{3.1}$$

where B is a $n \times n$ singular matrix. There is no explicit uncoupling of the differential and algebraic parts as in the semi-explicit form (2.1.1). This system can be converted easily to the semi-explicit form using the singular value decomposition of B. With regular Householder matrices U and V we obtain

$$UBVV^{-1}y' = Uf(VV^{-1}y), \quad V^{-1}y(0) = V^{-1}y_0$$

$$UBV = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \cdot & & & \\ & & \cdot & & 0 \\ & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right], \quad \sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0$$

The first r components of $V^{-1}y$ will be denoted by $\tilde{y}$, the remaining components $r+1,\ldots,n$ with $\tilde{z}$. After inversion of D we obtain the semi-explicit form (2.1.1)

$$\tilde{y}' = D^{-1}\left(Uf\left(V\begin{pmatrix}\tilde{y}\\ \tilde{z}\end{pmatrix}\right)\right)_{1,\ldots,r}$$

$$0 = \left(Uf\left(V\begin{pmatrix}\tilde{y}\\ \tilde{z}\end{pmatrix}\right)\right)_{r+1,\ldots,n} \tag{3.2}$$

with the initial values

$$[\tilde{y}(0), \tilde{z}(0)]^T = V^{-1}y_0$$

The numerical problems concerned with the solution of both form of index one problems using ROW (Rosenbrock-Wanner) methods are discussed in detail in [26] and [29]. For the application of the IDeC method to problems of the form (3.1.1) it is important whether the asymptotic expansions of the global error obtained for the semi-explicit form are changed or not. A simple answer to this question can be found using the Lemma 9.6 at p.79 in ([29]). The assertion of this lemma is:

If $[\tilde{y}_n, \tilde{z}_n]^T$ are solutions of the semi-explicit DAE (2.1.1) and y_n solutions of (3.1.1) by the ROW method, then

$$y_n = V\begin{pmatrix}\tilde{y}_n\\ \tilde{z}_n\end{pmatrix}.$$

Because the linearly implicit Euler method belongs to the class of ROW method (it is namely the simplest of this type of methods), the results for the asymptotic expansions of the global error is also valid for the DAE's of the form (3.1.1)

Linear Constant Coefficient Index Four DAE System

We consider the following artificial test example

$$\begin{aligned}
0 &= y_1 + t^4e^t & y_1(0) &= -1\\
y_1' &= y_2 & y_2(0) &= -1\\
y_2' &= y_3 & y_3(0) &= -1\\
y_3' &= y_4 & y_4(0) &= -1
\end{aligned}$$

Table 3.1: Order Pattern

	y_2	y_3	y_4
IE	1	1	1
1.DC	6	2	2
2.DC		5	3
3.DC			4

with the exact solutions

$$y_1 = -t^4 e^t, \quad y_2 = -4t^3 e^t, \quad y_3 = -12t^2 e^t, \quad y_4 = -24te^t$$

Global connection strategy and piecewise polynomials of degree $m = 6$ are used. The predicted orders of convergence for this problem can be derived using Theorem 1.1. Here IE denotes implicit Euler solutions and 1.DC, 2.DC, 3.DC the first, second and third defect correction solutions respectively.

The numerical results using constant step size and observed orders of convergence for each component are listed in the following tables at $t = 2.4$. The observed orders are computed using

$$p = \frac{\log \frac{e_n}{e_{n+1}}}{\log \frac{h_n}{h_{n+1}}}$$

where e_n and e_{n+1} are the global errors when the problem is solved with step sizes h_n and h_{n+1} respectively.

Table 3.2: Global error and observed orders for y_2

h	IE	1st DC
0.1	.389 D01	.121 D-05
0.05	.198 D01	.216 D-07
0.025	.995 D00	.362 D-09
0.0125	.449 D00	.725 D-11
	Observed Orders	
	0.9743	5.8078
	0.9927	5.8989
	0.9957	5.6419

Table 3.3: Global error and observed orders for y_3

h	IE	1st DC	2nd DC
0.1	.661 D01	.858 D-01	.590 D-04
0.05	.338 D01	.217 D-01	.211 D-05
0.025	.170 D01	.544 D-02	.709 D-07
0.0125	.855 D00	.136 D-02	.217 D-08
	Observed Orders		
	0.9785	1.9833	4.8054
	0.9915	1.9960	4.8953
	0.9915	2.0100	5.0300

Table 3.4: Global error and observed orders for y_4

h	IE	1st DC	2nd DC	3rd DC
0.1	.512 D01	.751 D-01	.268 D-02	.176 D-02
0.05	.259 D01	.197 D-01	.268 D-03	.127 D-03
0.0250	.130 D01	.504 D-02	.280 D-04	.866 D-05
0.0125	.665 D00	.128 D-02	.305 D-05	.468 D-06
		Observed Orders		
	0.9832	1.9306	3.3219	3.7927
	0.9944	1.9607	3.1587	3.8743
	0.9889	1.9773	3.1985	4.2098

For the first component we got an error .169 D-14 and it is not changed by defect correction steps because it was solved exactly by the linearly implicit Euler method. The observed orders are in good agreement with the predicted ones. With decreasing h the convergence to the fixed point is very fast.

Nonlinear index one artificial test problem

This test example is taken from ([26]).

$$\begin{aligned} y_1' &= \alpha z y_2^{\alpha\beta} & y_1(0) &= 1 \\ y_2' &= y_2 \cdot z^{\beta} & y_2(0) &= 1 \\ 0 &= z + \beta y_1 / y_2^{\alpha\beta} & z(0) &= -\beta \end{aligned}$$

with the exact solutions $y_1 = e^{-\alpha\beta t}$, $y_2 = e^{-t}$, $z = -\beta$.

Piecewise polynomials of degree $m = 3$ are used and the numerical results are listed at $t = 0.3$. We have applied the IDeC method for $\alpha = 0.5$, $\beta = 6.0$. LIE denotes here the numerical solutions obtained by the linearly implicit Euler method. The numerical results confirm the predicted orders of convergence by Theorem 2.1. Again on finer grids better results about the orders are obtained.

Table 3.5: Global error and observed orders for y_1

h	LIE	1st DC	2nd DC
0.1	.526 D-01	.133 D-01	.168 D-01
0.05	.302 D-01	.583 D-02	.971 D-03
0.01	.700 D-02	.318 D-03	.953 D-05
0.005	.357 D-02	.826 D-04	.225 D-05
0.001	.727 D-03	.340 D-05	.105 D-07
		Observed Orders	
	0.8005	1.8990	3.4754
	0.9083	1.8073	2.8374
	0.9714	1.9448	2.9305
	0.9888	1.9822	2.9697

Table 3.6: Global error and observed orders for y_2

h	LIE	1st DC	2nd DC
0.1	.284 D-01	.946 D-02	.113 D-02
0.05	.133 D-01	.236 D-02	.252 D-03
0.01	.255 D-02	.909 D-04	.194 D-05
0.005	.127 D-02	.226 D-04	.239 D-06
0.001	.252 D-03	.899 D-06	.188 D-08
		Observed Orders	
	1.0945	2.0031	2.1648
	1.0626	2.0235	3.0239
	1.0057	2.0000	3.0210
	1.0049	2.0034	3.0105

Implementation of IDeC methods to nonlinear index one DAE's

An efficient implementation of any ODE and DAE integrator requires the possibility to change the stepsize and the order of the method automatically during the integration . This will be done using local or global error estimations of the unknown exact solution. For the implementation of DAE's in general the techniques which work successfully for ODE's is taken with some modifications. Defect correction methods offer the possibility of both local and global error estimations, of changing the stepsize and order simultaneously (changing the the degree of the interpolation polynomials and the number of defect correction steps). For a discussion of problems encountered in the application of IDeC methods to stiff ODE's see ([30]). We also adopted this technique with slight modifications and will summarize it briefly in the following.
The first part of the control mechanism consists of calculation of the numerical approximation together with an error estimation. We will use the local error estimation ζ^j at the endpoints of each interpolation interval I_i. Here ζ^j denotes the defect correction solutions of the nonlinear index one problem in Chapter II, namely η^j and ξ^j. The local solution on I_i is defined as the exact solution of

$$\begin{aligned} y' &= f(y,z,t) \qquad y(t^{i-1}) = \eta_{i-1} \\ 0 &= g(y,z,t) \qquad z(t^{i-1}) = \xi_{i-1} \qquad t \in [t^{i-1}, t^i] \end{aligned} \tag{3.3}$$

The initial values of (3.3) are the last components of the most accurate defect correction solutions in the previous interval $I_{i-1} = (t^{i-2}, t^{i-1})$. The exact solutions of (3.3) are denoted by $y(t; t^{i-1}, \eta_{i-1})$ and $z(t; t^{i-1}, \xi_{i-1})$ respectively.
For the local error estimation we will use extrapolation; the integration from t^{i-1} to t^i is done twice. The one defect correction step with the interval length H_i produces ζ^j_m and the other with $H_i/2$ using two step produces the approximation $\bar{\zeta}^j_{2m}$. Then

$$(\zeta^j_m - \bar{\zeta}^j_{2m})/(2^j - 1) \tag{3.4}$$

is an estimate for the local errors $\bar{\eta}^j_{2m} - y(t; t^{i-1}, \eta_{i-1})$ and $\bar{\xi}^j_{2m} - z(t; t^{i-1}, \xi_{i-1})$. $\bar{\zeta}^j_{2m}$ is produced using the global connection strategy by proceeding from the first half of the interval I_i to the second one, because for the local connection strategy the computational effort is significantly larger. To avoid underestimation or overestimation of the observed errors a reliability factor Q_j is computed too. In situations where the estimates are correct, $\bar{\zeta}^j_{2m}$ and $\bar{\zeta}^{j+1}_{2m}$ should be identical. Therefore an obvious way of checking the assumptions of the asymptotic convergence orders is to control how much the following quotient varies from 1.

$$Q_j = \frac{\bar{\zeta}^j_{2m} - \bar{\zeta}^{j+1}_{2m}}{(\zeta^j_m - \bar{\zeta}^j_{2m})(2^j - 1)}$$

This requires one more defect correction step, but gives a reliable error estimate if $Q_j \approx 1$. If for all iterates $\bar{\zeta}^j_{2m}$ $(j = 0, \ldots, j_{\max})$ in the interval I_i $Q_j \approx 1$ is satisfied, then the new interval length can be determined according to

$$H_{new} = H_{old} \left(\frac{est}{tol}\right)^{-1/(2+j)} \tag{3.5}$$

where est denotes the local error estimation computed by (3.4) and tol the user specified tolerance. In practice H_{new} will be multiplied by a constant factor < 1 (here by 0.9) to make the stepsize selection conservative. To avoid large oscillations of the step size we restrict it in the following way

$$0.5 H_{old} \le H_{new} \le 1.5 H_{old}$$

The est is obtained using the following scaled norm

$$est = \max \left(\frac{1}{s_i}\right) \left| \frac{\zeta^j_{m,i} - \bar{\zeta}^j_{2m,i}}{2^j - 1} \right| \tag{3.6}$$

where $\zeta^j_{m,i}$ and $\bar{\zeta}^j_{2m,i}$ denote the the i th components of the vectors of ζ^j_m, $\bar{\zeta}^j_{2m}$. The scaling factor is defined as $s = (s_1, s_2, ..., s_n)^T$ by $s_i := \max(1, |\zeta^j_{m,i}|)$.
The scaling implies that for the solution components which are less then unity , absolute error control ; otherwise relative error control mechanism is applied (see [29]).
If $est < tol$ then the step is accepted and the integration can be done with the new stepsize. Otherwise the step is rejected, the integration must be repeated

with the new stepsize according to (3.5). This process is continued until the step is accepted or $H_{new} < H_{min}$, where H_{min} depends on the machine epsilon and the length of integration interval $[0, T]$.

The degree of the interpolation polynomials m_i can be changed during the integration too. The grids adapt better to the problem if polynomials of lower degree are chosen, whereas high degrees allow higher orders. A possible strategy to overcome this conflicting situation is given by [30] and we use it here :

The degree of the interpolation polynomial m_{i+1} in the next interval is chosen

(i) If the step was accepted :

if $j_{last} = j_{\max_i}$ then $m_{i+1} = m_i + 1$

if $j_{last} = j_{\max_i - 1}$ then $m_{i+1} = m_i$

if $j_{last} \leq j_{\max_i - 2}$ then $m_{i+1} = m_i - 1$

(ii) If the step was rejected :

if $j_{last} \leq j_{\max_i - 2}$ then $m_{i+1} = m_i - 1$

Two phase plug flow problem

This example is concerned with a semi-explicit nonlinear DAE of index one for a pipeline problem. It was solved in Bryne and Hindmarsh [3] using DASSL and in Hairer et. al. [14, pp. 106] using RADAU5.

The equations are

$$\pi\sqrt{\frac{R}{2\rho}}(R-y)^2\sqrt{-P'}\left(2.5\ln\left[\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\sqrt{-P'}-5\right]+10.5\right)$$
$$-bQ_{CO}-\frac{P_0}{P}Q_{CO}(1-b)=0 \quad (3.7a)$$

$$2\pi\sqrt{\frac{R}{2\rho}}\sqrt{-P'}\left((2.5Ry-1.25y^2)\ln\left[\sqrt{\frac{\rho R}{2}}\frac{y}{\mu}\sqrt{-P'}-5\right]+3Ry\right.$$
$$\left.-2.125y^2-13.6R\mu\sqrt{\frac{2}{\rho R}}\frac{1}{\sqrt{-P'}}\right)-Q_a=0 \quad (3.7b)$$

The physical meaning of the variables and parameters are described in the references above. $\sqrt{-P'}$ is substituted by u so that the differential equation

$$P'=-u^2$$

with the algebraic equations (3.7a) and (3.7b) together give a nonlinear index one DAE. P corresponds to the differential variable y in (0.2) , u and y to the algebraic variable z in (0.2).

This problem is solved for two set of data. The first one is the normal flow. The relevant data with consistent initial conditions is given in the references above. Figure (3.1) shows the numerical solution of this problem for $tol = 10^{-3}$.

The numerical results for a chocked flow is given in Figure (3.2) with $tol = 10^{-3}$. The data for this problem is given again the references above.

Figure 3.1: Two phase plug problem : normal flow

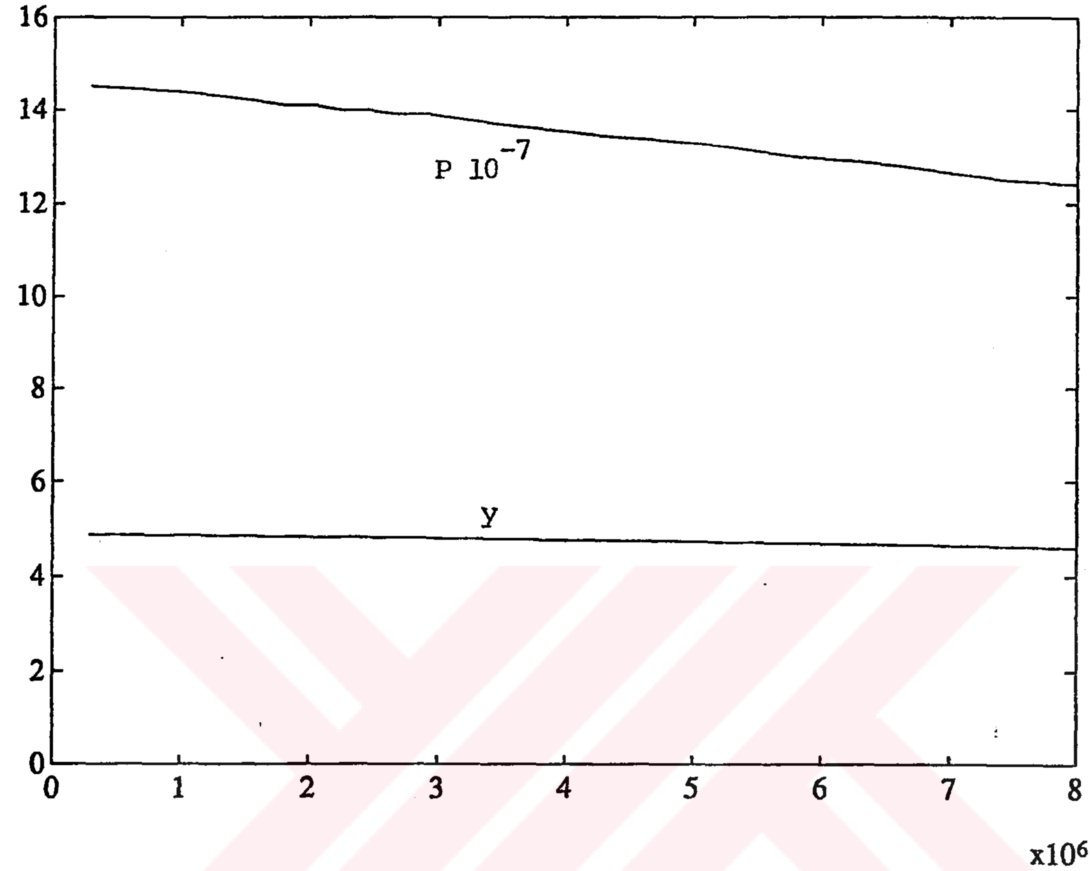

Figure 3.2: Two phase plug problem : chocked flow

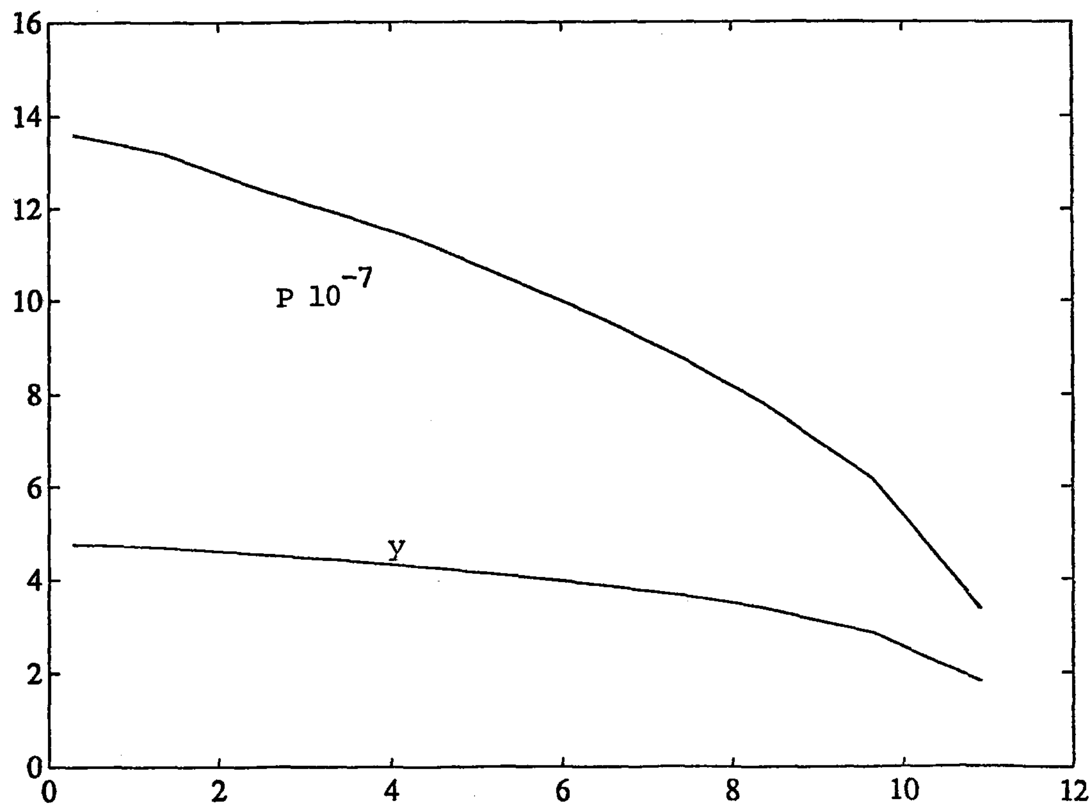

Table 3.7: Performance of the IDeC method for normal flow

tol	nsuc	nfail	njac	nfcn	nIDeC	jmax	mmin	mmax	hmin	hmax
10^0	3	0	6	81	6	1	3	3	1.0 D5	1.4 D6
10^{-1}	7	0	14	189	14	1	3	3	1.0 D5	9.1 D6
10^{-2}	22	0	44	1074	59	2	3	4	5.7 D4	1.6 D5

Table 3.8: Performance of the IDeC method for chocked flow

tol	nsuc	nfail	njac	nfcn	nIDeC	jmax	mmin	mmax	hmin	hmax
10^0	8	0	8	126	9	2	3	3	1.0 D5	1.2 D6
10^{-1}	13	1	28	662	35	3	3	4	1.0 D5	2.4 D6
10^{-2}	11	1	23	405	28	3	3	4	1.0 D5	3.5 D6

This problem is characterized by a singularity at $x = 1.0958 \cdot 10^7$. The numerical results show that this singularity is detected by the IDeC method with a modest accuracy.

A performance analysis of the IDeC methods is given in the tables above :

The meaning of the parameters in the tables above is :

nsuc : Number of successful steps

nfail : Number of failed steps

njac : Number of Jacobian evaluations

nfcn : Number of function (right hand side) evaluations

nIDeC : Number of defect correction steps including the OP solution

jmax : Maximum defect correction step

mnin : Minimum degree of the interpolation polynomials

mmax : Maximum degree of the interpolation polynomials

hmin : Minimum stepsize

hmax : Maximum stepsize

This problem is solved using the stepsize control mechanism described previously .

A comparison with the other codes not done yet. The other DAE codes like DASSL and RADAU5 are more flexible for stepsize and order changing and work for tighter tolerances. But the results presented here is still promising and justify further development of application of IDeC methods to DAE's .

Appendix

ASYMPTOTIC EXPANSIONS OF NON-LINEAR INDEX ONE DAE'S

As mentioned in the previous chapters the theoretical justification of the convergence analysis of IDeC methods requires the existence of the global error in powers of the step size h. The most elegant representation of the asymptotic expansions for DAE's and stiff ODE's is obtained using the way of analysis given by Hairer and Lubich in [15, pp. 211] . For index one and for some index two problems the asymptotic expansions of the semi-implicit methods like semi-implicit Euler and semi-implicit midpoint rule and for some implicit RK- methods are obtained using this concept (see [4] , [21], [22], [14]). For linearly implicit Euler method in case of nonlinear semi-explicit index one DAE's the so called perturbed asymptotic expansions are given first by Deufelhard et. al. ([4]). The perturbation terms are introduced to obtain a bounded remainder term in equation (0.10). The perturbed asymptotic expansions are of the following form

$$y_n - y(t_n) = a_1(t)h + (a_2(t) + \alpha_n^2)h^2 + \ldots + (a_N(t) + \alpha_n^N) + A(n,h)h^{N+1}$$

$$z_n - z(t_n) = (b_1(t) + \beta_n^1)h + (b_2(t) + \beta_n^2)h^2 + \ldots + (b_N(t) + \beta_n^N) + B(n,h)h^{N+1}$$

where

$$\begin{array}{lllll} a) & \alpha_n^2 = 0, & \alpha_n^3 = 0, \quad \alpha_n^4 = 0, \quad \beta_n^1 = 0, & n \geq 0 & \\ b) & \beta_n^2 = 0, & \beta_n^3 = 0, & n \geq 1 & \\ c) & \alpha_n^{j+1} = 0, & \beta_n^j = 0, & n \geq j-2 & \text{and } j \geq 4 \end{array}$$

Most of the perturbation terms vanish with increasing n, so that for IDeC methods with global connection strategy they don't affect the order of convergence with succeeding integration. In the following, the perturbed asymptotic expansions of the global error of the NP problem are derived for constant step sizes h , which give us the basis for the convergence analysis in Chapter 2.

Theorem A.1: Let the method (2.2.6) satisfy the consistency conditions, the solution $(y(t), z(t))$ of (2.2.1) is smooth and under the boundedness condition of g_z^{-1} , then the numerical solution of (2.2.6) possesses a perturbed asymptotic expansion of the form

$$y_\nu = y(t_\nu) + a_1(t_\nu)h + \sum_{l=2}^{m} h^l(a_l(t_\nu) + \alpha_\nu^l) + R_\nu \tag{A.1}$$

$$z_\nu = z(t_\nu) + b_1(t_\nu)h + \sum_{l=2}^{m} h^l(b_l(t_\nu) + \beta_\nu^l) + \tilde{R}_\nu \tag{A.2}$$

where $a_l(t)$ and $b_l(t)$ are smooth functions and all coefficients vanish for $\nu = 0$ i.e.

$$a_1(0) = 0, \;\; a_l(0) = -\alpha_0^l, \;\; b_l(0) = -\beta_0^l \tag{A.3}$$

and $\|R_\nu\| \leq const.\ h^{m+1}, \;\; \|\tilde{R}_\nu\| \leq const.\ h^{m+1}$ for $t \in [0, T]$.

Proof:

First Step : Let $y_\nu - a_1(t_\nu)h = y_\nu^*$ and $z_\nu - b_1(t_\nu)h - \beta_\nu^1 h = z_\nu^*$ can be interpreted as the numerical solution of a new method

$$B \begin{bmatrix} y_{\nu+1}^* - y_\nu^* \\ z_{\nu+1}^* - z_\nu^* \end{bmatrix} = h \begin{bmatrix} \Phi_\nu^*(y_\nu^*, z_\nu^*, h) \\ \psi_\nu^*(y_\nu^*, z_\nu^*, h) \end{bmatrix} \tag{A.4}$$

where

$$B = \begin{bmatrix} I - hA_1 & -hA_2 \\ -hA_3 & -hA_4 \end{bmatrix}$$

Substracting (A.4) from (2.2.6) we have;

$$B \begin{bmatrix} (y_{\nu+1} - y_{\nu+1}^*) - (y_\nu - y_\nu^*) \\ (z_{\nu+1} - z_{\nu+1}^*) - (z_\nu - z_\nu^*) \end{bmatrix} = h \begin{bmatrix} f(y_\nu, z_\nu) \\ g(y_\nu, z_\nu) \end{bmatrix} - h \begin{bmatrix} \Phi_\nu^*(y_\nu^*, z_\nu^*, h) \\ \psi_\nu^*(y_\nu^*, z_\nu^*, h) \end{bmatrix}$$

and

$$\begin{bmatrix} \Phi_\nu^*(y_\nu^*, z_\nu^*, h) \\ \psi_\nu^*(y_\nu^*, z_\nu^*, h) \end{bmatrix} = \begin{bmatrix} f(y_\nu^* + a_1(t_\nu)h,\ z_\nu^* + b_1(t_\nu)h + \beta_\nu^1 h) \\ g(y_\nu^* + a_1(t_\nu)h,\ z_\nu^* + b_1(t_\nu)h + \beta_\nu^1 h) \end{bmatrix}$$

$$-B \begin{bmatrix} a_1(t_{\nu+1}) - a_1(t_\nu) \\ b_1(t_{\nu+1}) - b_1(t_\nu) \end{bmatrix} - B \begin{bmatrix} 0 \\ \beta_{\nu+1}^1 - \beta_\nu^1 \end{bmatrix}$$

Our aim is to find smooth functions $a_1(t), b_1(t)$ and the sequence $\{\beta_\nu^1\}$ such that the new method is more accurate. By using the Taylor expansion;

$$B \begin{bmatrix} y(t_{\nu+1}) - y(t_\nu) \\ z(t_{\nu+1}) - z(t_\nu) \end{bmatrix} - h \begin{bmatrix} \Phi_\nu^*(y(t_\nu), z(t_\nu), h) \\ \psi_\nu^*(y(t_\nu), z(t_\nu), h) \end{bmatrix} =$$

$$B \begin{bmatrix} hy'(t_\nu) + \frac{h^2}{2!}y''(t_\nu) + \frac{h^3}{3!}y'''(t_\nu) + \cdots \\ hz'(t_\nu) + \frac{h^2}{2!}z''(t_\nu) + \frac{h^3}{3!}z'''(t_\nu) + \cdots \end{bmatrix}$$

$$-h \begin{bmatrix} f(y(t_\nu), z(t_\nu)) + f_y a_1(t_\nu)h + f_z(b_1(t_\nu) + \beta_\nu^1)h + \cdots \\ g(y(t_\nu), z(t_\nu) + g_y a_1(t_\nu)h + g_z(b_1(t_\nu) + \beta_\nu^1)h + \cdots \end{bmatrix}$$

$$+B \begin{bmatrix} h^2 a_1'(t_\nu) + \frac{h^3}{2!}a_1''(t_\nu) + \cdots \\ h^2 b_1'(t_\nu) + \frac{h^3}{2!}b_1''(t_\nu) + \cdots \end{bmatrix} + Bh \begin{bmatrix} 0 \\ \beta_{\nu+1}^1 - \beta_\nu^1 \end{bmatrix}$$

$$= h^2 \begin{bmatrix} \frac{1}{2!}y''(t_\nu) - A_1 y'(t_\nu) - A_2 z'(t_\nu) - f_y a_1(t_\nu) - f_z b_1(t_\nu) + a_1'(t_\nu) \\ -A_3 y'(t_\nu) - A_4 z'(t_\nu) - g_y a_1(t_\nu) - g_z b_1(t_\nu) \end{bmatrix} + \mathcal{O}(h^3)$$

$$-h^2 \begin{bmatrix} f_z \beta_\nu^1 \\ g_z \beta_\nu^1 \end{bmatrix} + hB \begin{bmatrix} 0 \\ \beta_{\nu+1}^1 - \beta_\nu^1 \end{bmatrix}$$

we find that the local error of the method (A.4) is $\mathcal{O}(h^3)$ provided that the function $a_1(t)$, $b_1(t)$ are smooth solutions of the following nonlinear semi-explicit DAE

$$a_1'(t) = f_y(y,z)a_1(t) + f_z(y,z)b_1(t) + A_1 y'(t) + A_2 z'(t) - \frac{1}{2!}y''(t) \qquad \text{(A.5a)}$$

$$0 = g_y(y,z)a_1(t) + g_z(y,z)b_1(t) + A_3 y'(t) + A_4 z'(t) \qquad \text{(A.5b)}$$

and the sequence $\{\beta_\nu^1\}$ are bounded solution of

$$B \begin{bmatrix} 0 \\ \beta_{\nu+1}^1 - \beta_\nu^1 \end{bmatrix} = h \begin{bmatrix} f_z \beta_\nu^1 \\ g_z \beta_\nu^1 \end{bmatrix} \qquad \text{(A.6)}$$

Written in the explicit form β_ν^1 satisfy the following equations

$$A_2\beta_{\nu+1}^1 = (A_2 - f_z)\beta_\nu^1$$
$$A_4\beta_{\nu+1}^1 = (A_4 - g_z)\beta_\nu^1$$

At $t = 0$, using the relations $b_1(0) = -\beta_0^1$, $a_1(0) = 0$ and the equation (A.5b) and the total differential of $g(y,z) = 0$, i.e. $g_y y' + g_z z' = 0$, we obtain

$$0 = A_3 a_1(0) + A_4 b_1(0) + A_3 y'(0) + A_4 z'(0)$$
$$0 = A_4 \beta_0^1$$

Since A_4^{-1} and exist, $\beta_0^1 = 0$ therefore recursively

$$A_4\beta_\nu^1 = 0 \text{ then } \beta_\nu^1 = 0 \text{ for all } \nu = 0, 1, \cdots, n \cdot m.$$

Second Step: In order to eliminate further error terms, we introduce

$$y_\nu^{**} = y_\nu - a_1(t_\nu)h - (a_2(t_\nu) + \alpha_\nu^2)h$$
$$z_\nu^{**} = z_\nu - b_1(t_\nu)h - (b_2(t_\nu) + \beta_\nu^2)h$$

which are the numerical solution of

$$B \begin{bmatrix} y_{\nu+1}^{**} - y_\nu^{**} \\ z_{\nu+1}^{**} - z_\nu^{**} \end{bmatrix} = h \begin{bmatrix} \Phi_\nu^{**}(y_\nu^{**}, z_\nu^{**}, h) \\ \psi_\nu^{**}(y_\nu^{**}, z_\nu^{**}, h) \end{bmatrix}$$

with the increment function Φ_ν^{**} and ψ_ν^{**}

$$h \begin{bmatrix} \Phi_\nu^{**}(y_\nu^{**}, z_\nu^{**}, h) \\ \psi_\nu^{**}(y_\nu^{**}, z_\nu^{**}, h) \end{bmatrix} =$$

$$h \begin{bmatrix} f(y_\nu^{**} + a_1(t_\nu)h + (a_2(t_\nu) + \alpha_\nu^2)h^2, \quad z_\nu^{**} + b_1(t_\nu)h + (b_2(t_\nu) + \beta_\nu^2)h^2) \\ g(y_\nu^{**} + a_1(t_\nu)h + (a_2(t_\nu) + \alpha_\nu^2)h^2, \quad z_\nu^{**} + b_1(t_\nu)h + (b_2(t_\nu) + \beta_\nu^2)h^2) \end{bmatrix}$$

$$-Bh \begin{bmatrix} a_1(t_{\nu+1}) - a_1(t_\nu) \\ b_1(t_{\nu+1}) - b_1(t_\nu) \end{bmatrix} - Bh^2 \begin{bmatrix} a_2(t_{\nu+1}) - a_2(t_\nu) \\ b_2(t_{\nu+1}) - b_2(t_\nu) \end{bmatrix}$$

$$-Bh^2\begin{bmatrix}\alpha^2_{\nu+1}-\alpha^2_\nu\\ \beta^2_{\nu+1}-\beta^2_\nu\end{bmatrix}$$

In a similar way, the local error will be of the form

$$B\begin{bmatrix}y(t_{\nu+1})-y(t_\nu)\\ z(t_{\nu+1})-z(t_\nu)\end{bmatrix}-h\begin{bmatrix}\Phi^{**}_\nu(y(t_\nu),z(t_\nu),h)\\ \psi^{**}_\nu(y(t_\nu),z(t_\nu),h)\end{bmatrix}$$

$$=h^2\begin{bmatrix}\frac{1}{2!}y''-A_1y'-A_2z'-f_ya_1-f_zb_1+a'_1\\ -A_3y'-A_4z'-g_ya_1-g_zb_1\end{bmatrix}_{t_\nu}$$

$$+h^3\begin{bmatrix}a'_2-f_ya_2-f_zb_2-\frac{1}{2!}A_1y''-\frac{1}{2!}A_2z''\\ -g_ya_2-g_zb_2-\frac{1}{2!}A_3y''-\frac{1}{2!}A_4z''-\frac{1}{2!}g_{yy}a_1^2\end{bmatrix}_{t_\nu}$$

$$+h^3\begin{bmatrix}\frac{1}{3!}y'''-\frac{1}{2!}f_{yy}a_1^2-f_{yz}a_1b_1-\frac{1}{2!}f_{zz}b_1^2+\frac{1}{2!}a_1''-A_1a'_1-A_2b'_1\\ -g_{yz}a_1b_1-\frac{1}{2!}g_{zz}b_1^2-A_3a'_1-A_4b'_1\end{bmatrix}_{t_\nu}$$

$$+h^2B\begin{bmatrix}\alpha^2_{\nu+1}-\alpha^2_\nu\\ \beta^2_{\nu+1}-\beta^2_\nu\end{bmatrix}-h^3\begin{bmatrix}f_y & f_z\\ g_y & g_z\end{bmatrix}_{t_\nu}\begin{bmatrix}\alpha^2_\nu\\ \beta^2_\nu\end{bmatrix}+O(h^4)$$

the first matrix is equal to zero matrix from the first step. The local error of this method is seen to be $O(h^4)$ if $a_2(t)$ and $b_2(t)$ are the smooth solutions of the following DAE.

$$\begin{aligned}a'_2(t)&=f_ya_2(t)+f_zb_2(t)+\frac{1}{2!}A_1y''(t)+\frac{1}{2!}A_2z''(t)-\frac{1}{3!}y'''(t)\\&+\frac{1}{2!}f_{yy}a_1^2(t)+f_{yz}a_1(t)b_1(t)+\frac{1}{2!}f_{zz}b_1^2(t)\\&-\frac{1}{2!}a_1''(t)+A_1a'_1(t)+A_2b'_1(t)\end{aligned}\tag{A.7a}$$

$$\begin{aligned}O&=g_ya_2(t)+g_zb_2(t)+\frac{1}{2!}A_3y''(t)+\frac{1}{2!}A_4z''(t)\\&+g_{yz}a_1(t)b_1(t)+\frac{1}{2!}g_{zz}b_1^2(t)+A_3a'_1(t)+A_4b'_1(t)+\frac{1}{2!}g_{yy}a_1^2(t)\end{aligned}\tag{A.7b}$$

and the sequences $\{\alpha^2_\nu\}$, $\{\beta^2_\nu\}$ are bounded solutions of

$$B\begin{bmatrix}\alpha^2_{\nu+1}-\alpha^2_\nu\\ \beta^2_{\nu+1}-\beta^2_\nu\end{bmatrix}=h\begin{bmatrix}f_y & f_z\\ g_y & g_z\end{bmatrix}_{t_\nu}\begin{bmatrix}\alpha^2_\nu\\ \beta^2_\nu\end{bmatrix}\tag{A.8}$$

where the functions f, g and all partial derivatives are the function of $y(t)$, $z(t)$.

Third Step:

$$y_\nu^{***} = y_\nu - a_1(t_\nu)h - (a_2(t_\nu) + \alpha_\nu^2)h^2 - (a_3(t_\nu) + \alpha_\nu^3)h^3$$

$$z_\nu^{***} = z_\nu - (b_1(t_\nu)h - (b_2(t_\nu) + \beta_\nu^2)h^2 - (b_3(t_\nu) + \beta_\nu^3)h^3$$

and the new system

$$B \begin{bmatrix} y_{\nu+1}^{***} - y_\nu^{***} \\ z_{\nu+1}^{***} - z_\nu^{***} \end{bmatrix} = h \begin{bmatrix} \Phi_\nu^{***}(y_\nu^{***}, z_\nu^{***}) \\ \psi_\nu^{***}(y_\nu^{***}, z_\nu^{***}) \end{bmatrix}$$

After using the same way as first and second step we conclude that

$$\begin{aligned} a'_3(t) &= f_y a_3(t) + f_z b_3(t) + \frac{1}{3!}A_1 y'''(t) + \frac{1}{3!}A_2 z'''(t) - \frac{1}{4!}y^{(4)}(t) \\ &+ f_{yy}a_1a_2(t) + f_{yz}b_2(t)a_1(t) + f_{zy}b_1(t)a_2(t) + f_{zz}b_1(t)b_2(t) \\ &+ f_{yyy}a_1^3(t) + f_{zzy}a_1(t)b_1^2(t) + f_{yyz}a_1^2(t)b_1(t) + f_{zzz}b_1^3(t) \\ &- \frac{1}{3!}a'''_1(t) + A_1\frac{1}{2!}a_1''(t) + A_2\frac{1}{2!}b''_1(t) + \frac{1}{2!}a_2''(t) \\ &- A_1a_2'(t) - A_2b_2' \end{aligned} \tag{A.9a}$$

$$\begin{aligned} O &= g_y a_3(t) + g_z b_3(t) + \frac{1}{3!}A_3 y'''(t) + \frac{1}{3!}A_4 z'''(t) \\ &+ g_{yy}a_1a_2(t) + g_{yz}b_2(t)a_1(t) + g_{zy}b_1(t)a_2(t) + g_{zz}b_1(t)b_2(t) \\ &+ g_{yyy}a_1^3(t) + g_{yyz}a_1^2(t)b_1(t) + g_{zzy}a_1(t)b_1(t)^2 \\ &+ A_3\frac{1}{2!}a_1''(t) + A_4\frac{1}{2!}b_1''(t) \end{aligned} \tag{A.9b}$$

$$\begin{aligned} B \begin{bmatrix} \alpha_{\nu+1}^3 - \alpha_\nu^3 \\ \beta_{\nu+1}^3 - \beta_\nu^3 \end{bmatrix} &= h \begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix}_{t_\nu} \begin{bmatrix} \alpha_\nu^3 \\ \beta_\nu^3 \end{bmatrix} \\ &- h \begin{bmatrix} f_{yy}a_1(t) + f_{zy}b_1(t) & f_{yz}a_1(t) + f_{zz}b_1(t) \\ g_{yy}a_1(t) + g_{zy}b_1(t) & g_{yz}a_1(t) + g_{zz}b_1(t) \end{bmatrix}_{t_\nu} \begin{bmatrix} \alpha_\nu^2 \\ \beta_\nu^2 \end{bmatrix} \end{aligned} \tag{A.10}$$

We can repeat this procedure in order to find $a_l(t), b_l(t)$ functions and $\{\alpha_\nu^l\}$, $\{\beta_\nu^l\}$ sequences. It can easily seen from the (A.5a,b, A.7a,b, A.9a,b) that $a_l(t)$ and $b_l(t)$ are smooth solutions of the following equations.

$$\begin{aligned} a'_l(t) &= f_y a_l(t) + f_z b_l(t) + \frac{1}{l!}A_1 y^{(l)}(t) + \frac{1}{l!}A_2 z^{(l)}(t) - \frac{1}{(l+1)!}y^{(l+1)}(t) \\ &+ S_l(a_1, a_2, \ldots, a_{l-1}, b_1, b_2, \ldots, b_{l-1}, y, z) \\ &+ \sum_{k=1}^{l-1} \Lambda_k a_{l-k} + \sum_{k=1}^{l-1} \tilde{\Lambda}_k b_{l-k} \end{aligned} \tag{A.11}$$

$$O = g_y(y,z)a_l(t) + g_z(y,z)b_l(t) + \frac{1}{l!}A_3 y^{(l)}(t) + \frac{1}{l!}A_4 z^{(l)}(t)$$
$$+ \tilde{S}_l(a_1,\ldots,a_{l-1},b_1,\ldots,b_{l-1},y,z) + \sum_{k=1}^{l-1}\Gamma_k a_{l-k} + \sum_{k=1}^{l-1}\tilde{\Gamma}_k b_{l-k} \quad \text{(A.12)}$$

where $\Lambda_k, \tilde{\Lambda}_k, \Gamma_k$ and $\tilde{\Gamma}_k$ are differential operators of order k with respect to t. S_l and $\tilde{S}_l$ are smooth functions.

Since g_z^{-1} exist, $b_l(t)$ is obtained from (A.12) and substituted in (A.11) and $a_l(t), b_l(t)$ are smooth functions for all l, using the implicit function theorem, after some substituting process, we get;

$$a'_l(t) = K_1(y,z)a_l(t) + \frac{1}{l!}K_2(y,z)y^{(l)}(t) + \frac{1}{l!}K_3(y,z)z^{(l)}(t)$$
$$+ \frac{1}{(l+1)!}y^{(l+1)}(t) + E_l(a_1,\ldots,a_{l-1},y,z) = \varphi_l(t,a_l(t)) \quad \text{(A.13)}$$

$$b_l(t) = K_4(y,z)a_l(t) + \frac{1}{l!}K_5(y,z)y^{(l)}(t) + \frac{1}{l!}K_6 z^{(l)}(t) + \tilde{E}_l(a_1,\ldots,a_{l-1},y,z)$$
$$= \phi_l(t,a_l(t)) \quad \text{(A.14)}$$

with the initial conditions;

$$a_l(0) = \begin{cases} 0 & l=1 \\ -\alpha_0^l & l=2,\ldots,m \end{cases} \quad \text{(A.15a)}$$

$$b_l(0) = \begin{cases} 0 & l=1 \\ -\beta_0^l & l=2,\ldots,m \end{cases} \quad \text{(A.15b)}$$

where

$$\begin{aligned} K_1(y,z) &= (f_y - f_z g_z^{-1} g_y)_{(y,z)} & K_4 &= (-g_z^{-1}g_y)_{(y,z)} \\ K_2(y,z) &= (A_1 - f_z g_z^{-1} A_3)_{(y,z)} & K_5 &= (-g_z^{-1}A_3)_{(y,z)} \\ K_3(y,z) &= (A_2 - f_z g_z^{-1} A_4)_{(y,z)} & K_6 &= (-g_z^{-1}A_4)_{(y,z)} \end{aligned} \quad \text{(A.15c)}$$

and E_l, $\tilde{E}_l$ are smooth functions with $E_1 = \tilde{E}_1 = 0$, φ_l and ϕ_l are smooth functions defined as follows,

$$\varphi_l : \mathbb{R}^{S_1+2} \to \mathbb{R}^{S_1+1}, \quad \phi_l : \mathbb{R}^{S_2+1} \to \mathbb{R}^{S_2}$$

Similarly the general system for the sequence $\{\alpha_\nu^l\}$ and $\{\beta_\nu^l\}$ is of the form

$$B\begin{bmatrix} \alpha_{\nu+1}^l - \alpha_\nu^l \\ \beta_{\nu+1}^l - \beta_\nu^l \end{bmatrix} = h\begin{bmatrix} f_y & f_z \\ g_y & g_z \end{bmatrix}_{t_\nu}\begin{bmatrix} \alpha_\nu^l \\ \beta_\nu^l \end{bmatrix} + hD_1(t_\nu)\begin{bmatrix} \alpha_\nu^{l-1} \\ \beta_\nu^{l-1} \end{bmatrix}$$

$$+hD_2(t_\nu)\begin{bmatrix}\alpha_\nu^{l-2}\\ \beta_\nu^{l-2}\end{bmatrix}+\ldots+hD_{l-2}(t_\nu)\begin{bmatrix}\alpha_\nu^{2}\\ \beta_\nu^{2}\end{bmatrix} \tag{A.16}$$

for $l = 2, 3, \ldots, m$.

The entries of the matrices D_i are the functions of f and g; and their partial derivatives at $t = t_\nu$.

Since (A.8) is the linear system, using back iteration we obtain

$$\begin{bmatrix}\alpha_\nu^{2}\\ \beta_\nu^{2}\end{bmatrix} = H_2(t_{\nu-1}, t_{\nu-2}, \ldots, t_0)\begin{bmatrix}\alpha_0^{2}\\ \beta_0^{2}\end{bmatrix}$$

and in similar manner,

$$\begin{bmatrix}\alpha_\nu^{3}\\ \beta_\nu^{3}\end{bmatrix} = H_3(t_0, \ldots, t_{\nu-1})\begin{bmatrix}\alpha_0^{3}\\ \beta_0^{3}\end{bmatrix} + \bar{H}_3(t_0, \ldots, t_{\nu-1})\begin{bmatrix}\alpha_0^{2}\\ \beta_0^{2}\end{bmatrix}$$

then (A.16) can be rewritten

$$\begin{bmatrix}\alpha_\nu^{l}\\ \beta_\nu^{l}\end{bmatrix} = M_l\begin{bmatrix}\alpha_0^{l}\\ \beta_0^{l}\end{bmatrix} + M_{l-1}\begin{bmatrix}\alpha_0^{l-1}\\ \beta_0^{l-1}\end{bmatrix} + \ldots + M_2\begin{bmatrix}\alpha_0^{2}\\ \beta_0^{2}\end{bmatrix} \tag{A.17}$$

where the entries of the matrices M_i, $(i = 2, \ldots, l)$ are the functions of f and g and their partial derivatives at the point $t_0, t_1, \ldots, t_{\nu-1}$.

We can now define α_ν^l and β_ν^l so that

$$\alpha_\nu^l = N_l(\alpha_0^2, \ldots, \alpha_0^l, \beta_0^2, \ldots, \beta_0^l, y(t_0), y(t_1), \ldots, y(t_{\nu-1}), z(t_0), z(t_1), \ldots, z(t_{\nu-1}))$$
$$\beta_\nu^l = \tilde{N}_l(\alpha_0^2, \ldots, \alpha_0^l, \beta_0^2, \ldots, \beta_0^l, y(t_0), y(t_1), \ldots, y(t_{\nu-1}), z(t_0), z(t_1), \ldots, z(t_{\nu-1}))$$

(A.18)

where N_l and $\tilde{N}_l$ are smooth functions with respect to the values $y(t_i)$ and $z(t_i)$ $(i = 1, 2, \ldots, n \cdot m)$.

Theorem A.2 : Under the assumptions of Theorem A.1, the numerical solution of (NP) (2.2.7) has a perturbed asymptotic expansion of the form,

$$\pi_\nu^{[j]} = P_h^{[j]}(t_\nu) + a_{1,h}^{[j]}(t_\nu)h + \sum_{l=2}^{m} h^l(a_{l,h}^{[j]}(t_\nu) + \alpha_{\nu,h}^{[j]l}) + R_{\nu,h}^{[j]} \tag{A.19a}$$

$$\omega_\nu^{[j]} = Q_h^{[j]}(t_\nu) + b_{1,h}^{[j]}(t_\nu)h + \sum_{l=2}^{m} h^l(b_{l,h}^{[j]}(t_\nu) + \beta_{\nu,h}^{[j]l}) + \tilde{R}_{\nu,h}^{[j]} \tag{A.19b}$$

for $\nu = 0, 1, \ldots, n \cdot m$ and $j = 0, 1, \ldots, j_{\max}$
where $\|R_{\nu,h}^{[j]}\| \leq$ *const.* h^{m+1}, $\|\tilde{R}_{\nu,h}^{[j]}\| \leq$ *const.* h^{m+1} and $P_h^{[j]}(t)$, $Q_h^{[j]}(t)$ are piecewise polynomial of fixed degree m, and the variational equations for

$$a_{l,h}^{[j]}(t) = a_{l,i}^{[j]}(t), \;\; b_{l,h}^{[j]}(t) = b_{l,i}^{[j]}(t), \quad t \in I_i, \;\; i = 1, 2, \ldots, n \tag{A.20}$$

and the sequences

$$\alpha_{\nu,h}^{[j]} = \alpha_{\nu_i}^{[j]l}, \;\; \beta_{\nu,h}^{[j]l} = \beta_{\nu_i}^{[j]l}, \quad \nu_i = (i-1) \cdot m, \ldots, i \cdot m$$

are given by

$$\begin{aligned} a_{l,i}^{[j]\prime}(t) &= K_{1,i}^{[j]}(P_i^{[j]}, Q_i^{[j]}) a_{l,i}^{[j]}(t) + \frac{1}{l!} K^{[j]} K_{2,i}(P_i^{[j]}, Q_i^{[j]}) P_i^{[j](l)}(t) + \frac{1}{l!} K_{3,i}^{[j]}(P_i^{[j]}, Q_i^{[j]}) Q_i^{[j](l)}(t) \\ &\quad + \frac{1}{(l+1)!} P_i^{[j](l+1)}(t) + E_{l,i}^{[j]}(a_{1,i}^{[j]}, a_{2,i}^{[j]}, \ldots, a_{l-1,i}^{[j]}, P_i^{[j]}, Q_i^{[j]}) \\ &= \varphi_{l,i}^{[j]}(t, a_{l,i}^{[j]}(t)) \end{aligned} \tag{A.21}$$

$$\begin{aligned} b_{l,i}^{[j]}(t) &= K_{4,i}^{[j]}(P_i^{[j]}, Q_i^{[j]}) a_{l,i}^{[j]}(t) + \frac{1}{l!} K_{5,i}^{[j]}(P^{[j]}, Q_i^{[j]}) P_i^{[j](l)}(t) + \frac{1}{l!} K_{6,i}^{[j]} Q_i^{[j](l)}(t) \\ &\quad + \tilde{E}_{l,i}^{[j]}(a_{1,i}^{[j]}, a_{2,i}^{[j]}, \ldots, a_{l-1,i}^{[j]}, P_i^{[j]}, Q_i^{[j]}) \\ &= \phi_{l,i}^{[j]}(t, a_{l,i}^{[j]}(t)) \end{aligned} \tag{A.22}$$

with the initial conditions

$$a_{l,i}^{[j]}(t^{i-1}) = \begin{cases} 0 & i = 1 \\ a_{l,i-1}^{[j]}(t^{i-1}) & i = 2, \ldots, n \end{cases} \tag{A.23a}$$

$$b_{l,i}^{[j]}(t^{i-1}) = \begin{cases} 0 & i = 1 \\ b_{l,i-1}(t^{i-1}) & i = 2, 3, \ldots, n \end{cases} \tag{A.23b}$$

and $E_{1,i}^{[j]} = 0$, $\tilde{E}_{1,i}^{[j]} = 0, \ldots, \varphi_{l,i}^{[j]}$ and $\phi_{l,i}^{[j]}$ are smooth functions.

$$\begin{aligned} \alpha_{\nu,h}^{[j]l} &= N_{l,h}^{[j]}(\alpha_0^{[j]2}, \alpha_0^{[j]3}, \ldots, \alpha_0^{[j]l}, \beta_0^{[j]2}, \beta_0^{[j]3}, \ldots, \beta_0^{[j]l}, P_h^{[j]}(t_0, t_1, \ldots, t_{\nu-1}), Q_h^{[j]}(t_0, t_1, \ldots, t_{\nu-1})) \\ \beta_{\nu,h}^{[j]l} &= \tilde{N}_{l,h}^{[j]}(\alpha_0^{[j]2}, \alpha_0^{[j]3}, \ldots, \alpha_0^{[j]l}, \beta_0^{[j]2}, \beta_0^{[j]3}, \ldots, \beta_0^{[j]l}, P_h^{[j]}(t_0, t_1, \ldots, t_{\nu-1}), Q_h^{[j]}(t_0, t_1, \ldots, t_{\nu-1})) \end{aligned} \tag{A.24}$$

where $N_{l,h}^{[j]}$ and $\tilde{N}_{l,h}^{[j]}$ are smooth functions in according to $P_h^{[j]}(t_i), Q_h^{[j]}(t_i)$ for $i = 1, \ldots, n \cdot m$. The proof is similar to the proof of Theorem 1.1.

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VITA

Şennur Somalı (Uzuner) was born in İstanbul, in 1960. She received the B.Sc. degree in mathematics from the Middle East Technical University in 1982. She obtained the M.Sc. degree in mathematics from the Middle East Technical University in 1985. She works as an assistant in the University of Dokuz Eylül in Izmir.