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RESEARCH ARTICLE

Optimum Distance Cyclic H-Orbit Full Flag Codes

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ABSTRACT Flag codes, introduced as a special class of nested subspace codes, have recently attracted significant attention in the research community. In this work, we present constructions of optimum-distance cyclic H-orbit full flag codes, where H is a cyclic subgroup of $\mathbb{F}_{q^n}^*$ of order at most $q^{n/2}+1$. Our aim is to extend certain results from (Horlemann-Trautmann and Rosenthal, 2018) to the framework of flag codes. The proposed constructions, unlike the work (Alonso-González and Navarro-Pérez, 2024), achieve full flags with optimum distance and either maximal or best-possible size, depending on the parity of the base field. We also provide explicit examples that demonstrate the structure of our codes. Furthermore, for the case of even base field size, we prove -via intricate arithmetical arguments- that our construction is unique under certain conditions.

INDEX TERMS Flag codes, orbit flag codes, subspace codes.

I. INTRODUCTION

The framework of random network coding, introduced in [1], reimagines data transmission by representing the network as a directed acyclic multigraph that can accommodate multiple sources and destinations. Unlike traditional communication systems where intermediate nodes merely forward incoming data, this model permits nodes to transmit random linear combinations of the received messages. While this flexibility improves throughput, it also amplifies the risk of error propagation throughout the network.

To address the increased susceptibility to errors inherent in such randomized transmission, Koetter and Kschischang proposed the use of subspace codes in [11] as an effective error-control mechanism within this context, together with a Sudan-style "list-1" minimum-distance decoding algorithm. These codes consist of sets of subspaces drawn from an n-dimensional vector space over a finite field $\mathbb{F}_q$, with a special distance metric different from [5] used to measure code performance. When all the codewords (i.e., subspaces) share the same dimension, the code is referred to as a constant-dimension code (see e.g. [13]).

Unlike conventional subspace codes -where each codeword corresponds to a single subspace and is transmitted in one channel use- a different strategy, known as multishot coding, was proposed by Nobrega and Uchôa-Filho in [15] and [16]. In this model, a codeword is made up of a sequence of r subspaces from $\mathbb{F}_q^n$, requiring r successive transmissions through the channel. This multishot approach allows for the construction of well-performing codes without altering fundamental parameters such as the field size q or the vector space dimension n. A specific instance of multishot codes is the concept of flag codes. Here, a flag is defined as an ordered sequence $\mathcal{F} = \left[V_{t_1} \subset V_{t_2} \subset \cdots \subset V_{t_r}\right]$ of nested subspaces within $\mathbb{F}_q^n$. The dimensions of these subspaces form a tuple $(\dim V_{t_1}, \dim V_{t_2}, \ldots, \dim V_{t_r})$, which determines the type of the flag. For any strictly increasing sequence of integers $0 < t_1 < t_2 < \cdots < t_r < n$, a flag code of type $(t_1, t_2, \ldots, t_r)$ consists of a non-empty set of flags adhering to that dimensional structure.

Flag codes were initially introduced in network coding applications in [14]. A more refined analysis appears in [2], where the focus is on optimum distance flag codes, which maximize the minimum distance between distinct codewords. In that study, flag codes are described in terms of their projected codes, i.e., the constant-dimension subcodes used at each shot. It is also shown that when one of these projected codes is a spread code, the resulting flag code not only achieves optimum distance but also attains the largest possible size under those constraints.

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In a communication setting, network coding has a significant importance as it allows the system to encode the intended message at each node through the network instead of transmitting it to the next node just as it is [8]. Using flag codes in network coding provides improved error detecting and correcting capability [2]. The transmitted subspace at each time step has a fixed dimension and must contain all subspaces sent in previous time steps. This nested structure enhances error-correction capability [14]. Several works, including [2], [3], [12], have since explored constructions of optimum-distance flag codes.

Cyclic orbit flag codes were investigated in [4], where it was shown that codes with full-length orbits generally achieve only quasi-optimum distance and are often not of full type.

In this work, we extend some results of [10] to flag code settlement. We focus on *optimum distance* H-orbit *full* flag codes with either *maximal* or *best possible* size, where H is a cyclic subgroup of $\mathbb{F}_{q^n}^*$ and n is even. More precisely, H has maximal size $q^{n/2}+1$ when q is even, and H has best possible size $(q^{n/2}+1)/m$ when q is odd. A full flag code has optimum distance if its flag distance is equal to $n^2/2$ and has quasi-optimum distance if its flag distance is $n^2/2-2$ while n is even. Below, we summarize our contributions followed by a concise comparison with the previous works on flag codes.

- For any even integer n and field size q, we provide full flag codes and prove that they have optimum distance,
- We determine the maximal size $q^{n/2}+1$ of the orbit created by H and our construction reaches the maximal size for even q,
- We show that for odd q, the maximal orbit size is not attainable. However, our construction attains the best possible size N for odd q, where $N=(q^{n/2}+1)/m$ with
$$m = 2^k \text{ such that } 2^k \mid (q^{n/2}+1) \text{ and } 2^{k+1} \nmid (q^{n/2}+1).$$
In other words, the best possible size of an optimum distance H-orbit full flag code is the largest odd divisor of $q^{n/2}+1$. Moreover, our construction yields an infinite family of flag codes since m increases with n.
- We also count the number of possible optimum distance cyclic H-orbit full flag codes using MAGMA [6], when H is fixed. Relying on the outcomes, we provide two conjectures on the number of such flag codes. Furthermore, we show that there is a *unique* optimum distance H-orbit full flag code with maximal size when q is even and $n=4$.

TABLE 1. The Comparison of our work with previous ones.

Work	Type	Size	Distance	Orbit Length
Our work	full	maximal/ best possible	optimum	non-full
[4]	non-full	NA	quasi-optimum	full length
[2]	full	maximal	optimum	NA
[3]	non-full	NA	optimum	NA

This paper is organized as follows: in Section II, we give the necessary background and definitions together with the notation used in the rest of the paper. Section III presents our contributions for even and odd field sizes, starting from our generalized main result. Then, the detailed analysis of the results are presented when $n=6$ for a better illustration. Finally, our experimental outcomes calculated via MAGMA are given in Table 1 in Section IV, together with two conjectures and the proof of a particular case of the first conjecture.

II. PRELIMINARIES

Throughout this paper, we use the following definitions and notations. Let $\mathbb{F}_q$ be the finite field with q elements, for an arbitrary prime power q, and let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. The set of all subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{P}_q(n)$. The set of all k-dimensional subspaces of $\mathbb{F}_q^n$ is called the *Grassmannian* over $\mathbb{F}_q$, which is denoted by $\mathcal{G}_q(k,n)$. The metric $d_S(U,V) := \dim_{\mathbb{F}_q}(U)+\dim_{\mathbb{F}_q}(V)-2\dim_{\mathbb{F}_q}(U\cap V)$ on $\mathcal{P}_q(n)$ is called the *subspace distance*. A non-empty subset $C \subseteq \mathcal{P}_q(n)$ with this metric is called a *subspace code*. If moreover $C \subseteq \mathcal{G}_q(k,n)$, then C is said to be a *constant dimension code*. We define the *minimum (subspace) distance* of C by $d(C) := \min\{d_S(U,V) : U, V \in \mathcal{C} \text{ and } U \neq V\}$.

Using the linear isomorphism between $\mathbb{F}_q^n$ and $\mathbb{F}_{q^n}$, we consider a k-dimensional subspace $\mathcal{U} \subseteq \mathbb{F}_{q^n}$. For $\alpha \in \mathbb{F}_{q^n}^*$, we have

$$\mathcal{U}\alpha = \{\, u\alpha \mid u \in \mathcal{U} \,\} \in \mathcal{G}_q(k,n). \tag{1}$$

In this way, $\mathbb{F}_{q^n}^*$ acts on $\mathcal{G}_q(k,n)$ and hence it allows us to build orbits of this multiplicative action. We can define the *cyclic orbit code* generated by $\mathcal{U}$ as $Orb(\mathcal{U}) := \left\{\mathcal{U}\alpha \mid \alpha \in \mathbb{F}_{q^n}^*\right\}$. The *stabilizer* of $\mathcal{U}$ is given as $Stab(\mathcal{U}) := \left\{\alpha \in \mathbb{F}_{q^n}^* \mid \mathcal{U}\alpha = \mathcal{U}\right\}$. The code $Orb(\mathcal{U})$ has cardinality $(q^n-1)/(q^m-1)$ if and only if $Stab(\mathcal{U}) := \mathbb{F}_{q^m}$, for some m dividing n.

Now let $n \geq 2$ be an integer and let $\{t_1, t_2, \ldots, t_r\} \subseteq \{1, \ldots, n-1\}$, where $1 \leq t_1 < t_2 < \cdots < t_r \leq (n-1)$, and set $T := (t_1, t_2, \ldots, t_r)$. By a *flag* $\mathcal{F}$ of type-T in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, we mean a chain of $\mathbb{F}_q$-linear subspaces

$$\mathbb{F}_q \subseteq V_{t_1} \subset V_{t_2} \subset \cdots \subset V_{t_r} \subseteq \mathbb{F}_q^n \tag{2}$$

such that $\mathcal{F} := \left[V_{t_1} \subset V_{t_2} \subset \cdots \subset V_{t_r}\right]$ and

$$\dim_{\mathbb{F}_q}(V_{t_1}) = t_1,\ \dim_{\mathbb{F}_q}(V_{t_2}) = t_2, \ldots,\ \dim_{\mathbb{F}_q}(V_{t_r}) = t_r.$$

The collection of all flags of type-T in $\mathbb{F}_{q^n}$ is called the *ambient space* of flags of type-T over $\mathbb{F}_q$, which is denoted by $\mathcal{S}$.

By the *(minimum) flag distance* $d_f(\mathcal{F}_1, \mathcal{F}_0)$ between two flags $\mathcal{F}_1 = \left[V_{t_1} \subset V_{t_2} \subset \cdots \subset V_{t_r}\right]$ and $\mathcal{F}_0 = \left[U_{t_1} \subset Ut_2 \subset \cdots \subset U_{t_r}\right]$ of type-T in $\mathbb{F}_{q^n}$, we mean

$$d_f(\mathcal{F}_1, \mathcal{F}_0) = \sum_{i=1}^{r} d_S\left(U_{t_i}, V_{t_i}\right).$$

Remark 1: If $r=1$ and $T=(t_1)$, then the ambient space of flags of type-T in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ is equal to the ambient space of constant t_1-dimensional $\mathbb{F}_q$-linear codes in $\mathbb{F}_{q^n}$, which is the Grassmannian $\mathcal{G}_q(t_1, n)$. Moreover, in this case, the flag distance is the same as the subspace distance (see [9], [10]). A *flag code of type-T* in $\mathbb{F}_{q^n}$ is a subset $\mathcal{C} \subseteq \mathcal{S}$ such that $|\mathcal{C}| \geq 2$. By the **minimum flag distance** $d_f(\mathcal{C})$ of the flag code $\mathcal{C}$, we mean

$$d_f(\mathcal{C}) := \min\left\{d_f(\mathcal{F}_1, \mathcal{F}_0) : \mathcal{F}_1, \mathcal{F}_0 \in \mathcal{C} \text{ and } \mathcal{F}_1 \neq \mathcal{F}_0\right\}. \quad (3)$$

Let $d \geq 1$ be an integer. We say that d is an *admissible minimum distance* for a flag code of type-T in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ if there exists a flag code $\mathcal{C}$ of type-T in $\mathbb{F}_{q^n}$ such that $d_f(\mathcal{C}) = d$.

We say that T is the *full type* in $\mathbb{F}_{q^n}$ if $r = n-1$ and $T = (1, 2, \ldots, n-1)$. We say $\mathcal{C}$ is a **full flag code**, if it is a flag code of the full type $(1, 2, \ldots, n-1)$ in $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

Let $d \geq 1$ be an admissible minimum distance for a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Let $\mathcal{C}$ be a flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ such that $d_f(\mathcal{C}) = d$. We say that $\mathcal{C}$ is a **maximal** flag code of type-T in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with $d_f(\mathcal{C}) = d$ if

$$|\mathcal{C}| = \mathcal{A}_{\mathbb{F}_q,n}(T; d),$$

where $\mathcal{A}_{\mathbb{F}_q,n}(T; d)$ is the largest possible size of a flag code of type$-T$ in $\mathbb{F}_q^n$ over $\mathbb{F}_q$ with $d_f(\mathcal{C}) \geq d$.

Remark 2: It is known that if d is an admissible minimum distance for a full flag code $\mathcal{C}$ in $\mathbb{F}_q^n$ over $\mathbb{F}_q$, then

$$d_f(C) \leq \begin{cases} \dfrac{n^2}{2}, & n \text{ is even,} \\ \dfrac{n^2-1}{2}, & n \text{ is odd.} \end{cases} \quad (4)$$

For any arbitrary full flag code $\mathcal{C}$, this upper bound is attainable only when n is even. We refer to [2] for the proof.

The concept of (1) can be extended to flags as follows. Given a flag $\mathcal{F}$ of type $T = (t_1, t_2, \ldots, t_r)$ on $\mathbb{F}_{q^n}$ and any element $\alpha \in \mathbb{F}_{q^n}^*$, we define

$$\mathcal{F}\alpha = [V_{t_1}\alpha \subset V_{t_2}\alpha \cdots \subset V_{t_r}\alpha].$$

Then, the *cyclic orbit flag code* generated by $\mathcal{F}$ is

$$Orb(\mathcal{F}) = \left\{\mathcal{F}\alpha \mid \alpha \in \mathbb{F}_{q^n}^*\right\}. \quad (5)$$

We summarize the given information here in the diagram given in the figure below.

III. OPTIMUM DISTANCE CYCLIC H-ORBIT FULL FLAG CODES

Let n be an arbitrary even integer. Let w be a generator of $\mathbb{F}_{q^n}$ and put $a = w^{(q^n-1)/N}$ for $N \mid q^n - 1$. We consider the cyclic subgroup $H := \langle a \rangle \subseteq \mathbb{F}_{q^n}^*$ with $|H| = N$. Also, let $V_1, \ldots, V_{n/2-1}$ be arbitrary $\mathbb{F}_q-$linear subspaces in $\mathbb{F}_{q^{n/2}}$ of dimension $1, \ldots, n/2-1$, respectively, forming the chain $0 \subset V_1 \subset \cdots \subset V_{n/2-1} \subset \mathbb{F}_{q^{n/2}}$. Let $V_{n/2+1}, \ldots, V_{n-1}$ be arbitrary $\mathbb{F}_q-$linear subspaces in $\mathbb{F}_{q^n}$ of dimension $n/2+1, \cdots, n-1$ forming the chain $\mathbb{F}_{q^{n/2}} \subset V_{n/2+1} \subset \cdots \subset V_{n-1} \subset \mathbb{F}_{q^n}$. Let $\mathcal{F}$ be the flag $\mathcal{F} = \left[V_1 \subset \cdots \subset V_{n/2-1} \subset \mathbb{F}_{q^{n/2}} \subset V_{n/2+1} \subset \cdots \subset V_{n-1}\right]$, then the code $\mathcal{C} = \left\{\mathcal{F}, a\mathcal{F}, a^2\mathcal{F}, \ldots, a^{N-1}\mathcal{F}\right\}$ is a cyclic H-orbit full flag code over $\mathbb{F}_q$ with respect to the flag distance given in (3).

We need the following number theoretic fact for our characterization of orbit sizes.

Lemma 1: Let q be a power of a prime number and let n be an even integer. Let $N = (q^{n/2}+1)/m$, where

$$m = 2^k \text{ such that } 2^k \mid (q^{n/2}+1) \text{ and } 2^{k+1} \nmid (q^{n/2}+1).$$

Then, we have $\gcd(N, m(q^{n/2}-1)) = 1$.

Proof: If q is even, then we obviously have $m = 1$ and $\gcd(q^{n/2}+1, q^{n/2}-1) = 1$. If q is odd, then N the largest odd divisor of $q^{n/2}+1$. Suppose that N and $m(q^{n/2}-1)$ have a common divisor p, which is an odd prime. In this case, we have $p \mid \gcd(q^{n/2}+1, q^{n/2}-1)$ and this implies $q^{n/2}+1 \equiv q^{n/2}-1 \equiv 0 \pmod p$. But then $2 \equiv 0 \pmod p$, which is impossible as p is odd. Hence, $\gcd(N, m(q^{n/2}-1)) = 1$. ■

Using Lemma 1 above, we utilize this particular choice of the orbit size $N = (q^{n/2}+1)/m$ so that the code $\mathcal{C} = \left\{\mathcal{F}, a\mathcal{F}, a^2\mathcal{F}, \ldots, a^{N-1}\mathcal{F}\right\}$ consists of N distinct flags with $a = w^{(q^n-1)/N} = w^{m(q^{n/2}-1)}$, which cannot be extended further since $Nm(q^{n/2}-1) = q^n - 1$. Now we formalize this notion as follows.

Definition 1: We will call the size of the cyclic H-orbit full flag code $\mathcal{C}$ is **maximal** if $N = q^{n/2}+1$. Otherwise, for $a = w^{(q^n-1)/N}$ and $|\mathcal{C}| = N$, the size of $\mathcal{C}$ is said to be **best possible** if N is the largest odd divisor of $q^{n/2}+1$. Moreover, C is said to be an **optimum distance** cyclic H-orbit full flag code if $d_f(C) = n^2/2$.

Having optimum distance for a full flag code means to attain the upper bound in (4). In the literature, there are several works that defines quasi-optimum distance or some other works prefer the notion optimal distance to express the distance $n^2/2 - 2$.

Now we are ready to state our main result. Our code construction yields maximal or best possible optimum distance cyclic H-orbit full flag codes, depending on the size of the subgroup H. More precisely, when q is even, our construction attains the maximal size. When q is odd, our construction yields the best possible size. Moreover, there is no maximal size cyclic orbit full flag code when q is odd.

In general, we have $N = (q^{n/2}+1)/m$, where

$$m = 2^k \text{ such that } 2^k \mid (q^{n/2}+1) \text{ and } 2^{k+1} \nmid (q^{n/2}+1). \quad (6)$$

Hence, the cyclic H-orbit full flag code $\mathcal{C}$ has maximal size if $m = 1$, and it has the best possible size otherwise. Notice that m increases with n. For the sake of simplicity, we will investigate some small parametrized cases.

We first focus on the cases $n = 4$ and $n = 6$ to illustrate our idea and we set some notation through this section.

Let w_1 be a generator of $\mathbb{F}_{q^4}^*$ and $a_1 \in \mathbb{F}_{q^4}^*$. We will consider the cyclic group $\langle a_1 \rangle \subseteq \mathbb{F}_{q^4}^*$. We set $|\langle a_1 \rangle| = N_1$ where $N_1 \mid q^4 - 1$. Also, let V_1 be an arbitrary $\mathbb{F}_q-$linear subspace in $\mathbb{F}_{q^2}$ of dimension 1 forming the chain $0 \subset V_1 \subset \mathbb{F}_{q^2}$. Let V_3 be an arbitrary $\mathbb{F}_q-$linear subspace in $\mathbb{F}_{q^4}$ of dimension 3 forming the chain $\mathbb{F}_{q^2} \subset V_3 \subset \mathbb{F}_{q^4}$. Let $\mathcal{F}_1$ be the flag $\mathcal{F}_1 = \left[V_1 \subset \mathbb{F}_{q^2} \subset V_3\right]$, then the code $\mathcal{C}_1 = \left\{\mathcal{F}_1, a_1\mathcal{F}_1, a_1^2\mathcal{F}_1, \ldots, a_1^{N_1-1}\mathcal{F}_1\right\}$ is a cyclic orbit flag code

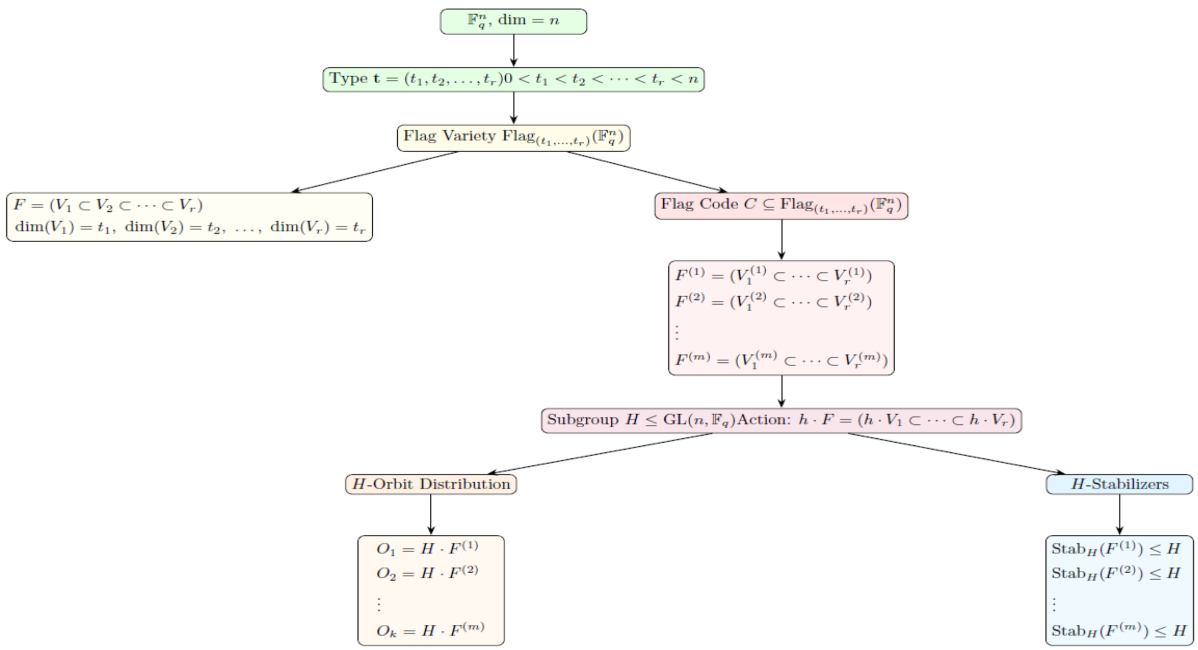

FIGURE 1. **The diagram of flag codes and the orbit flag codes.**

over $\mathbb{F}_q$ of type$-(1, 2, 3)$ with respect to the flag distance given in (3).

Example 1: Consider the flag, $\mathcal{F}_1 = \left[V_1 \subset \mathbb{F}_q^2 \subset V_3\right]$ with $dim_{\mathbb{F}_q}(V_3) = 3$ and let $\alpha \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$. We take

$$\alpha\mathcal{F}_1 = \left[\alpha V_1 \subset \alpha\mathbb{F}_q^2 \subset \alpha V_3\right].$$

The flag distance between these two flags is optimum. As we select $\alpha \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, then $d_S(V_1, \alpha V_1) = 2$ and $d_S(\mathbb{F}_{q^2}, \alpha\mathbb{F}_{q^2}) = 4$. We also have $d_S(V_3, \alpha V_3) = 3+3-2\cdot 2 = 2$, since $(V_3 + \alpha V_3) \supset \mathbb{F}_{q^2} + \alpha\mathbb{F}_{q^2} = \mathbb{F}_{q^4}$. This yields $d_f(\mathcal{F}_1, \alpha\mathcal{F}_1) = 2 + 4 + 2 = 8$, which is optimum.
The following results investigate the properties of the full flag code $\mathcal{C}_1$.

Theorem 1: Let $\mathbb{F}_q$ be a finite field of characteristic 2. Put $a_1 = w_1^{q^2-1}$. Let $\mathcal{C}_1$ be the flag code defined above, then $\mathcal{C}_1$ is a cyclic orbit flag code of type$-(1, 2, 3)$ of flag distance $d_f(\mathcal{C}_1) = 8$. In particular, $\mathcal{C}_1$ is optimum with respect to its flag distance. Moreover, the cardinality of $\mathcal{C}_1$ is maximal.

Theorem 2: Let $\mathbb{F}_q$ be a finite field of odd characteristic and $\langle w_1 \rangle = \mathbb{F}_{q^4}^*$. Let $\mathcal{C}_1$ be the flag code defined above, along with $a_1 = w_1^{2(q^2-1)}$ and $N_1 = (q^2+1)/2$. Then $\mathcal{C}_1$ is a cyclic orbit flag code of type$-(1, 2, 3)$ of flag distance $d_f(\mathcal{C}_1) = 8$. In particular, $\mathcal{C}_1$ is optimum with respect to its flag distance. Moreover, the cardinality of $\mathcal{C}$ is best possible.

Proofs of Theorems 1 and 2 :

Take any $\alpha_1, \alpha_2 \in \mathbb{F}_{q^4}^*$. If $\alpha_1/\alpha_2 \in \mathbb{F}_{q^2}$, then we get $(\alpha_1/\alpha_2)^{q^2-1} = \left(w_1^{m(q^2-1)(i_1-i_2)}\right)^{q^2-1} = 1$, which implies $(q^2+1) \mid m(q^2-1)(i_1-i_2)$.

- If q is even, then $m = 1$ and q^2+1 and q^2-1 are relatively prime. Hence, the size of the code is maximal.
- If q is odd, then $q^2+1 \equiv 2 \mod 4$.

$$\begin{aligned}
(\alpha_1/\alpha_2)^{q^2-1} = 1 &\Rightarrow \left(w_1^{2i_1(q^2-1)}/y^{2i_2(q^2-1)}\right)^{q^2-1} = 1 \\
&\Rightarrow w_1^{2(i_1-i_2)(q^2-1)^2} = 1 \\
&\Rightarrow q^4 - 1 \mid 2(q^2-1)^2(i_1-i_2) \\
&\Rightarrow q^2 + 1 \mid 2(q^2-1)(i_1-i_2) \\
&\Rightarrow \frac{q^2+1}{2} \,\Big|\, \frac{2(q^2-1)}{2}(i_1-i_2) \\
&\Rightarrow \frac{q^2+1}{2} \,\Big|\, (i_1-i_2)
\end{aligned}$$

Therefore, $\gcd(q^2+1, 2(q^2-1)) = 2$ as $q^2+1 \equiv 2 \mod 4$. Hence, $m = 2$.

■

Similarly, let w_2 be a generator of $\mathbb{F}_{q^6}^*$ and $a_2 \in \mathbb{F}_{q^6}^*$. We will consider the cyclic group $\langle a_2 \rangle \subseteq \mathbb{F}_{q^6}^*$. We set $|\langle a_2 \rangle| = N_2$ where $N_2 \mid q^6 - 1$. Also, let V_1, V_2 be arbitrary $\mathbb{F}_q-$linear subspaces in $\mathbb{F}_{q^3}$ of dimensions 1 and 2, respectively, forming the chain $0 \subset V_1 \subset V_2 \subset \mathbb{F}_{q^3}$. Let V_4, V_5 be arbitrary $\mathbb{F}_q-$linear subspaces in $\mathbb{F}_{q^6}$ of dimensions 4 and 5, respectively, forming the chain $\mathbb{F}_{q^3} \subset V_4 \subset V_5 \subset \mathbb{F}_{q^6}$. Let $\mathcal{F}_2$ be the flag $\mathcal{F}_2 = \left[V_1 \subset V_2 \subset \mathbb{F}_{q^3} \subset V_4 \subset V_5\right]$, then the code $\mathcal{C}_2 = \left\{\mathcal{F}_2, a_2\mathcal{F}_2, a_2^2\mathcal{F}_2, \ldots, a_2^{N_2-1}\mathcal{F}_2\right\}$ is a cyclic orbit flag code over $\mathbb{F}_q$ of type$-(1, 2, 3, 4, 5)$ with respect to the flag distance given in (3).

Example 2: Let $\langle w_2 \rangle = \mathbb{F}_{q^6}^*$ and take $w_2^{i_1} = \alpha_1$ and $w_2^{i_2} = \alpha_2$, for $i_1, i_2 \in \{0, \ldots, q^6-2\}$, such that $\alpha_1\mathbb{F}_{q^3} \cap \alpha_2\mathbb{F}_{q^3} =$

$\{0\}$, which implies $\alpha_1/\alpha_2 = w^{i_1-i_2} \notin \mathbb{F}^*_{q^3}$. For the full flag $\mathcal{F}_2 = \left[V_1 \subset V_2 \subset \mathbb{F}_{q^3} \subset V_4 \subset V_5\right]$, we consider

$$\alpha_1\mathcal{F}_2 = \left[\alpha_1 V_1 \subset \alpha_1 V_2 \subset \alpha_1\mathbb{F}_{q^3} \subset \alpha_1 V_4 \subset \alpha_1 V_5\right],$$
$$\alpha_2\mathcal{F}_2 = \left[\alpha_2 V_1 \subset \alpha_2 V_2 \subset \alpha_2\mathbb{F}_{q^3} \subset \alpha_2 V_4 \subset \alpha_2 V_5\right].$$

We claim that $d_f(\alpha_1\mathcal{F}_2, \alpha_2\mathcal{F}_2) = n^2/2 = 18$, which is the optimum distance for these parameters. Clearly, we have $\langle w_2^{q^3+1}\rangle = \mathbb{F}^*_{q^3}$. If $\alpha_1\mathbb{F}_{q^3} \cap \alpha_2\mathbb{F}_{q^3} \neq \{0\}$, then

$$\alpha_1 w_2^{(q^3+1)i} = \alpha_2 w_2^{(q^3+1)j}$$

for some $i, j \in \{0, \ldots, q^3-2\}$. But this implies

$$w_2^{i_1} w_2^{(q^3+1)i} = w_2^{i_2} w_2^{(q^3+1)j}$$

and therefore we have $q^3+1 \mid i_1 - i_2$, which is a contradiction as $\alpha_1/\alpha_2 = w_2^{i_1-i_2} \notin \mathbb{F}^*_{q^3}$. Hence, $\alpha_1\mathbb{F}_{q^3} \cap \alpha_2\mathbb{F}_{q^3} = \{0\}$.

So we have, $d_S(\alpha_1 V_1, \alpha_2 V_1) = 2$, $d_S(\alpha_1 V_2, \alpha_2 V_2) = 4$, and $d_S(\alpha_1\mathbb{F}_{q^3}, \alpha_2\mathbb{F}_{q^3}) = 6$. We also claim that $\dim(\alpha_1 V_4 \cap \alpha_2 V_4) = 2$, indeed $(\alpha_1 V_4 + \alpha_2 V_4) \supset \alpha_1\mathbb{F}_{q^3} + \alpha_2\mathbb{F}_{q^3} = \mathbb{F}_{q^6}$. Therefore, $d_S(\alpha_1 V_4, \alpha_2 V_4) = 4 + 4 - 2.2 = 4$. Similarly, $d_S(\alpha_1 V_5, \alpha_2 V_5) = 5 + 5 - 2.4 = 2$.
The following results investigate the properties of the full flag code $\mathcal{C}_2$.

Theorem 3: Let $\mathbb{F}_q$ be a finite field of characteristic 2. Let $\mathcal{C}_2$ be the flag code defined above, then $\mathcal{C}_2$ is a cyclic H-orbit flag code of type$-(1, 2, 3, 4, 5)$ of flag distance $d_f(\mathcal{C}_2) = 18$. In particular, $\mathcal{C}_2$ is optimum with respect to its flag distance. Moreover, the cardinality of $\mathcal{C}_2$ is maximal.

Theorem 4: Let $\mathbb{F}_q$ be a finite field of odd characteristic. Let $\mathcal{C}_2$ be the flag code defined above, such that

1) if $q^3 + 1 \equiv 2 \mod 4$, then $a_2 = w_2^{2(q^3-1)}$, $N_2 = (q^3+1)/2$,
2) if $q^3 + 1 \equiv 4 \mod 8$, then $a_2 = w_2^{4(q^3-1)}$, $N_2 = (q^3+1)/4$,
3) if $q^3 + 1 \equiv 0 \mod 8$, then $a_2 = w_2^{8(q^3-1)}$, $N_2 = (q^3+1)/8$.

Proofs of Theorems 3 and 4 :
Take any $\alpha_1, \alpha_2 \in \mathbb{F}^*_{q^6}$. If $\alpha_1/\alpha_2 \in \mathbb{F}_{q^3}$, then we get $(\alpha_1/\alpha_2)^{q^3-1} = \left(w_2^{m(q^3-1)(i_1-i_2)}\right)^{q^3-1} = 1$, which implies $(q^3+1) \mid m(q^3-1)(i_1-i_2)$.

- If q is even, then $m = 1$ and q^3+1 and q^3-1 are relatively prime. Hence, the size of the code is maximal.
- If q is odd, then there are 3 cases to consider.
  i. If q is odd, then $q = 4k+1$ or $q = 4k+3$. If $q = 4k+1$, then $q^3+1 = 64k^3 + 48k^2 + 12k + 2$. Hence, $q^3+1 \equiv 6 \mod 8$, when k is odd, and $q^3+1 \equiv 2 \mod 8$, when k is even. Therefore, $\gcd(2(q^3-1), (q^3+1)/2) = 1$.
  ii. If $q = 4k+3$, then $q^3+1 = 64k^3 + 144k^2 + 108k + 28$. Hence, $q^3+1 \equiv 4 \mod 8$, when k is even, and $\gcd(4(q^3+1), (q^3-1)/4) = 1$.
  iii. For $q = 4k+3$, we have $q^3+1 \equiv 0 \mod 8$, when k is odd, and hence $\gcd(8(q^3+1), (q^3-1)/8) = 1$.

Hence, when q is odd, we obtain the following.

Case 1: q is odd and $\mathbf{q^3+1 \equiv 2 \mod 4}$. Here we have $(q^3+1)/2 \big| (i_1-i_2)$. Hence, $m = 2$ and note that this case covers $q^3+1 \equiv 6 \mod 8$ and $q^3+1 \equiv 2 \mod 8$ by item i. above.

Case 2: q is odd and $\mathbf{q^3+1 \equiv 4 \mod 8}$. Using item ii., we get $(q^3+1)/4 \big| (i_1-i_2)$. Therefore, $m = 4$.

Case 3: q is odd and $\mathbf{q^3+1 \equiv 0 \mod 8}$. Item iii. implies $(q^3+1)/8 \big| (i_1-i_2)$ and $m = 8$. ∎

Then $\mathcal{C}_2$ is a cyclic H-orbit flag code of type$-(1, 2, 3, 4, 5)$ of flag distance $d_f(\mathcal{C}_2) = 18$. In particular, $\mathcal{C}_2$ is optimum with respect to its flag distance. Moreover, the cardinality of $\mathcal{C}_2$ is best possible.
Now we are ready to give our main theorem. The following result gives an infinite family of full flag codes.

Theorem 5: Let $\mathbb{F}_q$ be a finite field. Put $a = w^{(q^n-1)/N}$. Let $\mathcal{F} = \left[V_1 \subset \cdots \subset \mathbb{F}_{q^{n/2}} \subset \cdots \subset V_{n-1}\right]$ be a flag under $\mathbb{F}_{q^n}$, then $\mathcal{C} = \left\{\mathcal{F}, a\mathcal{F}, a^2\mathcal{F}, \ldots, a^{N-1}\mathcal{F}\right\}$ is an optimum distance cyclic H-orbit full flag code. While calculating the size of the code, the value of N is crucial and it is exactly $N = (q^{n/2}+1)/m$. Hence, $\mathcal{C}$ is maximal if $q = 2$, otherwise its size is said to be best possible.

A direct generalization of the last proof above can be applied for the proof of Theorem 5.

IV. NUMBER OF DISTINCT OPTIMUM DISTANCE CYCLIC *H*-ORBIT FULL FLAG CODES

There is a natural question that arises: how many other $n/2-$dimensional subspaces U create an optimum distance cyclic H-orbit full flag code when we fix $H = \langle w^{(q^n-1)/N}\rangle$ with $\mathbb{F}^*_{q^n} = \langle w\rangle$, $|H| = N$ and $N \mid q^n - 1$?

To study this question, we take any $n/2-$dimensional subspace U of $\mathbb{F}_q^n$ instead of $\mathbb{F}_{q^{n/2}}$. For $\alpha_1, \alpha_2 \in H$, we consider the distinct flags

$$\alpha_1\mathcal{F} = [\alpha_1 V_1 \subset \alpha_1 V_2 \subset \cdots \subset \alpha_1 U \subset \cdots \subset \alpha_1 V_{n-2} \subset \alpha_1 V_{n-1}]$$
$$\alpha_2\mathcal{F} = [\alpha_2 V_1 \subset \alpha_2 V_2 \subset \cdots \subset \alpha_2 U \subset \cdots \subset \alpha_2 V_{n-2} \subset \alpha_2 V_{n-1}].$$

What we want here is

$$\alpha_1, \alpha_2 \in \mathbb{F}_{q^n}, \qquad \alpha_1 U \cap \alpha_2 U = \{0\}. \tag{7}$$

Such a subset H must satisfy the condition (7) above and we need $|H| = N = (q^{n/2}+1)/m$, where m was determined in (6). Using MAGMA, we count all the possible $n/2-$dimensional subspaces U that create an optimum distance cyclic H-orbit full flag code for given n and q. These arithmetic calculations took about two-weeks at most using a MAGMA installed computer with 2,9 GHz Dual-Core CPU and 8 GB memory.

Based on the outcomes presented in Table 1 above, we pose the following two conjectures for the interested researchers in

TABLE 2. **The Numbers of distinct maximal cyclic *H*-orbit full flag codes.**

q	n	Number of U's	Orbit size	Number of Distinct Flags
2	4	5	q^2+1	1
3	4	50	$\frac{q^2+1}{2}$	q^2+1
5	4	338	$\frac{q^2+1}{2}$	q^2+1
7	4	1250	$\frac{q^2+1}{2}$	q^2+1
2	6	135	q^3+1	q^3+q^2+q+1
3	6	5488	$\frac{q^3+1}{4}$	q^6+2q^3+1
5	6	15750	$\frac{q^3+1}{2}$	$2q^3$
2	8	17	q^4+1	1
3	8	9922	$\frac{q^4+1}{2}$	q^5-1
2	10	2079	q^5+1	$q^5+q^4+q^3+q^2+q+1$

the field. We believe in that they are valuable for arithmetic of finite fields and finite geometry as well.

Conjecture 1: When $q = 2$ and $H = \langle z^{(q^{n/2}-1)}\rangle$, there is a unique maximal optimum distance cyclic H-orbit code if $n \equiv 0 \mod 4$, and there are $q^{n/2}+q^{n/2-1}+\cdots+q+1$ distinct maximal optimum distance cyclic H-orbit codes if $n \equiv 2 \mod 4$.

Conjecture 2: When q is odd, $n = 4$ and $H = \langle z^{(q^4-1)/N}\rangle$, the number of distinct best possible optimum distance cyclic H-orbit codes is $q^2 + 1$.

Note that the first part of Conjecture 1 is equivalent to the following statement: for even q, let $U \subseteq \mathbb{F}_{q^4}$ be $\mathbb{F}_2$-linear such that $|U| = q^2$. If $\alpha_1 U \cap \alpha_2 U \neq \{0\}$ implies $\alpha_1 = \alpha_2$, for any $\alpha_1, \alpha_2 \in H$, then U is $\mathbb{F}_{q^2}$-linear. Below we will show that if U is $\mathbb{F}_q$-linear, then U is $\mathbb{F}_{q^2}$-linear, using some intricate arithmetical methods. The general case when U is $\mathbb{F}_{2^\ell}$-linear with $\mathbb{F}_2 \subseteq \mathbb{F}_{2^\ell} \subseteq \mathbb{F}_q$ is still open.

Theorem 6: Let q be even and $H \subseteq \mathbb{F}_{q^4}^*$ be the subgroup such that $H = \langle z^{q^2-1}\rangle$ and $|H| = q^2+1$, where z is a primitive element of $\mathbb{F}_{q^4}$. Let $U \subseteq \mathbb{F}_{q^4}$ be $\mathbb{F}_q$-linear such that $|U| = q^2$. If $\alpha_1 U \cap \alpha_2 U \neq \{0\}$ implies $\alpha_1 = \alpha_2$, for any $\alpha_1, \alpha_2 \in H$, then U is $\mathbb{F}_{q^2}$-linear.

Proof: Consider

$$\bigcup_{\alpha \in H} \alpha(U \setminus \{0\}),$$

which is a disjoint union that is equal to $\mathbb{F}_{q^4}^*$ by the condition (7). Therefore, there exists $\alpha_0 \in H$ such that $1 \in \alpha_0 U$. If we set $U_1 := \alpha_0 U$, then the condition is equivalent to

$$U_1 \cap \alpha U_1 = \{0\}, \text{ for all } \alpha \in H \setminus \{1\}. \quad (8)$$

Hence, we will show that if U_1 is not $\mathbb{F}_{q^2}$-linear, then the condition (8) above does not hold.

Let $\theta \in U_1$ such that $U_1 = \langle 1, \theta\rangle$. Suppose that U_1 is $\mathbb{F}_q$-linear but not $\mathbb{F}_{q^2}$-linear. Without loss of generality, $\theta \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, since U_1 would be $\mathbb{F}_{q^2}$-linear if $\theta \in \mathbb{F}_{q^2}$. It is enough to prove that there exist $a_0, a_1, b_0, b_1 \in \mathbb{F}_q$ and $\alpha \in H \setminus \{1\}$ such that $a_0 + a_1\theta = \alpha(b_0 + b_1\theta)$ when U_1 is not $\mathbb{F}_{q^2}$-linear.

Indeed, if this is the case that $a_0 + a_1\theta \in U_1$ and $\alpha(b_0 + b_1\theta) \in \alpha U_1$, then

$$\left(\frac{a_0 + a_1\theta}{b_0 + b_1\theta}\right)^{q^2+1} = 1 \text{ and } \frac{a_0 + a_1\theta}{b_0 + b_1\theta} \neq 0$$

with $(a_0, a_1), (b_0, b_1) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}$. We have

$$(a_0 + a_1\theta)^{q^2+1} = a_0^2 + a_0a_1(\theta + \theta^{q^2}) + a_1^2\theta^{q^2+1}.$$

Note that $\left\{1, \theta + \theta^{q^2}, \theta^{q^2+1}\right\} \subseteq \mathbb{F}_{q^2}$ is a linearly dependent set over $\mathbb{F}_q$. Hence, there exist $A, B, C \in \mathbb{F}_q$ such that $(A, B, C) \neq (0, 0, 0)$ and

$$A + B(\theta + \theta^{q^2}) + C\theta^{q^2+1} = 0. \quad (9)$$

Next, we will find solutions for $a_0, a_1, b_0, b_1 \in \mathbb{F}_q$ in terms of $A, B, C \in \mathbb{F}_q$ with $(A, B, C) \neq (0, 0, 0)$.

Case 1: C = 0.

By (9), we have $\theta + \theta^{q^2} = \mu \in \mathbb{F}_q \setminus \{0\}$ as $\theta \notin \mathbb{F}_{q^2}$. Consider

$$\left(\frac{\theta}{\theta + \mu}\right)^{q^2+1}$$

with $a_0 = 0, a_1 = 1, b_0 = \mu$ and $b_1 = 1$. Note that we have $\theta \neq 0, \theta + \mu \neq 0$ and $\theta \neq \theta + \mu$. Direct computation gives $\theta^{q^2+1} = (\theta + \mu)^{q^2+1}$.

Case 2: C ≠ 0.

Without loss of generality, we assume that $C = 1$. By (9), we have $A+B(\theta+\theta^{q^2}) = \theta^{q^2+1}$. Assume that $A = 0$. Then we get $B(\theta + \theta^{q^2}) = \theta^{q^2+1}$. Note that $U_1 = \langle 1, \theta\rangle = \langle 1, 1 + \theta\rangle$. Put $\theta_1 = 1+\theta$, then we have $\theta_1^{q^2+1} = 1+\theta+\theta^{q^2}+\theta^{q^2+1}$ and $\theta_1+\theta_1^{q^2} = \theta+\theta^{q^2}$. Therefore, $\theta_1^{q^2+1} = 1+(B+1)(\theta_1+\theta_1^{q^2}) = A_1+B_1(\theta_1+\theta_1^{q^2})$. Hence, we can assume that $A \neq 0$, without loss of generality.

We have $\theta^{q^2+1} = A + B(\theta_1 + \theta_1^{q^2})$ with $A, B \in \mathbb{F}_q$ and $A \neq 0$. Consider $a = 1 + (B/A)\theta \in U_1$ and $b = \left(1/\sqrt{A} + B/A\right)\theta \in U_1$ (i.e. we take $a_0 = 1, a_1 = B/A, b_0 = 0$ and $b_1 = 1/\sqrt{A} + B/A$).

We get $a^{q^2+1} = b^{q^2+1}$ by direct computation. If $b \neq 0$, then $a \neq b$ and we are done. Observe that $b = 0$ if and only if $A = B^2$. In this case, we have $\theta^{q^2+1} = B^2 + B(\theta_1 + \theta_1^{q^2})$ with $B \in F_q^*$. But this case is impossible. Indeed, otherwise $a^{q^2+1} = (1 + \theta/B)^{q^2+1} = 0$, which implies $a = 0$ and therefore $\theta = B$, but this is a contradiction as $\theta \notin \mathbb{F}_q$. Hence, we have $a \neq 0, b \neq 0$ and $a \neq b$. This completes the proof. ■

V. CONCLUSION

In this work, we have constructed a novel class of cyclic H-orbit flag codes that are of full type and achieve optimum distance with respect to their structural parameters. These constructions not only meet theoretical bounds on minimum distance but also exhibit maximal or best possible cardinality, depending on the characteristic (even or odd) of the finite field. This demonstrates that our approach yields highly efficient codes in terms of both error-correcting capability and code size.

To highlight the distinctive features of our constructions, we provide a comparative table outlining key conceptual differences between our approach and several recent works from the literature. In particular, the table emphasizes that while our underlying orbit code is not of full length, the resulting flag code is of full type and achieves maximal or best possible size, depending on the parity of the base field. This contrasts with other approaches that may prioritize orbit length but do not attain similar optimality in flag code parameters. The comparison clarifies the novel contributions of our method, especially the role of cyclic group actions, and positions our work within the broader context of current research on flag codes.

Beyond the construction itself, we have contributed to the theoretical understanding of such codes by proposing two conjectures concerning the enumeration of distinct cyclic H-orbit full flag codes. Notably, we provide a partial resolution to the first conjecture, offering insight into the underlying combinatorial structures and indicating promising directions for further research.

Overall, our results highlight the potential of cyclic group actions in the design of structured flag codes with optimal properties, laying the groundwork for future advancements in the theory and applications of subspace and flag codes in network coding and related areas. In the study of flag code construction and network communication robustness, recent research has also explored novel information security methods that combine chaotic dynamics with nonlinear components. For example, [7] employs discrete chaotic systems for video segment encryption, which may provide valuable insights for future integration of optimum distance flag codes with multimedia security and network transmission.

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