COMPACTNESS AND SPECTRA OF HANKEL OPERATORS

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ABSTRACT

COMPACTNESS AND SPECTRA OF HANKEL OPERATORS

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Spaces of analytic functions and their operators are widely studied. Two of those operators are Toeplitz and Hankel operators. In analytic function spaces that are complemented in a larger space, the Toeplitz operator T_ϕ takes the part of the multiplication operator $M_\phi f := \phi f$ that remains in said space, and the Hankel operator H_ϕ takes the rest. Namely, if P is the continuous projection onto said space, we define $T_\phi = PM_\phi$ and $H_\phi = (I-P)M_\phi$. In this thesis, we will introduce Toeplitz and Hankel operators in various spaces and provide a foundation, after which we will narrow our focus to the Hankel operators of the Bergman space $A^2(\Omega)$. We will elaborate on the articles [1] and [2], the first one being about smooth symbols of compact Hankel operators on $A^2(\Omega)$ where Ω is a pseudoconvex domain in $\mathbb{C}^n$ and the relation between these symbols and analytic discs of the boundary $b\Omega$, and the second one developing an understanding about the spectra and essential spectra of the Hermitian square ${H_\phi}^* H_\phi$ of Hankel operators H_ϕ on the Bergman space $A^2(\mathbb{D}^n)$ of the unit polydisc. We will follow the proofs given in these articles and provide explanations for some claims left to the reader in the original papers.

Keywords: Hankel operator, analytic function spaces, essential spectrum, compact operators, pseudoconvex domain

ÖZ

HANKEL OPERATÖRLERİNİN TIKIZLIĞI VE SPEKTRUMLARI

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Analitik fonksiyon uzayları ve bu uzayların operatörleri yoğun bir biçimde incelenmektedir. Toeplitz ve Hankel operatörleri bu bahsi geçen operatörlerden iki tanesidir. Daha büyük bir uzayın içinde tümlenebilen analitik fonksiyon uzayları için, P bu uzayın sürekli izdüşümü olmak üzere, ϕ sembollü Toeplitz operatorü T_ϕ, ϕ sembollü M_ϕ çarpım operatörünün bahsi geçen uzayın içinde kalan kısmını, ϕ sembollü Hankel operatörü H_ϕ ise geri kalan kısmını alır. Diğer bir deyişle, $T_\phi = PM_\phi$ ve $H_\phi = (I - P)M_\phi$ olarak ifade edilebilir. Bu tezde, Toeplitz ve Hankel operatörlerini farklı analitik fonksiyon uzayları için tanımlayacak ve temel bilgileri vereceğiz. Sonrasında tamamen Bergman uzayı $A^2(\Omega)$'nın Hankel operatörleri üzerine yoğunlaşacak ve [1] ve [2] makalelerini detaylandıracağız. Bu makalelerin ilki sözde dışbükey bir $\Omega \subseteq \mathbb{C}^n$ bölgesi için $A^2(\Omega)$'da tıkız Hankel operatörlerinin C^∞ sembolleri ve bu sembollerin Ω bölgesinin sınırı $b\Omega$'daki analitik disklerle ilişkisi üzerine, ikincisi ise $\mathbb{D}^n$ çoklu diskinin Bergman uzayı $A^2(\mathbb{D}^n)$ üzerindeki H_ϕ Hankel operatörünün Hermisyen karesi ${H_\phi}^* H_\phi$ operatörünün spektrum ve esas spektrumu ile ilgili bir anlayış geliştirmek üzerinedir. Bu makalelerde verilen kanıtların üzerinden geçecek ve okuyucuya bırakılmış bazı savları açıklayacağız.

Anahtar Kelimeler: Hankel operatörü, analitik fonksiyon uzayları, esas spektrum, tıkız operatörler, sözde dışbükey bölge

To my family

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CHAPTER 1

INTRODUCTION

After a study of general theory of functional analysis, working on concrete examples is a natural continuation. Examples of Banach spaces mainly arise from a desire to understand sequences and functions. Thence, the study of analytic function spaces was born. People worked on classes of analytic functions having different bound or growth conditions. Of course, their operators were of great interest as well. In different settings, it is possible to see the multiplication, composition, Toeplitz, and Hankel operators being studied.

An indispensable tool and an advantage of working with a concrete Banach or Hilbert space, namely an analytic function space, is the tools that come from the function theory of a single or several complex variables. These studies generally started with the examination of the unit disc $\mathbb{D}$ of $\mathbb{C}$, or the unit ball B_n or the unit polydisc $\mathbb{D}^n$ of $\mathbb{C}^n$. With tools coming from the function theory in several complex variables or pluripotential theory, people were able to work with more general, usually pseudoconvex, domains as well. To give an example, [3] introduces a new class of H^p spaces in hyperconvex domains with the help of plurisubharmonic exhaustion functions and the Monge-Ampère operator.

Here, we will focus on Toeplitz and Hankel operators with the dominating concept being Hankel operators. These two operators have a close relationship in the sense that they are both derived from multiplication operators. We can look at these operators from different angles. Without going into too much detail, in one of the spaces on which these operators are defined, Toeplitz and Hankel operators appear as the operators that have "nice" matrices. It turns out that this form of "niceness" is exclusive to these operators. In another space, we will see that Hankel operators are very closely

related to the complex PDE $\bar{\partial}u = f$ due to the orthogonal projection to this space having close relations with the $\bar{\partial}$-Neumann operator.

This thesis consists of three main topics. In Chapter 2, we will provide the necessary machinery to study the rest of the material. In Chapter 3, we will provide the definitions and basic facts regarding Toeplitz and Hankel operators in various analytic function spaces. The rest of the thesis will focus only on the Hankel operators on the Bergman space $A^2(\Omega)$. In Chapter 4, we will follow the work of [1], examining smooth symbols of Hankel operators in pseudoconvex domains. Finally, in Chapter 5, we will examine the work in [2], dealing with Hankel operators H_ϕ on $A^2(\mathbb{D}^n)$ and trying to understand when the essential spectrum of ${H_\phi}^* H_\phi$ contains an open interval.

CHAPTER 2

PRELIMINARIES

2.1 General Operator Theory and Spectral Theory

This section is about providing necessary facts from general operator theory of Banach or Hilbert spaces and spectral theory. Throughout, $\mathcal{D}(T)$ is going to denote the domain of a densely defined operator T.

Let us prove the following well-known fact:

Theorem 2.1.1. *Let E, F, G, H be Banach spaces and $S : E \to F$, $T : F \to G$, $R : G \to H$ be linear operators. If T is compact and R, S are bounded, then $RTS : E \to H$ is compact.*

Proof. Suppose that S and R are bounded and T is compact. Note that S, T, and R are all continuous functions. Let $A \subseteq E$ be a bounded set. That is, $\sup_{x\in A} \|x\| < \infty$. Owing to the boundedness of S, we have

$$\sup_{y\in S(A)} \|y\| = \sup_{x\in A} \|Sx\| \leq \|S\| \sup_{x\in A} \|x\| < \infty;$$

in other words, $S(A)$ is bounded. Since T is compact and $S(A)$ is bounded, it follows that $T(S(A))$ is relatively compact; that is, $\overline{T(S(A))}$ is compact. Then, the continuity of R yields that $R(\overline{T(S(A))})$ is compact, whence it follows that $\overline{R(T(S(A)))} \subseteq R(\overline{T(S(A))})$ is compact, so $R(T(S(A)))$ is relatively compact. Hence, the operator RTS is a compact one. □

The definition below is given in [4] in its most general case. While we will not go into this extent of generality, it is still good practice to include this version.

Definition 2.1.2. Let X be a Banach space and $T : \mathcal{D}(T) \to X$ be a linear operator, where $\mathcal{D}(T)$ is dense in X.

(1) The *discrete spectrum* of T, denoted $\sigma_{\mathrm{d}}(T)$, consists of all $\lambda \in \sigma_{\mathrm{p}}(T)$ of finite algebraic multiplicity that are also isolated points of $\sigma(T)$.

(2) The *essential spectrum* of T, denoted $\sigma_{\mathrm{ess}}(T)$, is the complement of $\sigma_{\mathrm{d}}(T)$ in $\sigma(T)$; that is, $\sigma_{\mathrm{ess}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$.

The essential spectrum of a densely defined linear operator remains unchanged under compact perturbations; that is, $\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(T + K)$ for any compact operator where $T + K$ is applicable (see [4]).

When the space of interest is a Hilbert space, we have the following indispensable theorem:

Theorem 2.1.3 (Riesz Representation Theorem)**.** *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let H' denote its topological dual. For any $f \in H'$, there is a unique $y \in H$ such that $f(\cdot) = \langle \cdot, y \rangle$. Furthermore, $\|f\|_{H'} = \|y\|_H$.*

A proof of Theorem 2.1.3 can be found in any functional analysis textbook.

Remark 2.1.4. Notice that Theorem 2.1.3 implies that any Hilbert space can be identified with its own topological dual.

Let E, F be Banach spaces and $T : E \to F$ a linear operator. The *dual operator* of T is the operator $T' : F' \to E'$ defined by $T'v = u$ for all $v \in F'$ so that $u(x) = v(Tx)$ for each $x \in E$. Owing to Theorem 2.1.3, this dual operator can be identified with the so-called *adjoint operator* in Hilbert spaces.

Definition 2.1.5. Let H_1, H_2 be Hilbert spaces and $T : H_1 \to H_2$ a linear operator. The *adjoint* of T, denoted T^*, is the operator $T^* : H_2 \to H_1$ defined in such a way that $\langle x, T^* y \rangle_{H_1} = \langle Tx, y \rangle_{H_2}$ for any $x \in H_1, y \in H_2$. An operator $T : H \to H$ where H is a Hilbert space is called *self-adjoint* if $T = T^*$.

The following theorem is a well-known characterisation of the elements of spectra and essential spectra of densely defined self-adjoint operators on Hilbert spaces.

Theorem 2.1.6 (Weyl's Criterion)**.** *Let H be a Hilbert space and $\mathcal{D}(T)$ be dense in H. Let $T : \mathcal{D}(T) \to H$ be a densely defined, self-adjoint linear operator on H. Then, the following hold:*

(1) A complex number λ is a spectral value if and only if there is a sequence $\{u_n\}$ in $\mathcal{D}(T)$ with $\|u_n\| = 1$, such that $\lim_{n\to\infty} \|(T - \lambda I)u_n\| = 0$,

(2) $\lambda \in \sigma_{\text{ess}}(T)$ if and only if there is a sequence $\{u_n\}$ in $\mathcal{D}(T)$ with $\|u_n\| = 1$, such that $(T - \lambda I)u_n \to 0$, and $u_n \rightharpoonup 0$.

A proof of Theorem 2.1.6 can be found in [4].

It is a well-known fact that for any bounded linear operator $T : H \to H$ where H is a Hilbert space, the spectrum lies inside the set $\{z \in \mathbb{C} : |z| \leq \|T\|\}$. It is not always the case that this enclosing set is fully occupied by $\sigma(T)$, motivating us to define the notion known as the *spectral radius*.

Definition 2.1.7. Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. The *spectral radius* of T is the number

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

In a sense, the spectral radius measures the smallest disc centred at the origin that can contain the spectrum of T. The following theorem from [4] asserts that the spectral radius assumes a particularly memorable form in the event that the operator of discourse is a self-adjoint one.

Theorem 2.1.8. *Let H be a Hilbert space and $T \in B(H)$ be self-adjoint. Then,*

$$r(T) = \|T\|.$$

The following proposition appeared in [5], characterising the compactness of a linear operator between Hilbert spaces.

Proposition 2.1.9. *Let H_1, H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a linear operator. Then, the following are equivalent:*

(1) T is compact.

(2) For every $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ and a $K_\varepsilon \in K(H_1, H_2)$ such that for each $x \in H_1$, one has

$$\|Tx\|_{H_2} \leq \varepsilon \|x\|_{H_1} + C_\varepsilon \|K_\varepsilon x\|_{H_2}.$$

(3) For every $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ and a compact operator K_ε such that for each $x \in H_1$, one has

$$\|Tx\|_{H_2}^2 \leq \varepsilon \|x\|_{H_1}^2 + C_\varepsilon \|K_\varepsilon x\|_{H_2}^2.$$

2.2 Functions of Several Complex Variables, Pseudoconvexity, and the Levi Form

In this section, we present some important notions from function theory of a single and several complex variables. We will talk about pseudoconvexity through the eyes of subharmonic and plurisubharmonic functions, and introduce the Levi form, which is a tool to understand pseudoconvex domains better.

Lemma 2.2.1. *Let f be analytic in a ball $B = B(0; \varepsilon) \subseteq \mathbb{C}$ and χ be an integrable function on $|B| := \{|z| : z \in B\}$ such that $C = \int_{|B|} \chi(t)\, dt > 0$. Then, the weighted mean value property*

$$f(0) = \frac{1}{C} \int_B \chi(|z|) f(z)\, dA(z)$$

holds.

Proof. Since f is analytic in B, the regular mean value property

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, d\theta$$

holds for any r with $0 \leq r < \varepsilon$. Then, we have

$$\begin{aligned}
\int_B \chi(|z|) f(z)\, dA(z) &= \int_0^{\varepsilon} \int_0^{2\pi} \chi(r) f(re^{i\theta}) r\, d\theta\, dr \\
&= \int_0^{\varepsilon} \chi(r) r \int_0^{2\pi} f(re^{i\theta})\, d\theta\, dr \\
&= f(0) \int_0^{\varepsilon} 2\pi \chi(r) r\, dr \\
&= f(0) \int_0^{2\pi} \int_0^{\varepsilon} \chi(r) r\, dr\, d\theta \\
&= f(0) \int_B \chi(|z|)\, dA(z).
\end{aligned}$$

Thus, we have the desired result. □

To understand what constitutes a plurisubharmonic function, we need to first look at what a subharmonic function is. To define subharmonic and plurisubharmonic functions in the most general case, we introduce the notion of upper semicontinuity.

Definition 2.2.2. A function $\phi : \Omega \to [-\infty, \infty]$ is said to be *upper semicontinuous* if the set $\{z \in \Omega : \phi(z) < r\}$ is open for all $r \in \mathbb{R}$.

Definition 2.2.3. Let $\Omega \subseteq \mathbb{C}$ be an open set. An upper semicontinuous function $u : \overline{\Omega} \to [-\infty, \infty)$ is said to be *subharmonic* if for any harmonic function h such that $u(z) \leq h(z)$ for all $z \in b\Omega$, one has $u(z) \leq h(z)$ for all $z \in \Omega$.

If u is C^2, then subharmonicity can be determined using the Laplacian. In [6], we see that such functions u are subharmonic if and only if $\Delta u \geq 0$. In fact, they define subharmonicity in a different way, namely as in the following theorem, and then reach Definition 2.2.3 as an equivalent condition.

Theorem 2.2.4. *Let $\Omega \subseteq \mathbb{C}$ be an open set. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is subharmonic if and only if the* local submean inequality *is satisfied; that is, for any $z \in \Omega$, there exists $r > 0$ such that*

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + \rho e^{i\theta})\, d\theta$$

holds for any $0 \leq \rho < r$.

The trivial and most basic example of a subharmonic function is the following.

Example 2.2.5. For any domain $\Omega \subseteq \mathbb{C}$, the function $u \equiv -\infty$ is subharmonic.

Remark 2.2.6. Because this may create some problems in the proofs of most theorems, it is also a common practice to exclude the case $u \equiv -\infty$ when defining subharmonic functions, or plurisubharmonic functions, whose definition will come later in this section.

Again, from [6], we see ways to generate new subharmonic functions from existing ones.

Example 2.2.7. Say u, v are subharmonic and f is holomorphic on an open set $\Omega \subseteq \mathbb{C}$. Then,

- $w_1(z) = \max\{u(z), v(z)\}$,
- $w_2(z) = c_1 u(z) + c_2 v(z)$ where $c_1, c_2 \geq 0$,
- $w_3(z) = \log(|f(z)|)$,
- $w_4(z) = |f(z)|^p$ where $1 \leq p < \infty$

are all subharmonic functions on Ω.

The last class of examples above can be seen as follows: Let $\Omega \subseteq \mathbb{C}$ be a domain and f be holomorphic on Ω. Then, by the mean value property of holomorphic functions, for some $r > 0$, we have

$$\begin{aligned} |f(z)|^p &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta})\, d\theta \right|^p \\ &= \left(\frac{1}{2\pi} \right)^p \left| \int_0^{2\pi} f(z + \rho e^{i\theta})\, d\theta \right|^p \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z + \rho e^{i\theta}) \right|^p \, d\theta \end{aligned}$$

for any ρ with $0 \leq \rho < r$. Thus, the local submean inequality is satisfied.

The following definition can be found in [7].

Definition 2.2.8. An upper semicontinuous function $\phi : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if for any $z, w \in \mathbb{C}^n$, the function $\tau \mapsto \phi(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

Plurisubharmonicity is important as it characterises *pseudoconvex domains*. [8] introduces pseudoconvex domains and demonstrates that pseudoconvexity is equivalent to the following:

Definition 2.2.9. Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A function $\rho : \Omega \to \mathbb{R}$ such that the set $\{z : \rho(z) < a\}$ is relatively compact for each $a \in \mathbb{R}$ is called a *defining function* for Ω. The domain Ω is said to be *pseudoconvex* if it has a continuous plurisubharmonic defining function.

Another notion is *Levi pseudoconvexity*. In [9], one can find the following:

Definition 2.2.10. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain where $n \geq 2$. Let ρ be a C^2 defining function for Ω. Then, Ω is called *Levi pseudoconvex* at $p \in b\Omega$ if the *Levi form* satisfies

$$\mathcal{L}_p(\rho; t) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k \geq 0$$

for any $t \in T_p^{1,0}(b\Omega)$. If Ω is Levi pseudoconvex at every $p \in b\Omega$, then Ω is called *Levi pseudoconvex*.

One important aspect of the Levi form is that its rank is a biholomorphic invariant, as mentioned in [10].

If the Levi form of the defining function is positive definite; that is, if in the above definition, we have strict inequality for any non-zero vector, then the domain Ω is said to be *strictly pseudoconvex* or *strongly pseudoconvex*.

Using [8, 7], we can observe the equivalence of these two notions of pseudoconvexity.

Theorem 2.2.11. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and u be a real-valued C^2 function. Then, u is plurisubharmonic if and only if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k \geq 0$$

for any choice of $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$ and $z \in \Omega$.

Also from [8], we have the following.

Proposition 2.2.12. *If $\Omega \subseteq \mathbb{C}^n$ is a strictly pseudoconvex domain with C^2 boundary, then there exists a defining function ρ such that for some $C > 0$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) t_j \bar{t}_k \geq C \left|t\right|^2$$

for any $t \in \mathbb{C}^n$ and $p \in b\Omega$.

Another important concept is that of a *Reinhardt domain*.

Definition 2.2.13. An open set $\Omega \subseteq \mathbb{C}^n$ is called a *Reinhardt domain* centred at 0 if whenever $z = (z_1, \ldots, z_n) \in \Omega$, so is $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$ for any $\theta_1, \ldots, \theta_n \in \mathbb{R}$. It is called a *complete Reinhardt domain* if whenever $z \in \Omega$, the closed polydisc $\overline{\Delta}(0, |z|) = \{w \in \mathbb{C}^n : |w_j| \leq |z_j| \text{ for each } j = 1, \ldots, n\}$ centred at 0 of multiradius $|z| = (|z_1|, \ldots, |z_n|)$ lies inside Ω.

2.3 Analytic Function Spaces

In this section, we will introduce two analytic function spaces. These spaces have been thoroughly examined since at least the second half of the 20th century. While the only space that is inside our scope is going to be the Bergman space $A^2(\Omega)$, we see that the Toeplitz and Hankel operators are being studied in these particular spaces and more.

Let us begin with Hardy spaces. Inspired by the work of G. H. Hardy and J. E. Littlewood in the 1920s, people have worked on these spaces, looking from many angles, including characterising functions in these "Hardy classes", working on the classes regarding them as linear spaces, and finally working on them regarding them as Banach spaces. For more information on Hardy spaces, we refer the reader to the canonical text [11].

Definition 2.3.1. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function. We define the integral p-mean $M_p(r, f)$ of f where $0 < p < \infty$ and $0 \leq r < 1$ as

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta \right)^{1/p},$$

and $M_\infty(r,f) = \max\limits_{0\le\theta<2\pi} \left|f(re^{i\theta})\right|$ The *Hardy class* $H^p(\mathbb{D})$ of the unit disc for $0 < p \le \infty$ is defined as the set of all analytic functions f such that the quantity $M_p(r,f)$ remains bounded as $r \to 1^-$.

When $1 \le p \le \infty$, with the norm $\|f\| = \lim\limits_{r\to 1} M_p(r,f)$ the pair $(H^p(\mathbb{D}), \|\cdot\|)$ is a Banach space.

Functions of class H^p on $\mathbb{D}$ and the Banach spaces $H^p(\mathbb{D})$ have many interesting properties themselves, so people have generalised them to various domains. In [11], there is a chapter dedicated to Hardy spaces $H^p(\mathbb{H}^+)$ of the upper half-plane $\mathbb{H}^+$. In 2008, the article [3] was published, which allowed people to characterise and understand Hardy-like spaces in hyperconvex domains via the use of the Monge-Ampère operator and what is called a *Monge-Ampère measure*.

Next, we will look at Bergman spaces. The definition does not require much machinery, allowing us to define these spaces in the most general sense.

Definition 2.3.2. Let $\Omega \subseteq \mathbb{C}^n$. The *Bergman space* $A^p(\Omega)$ on Ω consists of all functions f that are holomorphic on Ω and that satisfy $f \in L^p(\Omega)$. In other words,

$$A^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} : f \text{ is holomorphic in } \Omega, \int\limits_\Omega |f(z)|^p \; dV(z) < \infty \right\}$$

where $dV(z)$ denotes the usual Lebesgue measure on $\mathbb{C}^n$.

With the norm

$$\|f\|_p = \left(\int_\Omega |f(z)|^p \, dV(z) \right)^{1/p},$$

the pair $(A^p(\Omega), \|\cdot\|_p)$ is a Banach space.

The following lemma is one that we will use in several different places.

Lemma 2.3.3. *Let $\{f_j\}$ be a bounded sequence in $A^2(\Omega)$ for a domain $\Omega \subseteq \mathbb{C}^n$. Then, $\{f_j\}$ has a subsequence that is locally uniformly convergent in Ω.*

Proof. Suppose that $\|f_j\| \le M$ for all j. Let K be a compact set in Ω. Let $\Delta_r^n(z)$ denote the polydisc centred at $z \in \mathbb{C}^n$ of dimension n and multiradius $(r, \ldots, r)$. Notice that

$$\mathcal{U} = \{\Delta_r^n(z) : z \in K, \Delta_r^n(z) \subset\subset \Omega\}$$

is an open covering of K and hence has a finite subcover

$$\left\{\Delta^n_{r_k}(z_k) : k = 1, \ldots, k_0\right\}.$$

Fix $k \in \{1, \ldots, k_0\}$. For any $f \in A^2(\Omega)$, by Cauchy integral formula, we have

$$\begin{aligned}
|f(z)| &\leq \int\limits_{|z_1 - z_1^{(k)}| = r_k} \cdots \int\limits_{|z_n - z_n^{(k)}| = r_k} \left| \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \right| \, d\zeta_1 \wedge \cdots \wedge d\zeta_n \\
&= \frac{1}{r_k^n} \int\limits_{|z_1 - z_1^{(k)}| = r_k} \cdots \int\limits_{|z_n - z_n^{(k)}| = r_k} |f(\zeta_1, \ldots, \zeta_n)| \, d\zeta_1 \wedge \cdots \wedge d\zeta_n \\
&\leq \frac{1}{r_k^n} \left(\int\limits_{|z_1 - z_1^{(k)}| = r_k} \cdots \int\limits_{|z_n - z_n^{(k)}| = r_k} |f(\zeta_1, \ldots, \zeta_n)|^2 \, d\zeta_1 \wedge \cdots \wedge d\zeta_n \right)^{1/2} \cdot V(\Delta^n_{r_k}) \\
&= \frac{V(\Delta^n_{r_k})}{r_k^n} \|f\|_2
\end{aligned}$$

where the third line happens due to Hölder's inequality, and $V(\cdot)$ denotes the volume in the usual Lebesgue measure sense. Consequently, for any j and $z \in K$, one has

$$|f_j(z)| \leq \max_{1 \leq k \leq k_0} \frac{V(\Delta^n_{r_k})}{r_k^n} \|f_j\|_2 \leq \frac{V(\Delta^n_{r_k})}{r_k^n} M,$$

showing that $\{f_j\}$ is a locally uniformly bounded sequence in Ω. By Montel's Theorem, the statement follows. □

Remark 2.3.4. Notice that the result is independent of p. In other words, the proof may be adjusted to show that if $\{f_j\}$ is a bounded sequence in $A^p(\Omega)$, then a subsequence of $\{f_j\}$ converges locally uniformly on Ω.

The following lemma from [2] characterises weak convergence of functions in $A^2(\Omega)$, which will be particularly useful when we need to use Theorem 2.1.6.

Lemma 2.3.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A sequence $\{f_j\}$ in $A^2(\Omega)$ converges weakly to 0 if and only if it is bounded in the A^2-norm and locally uniformly converges to 0.*

Definition 2.3.6. Let $A \neq \emptyset$. Let $\mathcal{F}(A) = \{f \mid f : A \to \mathbb{C}\}$. A *reproducing kernel Hilbert space*, often abbreviated as an RKHS, on A is a Hilbert space $H \subseteq \mathcal{F}(A)$ with a function $K : A \times A \to \mathbb{C}$, such that

(1) $K_p := K(\cdot, p) \in H$ for all $p \in A$,

(2) $\langle f, K_p \rangle = f(p)$ for each $p \in A$.

Such a function K is called the *reproducing kernel* of H.

We may think of reproducing kernels as follows: On an RKHS H, the point evaluation map ε_a is continuous due to the Cauchy-Schwarz inequality for any $a \in A$. By Theorem 2.1.3, the continuous linear functional ε_a has a Riesz representative $g_a \in H$ such that $\varepsilon_a(\cdot) = \langle \cdot, g_a \rangle$. By the uniqueness of this representative, we see that the point evaluation ε_a is represented in H exactly by K_a.

Example 2.3.7. In the Hardy-Hilbert space $H^2(\mathbb{D})$, [12] demonstrates that all point evaluations ε_a are continuous, so they all have a unique Riesz representative $g_a \in H$, with properties described above. Letting $K(\cdot, w) = g_w$, we have the reproducing kernel K. In fact, in the case of $H^2(\mathbb{D})$, we can calculate the reproducing kernel, called the *Szegő kernel*. The Szegő kernel function on $H^2(\mathbb{D})$ is given in [12] as

$$K(z, w) = \sum_{n=0}^{\infty} \overline{w}^n z^n = \frac{1}{1 - \overline{w}z}.$$

Example 2.3.8. In the Bergman space $A^2(\mathbb{D})$, once more, the point evaluations are continuous. The proof of this fact goes similarly to that of Lemma 2.3.3, and as a result, the Bergman space $A^2(\mathbb{D})$ is an RKHS. In fact, the Bergman space $A^2(\Omega)$ for any domain $\Omega \subseteq \mathbb{C}^n$ is an RKHS by Cauchy integral formula, using the same argument in Lemma 2.3.3 to reach a norm estimate in a neighbourhood of every point of Ω. The reproducing kernel for the Bergman space is called the *Bergman kernel*. It has several properties that are of interest to us. The Bergman kernel for the unit disc $\mathbb{D}$ is shown in [13] to be

$$K(z, w) = \frac{1}{(1 - \overline{w}z)^2}.$$

More generally, in the n-dimensional unit ball, the Bergman kernel is

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}$$

where $\langle z, w \rangle$ denotes the standard inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j$ in $\mathbb{C}^n$, according to [14].

2.4 The Bergman Kernel

The Bergman kernel is going to play an indispensable role in our work. In this section, we will discuss several facts about this object. According to [8], the square-integrability and analyticity, the conjugate-symmetry, and the reproducing property uniquely determine the Bergman kernel function.

We will calculate the Bergman kernel for an n-dimensional ball of arbitrary radius, using the following proposition from [8].

Proposition 2.4.1. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be two domains and $F : \Omega_1 \to \Omega_2$ be a biholomorphism. Let*

$$J_{\mathbb{C}}F(z) = \frac{\partial(F_1(z), \ldots, F_n(z))}{\partial(z_1, \ldots, z_n)}$$

be the complex Jacobian of F. Then,

$$\det J_{\mathbb{C}}F(z_1) K_{\Omega_2}(F(z_1), F(z_2)) \overline{\det J_{\mathbb{C}}F(z_2)} = K_{\Omega_1}(z_1, z_2).$$

Let $B_n(0, r)$ denote the n-dimensional ball of radius $r > 0$, centred at the origin. It is evident that $F : B_n(0, 1) \to B_n(0, r)$ given by $F(z) = rz$ is a biholomorphism. More specifically, we have $(F_1(z_1, \ldots, z_n), \ldots, F_n(z_1, \ldots, z_n)) = (rz_1, \ldots, rz_n)$. Then, $J_{\mathbb{C}}F(z)$ looks like

$$J_{\mathbb{C}}F(z) = \begin{bmatrix} r & 0 & 0 & 0 & \cdots & 0 \\ 0 & r & 0 & 0 & \cdots & 0 \\ 0 & 0 & r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & r & 0 \\ 0 & 0 & 0 & \cdots & 0 & r \end{bmatrix} = rI,$$

so its determinant is precisely r^n. Let $K^r(z, w)$ denote the Bergman kernel for $B_n(0, r)$. By Proposition 2.4.1, we have $r^n K^r(z, w) r^n = \dfrac{1}{(1 - \frac{1}{r^2}\langle z, w\rangle)^{n+1}}$. In other words,

$$K^r(z, w) = \frac{r^{2n+2}}{r^{2n}(1 - \langle z, w\rangle)^{n+1}} = \frac{r^2}{(1 - \langle z, w\rangle)^{n+1}}.$$

Remark 2.4.2. The above argument shows that on the diagonal of $B_n(0, r) \times B_n(0, r)$, the Bergman kernel is directly correlated to the distance between $z \in B_n(0, r)$ and the boundary $bB_n(0, r)$.

If the Bergman kernel is not explicitly known to us, we may still obtain some information about it. For instance, the Bergman kernel always satisfies the *conjugate-symmetry* property. In other words, $K(z,w) = \overline{K(w,z)}$ for any z, w, since

$$K(z,w) = \langle K(\cdot,w), K(\cdot,z)\rangle = \overline{\langle K(\cdot,z), K(\cdot,w)\rangle} = \overline{K(w,z)}.$$

If the domain at hand is a product of two domains, then we may acquire some idea regarding the Bergman kernel on that domain via the kernels of the factor domains.

Proposition 2.4.3. *Let $\Omega_1 \subseteq \mathbb{C}^k$ and $\Omega_2 \subseteq \mathbb{C}^\ell$ be domains. Then, for $\Omega = \Omega_1 \times \Omega_2$, we have*

$$K^\Omega = K^{\Omega_1} K^{\Omega_2}.$$

Proof. Assume the above setup and let $f \in A^2(\Omega)$. Then, we have

$$\begin{aligned}
\left\langle f, K^{\Omega_1}_{z'} K^{\Omega_2}_{z''} \right\rangle &= \int_\Omega f(w',w'') K^{\Omega_1}(z',w') K^{\Omega_2}(z'',w'')\, dV(w) \\
&= \int_{\Omega_1\times\Omega_2} f(w',w'') K^{\Omega_1}(z',w') K^{\Omega_2}(z'',w'')\, dV(w) \\
&= \int_{\Omega_1}\int_{\Omega_2} f(w',w'') K^{\Omega_1}(z',w') K^{\Omega_2}(z'',w'')\, dV(w'')\, dV(w') \\
&= \int_{\Omega_1} K^{\Omega_1}(z',w') \int_{\Omega_2} f(w',w'') K^{\Omega_2}(z'',w'')\, dV(w'')\, dV(w') \\
&= \int_{\Omega_1} f(w',z'') K^{\Omega_1}(z',w')\, dV(w') = f(z',z'')
\end{aligned}$$

for any $z = (z', z'') \in \Omega$ with $z' \in \Omega_1$ and $z'' \in \Omega_2$. In other words, for any $f \in A^2(\Omega)$, we have $f(z) = \left\langle f, K^\Omega_z \right\rangle = \left\langle f, K^{\Omega_1}_{z'} K^{\Omega_2}_{z''} \right\rangle$. Hence, the proposition holds. □

We will use the following proposition to divide the Bergman projection.

Proposition 2.4.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let P denote the Bergman projection, namely the orthogonal projection onto $A^2(\Omega)$. Then, Pf can be expressed via the integral*

$$Pf(z) = \int_\Omega f(w) K^\Omega(z,w)\, dV(w)$$

for any $f \in L^2(\Omega)$.

Proof. Note that $A^2(\Omega)$ itself is a Hilbert space, so we may decompose it as the topological direct sum $L^2(\Omega) = A^2(\Omega) \oplus (A^2(\Omega))^\perp$. For legibility concerns, let us drop the domain. Let $f \in L^2$. Then, $f = f_1 + f_2$ where $f_1 \in A^2$ and $f_2 \in (A^2)^\perp$. By definition, we have $Pf = f_1$, so

$$\begin{aligned}\int_\Omega f(w)K^\Omega(z,w)\,dV(w) &= \left\langle f, K_z^\Omega \right\rangle = \left\langle f_1 + f_2, K_z^\Omega \right\rangle \\ &= \left\langle f_1, K_z^\Omega \right\rangle + \left\langle f_2, K_z^\Omega \right\rangle = f_1(z) = Pf(z)\end{aligned}$$

since $f_1 \in A^2$ and $K_z^\Omega \perp f_2$. Thus, we have the claim. □

Now, with divided kernels and integral representations, the following proposition follows immediately.

Proposition 2.4.5. *Let P_Ω denote the Bergman projection; that is, the orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)$ for a domain $\Omega \subseteq \mathbb{C}^n$. Then, for functions $f \in L^2(\Omega_1)$ and $g : L^2(\Omega_2)$, one has*

$$P_{\Omega_1 \times \Omega_2}(fg) = (P_{\Omega_1} f)(P_{\Omega_2} g)$$

where $\Omega_1 \subseteq \mathbb{C}^k, \Omega_2 \subseteq \mathbb{C}^\ell$ are arbitrary domains.

While for arbitrary domains we do not have much information, according to [1], for a pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$, the Bergman kernel K^Ω satisfies

$$K^\Omega(z,z) = \sup\left\{ |f(z)|^2 : f \in A^2(\Omega), \|f\|_{L^2(\Omega)} = 1 \right\}.$$

They also give the following proposition regarding the estimation of K^Ω.

Proposition 2.4.6. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that the boundary is C^2 near $p \in b\Omega$, and that the Levi form of $b\Omega$ is of rank k at p. Then, there exists a neighbourhood U of p and a constant $C > 0$ such that the following inequality is satisfied:*

$$K^\Omega(z,z) \geq \frac{C}{(d(z, b\Omega))^{k+2}},$$

where $d(z, b\Omega)$ denotes the distance from $z \in \Omega$ to the boundary $b\Omega$.

In [15], it is stated that if an orthonormal basis $\{e_j\}$ for $A^2(\Omega)$ is known, then the Bergman kernel is easily calculated as

$$K(z,w) = \sum_{j=1}^{\infty} e_j(z)\overline{e_j(w)}.$$

Unfortunately, for most domains, we can only ensure that such an orthonormal basis exists, with no way to construct it. However, when Ω is a complete Reinhardt domain, [16] states that the monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ where $\alpha \in \mathbb{N}_0^n$ such that $z^\alpha \in A^2(\Omega)$ form an orthogonal basis for $A^2(\Omega)$. Similarly, in [2], we can see the kernel being explicitly presented as

$$K(z,w) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha \bar{w}^\alpha \tag{2.1}$$

where the coefficients c_α are given as

$$c_\alpha = \begin{cases} 0 & \text{if } z^\alpha \notin A^2(\Omega) \\ \dfrac{1}{\|z^\alpha\|^2} & \text{if } z^\alpha \in A^2(\Omega) \end{cases}$$

2.5 The $\bar{\partial}$-Neumann Operator

This thesis mainly focuses on Hankel operators in different domains, which will be defined later, and these operators have a fundamental relationship with the main object of discourse of this section when the space is the Bergman space $A^2(\Omega)$, namely the $\bar{\partial}$-Neumann operator.

Let us define the problem first. Following [9], we introduce the Dolbeault differential operators ∂ and $\bar{\partial}$ as follows:

Let f be a complex-valued continuously differentiable function on a complex manifold M of dimension n. The exterior derivative of f is the 1-form

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j.$$

Using the Wirtinger derivatives $\dfrac{\partial}{\partial z_j}$ and $\dfrac{\partial}{\partial \bar{z}_j}$ with $dz_j = dx_j + i dy_j$ and $d\bar{z}_j = dx_j - i dy_j$, we may rewrite this as

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

Then, we set

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \text{ and } \bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

so that $d = \partial + \bar{\partial}$.

More generally, we may define the $\bar{\partial}$-operator on the space of so-called (p, q)-*forms*.

Definition 2.5.1. A complex differential form of *bidegree* (p, q), or a (p, q)-*form* for short, is a differential form

$$\omega = \sum_{|I|=p, |J|=q} f_{I,J} dz_I \wedge d\bar{z}_J$$

where $I = (i_1, \ldots, i_k), J = (j_1, \ldots, j_\ell)$ denote *multi-indices*, $|I|$ denotes the number of indices I contains, and $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_k}$.

Spaces of forms adopt a notation which allows the coefficient functions to show themselves. For instance, the space $L^2_{(p,q)}(\Omega)$ denotes the space of differential forms of bidegree (p, q) such that, adopting the notation from Definition 2.5.1, each $f_{I,J}$ belongs to the space $L^2(\Omega)$.

Then, we have

$$\bar{\partial}\omega = \sum_{j=1}^{n} \sum_{|I|=p, |J|=q} \frac{\partial f_{I,J}}{\partial \bar{z}_j} \, d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J$$

and we may see the $\bar{\partial}$ operator as $\bar{\partial} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q+1)}(\Omega)$.

Now, it is easy to see that $\bar{\partial} u = 0$ corresponds exactly to the Cauchy-Riemann equations in any domain, so this homogeneous equation has solutions wherever there are analytic functions. A natural follow-up question is whether we can find solutions for the non-homogeneous $\bar{\partial} u = f$.

In every bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$, the equation

$$\bar{\partial} u = f$$

has a solution $u \in L^2(\Omega)$ whenever $f \in L^2(\Omega)$. For a demonstration, see either [7] or [9]. This motivates us to define the $\bar{\partial}$-Neumann operator. Boas and Straube [17] cite Hörmander's influential paper on L^2 estimates, dated 1965, for the existence of what we will call the $\bar{\partial}$-Neumann operator on the space of $(0, q)$-forms in the shape of the following theorem.

Theorem 2.5.2. *Let $n \geq 2$ and $\Omega \subseteq \mathbb{C}^n$ be a bounded pseudoconvex domain. If $1 \leq q \leq n$, then the complex Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is an unbounded, surjective, and self-adjoint operator on $L^2_{(0,q)}(\Omega)$ with a bounded inverse $N_{(0,q)}$ which satisfies the estimates*

$$\begin{aligned} \left\|N_{(0,q)}u\right\|_2 &\leq \left(\frac{\operatorname{diam}(\Omega)^2 \cdot e}{q}\right) \|u\|_2 , \\ \left\|\bar{\partial}N_{(0,q)}u\right\|_2 &\leq \left(\frac{\operatorname{diam}(\Omega)^2 \cdot e}{q}\right)^{1/2} \|u\|_2 , \\ \left\|\bar{\partial}N_{(0,q)}u\right\|_2 &\leq \left(\frac{\operatorname{diam}(\Omega)^2 \cdot e}{q}\right)^{1/2} \|u\|_2 . \end{aligned} \tag{2.2}$$

Furthermore, for any $\bar{\partial}$-closed $(0,q)$-form f, the canonical solution to $\bar{\partial}u = f$ is given by $\bar{\partial}^* N_{(0,q)} f$.

The $\bar{\partial}$-Neumann operator connects to our work in the form of the following theorem, known as Kohn's formula, which appears in [9].

Theorem 2.5.3 (Kohn's Formula)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded pseudoconvex domain and $f \in L^2(\Omega)$. Then, the Bergman projection P is given by*

$$Pf = f - \bar{\partial}^* N_{(0,1)} \bar{\partial} f.$$

CHAPTER 3

BASIC THEORY OF TOEPLITZ AND HANKEL OPERATORS ON ANALYTIC FUNCTION SPACES

3.1 Introduction

Toeplitz and Hankel operators are extensively studied. These operators arise from multiplication operators, and appear in many places. In the Hardy-Hilbert space $H^2(\mathbb{D})$, or H^2 for short, because of the behaviour of the unilateral shift operator, the matrix representations of Toeplitz and Hankel operators are constant in the diagonal and the skew-diagonal, respectively. In contrast, examining Toeplitz and Hankel operators in Bergman spaces, especially when the domain is not $\mathbb{D}$ anymore, leads to a completely different theory. In this chapter, we are mainly going to focus on what happens in spaces of analytic functions over the open unit disc $\mathbb{D}$, but assume the more general cases whenever the generality of the results allow.

3.2 Hardy Spaces

This section will focus on Toeplitz and Hankel operators of the Hardy-Hilbert space $H^2(\mathbb{D})$, following the book [12]. In the Hardy-Hilbert space, many discussions revolve around matrices. This is because of the fact that in H^2, each Toeplitz operator is represented by a Toeplitz matrix, and vice versa. Similarly, each Hankel operator is represented by a Hankel matrix, and vice versa.

In the Hardy-Hilbert space $H^2(\mathbb{D})$, the Toeplitz and Hankel operators are defined by seeing $H^2(\mathbb{D})$ as a subspace of $L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle, or, to be more context-appropriate, the boundary of $\mathbb{D}$. Indeed, functions of class $H^2(\mathbb{D})$ have radial

limits almost everywhere, and the map $f \mapsto f^*$ where $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ is an isometric embedding of $H^2(\mathbb{D})$ into $L^2(\mathbb{T})$.

Definition 3.2.1. Let $A = (a_{i,j})$ be a matrix, possibly infinite, or doubly infinite; that is, the entries can be indexed by a finite set, $\mathbb{N} \times \mathbb{N}$, or $\mathbb{Z} \times \mathbb{Z}$.

(1) A is called a *Toeplitz matrix* if its entries remain unchanged along any diagonal; that is, if $a_{i,j} = a_{i+k,j+k}$ for each i, j and any integer k.

(2) A is called a *Hankel matrix* if its entries remain unchanged along any skew-diagonal; that is, if $a_{i,j} = a_{i+k,j-k}$ for each i, j and any integer k.

The following theorem was stated in [18].

Theorem 3.2.2. *Let* $\mathbf{a} = \{a_j\}_{j\in\mathbb{Z}}$. *The Toeplitz matrix*

$$T(\mathbf{a}) = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & a_{-5} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_3 & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_4 & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

defines a bounded operator on ℓ^2 *if and only if there is a* $\phi \in L^\infty(\mathbb{T})$ *such that* $a_j = \hat{\phi}(j)$ *for all* $j \in \mathbb{Z}$, *where* $\hat{\phi}(j)$ *denotes the* j^{th} *Fourier coefficient of* ϕ.

After this fact, Toeplitz operators arise.

Definition 3.2.3. Let $\widehat{H}^2$ denote the image of the above-mentioned identification and $\phi \in L^\infty(\mathbb{T})$. The *Toeplitz operator* with *symbol* ϕ, denoted by T_ϕ, is the linear operator $T_\phi : \widehat{H}^2 \to \widehat{H}^2$ defined as $T_\phi(f) = P(\phi f)$ where $P : L^2(\mathbb{T}) \to \widehat{H}^2$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $\widehat{H}^2$.

It is also possible to define Toeplitz operators for an unbounded symbol; however, the boundedness of Toeplitz operators is completely characterised by the boundedness of the symbol in $\widehat{H}^2$.

We have that $\{e^{in\theta} : n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{T})$, and $\{e^{in\theta} : n \in \mathbb{N}_0\}$ is an orthonormal basis for $\widehat{H}^2$. We will call these the standard bases for these spaces from this point on. Because of this fact, we get the nice correspondence between Toeplitz matrices and Toeplitz operators, and for Hankel matrices and Hankel operators.

We combine two theorems from [12] to present the following.

Theorem 3.2.4. *The matrix of T_ϕ with respect to the standard basis, where $\phi \in L^\infty$, is the Toeplitz matrix $(a_{i,j}) = (\phi_{i-j})$ where ϕ_n denotes the n^{th} Fourier coefficient of ϕ. Conversely, a Toeplitz matrix $(a_{i,j}) = (a_{i-j})$ induces a Toeplitz operator with symbol $\phi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$.*

The unilateral shift operator on ℓ^2 presents itself as M_z in $L^2(\mathbb{T})$. Toeplitz operators on H^2 have a close relationship with this operator. We see this in [12] in the form of the following theorem.

Theorem 3.2.5. *Let $T : H^2 \to H^2$ be a bounded linear operator. Then, T is a Toeplitz operator if and only if $S^*TS = T$, where $S = M_z$ denotes the unilateral shift operator.*

Another fact that make these operators fascinating is their relationship with their symbols, as can be seen from the following theorem in [12].

Theorem 3.2.6. *The map $T : L^\infty \to B(H^2)$ taking ϕ to T_ϕ is a linear, adjoint-preserving isometry.*

It immediately follows that a Toeplitz operator T_ϕ is self-adjoint if and only if $\phi(e^{i\theta}) \in \mathbb{R}$ for almost every $\theta \in \mathbb{R}$.

In the case that the symbol of a Toeplitz operator is analytic, we have the following fact from [12].

Theorem 3.2.7. *If $\phi \in H^\infty$, then the matrix of T_ϕ with respect to the standard basis is lower triangular.*

Remark 3.2.8. Notice that Theorem 3.2.7 is an easy corollary of the fact that, due to the standard basis, the entries of the matrix representation of any Toeplitz operator

are determined by the Fourier coefficients of its symbol. If that symbol happens to be analytic, then there will be no negatively indexed coefficients.

Again from [12], we have a characterisation of the spectrum of a Toeplitz operator.

Theorem 3.2.9. *For every* $\phi \in L^\infty$, $\sigma(T_\phi) = \overline{\phi(\mathbb{D})}$.

Remark 3.2.10. Theorem 3.2.9 also implies that $r(T_\phi) = \|\phi\|_\infty = \|T_\phi\|$.

The following theorem from [12] shows us that the compactness of Toeplitz operators on $\widehat{H}^2$ is not so intriguing.

Theorem 3.2.11. *Let* T_ϕ *be a compact Toeplitz operator on* $\widehat{H}^2$. *Then,* $\phi \equiv 0$.

There are two ways to define the Hankel operators in $\widehat{H}^2$. One requires the *flip operator* $J : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, defined by $Jf(e^{i\theta}) = f(e^{-i\theta})$. The difference between the two definitions is that one sends elements of $\widehat{H}^2$ to itself, and the other one to $L^2(\mathbb{T}) \ominus \widehat{H}^2$.

Definition 3.2.12. Let $\phi \in L^\infty(\mathbb{T})$.

(1) A *Hankel operator* is an operator of the form PJM_ϕ, where $P : L^2(\mathbb{T}) \to \widehat{H}^2$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $\widehat{H}^2$, J is the flip operator, and $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the multiplication operator $M_\phi f = \phi f$ by ϕ.

(2) A *Hankel operator* is an operator of the form $(I - P)M_\phi$ where $P : L^2(\mathbb{T}) \to \widehat{H}^2$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $\widehat{H}^2$.

Usually, we see the second definition in use; however, the first definition allows us to see Hankel operators as endomorphisms and lets us work with their spectra as well.

Remark 3.2.13. Sometimes the notations P_+ and P_- are used to denote the orthogonal projections to $\widehat{H}^2$ and its orthogonal complement, respectively. Therefore, with this notation, it is also written $H_\phi f = P_-(\phi f)$.

In any case, Hankel operators on H^2 are identified with Hankel matrices, by the well-celebrated Nehari's Theorem. For a proof, we refer the reader to [19] or [12], where, using the negative-indexed Fourier coefficients, the matrix is constructed.

Theorem 3.2.14 (Nehari's Theorem)**.** *For a Hankel matrix $(a_{i,j}) = (\phi_{-(i+j)})$ to correspond to a bounded linear operator T on H^2, it is necessary and sufficient that there is $\phi \in L^\infty$ with $T = H_\phi$. Furthermore, there is a symbol ψ of H_ϕ such that $H_\phi = H_\psi = \|\psi\|$.*

As can be inferred from the wording of the above theorem, Hankel operators admit several symbols instead of only one as in the case of Toeplitz operators. In fact, if $\psi \in H^\infty$, then $H_\phi = H_{\phi+\psi}$ for any $\phi \in L^\infty$.

The compactness of Hankel operators is completely characterised in the case of H^2. Indeed, we have the following from [20].

Theorem 3.2.15 (Hartman Compactness Criterion)**.** *A Hankel operator H is compact if and only if it admits a continuous symbol.*

Equivalent to the Hartman compactness criterion, the compactness of Hankel operators is sometimes formulated as follows: a Hankel operator H is compact if and only if each symbol of H belongs to $H^\infty + C$, namely the space of functions that are the sum of a bounded analytic and a continuous function.

3.3 Bergman Spaces

In the Bergman space $A^2(\Omega)$, we lose the matrix correspondence that was prominent in the Hardy space case. However, it was established in Section 2.5 that when Ω is a pseudoconvex domain, the problem $\bar\partial u = f$ has an L^2-solution whenever $f \in L^2(\Omega)$. Thence, the $\bar\partial$-Neumann operator N is defined. By Theorem 2.5.3, we have

$$I - P = \bar\partial^* N \bar\partial$$

where P denotes the Bergman projection, which is to say that Toeplitz and Hankel operators and the $\bar\partial$-Neumann problem have an extremely close relationship in this case.

We again define these operators in a similar fashion for $A^2(\Omega)$ for a domain $\Omega \subseteq \mathbb{C}^n$.

Definition 3.3.1. Let $\phi \in L^\infty(\Omega)$. The *Toeplitz operator* with *symbol* ϕ is the operator T_ϕ on $A^2(\Omega)$ such that $T_\phi f = P(\phi f)$.

We can see some basic facts about Toeplitz operators in [21].

Proposition 3.3.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\phi \in L^\infty(\Omega)$. Then,*

(1) $\|T_\phi\| \leq \|\phi\|_\infty$,

(2) $T_\phi^* = T_{\bar{\phi}}$.

Furthermore, if ϕ is analytic and $\psi \in L^\infty(\Omega)$, then

(3) $T_\psi T_\phi = T_{\psi\phi}$,

(4) $T_{\bar{\phi}} T_\psi = T_{\bar{\phi}\psi}$.

In addition to these, it is also easy to see that $T_{a\phi+b\psi} = aT_\phi + bT_\psi$ for any $a, b \in \mathbb{C}$ and $\phi, \psi \in L^\infty(\Omega)$.

Because the kernel is explicitly known in the case that the underlying set is one of these, most textbooks restrict their scope to the Hardy and Bergman spaces of the unit disc, the unit n-ball, or the unit polydisc in $\mathbb{C}^n$. We also see weighted Bergman spaces in works such as [22], but we will not go into detailed discussions on weighted Bergman spaces.

Zhu [22] also shows that on $A^2(\mathbb{D})$, we have a characterisation for Toeplitz operators in terms of integrable symbols.

Proposition 3.3.3. *Let $\phi \in L^1(\mathbb{D})$. If ϕ is harmonic, then the boundedness of T_ϕ is equivalent to the boundedness of ϕ. Furthermore, T_ϕ is compact if and only if $\phi \equiv 0$.*

The compactness of T_ϕ for continuous symbols is as follows. Compared to the Hardy space case, in Bergman spaces, we get many more compact Toeplitz operators.

Proposition 3.3.4. *Suppose that ϕ is continuous on $\overline{\mathbb{D}}$. Then, T_ϕ is compact on $A^2(\Omega)$ if and only if ϕ can be uniformly approximated on $\mathbb{D}$ via a sequence of compactly supported continuous functions on $\mathbb{D}$.*

Definition 3.3.5. Let $\Omega \subseteq \mathbb{C}^n$. An operator $H_\phi : A^2(\Omega) \to L^2(\Omega) \ominus A^2(\Omega)$ of the form $H_\phi f = (I - P)(\phi f)$ where $\phi \in L^\infty(\Omega)$ and $P : L^2(\Omega) \to A^2(\Omega)$ denotes the

Bergman projection on Ω, is called a *Hankel operator*. Such a function ϕ is called a *symbol* for H_ϕ.

Indeed, Kohn's Formula allows us to rewrite the definition of a Hankel operator on $A^2(\Omega)$ as $H_\phi f = \bar{\partial}^* N \bar{\partial}(\phi f)$. We will benefit greatly from this identity.

The Toeplitz and Hankel operators are simply components of multiplication operators arising from the space decomposition. However, they are also related to each other via the following proposition, which was left as an exercise in [22] for the weighted Bergman space of the unit disc. We will prove this fact for the Bergman spaces of arbitrary domains.

Proposition 3.3.6. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\phi, \psi \in L^\infty(\Omega)$. Then,*

$$H_\phi^* H_\psi = T_{\bar{\phi}\psi} - T_{\bar{\phi}} T_\psi.$$

Proof. Let $f, g \in A^2(\Omega)$. Then,

$$\begin{aligned}
\left\langle H_\phi^* H_\psi f, g \right\rangle &= \langle H_\psi f, H_\phi g \rangle = \langle (I-P)(\psi f), (I-P)(\phi g) \rangle \\
&= \left\langle (I-P)^2(\psi f), \phi g \right\rangle = \langle (I-P)(\psi f), \phi g \rangle \\
&= \langle \psi f, \phi g \rangle - \langle P(\psi f), \phi g \rangle \\
&= \left\langle \bar{\phi}\psi f, g \right\rangle - \left\langle P^2(\psi f), \phi g \right\rangle \\
&= \left\langle \bar{\phi}\psi f, Pg \right\rangle - \langle P(\psi f), P(\phi g) \rangle \\
&= \left\langle P(\bar{\phi}\psi f), g \right\rangle - \langle P(\psi f), P(\phi g) \rangle \\
&= \left\langle T_{\bar{\phi}\psi} f, g \right\rangle - \langle T_\psi f, T_\phi g \rangle \\
&= \left\langle (T_{\bar{\phi}\psi} - T_\phi^* T_\psi) f, g \right\rangle
\end{aligned}$$

and hence, $H_\phi^* H_\psi = T_{\bar{\phi}\psi} - T_\phi^* T_\psi = T_{\bar{\phi}\psi} - T_{\bar{\phi}} T_\psi$. □

Proposition 3.3.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\phi \in L^\infty(\Omega)$. Then, $\|H_\phi\| \leq \|\phi\|_\infty$.*

Proof. Observe that

$$\|H_\phi f\| = \|(I-P)(\phi f)\| \leq \|\phi f\| \leq \|\phi\|_\infty \|f\|.$$

Thus, we have the desired result. □

CHAPTER 4

COMPACTNESS OF HANKEL OPERATORS ON PSEUDOCONVEX DOMAINS

4.1 Introduction

This chapter will be a thorough explanation of [1]. The theorems and proofs belong to them unless otherwise stated, with a few expansions for claims that were left to the reader. This paper deals with how the smooth symbols of compact Hankel operators are holomorphic "along" the analytic discs of the boundary when the underlying domain is pseudoconvex and through some restrictions on the Levi form when necessary, and provides a proof for the converse of this statement in two dimensions when the domain happens to be convex.

4.2 Preparatory Work

Before we begin, let us remark that the notation $A \lesssim B$ is used when quantities A and B depend on the same variable(s), and $A \leq CB$ for some $C > 0$ that is independent of this/these variable(s). For instance, for a bounded operator T, we have $\|Tx\| \lesssim \|x\|$.

The following is a well-known fact in analysis and will prove useful in the proof of Theorem 4.3.6.

Proposition 4.2.1 (Young's Inequality)**.** *Let $a, b \geq 0$ and p, q be positive Hölder conjugates; that is, $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The first proposition in [1] asserts that when we look at compact Hankel operators on $A^2(\Omega)$ where Ω is a bounded pseudoconvex domain locally, we obtain another compact Hankel operator. Furthermore, if the boundary $b\Omega$ is made up of parts that cover it such that a Hankel operator is compact when restricted to each piece, then the whole has to be compact as well.

Notation 4.2.2. Let $f : \Omega \to \mathbb{C}$ and $A \subseteq \Omega \subseteq \mathbb{C}^n$. Then, the operator R_A denotes the restriction operator onto A. That is, $R_A : \mathcal{F}(\Omega) \to \mathcal{F}(A)$, defined as $R_A(f) = f \mid_A$.

Proposition 4.2.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $\phi \in L^\infty(\Omega)$. Then,*

(1) *if H^Ω_ϕ is compact on $A^2(\Omega)$, then for every $p \in b\Omega$ and open neighbourhood U of p that makes $U \cap \Omega$ a domain, the operator $H^{U\cap\Omega}_{R_{U\cap\Omega}(\phi)} R_{U\cap\Omega}$ is compact on $A^2(\Omega)$.*

(2) *if for every $p \in b\Omega$, there is a neighbourhood U of p such that $U \cap \Omega$ is a domain and $H^{U\cap\Omega}_{R_{U\cap\Omega}(\phi)} R_{U\cap\Omega}$ is compact on $A^2(\Omega)$, then so is H^Ω_ϕ.*

Proof. Starting with (1), for $f \in A^2(\Omega)$, we have

$$\begin{aligned}(I_{U\cap\Omega} - P_{U\cap\Omega}) R_{U\cap\Omega} H^\Omega_\phi(f) &= (I_{U\cap\Omega} - P_{U\cap\Omega}) R_{U\cap\Omega}(\phi f - P_\Omega(\phi f)) \\ &= R_{U\cap\Omega}(\phi f) - R_{U\cap\Omega}(P_\Omega(\phi f)) - P_{U\cap\Omega} R_{U\cap\Omega}(\phi f) + P_{U\cap\Omega} R_{U\cap\Omega} P_\Omega(\phi f) \\ &= H^{U\cap\Omega}_{R_{U\cap\Omega}(\phi)} R_{U\cap\Omega}(f) + P_{U\cap\Omega} R_{U\cap\Omega} P_\Omega(\phi f) - R_{U\cap\Omega} P_\Omega(\phi f).\end{aligned}$$

Notice that $P_{U\cap\Omega} R_{U\cap\Omega} P_\Omega = R_{U\cap\Omega} P_\Omega$ on $L^2(\Omega)$ and hence,

$$\begin{aligned}H^{U\cap\Omega}_{R_{U\cap\Omega}(\phi)} R_{U\cap\Omega}(f) + P_{U\cap\Omega} R_{U\cap\Omega} P_\Omega(\phi f) - R_{U\cap\Omega} P_\Omega(\phi f) \\ = H^{U\cap\Omega}_{R_{U\cap\Omega}(\phi)} R_{U\cap\Omega}(f).\end{aligned}$$

Thus, we have $(I_{U\cap\Omega} - P_{U\cap\Omega}) = H^{U\cap\Omega}_{R_{U\cap\Omega}(\phi)} R_{U\cap\Omega}$, which shows that if H^Ω_ϕ is assumed to be compact, then the right-hand side of this equation is necessarily compact as well.

To prove (2), choose $\{p_1, \ldots, p_m\} \subseteq b\Omega$ and open sets $U_1, \ldots, U_m$ such that

(a) U_j is a neighbourhood of p_j and $U_j \cap \Omega$ is a domain for $j = 1, \ldots, m$,

(b) $b\Omega \subseteq \bigcup_{j=1}^{m} U_j$,

(c) $S_j = H^{U_j\cap\Omega}_{R_{U_j\cap\Omega}(\phi)} R_{U_j\cap\Omega}$ is compact for all $j = 1, \ldots, m$.

Let $X = \Omega \setminus \bigcup_{j=1}^{m} U_j$. Notice that X is compact. Take a neighbourhood U_0 of X that is a domain such that $U_0 \subset\subset \Omega$ and let $S_0 = H^{U_0}_{R_{U_0}(\phi)} R_{U_0}$, and $\{\chi_j : j_0, \ldots, m\}$ be a C^∞ partition of unity subject to the open cover $\{U_j : j = 0, \ldots, m\}$. Then, for $f \in A^2(\Omega)$, one has

$$\begin{aligned}
\bar\partial\left(\sum_{j=0}^{m} \chi_j S_j(f)\right) &= \sum_{j=0}^{m} (\bar\partial\chi_j) S_j(f) + \sum_{j=0}^{m} \chi_j \bar\partial S_j(f) \\
&= \sum_{j=0}^{m} (\bar\partial\chi_j) S_j(f) + \sum_{j=0}^{m} \chi_j \bar\partial((I - P_{U_j\cap\Omega})(\phi f)) \\
&= \sum_{j=0}^{m} (\bar\partial\chi_j) S_j(f) + \sum_{j=0}^{m} \chi_j \bar\partial(\phi f) \\
&= \sum_{j=0}^{m} \bar\partial(\chi_j) S_j(f) + \sum_{j=0}^{m} \chi_j (\bar\partial\phi) f \\
&= \sum_{j=0}^{m} \bar\partial(\chi_j) S_j(f) + \left(\sum_{j=0}^{m} \chi_j\right) (\bar\partial\phi) f \\
&= \sum_{j=0}^{m} \bar\partial(\chi_j) S_j(f) + (\bar\partial\phi) f.
\end{aligned}$$

In other words, we have

$$\sum_{j=0}^{m} \bar\partial(\chi_j) S_j(f) = \sum_{j=0}^{m} \bar\partial(\chi_j) S_j(f) - (\bar\partial\phi) f.$$

Notice that the right-hand side of the equation is $\bar\partial$-closed and thus, so is the sum $\sum_{j=0}^{m} \bar\partial(\chi_j) S_j(f)$. Define

$$S = \sum_{j=0}^{m} \chi_j S_j - \bar\partial^* N^\Omega \sum_{j=0}^{m} \bar\partial(\chi_j) S_j.$$

We claim that $\bar\partial S(f) = (\bar\partial\phi) f$. Indeed, we have

$$\bar\partial S(f) = (\bar\partial\phi) f + \sum_{j=0}^{m} (\bar\partial\chi_j) S_j(f) - \bar\partial\bar\partial^* N^\Omega \sum_{j=0}^{m} (\bar\partial\chi_j) S_j(f).$$

It suffices to show that $\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f) = \bar{\partial}\bar{\partial}^* N^{\Omega}\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f)$. Note that both sides of the equality are $(0,1)$-forms. Since the space of harmonic $(0,1)$-forms is trivial on a bounded pseudoconvex domain, the complex Laplacian $\Box$ and N^{Ω} are exactly the inverse of each other. Thus, we have

$$\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f) = N^{\Omega}(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f) = N^{\Omega}\bar{\partial}\bar{\partial}^*\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f)$$

where the last equality follows from the fact that $\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f)$ is $\bar{\partial}$-closed. Since both $\bar{\partial}N^{\Omega} = N^{\Omega}\bar{\partial}$ and $\bar{\partial}^*N^{\Omega} = N^{\Omega}\bar{\partial}^*$ hold, we have the desired equality. In other words, it holds that

$$\bar{\partial}S(f) = (\bar{\partial}\phi)f + \sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f) - \bar{\partial}\bar{\partial}^*N^{\Omega}\sum_{j=0}^{m}(\bar{\partial}\chi_j)S_j(f) = (\bar{\partial}\phi)f.$$

Take a bounded sequence $\{f_j\}$ in $A^2(\Omega)$ and $K = \overline{U_0}$. Say $\|f_j\|_2 \leq M$ for all j. This means that $\|P_{\Omega}(\phi f_j)\|_2 \leq \|\phi f_j\|_2 \leq \|\phi\|_{\infty}\|f_j\|_2 \leq M\|\phi\|_{\infty}$. By Lemma 2.3.3, it follows that the sequence $\{f_j\}$ has a subsequence $\{f_{j_k}\}$ that is locally uniformly convergent on Ω, say to f. Moreover, the sequence $\{P_{\Omega}(\phi f_{j_k})\}$ has a subsequence $\{P_{\Omega}(\phi f_{j_{k_\ell}})\}$ that converges uniformly on K. It follows that

$$\{\chi_0 f_{j_{k_\ell}} - \chi_0 P_{\Omega}(\phi f_{j_{k_\ell}})\} = \{\chi_0 S_0(f_{j_{k_\ell}})\}$$

is uniformly convergent on K. Thus,

$$\begin{aligned}
&\left\|\chi_0 S_0 f_{j_{k_\ell}} - \chi_0 S_0 f\right\|_2^2 \\
&= \int_{\Omega\setminus K}\left|\chi_0 S_0 f_{j_{k_\ell}}(z) - \chi_0 S_0 f(z)\right|^2 dV(z) + \int_K \left|\chi_0 S_0 f_{j_{k_\ell}}(z) - \chi_0 S_0 f(z)\right|^2 dV(z) \\
&= \int_K \left|\chi_0 S_0 f_{j_{k_\ell}}(z) - \chi_0 S_0 f(z)\right|^2 dV(z)
\end{aligned}$$

as the support of χ_0 is contained in K. Letting $\ell \to \infty$, the integral over K tends to 0 by uniform convergence. It follows that $\chi_0 S_0$ is compact on $A^2(\Omega)$. Notice that $\operatorname{supp}\left(\bar{\partial}\chi_0\right) \subseteq K$ as well. Hence, the same argument asserts that $(\bar{\partial}\chi_0)S_0$ is compact. Note that $\bar{\partial}^*N^{\Omega}$ is continuous since Ω is a bounded pseudoconvex domain. Hence, S

is compact. We have already established that $\bar{\partial}S(f) = (\bar{\partial}\phi)f$. Thus, we have

$$\begin{aligned} H^{\Omega}_{\phi} f &= (I - P_{\Omega})(\phi f) = \bar{\partial}^* N^{\Omega} \bar{\partial}(\phi f) \\ &= \bar{\partial}^* N^{\Omega}((\bar{\partial}\phi) f) = \bar{\partial}^* N^{\Omega}(\bar{\partial} S(f)) \\ &= (I - P_{\Omega}) S(f), \end{aligned}$$

showing that H^{Ω}_{ϕ} is the composition of a bounded operator and a compact one. By Theorem 2.1.1, the Hankel operator H^{Ω}_{ϕ} is compact on $A^2(\Omega)$. □

The following lemma shows that the compactness of a Hankel operator is preserved under biholomorphisms.

Lemma 4.2.4. *Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n$. Take a function $\phi \in C^{\infty}(\overline{\Omega}_2)$, and let $F : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphism that has a smooth extension to the boundary. Assume that $H^{\Omega_2}_{\phi}$ is compact on $A^2(\Omega_2)$. Then, $H^{\Omega_1}_{\phi \circ F}$ is compact on $A^2(\Omega_1)$.*

Proof. Let $g \in A^2(\Omega_1)$. Define $f = g \circ F^{-1}$, $u = \bar{\partial}^* N^{\Omega_2} \bar{\partial}(\phi f)$, and $w = u \circ F = F^*(u)$. Then, $f \in A^2(\Omega_2)$, $u = H^{\Omega_2}_{\phi}(f)$, and by commutativity of the pullback and $\bar{\partial}$, we have

$$\bar{\partial} w = \bar{\partial} F^*(u) = F^*(\bar{\partial} u) = F^*(f \bar{\partial}\phi) = (f \circ F)\bar{\partial}(\phi \circ F)$$

since u solves $\bar{\partial} u = \bar{\partial}(\phi f)$.

Note that for any suitable function g, by Theorem 2.5.2, the canonical solution of the equation $\bar{\partial} u = g$ is given by $u = \bar{\partial}^* N g$ where N denotes the $\bar{\partial}$-Neumann operator. This fact can be applied to the situation at hand to yield the fact that

$$\bar{\partial}^* N^{\Omega_1}(f \circ F)\bar{\partial}(\phi \circ F) = \bar{\partial}^* N^{\Omega_1} \bar{\partial} w = \bar{\partial}^* N^{\Omega_1} \bar{\partial}(u \circ F)$$

is the canonical solution to the equation

$$\bar{\partial} w = (f \circ F)\bar{\partial}(\phi \circ F)$$

on Ω_1, with the commutativity of the pullback and the operators $\bar{\partial}^*, N^{\Omega_1}, \bar{\partial}$ being implied by the fact that F is a biholomorphism. Then,

$$H^{\Omega_1}_{\phi \circ F}(g) = H^{\Omega_1}_{\phi \circ F}(f \circ F) = \bar{\partial}^* N^{\Omega_1} \bar{\partial}(u \circ F) = \bar{\partial}^* N^{\Omega_1} \bar{\partial} \left(F^* H^{\Omega_2}_{\phi} (F^{-1})^*(g) \right).$$

By Theorem 2.1.1, the operator $H^{\Omega_1}_{\phi \circ F}$ is compact on $A^2(\Omega_1)$. □

Now, the following lemma from [23] lets us work with a simpler domain in Theorem 4.3.1.

Lemma 4.2.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a smooth, bounded, pseudoconvex domain, $M \subseteq b\Omega$ a k-dimensional complex manifold, and $p \in M$. Then, there is a ball B whose centre is p, such that there is a biholomorphic map $G : B \to G(B)$ mapping p to 0 and $M \cap B$ to the first k coordinates, so that the real normal to $G(b\Omega \cap B)$ on $G(M \cap B)$ goes in the same direction as the x_n-axis.*

Since it will be helpful to the discussion in Theorem 4.3.6, we make a detour here and mention Sobolev spaces and their duals. On $\Omega \subseteq \mathbb{C}^n$, the Sobolev space $W^{k,p}(\Omega)$ is the space of p-integrable functions that have all weak derivatives/partial derivatives up to k^{th} order. For the dual of $W^{1,p}(\Omega)$, we use the notation $W^{-1,q}(\Omega)$ where q is the Hölder conjugate for p. After the following theorem from [24], we will discuss the norm of $W^{-1,2}(\Omega)$.

Theorem 4.2.6 (Rellich-Kondrachov)**.** *Let Ω be a bounded subset of $\mathbb{R}^n$ that is of class C^1. Then, the inclusion map $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for any q with $1 \leq q < p^*$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. If $p = n$, then this inclusion is compact for any $q \geq p$.*

Let us look at the dual map of this inclusion. If $n \geq 2$ and $p = 2$, then $p^* > 2$, so the dual of the inclusion

$$L^2(\Omega) \hookrightarrow W^{-1,2}(\Omega)$$

is compact, and hence continuous. This shows that for any $f \in L^2(\Omega)$, we have $\|f\|_{L^2(\Omega)} \lesssim \|f\|_{-1}$.

4.3 Main Results

In this section, we present the three main theorems that appear in [1] along with their proofs, and follow them while explaining some parts that were left to the reader.

Theorem 4.3.1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$ and $\beta \in C^\infty(\overline{\Omega})$. Assume that the Levi form of $b\Omega$ is of rank at least $n - 2$. If H_β is compact on $A^2(\Omega)$, then $\beta \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

The following proof argues for a more general result than that of Theorem 4.3.1. Namely, rather than the Levi form having rank at least $n-2$, the authors argue that the statement of the theorem also holds on the condition that the Levi form of $b\Omega$ is of rank k where k is between 1 and $n-1$, and there is an $(n-k-1)$-dimensional complex manifold in $b\Omega$ that contains p.

Proof. We aim for a contradiction by assuming that

(i) Ω is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and $p \in b\Omega$,

(ii) the Levi form of $b\Omega$ is of rank k at p and there is an $(n-k-1)$-dimensional complex manifold $M \subseteq b\Omega$ to which p belongs,

(iii) there exists a non-constant $f : \mathbb{D} \to b\Omega$ which is analytic and $q \in \mathbb{D}$ such that $f(q) = p$, $Df(q)$ is full rank, at which $\beta \circ f$ is not holomorphic,

(iv) H_β is compact.

In this proof, we will first carry p to the origin with the part of Ω and $b\Omega$ near p using a change of coordinates; then we will work with the Hankel operator of the appropriate symbol in this new domain. We will find that this Hankel operator is compact. On the other hand, for a particular bounded sequence of functions, we will observe that the image of this sequence under the Hankel operator cannot have a convergent subsequence, reaching a contradiction.

By Lemma 4.2.5, we may carry a neighbourhood V of p via a biholomorphism G so that $G(p) = 0$ and that the positive x_n-axis is the outward normal direction to the boundary of $\Omega_1 = G(V \cap \Omega)$ at every point of

$$M = \{z \in \mathbb{D}^n : z_{n-k} = \ldots = z_n = 0\},$$

with the additional assumption that $M \subseteq b\Omega_1$. Denote z by (z', z'') where $z' = (z_1, \ldots, z_{n-k-1})$, and $z'' = (z_{n-k}, \ldots, z_n)$. Define L to be the slice of Ω_1 that passes through the origin and is orthogonal to M. In other words, we define L to be the set $\{z'' \in \mathbb{C}^{k+1} : (0, z'') \in \Omega_1\}$. Since the rank of the Levi form is a biholomorphic invariant, the rank of the Levi form of the boundary of the $(k+1)$-dimensional complex manifold L at the origin is k, showing that L is strictly pseudoconvex at the

origin when $k \geq 1$, and is a domain in $\mathbb{C}$ when $k = 0$. Then, there exists a scalar $0 < \lambda < 1$ such that $M_1 \times L_1 \subseteq \Omega_1$, where L_1 is the ball in $\mathbb{C}^{k+1}$ of radius λ and centre $(0, \ldots, 0, -\lambda)$, and $M_1 = \frac{1}{2}M$. For every j, choose $p_j = (0, \ldots, -1/j) \in M_1 \times L_1$. We will also consider $p_j = (0, \ldots, 0, -1/j) \in L_1$ and use the name interchangeably. Now, we define $q_j = G^{-1}(p_j) \in V \cap \Omega$ and

$$f_j(z) = \frac{K^\Omega(z, q_j)}{\sqrt{K^\Omega(q_j, q_j)}}.$$

First, notice that

$$\|K^\Omega(z, q_j)\|_{A^2(\Omega)} = \left\langle K^\Omega(z, q_j), K^\Omega(z, q_j) \right\rangle^{1/2} = \sqrt{K^\Omega(q_j, q_j)}$$

as $K_\Omega(z, q_j)$ is the Riesz representative of the evaluation functional $f \mapsto f(q_j)$, so $\{f_j\}$ is a normalised sequence of square-integrable functions in Ω. By Lemma 2.3.3, we see that $\{f_j\}$ is locally uniformly convergent. Furthermore, for a fixed $z \in \Omega$,

$$f_j(z) = \frac{K^\Omega(z, q_j)}{\sqrt{K^\Omega(q_j, q_j)}} \leq \frac{K^\Omega(z, q_j)}{C}(d(q_j, b\Omega))^{\frac{k+2}{2}} \to 0 \text{ as } j \to \infty,$$

so $f_j \to 0$ pointwise as $j \to \infty$, whence our claim follows. Define $\alpha_j = f_j \circ G^{-1}$ and $\beta_1 = \beta \circ G^{-1}$. Then, $H^{V\cap\Omega}_{R_{V\cap\Omega}(\beta)} R_{V\cap\Omega}$ is compact by Proposition 4.2.3. Using the connection via G and Lemma 4.2.4, we then obtain $H^{\Omega_1}_{\beta_1}$ is compact. Since $\{\alpha_j\}$ is a bounded sequence and $H^{\Omega_1}_{\beta_1}$ is a compact operator, one would expect that the sequence $\{H^{\Omega_1}_{\beta_1}(\alpha_j)\}$ has a convergent subsequence. In the remainder of the proof, this will be demonstrated to be not the case, arriving at a contradiction.

Per our assumption, we have $\bar{\partial}(\beta \circ f) \neq 0$. A change of coordinates allows us to assume without loss of generality that $\left|\frac{\partial \beta_1}{\partial \bar{z}_1}\right| > 0$ at the origin. Consequently, there exists $0 < r < 1$, and a smooth function $0 \leq \chi \leq 1$ on real numbers such that the followings hold:

(1) $\Delta_r^{n-k-1} \subseteq M_1$,

(2) $\chi(t) = 1$ for $|t| \leq r/2$ and $\chi(t) = 0$ for $|t| \geq 3r/5$,

(3) $\left|\frac{\partial \beta_1}{\partial \bar{z}_1}\right| > 0$ on Δ_r^n.

Then, $C = \int_{|z_1|<3r/4} \chi(|z_1|)\, dV(z_1) > 0$. Define a function γ on Ω_1 to satisfy the equality

$$\gamma(z)\frac{\partial \beta_1(z)}{\partial \bar{z}_1} = \chi(|z_1|) \cdots \chi(|z_n|)$$

and let $\langle ., . \rangle$ denote the pointwise inner product on forms in $\mathbb{C}^n$ given by the declaration

$$\langle dz_I \wedge d\bar{z}_J, dz_K \wedge d\bar{z}_L \rangle = \delta_{IK}\delta_{JL}$$

for multi-indices $I, J, K, L \subseteq \{1, \ldots, n\}$. For the following calculation, we only want one variable in action, so we write $z = (z_1, w)$ where $w = (z_2, \ldots, z_n)$. Using Proposition 2.2.1 for an analytic function $\alpha \in A^2(\Omega)$ and fixed $w = (w_2, \ldots, w_n) \in \Delta_{3r/4}^{n-1}$ so that $\Delta_r \times \{w\} \subseteq M_1 \times L_1$, we obtain, letting $B = \chi(|w_2|) \cdots \chi(|w_n|)$,

$$\begin{aligned}
BC\alpha(0, w) &= B \int\limits_{|z_1|<3r/4} \chi(|z_1|)\alpha(z_1, w)\, dV(z_1) \\
&= \int\limits_{|z_1|<3r/4} \gamma(z_1, w) \frac{\partial \beta_1(z_1, w)}{\partial \bar{z}_1} \alpha(z_1, w)\, dV(z_1) \\
&= \int\limits_{|z_1|<3r/4} \langle \alpha \bar{\partial} \beta_1, \bar{\gamma} d\bar{z}_1 \rangle\, dV(z_1) \\
&= \int\limits_{|z_1|<3r/4} \langle \bar{\partial}\bar{\partial}^* N^{\Omega_1}(\alpha \bar{\partial} \beta_1), \bar{\gamma} d\bar{z}_1 \rangle\, dV(z_1) \\
&= \int\limits_{|z_1|<3r/4} \langle \bar{\partial} H_{\beta_1}^{\Omega_1}(\alpha), \bar{\gamma} d\bar{z}_1 \rangle\, dV(z_1) \\
&= \int\limits_{|z_1|<3r/4} \frac{\partial H_{\beta_1}^{\Omega_1}(\alpha)}{\partial \bar{z}_1} \gamma\, dV(z_1) \\
&= \int\limits_{|z_1|<3r/4} \frac{\partial}{\partial \bar{z}_1} \left(H_{\beta_1}^{\Omega_1}(\alpha)\gamma \right)\, dV(z_1) - \int\limits_{|z_1|<3r/4} H_{\beta_1}^{\Omega_1}(\alpha) \frac{\partial \gamma}{\partial \bar{z}_1}\, dV(z_1) \\
&= \frac{1}{2i} \int\limits_{|z_1|=3r/4} H_{\beta_1}^{\Omega_1}(\alpha)\gamma\, dz_1 - \int\limits_{|z_1|<3r/4} H_{\beta_1}^{\Omega_1}(\alpha) \frac{\partial \gamma}{\partial \bar{z}_1}\, dV(z_1) \\
&= - \int\limits_{|z_1|<3r/4} H_{\beta_1}^{\Omega_1}(\alpha) \frac{\partial \gamma}{\partial \bar{z}_1}\, dV(z_1)
\end{aligned}$$

where the fourth line is due to the fact that both α and $\bar{\partial}\beta_1$ are $\bar{\partial}$-closed and that $\square$ and N^{Ω_1} are inverses of each other, and the second-to-last line follows from Green's

Theorem. By Cauchy-Schwarz inequality, it is easy to see that

$$\int\limits_{|z_1|<3r/4} H^{\Omega_1}_{\beta_1}(\alpha)\frac{\partial\gamma}{\partial\bar{z}_1}\,dV(z_1)$$
$$\leq \left(\int\limits_{|z_1|<3r/4} \left|H^{\Omega_1}_{\beta_1}(\alpha)\right|^2\,dV(z_1)\right)^{1/2}\left(\int\limits_{|z_1|<3r/4}\left|\frac{\partial\gamma}{\partial\bar{z}_1}\right|^2\,dV(z_1)\right)^{1/2}.$$

Combining everything together, we have

$$|\alpha(0,w)| \lesssim \left(\int\limits_{|z_1|<3r/4} \left|H^{\Omega_1}_{\beta_1}(\alpha)\right|^2\,dV(z_1)\right)^{1/2}.$$

Squaring both sides yields

$$|\alpha(0,w)|^2 \lesssim \int\limits_{|z_1|<3r/4} \left|H^{\Omega_1}_{\beta_1}(\alpha)\right|^2\,dV(z_1).$$

Since α is analytic, the function $|\alpha(0,w)|^2$ is subharmonic, so

$$|\alpha(0,z'')|^2 \lesssim \int\limits_{z\in\Delta^{n-k-1}_{3r/4}} |H^{\Omega_1}_{\beta_1}(\alpha)(z',z'')|^2\,dV(z'). \tag{4.1}$$

The inequality (4.1) applied to α_j implies that

$$\begin{aligned}
\|\alpha_j\mid_{L_1}\|^2_{L^2(L_1)} &= \int\limits_{L_1} |\alpha_j(0,z'')|^2\,dV(z'')\\
&\lesssim \int\limits_{L_1}\int\limits_{z\in\Delta^{n-k-1}_{3r/4}} |H^{\Omega_1}_{\beta_1}(\alpha_j)(z',z'')|^2\,dV(z')\,dV(z'')\\
&\leq \int\limits_{L_1}\int\limits_{M_1} |H^{\Omega_1}_{\beta_1}(\alpha_j)(z',z'')|^2\,dV(z')\,dV(z'')\\
&= \left\|H^{\Omega_1}_{\beta_1}(\alpha_j)\right\|_{L^2(M_1\times L_1)} \leq \left\|H^{\Omega_1}_{\beta_1}(\alpha_j)\right\|_{L^2(\Omega_1)}.
\end{aligned}$$

Thus, $\alpha_j\mid_{L_1}\in L^2(L_1)$. Utilising the Bergman kernel on L_1, we obtain

$$\alpha_j(p_j) = \int\limits_{L_1} K^{L_1}(p_j,z)\alpha_j\mid_{L_1}(z)\,dV(z).$$

By Cauchy-Schwarz inequality, one has

$$|\alpha_j(p_j)| \leq \|\alpha_j\mid_{L_1}\|_{L^2(L_1)}\left\|K^{L_1}(p_j,\cdot)\right\|_{L^2(L_1)} = \|\alpha_j\mid_{L_1}\|_{L^2(L_1)}\sqrt{K^{L_1}(p_j,p_j)}.$$

It follows that

$$\|\alpha_j \mid_{L_1}\|_{L^2(L_1)} \geq \frac{|\alpha_j(p_j)|}{\sqrt{K^{L_1}(p_j,p_j)}} = \sqrt{\frac{K^{\Omega}(q_j,q_j)}{K^{L_1}(p_j,p_j)}}.$$

Since L_1 is a ball in $\mathbb{C}^{k+1}$, by Proposition 2.4.6 and Remark 2.4.2, we have

$$\frac{1}{(d(p_j,bL_1))^{k+2}} \lesssim K^{L_1}(p_j,p_j) \lesssim \frac{1}{(d(p_j,bL_1))^{k+2}}$$

and

$$\frac{1}{(d(q_j,b\Omega))^{k+2}} \lesssim K^{\Omega}(q_j,q_j).$$

By definition, we have, for sufficiently large j, $d(p_j,b\Omega_1) = d(p_j,bL_1)$. Furthermore, since $p_j = G(q_j)$ and G is a diffeomorphism, $d(p_j,b\Omega_1) = d(p_j,bL_1)$ is comparable to $d(q_j,b\Omega)$. This shows that $\sqrt{\frac{K^{\Omega}(q_j,q_j)}{K^{L_1}(p_j,p_j)}}$ and thus $\|\alpha_j \mid_{L_1}\|_{L^2(L_1)} > \varepsilon > 0$ for some ε. We have established before that $\alpha_j \to 0$ locally uniformly, so if their norms are away from 0, the sequence $\{\alpha_j \mid_{L_1}\}$ cannot have a subsequence that converges in $L^2(L_1)$. By (4.1), one has

$$\|\alpha_j \mid_{L_1} - \alpha_k \mid_{L_1}\|_{L^2(L_1)} \lesssim \left\|H^{\Omega_1}_{\beta_1}(\alpha_j - \alpha_k)\right\|_{L^2(\Omega_1)}.$$

Thus, $\left\{H^{\Omega_1}_{\beta_1}(\alpha_j)\right\}$ cannot have any Cauchy subsequences. This yields the contradiction we need. □

Corollary 4.3.2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ and $\beta \in C^{\infty}(\overline{\Omega})$. If H_β is compact on $A^2(\Omega)$, then $\beta \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

Proof. Since the Levi form is of rank at least $n-2 = 0$ no matter what, the statement immediately follows. □

Another major result found in [1] was that we can get rid of the restriction on the Levi form altogether, provided that the domain is convex.

Theorem 4.3.3. *Let Ω be a smooth bounded convex domain in $\mathbb{C}^n$ for $n \geq 2$ and $\beta \in C^{\infty}(\overline{\Omega})$. Assume that H_β is compact on $A^2(\Omega)$. Then, $\beta \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

The proof of this result follows from a parallel argument with Theorem 4.3.1, and will be omitted.

Let us introduce a definition from [25].

Definition 4.3.4. Let $\Omega \subseteq \mathbb{C}^n$ be a domain. Then, $b\Omega$ is said to satisfy property (P) if for every $M > 0$, there is a plurisubharmonic function $u \in C^\infty(\Omega)$ ranging in $[0, 1]$, such that for any $z \in b\Omega$ and $w \in \mathbb{C}^n$, one has

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k \geq M \left|w\right|^2 .$$

Property (P) is important as the $\bar{\partial}$-Neumann problem satisfies the following *compactness estimate* on the Bergman space of domains that satisfy property (P), as shown in [17]. Consequenctly, on such domains, the $\bar{\partial}$-Neumann operator N is compact.

Proposition 4.3.5. *If $\Omega \subseteq \mathbb{C}^n$ satisfies property (P), then for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that for any $u \in \mathcal{D}(\bar{\partial}^*) \cap \mathcal{D}(\bar{\partial})$,*

$$\|u\|^2_{L^2(\Omega)} \leq \varepsilon \left(\left\|\bar{\partial} u\right\|^2_{L^2(\Omega)} + \left\|\bar{\partial}^* u\right\|^2_{L^2(\Omega)} \right) + C_\varepsilon \|u\|^2_{-1} .$$

In a convex domain in $\mathbb{C}^2$, we have the following.

Theorem 4.3.6. *Let Ω be a smooth bounded convex domain in $\mathbb{C}^2$ and $\beta \in C^\infty(\overline{\Omega})$. If $\beta \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$, then H_β is compact on $A^2(\Omega)$.*

Proof. Set

$$K = \overline{\bigcup_{\substack{f:\mathbb{D}\to b\Omega \\ f \text{ is analytic}}} f(\mathbb{D})}.$$

Pick a defining function ρ whose gradient has norm 1 at each point. Write β as $\beta_1 + i\beta_2$ for real-valued functions β_1 and β_2. Define

$$\nu = \sum_{j=1}^{2} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial x_j} + \frac{\partial \rho}{\partial y_j} \frac{\partial}{\partial y_j} \text{ and } \tau = \sum_{j=1}^{2} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial y_j} - \frac{\partial \rho}{\partial y_j} \frac{\partial}{\partial x_j}.$$

Notice that ν is in the outward normal direction and τ is orthogonal to it, so it is

tangent to the boundary. Let us observe that

$$\begin{aligned}
\nu + i\tau &= \sum_{j=1}^{2} \frac{\partial\rho}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial\rho}{\partial y_j}\frac{\partial}{\partial y_j} + i\frac{\partial\rho}{\partial x_j}\frac{\partial}{\partial y_j} - i\frac{\partial\rho}{\partial y_j}\frac{\partial}{\partial x_j} \\
&= \sum_{j=1}^{2} \frac{\partial\rho}{\partial x_j}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) + \frac{\partial\rho}{\partial y_j}\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial x_j}\right) \\
&= \sum_{j=1}^{2} \frac{\partial\rho}{\partial x_j}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) - i\frac{\partial\rho}{\partial y_j}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \\
&= \sum_{j=1}^{2} 2\frac{\partial\rho}{\partial x_j}\frac{\partial}{\partial \bar{z}_j} - 2i\frac{\partial\rho}{\partial y_j}\frac{\partial}{\partial \bar{z}_j} \\
&= 4\sum_{j=1}^{2} \frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial \bar{z}_j}.
\end{aligned}$$

Thus, we may interpret $\nu + i\tau$ as measuring analyticity at the boundary.

For sufficiently small ε and $p \in b\Omega$, we define

$$\widetilde{\beta}_1(p + \varepsilon\nu(p)) = \beta_1(p) + \varepsilon\tau(\beta_2)(p) \text{ and } \widetilde{\beta}_2(p + \varepsilon\nu(p)) = \beta_2(p) - \varepsilon\tau(\beta_1)(p).$$

In other words, these two functions can be regarded as defined on a tube following the trajectory of $b\Omega$. We see that $\widetilde{\beta} = \widetilde{\beta}_1 + i\widetilde{\beta}_2$ is a smooth function in a neighbourhood of $b\Omega$ and $\widetilde{\beta}\,|_{b\Omega} = \beta$. In other words, we have $\widetilde{\beta}(p + \varepsilon\nu(p)) = \beta(p) - i\varepsilon\tau(\beta)(p)$. Extend $\widetilde{\beta}$ as a smooth function on $\overline{\Omega}$ with the same name. Notice that $\nu(\widetilde{\beta})(p) = \nabla\rho \cdot \nabla\widetilde{\beta}$ for any $p \in b\Omega$. In other words, it is the derivative of $\widetilde{\beta}$ in the direction of $\nabla\rho$ at p. Note that $\nabla\rho$ is the direction normal to the boundary. Then,

$$\nu(\widetilde{\beta})(p) = \frac{d}{d\varepsilon}\widetilde{\beta}(p + \varepsilon\nu(p))\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\beta(p) - i\varepsilon\tau(\beta)(p)\right) = -i\tau(\beta)(p).$$

Notice that τ is in the tangent space of $b\Omega$ at each point of the boundary. Therefore, we need to have $\tau(\widetilde{\beta})(p) = \tau(\beta)(p)$ for any $p \in b\Omega$. What follows is that $(\nu + iT)(\widetilde{\beta}) = 0$ on $b\Omega$. On $\overline{\Omega}$, define $\widehat{\beta} = \beta - \widetilde{\beta}$. Since both β and $\widetilde{\beta}$ are smooth on $\overline{\Omega}$, so is $\widehat{\beta}$. Furthermore, since $\beta = \widetilde{\beta}$ on $b\Omega$, we have $\widehat{\beta} = 0$ on $b\Omega$, and $\widehat{\beta}$ is holomorphic on K per our assumption that β is holomorphic along analytic discs of $b\Omega$ and the fact that $\widetilde{\beta}$ is holomorphic on the boundary. Take a nested sequence K_m of compact sets such that $\bigcup_{m=1}^{\infty} K_m = \Omega$ and let χ_m be compactly supported smooth functions such that $0 \leq \chi_m \leq 1$ and $\chi_m(z) = 1$ for all $z \in K_m$. Then, $\beta_m = \chi_m\widehat{\beta}$ yields a sequence of smooth functions converging to $\widehat{\beta}$ locally uniformly such that $\operatorname{supp}(\beta_m) \subset\subset \Omega$.

Take a sequence $\{f_j\}$ such that $f_j \rightharpoonup 0$. We wish to demonstrate that $H^{\Omega}_{\widehat{\beta}} f_j \to 0$ as $j \to \infty$. By the uniform boundedness principle, it is easy to see that $\{f_j\}$ is bounded in $A^2(\Omega)$. By Lemma 2.3.3, we may deduce that it is locally uniformly convergent on Ω. Then, both $\{\widehat{\beta} f_j\}$ and $\{P(\widehat{\beta} f_j)\}$ have locally uniformly convergent subsequences, so $\{H_{\widehat{\beta}} f_j\}$ has a locally uniformly convergent subsequence converging to 0. Let $A \subseteq \Omega$ be a compact set. Then, we have

$$\int\limits_A \left| H_{\widehat{\beta}} f_j(z) \right|^2 \, dV(z) \to 0.$$

By definition, we have $\beta_m(z) \nearrow \widehat{\beta}(z)$ for each $z \in \Omega$ and also that $\widehat{\beta} \in L^2(\Omega)$. By monotone convergence theorem, we have

$$\begin{aligned}
\int\limits_{\Omega \setminus A} \left| H^{\Omega}_{\widehat{\beta}} f_j \right|^2 \, dV(z) &= \int\limits_{\Omega \setminus A} \left| \bar{\partial}^* N \bar{\partial}(\widehat{\beta}) f_j \right|^2 \, dV(z) \\
&\leq \int\limits_{\Omega \setminus A} \left| \widehat{\beta} f_j \right|^2 \, dV(z) \\
&= \lim_{m \to \infty} \int\limits_{\Omega \setminus A} |\beta_m f_j|^2 \, dV(z).
\end{aligned}$$

Now,

$$\int\limits_{\Omega \setminus A} |\beta_m f_j|^2 \, dV(z) \to 0$$

for each m as $j \to \infty$ since $\{f_j\}$ is a bounded sequence due to weak convergence and $\beta_m f_j$ is supported inside $\operatorname{supp}(\beta_m)$ for each j. It follows that $H^{\Omega}_{\widehat{\beta}}$ is compact on $A^2(\Omega)$. It remains to show that $H^{\Omega}_{\widetilde{\beta}}$ is compact on $A^2(\Omega)$, and the result will follow.

Take a sequence $\{\psi_j\}$ of functions that vanish in a neighbourhood containing K and that converge to $\bar{\partial}\widetilde{\beta}$ on $\overline{\Omega}$. This means that $\operatorname{supp}(\psi_j)$ does not contain any analytic discs, or in the language of [25], analytic varieties of dimension 1. By convexity, this is equivalent to the $\bar{\partial}$-Neumann operator on $\operatorname{supp}(\psi_j)$ being compact on $(0,1)$-forms by [25]. Let $g \in A^2(\Omega)$. Then,

$$\begin{aligned}
\left\langle \bar{\partial}^* N(g\bar{\partial}\widetilde{\beta}), \bar{\partial}^* N(g\bar{\partial}\widetilde{\beta}) \right\rangle_{L^2(\Omega)} &= \left\langle N(g\bar{\partial}\widetilde{\beta}), \bar{\partial}\bar{\partial}^* N(g\bar{\partial}\widetilde{\beta}) \right\rangle_{L^2(\Omega)} \\
&= \left\langle N(g\bar{\partial}\widetilde{\beta}), g\bar{\partial}\widetilde{\beta} \right\rangle \qquad (4.2) \\
&= \left\langle N(g\bar{\partial}\widetilde{\beta}), g(\bar{\partial}\widetilde{\beta} - \psi_j) \right\rangle + \left\langle N(g\bar{\partial}\widetilde{\beta}), g\psi_j \right\rangle
\end{aligned}$$

Fix j and pick $\psi \in C^\infty(\Omega)$ such that $0 < \psi < 1$, that it is identically 1 on $\operatorname{supp}(\psi_j)$, and that $\operatorname{supp}(\psi) \cap K = \emptyset$. For a $g \in A^2(\Omega)$, one has

$$\begin{aligned}\left|\left\langle N(g\bar{\partial}\widetilde{\beta}), g\psi_j \right\rangle\right| &= \left|\left\langle \psi N(g\bar{\partial}\widetilde{\beta}), g\psi_j \right\rangle\right| \\ &\leq \left\|\psi N(g\bar{\partial}\widetilde{\beta})\right\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.\end{aligned}$$

Take finitely many balls $B_1, \ldots, B_m$ such that $\Omega \cap B_j$ is a domain for each j, that

$$\bigcup_{j=1}^{m} B_j \supseteq \operatorname{supp}(\psi) \cap b\Omega,$$

and that $\Omega \cap B_j$ has a compact $\bar{\partial}$-Neumann operator for each j. Once more, the existence of such sets is guaranteed by [25] and the fact that ψ is supported away from K, and all analytic discs in particular. Take $\phi_j \in C_0^\infty(B_j)$ such that $\sum_{j=0}^{m} \phi_j = \psi$ by simply declaring $B_0 = \Omega$ taking a smooth partition of unity with compact support subject to $\{B_j : j = 0, \ldots, m\}$ and multiplying each element of said partition with ψ. Note that the domain of $\bar{\partial}^*$ is not affected by multiplication by smooth functions and $B_j \cap \Omega$ is a domain on which the $\bar{\partial}$-Neumann operator is compact for each j.

Here we wish to digress and provide some calculations that will be necessary in the next step. For a smooth function $\phi \in C^\infty(\overline{\Omega})$ and $\omega \in L^2_{(0,1)}(\Omega)$, we have

$$\begin{aligned}\left\|\bar{\partial}(\phi\omega)\right\|_2 &= \left\|(\bar{\partial}\phi) \wedge \omega + \phi\bar{\partial}(\omega)\right\|_2 \\ &\leq \left\|(\bar{\partial}\phi) \wedge \omega\right\|_2 + \left\|\phi\bar{\partial}\omega\right\|_2 \\ &\leq \left\|\bar{\partial}\phi\right\|_\infty \|\omega\|_2 + \|\phi\|_\infty \left\|\bar{\partial}\omega\right\|_2 \\ &\lesssim \left\|\bar{\partial}\omega\right\|_2 + \|\omega\|_2\end{aligned}$$

and similarly,

$$\left\|\bar{\partial}^*(\phi\omega)\right\|_2 \lesssim \left\|\bar{\partial}^*\omega\right\|_2 + \|\omega\|_2.$$

We refer the reader to the calculations in [26] for details and an explicit product rule for $\bar{\partial}^*$.

By Proposition 4.3.5, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any form

$h \in L^2_{(0,1)}(\Omega) \cap \mathcal{D}(\bar{\partial}) \cap \mathcal{D}(\bar{\partial}^*)$, one has

$$\begin{aligned}
\|\psi h\|_{L^2(\Omega)} &= \left\| \sum_{j=0}^{m} \phi_j h \right\|_{L^2(\Omega)} \leq \sum_{j=0}^{m} \|\phi_j h\|_{L^2(\Omega)} \\
&\lesssim \sum_{j=0}^{m} \varepsilon \left(\left\|\bar{\partial}(\phi_j h)\right\|_{L^2(\Omega)} + \left\|\bar{\partial}^*(\phi_j h)\right\|_{L^2(\Omega)} \right) + C_\varepsilon \|\phi_j h\|_{-1} \\
&\lesssim \varepsilon \left(\left\|\bar{\partial} h\right\|_{L^2(\Omega)} + \left\|\bar{\partial}^* h\right\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \right) + C_\varepsilon \|h\|_{-1} .
\end{aligned}$$

When Nh takes the place of h in the above calculation, by the estimates (2.2), we have

$$\|\psi N h\|_2 \lesssim \varepsilon \|h\|_2 + C_\varepsilon \|Nh\|_{-1} \lesssim \|h\|_2 + C_\varepsilon \|Nh\|_2$$

for $h \in L^2_{(0,1)}(\Omega)$. By Proposition 2.1.9, we have that ψN is compact on $L^2_{(0,1)}(\Omega)$. By an immediate consequence of Proposition 4.2.1, namely that $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ for any $a, b, \varepsilon > 0$, and by (4.2), we have for any $\varepsilon > 0$,

$$\begin{aligned}
\left| \left\langle N(g\bar{\partial}\widetilde{\beta}), g\psi_j \right\rangle \right| &\leq \left\| \psi N(g\bar{\partial}\widetilde{\beta}) \right\| \|g\| \\
&\leq \varepsilon \|g\|^2 + \frac{1}{\varepsilon} \left\| \psi N(g\bar{\partial}\widetilde{\beta}) \right\|
\end{aligned}$$

for any $g \in A^2(\Omega)$. Let $\varepsilon > 0$ and choose $j \in \mathbb{N}$ so that $\left\| \psi_j - \bar{\partial}\widetilde{\beta} \right\|_\infty < \frac{\varepsilon}{\left\| N(g\bar{\partial}\widetilde{\beta}) \right\|}$. Then,

$$\left| \left\langle N(g\bar{\partial}\widetilde{\beta}), g(\bar{\partial}\widetilde{\beta} - \psi_j \right\rangle) \right| < \varepsilon \|g\| .$$

It follows that

$$\begin{aligned}
\left\| H^\Omega_{\widetilde{\beta}}(g) \right\|^2 &= \left\| \bar{\partial}^* N(g\bar{\partial}\widetilde{\beta}) \right\|^2 \\
&= \left\langle N(g\bar{\partial}\widetilde{\beta}), g(\bar{\partial}\widetilde{\beta} - \psi_j) \right\rangle + \left\langle N(g\bar{\partial}\widetilde{\beta}), g\psi_j \right\rangle \\
&\lesssim \varepsilon \|g\|^2 + C_\varepsilon \left\| N(g\bar{\partial}\widetilde{\beta}) \right\|^2_{-1} \\
&\lesssim \varepsilon \|g\|^2 + C_\varepsilon \left\| N(g\bar{\partial}\widetilde{\beta}) \right\|^2_{2} .
\end{aligned}$$

Thus, we see that $H^\Omega_{\widetilde{\beta}}$ is compact on $A^2(\Omega)$. It follows that $H^\Omega_\beta = H^\Omega_{\widetilde{\beta}} + H^\Omega_{\widetilde{\beta}}$ is compact on $A^2(\Omega)$. □

Corollary 4.3.7. *Let Ω be a smooth bounded convex domain in $\mathbb{C}^2$ and $\beta \in C^\infty(\overline{\Omega})$. Then, H_β is compact if and only if $\beta \circ f$ is holomorphic for any holomorphic function $f : \mathbb{D} \to b\Omega$.*

CHAPTER 5

ESSENTIAL SPECTRA OF HERMITIAN SQUARES OF HANKEL OPERATORS ON THE POLYDISC

There are a lot of studies on compactness of Hankel operators and their consequences, starting with Axler in [27], where he completely characterised the bounded and compact Hankel operators with L^2 symbols of $A^2(\mathbb{D})$. We note that when an operator is compact, its spectrum is countable. When H_ϕ is compact, it is automatically the case that so is $H_\phi^* H_\phi$, which means that $\sigma\left(H_\phi^* H_\phi\right)$ cannot contain an open interval in this case. In this chapter, following [2], we will discuss the spectra and essential spectra of Hermitian squares of Hankel operators, namely operators of the form $H_\phi^* H_\phi$, and under which conditions $\sigma_{\text{ess}}\left(H_\phi^* H_\phi\right)$ contains an interval or not. Furthermore, the spectrum will be explicitly given in the case that the symbol is a monomial in the variables z and $\bar{z}$. The proofs of the main theorems are taken from [2] and some parts that were left to the reader were expanded.

5.1 Necessary Facts

We have already covered most of the background work on our way up here; however, there are still a few statements that do not seem to fit in the scope of the previous chapters and sections, yet are used in proofs. The following proposition appears inside the proof of the first theorem of [2], namely, what is known to us in this thesis as Theorem 5.2.2. In addition to Theorem 5.2.2, we see it being used in the proof of Theorem 5.2.4 as well.

Proposition 5.1.1. *$f \in L^2(\mathbb{D}^{n-1})$ for some $n \in \mathbb{N}$. Say $q \in b\mathbb{D}$ and $\psi \in C(\overline{\mathbb{D}^n})$ such that $\psi(z', z_n) = \phi(z')\chi(z_n)$ for functions $\phi \in C(\overline{\mathbb{D}^{n-1}}), \chi \in C(\overline{\mathbb{D}})$. Define ψ_q by*

$\psi_q(z', z_n) = \psi(z', q)$. *Then,* $\left\|(\psi - \psi_q) f k_p^{\mathbb{D}}\right\| \to 0$ *as* $p \to q$.

The following lemma from [2] characterises the spectrum in the case that the eigenvectors of an operator form an orthogonal basis.

Lemma 5.1.2. *Let H be a Hilbert space and $T \in B(H)$. Let λ_j be an eigenvalue of T and assume that u_j is an eigenvector corresponding to λ_j for each $j \in \mathbb{N}$. Assume moreover that the set $\{u_j : j \in \mathbb{N}\}$ is an orthogonal basis for H. Then, $\sigma(T) = \overline{\{\lambda_j : j \in \mathbb{N}\}}$. Furthermore, $\sigma_{\text{ess}}(T)$ consists precisely of cluster points of the sequence $\{\lambda_j\}$.*

The following proposition appeared in [28].

Proposition 5.1.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $u, v \in L^\infty(\Omega)$. Then,*

$$H_u^* H_v = P M_{\bar{u}} H_v = P M_{\bar{u}} (I - P) M_v.$$

Let us present an alternative proof for Proposition 5.1.3 to what was provided in [28] using Proposition 3.3.6.

Proof. By Proposition 3.3.6, we know that

$$\begin{aligned} H_u^* H_v &= T_{\bar{u}v} - T_{\bar{u}} T_v = P M_{\bar{u}} M_v - P M_{\bar{u}} P M_v \\ &= P M_{\bar{u}} \left(M_v - P M_v\right) = P M_{\bar{u}} (I - P) M_v. \end{aligned}$$

□

5.2 The Essential Spectrum of $H_\psi^* H_\psi$

In this section, we are going to give two main results, namely the ones in the first half of [2].

Notation 5.2.1. We are going to use k^Ω to denote the normalised Bergman kernel on Ω. For instance,

$$k^{\mathbb{D}}(z, p) = k_p^{\mathbb{D}}(z) = \frac{\pi(1 - |p|^2)}{(1 - \bar{p}z)^2}.$$

Theorem 5.2.2. *Let $\phi \in C(\overline{\mathbb{D}^{n-1}}),\ \chi \in C(\overline{\mathbb{D}})$ and $\psi(z', z_n) = \phi(z')\chi(z_n)$. Then,*

$$\sigma_{\text{ess}}\left(H_\psi^* H_\psi\right) \supseteq \left\{|\chi(q)|^2\mu : q \in b\mathbb{D}, \mu \in \sigma\left(H_\phi^{\mathbb{D}^{n-1}*} H_\phi^{\mathbb{D}^{n-1}}\right)\right\}.$$

Proof. Let $q \in b\mathbb{D}$ and $\mu \in \sigma(H_\phi^{\mathbb{D}^{n-1}*} H_\phi^{\mathbb{D}^{n-1}})$. Theorem 2.1.6 allows us to pick a sequence $\{g_j\}$ in $A^2(\mathbb{D})$ so that $\|g_j\| = 1$ and

$$\left\| H_\phi^{\mathbb{D}^{n-1}*} H_\phi^{\mathbb{D}^{n-1}} g_j - \mu g_j \right\| \to 0.$$

Define the function $f_{j,p}(z', z_n)$ to be

$$f_{j,p}(z', z_n) = g_j(z') k_p^{\mathbb{D}}(z_n).$$

Take a sequence $\{p_j\}$ that converges to q. The remainder of the proof is to simply verify that $\{f_{j,p_j}\}$ is a sequence for which the conditions of Theorem 2.1.6 are satisfied for $\lambda = |\chi(q)|^2\mu$, and therefore $\lambda \in \sigma_{\text{ess}}\left(H_\psi{}^* H_\psi\right)$. Note that by Lemma 2.3.3, the sequence $\{g_j\}$ is locally uniformly bounded on $\mathbb{D}^{n-1}$. Furthermore, the normalised kernels $k_p^{\mathbb{D}}(z) = \dfrac{\pi(1-|p|^2)}{(1-\bar{p}z)^2}$ tend to 0 locally uniformly as $p \to q$. Thus,

$$\sup\{|f_{j,p}(z)| : j \in \mathbb{N}, z \in K\} \to 0$$

as $p \to q$ for any compact set K. It follows from Lemma 2.3.5 that $f_{j,p} \rightharpoonup 0$ as $p \to q$.

Fix $j \in \mathbb{N}$. Define ψ_q to be the function $\psi_q(z', z_n) = \psi(z', q)$ for all $z' \in \overline{\mathbb{D}^{n-1}}, z_n \in \overline{\mathbb{D}}$. By Proposition 2.4.5,

$$\begin{aligned} H_{\psi_q} f_{j,p} &= \psi_q g_j k_p^{\mathbb{D}} - P(\psi_q g_j k_p^{\mathbb{D}}) \\ &= \chi(q)\phi g_j k_p^{\mathbb{D}} - \chi(q) P_{\mathbb{D}^{n-1}}(\phi g_j) P(k_p^{\mathbb{D}}) \\ &= \chi(q)\left(\phi g_j - P_{\mathbb{D}^{n-1}}(\phi g_j)\right) k_p^{\mathbb{D}} \\ &= \chi(q) H_\phi^{\mathbb{D}^{n-1}}(g_j) k_p^{\mathbb{D}}. \end{aligned}$$

As a consequence,

$$\begin{aligned}
\left\|H_{\psi_q} f_{j,p}\right\| &= \left\|\chi(q) H_\phi^{\mathbb{D}^{n-1}}(g_j) k_p^{\mathbb{D}}\right\| \\
&= |\chi(q)| \left\|H_\phi^{\mathbb{D}^{n-1}}(g_j) k_p^{\mathbb{D}}\right\|_{L^2(\mathbb{D}^n)} \\
&= |\chi(q)| \left(\int\limits_{\mathbb{D}^{n-1}\times\mathbb{D}} \left| \left(H_\phi^{\mathbb{D}^{n-1}}(g_j) k_p^{\mathbb{D}} \right)(z) \right|^2 \, dV(z) \right)^{1/2} \\
&= |\chi(q)| \left(\int\limits_{\mathbb{D}^{n-1}} \int\limits_{\mathbb{D}} \left| H_\phi^{\mathbb{D}^{n-1}}(g_j)(z') k_p^{\mathbb{D}}(z_n) \right| \, dV(z') \, dV(z_n) \right)^{1/2} \\
&= |\chi(q)| \left\|H_\phi^{\mathbb{D}^{n-1}}(g_j)\right\|_{L^2(\mathbb{D}^{n-1})} \left\|k_p^{\mathbb{D}}\right\| \\
&= |\chi(q)| \left\|H_\phi^{\mathbb{D}^{n-1}}(g_j)\right\|_{L^2(\mathbb{D}^{n-1})} .
\end{aligned}$$

It is easy to see that $H_{\alpha+\beta} = H_\alpha + H_\beta$ for any function α, β. Following this, we have

$$H_\psi{}^* H_\psi = H_{\psi_q}{}^* H_{\psi_q} + H_\psi^* H_{\psi-\psi_q} + H_{\psi-\psi_q}^* H_{\psi_q},$$

and noting the triangle inequality in addition, one has

$$\begin{aligned}
&\left\|H_{\psi_q}{}^* H_{\psi_q} f_{j,p} - \xi f_{j,p}\right\| - \left\|H_\psi^* H_{\psi-\psi_q} f_{j,p} + H_{\psi-\psi_q}^* H_\psi f_{j,p}\right\| \\
&\leq \left\|H_\psi{}^* H_\psi f_{j,p} - \xi f_{j,p}\right\| \\
&\leq \left\|H_{\psi_q}{}^* H_{\psi_q} f_{j,p} - \xi f_{j,p}\right\| + \left\|H_\psi^* H_{\psi-\psi_q} f_{j,p} + H_{\psi-\psi_q}^* H_\psi f_{j,p}\right\|
\end{aligned} \tag{5.1}$$

for any $\xi \in \mathbb{R}$.

Notice that by Proposition 5.1.1, letting $f = g_j$, we have

$$\begin{aligned}
\left\|H_\psi^* H_{\psi-\psi_q} f_{j,p}\right\| &= \left\|H_\psi^* (I-P)(\psi-\psi_q) f_{j,p}\right\| \\
&\leq \left\|H_\psi^*\right\| \|I-P\| \left\|(\psi-\psi_q) f_{j,p}\right\| \\
&= \left\|H_\psi^*\right\| \left\|(\psi-\psi_q) g_j k_p^{\mathbb{D}}\right\| \to 0
\end{aligned}$$

as $p \to q$.

We will use Proposition 5.1.1 once more, combining with Proposition 5.1.3, with

$f = H_\phi^{\mathbb{D}^{n-1}} g_j$, to obtain

$$\begin{aligned}
\left\| H^*_{\psi-\psi_q} H_{\psi_q} f_{j,p} \right\| &= \left\| P M_{\overline{\psi}-\overline{\psi}_q} (I-P) M_{\psi_q} f_{j,p} \right\| \\
&= \left\| P\left((\overline{\psi}-\overline{\psi}_q) H_{\psi_q} f_{j,p} \right) \right\| \\
&= \left\| P\left((\overline{\psi}-\overline{\psi}_q) \chi(q) H_\phi^{\mathbb{D}^{n-1}} g_j k_p^{\mathbb{D}} \right) \right\| \\
&= |\chi(q)| \left\| P\left((\overline{\psi}-\overline{\psi}_q) H_\phi^{\mathbb{D}^{n-1}} (g_j) k_p^{\mathbb{D}} \right) \right\| \to 0
\end{aligned}$$

as $p \to q$, and

$$\begin{aligned}
{H_{\psi_q}}^* H_{\psi_q} f_{j,p} &= P\left(\overline{\psi_q} \chi(q) \left(H_\phi^{\mathbb{D}^{n-1}} g_j \right) k_p^{\mathbb{D}} \right) \\
&= P\left(\overline{\chi(q)} \chi(q) \overline{\phi} \left(H_\phi^{\mathbb{D}^{n-1}} g_j \right) k_p^{\mathbb{D}} \right) \\
&= |\chi(q)|^2 P_{\mathbb{D}^{n-1}} \left(\overline{\phi} \left(H_\phi^{\mathbb{D}^{n-1}} g_j \right) \right) k_p^{\mathbb{D}} \\
&= |\chi(q)|^2 ({H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} g_j) k_p^{\mathbb{D}}.
\end{aligned}$$

Let us remind the fact that

$$\left\| {H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} g_j - \mu g_j \right\| \to 0$$

and that $\lambda = |\chi(q)|^2 \mu$. It follows that

$$\left\| |\chi(q)|^2 {H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} g_j - \lambda g_j \right\| = |\chi(q)|^2 \left\| {H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} g_j - \mu g_j \right\| \to 0$$

as $j \to \infty$. By (5.1), we also see that

$$\| {H_\psi}^* H_\psi f_{j,p} - \xi f_{j,p} \| \to \left\| |\chi(q)|^2 {H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} g_j - \xi g_j \right\|$$

as $p \to q$, for any $\xi \in \mathbb{R}$, due to the fact that $k_p^{\mathbb{D}}$ can be taken as a common factor of the difference and then separated as an integral in the norm calculations as well.

The limit calculations above allow us to pick a sequence $\{p_j\}$ in $\mathbb{D}$ such that $p_j \to q$ and

$$\left\| {H_\psi}^* H_\psi f_{j,p} - \lambda f_{j,p_j} \right\| < \left\| |\chi(q)|^2 {H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} g_j - \lambda g_j \right\| + \frac{1}{j}$$

by taking $\varepsilon = 1/j$. It follows that

$$\left\| {H_\psi}^* H_\psi f_{j,p} - \lambda f_{j,p_j} \right\|.$$

By Theorem 2.1.6, it is now established that $\lambda = |\chi(q)|^2 \mu \in \sigma_{\text{ess}} \left({H_\psi}^* H_\psi \right)$ whenever $q \in b\mathbb{D}$ and $\mu \in \sigma \left({H_\phi^{\mathbb{D}^{n-1}}}^* H_\phi^{\mathbb{D}^{n-1}} \right)$. □

Corollary 5.2.3. *Let $\chi_j \in C(\overline{\mathbb{D}})$ for $j = 1, \ldots, n$ and let ψ be their product; that is, $\psi(z_1, \ldots, z_n) = \chi_1(z_1) \cdots \chi_n(z_n)$. Then,*

$$\sigma_{\text{ess}}\left(H_\psi^* H_\psi\right) \supseteq \bigcup_{j=1}^{n} \left\{ \mu \prod_{k \neq j} |\chi_k(q_k)|^2 : q_k \in b\mathbb{D}, \mu \in \sigma\left(H_{\chi_j}^{\mathbb{D}*} H_{\chi_j}^{\mathbb{D}}\right) \right\}.$$

Proof. Let $\mu \in \sigma\left(H_{\chi_1}^* H_{\chi_1}\right)$ and $q \in b\mathbb{D}$. Then, by Theorem 5.2.2, we have

$$|\chi_2(q)|^2 \mu \in \sigma_{\text{ess}}\left(H_{\chi_1\chi_2}^* H_{\chi_1\chi_2}\right) \subseteq \sigma\left(H_{\chi_1\chi_2}^* H_{\chi_1\chi_2}\right).$$

Keeping this process going exhausts all χ_j's and the corollary follows. □

Theorem 5.2.4. *Let $\psi \in C(\overline{\mathbb{D}^n})$. Let $q \in b\mathbb{D}$, and $\psi_q(z') = \psi(z', q)$ for $z' \in \mathbb{D}^{n-1}$. If the map $q \mapsto \left\| H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|$ is not constant, then $\sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right)$ contains an open interval.*

Proof. Let $q \in b\mathbb{D}$ and

$$\lambda_q = \left\| {H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|_{L^2(\mathbb{D}^{n-1})} = \left\| H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|_{L^2(\mathbb{D}^{n-1})}^2.$$

Let us outline the proof. Some arguments are similar to those of Theorem 5.2.2, so for the sake of legibility, they will only be stated. At the end of the proof, we aim to reach two facts, namely that $\lambda_q \in \sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right)$, and that the set $\Lambda = \{\lambda_q : q \in b\mathbb{D}\}$ is a connected one. Since ${H_\psi}^* H_\psi$ is a self-adjoint operator, its spectrum consists only of real numbers. If

$$\Lambda = \{\lambda_q : q \in b\mathbb{D}\} \subseteq \sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right) \subseteq \sigma\left({H_\psi}^* H_\psi\right) \subseteq \mathbb{R}$$

is connected, then Λ has to be an interval. By the assumption that $q \mapsto \left\| H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|$ is not constant, the interval Λ consists of more than two elements, and hence, we will have an open interval inside this set and consequently in $\sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right)$.

Note that ${H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}}$ is self-adjoint and furthermore, positive, so $|\lambda| = \lambda$ for each $\lambda \in \sigma\left({H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}}\right)$. By Theorem 2.1.8, we have

$$r\left({H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}}\right) = \lambda_q = \sup\left\{|\lambda| : \lambda \in \sigma\left({H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}}\right)\right\}.$$

Since the spectrum of a bounded operator is a compact set, we see that the spectral radius is indeed a spectral value; that is, $\lambda_q \in \sigma\left({H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}}\right)$. Through Theorem 2.1.6, we find a sequence $\{g_{j,q}\}$ in $A^2(\mathbb{D}^{n-1})$ such that $\|g_{j,q}\| = 1$ and

$$\left\| {H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q} - \lambda_q g_{j,q} \right\| \to 0$$

as $j \to \infty$.

Define $f_{j,q,p}(z', z_n) = g_{j,q}(z')k_p^{\mathbb{D}}(z_n)$. One can show in a similar fashion to Theorem 5.2.2 that $f_{j,q,p_j} \rightharpoonup 0$ for any sequence $\{p_j\}$ in $\mathbb{D}$ converging to q. Observe that

$$H_{\psi_q} f_{j,q,p} = H_{\psi_q}^{\mathbb{D}^{n-1}}(g_{j,q})k_p^{\mathbb{D}}.$$

Once more, this implies $\left\|H_{\psi_q} f_{j,q,p}\right\| = \left\|H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}\right\|_{L^2(\mathbb{D}^{n-1})}$.

Just as before, we may bound $\|H_\psi{}^* H_\psi f_{j,q,p} - \lambda f_{j,q,p}\|$ using symbol manipulation and triangle inequality as

$$\begin{aligned}
&\left\|H_{\psi_q}{}^* H_{\psi_q} f_{j,q,p} - \lambda_q f_{j,q,p}\right\| - \left\|H_\psi^* H_{\psi-\psi_q} f_{j,q,p} + H_{\psi-\psi_q}^* H_{\psi_q} f_{j,q,p}\right\| \\
&\leq \|H_\psi{}^* H_\psi f_{j,q,p} - \lambda_q f_{j,q,p}\| \\
&\leq \left\|H_{\psi_q}{}^* H_{\psi_q} f_{j,q,p} - \lambda_q f_{j,q,p}\right\| + \left\|H_\psi^* H_{\psi-\psi_q} f_{j,q,p} + H_{\psi-\psi_q}^* H_{\psi_q} f_{j,q,p}\right\|.
\end{aligned}$$

Using Proposition 5.1.1, we observe that

$$\begin{aligned}
\left\|H_\psi^* H_{\psi-\psi_q} f_{j,q,p}\right\| &= \left\|H_\psi^*(I-P)(\psi-\psi_q) f_{j,q,p}\right\| \\
&\leq \|H_\psi\| \, \|(\psi-\psi_q) f_{j,q,p}\| \to 0
\end{aligned}$$

as $p \to q$. By the very same fact and Proposition 5.1.3, we also have

$$\left\|H_{\psi-\psi_q}^* H_{\psi_q} f_{j,q,p}\right\| = \left\|P\left(H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}(\overline{\psi} - \overline{\psi_q})k_p^{\mathbb{D}}\right)\right\| \to 0$$

as $p \to q$. Referring the reader to the previous calculations once more, we see that

$$\begin{aligned}
H_{\psi_q}{}^* H_{\psi_q} f_{j,q,p} &= P\left(\overline{\psi_q}\left(H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}\right)k_p^{\mathbb{D}}\right) \\
&= P_{\mathbb{D}^{n-1}}\left(\overline{\psi_q}\left(H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}\right)\right)k_p^{\mathbb{D}} \\
&= \left(H_{\psi_q}^{\mathbb{D}^{n-1}{}^*} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q}\right)k_p^{\mathbb{D}}.
\end{aligned}$$

Consequently, for any fixed j, we have

$$\|H_\psi{}^* H_\psi f_{j,q,p} - \lambda_q f_{j,q,p}\| \to \left\|H_{\psi_q}^{\mathbb{D}^{n-1}{}^*} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j,q} - \lambda_q g_{j,q}\right\|_{L^2(\mathbb{D}^{n-1})} \tag{5.2}$$

as $p \to q$. By choice of $\{g_{j,q}\}$, there is a subsequence $\{g_{j_k,q}\}$ of $\{g_{j,q}\}$ such that

$$\left\|H_{\psi_q}^{\mathbb{D}^{n-1}{}^*} H_{\psi_q}^{\mathbb{D}^{n-1}} g_{j_k,q} - \lambda_q g_{j_k,q}\right\| < \frac{1}{k}$$

for all k. For the sake of legibility, assume, without loss of generality, that this subsequence is $\{g_{j,q}\}$ itself.Take a sequence $\{p_j\}$ in $\mathbb{D}$ such that $p_j \to q$ and

$$\left\| {H_\psi}^* H_\psi f_{j,q,p_j} - \lambda_q f_{j,q,p_j} \right\| < \frac{2}{j},$$

whose existence is ensured by (5.2) with $\varepsilon = \frac{1}{j}$. This shows that

$$\left\| {H_\psi}^* H_\psi f_{j,q,p_j} - \lambda_q f_{j,q,p} \right\| \to 0 \tag{5.3}$$

As $j \to \infty$. All in all, the sequence $\{f_{j,q,p_j}\}$ converges weakly to 0, each of its elements has norm 1, and satisfies (5.3). By Theorem 2.1.6, we have $\lambda_q \in \sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right)$ for each $q \in b\mathbb{D}$.

It remains to show that the set Λ is connected. We claim that the map

$$q \mapsto \left\| {H_{\psi_q}^{\mathbb{D}^{n-1}}}^* H_{\psi_q}^{\mathbb{D}^{n-1}} \right\| = \left\| H_{\psi_q}^{\mathbb{D}^{n-1}} \right\|$$

is a continuous function. It suffices to show that $q \mapsto H_{\psi_q}^{\mathbb{D}^{n-1}}$ is continuous. Let $\varepsilon > 0$. For $q_1, q_2 \in b\mathbb{D}$, we have

$$\left\| H_{\psi_{q_1}} - H_{\psi_{q_2}} \right\| = \left\| H_{\psi_{q_1} - \psi_{q_2}} \right\| \leq \left\| \psi_{q_1} - \psi_{q_2} \right\|_\infty$$

by Proposition 3.3.7. We remind the reader that $\psi \in C(\overline{\mathbb{D}^n})$ and hence, there exists a $\delta_1 > 0$ such that whenever $|q_1 - q_2| < \delta_1$, we have $|\psi_{q_1}(z') - \psi_{q_2}(z')| < \varepsilon/2$ for any $z' \in \mathbb{D}^{n-1}$. Furthermore, since ψ_{q_1} is continuous, there exists $\delta_2 > 0$ such that $|\psi_{q_1}(z_1') - \psi_{q_1}(z_2')| < \varepsilon/2$ whenever $|z_1' - z_2'| < \delta_2$. Taking $\delta = \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} |\psi_{q_1}(z_1') - \psi_{q_2}(z_2')| &= |\psi_{q_1}(z_1') - \psi_{q_1}(z_2') + \psi_{q_1}(z_2') - \psi_{q_2}(z_2')| \\ &\leq |\psi_{q_1}(z_1') - \psi_{q_1}(z_2')| + |\psi_{q_1}(z_2') - \psi_{q_2}(z_2')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for any $z_1', z_2' \in \mathbb{D}^{n-1}$. This shows that $q \mapsto \psi_q$ is a continuous function of q and hence, so is $q \mapsto H_{\psi_q}$. Since $T \mapsto \|T\|^2$ is also continuous on $B\left(A^2(\mathbb{D}^{n-1})\right)$, we have the claim. Notice that the domain of this map is $b\mathbb{D}$, which is connected. Ergo, its image, namely Λ as we have called it, is also connected. Thus, Λ contains an open interval and consequently, so does $\sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right)$. □

Corollary 5.2.5. *Let $\psi \in C(\overline{\mathbb{D}^2})$ and define $\psi_{1,\theta}(\zeta) = \psi(e^{i\theta}, \zeta)$ and $\psi_{2,\theta}(\zeta) = \psi(\zeta, e^{i\theta})$ for $\zeta \in \mathbb{D}$. Then,*

$$\left\{ \left\| H_{\psi_{1,\theta}} \right\|^2_{L^2(\mathbb{D})} : \theta \in [0, 2\pi) \right\} \cup \left\{ \left\| H_{\psi_{2,\theta}} \right\|^2_{L^2(\mathbb{D})} : \theta \in [0, 2\pi) \right\} \subseteq \sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right).$$

Proof. Notice that the set $\left\{ \left\| H_{\psi_{2,\theta}} \right\|^2_{L^2(\mathbb{D})} : \theta \in [0, 2\pi) \right\}$ is nothing but the set Λ in the proof of Theorem 5.2.4. Furthermore, for the set $\left\{ \left\| H_{\psi_{1,\theta}} \right\|^2_{L^2(\mathbb{D})} : \theta \in [0, 2\pi) \right\}$, the proof can be effortlessly adjusted for the definition $\psi_q(z') = \psi(q, z')$ due to the symmetry of $\mathbb{D}^2$. Hence, the statement follows. □

Corollary 5.2.6. *Let $\phi \in C(\overline{\mathbb{D}^{n-1}})$ and $\chi \in C(\overline{\mathbb{D}})$ such that $|\chi|$ is not constant on the unit circle and ϕ is not holomorphic. Then, $\sigma_{\text{ess}}\left({H_\psi}^* H_\psi\right)$ contains an open interval where $\psi(z', z_n) = \phi(z')\chi(z_n)$ for $z' \in \mathbb{D}^{n-1}$ and $z_n \in \mathbb{D}$.*

5.3 The Spectrum of $H^*_\psi H_\psi$ for Monomial Symbols

In this section, we are going to examine the spectrum of $H^*_\psi H_\psi$ when the symbol ψ is a monomial in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$. We will see that in each case presented here, the spectrum is countable, so we do not worry about open intervals inside the spectrum or the essential spectrum. Instead, the focus of this section is going to be to characterise these spectra.

Definition 5.3.1. A complex-valued function ψ of n complex variables is said to be *quasi-homogeneous* if for some $f : [0, \infty) \to \mathbb{C}$ and $k = (k_1, \ldots, k_n)$, we have $\psi\left(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}\right) = f(r_q, \ldots, r_n) e^{i(k_1\theta_1 + \cdots + k_n\theta_n)}$ for any $r_j \geq 0$ and $\theta_j \in \mathbb{R}$.

We refer the reader to [2] to reach the following lemma.

Lemma 5.3.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a complete Reinhardt domain and ψ be a quasi-homogeneous function on Ω. Then, the operator $H^*_\psi H_\psi$ has eigenvectors z^α where $\alpha \in \mathbb{N}^n_0$ corresponding to the eigenvalue*

$$\lambda_\alpha = \begin{cases} \dfrac{\|z^\alpha \psi\|_2^2}{\|z^\alpha\|_2^2} & \text{if } \alpha + k \notin \mathbb{N}^n_0 \\ \dfrac{\|z^\alpha \psi\|_2^2}{\|z^\alpha\|_2^2} - \dfrac{\left| \int_\Omega |w|^{2\alpha + k} f(|w|)\, dV(w) \right|^2}{\|z^\alpha\|_2^2 \, \|z^{\alpha+k}\|_2^2} & \text{if } \alpha + k \in \mathbb{N}^n_0. \end{cases}$$

Furthermore, $\sigma\left(H^*_\psi H_\psi\right) = \overline{\sigma_{\text{p}}\left(H^*_\psi H_\psi\right)}$.

Lemma 5.3.2 will be used in Theorem 5.3.4 to get an insight on the spectrum.

Notation 5.3.3. Before stating the next theorem, we need to introduce a new quantity. We will denote by B_n the set $\{1, \ldots, n\}$. Let $a, b, \alpha \in \mathbb{N}_0$ be multi-indices and $B \subseteq B_n$. With this setup, let us define $\lambda_{a,b,\alpha,B}$ as

$$\lambda_{a,b,\alpha,B} = \prod_{k \in B} \frac{\alpha_k + 1}{\alpha_k + a_k + b_k + 1}$$

when $\alpha_k < b_k - a_k$ for some $k \in B$ and

$$\lambda_{a,b,\alpha,B} = \prod_{k \in B} \frac{\alpha_k + 1}{\alpha_k + a_k + b_k + 1} - \prod_{k \in B} \frac{(\alpha_k + 1)(\alpha_k + a_k - b_k + 1)}{(\alpha_k + a_k + 1)^2}$$

when $\alpha_k \geq b_k - a_k$ for each $k \in B$.

The following theorem explicitly provides the spectra of Hermitian squares of Hankel operators in the case that the symbol is a monomial in z and $\bar{z}$.

Theorem 5.3.4. *Let $\psi(z) = z^a \bar{z}^b$ for multi-indices $a, b \in \mathbb{N}_0^n$. Then,*

$$\sigma\left({H_\psi}^* H_\psi\right) = \{0\} \cup \left\{\lambda_{a,b,\alpha,B} : \alpha \in \mathbb{N}_0, \emptyset \neq B \subseteq B_n\right\}.$$

Remark 5.3.5. Notice that the case $b = 0$ implies that the symbol is holomorphic and thus, the operator H_ψ is the 0 operator. Therefore, we will assume in the proof that $b \neq 0$.

Remark 5.3.6. Notice that Theorem 5.3.4 implies that the spectra for monomial symbols of this form are countable. This is to be expected as [29] showed that H_ψ is compact on $A^2(\mathbb{D}^n)$ if and only if $\psi = \psi_1 + \psi_2$ where ψ_1 is square-integrable and admits the condition that the mean square of f tends to 0 on balls of some fixed radius as their centres approach the boundary of $\mathbb{D}^n$, and ψ_2 satisfies

$$\frac{\partial \psi_2}{\partial \bar{z}_j}(1 - |z_j|^2) \to 0$$

as $z \to b\mathbb{D}^n$ for each j. Here, choosing $\psi_1 \equiv 0$ and $\psi_2(z) = z^a \bar{z}^b$ gives us exactly what we need.

Proof. Since $z^a \bar{z}^b = |z|^{a+b} e^{i(a-b)\cdot\theta} = f(|z|)e^{ik\cdot\theta}$ and

$$\begin{aligned}\int_\Omega |w|^{2\alpha+k} f(|w|)\, dV(w) &= \int_{\mathbb{D}^n} |w|^{2\alpha+a-b} |w|^a |\bar{w}|^b \, dV(w) \\ &= \left|w^{2\alpha+2a-b}\bar{w}^b\right| \, dV(w) = \left\|z^{2\alpha+a-b}\bar{z}^b\right\|_1,\end{aligned}$$

by Lemma 5.3.2, we have

$$H_\psi^* H_\psi z^\alpha = \begin{cases} \dfrac{\left\|z^{\alpha+a}\bar{z}^b\right\|_2^2}{\left\|z^\alpha\right\|_2^2} z^\alpha & , \alpha+a-b \notin \mathbb{N}_0^n \\ \left(\dfrac{\left\|z^{\alpha+a}\bar{z}^b\right\|_2^2}{\left\|z^\alpha\right\|_2^2} - \dfrac{\left\|z^{2\alpha+a-b}\bar{z}^b\right\|_1^2}{\left\|z^\alpha\right\|_2^2 \left\|z^{\alpha+a-b}\right\|_2^2} \right) z^\alpha & , \alpha+a-b \in \mathbb{N}_0^n. \end{cases}$$

Let $\beta \in \mathbb{N}_0^n$. Observe that

$$\begin{aligned} \left\|z^\beta\right\|_2^2 = \left\|z^{2\beta}\right\|_1 &= \int_{\mathbb{D}^n} \left|\zeta^\beta\right|^2 \, dV(\zeta) \\ &= \int_{\mathbb{D}} \cdots \int_{\mathbb{D}} \left|\zeta_1^{\beta_1}\right|^2 \cdots \left|\zeta_n^{\beta_n}\right|^2 \, dA(\zeta_1) \cdots dA(\zeta_n) \\ &= \prod_{k=1}^n \int_{\mathbb{D}} \left|\zeta_k^{\beta_k}\right|^2 \, dA(\zeta_k) \\ &= \prod_{k=1}^n \int_0^{2\pi} \int_0^1 r^{2\beta_k} r \, dr \, d\theta \\ &= \prod_{k=1}^n \int_0^{2\pi} \int_0^1 r^{2\beta_k+1} \, dr \, d\theta \\ &= \prod_{k=1}^n \int_0^{2\pi} \frac{1}{2\beta_k+2} \, d\theta \\ &= \prod_{k=1}^n \frac{2\pi}{2(\beta_k+1)} = \frac{\pi^n}{\prod_{k=1}^n (\beta_k+1)}. \end{aligned}$$

Returning back to $H_\psi^* H_\psi$, since $\left\|z^\beta \bar{z}^\gamma\right\|_2 = \left\|z^{\beta+\gamma}\right\|_2$, we have

$$H_\psi^* H_\psi z^\alpha = \begin{cases} \left(\prod_{k=1}^n \frac{\alpha_k+1}{\alpha_k+a_k+b_k+1} \right) z^\alpha & , \alpha+a-b \notin \mathbb{N}_0^n \\ \left(\prod_{k \in B} \frac{\alpha_k+1}{\alpha_k+a_k+b_k+1} - \prod_{k \in B} \frac{(\alpha_k+1)(\alpha_k+a_k-b_k+1)}{(\alpha_k+a_k+1)^2} \right) z^\alpha & , \alpha+a-b \in \mathbb{N}_0^n \end{cases}$$

Thus, $H_\psi^* H_\psi z^\alpha = \lambda_{a,b,\alpha,B_n}$. This shows that z^α is an eigenvector of $H_\psi^* H_\psi$ with corresponding eigenvalue λ_{a,b,α,B_n}. Notice that $\mathbb{D}^n$ is a complete Reinhardt domain. Let us remind the fact that this, with the fact that $\mathbb{D}^n$ is bounded, implies that $\{z^\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthogonal basis for $A^2(\mathbb{D}^n)$. By Lemma 5.1.2, we see that $\sigma\left(H_\psi^* H_\psi\right) = \overline{\{\lambda_{a,b,\alpha,B_n} : \alpha \in \mathbb{N}_0^n\}}$. In other words, each spectral value is a limit of a sequence from the set of λ_{a,b,α,B_n}'s. Take $\lambda \in \sigma\left(H_\psi^* H_\psi\right)$ such that

$$\lambda = \lim_{j \to \infty} \lambda_{a,b,\alpha^{(j)},B_n}$$

for a sequence of elements $\alpha^{(j)}$ of $\mathbb{N}_0^n$. We aim to prove that either $\lambda = 0$ or $\lambda = \lambda_{a,b,\alpha,B}$ for some non-empty $B \subseteq B_n$. Assume that $\lambda \neq 0$. For the time being, let

us assume moreover that for each j, there is a k_j such that $\alpha_{k_j}^{(j)} < a_{k_j} - b_{k_j}$. In other words, we want each $\lambda_{a,b,\alpha^{(j)},B}$ to be of the form

$$\lambda_{a,b,\alpha^{(j)},B_n} = \prod_{k=1}^{n} \frac{\alpha_k^{(j)} + 1}{\alpha_k^{(j)} + a_k + b_k + 1}.$$

Since B_n is a finite set, there is $k \in B_n$ such that $\alpha_k^{(j)} < a_k - b_k$ for infinitely many j. Without loss of generality, say $\alpha_k^{(j)} < a_k - b_k$ for all j. Notice that this means $\{\alpha_k^{(j)}\}_{j\in\mathbb{N}}$ is bounded, so it becomes constant after some index j. Let k_1 be the smallest index so that $\alpha_{k_1}^{(j)}$ has a bounded subsequence. Again, we may safely assume that that sequence is $\alpha_{k_1}^{(j)}$, passing to the bounded subsequence if necessary. By boundedness, we see that $\{\alpha_{k_1}^{(j)}\}$ is ultimately constant. Therefore, we take the subsequence that makes $\{\alpha_{k_1}^{(j)}\}$ constant as our $\{\alpha^{(j)}\}$. If $\{\alpha_k^{(j)}\}$ does not remain bounded for any other $k \in B_n$, then we take $B = \{k_1\}$. If not, we look at the smallest $k_2 > k_1$ such that $\{\alpha_{k_2}^{(j)}\}$ is bounded. This process is bound to stop at some point, say at $B = \{k_1, \ldots, k_p\}$. At the end, we have reached a B and a sequence $\alpha^{(j)}$ such that $\{\alpha_k^{(j)}\}$ is constant for each $k \in B$ and approaches ∞ whenever $k \notin B$. Notice that

$$\lim_{j\to\infty} \frac{\alpha_k^{(j)} + 1}{\alpha_k^{(j)} + a_k + b_k + 1} = \begin{cases} 1 & \text{if } k \notin B \\ \frac{\alpha_k+1}{\alpha_k+a_k+b_k+1} & \text{if } k \in B. \end{cases}$$

It follows that

$$\begin{aligned} \lambda = \lim_{j\to\infty} \lambda_{a,b,\alpha^{(j)},B_n} &= \lim_{j\to\infty} \prod_{k=1}^{n} \frac{\alpha_k^{(j)} + 1}{\alpha_k^{(j)} + a_k + b_k + 1} \\ &= \prod_{k\in B} \frac{\alpha_k + 1}{\alpha_k + a_k + b_k + 1} = \lambda_{a,b,\alpha,B}. \end{aligned}$$

If we focus on the case $\alpha_k^{(j)} \geq b_k - a_k$ for each k and j, then for at least one $k \in B_n$ we need to have a bounded $\{\alpha_k^{(j)}\}$ since otherwise we would have $B = \emptyset$ and thus $\lambda = 0$, which we excluded in the beginning. Similar to the case above, here we also get

$$\lambda = \lim_{j\to\infty} \lambda_{a,b,\alpha^{(j)},B_n} = \lambda_{a,b,\alpha^{(j)},B}.$$

This argument proves that $\sigma\left(H_\psi^* H_\psi\right)$ consists of nothing but $\lambda_{a,b,\alpha,B}$'s and 0. To complete the proof, we will now show that $\sigma\left(H_\psi^* H_\psi\right)$ exhausts this set. We start by showing that 0 is a spectral value. We may observe this using Remark 5.3.6 and the fact that 0 is always in the spectrum of a compact operator in an infinite-dimensional

Banach space. In [2], the claim $0 \in \sigma\left(H_\psi^* H_\psi\right)$ was reached by taking a sequence $\alpha^{(j)} = (j, \ldots, j)$ and looking at $\lambda_{a,b,\alpha^{(j)},B_n}$. For a sufficiently large index j, we have $\alpha_k^{(j)} \geq b_k - a_k$ for all k, so we use the according definition for $\lambda_{a,b,\alpha^{(j)},B_n}$. As $j \to \infty$, we see both products tend to 1 and hence, their difference tends to 0. This shows that $0 \in \sigma\left(H_\psi^* H_\psi\right)$ since 0 is a limit of the eigenvalues. Now, take a non-empty $B \subseteq B_n$ such that $\alpha_k < b_k - a_k$ for some $k \in B$. Pick

$$\alpha^{(j)} = \begin{cases} \alpha_k & \text{if } k \in B \\ j & \text{if } k \notin B. \end{cases} \tag{5.4}$$

Once more, the fractions indexed by $k \in B$ will tend to 1 and thus, we see that

$$\lim_{j \to \infty} \lambda_{a,b,\alpha^{(j)},B_n} = \lambda_{a,b,\alpha^{(j)},B},$$

so $\lambda_{a,b,\alpha^{(j)},B} \in \sigma\left(H_\psi^* H_\psi\right)$.

The case $\alpha_k \geq b_k - a_k$ for all $k \notin B$ is similar. We again choose α^j as in (5.4). The terms that are excluded from B tend to 1 as $j \to \infty$, and again $\lambda_{a,b,\alpha^{(j)},B_n} \to \lambda_{a,b,\alpha^{(j)},B}$ as $j \to \infty$. Thus, we have the theorem. □

As we have established, the spectrum is countable. Therefore, it can contain no neighbourhoods.

Corollary 5.3.7. *Let $\psi(z) = z^a \bar{z}^b$ for some $a, b \in \mathbb{N}_0^n$. Then,* $\operatorname{int}\left(\sigma\left(H_{\psi}{}^* H_\psi\right)\right) = \emptyset$.

Proof. We may observe this fact explicitly using the characterisation of the spectrum following from Theorem 5.3.4, or implicitly by Remark 5.3.6, since compactness implies that the spectrum is countable, and hence must have empty interior. □

If we turn our focus to the bidisc $\mathbb{D}^2$ and if the symbol is purely coanalytic, we obtain the following corollary from Theorem 5.3.4.

Corollary 5.3.8. *Let $\psi(z_1, z_2) = \bar{z}_1^n \bar{z}_2^m$ where m, n are positive integers. Then,*

$\sigma\left({H_\psi}^* H_\psi\right) = \{0\} \cup \bigcup_{j=1}^{7} U_j$ *where*

$$
\begin{aligned}
U_1 &= \left\{ \frac{\alpha_1 + 1}{\alpha_1 + n + 1} : \alpha_1 \in \mathbb{N}_0, 0 \leq \alpha_1 \leq n - 1 \right\} \\
U_2 &= \left\{ \frac{\alpha_2 + 1}{\alpha_2 + m + 1} : \alpha_2 \in \mathbb{N}_0, 0 \leq \alpha_2 \leq m - 1 \right\} \\
U_3 &= \left\{ \frac{n^2}{(\alpha_1 + n + 1)(\alpha_1 + 1)} : \alpha_1 \in \mathbb{N}_0, \alpha_1 \geq n \right\} \\
U_4 &= \left\{ \frac{m^2}{(\alpha_2 + m + 1)(\alpha_2 + 1)} : \alpha_2 \in \mathbb{N}_0, \alpha_2 \geq m \right\} \\
U_5 &= \left\{ \frac{n^2(\alpha_2 + 1)^2 + m^2(\alpha_1 + 1)^2 - n^2 m^2}{(\alpha_1 + n + 1)(\alpha_2 + m + 1)(\alpha_1 + 1)(\alpha_2 + 1)} : \alpha_1 \geq n, \alpha_2 \geq m \right\} \\
U_6 &= \left\{ \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{(\alpha_1 + n + 1)(\alpha_2 + m + 1)} : \alpha_1, \alpha_2 \in \mathbb{N}_0, 0 \leq \alpha_1 \leq n - 1 \right\} \\
U_7 &= \left\{ \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{(\alpha_1 + n + 1)(\alpha_2 + m + 1)} : \alpha_1, \alpha_2 \in \mathbb{N}_0, 0 \leq \alpha_2 \leq m - 1 \right\}.
\end{aligned}
$$

In addition to the conditions of Corollary 5.3.8, if the symbol does not depend on the second variable as well, we observe the following, even more simplified, spectrum.

Corollary 5.3.9. *Let $\psi(z_1, z_2) = \bar{z}_1^n$ on $\mathbb{D}^2$ for some positive integer n. Then,*

$$
\sigma\left({H_\psi}^* H_\psi\right) = \{0\} \cup \bigcup_{\alpha_1=0}^{n-1} \left\{ \frac{\alpha_1 + 1}{\alpha_1 + n + 1} \right\} \cup \bigcup_{\alpha_1=n}^{\infty} \left\{ \frac{n^2}{(\alpha_1 + n + 1)(\alpha_1 + 1)} \right\}.
$$

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