ACQUISITION PRICING AND COORDINATION IN CLOSED-LOOP SUPPLY CHAINS WITH QUALITY-DIFFERENTIATED RETURNS

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ABSTRACT

ACQUISITION PRICING AND COORDINATION IN CLOSED-LOOP SUPPLY CHAINS WITH QUALITY-DIFFERENTIATED RETURNS

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This thesis studies a closed-loop supply chain with an independent collector and remanufacturer. The collector acquires used products, referred to as cores, at quality-dependent acquisition prices and transfers them to the remanufacturer at negotiated transfer prices; the remanufacturer processes the cores and sells the resulting products at a market-clearing price. We first analyze a single-quality class model in which the collector acts as Stackelberg leader and the remanufacturer as follower. The centralized benchmark and the decentralized equilibrium are derived in closed form, and the latter is strictly inefficient due to double marginalization. A two-part tariff always coordinates the supply chain. A revenue sharing implementable contract, with the revenue share and transfer price agreed jointly in advance, coordinates over a feasible range of shares satisfying individual rationality for both parties. We then extend the analysis to a two-quality class model in which cores differ in remanufacturing cost, deriving the centralized and decentralized solutions and both coordination contracts in closed form. A computational study establishes two further results. Quality differentiation strictly increases the efficiency loss of decentralization, most severely where high quality supply is least elastic. The revenue sharing contract fails whenever the

low quality class is remanufacturer intensive, that is, when its cores are cheap to acquire but costly to process. This is the empirically motivated configuration, and over the ranges examined the contract becomes feasible only where the centralized system ceases to acquire low quality cores. The two-part tariff therefore remains the only robust coordinating mechanism under quality differentiation.

Keywords: Closed-loop Supply Chain Coordination, Remanufacturing, Acquisition Pricing, Stackelberg Game, Quality Differentiated Returns

ÖZ

FARKLI KALİTEDE İADELERİN OLDUĞU KAPALI DÖNGÜ TEDARİK ZİNCİRLERİNDE GERİ KAZANIM FİYATLANDIRMASI VE KOORDİNASYON

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Bu tezde, bağımsız bir toplayıcı ile yeniden üreticiden oluşan bir kapalı döngü tedarik zinciri incelenmektedir. Toplayıcı, çekirdek ürün olarak adlandırılan kullanılmış ürünleri kaliteye bağlı geri kazanım fiyatları üzerinden temin etmekte ve bunları üzerinde uzlaşılan transfer fiyatlarıyla yeniden üreticiye devretmektedir. Yeniden üretici ise çekirdek ürünleri yeniden üretim sürecinden geçirerek elde ettiği ürünleri piyasa denge fiyatından satışa sunmaktadır. Çalışmada öncelikle, toplayıcının Stackelberg lideri, yeniden üreticinin ise takipçi olduğu tek kalite sınıflı bir model analiz edilmektedir. Merkezi sistem için referans çözüm ile merkezi olmayan sistem dengesi kapalı formda elde edilmiş ve merkezi olmayan yapının çifte marjinalizasyon nedeniyle verimsiz olduğu gösterilmiştir. İki parçalı tarife sözleşmesinin tedarik zincirini her koşulda koordine ettiği gösterilmiştir. Gelir paylaşımı sözleşmesinin ise gelir paylaşım oranı ile transfer fiyatının önceden birlikte belirlendiği durumda, her iki tarafın bireysel rasyonellik koşullarını sağlayan belirli bir gelir paylaşım oranı aralığında koordinasyon sağlayabildiği ortaya konmuştur. Daha sonra analiz, çekirdek ürünlerin

yeniden üretim maliyetleri bakımından farklılaştığı iki kalite sınıflı modele genişletilmiştir. Bu model için merkezi ve merkezi olmayan çözümler ile her iki koordinasyon sözleşmesi de kapalı formda elde edilmiştir. Hesaplamalı analiz iki ek bulgu ortaya koymaktadır. İlk olarak, kalite farklılaştırmasının merkezi olmayan yapının yol açtığı verimlilik kaybını artırdığı ve bu artışın yüksek kaliteli çekirdek ürün arzının en düşük esnekliğe sahip olduğu durumlarda en belirgin düzeye ulaştığı gösterilmiştir. İkinci olarak, gelir paylaşımı sözleşmesinin düşük kalite sınıfının yeniden üretici yoğun olduğu, yani çekirdek ürünlerin edinilmesinin düşük maliyetli, buna karşılık yeniden üretim maliyetlerinin yüksek olduğu durumlarda koordinasyon sağlayamadığı belirlenmiştir. Bu durum ampirik olarak gözlenen yapıyı temsil etmektedir. İncelenen parametre aralıklarında sözleşme ancak merkezi sistemin düşük kaliteli çekirdek ürünleri toplamayı bıraktığı durumlarda uygulanabilir hâle gelmektedir. Sonuç olarak, kalite farklılaştırmasının bulunduğu ortamda iki parçalı tarife tedarik zincirini koordine edebilen tek sağlam mekanizma olarak öne çıkmaktadır.

Anahtar Kelimeler: Kapalı Döngü Tedarik Zinciri Koordinasyonu, Yeniden Üretim, Geri Kazanım Fiyatlandırması, Stackelberg Oyunu, Farklı Kalitede İadeler

To my family...

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CHAPTER 1

INTRODUCTION

The transition toward more sustainable production systems has made remanufacturing an increasingly important alternative to traditional manufacturing. Remanufacturing is a key enabler of the circular economy, in which materials and products are kept in use for as long as possible through recovery, restoration, and reuse rather than disposal. By recovering used products, often referred to as cores, and restoring them to a serviceable condition, remanufacturing reduces both the consumption of raw materials and the environmental burden associated with disposal. Closed-loop supply chains, in which forward flows of new and remanufactured products are integrated with reverse flows of returned cores, have therefore attracted sustained attention from both industry and academia over the past three decades.

A central operational challenge in any closed-loop supply chain is product acquisition management, that is, the design of pricing mechanisms that attract a sufficient quantity and quality of returns to support profitable remanufacturing. Because cores vary widely in condition, attracting only high-quality returns may be costly, while attracting returns indiscriminately may yield cores that are too expensive to remanufacture. The price offered for returns therefore plays a central role in determining both the profitability and the environmental impact of a remanufacturing operation.

A further complication arises when the collector and the remanufacturer are independent firms rather than a single integrated decision maker. In this decentralized setting, the collector sets acquisition and transfer prices to maximize its own profit, taking into account the remanufacturer's response. The remanufacturer, in turn, chooses remanufacturing quantities to maximize its own profit given the prices it faces. This sequential decision making is a Stackelberg game, and it generally leads to a decen-

tralized equilibrium that is inefficient relative to the centralized benchmark due to double marginalization: the collector marks up the transfer price above the acquisition cost, which inflates the remanufacturer's effective costs and causes it to order less than the system optimum. Coordination contracts, that is, agreements between the collector and the remanufacturer designed to restore system-wide efficiency, are therefore of both practical and theoretical interest.

This thesis studies a closed-loop supply chain consisting of two independent decision makers, a collector and a remanufacturer. The collector acquires used products, referred to as cores, from the supply market at quality-dependent acquisition prices and transfers them to the remanufacturer at negotiated transfer prices. The remanufacturer processes the cores and sells the resulting products to consumers at a market-clearing price. We examine this system under acquisition pricing and quality differentiation, first with a single quality class of returns and then with two quality classes that differ in their acquisition and remanufacturing costs. In each setting, we characterize the centralized benchmark, the decentralized equilibrium, and coordination contracts designed to restore system-wide efficiency.

The analysis yields several analytical results. For the single-quality model, the relative efficiency loss, defined as the percentage reduction in total profit due to double marginalization, is shown to depend only on supply and demand sensitivity parameters, and not on remanufacturing cost or market size. Both models are analyzed under two coordination contracts. A two-part tariff always coordinates the supply chain. A revenue sharing implementable contract, in which the revenue share and the quality-specific transfer prices are fixed jointly in advance before the Stackelberg game, is shown to coordinate the supply chain over a feasible range of revenue shares satisfying individual rationality for both parties.

The remainder of this thesis is organized as follows. Chapter 2 reviews the relevant literature on acquisition pricing and closed-loop supply chain coordination, and formally defines the problem studied in this thesis. Chapter 3 analyzes the single-quality class model, characterizing the centralized and decentralized equilibria and the coordination contracts. Chapter 4 extends the analysis to the two-quality class setting, deriving the centralized and decentralized solutions and the corresponding coordina-

tion contracts. Chapter 5 presents a computational study of both models, examining the sensitivity of equilibrium quantities, profits, consumer surplus, efficiency loss, and coordination contract feasibility to changes in cost and market parameters. Chapter 6 concludes with a summary of findings and directions for future research.

CHAPTER 2

LITERATURE REVIEW AND THE PROBLEM DEFINITION

This chapter provides the background and formal setup for the thesis. Section 2.1 reviews the relevant literature on product acquisition management in closed-loop supply chains, covering both centralized and decentralized settings. Section 2.2 formally defines the problem studied in this thesis, introducing the model assumptions, decision variables, and the structure of the Stackelberg game between the collector and the remanufacturer for both the single quality class and two quality class models.

2.1 Literature Review

Closed-loop supply chain (CLSC) management has been an active area of research in operations research and management science for nearly three decades. Early work established the fundamental structure of reverse logistics systems and identified the key challenges of integrating forward and reverse flows; including return flow uncertainty, inspection and sorting operations, and the matching of recovered supply with remanufacturing demand. Since then, the literature has expanded to cover a broad range of topics: inventory management under return uncertainty, production planning in hybrid manufacturing-remanufacturing systems, pricing of recovered products, reverse channel structure design, and coordination among supply chain actors. Foundational treatments of these themes can be found in Fleischmann et al. [1], Guide and Van Wassenhove [2], and Bansal et al. [3], and a comprehensive review of analytical remanufacturing research across strategic, operational, and marketing dimensions is provided by Atasu et al. [4].

This chapter focuses on the stream of literature most directly relevant to the present

thesis: product acquisition management in closed-loop supply chains. Product acquisition management refers to the active control of the quantity and quality of returned products through pricing incentives, that is, the strategic setting of acquisition prices to attract returns from consumers or intermediaries. This is a central operational challenge in remanufacturing systems, as profitability depends critically on obtaining the right quantity and quality of cores at the right cost.

The studies reviewed in this chapter span two structural settings. In centralized systems, a single decision maker jointly optimizes all system variables to maximize total supply chain profit, a global optimization problem. In decentralized systems, independent parties each optimize their own profit, leading to outcomes that are generally inferior to the centralized benchmark. This inefficiency arises from double marginalization: when the collector and the remanufacturer each apply markups independently, the resulting equilibrium quantity and profit fall short of what a centralized planner could achieve. Supply chain coordination contracts are mechanisms designed to eliminate this inefficiency by modifying the transfer prices or payment structures faced by each party, aligning their individual incentives with the system-wide optimum. The theoretical foundations for such contracts, including wholesale price contracts, two-part tariffs, and revenue sharing contracts, are surveyed by Cachon [5] and analyzed in detail by Cachon and Lariviere [6] in the context of forward supply chains.

The section is organized as follows. Section 2.1.1 reviews centralized CLSC models with acquisition management. Section 2.1.2 reviews decentralized models, including game-theoretic analyses and coordination contract studies.

2.1.1 Centralized CLSCs with Acquisition Management

The studies reviewed in this section analyze acquisition management decisions in closed-loop supply chains under centralized control, where a single decision maker jointly optimizes all system variables. The central questions are how acquisition prices or quantities should be set to attract returns of the right quantity and quality, and how these decisions interact with production and selling decisions.

Guide et al. [7] provide the foundational framework for quality-differentiated acquisition pricing in remanufacturing systems. In a single-period model, a single integrated firm simultaneously determines quality-specific acquisition prices for each return class and the selling price for remanufactured products. The supply of returns in each quality class is increasing in the acquisition price offered for that class and independent of prices offered for other classes. Demand for remanufactured products is deterministic and decreasing in the selling price. Market clearing requires that total supply across all quality classes equals demand, so the selling price is fully determined by the acquisition prices. The objective is to maximize total profit, defined as sales revenue minus acquisition costs and remanufacturing costs, where each quality class carries a distinct unit remanufacturing cost. All quality classes are remanufactured to a single uniform standard. Guide et al.[7] characterize the optimal solution to this problem: the optimality condition equates the marginal revenue from sales to the marginal acquisition and remanufacturing cost of each quality class, yielding a clean characterization of the quality-differentiated acquisition problem. Guide et al. [7] illustrate their framework with an application from the cellular telephone industry. The present thesis builds directly on this framework, extending it to a decentralized two-quality setting with a collector-remanufacturer Stackelberg game and coordination contract analysis.

Bakal and Akcali [8] study a single-period remanufacturing system in which a single integrated firm acquires end-of-life vehicles from consumers at a quality-dependent acquisition price, disassembles them, and sells remanufactured parts in the secondary market. The firm determines the acquisition price and the selling price to maximize total profit, defined as sales revenue minus acquisition costs and remanufacturing costs. A key feature is that the recovery yield, the fraction of parts that are remanufacturable, is random and increasing in the acquisition price, since higher prices attract vehicles in better condition. Both supply and demand are deterministic linear functions of their respective prices. Three models are analyzed according to the timing of pricing decisions: a deterministic benchmark in which yield is known, a postponed pricing model in which the selling price is set after yield realization, and a simultaneous model in which both prices are set before yield realization. Their analysis shows that postponing the selling price decision until after yield realization is always

beneficial, with the gain increasing for lower yield rates and higher yield variability. The present thesis adopts the deterministic supply and demand structure of Bakal and Akcali [8] but focuses on a two-quality setting with a decentralized channel structure rather than yield uncertainty.

Pokharel and Liang [9] develop a single-period cost minimization model from the perspective of a consolidation center. The center acquires used products of multiple quality levels from collection centers by offering quality-dependent acquisition prices. It determines the acquisition price and order quantity for each quality level to minimize total costs. These costs comprise acquisition payments to collection centers, the cost of replacement parts required for each quality level, penalty costs incurred when collected quantity falls short of the remanufacturer's fixed demand, and salvage revenues from surplus used products. The supply of used products at each quality level is stochastic with known mean and standard deviation. Using the Kuhn–Tucker conditions, they characterize the optimal acquisition price and quantity for each quality level in terms of a single Lagrange multiplier associated with the demand constraint. They then develop a numerical search procedure over this multiplier to identify the acquisition price and quantity that jointly satisfy the total demand requirement. Unlike the present thesis, this work focuses on cost minimization under supply uncertainty from the perspective of an intermediary, rather than profit maximization with price-sensitive supply and demand in a strategic collector-remanufacturer setting.

Cai et al. [10] study a hybrid manufacturing and remanufacturing system in which an integrated firm acquires cores classified into high- and low-quality levels with quality-dependent acquisition prices. In a multi-period setting formulated as a Markov decision process, the firm simultaneously determines the acquisition price for each quality class, the remanufacturing quantity for each class, and the manufacturing quantity for new products in each period. Demand is stochastic with a known distribution. The objective is to minimize total discounted expected cost over the planning horizon, comprising acquisition costs, holding costs for core inventories and serviceable products, manufacturing and remanufacturing costs, and backordering costs for unsatisfied demand. The optimal production policy is characterized by three threshold parameters: a manufacture-up-to level and two remanufacture-up-to levels for high-

and low-quality cores. Their values depend critically on the cost differences among manufacturing new products and remanufacturing high- and low-quality cores. The present thesis differs from Cai et al. [10] in that it does not include a manufacturing option. The system is a pure remanufacturing system and it focuses on the strategic interaction between a collector and a remanufacturer as separate decision makers rather than an integrated firm operating over multiple periods.

Bulmuş et al. [11] study a single-period hybrid manufacturing and remanufacturing system in which an integrated original equipment manufacturer simultaneously determines acquisition prices for each return type, the selling price for new products, and the selling price for remanufactured products. The return rate for each type is deterministic and linear increasing in the acquisition price offered for that type. New and remanufactured products are sold in partially overlapping markets, with consumers having a lower willingness to pay for remanufactured products; demand for both product types is deterministic and decreasing in their respective selling prices. The objective is to maximize total profit, defined as sales revenue from new and remanufactured products minus manufacturing costs, acquisition costs, and remanufacturing costs for each return type. A closed-form optimal solution procedure is derived, and a key managerial finding is that the optimal profit per remanufactured item is higher for more preferred return types, that is, types with lower total acquisition and remanufacturing costs, rather than being equal across all types as practitioners might intuitively assume. The present thesis differs from Bulmuş et al. [11] in that it does not include a manufacturing option and focuses on the strategic interaction between a collector and a remanufacturer as separate decision makers rather than an integrated original equipment manufacturer, and does not consider cannibalization between new and remanufactured products.

Sun and Li [12] study the joint acquisition and production planning problem of a third-party remanufacturer who acquires unsorted cores at a fixed price per unit, sorts them into graded quality classes, and decides how many cores of each grade to remanufacture in a single period before demand is realized. The remanufacturer determines the acquisition quantity and the production quantity to maximize expected profit, defined as sales revenue minus a piecewise-linear convex production cost, which increases with the grade of cores used, and a linear acquisition cost. Demand is

stochastic with a known distribution; the deterministic demand case is analyzed first as a special case. Analytically tractable closed-form solutions are derived under deterministic demand, and the structure of the optimal policy under stochastic demand is fully characterized, showing that quality grading significantly affects both acquisition quantities and expected profits. The present thesis shares the quality-graded core structure of Sun and Li [12] but differs in that acquisition is controlled through strategically determined prices rather than quantities, and involves a decentralized setting with an independent collector acting as a Stackelberg leader rather than an integrated remanufacturer.

2.1.2 Decentralized CLSCs with Acquisition Management

The analysis of decentralized CLSCs as strategic games was pioneered by Savaskan et al. [13] and extended by Savaskan and Van Wassenhove [14], who model independent supply chain members as Stackelberg players and characterize equilibrium collection and pricing decisions under various channel structures. Savaskan et al. [13] consider three reverse channel structures that differ in the supply chain partner responsible for collection: the manufacturer, the retailer, or a third-party collector. It shows that the retailer-collects structure yields the highest system profit among the three decentralized alternatives, because the retailer's proximity to consumers allows efficient reflection of remanufacturing cost savings into retail prices. They further show that a two-part tariff contract can coordinate the retailer-collects channel and restore the centralized profit level. Savaskan and Van Wassenhove [14] extend this study to a setting with two competing retailers and show that the preferred collection structure depends on the degree of product substitutability between retailers. Although these studies model collection as an effort-dependent return rate rather than through price-sensitive acquisition prices, they establish the game-theoretic framework. The manufacturer acts as Stackelberg leader while the retailer is the follower. The problem is solved via backward induction, and the optimal solutions are compared to the centralized benchmark. The studies reviewed in this section adopt this framework but focus specifically on settings where acquisition prices are the primary instrument for controlling the quantity or quality of returns.

Kaya [15] analyzes a single-period model of incentive and production decisions for remanufacturing operations in both centralized and decentralized settings. A manufacturer produces both original products using virgin materials and remanufactured products using consumer returns, and must offer a financial incentive to consumers to collect the right quantity of used products. Demand is deterministic. In the decentralized setting, a third-party collection agency collects used products and sells them to the manufacturer at a transfer price; the collection agency determines the incentive level to maximize its own profit, defined as transfer revenue minus incentive payments and collection costs, while the manufacturer determines the production quantities to maximize its own profit, defined as sales revenue minus production costs, remanufacturing costs, and transfer payments. Three models are developed: a pure remanufacturing model, a model with perfectly substitutable new and remanufactured products, and a model with partially substitutable products. A linear contract consisting of a unit transfer price and a fixed payment is shown to coordinate the decentralized system and achieve centralized profits across all three models. The present thesis shares the focus on incentive design in a collector-remanufacturer system but differs in that it analyzes quality-differentiated returns with deterministic demand and supply, and derives closed-form equilibrium solutions rather than relying on numerical analysis.

Ünal [16] examines a single-period two-echelon closed-loop supply chain consisting of a collector and a remanufacturer, where returned products are classified into two quality types after inspection. The supply of cores is deterministic and increasing in the acquisition fee set by the collector; demand for recovered products is deterministic and decreasing in the selling price. The quality proportion, that is the fraction of superior quality returns within the total supply, is stochastic with a known distribution. In the decentralized setting, the collector determines the acquisition fee and, depending on the model, the transfer prices for each quality type, to maximize its own profit, defined as transfer revenue minus acquisition costs and inspection costs; the remanufacturer determines the selling price and the remanufacturing quantities for each type to maximize its own profit, defined as sales revenue minus transfer payments and remanufacturing costs. Surplus cores are salvaged at a fixed unit revenue by the collector. Four models are developed according to the timing of pricing decisions and the degree of centralization: simultaneous pricing, sequential pricing, collector pric-

ing before inspection, and collector pricing after inspection. A key finding is that postponing pricing decisions until after inspection always benefits the decision maker with pricing authority, and that centralized decision making yields strictly higher system profits than decentralized decision making due to double marginalization. The present thesis shares the two-quality collector-remanufacturer structure of Ünal [16] but differs in that quality proportions are deterministic rather than stochastic, and the analysis focuses on closed-form equilibrium characterization and coordination contract design rather than the value of quality information.

De Giovanni and Zaccour [17] study a two-period closed-loop supply chain in which a manufacturer decides whether to collect used products itself, outsource collection to the retailer, or outsource to an external third-party service provider. The game is played non-cooperatively à la Stackelberg with the manufacturer as the leader and the retailer as the follower. Demand is deterministic and decreasing in price. The return rate depends on green activity investments undertaken by the collecting party. The manufacturer determines the collection mode and the wholesale price to maximize total two-period profit, defined as sales revenue minus production costs and collection costs; the retailer determines the retail price and the level of green activity investment to maximize its own two-period margin. A key finding is that the manufacturer should generally collect used products itself, and outsource only when the outsourcee performs substantially better in terms of collection cost or green activity effectiveness which is a result contrasting with Savaskan et al. [13], who find that the retailer should always manage collection. The present thesis differs from De Giovanni and Zaccour [17] in that it focuses on a single-period model with quality-differentiated returns and quality-specific transfer prices rather than a dynamic model with return-rate-dependent green activity investments.

Govindan and Popiuc [18] apply revenue sharing contracts to reverse supply chain coordination in the personal computers industry in a single-period setting. A retailer collects used products from consumers by offering a discount and transfers them downstream for remanufacturing. The volume of returns is deterministic and increasing in the discount offered. In the decentralized setting, the retailer determines the consumer discount to maximize its own profit, defined as the recycling fee received from the manufacturer minus collection and processing costs; the manufacturer de-

termines the recycling fee to maximize its own profit, defined as remanufacturing revenue minus the recycling fee paid to the retailer. Under a revenue sharing contract, the manufacturer shares a fraction of remanufacturing revenue with the retailer, inducing the retailer to set a discount that maximizes total supply chain profit. A numerical study based on Apple Inc. data confirms the practical relevance of the results. The present thesis differs from Govindan and Popiuc [18] in that it focuses on quality-differentiated returns with quality-specific transfer prices rather than a single homogeneous return stream. While a freely-optimizing revenue sharing contract cannot coordinate the system when remanufacturing costs differ across quality types, the present thesis shows that an implementable contract, in which the revenue share and transfer prices are agreed jointly in advance, restores coordination.

He [19] models a single-period closed-loop supply chain consisting of a manufacturer, a recycle channel supplier, and a reliable supply channel. The manufacturer may reuse recycled core components and remanufacture the product; when the recycle channel yields insufficient supply, the shortfall is purchased from the reliable channel at a higher unit cost. The recycle channel exhibits stochastic recovery, that is the realized quantity depends on the acquisition price offered but is also subject to random yield. Both deterministic and stochastic demand cases are analyzed. In the centralized setting, the manufacturer controls the recycle channel directly and determines the acquisition price and remanufacturing quantity to maximize total expected profit, defined as sales revenue minus remanufacturing costs, reliable channel procurement costs for any shortfall, and acquisition costs. In the decentralized setting, the recycle channel supplier independently determines the acquisition price to maximize its own profit, defined as transfer revenue received from the manufacturer minus acquisition costs paid to consumers; the manufacturer then determines the remanufacturing quantity to maximize its own profit, defined as sales revenue minus remanufacturing costs, transfer payments to the recycle supplier, and reliable channel procurement costs for any shortfall. He [19] shows that under decentralization, the optimal acquisition price is always lower than under the centralized benchmark, resulting in reduced recycle quantity and lower remanufactured output, a double marginalization effect analogous to that in forward supply chains. Two coordination contracts are proposed: a complete compensation contract and a partial compensation contract. Both of them are

shown to restore the centralized acquisition price and achieve Pareto improvements for both parties under certain conditions. This result directly motivates the coordination contract analysis of the present thesis.

Heydari and Ghasemi [20] study single-period reverse supply chain coordination under two simultaneous sources of uncertainty: stochastic quality of returned products and uncertain remanufacturing capacity. A single remanufacturer and a single collector operate in a two-echelon reverse supply chain. The collector offers a fixed reward to consumers to incentivize returns and inspects returned items, accepting only those that meet a minimum quality threshold. The remanufacturer refurbishes accepted items subject to uncertain capacity, with excess items sold in a secondary market at a salvage value. In the decentralized setting, the collector determines the reward amount to maximize its own profit, defined as the recycling fee received from the remanufacturer minus the reward paid to consumers, without internalizing the consequences of capacity uncertainty at the remanufacturer. The remanufacturer determines the recycling fee and the revenue sharing parameter under a customized revenue sharing contract in which it offers the collector a reduced recycling fee and simultaneously shares a fraction of remanufacturing revenue, aligning the collector's reward decision with the system-wide optimum. The present thesis shares the collector-remanufacturer structure of Heydari and Ghasemi [20] but differs in that supply and demand are deterministic and quality heterogeneity, rather than capacity uncertainty, is the primary source of coordination difficulty.

The present thesis sits at the intersection of two streams of literature reviewed in this chapter: quality-differentiated acquisition pricing in remanufacturing systems, and decentralized closed-loop supply chain analysis with coordination contract design. While each of these streams is well developed individually, their joint treatment in a unified analytical framework has received limited attention.

The centralized acquisition pricing stream, initiated by Guide et al. [7] and extended by Bakal and Akcali [8], Pokharel and Liang [9], Cai et al. [10], Bulmuş et al. [11], and Sun and Li [12], establishes that quality heterogeneity in returns has important implications for acquisition pricing and system profitability. However, these studies either analyze only the centralized setting, in which a single integrated firm makes

all decisions, or treat quality heterogeneity in terms of yield uncertainty rather than strategically determined quality-specific transfer prices between independent parties. The decentralized dimension, in which a collector and a remanufacturer operate as independent strategic agents with quality-specific acquisition and transfer prices, is not addressed in this stream.

The decentralized CLSC stream, anchored by the game-theoretic framework of Savaskan et al. [13] and extended by Kaya [15], De Giovanni and Zaccour [17], Govindan and Popiuc [18], He [19], and Heydari and Ghasemi [20], provides a rich framework for analyzing strategic interactions in reverse supply chains. However, these studies either assume a single quality type of returned products, abstracting away the quality heterogeneity that characterizes real remanufacturing systems, or focus on stochastic quality proportions and the value of quality information rather than on strategic equilibrium characterization with quality-specific transfer prices. Ünal [16] extends the two-quality CLSC structure to a decentralized setting but focuses on the value of quality information under stochastic quality proportions rather than closed-form equilibrium characterization and coordination contract design.

The coordination contract literature, surveyed by Cachon [5] and Cachon and Lariviere [6], establishes that two-part tariffs and revenue sharing can coordinate simple supply chains, and identifies structural conditions under which each contract type succeeds or fails. Govindan and Popiuc [18] and Heydari and Ghasemi [20] demonstrate that revenue sharing can coordinate reverse supply chains under various forms of uncertainty, but both studies assume a single homogeneous quality type. The interaction between quality heterogeneity and contract design, specifically whether standard coordination mechanisms remain effective when returns differ in remanufacturing cost, has not been analyzed in the existing literature.

Table 2.1 summarizes the reviewed studies and their relation to the key dimensions of the present thesis.

Table 2.1: Summary of Reviewed Studies and Related Topics

Study	Quality Differentiation	Stackelberg Game	Uncertain Return Quantity	Uncertain Return Quality	Uncertain Demand	Two-Part Tariff Contract	Revenue Sharing Contract
Guide et al. (2003) [7]	✓						
Bakal & Akcali (2006) [8]	✓			✓			
Kaya (2009) [15]		✓				✓	
Ünal (2010) [16]	✓	✓		✓			
Pokharel & Liang (2012) [9]	✓		✓	✓			
Cai et al. (2013) [10]	✓				✓		
De Giovanni & Zaccour (2013) [17]		✓					
Govindan & Popiuc (2013) [18]		✓					✓
Bulmuş et al. (2014) [11]	✓						
He (2015) [19]		✓	✓		✓		
Heydari & Ghasemi (2018) [20]		✓		✓			✓
Sun & Li (2023) [12]	✓				✓		
Present thesis	✓	✓				✓	✓

2.2 Problem Definition

This thesis studies a two-echelon closed-loop supply chain in which a collector and a remanufacturer operate as the primary decision makers. The basic setting is adopted from Guide et al. [7], who develop a centralized single-period model of quality-differentiated acquisition pricing in remanufacturing systems. The supply structure, demand function, quality ordering assumption, and remanufacturing cost structure used throughout this thesis follow their framework. The present thesis extends this setting to a decentralized channel structure with a collector-remanufacturer Stackelberg game and coordination contract analysis.

Used products, referred to as cores, are obtained from the supply market by the collector, who classifies them by quality and transfers them to the remanufacturer at prices set by the collector. The remanufacturer processes the acquired cores and offers the resulting recovered products for sale in the consumer market at selling price p. The system is a pure remanufacturing system: cores constitute the sole input to production and no manufacturing from virgin materials takes place.

Both the supply of cores and the demand for recovered products are modeled as deterministic functions of price, following Guide et al. [7]. Demand for recovered products is a linear decreasing function of the selling price p:

$$D(p) = \alpha - \gamma p,$$

where $\alpha > 0$ is the potential market size (the maximum attainable demand when $p = 0$) and $\gamma > 0$ is the price sensitivity of demand. Hence, the maximum willingness to pay is α/γ. Supply of cores is a linear increasing function of the acquisition price offered by the collector, denoted $R(a)$, with $R(a) = 0$ for prices below some minimum threshold. The precise form of $R(a)$ is specified for single and two quality class models in the remainder of this section. Let Q denote the total remanufacturing quantity, that is, the quantity of cores processed by the remanufacturer and sold in the consumer market, so that $Q = D(p) = \alpha - \gamma p$. Since remanufactured products cannot be stored or salvaged, the model requires that the quantity collected be at least as large as the quantity sold: $R(a) \geq Q$.

This thesis considers a single selling period throughout both models developed in the following chapters. Several further assumptions shape the environment in which the collector and the remanufacturer operate. The remanufacturer is assumed to face no production capacity limitations, so that its quantity decisions are governed entirely by the costs and prices in the model rather than by any binding physical constraint. Cores can be perfectly classified into quality categories, ensuring that the quality-specific prices and costs introduced later in the thesis apply cleanly to each unit collected.

We analyze the system under two models, described below. In each model, we develop and compare three frameworks: a centralized benchmark in which a single decision maker maximizes total system profit, a decentralized Stackelberg game in which the collector acts as the leader and the remanufacturer acts as the follower, and a coordination contract analysis in which mechanisms are designed to restore system-wide efficiency under decentralized decision making.

In the model with a single quality class, all cores are of uniform quality (Chapter 3). The collector offers a single acquisition price $a \geq 0$ to the supply market, where a is the unit payment made to core holders. The supply of cores responds linearly:

$$R(a) = \delta(a - \beta) \quad \text{for } a \geq \beta,$$

and zero otherwise, where $\beta > 0$ is the minimum acquisition price below which no cores are supplied and $\delta > 0$ is the supply elasticity. The unit remanufacturing cost is $c > 0$. The collector charges a single unit transfer price w for each core collected to the remanufacturer.

For remanufacturing to be economically viable, we assume:

$$\frac{\alpha}{\gamma} > \beta + c.$$

The left-hand side α/γ is the customer's maximum willingness to pay, while $\beta + c$ is the minimum total unit cost: the lowest possible acquisition price of β plus the remanufacturing cost of c. If this condition fails, the market price can never cover even the minimum cost of producing a unit, and no positive quantity would be profitably remanufactured.

In the decentralized setting, the sequence of events is as follows:

1. The collector sets the acquisition price a and the transfer price w.
2. Given w, the remanufacturer determines the remanufacturing quantity Q and the selling price p, subject to market clearing.

In the model with two quality classes of returns, cores are classified into two quality classes: high quality cores (class h) and low quality cores (class l) (Chapter 4). The collector offers separate acquisition prices a_h and a_l to the supply market to acquire high and low quality class of cores, respectively . The supply of each quality class, i, responds linearly and independently to its own acquisition price:

$$R_i(a_i) = \delta_i(a_i - \beta_i) \quad \text{for } a_i \geq \beta_i, \quad i \in \{h, l\},$$

and zero otherwise, where $\beta_i > 0$ is the minimum acquisition price and $\delta_i > 0$ is the supply elasticity for quality class $i \in \{h, l\}$. The acquisition price for one quality class does not affect the supply of the other quality class. Although both quality classes are remanufactured to a single uniform quality standard and sold at the same selling price p, their unit remanufacturing costs differ: c_h for high-quality cores and c_l for low-quality cores, with $c_h < c_l$ since high-quality cores require less processing. The collector charges separate transfer prices w_h and w_l to the remanufacturer for class h and class l cores, respectively. Let Q_h and Q_l denote the remanufacturing quantities of class h and class l cores. The market clearing condition becomes:

$$Q_h + Q_l = D(p) = \alpha - \gamma p.$$

The two quality class model imposes two additional assumptions. First, the quality ordering condition $\beta_h + c_h \leq \beta_l + c_l$ requires that the total minimum cost of acquiring and remanufacturing a high quality core does not exceed that of a low quality core, ensuring that high quality returns are always less costly to process than low quality returns. Second, for remanufacturing to be economically viable for both quality classes, we assume $\alpha/\gamma > \beta_l + c_l$. The maximum willingness to pay must exceed the minimum total unit cost for low quality cores. Since low quality cores are the most costly to process, ensuring viability for quality class l automatically guarantees viability for class h as well, given the quality ordering assumption.

In the decentralized setting, the sequence of events is as follows:

1. The collector sets acquisition prices a_h, a_l and transfer prices w_h, w_l.

2. Given w_h and w_l, the remanufacturer determines quantities Q_h, Q_l and the selling price p, subject to market clearing.

The present thesis contributes to the literature reviewed in Section 2.1 in three main respects. First, while Guide et al. [7] establish the centralized acquisition pricing framework with quality differentiation, this thesis extends it to a decentralized setting in which the collector and the remanufacturer are independent strategic agents. The collector acts as Stackelberg leader and sets quality-specific acquisition and transfer prices anticipating the remanufacturer's response. Second, the decentralized Stackelberg equilibrium is characterized in closed form for both the single quality class and two quality class models, including a complete piecewise analysis of the collector's profit function across all operational regions. Third, two coordination contracts, a two-part tariff and a revenue sharing implementable contract, are analyzed in a setting with quality-differentiated returns. While prior work such as Govindan and Popiuc [18] and Heydari and Ghasemi [20] study revenue sharing in reverse supply chains, they consider a single homogeneous quality type and do not address quality-differentiated acquisition pricing. This thesis shows that when returns differ in quality, coordination requires fixing both the revenue share and the quality-specific transfer prices jointly in advance, following Cachon [5]. By jointly addressing quality differentiation, decentralized Stackelberg pricing, and coordination contract analy-

sis, this thesis aims to provide a more complete treatment of acquisition pricing in closed-loop supply chains than has been offered in the existing literature.

The notation used throughout the thesis is summarized in Table 2.2.

Table 2.2: Notation

Symbol	Description
α	Potential market size
γ	Price sensitivity of demand
p	Selling price
$D(p)$	Demand: $\alpha - \gamma p$
β, β_h, β_l	Minimum acquisition prices
δ, δ_h, δ_l	Supply elasticities
c, c_h, c_l	Remanufacturing costs, $c_h < c_l$
a, a_h, a_l	Acquisition prices
w, w_h, w_l	Transfer prices
Q	Total remanufacturing quantity
Q_h, Q_l	Remanufacturing quantities
$R(a)$	Supply function (single quality class): $\delta(a - \beta)$
$R_i(a_i)$	Supply function (two quality class): $\delta_i(a_i - \beta_i)$
$i \in \{h, l\}$	Quality class index
$\Pi^C(a^{C*}, p^{C*})$	Centralized total profit function
$\Pi^D(a^{D*}, w^{D*})$	Decentralized total supply chain profit function

A superscript $*$ denotes a quantity evaluated at the optimal decision variables (e.g., a_h^{C*}, w_l^{D*}).

CHAPTER 3

MODEL WITH A SINGLE QUALITY CLASS OF RETURNS

This chapter analyzes the closed-loop supply chain under a single quality class of returns, serving as a tractable foundation before the more complex two-quality class model is introduced in subsequent chapters. All cores are of uniform quality, so the collector offers a single acquisition price a and a single transfer price w to the remanufacturer. The chapter is organized as follows. Section 3.1 develops the centralized benchmark, in which a single integrated firm maximizes total system profit and derives the optimal acquisition price, quantity, selling price, and profit. Section 3.2 analyzes the decentralized setting as a Stackelberg game, in which the collector acts as the leader and the remanufacturer acts as the follower, characterizes the unique decentralized equilibrium, and establishes that the decentralized equilibrium is strictly inefficient relative to the centralized benchmark due to double marginalization. Section 3.3 analyzes two coordination contracts,a two-part tariff and a revenue sharing contract, and characterizes the contract parameters under which both mechanisms can eliminate the efficiency loss and restore system-wide optimality while satisfying individual rationality for both parties.

3.1 Centralized Setting

3.1.1 Model Formulation

In the centralized setting, a single integrated firm makes all decisions to maximize total system profit. The firm chooses the acquisition price a and the selling price p simultaneously. Throughout this chapter, all model parameters $\alpha, \gamma, \delta, \beta, c$ are assumed to be strictly positive. Recalling that $D(p) = \alpha - \gamma p$ is the demand function and

$R(a) = \delta(a - \beta)$ is the supply function, the total system profit consists of revenue from selling remanufactured products, minus remanufacturing costs, minus acquisition costs paid to core holders:

$$\Pi^C(a, p) = p \cdot D(p) - c \cdot D(p) - a \cdot R(a). \tag{3.1}$$

Substituting $D(p) = \alpha - \gamma p$ and $R(a) = \delta(a - \beta)$ into (3.1), the profit as a function of a and p can be expressed as:

$$\Pi^C(a, p) = p(\alpha - \gamma p) - c(\alpha - \gamma p) - a\,\delta(a - \beta).$$

Then, the centralized problem is:

$$\begin{aligned}
\max_{a,\,p} \quad & \Pi^C(a, p) \\
\text{s.t.} \quad & \alpha - \gamma p \geq 0 && (3.2)\\
& p \geq 0 && (3.3)\\
& a \geq \beta && (3.4)\\
& \delta(a - \beta) - (\alpha - \gamma p) \geq 0 && (3.5)
\end{aligned}$$

Constraint (3.2) ensures that market demand is non-negative. Since $D(p) = \alpha - \gamma p \geq 0$ requires $p \leq \alpha/\gamma$, this constraint bounds the selling price from above. Constraint (3.3) ensures that the selling price is non-negative. Constraints (3.2) and (3.3) together bound the selling price to $0 \leq p \leq \alpha/\gamma$. Constraint (3.4) ensures that the acquisition price is sufficient to attract any supply. Since $R(a) = \delta(a - \beta) \geq 0$ requires $a - \beta \geq 0$, we need $a \geq \beta$. Constraint (3.5) states that the firm cannot sell more units than it collects: the quantity sold, $D(p) = \alpha - \gamma p$, cannot exceed the quantity collected, $R(a) = \delta(a - \beta)$. We write this as an inequality, rather than imposing $D(p) = R(a)$ directly, because the firm is in principle free to collect more cores than it sells. Below, we show that the optimal solution to the centralized problem satisfies this constraint with equality, so that the firm never collects more than it sells.

Lemma 3.1. $\Pi^C(a, p)$ *is strictly jointly concave in* (a, p).

Proof. The first-order partial derivatives of $\Pi^C(a,p)$ are:

$$\frac{\partial \Pi^C(a,p)}{\partial a} = -\delta(2a-\beta),$$
$$\frac{\partial \Pi^C(a,p)}{\partial p} = \alpha + \gamma c - 2\gamma p.$$

The Hessian of $\Pi^C(a,p)$ with respect to (a,p) is:

$$H = \begin{pmatrix} \dfrac{\partial^2 \Pi^C(a,p)}{\partial a^2} & \dfrac{\partial^2 \Pi^C(a,p)}{\partial a\,\partial p} \\ \dfrac{\partial^2 \Pi^C(a,p)}{\partial p\,\partial a} & \dfrac{\partial^2 \Pi^C(a,p)}{\partial p^2} \end{pmatrix} = \begin{pmatrix} -2\delta & 0 \\ 0 & -2\gamma \end{pmatrix}.$$

Since $\delta, \gamma > 0$, the diagonal entries are both negative and the off-diagonal entries are zero, so H is negative definite. Therefore $\Pi^C(a,p)$ is jointly strictly concave in (a,p). □

Lemma 3.2. *At the optimal solution to the centralized problem, constraint* (3.5) *is always binding, that is, the system collects exactly as many cores as it sells:* $\delta(a^{C*} - \beta) = \alpha - \gamma p^{C*}$.

Proof. Since $\Pi^C(a,p)$ is strictly jointly concave by Lemma 3.1 and the feasible region defined by constraints (3.2)–(3.5) is convex, any point satisfying the KKT conditions is the unique global maximum. The Lagrange function is:

$$\mathcal{L}(a,p,\lambda) = \Pi^C(a,p) + \lambda\big[\delta(a-\beta) - (\alpha - \gamma p)\big], \tag{3.6}$$

where $\lambda \geq 0$ is the multiplier associated with constraint (3.5). Constraints (3.2)–(3.4) are satisfied with strict inequality at the optimum and their multipliers are zero by complementary slackness. Differentiating (3.6) with respect to p and setting the result to zero:

$$\frac{\partial \mathcal{L}(a,p,\lambda)}{\partial p} = \alpha - 2\gamma p + \gamma c + \lambda\gamma = 0. \tag{3.7}$$

Solving (3.7) for p:

$$p = \frac{\alpha + \gamma c + \lambda\gamma}{2\gamma}. \tag{3.8}$$

Differentiating (3.6) with respect to a and setting the result to zero:

$$\frac{\partial \mathcal{L}(a,p,\lambda)}{\partial a} = -\delta(2a-\beta) + \lambda\delta = 0. \tag{3.9}$$

Solving (3.9) for a:

$$a = \frac{\beta + \lambda}{2}. \tag{3.10}$$

Solving (3.9) and (3.7) simultaneously with complementary slackness gives either $\lambda = 0$ or constraint (3.5) binds. Suppose $\lambda = 0$. Then from (3.9), $a = \beta/2 < \beta$, which violates constraint (3.4). Hence $\lambda > 0$, and by complementary slackness, constraint (3.5) holds with equality at the optimum. □

Proposition 3.1. *The optimal solution to the centralized problem is:*

$$a^{C*} = \frac{\alpha + 2\delta\beta + \gamma\beta - \gamma c}{2(\delta + \gamma)}, \tag{3.11}$$

$$Q^{C*} = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(\delta + \gamma)}, \tag{3.12}$$

$$p^{C*} = \frac{(\delta + 2\gamma)\alpha + \delta\gamma(\beta + c)}{2\gamma(\delta + \gamma)}, \tag{3.13}$$

$$\Pi^{C*}(a^{C*}, p^{C*}) = \frac{\delta[\alpha - \gamma(\beta + c)]^2}{4\gamma(\delta + \gamma)}. \tag{3.14}$$

Proof. By Lemma 3.2, $\lambda^* > 0$ and $\delta(a^* - \beta) = \alpha - \gamma p^*$. We solve for λ^* by substituting (3.8) and (3.10) into this binding constraint:

$$\delta\left(\frac{\lambda - \beta}{2}\right) = \alpha - \frac{\alpha + \gamma c + \lambda\gamma}{2}$$

$$\delta(\lambda - \beta) = \alpha - \gamma c - \lambda\gamma$$

$$\lambda(\delta + \gamma) = \alpha + \delta\beta - \gamma c.$$

Solving for λ:

$$\lambda^* = \frac{\alpha + \delta\beta - \gamma c}{\delta + \gamma}. \tag{3.15}$$

Substituting λ^* from (3.15) into (3.10):

$$a^{C*} = \frac{\beta + \lambda^*}{2} = \frac{1}{2}\left(\beta + \frac{\alpha + \delta\beta - \gamma c}{\delta + \gamma}\right) = \frac{\beta(\delta + \gamma) + \alpha + \delta\beta - \gamma c}{2(\delta + \gamma)},$$

which simplifies, since $\beta(\delta + \gamma) + \delta\beta = 2\delta\beta + \gamma\beta$, to (3.11).

We verify $a^{C*} \geq \beta$. Computing $a^{C*} - \beta$ from (3.11):

$$a^{C*} - \beta = \frac{\alpha + 2\delta\beta + \gamma\beta - \gamma c}{2(\delta + \gamma)} - \beta = \frac{\alpha - \gamma(\beta + c)}{2(\delta + \gamma)} > 0,$$

where the strict inequality follows since the numerator $\alpha - \gamma(\beta + c) > 0$ by market viability, and the denominator $2(\delta + \gamma) > 0$ since $\delta, \gamma > 0$ by assumption. Hence constraint (3.4) is satisfied with strict inequality. We can also confirm $\lambda^* > 0$ directly: solving (3.10) for λ gives $\lambda = 2a^{C*} - \beta$. Since $a^{C*} - \beta > 0$ as shown above, we

have $a^{C*} > \beta$, and therefore $\lambda^* = 2a^{C*} - \beta > a^{C*} > \beta > 0$, confirming $\lambda^* > 0$ as required by Lemma 3.2.

By Lemma 3.2, $Q^{C*} = \delta(a^{C*} - \beta)$. Substituting (3.11):

$$Q^{C*} = \delta\left(\frac{\alpha + 2\delta\beta + \gamma\beta - \gamma c}{2(\delta+\gamma)} - \beta\right) = \delta\left(\frac{\alpha - \gamma\beta - \gamma c}{2(\delta+\gamma)}\right) = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(\delta+\gamma)}.$$

Since $\delta > 0$ and $\alpha > \gamma(\beta + c)$ by market viability, $Q^{C*} > 0$, so constraint (3.2) is satisfied with strict inequality.

By Lemma 3.2, $p^{C*} = (\alpha - Q^{C*})/\gamma$. Substituting (3.12):

$$p^{C*} = \frac{\alpha - \dfrac{\delta[\alpha - \gamma(\beta + c)]}{2(\delta+\gamma)}}{\gamma} = \frac{2(\delta+\gamma)\alpha - \delta[\alpha - \gamma(\beta + c)]}{2\gamma(\delta+\gamma)} = \frac{(\delta + 2\gamma)\alpha + \delta\gamma(\beta + c)}{2\gamma(\delta+\gamma)}.$$

Since all parameters are strictly positive, $p^{C*} > 0$, so constraint (3.3) is satisfied with strict inequality.

Finally, by Lemma 3.2, $\Pi^C(a^{C*}, p^{C*}) = (p^{C*} - c - a^{C*})\, Q^{C*}$. Substituting (3.13) and (3.11):

$$p^{C*} - c - a^{C*} = \frac{(\delta + 2\gamma)\alpha + \delta\gamma(\beta + c)}{2\gamma(\delta+\gamma)} - c - \frac{\alpha + 2\delta\beta + \gamma\beta - \gamma c}{2(\delta+\gamma)} = \frac{\alpha - \gamma(\beta + c)}{2\gamma}.$$

Multiplying by Q^{C*} from (3.12):

$$\Pi^C(a^{C*}, p^{C*}) = \frac{\alpha - \gamma(\beta + c)}{2\gamma} \cdot \frac{\delta[\alpha - \gamma(\beta + c)]}{2(\delta+\gamma)} = \frac{\delta[\alpha - \gamma(\beta + c)]^2}{4\gamma(\delta+\gamma)}.$$

□

3.2 Decentralized Setting

In the decentralized setting, the collector and the remanufacturer are independent profit-maximizing firms. The collector acts as the Stackelberg leader, setting the acquisition price a and the transfer price w anticipating the remanufacturer's best response. The remanufacturer acts as the Stackelberg follower, choosing the remanufacturing quantity Q to maximize its own profit given the transfer price w set by the collector. Since the collector and the remanufacturer make sequential decisions, the game is solved by backward induction: we first characterize the remanufacturer's optimal quantity for given (a, w) in Stage 2, and we then solve the collector's problem in Stage 1, incorporating the remanufacturer's Stage 2 response.

3.2.1 Remanufacturer's Problem

For given acquisition price a and transfer price w set by the collector, the available supply of cores is $R(a) = \delta(a - \beta)$. The remanufacturer chooses quantity Q to maximize its profit:

$$\Pi^R(Q) = p \cdot Q - (w + c) \cdot Q = \frac{\alpha - Q}{\gamma} \cdot Q - (w + c) \cdot Q. \tag{3.16}$$

Since remanufactured products cannot be stored or salvaged, the remanufacturer has no reason to order a unit it does not intend to sell. The quantity it orders from the collector and the quantity it sells to the market are therefore the same Q, and the price p at which this quantity sells is given directly by the inverse demand function, $p = (\alpha - Q)/\gamma$. The remanufacturer's problem is:

$$\begin{aligned} \max_{Q} \quad & \Pi^R(Q) \\ \text{s.t.} \quad & 0 \leq Q \leq R(a) && (3.17) \\ & \alpha - Q \geq 0. && (3.18) \end{aligned}$$

Constraint (3.17) ensures that the remanufacturing quantity is non-negative and does not exceed the available supply $R(a)$ provided by the collector. Constraint (3.18) ensures that the resulting selling price $p = (\alpha - Q)/\gamma$ is non-negative.

Lemma 3.3. *The remanufacturer's optimal quantity is:*

$$Q^*(a, w) = \min \left\{ \frac{\alpha - \gamma(w + c)}{2},\ R(a) \right\}. \tag{3.19}$$

Proof. The first-order condition of (3.16) with respect to Q is:

$$\frac{d\Pi^R(Q)}{dQ} = \frac{\alpha - 2Q}{\gamma} - w - c = 0,$$

which gives the unconstrained optimal quantity:

$$Q^{unc} = \frac{\alpha - \gamma(w + c)}{2}.$$

The second-order derivative of (3.16) with respect to Q is:

$$\frac{d^2\Pi^R(Q)}{dQ^2} = -\frac{2}{\gamma} < 0,$$

since $\gamma > 0$, so $\Pi^R(Q)$ is strictly concave in Q. Setting the first-order derivative of (3.16) with respect to Q equal to zero gives the unconstrained first-order condition:

$$\frac{d\Pi^R(Q)}{dQ} = \frac{\alpha - 2Q}{\gamma} - (w + c) = 0. \tag{3.20}$$

Solving (3.20) for Q:

$$Q^{unc}(w) = \frac{\alpha - \gamma(w + c)}{2}.$$

Since the objective is strictly concave, the constrained maximum over $[0, R(a)]$ is the smaller of the unconstrained maximizer and the upper bound of the feasible interval, that is, $Q^* = \min\{Q^{unc}(w),\ R(a)\}$, giving (3.19). □

3.2.2 Collector's Problem

Anticipating the remanufacturer's best response (3.19), the collector's problem depends on whether the remanufacturer's unconstrained optimal quantity exceeds the available supply or not. We refer to the value of w at which the two are exactly equal as the threshold transfer price. Setting $R(a) = Q^{unc}(w)$ and solving for w:

$$\delta(a - \beta) = \frac{\alpha - \gamma(w + c)}{2} \quad \text{gives} \tag{3.21}$$

$$\bar{w}(a) = \frac{\alpha - 2\delta(a - \beta)}{\gamma} - c.$$

When $w \leq \bar{w}(a)$, the remanufacturer's demand under the unconstrained solution exceeds the available supply, and by (3.19) the remanufacturer is supply-constrained: $Q^* = R(a)$. We refer to this case as Region 1. When $w > \bar{w}(a)$, the remanufacturer's unconstrained demand falls short of the available supply, and the remanufacturer is demand-constrained: $Q^* = \frac{\alpha - \gamma(w+c)}{2}$. We refer to this case as Region 2.

The collector's profit is therefore a piecewise function of (a, w):

$$\Pi^{Col}(a, w) = \begin{cases} \Pi^{Col,1}(a, w) & \text{if } w \leq \bar{w}(a), \\ \Pi^{Col,2}(a, w) & \text{if } w > \bar{w}(a), \end{cases}$$

where

$$\Pi^{Col,1}(a, w) = w\,\delta(a - \beta) - a\,\delta(a - \beta) = \delta(a - \beta)(w - a), \tag{3.22}$$

$$\Pi^{Col,2}(a, w) = w\,\frac{\alpha - \gamma(w + c)}{2} - a\,\delta(a - \beta). \tag{3.23}$$

In Region 1, the collector's revenue is w per unit on the full supply $R(a) = \delta(a - \beta)$, since the remanufacturer purchases all available cores. In Region 2, the collector's revenue is w per unit on only the remanufacturer's unconstrained demand $Q^{unc}(w)$, since the remanufacturer does not purchase the full supply. In both regions, the collector pays acquisition cost a on every unit collected, $R(a) = \delta(a - \beta)$. The collector's problem is:

$$\begin{aligned} \max_{a,\, w} \quad & \Pi^{Col}(a, w) \\ \text{s.t.} \quad & a \geq \beta && (3.24) \\ & w \geq 0. && (3.25) \end{aligned}$$

Constraint (3.24) ensures that the acquisition price is sufficient to attract any supply. Constraint (3.25) ensures that the transfer price is non-negative.

Lemma 3.4. *$\Pi^{Col}(a, w)$ is continuous at $w = \bar{w}(a)$ for all $a \geq \beta > 0$.*

Proof. At $w = \bar{w}(a)$, the defining relation (3.21) gives $\delta(a - \beta) = \frac{\alpha - \gamma(\bar{w}(a) + c)}{2}$. Evaluating $\Pi^{Col,1}(a, w)$ at $w = \bar{w}(a)$:

$$\Pi^{Col,1}(a, \bar{w}(a)) = \bar{w}(a) \cdot \delta(a - \beta) - a \cdot \delta(a - \beta) = \delta(a - \beta)(\bar{w}(a) - a).$$

Evaluating $\Pi^{Col,2}(a, w)$ at $w = \bar{w}(a)$ and substituting $\frac{\alpha - \gamma(\bar{w}(a) + c)}{2} = \delta(a - \beta)$:

$$\Pi^{Col,2}(a, \bar{w}(a)) = \bar{w}(a) \cdot \delta(a - \beta) - a \cdot \delta(a - \beta) = \delta(a - \beta)(\bar{w}(a) - a).$$

Since $\Pi^{Col,1}(a, \bar{w}(a)) = \Pi^{Col,2}(a, \bar{w}(a))$, $\Pi^{Col}(a, w)$ is continuous at $w = \bar{w}(a)$. □

Lemma 3.5. *$\Pi^{Col}(a, w)$ is not continuously differentiable at $w = \bar{w}(a)$ for all $a > \beta$. Specifically, the partial derivative with respect to w has a jump discontinuity at $w = \bar{w}(a)$.*

Proof. We compute the left-hand and right-hand limits of $\partial \Pi^{Col}(a, w)/\partial w$ at $w = \bar{w}(a)$ separately. Approaching from Region 1:

$$\lim_{w \uparrow \bar{w}(a)} \frac{\partial \Pi^{Col}(a, w)}{\partial w} = \left. \frac{\partial \Pi^{Col,1}(a, w)}{\partial w} \right|_{w = \bar{w}(a)} = \delta(a - \beta).$$

Approaching from Region 2:

$$\lim_{w \downarrow \bar{w}(a)} \frac{\partial \Pi^{Col}(a, w)}{\partial w} = \left. \frac{\partial \Pi^{Col,2}(a, w)}{\partial w} \right|_{w = \bar{w}(a)} = \frac{\alpha - \gamma(2\bar{w}(a) + c)}{2}.$$

Substituting $\delta(a-\beta) = \frac{\alpha-\gamma(\bar{w}(a)+c)}{2}$ into the right-hand limit:

$$\left.\frac{\partial \Pi^{Col,2}(a,w)}{\partial w}\right|_{w=\bar{w}(a)} = \delta(a-\beta) - \frac{\gamma\bar{w}(a)}{2}.$$

The jump in the partial derivative at $w = \bar{w}(a)$ is therefore:

$$\lim_{w\uparrow\bar{w}(a)} \frac{\partial \Pi^{Col}(a,w)}{\partial w} - \lim_{w\downarrow\bar{w}(a)} \frac{\partial \Pi^{Col}(a,w)}{\partial w} = \frac{\gamma\bar{w}(a)}{2},$$

which is strictly positive since $\bar{w}(a) > 0$ for $a > \beta$ under market viability. Therefore $\Pi^{Col}(a,w)$ is not continuously differentiable at $w = \bar{w}(a)$. □

Since $\Pi^{Col}(a,w)$ is not differentiable at the boundary between the two regions, we cannot apply a single joint first-order condition across the whole domain. We therefore analyze each region separately, checking in each case whether a joint maximum in (a,w) exists within the region, before combining the two regional solutions.

In Region 1, when $w \leq \bar{w}(a)$, the remanufacturer's best response (3.19) gives $Q^* = R(a)$: the supply constraint is active and the remanufacturer acquires all available cores. The collector's profit is given by (3.22).

Lemma 3.6. $\Pi^{Col,1}(a,w)$ *has no joint maximum within Region 1.*

Proof. The second-order partial derivatives of $\Pi^{Col,1}(a,w) = \delta(a-\beta)(w-a)$ are:

$$\frac{\partial^2 \Pi^{Col,1}(a,w)}{\partial a^2} = -2\delta, \quad \frac{\partial^2 \Pi^{Col,1}(a,w)}{\partial w^2} = 0, \quad \frac{\partial^2 \Pi^{Col,1}(a,w)}{\partial a \partial w} = \frac{\partial^2 \Pi^{Col,1}(a,w)}{\partial w \partial a} = \delta.$$

The Hessian matrix is:

$$H_1 = \begin{pmatrix} -2\delta & \delta \\ \delta & 0 \end{pmatrix},$$

with determinant $\det(H_1) = -\delta^2 < 0$. Since H_1 is indefinite, $\Pi^{Col,1}(a,w)$ has no joint maximum within Region 1. □

As a increases, $\Pi^{Col,1}(a,w)$ is subject to two opposing effects: a larger quantity is collected, $\delta(a-\beta)$, which raises revenue if the markup $w-a$ remains positive, but the markup itself shrinks as a rises toward w, which lowers profit per unit. These competing effects give $\Pi^{Col,1}(a,w)$ an optimum in a alone. In contrast, raising w while holding a fixed always increases the markup with no offsetting cost, so $\Pi^{Col,1}(a,w)$

is simply increasing in w throughout Region 1, with no optimum in w. The collector therefore wants to raise w without bound within Region 1, and is stopped only by leaving the region altogether, not by any internal tradeoff, which is exactly why no joint maximum in (a, w) exists within Region 1.

Since no joint maximum exists within Region 1, we instead examine how $\Pi^{Col,1}(a, w)$ behaves along the boundary of Region 1.

Lemma 3.7. $\Pi^{Col,1}(a, w)$ *is strictly increasing in* w *for all* $a > \beta$.

Proof. Differentiating (3.22) with respect to w:

$$\frac{\partial \Pi^{Col,1}(a, w)}{\partial w} = \delta(a - \beta),$$

which is strictly positive for all $a > \beta$. □

By Lemma 3.7, the collector always benefits from raising w within Region 1, so the optimal w in Region 1 lies at the upper boundary of the region, $w = \bar{w}(a)$. We substitute $w = \bar{w}(a)$, given by (3.21), into (3.22):

$$\begin{aligned}\Pi^{Col,1}(a, \bar{w}(a)) &= \delta(a - \beta)\left[\frac{\alpha - 2\delta(a - \beta)}{\gamma} - c - a\right] \\ &= \frac{\delta(a - \beta)}{\gamma}\big[\alpha + 2\delta\beta - \gamma c - (2\delta + \gamma)a\big].\end{aligned}$$

This reduces the problem in Region 1 to a single-variable optimization in a:

$$\Pi^{Col,1}(a) = \frac{\delta(a - \beta)}{\gamma}\big[\alpha + 2\delta\beta - \gamma c - (2\delta + \gamma)a\big]. \tag{3.26}$$

Lemma 3.8. $\Pi^{Col,1}(a)$ *is strictly concave in* a.

Proof. Differentiating (3.26) once with respect to a:

$$\frac{d\Pi^{Col,1}(a)}{da} = \frac{\delta}{\gamma}\big[\alpha + 4\delta\beta + \gamma\beta - \gamma c - 2(2\delta + \gamma)a\big].$$

Differentiating once more, the second derivative is:

$$\frac{d^2\Pi^{Col,1}(a)}{da^2} = -\frac{2\delta(2\delta + \gamma)}{\gamma} < 0.$$

Therefore $\Pi^{Col,1}(a)$ is strictly concave in a. □

Lemma 3.9. *The optimal solution in Region 1 is:*

$$a^{1*} = \frac{\alpha + 4\delta\beta + \gamma\beta - \gamma c}{2(2\delta + \gamma)}, \tag{3.27}$$

$$w^{1*} = \frac{(\delta + \gamma)(\alpha - \gamma c) + \delta\gamma\beta}{\gamma(2\delta + \gamma)}, \tag{3.28}$$

$$Q^{1*} = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(2\delta + \gamma)}, \tag{3.29}$$

$$p^{1*} = \frac{(3\delta + 2\gamma)\alpha + \delta\gamma(\beta + c)}{2\gamma(2\delta + \gamma)}, \tag{3.30}$$

$$\Pi^{Col,1}(a^{1*}, w^{1*}) = \frac{\delta[\alpha - \gamma(\beta + c)]^2}{4\gamma(2\delta + \gamma)}, \tag{3.31}$$

$$\Pi^{R}(Q^{1*}) = \frac{\delta^2[\alpha - \gamma(\beta + c)]^2}{4\gamma(2\delta + \gamma)^2}. \tag{3.32}$$

Proof. By Lemma 3.8, $\Pi^{Col,1}(a)$ is strictly concave, so the unique maximizer is found by setting the first-order condition equal to zero:

$$\frac{d\Pi^{Col,1}(a)}{da} = \alpha + 4\delta\beta + \gamma\beta - \gamma c - 2(2\delta + \gamma)a = 0.$$

Solving for a:

$$a^{1*} = \frac{\alpha + 4\delta\beta + \gamma\beta - \gamma c}{2(2\delta + \gamma)}.$$

We verify $a^{1*} \geq \beta$. Computing $a^{1*} - \beta$:

$$a^{1*} - \beta = \frac{\alpha + 4\delta\beta + \gamma\beta - \gamma c}{2(2\delta + \gamma)} - \beta = \frac{\alpha - \gamma(\beta + c)}{2(2\delta + \gamma)} > 0,$$

where the strict inequality follows since the numerator $\alpha - \gamma(\beta + c) > 0$ by market viability, and the denominator $2(2\delta + \gamma) > 0$ since $\delta, \gamma > 0$ by assumption. Hence constraint (3.24) is satisfied with strict inequality.

For Q^{1*}, we substitute a^{1*} into $Q = \delta(a - \beta)$:

$$Q^{1*} = \delta \cdot \frac{\alpha + 4\delta\beta + \gamma\beta - \gamma c - 2(2\delta + \gamma)\beta}{2(2\delta + \gamma)} = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(2\delta + \gamma)}.$$

For w^{1*}, we substitute Q^{1*} into the threshold condition $\bar{w}(a) = \frac{\alpha - 2\delta(a - \beta)}{\gamma} - c$, recalling that $2Q^{1*} = \delta[\alpha - \gamma(\beta + c)]/(2\delta + \gamma)$:

$$w^{1*} = \frac{\alpha - \frac{\delta[\alpha - \gamma(\beta + c)]}{2\delta + \gamma}}{\gamma} - c = \frac{(\delta + \gamma)(\alpha - \gamma c) + \delta\gamma\beta}{\gamma(2\delta + \gamma)}.$$

We verify $w^{1*} \geq 0$. Since all parameters are strictly positive and $\alpha > \gamma(\beta + c) > \gamma c$ by market viability, both terms $(\delta + \gamma)(\alpha - \gamma c)$ and $\delta\gamma\beta$ in the numerator are strictly positive, so $w^{1*} > 0$. Hence constraint (3.25) is satisfied with strict inequality.

For p^{1*}, we substitute Q^{1*} into $p = (\alpha - Q)/\gamma$:

$$p^{1*} = \frac{2(2\delta + \gamma)\alpha - \delta[\alpha - \gamma(\beta + c)]}{2\gamma(2\delta + \gamma)} = \frac{(3\delta + 2\gamma)\alpha + \delta\gamma(\beta + c)}{2\gamma(2\delta + \gamma)}.$$

For $\Pi^{Col,1}(a^{1*}, w^{1*})$, the bracketed term of (3.26) evaluates to $\frac{\alpha - \gamma(\beta+c)}{2}$ at a^{1*}, giving:

$$\Pi^{Col,1}(a^{1*}, w^{1*}) = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(2\delta + \gamma)} \cdot \frac{\alpha - \gamma(\beta + c)}{2\gamma} = \frac{\delta[\alpha - \gamma(\beta + c)]^2}{4\gamma(2\delta + \gamma)}.$$

For $\Pi^R(Q^{1*})$, we use $\Pi^R(Q) = (p - w - c)Q$, computing $p^{1*} - w^{1*} - c = \frac{\delta[\alpha - \gamma(\beta+c)]}{2\gamma(2\delta+\gamma)}$:

$$\Pi^R(Q^{1*}) = \frac{\delta[\alpha - \gamma(\beta + c)]}{2\gamma(2\delta + \gamma)} \cdot \frac{\delta[\alpha - \gamma(\beta + c)]}{2(2\delta + \gamma)} = \frac{\delta^2[\alpha - \gamma(\beta + c)]^2}{4\gamma(2\delta + \gamma)^2}.$$

□

In Region 2 ($w > \bar{w}(a)$), the remanufacturer's best response (3.19) gives $Q^* = \frac{\alpha - \gamma(w+c)}{2}$: the remanufacturer is demand-constrained and does not purchase all available cores. The collector's profit is given by (3.23).

Lemma 3.10. *$\Pi^{Col,2}(a, w)$ is jointly strictly concave in (a, w).*

Proof. The second-order partial derivatives of $\Pi^{Col,2}(a, w)$ are:

$$\frac{\partial^2\Pi^{Col,2}(a,w)}{\partial a^2} = -2\delta, \quad \frac{\partial^2\Pi^{Col,2}(a,w)}{\partial w^2} = -\gamma, \quad \frac{\partial^2\Pi^{Col,2}(a,w)}{\partial a \partial w} = \frac{\partial^2\Pi^{Col,2}(a,w)}{\partial w \partial a} = 0.$$

The Hessian matrix is:

$$H_2 = \begin{pmatrix} -2\delta & 0 \\ 0 & -\gamma \end{pmatrix}.$$

Since $\delta, \gamma > 0$, the diagonal entries are both negative and the off-diagonal entries are zero, so H_2 is negative definite. Therefore $\Pi^{Col,2}(a, w)$ is jointly strictly concave in (a, w). □

Although $\Pi^{Col,2}(a, w)$ is jointly concave by Lemma 3.10, its unconstrained joint maximum does not lie within the feasible region. Setting both partial derivatives

of $\Pi^{Col,2}(a, w)$ to zero:

$$\frac{\partial \Pi^{Col,2}(a,w)}{\partial a} = -\delta(2a - \beta) = 0 \quad \text{gives} \quad a = \frac{\beta}{2},$$
$$\frac{\partial \Pi^{Col,2}(a,w)}{\partial w} = \frac{\alpha - \gamma(2w + c)}{2} = 0 \quad \text{gives} \quad w = \frac{\alpha - \gamma c}{2\gamma}.$$

Since $\beta > 0$, the unconstrained solution $a = \beta/2$ violates constraint (3.24), which requires $a \geq \beta$. The unconstrained maximizer is therefore infeasible, and the true maximum over the feasible region must lie on its boundary. We establish this directly below by examining the sign of $\partial \Pi^{Col,2}(a, w)/\partial a$ throughout the feasible region.

Lemma 3.11. $\Pi^{Col,2}(a, w)$ *is strictly decreasing in* a *for all* $a > \beta$.

Proof. Differentiating (3.23) with respect to a:

$$\frac{\partial \Pi^{Col,2}(a,w)}{\partial a} = -\delta(2a - \beta),$$

which is strictly negative for all $a > \beta$. □

By Lemma 3.11, the collector always benefits from lowering a within Region 2: any unit collected beyond the remanufacturer's unconstrained demand $Q^{unc}(w)$ raises acquisition cost without raising revenue, since the remanufacturer purchases only $Q^{unc}(w)$ in this region. The optimal a in Region 2 therefore lies at the lower boundary of the region, $a = \bar{a}(w)$, where $\bar{a}(w)$ is obtained by solving (3.21) for a:

$$\bar{a}(w) = \beta + \frac{\alpha - \gamma(w + c)}{2\delta}. \tag{3.33}$$

Substituting (3.33) into (3.23) reduces the problem in Region 2 to a single-variable optimization in w:

$$\Pi^{Col,2}(w) = \frac{\alpha - \gamma(w + c)}{2} \left[w - \beta - \frac{\alpha - \gamma(w + c)}{2\delta} \right]. \tag{3.34}$$

Lemma 3.12. $\Pi^{Col,2}(w)$ *is strictly concave in* w.

Proof. Differentiating (3.34) once with respect to w:

$$\frac{d\Pi^{Col,2}(w)}{dw} = -\frac{\gamma}{2} \left[w - \beta - \frac{\alpha - \gamma(w + c)}{2\delta} \right] + \frac{\alpha - \gamma(w + c)}{2} \cdot \frac{2\delta + \gamma}{2\delta}.$$

Differentiating once more, the second derivative is:

$$\frac{d^2\Pi^{Col,2}(w)}{dw^2} = -\frac{\gamma(2\delta + \gamma)}{2\delta} < 0.$$

Therefore $\Pi^{Col,2}(w)$ is strictly concave in w. □

Lemma 3.13. *The optimal solution in Region 2 is:*

$$w^{2*} = \frac{(\delta+\gamma)(\alpha-\gamma c)+\delta\gamma\beta}{\gamma(2\delta+\gamma)}, \tag{3.35}$$

$$a^{2*} = \frac{\alpha+4\delta\beta+\gamma\beta-\gamma c}{2(2\delta+\gamma)}, \tag{3.36}$$

$$Q^{2*} = \frac{\delta[\alpha-\gamma(\beta+c)]}{2(2\delta+\gamma)}, \tag{3.37}$$

$$p^{2*} = \frac{(3\delta+2\gamma)\alpha+\delta\gamma(\beta+c)}{2\gamma(2\delta+\gamma)}, \tag{3.38}$$

$$\Pi^{Col,2}(a^{2*},w^{2*}) = \frac{\delta[\alpha-\gamma(\beta+c)]^2}{4\gamma(2\delta+\gamma)}, \tag{3.39}$$

$$\Pi^{R}(Q^{2*}) = \frac{\delta^2[\alpha-\gamma(\beta+c)]^2}{4\gamma(2\delta+\gamma)^2}. \tag{3.40}$$

Proof. By Lemma 3.12, $\Pi^{Col,2}(w)$ is strictly concave, so the unique maximizer is found by setting the first-order condition equal to zero. Multiplying through by 4δ:

$$-2\delta\gamma\left[w-\beta-\frac{\alpha-\gamma(w+c)}{2\delta}\right]+[\alpha-\gamma(w+c)](2\delta+\gamma)=0$$

$$-2\delta\gamma(w-\beta)+[\alpha-\gamma(w+c)](2\delta+2\gamma)=0$$

$$(\delta+\gamma)[\alpha-\gamma(w+c)]=\delta\gamma(w-\beta).$$

Solving for w:

$$w^{2*} = \frac{(\delta+\gamma)(\alpha-\gamma c)+\delta\gamma\beta}{\gamma(2\delta+\gamma)}.$$

For a^{2*}, we substitute w^{2*} into $\bar{a}(w) = \beta + \frac{\alpha-\gamma(w+c)}{2\delta}$ from (3.33). Computing $\alpha - \gamma(w^{2*}+c) = \frac{\delta[\alpha-\gamma(\beta+c)]}{2\delta+\gamma}$ first, and then substituting:

$$a^{2*} = \beta + \frac{\alpha-\gamma(\beta+c)}{2(2\delta+\gamma)} = \frac{\alpha+4\delta\beta+\gamma\beta-\gamma c}{2(2\delta+\gamma)}.$$

The expressions for Q^{2*}, p^{2*}, $\Pi^{Col,2}(a^{2*},w^{2*})$, and $\Pi^{R}(Q^{2*})$ follow by the same substitutions used in the proof of Lemma 3.9, since $a^{2*} = a^{1*}$ as shown below. □

Comparing (3.27)–(3.32) with (3.35)–(3.40) shows that both regions yield the same optimal solution:

$$a^{1*} = a^{2*}, \quad w^{1*} = w^{2*}, \quad Q^{1*} = Q^{2*}, \quad p^{1*} = p^{2*},$$

$$\Pi^{Col,1}(a^{1*},w^{1*}) = \Pi^{Col,2}(a^{2*},w^{2*}), \quad \Pi^{R}(Q^{1*}) = \Pi^{R}(Q^{2*}).$$

The fact that both regions yield the same optimal solution has a clear economic interpretation. In Region 1, the remanufacturer is supply-constrained and purchases all available cores. The collector pays acquisition costs on all collected units regardless of the transfer price, so raising w increases revenue without increasing cost; the collector therefore raises w until supply exactly meets the remanufacturer's unconstrained demand at the boundary. In Region 2, the remanufacturer is demand-constrained and does not purchase all available cores. The collector pays acquisition costs on units the remanufacturer does not buy, so reducing a lowers cost without reducing transfer revenue; the collector therefore reduces a until supply falls to meet the remanufacturer's unconstrained demand at the boundary. Both adjustments drive the system to the same boundary point, where supply exactly equals the remanufacturer's unconstrained optimal demand, yielding a unique decentralized equilibrium.

Proposition 3.2. *Since both regions yield the same solution, the unique decentralized equilibrium* $(a^{D*}, w^{D*}, Q^{D*}, p^{D*})$ *is given by:*

$$a^{D*} = (3.27),$$
$$w^{D*} = (3.28),$$
$$Q^{D*} = (3.29),$$
$$p^{D*} = (3.30),$$

with collector profit $\Pi^{Col}(a^{D*}, w^{D*})$ *given by* (3.31)*, remanufacturer profit* $\Pi^{R}(Q^{D*})$ *given by* (3.32)*, and total profit* $\Pi^{D}(a^{D*}, w^{D*}) = \Pi^{Col}(a^{D*}, w^{D*}) + \Pi^{R}(Q^{D*})$.

In the centralized setting, a single integrated firm internalizes all costs and revenues and therefore chooses the system-wide optimal quantity Q^{C*}. In the decentralized setting, the collector and the remanufacturer each maximize their own profit, leading to a quantity Q^{D*} that is strictly less than Q^{C*}. This underproduction arises from double marginalization: the collector marks up the transfer price above acquisition cost, which raises the remanufacturer's effective marginal cost and leads it to order less than the system optimum.

Proposition 3.3. *The decentralized equilibrium produces strictly less quantity and strictly lower total profit than the centralized optimum:*

$$Q^{D*} < Q^{C*} \quad \textit{and} \quad \Pi^{D}(a^{D*}, w^{D*}) < \Pi^{C}(a^{C*}, p^{C*}).$$

Proof. For quantities, since $2\delta + \gamma > \delta + \gamma$ for all $\delta > 0$:

$$Q^{D*} = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(2\delta + \gamma)} < \frac{\delta[\alpha - \gamma(\beta + c)]}{2(\delta + \gamma)} = Q^{C*}.$$

For profits, we compute $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})$ directly:

$$\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) = \frac{\delta[\alpha - \gamma(\beta + c)]^2}{4\gamma} \cdot \frac{(2\delta + \gamma)^2 - (\delta + \gamma)(3\delta + \gamma)}{(\delta + \gamma)(2\delta + \gamma)^2}.$$

Expanding the numerator of the second factor:

$$(2\delta + \gamma)^2 - (\delta + \gamma)(3\delta + \gamma) = (4\delta^2 + 4\delta\gamma + \gamma^2) - (3\delta^2 + 4\delta\gamma + \gamma^2) = \delta^2.$$

Since $\delta^2 > 0$, we obtain:

$$\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) = \frac{\delta^3[\alpha - \gamma(\beta + c)]^2}{4\gamma(\delta + \gamma)(2\delta + \gamma)^2} > 0. \tag{3.41}$$

The source of this inefficiency is the markup $w^{D*} - a^{D*} = \frac{\alpha - \gamma(\beta + c)}{2\gamma} > 0$, which inflates the remanufacturer's effective marginal cost from $a^{D*} + c$ to $w^{D*} + c$, causing it to order strictly less than the system optimum. □

Corollary 3.1. *The relative efficiency loss that is the efficiency loss as a fraction of the optimized centralized profit is:*

$$\frac{\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})}{\Pi^C(a^{C*}, p^{C*})} = \frac{\delta^2}{(2\delta + \gamma)^2},$$

which depends only on δ and γ, not on α, β, or c. This ratio is strictly positive and bounded above by $1/4$:

$$\frac{\delta^2}{(2\delta + \gamma)^2} < \frac{\delta^2}{4\delta^2} = \frac{1}{4},$$

since $(2\delta + \gamma)^2 > 4\delta^2$ for all $\gamma > 0$. The bound is approached asymptotically as supply becomes infinitely elastic:

$$\lim_{\delta \to \infty} \frac{\delta^2}{(2\delta + \gamma)^2} = \frac{1}{4}.$$

Proof. Dividing (3.41) by $\Pi^C(a^{C*}, p^{C*}) = \delta[\alpha - \gamma(\beta + c)]^2/[4\gamma(\delta + \gamma)]$:

$$\frac{\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})}{\Pi^C(a^{C*}, p^{C*})} = \frac{\delta^3[\alpha - \gamma(\beta + c)]^2}{4\gamma(\delta + \gamma)(2\delta + \gamma)^2} \cdot \frac{4\gamma(\delta + \gamma)}{\delta[\alpha - \gamma(\beta + c)]^2} = \frac{\delta^2}{(2\delta + \gamma)^2}.$$

The bound follows since $(2\delta + \gamma)^2 - 4\delta^2 = \gamma(4\delta + \gamma) > 0$ for all $\gamma > 0$, so $(2\delta + \gamma)^2 > 4\delta^2$, giving $\delta^2/(2\delta + \gamma)^2 < 1/4$. The limit $\lim_{\delta \to \infty} \delta^2/(2\delta + \gamma)^2 = 1/4$ follows by dividing numerator and denominator by δ^2. □

3.3 Coordination

A contract *coordinates* the supply chain if it induces the decentralized system to achieve the centralized quantity Q^{C*}, given in (3.12), and the total profit $\Pi^C(a^{C*}, p^{C*})$, given in (3.14). A contract satisfies *individual rationality* if both the collector and the remanufacturer earn at least their decentralized equilibrium profits, $\Pi^{Col}(a^{D*}, w^{D*})$ and $\Pi^R(Q^{D*})$ respectively, so that both parties voluntarily participate. These definitions follow the coordination framework of Cachon [5]. Since $\Pi^C(a^{C*}, p^{C*}) > \Pi^D(a^{D*}, w^{D*})$ by Proposition 3.3, the coordination surplus $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) > 0$ is always available to satisfy individual rationality. We analyze two coordination contracts: a two-part tariff and a revenue sharing implementable contract.

3.3.1 Two-Part Tariff

Under a two-part tariff contract, the collector charges a per-unit transfer price w and a fixed fee $F \geq 0$ paid by the remanufacturer to the collector regardless of the quantity sold. The fixed fee does not affect the remanufacturer's marginal cost and therefore does not distort the quantity decision; it serves purely as a profit allocation mechanism. The profit functions under the two-part tariff are:

$$\Pi^R_{TPT}(Q) = \frac{\alpha - Q}{\gamma} \cdot Q - (w + c) \cdot Q - F,$$
$$\Pi^{Col}_{TPT}(a, w, F) = (w - a) \cdot R(a) + F.$$

We present below the equilibrium under the two-part tariff contract, characterized by backward induction as in Section 3.2.

3.3.1.1 Remanufacturer's Problem

For given (a, w, F) set by the collector, the remanufacturer chooses Q to maximize $\Pi^R_{TPT}(Q)$:

$$\begin{aligned} \max_{Q} \quad & \Pi^R_{TPT}(Q) \\ \text{where} \quad & 0 \leq Q \leq R(a). \end{aligned} \tag{3.42}$$

Since F does not appear in the first-order condition with respect to Q, the remanufacturer's problem here is structurally identical to the standard decentralized problem of Section 3.2. By the same argument as Lemma 3.3, the remanufacturer's optimal quantity is:

$$Q^*_{TPT}(a, w) = \min\left\{\frac{\alpha - \gamma(w + c)}{2},\ R(a)\right\}. \tag{3.43}$$

3.3.1.2 Collector's Problem

The collector chooses (a, w, F) to maximize $\Pi^{Col}_{TPT}(a, w, F)$:

$$\begin{aligned}
\max_{a,\,w,\,F} \quad & \Pi^{Col}_{TPT}(a, w, F) \\
\text{where} \quad & a \geq \beta && (3.44)\\
& w \geq 0 && (3.45)\\
& F \geq 0 && (3.46)\\
& \Pi^R_{TPT}(Q^*_{TPT}) \geq \Pi^R(Q^{D*}). && (3.47)
\end{aligned}$$

Constraint (3.44) ensures the acquisition price attracts supply. Constraint (3.45) ensures the transfer price is non-negative. Constraint (3.46) ensures the fixed fee is non-negative, since the remanufacturer pays the collector, not the other way around. Constraint (3.47) is the individual rationality constraint for the remanufacturer: the remanufacturer must earn at least its decentralized profit $\Pi^R(Q^{D*})$, or it would prefer the decentralized contract instead.

Since the collector sets F as large as possible subject to the remanufacturer's individual rationality constraint, and the remanufacturer's gross profit before the fixed fee is $\Pi^R_{gross}(Q^{C*}) = \Pi^R_{TPT}(Q^{C*}) + F$, setting constraint (3.47) to equality gives the coordinating fee interval in Proposition 3.4.

Proposition 3.4. *The two-part tariff contract that coordinates the supply chain sets the acquisition price at $a = a^{C*}$, the transfer price at*

$$w^{TPT} = \frac{\alpha + \delta\beta - \gamma c}{\delta + \gamma}, \tag{3.48}$$

and the fixed fee at

$$F = \frac{\delta^3(3\delta + 2\gamma)[\alpha - \gamma(\beta + c)]^2}{4\gamma(\delta + \gamma)^2(2\delta + \gamma)^2}. \tag{3.49}$$

Under this contract, the collector earns strictly more than under the decentralized equilibrium, and the remanufacturer earns exactly its decentralized profit.

Proof. Since the collector effectively maximizes total system profit when he chooses $a = a^{C*}$, the centralized optimal acquisition price, together with a transfer price w^{TPT} that induces the remanufacturer to choose Q^{C*}. For the remanufacturer to choose Q^{C*}, its unconstrained optimal quantity must equal Q^{C*}:

$$Q^{unc}(w^{TPT}) = \frac{\alpha - \gamma(w^{TPT} + c)}{2} = Q^{C*} = \frac{\delta[\alpha - \gamma(\beta + c)]}{2(\delta + \gamma)}. \tag{3.50}$$

Multiplying (3.50) by 2 and rearranging:

$$\gamma w^{TPT} = \alpha - \gamma c - \frac{\delta[\alpha - \gamma(\beta + c)]}{\delta + \gamma} = \frac{\gamma(\alpha + \delta\beta - \gamma c)}{\delta + \gamma}.$$

Dividing both sides by γ gives (3.48).

We verify $w^{TPT} \geq 0$. Since $\alpha > \gamma c$ by market viability and $\delta\beta > 0$, (3.48) gives $w^{TPT} > 0$.

We also verify that the remanufacturer can actually reach Q^{C*} under constraint (3.42), that is, that it is not rationed by the supply available. Since the collector sets $a = a^{C*}$, Lemma 3.2 gives $R(a^{C*}) = Q^{C*}$, so the minimum in (3.43) is $Q^*_{TPT} = \min\{Q^{C*}, Q^{C*}\} = Q^{C*}$, as required.

We now find the fixed fee F. Substituting a^{C*} and w^{TPT} into (3.13), the gross margin per unit for the remanufacturer is $p^{C*} - w^{TPT} - c = \delta[\alpha - \gamma(\beta + c)]/[2\gamma(\delta + \gamma)]$. Multiplying this margin by Q^{C*}, the remanufacturer's gross profit is:

$$\Pi^R_{gross}(Q^{C*}) = \frac{\delta^2[\alpha - \gamma(\beta + c)]^2}{4\gamma(\delta + \gamma)^2}.$$

Substituting $\Pi^R_{gross}(Q^{C*})$ derived above and $\Pi^R(Q^{D*})$ from Proposition 3.2, the fixed fee is given by (3.49).

We verify $F \geq 0$. Since $(\delta + \gamma)^2 < (2\delta + \gamma)^2$ for $\delta > 0$, we have $\Pi^R_{gross}(Q^{C*}) > \Pi^R(Q^{D*})$, so $F > 0$ by (3.49).

Under w^{TPT}, the remanufacturer chooses Q^{C*} by construction of (3.50), verified feasible above. The collector sets $a = a^{C*}$, since maximizing total system profit is equivalent to choosing $a = a^{C*}$. Therefore total system profit equals $\Pi^C(a^{C*}, p^{C*})$.

The remanufacturer earns $\Pi^R_{gross}(Q^{C*}) - F = \Pi^R(Q^{D*})$ by construction of F. The collector earns the remainder, $\Pi^C(a^{C*}, p^{C*}) - \Pi^R(Q^{D*})$, and since $\Pi^C(a^{C*}, p^{C*}) - \Pi^R(Q^{D*}) - \Pi^{Col}(a^{D*}, w^{D*}) = \Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) > 0$ by Proposition 3.3, the collector earns strictly more than $\Pi^{Col}(a^{D*}, w^{D*})$. □

The fixed fee F is the profit allocation instrument. The upper bound $\bar{F}$ ensures the remanufacturer earns at least its decentralized profit $\Pi^R(Q^{D*})$:

$$\bar{F} = (p^{C*} - w^{TPT} - c) \cdot Q^{C*} - \Pi^R(Q^{D*}).$$

The lower bound $\underline{F}$ ensures the collector earns at least its decentralized profit $\Pi^{Col}(a^{D*}, w^{D*})$:

$$\underline{F} = \Pi^{Col}(a^{D*}, w^{D*}) - (w^{TPT} - a^{C*}) \cdot Q^{C*}.$$

Theorem 3.1. *A feasible fixed fee $F \in [\underline{F},\ \bar{F}]$ satisfying both individual rationality constraints always exists.*

Proof. $\bar{F} - \underline{F} = \Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) > 0$ by Proposition 3.3, so $[\underline{F},\ \bar{F}]$ is non-empty. □

3.3.2 Revenue Sharing

Under a revenue sharing contract, the remanufacturer retains a fraction $\phi \in (0, 1)$ of sales revenue and pays a per-unit transfer price w_{rs} to the collector; the collector receives the remaining fraction $1 - \phi$ of revenue plus the transfer payment. Following Cachon [5], we treat (ϕ, w_{rs}) as a pair agreed jointly in advance, fixed before the game is played.

3.3.2.1 Remanufacturer's Problem

For given (a, w_{rs}) and agreed ϕ, the remanufacturer chooses Q to maximize its profit:

$$\Pi^R_{RS}(Q) = \phi \cdot \frac{\alpha - Q}{\gamma} \cdot Q - (w_{rs} + c) \cdot Q. \tag{3.51}$$

The remanufacturer's problem is:

$$\begin{aligned} \max_{Q} \quad & \Pi^{R}_{RS}(Q) \\ \text{where} \quad & 0 \le Q \le R(a). \end{aligned}$$

Lemma 3.14. *The optimal quantity is:*

$$Q^{*}_{RS}(a, w_{rs}) = \min\left\{ \frac{\phi\alpha - \gamma(w_{rs} + c)}{2\phi},\ R(a) \right\}. \tag{3.52}$$

Proof. The second-order derivative of (3.51) with respect to Q is $-2\phi/\gamma < 0$, so $\Pi^{R}_{RS}(Q)$ is strictly concave in Q. Setting the first-order derivative equal to zero:

$$\frac{d\Pi^{R}_{RS}(Q)}{dQ} = \phi \cdot \frac{\alpha - 2Q}{\gamma} - (w_{rs} + c) = 0.$$

Solving for Q:

$$Q^{unc}_{RS}(w_{rs}) = \frac{\phi\alpha - \gamma(w_{rs} + c)}{2\phi}. \tag{3.53}$$

Since the objective is strictly concave, the constrained maximum over $[0, R(a)]$ is $Q^{*}_{RS} = \min\{Q^{unc}_{RS}(w_{rs}),\ R(a)\}$, giving (3.52). □

3.3.2.2 Coordination via an Implementable Contract

Coordination requires the quantity actually transacted, $Q^{*}_{RS} = \min\{Q^{unc}_{RS}(w_{rs}),\ R(a)\}$ from (3.52), to equal Q^{C*}.

Theorem 3.2. *For any $\phi \in (0, 1)$, the revenue sharing contract with acquisition price $a = a^{C*}$ and transfer price $w^{coord}_{rs} = \phi(a^{C*} + c) - c$ coordinates the supply chain, achieving the centralized quantity Q^{C*} and total profit $\Pi^{C}(a^{C*}, p^{C*})$.*

Proof. We establish the contract in three steps: first, that coordination requires $a = a^{C*}$; second, that coordination alone restricts w_{rs} only to an interval; third, that the requirement that the remanufacturer receive share ϕ of system profit pins w_{rs} uniquely within that interval.

Step 1: the acquisition price. Suppose $R(a) > Q^{C*}$. Since remanufactured products cannot be stored or salvaged, units collected beyond Q^{C*} are never sold under a coordinating contract, and the collector pays acquisition cost on them without earning any offsetting revenue or revenue share. Reducing a slightly lowers

acquisition cost without reducing the transacted quantity, so $R(a) > Q^{C*}$ cannot hold under an optimal coordinating contract. Suppose instead $R(a) < Q^{C*}$. Then $Q^*_{RS} = \min\{Q^{unc}_{RS}(w_{rs}), R(a)\} \leq R(a) < Q^{C*}$ regardless of w_{rs}, so the remanufacturer is rationed below Q^{C*} and the contract fails to coordinate. Hence $R(a) = Q^{C*}$ exactly, and by Lemma 3.2 this requires $a = a^{C*}$.

Step 2: coordination restricts w_{rs} to an interval. With $a = a^{C*}$ we have $R(a^{C*}) = Q^{C*}$, so by (3.52)

$$Q^*_{RS} = \min\big\{Q^{unc}_{RS}(w_{rs}),\ Q^{C*}\big\},$$

and coordination, $Q^*_{RS} = Q^{C*}$, holds if and only if $Q^{unc}_{RS}(w_{rs}) \geq Q^{C*}$. Substituting (3.53), this is equivalent to

$$w_{rs} \ \leq\ \phi \cdot \frac{\alpha - 2Q^{C*}}{\gamma} - c. \tag{3.54}$$

Coordination therefore constrains w_{rs} from above but does not determine it uniquely: any transfer price satisfying (3.54) induces the centralized quantity, because the supply available at a^{C*} rations the remanufacturer to exactly Q^{C*}.

Step 3: the profit split determines w_{rs}. Under any contract satisfying Steps 1 and 2, the transacted quantity is Q^{C*} and the market price is p^{C*}, so

$$\begin{aligned}
\Pi^R_{RS} &= \phi\, p^{C*}Q^{C*} - (w_{rs} + c)\, Q^{C*},\\
\Pi^{Col}_{RS} &= (1-\phi)\, p^{C*}Q^{C*} + w_{rs}Q^{C*} - a^{C*}R(a^{C*}),
\end{aligned}$$

whose sum is $(p^{C*} - c - a^{C*})Q^{C*} = \Pi^C(a^{C*}, p^{C*})$ for every w_{rs}: the transfer price redistributes profit between the two parties without affecting the total. Requiring the remanufacturer to receive exactly share ϕ of system profit, $\Pi^R_{RS} = \phi\,\Pi^C(a^{C*}, p^{C*}) = \phi\,(p^{C*} - c - a^{C*})Q^{C*}$, and dividing by $Q^{C*} > 0$:

$$\phi\, p^{C*} - (w_{rs} + c) = \phi\,\big(p^{C*} - c - a^{C*}\big),$$

which gives $w_{rs} = \phi(a^{C*} + c) - c = w^{coord}_{rs}$. The collector then receives the complementary share $1 - \phi$.

Step 4: the contract satisfies (3.54). From (3.11) and (3.12),

$$2a^{C*} - \beta = \frac{\alpha + \delta\beta - \gamma c}{\delta + \gamma} = w^{TPT}, \qquad \frac{\alpha - 2Q^{C*}}{\gamma} = \frac{\alpha + \delta(\beta + c)}{\delta + \gamma} = w^{TPT} + c, \tag{3.55}$$

where w^{TPT} is the coordinating two-part tariff transfer price (3.48). Since $a^{C*} > \beta$, we have $a^{C*} + c < 2a^{C*} - \beta + c = w^{TPT} + c$, and therefore

$$w_{rs}^{coord} = \phi(a^{C*} + c) - c \;<\; \phi\big(w^{TPT} + c\big) - c = \phi \cdot \frac{\alpha - 2Q^{C*}}{\gamma} - c,$$

so (3.54) holds strictly. Consequently $Q_{RS}^{unc}(w_{rs}^{coord}) > Q^{C*}$: the remanufacturer would purchase more than Q^{C*} if it could, and is held to Q^{C*} by the supply $R(a^{C*})$. The transacted quantity is Q^{C*}, total system profit is $\Pi^C(a^{C*}, p^{C*})$, and the profits split as $(\phi, 1-\phi)$. □

Remark 3.1. Coordination alone does not determine the transfer price: every w_{rs} satisfying (3.54) induces the centralized quantity. What selects w_{rs}^{coord} is the requirement that the revenue share ϕ translate into a profit share ϕ. Under any smaller transfer price the chain still coordinates, but the collector retains less than $1 - \phi$ of system profit.

Remark 3.2. Setting $w_{rs}^{coord} = 0$ identifies a specific value of the revenue share at which coordination is achieved with no per-unit transfer payment at all; the collector then recovers its acquisition cost solely through the revenue share $1 - \phi$:

$$0 = \phi(a^{C*} + c) - c \quad \text{gives} \quad \phi = \frac{c}{a^{C*} + c}.$$

This value coincides exactly with ϕ_{min}, defined below in (3.57), the smallest revenue share for which $w_{rs}^{coord} \geq 0$. That is, $w_{rs}^{coord} = 0$ occurs precisely at the lower feasibility boundary of the coordinating contract family: for $\phi < \phi_{min}$, coordination would require a negative transfer price, which is infeasible by constraint (3.56) below.

We require $w_{rs} \geq 0$:

$$w_{rs} \geq 0. \tag{3.56}$$

We verify $w_{rs}^{coord} \geq 0$ directly. This requires $\phi(a^{C*} + c) \geq c$, that is:

$$\phi \geq \phi_{min} \equiv \frac{c}{a^{C*} + c}. \tag{3.57}$$

Since $a^{C*} > 0$, we have $\phi_{min} \in (0, 1)$.

Under the coordinating contract, the profits are:

$$\Pi_{RS}^{R}(\phi) = \phi \cdot \Pi^C(a^{C*}, p^{C*}),$$
$$\Pi_{RS}^{Col}(\phi) = (1 - \phi) \cdot \Pi^C(a^{C*}, p^{C*}).$$

Individual rationality requires both parties to earn at least their decentralized profits. For the remanufacturer, $\Pi^R_{RS}(\phi) \geq \Pi^R(Q^{D*})$ requires:

$$\phi \geq \frac{\Pi^R(Q^{D*})}{\Pi^C(a^{C*}, p^{C*})} \equiv \underline{\phi}. \tag{3.58}$$

For the collector, $\Pi^{Col}_{RS}(\phi) \geq \Pi^{Col}(a^{D*}, w^{D*})$ requires:

$$\phi \leq 1 - \frac{\Pi^{Col}(a^{D*}, w^{D*})}{\Pi^C(a^{C*}, p^{C*})} \equiv \bar{\phi}. \tag{3.59}$$

Theorem 3.3. *A feasible range of coordinating revenue shares $\phi \in [\underline{\phi}, \bar{\phi}]$ satisfying both individual rationality constraints always exists.*

Proof. Subtracting (3.58) from (3.59):

$$\bar{\phi} - \underline{\phi} = \frac{\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})}{\Pi^C(a^{C*}, p^{C*})}.$$

By Proposition 3.3, $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) > 0$, so $\bar{\phi} - \underline{\phi} > 0$, and the interval $[\underline{\phi}, \bar{\phi}]$ is non-empty. □

Proposition 3.5. *If, instead, the collector were free to optimize the transfer price w_{rs} as Stackelberg leader for a fixed, exogenously agreed $\phi \in (0, 1)$, the resulting equilibrium quantity $Q^*_{RS,eq}(\phi)$ would satisfy $Q^*_{RS,eq}(\phi) < Q^{C*}$ for every $\phi \in (0, 1)$. This freely-optimizing revenue sharing contract cannot coordinate the supply chain for any $\phi \in (0, 1)$, which is why (ϕ, w_{rs}) are instead fixed jointly in advance, as above.*

The full derivation of $Q^*_{RS,eq}(\phi)$, characterizing the collector's Stackelberg equilibrium across both regions of the remanufacturer's best response, and the proof of this proposition, are given in Appendix A. The source of this failure mirrors the inefficiency of the plain decentralized equilibrium (Proposition 3.3): even after the revenue share ϕ reduces the remanufacturer's effective margin, the collector, optimizing w_{rs} freely, still marks up the transfer price above acquisition cost, reproducing a form of double marginalization that no choice of ϕ alone can undo.

Both the two-part tariff and the revenue sharing contract coordinate the single-quality supply chain. The two-part tariff coordinates through a unique w^{TPT} together with a fixed fee $F > 0$ transferring profit from the remanufacturer to the collector. The

revenue sharing contract coordinates through a family of implementable contracts (ϕ, w_{rs}^{coord}) parametrized by $\phi \in [\underline{\phi}, \bar{\phi}]$, where for each ϕ the transfer price is uniquely determined by $w_{rs}^{coord} = \phi(a^{C*}+c)-c$, and the acquisition price is fixed at a^{C*}. Higher ϕ allocates more profit to the remanufacturer, and lower ϕ allocates more to the collector. Both contracts exploit the coordination surplus $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*}) > 0$ to satisfy individual rationality.

CHAPTER 4

MODEL WITH TWO QUALITY CLASSES OF RETURNS

This chapter extends the model with a single quality class of returns studied in Chapter 3 to a setting where there are two quality classes of used products, namely high quality and low quality. Cores from both quality classes are remanufactured into identical finished products.The model captures the additional complexity arising from quality-specific acquisition decisions and heterogeneous remanufacturing costs.

Section 4.1 derives the centralized solution, where a single decision maker jointly optimizes the acquisition prices (a_h, a_l), with the selling price p determined endogenously by market clearing, to maximize total supply chain profit. It further establishes the conditions under which both quality classes are actively acquired at the optimum.The centralized solution serves as the efficiency benchmark throughout the chapter. Section 4.2 analyzes the decentralized setting, where the collector and the remanufacturer are independent profit-maximizing firms interacting through a Stackelberg game. First, the collector sets acquisition prices (a_h, a_l) and transfer prices (w_h, w_l) for the high and low quality classes respectively as the leader. Then the remanufacturer chooses quantities (Q_h, Q_l) for high and low quality classes respectively as the follower. After the sequential decisions of both parties are resolved through backward induction, the decentralized equilibrium is characterized.

Section 4.3 designs contracts that eliminate the inefficiency of the decentralized equilibrium. Two contracts that coordinate the supply chain are characterized. A two-part tariff, consisting of quality-specific transfer prices and a fixed fee, achieves the centralized optimum and satisfies individual rationality for both parties. A revenue sharing contract, consisting of quality-specific transfer prices and a revenue share parameter, likewise achieves the centralized optimum and satisfies individual rationality

for both parties.

4.1 Centralized Setting

In the centralized setting, a single decision maker jointly optimizes the acquisition prices (a_h, a_l) and the selling price $p(a_h, a_l)$ to maximize total supply chain profit. The selling price $p(a_h, a_l)$ is determined endogenously by market clearing, as established below. The centralized solution serves as the efficiency benchmark throughout the chapter.

4.1.1 Model Formulation

The integrated firm collects cores of both quality classes and remanufactures all acquired units into identical finished products sold in a single market at price p. Recalling that $D(p) = \alpha - \gamma p$, $R_h(a_h) = \delta_h(a_h - \beta_h)$, and $R_l(a_l) = \delta_l(a_l - \beta_l)$, the total system profit consists of revenue from selling remanufactured products, minus remanufacturing costs, minus acquisition costs paid to core holders. Therefore the profit can be expressed as:

$$\Pi^C(a_h, a_l, p) = p \cdot D(p) - c_h R_h(a_h) - c_l R_l(a_l) - a_h R_h(a_h) - a_l R_l(a_l).$$

Substituting $D(p) = \alpha - \gamma p$, $R_h(a_h) = \delta_h(a_h - \beta_h)$, and $R_l(a_l) = \delta_l(a_l - \beta_l)$:

$$\Pi^C(a_h, a_l, p) = p(\alpha - \gamma p) - c_h\,\delta_h(a_h - \beta_h) - c_l\,\delta_l(a_l - \beta_l) - a_h\,\delta_h(a_h - \beta_h) - a_l\,\delta_l(a_l - \beta_l). \tag{4.1}$$

The centralized problem is:

$$\max_{a_h,\, a_l,\, p} \Pi^C(a_h, a_l, p)$$

subject to

$$\alpha - \gamma p \geq 0, \tag{4.3}$$

$$p \geq 0, \tag{4.4}$$

$$a_h \geq \beta_h, \tag{4.5}$$

$$a_l \geq \beta_l, \tag{4.6}$$

$$\delta_h(a_h - \beta_h) + \delta_l(a_l - \beta_l) - (\alpha - \gamma p) \geq 0. \tag{4.7}$$

Constraint (4.3) ensures that market demand $D(p) = \alpha - \gamma p$ is non-negative, bounding the selling price from above by α/γ. Constraint (4.4) ensures the selling price is non-negative. Together, (4.3) and (4.4) restrict the selling price to $0 \leq p \leq \alpha/\gamma$. Constraint (4.5) ensures the high-quality acquisition price is sufficient to attract any supply, since $R_h(a_h) = \delta_h(a_h - \beta_h) \geq 0$ requires $a_h \geq \beta_h$. Constraint (4.6) imposes the same requirement for low-quality returns. Constraint (4.7) states that the firm cannot sell more units than it collects: the quantity sold, $D(p) = \alpha - \gamma p$, cannot exceed the total quantity collected from both quality classes, $R_h(a_h) + R_l(a_l)$. We write this as an inequality rather than imposing equality directly, because the firm is in principle free to collect more cores than it sells. Below, we show that the optimal solution satisfies this constraint with equality, so that the firm never collects more than it sells.

Lemma 4.1. *At the optimal solution to the centralized problem, constraint* (4.7) *is always binding, that is, the firm collects exactly as many cores as it sells:* $R_h(a_h^*) + R_l(a_l^*) = D(p^*)$.

Proof. The Lagrange function is:

$$\mathcal{L}(a_h, a_l, p, \lambda, \mu_h, \mu_l, \mu_p) = \Pi^C(a_h, a_l, p) + \lambda\big[\delta_h(a_h - \beta_h) + \delta_l(a_l - \beta_l) - (\alpha - \gamma p)\big] + \mu_h(a_h - \beta_h) + \mu_l(a_l - \beta_l) + \mu_p \cdot p,$$

where $\lambda \geq 0$ is the multiplier for constraint (4.7), and $\mu_h, \mu_l, \mu_p \geq 0$ are the multipliers for constraints (4.5), (4.6), and (4.4) respectively. Constraint (4.3) is satisfied

strictly at the optimum under market viability and does not require an explicit multiplier. The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial a_h} = -\delta_h(2a_h - \beta_h + c_h) + \lambda\delta_h + \mu_h = 0, \tag{4.8}$$

$$\frac{\partial \mathcal{L}}{\partial a_l} = -\delta_l(2a_l - \beta_l + c_l) + \lambda\delta_l + \mu_l = 0, \tag{4.9}$$

$$\frac{\partial \mathcal{L}}{\partial p} = \alpha - 2\gamma p - \lambda\gamma + \mu_p = 0. \tag{4.10}$$

Solving (4.8) for a_h:

$$a_h = \frac{\beta_h - c_h + \lambda + \mu_h/\delta_h}{2}. \tag{4.11}$$

Solving (4.9) for a_l:

$$a_l = \frac{\beta_l - c_l + \lambda + \mu_l/\delta_l}{2}. \tag{4.12}$$

Solving (4.10) for p:

$$p = \frac{\alpha - \lambda\gamma + \mu_p}{2\gamma}.$$

The complementary slackness conditions are:

$$\lambda \cdot \big[D(p) - R_h(a_h) - R_l(a_l)\big] = 0, \quad \lambda \geq 0, \tag{4.13}$$

$$\mu_h \cdot (a_h - \beta_h) = 0, \quad \mu_h \geq 0, \tag{4.14}$$

$$\mu_l \cdot (a_l - \beta_l) = 0, \quad \mu_l \geq 0, \tag{4.15}$$

$$\mu_p \cdot p = 0, \quad \mu_p \geq 0. \tag{4.16}$$

By complementary slackness, either $\lambda = 0$ or constraint (4.7) binds. Suppose $\lambda = 0$. From (4.11) and complementary slackness (4.14), either $\mu_h = 0$ giving $a_h = (\beta_h - c_h)/2 < \beta_h$ (violating constraint (4.5) since $c_h > 0$), or $a_h = \beta_h$ and $R_h(\beta_h) = 0$. In the latter case, the same argument applied to (4.12) gives either $a_l = (\beta_l - c_l)/2 < \beta_l$ (violating constraint (4.6)), or $a_l = \beta_l$ and $R_l(\beta_l) = 0$. But then total supply is zero and $\Pi^C(\beta_h, \beta_l, p) = 0$ for all p, while market viability $\alpha > \gamma(\beta_l + c_l)$ guarantees strictly positive profit is achievable, a contradiction. Hence $\lambda > 0$, and by complementary slackness (4.13), constraint (4.7) holds with equality at the optimum. □

The signs of μ_h, μ_l, and μ_p are determined by the case analysis in Propositions 4.1 and 4.2: in both cases $a_h > \beta_h$, $a_l \geq \beta_l$, and $p > 0$, so by complementary slackness (4.14)–(4.16), $\mu_h = \mu_l = \mu_p = 0$.

Since market clearing holds at the optimum, the selling price $p(a_h, a_l)$ is determined endogenously by (a_h, a_l):

$$p(a_h, a_l) = \frac{\alpha - \delta_h(a_h - \beta_h) - \delta_l(a_l - \beta_l)}{\gamma}. \tag{4.17}$$

Substituting $p(a_h, a_l)$ from (4.17) into $\Pi^C(a_h, a_l, p)$ eliminates p as a decision variable. Constraint (4.7) is eliminated by Lemma 4.1. Constraint (4.4), after substituting $p(a_h, a_l)$, becomes $\delta_h(a_h - \beta_h) + \delta_l(a_l - \beta_l) \leq \alpha$, which is satisfied strictly at the optimum (shown in the proof of Proposition 4.1) and is therefore redundant. The problem thus reduces to two decision variables (a_h, a_l):

$$\begin{aligned} &\max_{a_h,\, a_l} \Pi^C(a_h, a_l) \\ \text{subject to} \quad & a_h \geq \beta_h, \\ & a_l \geq \beta_l, \end{aligned}$$

where the reformulated profit function $\Pi^C(a_h, a_l)$ is:

$$\begin{aligned} \Pi^C(a_h, a_l) = {} & \frac{\delta_h(a_h - \beta_h)}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) - (\delta_h + \gamma)(a_h - \beta_h) - \delta_l(a_l - \beta_l)\big] \\ & + \frac{\delta_l(a_l - \beta_l)}{\gamma}\big[\alpha - \gamma(\beta_l + c_l) - \delta_h(a_h - \beta_h) - (\delta_l + \gamma)(a_l - \beta_l)\big]. \end{aligned}$$

Since $\Pi^C(a_h, a_l)$ is a polynomial in (a_h, a_l) it is C^∞, and the following lemma confirms it is strictly concave.

Lemma 4.2. *$\Pi^C(a_h, a_l)$ is strictly jointly concave in (a_h, a_l).*

Proof. The first-order partial derivatives of $\Pi^C(a_h, a_l)$ are:

$$\begin{aligned} \frac{\partial \Pi^C(a_h, a_l)}{\partial a_h} &= \frac{\delta_h}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) - 2(\delta_h + \gamma)(a_h - \beta_h) - 2\delta_l(a_l - \beta_l)\big], \\ \frac{\partial \Pi^C(a_h, a_l)}{\partial a_l} &= \frac{\delta_l}{\gamma}\big[\alpha - \gamma(\beta_l + c_l) - 2\delta_h(a_h - \beta_h) - 2(\delta_l + \gamma)(a_l - \beta_l)\big]. \end{aligned}$$

The second-order partial derivatives of $\Pi^C(a_h, a_l)$ are:

$$\frac{\partial^2 \Pi^C}{\partial a_h^2} = -\frac{2\delta_h(\delta_h + \gamma)}{\gamma} < 0, \qquad \frac{\partial^2 \Pi^C}{\partial a_l^2} = -\frac{2\delta_l(\delta_l + \gamma)}{\gamma} < 0, \qquad \frac{\partial^2 \Pi^C}{\partial a_h \partial a_l} = -\frac{2\delta_h\delta_l}{\gamma}.$$

The Hessian matrix is:

$$H = -\frac{2}{\gamma}\begin{pmatrix} \delta_h(\delta_h + \gamma) & \delta_h\delta_l \\ \delta_h\delta_l & \delta_l(\delta_l + \gamma) \end{pmatrix}.$$

Its leading principal minors are $|H_1| = -2\delta_h(\delta_h+\gamma)/\gamma < 0$ and $\det(H) = 4\delta_h\delta_l\gamma(\delta_h+\delta_l+\gamma)/\gamma^2 > 0$, so H is negative definite and $\Pi^C(a_h, a_l)$ is strictly jointly concave. Any local maximizer is the unique global maximizer. □

Since $\Pi^C(a_h, a_l)$ is strictly concave by Lemma 4.2, the KKT conditions are necessary and sufficient for a global maximum. The complementary slackness conditions (4.14)–(4.15) identify the active constraints at the optimum and yield the following case analysis. Under quality ordering $\beta_h + c_h \leq \beta_l + c_l$, the low-quality-only case ($a_h^* = \beta_h$, $a_l^* > \beta_l$) is ruled out since high quality always has a weakly lower total minimum cost. Under market viability $\alpha > \gamma(\beta_l + c_l)$, the no-returns case ($a_h^* = \beta_h$, $a_l^* = \beta_l$) is also ruled out. The two remaining feasible cases are characterized below.

Proposition 4.1 (Hybrid)**.** *When both quality classes are acquired at positive quantities, that is $a_h > \beta_h$ and $a_l > \beta_l$, complementary slackness conditions* (4.14) *and* (4.15) *give $\mu_h = \mu_l = 0$. The unique optimal solution is:*

$$a_h^{1*} = \beta_h + \frac{(\delta_l + \gamma)[\alpha - \gamma(\beta_h + c_h)] - \delta_l[\alpha - \gamma(\beta_l + c_l)]}{2\gamma(\delta_h + \delta_l + \gamma)}, \tag{4.18}$$

$$a_l^{1*} = \beta_l + \frac{(\delta_h + \gamma)[\alpha - \gamma(\beta_l + c_l)] - \delta_h[\alpha - \gamma(\beta_h + c_h)]}{2\gamma(\delta_h + \delta_l + \gamma)}, \tag{4.19}$$

$$p_1^*(a_h^{1*}, a_l^{1*}) = \frac{\alpha - \delta_h(a_h^{1*} - \beta_h) - \delta_l(a_l^{1*} - \beta_l)}{\gamma}, \tag{4.20}$$

$$Q_h^{1*} = R_h(a_h^{1*}) = \delta_h(a_h^{1*} - \beta_h), \tag{4.21}$$

$$Q_l^{1*} = R_l(a_l^{1*}) = \delta_l(a_l^{1*} - \beta_l). \tag{4.22}$$

The total system profit is $\Pi^{C,1*} = \Pi^C(a_h^{1*}, a_l^{1*}, p_1^*(a_h^{1*}, a_l^{1*}))$, obtained by substituting (4.18)–(4.20) *into* (4.1)*. This solution is feasible if and only if:*

$$(\delta_h + \gamma)[\alpha - \gamma(\beta_l + c_l)] > \delta_h[\alpha - \gamma(\beta_h + c_h)], \tag{4.23}$$

which ensures $a_l^{1*} > \beta_l$. Under quality ordering, $a_h^{1*} > \beta_h$ then follows automatically.

Proof. With $\mu_h = \mu_l = 0$, the first-order conditions of $\Pi^C(a_h, a_l)$ give:

$$\frac{\partial \Pi^C(a_h, a_l)}{\partial a_h} = \frac{\delta_h}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) - 2(\delta_h + \gamma)(a_h - \beta_h) - 2\delta_l(a_l - \beta_l)\big] = 0, \tag{4.24}$$

$$\frac{\partial \Pi^C(a_h, a_l)}{\partial a_l} = \frac{\delta_l}{\gamma}\big[\alpha - \gamma(\beta_l + c_l) - 2\delta_h(a_h - \beta_h) - 2(\delta_l + \gamma)(a_l - \beta_l)\big] = 0. \tag{4.25}$$

Cancelling the positive factors δ_h/γ and δ_l/γ and rearranging yields the linear system:

$$2(\delta_h + \gamma)(a_h - \beta_h) + 2\delta_l(a_l - \beta_l) = \alpha - \gamma(\beta_h + c_h), \tag{4.26}$$

$$2\delta_h(a_h - \beta_h) + 2(\delta_l + \gamma)(a_l - \beta_l) = \alpha - \gamma(\beta_l + c_l). \tag{4.27}$$

The coefficient matrix has determinant $4[(\delta_h + \gamma)(\delta_l + \gamma) - \delta_h\delta_l] = 4\gamma(\delta_h + \delta_l + \gamma) > 0$, confirming a unique solution. Solving (4.26)–(4.27) gives (4.18)–(4.19). The optimal price $p_1^*(a_h^{1*}, a_l^{1*})$ follows from market clearing (4.17), giving (4.20). Since all collected cores are remanufactured, the quantities follow directly from $R_h(a_h) = \delta_h(a_h - \beta_h)$ and $R_l(a_l) = \delta_l(a_l - \beta_l)$, giving (4.21)–(4.22).

Feasibility requires $a_l^{1*} > \beta_l$, which from (4.19) is equivalent to condition (4.23). Under quality ordering $\alpha - \gamma(\beta_h + c_h) \geq \alpha - \gamma(\beta_l + c_l)$, so from (4.18):

$$a_h^{1*} - \beta_h \geq \frac{(\delta_l + \gamma)[\alpha - \gamma(\beta_l + c_l)] - \delta_l[\alpha - \gamma(\beta_l + c_l)]}{2\gamma(\delta_h + \delta_l + \gamma)} = \frac{\alpha - \gamma(\beta_l + c_l)}{2(\delta_h + \delta_l + \gamma)} > 0,$$

so $a_h^{1*} > \beta_h$ follows automatically. The first-order conditions (4.24)–(4.25) can be rewritten as the optimality condition:

$$2a_h^{1*} + c_h - \beta_h = 2a_l^{1*} + c_l - \beta_l = 2p_1^*(a_h^{1*}, a_l^{1*}) - \frac{\alpha}{\gamma}, \tag{4.28}$$

stating that the marginal cost of acquiring and remanufacturing each quality class equals the marginal revenue from sales at the optimum. □

Proposition 4.2 (High Quality Only)**.** *When only the high quality class is acquired at positive quantities, that is $a_h > \beta_h$ and $a_l = \beta_l$, complementary slackness conditions*

(4.14) *and* (4.15) *give* $\mu_h = 0$ *and* $\mu_l \geq 0$. *The unique optimal solution is:*

$$a_h^{2*} = \beta_h + \frac{\alpha - \gamma(\beta_h + c_h)}{2(\delta_h + \gamma)}, \tag{4.29}$$

$$a_l^{2*} = \beta_l, \tag{4.30}$$

$$p_2^*(a_h^{2*}, \beta_l) = \frac{\alpha - \delta_h(a_h^{2*} - \beta_h)}{\gamma}, \tag{4.31}$$

$$Q_h^{2*} = R_h(a_h^{2*}) = \delta_h(a_h^{2*} - \beta_h), \tag{4.32}$$

$$Q_l^{2*} = 0. \tag{4.33}$$

This case applies when condition (4.23) *fails. The proof follows directly from Chapter 3 with parameters* (δ_h, β_h, c_h).

Since Case 2 reduces entirely to the single quality class analysis of Chapter 3, the remainder of this chapter focuses on Case 1. To simplify notation, we write a_h^{C*}, a_l^{C*}, p^{C*}, Q_h^{C*}, Q_l^{C*}, and Π^{C*} in place of a_h^{1*}, a_l^{1*}, $p_1^*(a_h^{1*}, a_l^{1*})$, Q_h^{1*}, Q_l^{1*}, and $\Pi^{C,1*}$ respectively, where the superscript C denotes the centralized optimum.

Table 4.1 summarizes both cases.

Table 4.1: Summary of Centralized Solution

Case	**Feasibility Condition**	**Optimal Solution**
Hybrid	$(\delta_h+\gamma)[\alpha-\gamma(\beta_l+c_l)]$ $> \delta_h[\alpha-\gamma(\beta_h+c_h)]$	a_h^{C*} (4.18), a_l^{C*} (4.19), p^{C*} (4.20), Q_h^{C*} (4.21), Q_l^{C*} (4.22)
High Quality Only	$(\delta_h+\gamma)[\alpha-\gamma(\beta_l+c_l)]$ $\leq \delta_h[\alpha-\gamma(\beta_h+c_h)]$	a_h^{2*} (4.29), $a_l^{2*} = \beta_l$, p_2^* (4.31), Q_h^{2*} (4.32), $Q_l^{2*} = 0$

4.2 Decentralized Setting

In the decentralized setting, the collector and the remanufacturer are independent profit-maximizing firms. The collector acts as the Stackelberg leader, setting acquisition prices for high and low quality cores a_h and a_l and transfer prices w_h and w_l

respectively, anticipating the remanufacturer's best response. The remanufacturer acts as the Stackelberg follower, choosing quantities Q_h and Q_l to maximize its own profit given the transfer prices set by the collector. The game is solved by backward induction: we first characterize the remanufacturer's best response for given (a_h, a_l, w_h, w_l) in Stage 2, and then solve the collector's problem in Stage 1 anticipating this response.

4.2.1 Remanufacturer's Problem

For given (a_h, a_l, w_h, w_l), the available return quantities are $R_h(a_h) = \delta_h(a_h - \beta_h)$ and $R_l(a_l) = \delta_l(a_l - \beta_l)$. The remanufacturer chooses $Q_h \in [0, R_h(a_h)]$, $Q_l \in [0, R_l(a_l)]$, and p to maximize his profit. Since the remanufacturer sells all units it produces, holding unsold inventory is never optimal when demand is downward-sloping and there are no salvage revenues. Market clearing $Q_h + Q_l = D(p) = \alpha - \gamma p$ holds at the optimum, by the same argument as in the centralized setting (Lemma 4.1). Using $p(Q_h, Q_l) = (\alpha - Q_h - Q_l)/\gamma$ from market clearing, the remanufacturer's profit is:

$$\Pi^R(Q_h, Q_l) = \frac{\alpha - Q_h - Q_l}{\gamma}(Q_h + Q_l) - (w_h + c_h)Q_h - (w_l + c_l)Q_l.$$

The remanufacturer solves:

$$\begin{aligned} \max_{Q_h, Q_l} \ & \Pi^R(Q_h, Q_l) \\ \text{subject to} \quad & 0 \le Q_h \le R_h(a_h), \\ & 0 \le Q_l \le R_l(a_l). \end{aligned}$$

Since the aggregate cost of a remanufactured unit is $w_i + c_i$ for quality class i, the remanufacturer always prefers the cheaper class, using it first before supplementing with the more expensive class. This gives rise to two cases:

- **Case 1:** $w_h + c_h \le w_l + c_l$ (high quality class is more economical)
- **Case 2:** $w_l + c_l \le w_h + c_h$ (low quality class is more economical)

Lemma 4.3 (Remanufacturer's Best Response)**.** $\Pi^R(Q_h, Q_l)$ *is jointly concave in* (Q_h, Q_l)*. The optimal quantities are as follows. The optimal selling price follows from market clearing:* $p^* = (\alpha - Q_h^* - Q_l^*)/\gamma$.

Case 1 *($w_h + c_h \leq w_l + c_l$):*

$$(Q_h^*, Q_l^*) = \begin{cases} \left(\dfrac{\alpha - \gamma(w_h + c_h)}{2},\ 0\right) & \text{if } R_h(a_h) \geq \dfrac{\alpha - \gamma(w_h + c_h)}{2}, \\ \left(R_h(a_h),\ 0\right) & \text{if } \dfrac{\alpha - \gamma(w_l + c_l)}{2} \leq R_h(a_h) < \dfrac{\alpha - \gamma(w_h + c_h)}{2}, \\ \left(R_h(a_h),\ \dfrac{\alpha - \gamma(w_l + c_l)}{2} - R_h(a_h)\right) & \text{if } R_h(a_h) < \dfrac{\alpha - \gamma(w_l + c_l)}{2} \leq R_h(a_h) + R_l(a_l), \\ \left(R_h(a_h),\ R_l(a_l)\right) & \text{if } R_h(a_h) + R_l(a_l) < \dfrac{\alpha - \gamma(w_l + c_l)}{2}. \end{cases} \tag{4.34}$$

Case 2 *($w_l + c_l \leq w_h + c_h$):*

$$(Q_h^*, Q_l^*) = \begin{cases} \left(0,\ \dfrac{\alpha - \gamma(w_l + c_l)}{2}\right) & \text{if } R_l(a_l) \geq \dfrac{\alpha - \gamma(w_l + c_l)}{2}, \\ \left(0,\ R_l(a_l)\right) & \text{if } \dfrac{\alpha - \gamma(w_h + c_h)}{2} \leq R_l(a_l) < \dfrac{\alpha - \gamma(w_l + c_l)}{2}, \\ \left(\dfrac{\alpha - \gamma(w_h + c_h)}{2} - R_l(a_l),\ R_l(a_l)\right) & \text{if } R_l(a_l) < \dfrac{\alpha - \gamma(w_h + c_h)}{2} \leq R_h(a_h) + R_l(a_l), \\ \left(R_h(a_h),\ R_l(a_l)\right) & \text{if } R_h(a_h) + R_l(a_l) < \dfrac{\alpha - \gamma(w_h + c_h)}{2}. \end{cases}$$

Proof. We prove Case 1; Case 2 follows symmetrically by swapping $h \leftrightarrow l$.

The first-order conditions of $\Pi^R(Q_h, Q_l)$ are:

$$\frac{\partial \Pi^R(Q_h, Q_l)}{\partial Q_h} = \frac{\alpha - 2Q_h - 2Q_l}{\gamma} - (w_h + c_h) = 0,$$
$$\frac{\partial \Pi^R(Q_h, Q_l)}{\partial Q_l} = \frac{\alpha - 2Q_h - 2Q_l}{\gamma} - (w_l + c_l) = 0.$$

The second-order partial derivatives are:

$$\frac{\partial^2 \Pi^R(Q_h, Q_l)}{\partial Q_h^2} = -\frac{2}{\gamma}, \qquad \frac{\partial^2 \Pi^R(Q_h, Q_l)}{\partial Q_l^2} = -\frac{2}{\gamma}, \qquad \frac{\partial^2 \Pi^R(Q_h, Q_l)}{\partial Q_h \partial Q_l} = -\frac{2}{\gamma}.$$

The Hessian is:

$$H^R = -\frac{2}{\gamma}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which is negative semidefinite ($\det(H^R) = 0$), confirming that $\Pi^R(Q_h, Q_l)$ is jointly concave. It is strictly concave in total quantity $Q = Q_h + Q_l$, which uniquely determines optimal total output.

From the first-order conditions, the unconstrained optimal quantities when only quality class h or l is acquired are:

$$Q_h^{unc}(w_h) = \frac{\alpha - \gamma(w_h + c_h)}{2}, \qquad Q_l^{unc}(w_l) = \frac{\alpha - \gamma(w_l + c_l)}{2}.$$

Under Case 1, $w_h + c_h \leq w_l + c_l$ implies $Q_h^{unc}(w_h) \geq Q_l^{unc}(w_l)$. The remanufacturer uses high quality first and the four cases follow from comparing $R_h(a_h)$ and $R_h(a_h) + R_l(a_l)$ against $Q_h^{unc}(w_h)$ and $Q_l^{unc}(w_l)$:

When $R_h(a_h) \geq Q_h^{unc}(w_h)$: the supply constraint on high quality does not bind and $Q_h^* = Q_h^{unc}(w_h)$. Low quality is not needed.

When $Q_l^{unc}(w_l) \leq R_h(a_h) < Q_h^{unc}(w_h)$: $Q_h^* = R_h(a_h)$. The marginal value of the first unit of low quality at $Q_l = 0$ is $(\alpha - 2R_h(a_h))/\gamma - (w_l + c_l) \leq 0$ since $R_h(a_h) \geq Q_l^{unc}(w_l)$, so $Q_l^* = 0$.

When $R_h(a_h) < Q_l^{unc}(w_l) \leq R_h(a_h) + R_l(a_l)$: all high quality is used and low quality supplements to reach $Q_l^{unc}(w_l)$, giving $Q_l^* = Q_l^{unc}(w_l) - R_h(a_h)$.

When $R_h(a_h) + R_l(a_l) < Q_l^{unc}(w_l)$: total supply is insufficient and both constraints bind, giving $Q_h^* = R_h(a_h)$ and $Q_l^* = R_l(a_l)$.

The four conditions are mutually exclusive and collectively exhaustive by construction.

For subsequent reference, Table 4.2 names the four regions and describes their economic content.

Remark 4.1. The analysis below is carried out under Case 1, in which the high quality class is the more economical one for the remanufacturer. Case 2 is obtained by interchanging the indices h and l throughout: the collector's problem, the candidate

Table 4.2: Remanufacturer's optimal quantities under Case 1 ($w_h + c_h \leq w_l + c_l$).

Name	**Condition**	**Interpretation**
High Quality Unconstrained	$R_h(a_h) \geq Q_h^{unc}(w_h)$	Supply exceeds unconstrained optimum; constraint does not bind, low quality not used.
High Quality Constrained	$Q_l^{unc}(w_l) \leq R_h(a_h) < Q_h^{unc}(w_h)$	Supply binds but low quality unprofitable at the margin; only high quality used.
Partially Utilized	$R_h(a_h) < Q_l^{unc}(w_l) \leq R_h(a_h) + R_l(a_l)$	High quality exhausted; low quality supplements to meet optimal total demand.
Hybrid	$R_h(a_h) + R_l(a_l) < Q_l^{unc}(w_l)$	Total supply insufficient; all returns remanufactured.

solutions, and the coordination contracts all follow by symmetry, with no new structure. Under the quality ordering assumption $\beta_h + c_h \leq \beta_l + c_l$, Case 1 is also the empirically relevant one, since the high quality class is cheaper to bring to market. All numerical results in Chapter 5 are reported under Case 1.

4.2.2 Collector's Problem

The collector acts as the Stackelberg leader and anticipates the remanufacturer's best response characterized in Lemma 4.3. The collector chooses acquisition prices (a_h, a_l) and transfer prices (w_h, w_l) to maximize its profit:

$$\Pi^{Col}(a_h, a_l, w_h, w_l) = (w_h - a_h)Q_h^* + (w_l - a_l)Q_l^*,$$

subject to $a_h \geq \beta_h$, $a_l \geq \beta_l$, $w_h \geq 0$, $w_l \geq 0$, where (Q_h^*, Q_l^*) is the remanufacturer's best response from Lemma 4.3.

Since (Q_h^*, Q_l^*) takes a different form in each of the four cases of Table 4.2, Π^{Col} is piecewise in (a_h, a_l, w_h, w_l). We first establish continuity and concavity, then derive the optimal solution in each case.

Lemma 4.4. *$\Pi^{Col}(a_h, a_l, w_h, w_l)$ is continuous at all boundaries of all cases.*

Proof. We verify continuity at each boundary by substituting the boundary condition into both adjacent profit expressions and confirming they are equal.

At the High Quality Unconstrained – High Quality Constrained boundary, $R_h(a_h) = Q_h^{unc}(w_h)$. Substituting into the High Quality Unconstrained profit $\Pi^{Col}(a_h, a_l, w_h, w_l) = w_h Q_h^{unc}(w_h) - a_h R_h(a_h) - a_l R_l(a_l)$ and using $Q_h^{unc}(w_h) = R_h(a_h)$:

$$\begin{aligned}\Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{HQU}} &= w_h R_h(a_h) - a_h R_h(a_h) - a_l R_l(a_l) \\ &= R_h(a_h)(w_h - a_h) - a_l R_l(a_l).\end{aligned}$$

The High Quality Constrained profit is $\Pi^{Col}(a_h, a_l, w_h, w_l) = R_h(a_h)(w_h - a_h) - a_l R_l(a_l)$ throughout, so:

$$\Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{HQC}} = R_h(a_h)(w_h - a_h) - a_l R_l(a_l).$$

The two expressions are equal at this boundary.

At the High Quality Constrained – Partially Utilized boundary, $R_h(a_h) = Q_l^{unc}(w_l)$. The Partially Utilized profit is:

$$\begin{aligned}\Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{PU}} &= w_h R_h(a_h) + w_l Q_l^*(w_l, a_h) \\ &\quad - a_h R_h(a_h) - a_l R_l(a_l),\end{aligned}$$

where $Q_l^*(w_l, a_h) = Q_l^{unc}(w_l) - R_h(a_h)$. Substituting $R_h(a_h) = Q_l^{unc}(w_l)$:

$$Q_l^*(w_l, a_h) = Q_l^{unc}(w_l) - Q_l^{unc}(w_l) = 0.$$

Therefore:

$$\begin{aligned}\Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{PU}} &= w_h R_h(a_h) + w_l \cdot 0 \\ &\quad - a_h R_h(a_h) - a_l R_l(a_l) \\ &= R_h(a_h)(w_h - a_h) - a_l R_l(a_l) \\ &= \Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{HQC}}.\end{aligned}$$

At the Partially Utilized – Hybrid boundary, $R_h(a_h) + R_l(a_l) = Q_l^{unc}(w_l)$. The Partially Utilized profit has $Q_l^*(w_l, a_h) = Q_l^{unc}(w_l) - R_h(a_h)$. Substituting $Q_l^{unc}(w_l) = R_h(a_h) + R_l(a_l)$:

$$Q_l^*(w_l, a_h) = R_h(a_h) + R_l(a_l) - R_h(a_h) = R_l(a_l).$$

Therefore:

$$\begin{aligned}\Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{PU}} &= w_h R_h(a_h) + w_l R_l(a_l) \\ &\quad - a_h R_h(a_h) - a_l R_l(a_l) \\ &= R_h(a_h)(w_h - a_h) \\ &\quad + R_l(a_l)(w_l - a_l) \\ &= \Pi^{Col}(a_h, a_l, w_h, w_l)\big|_{\text{Hyb}}.\end{aligned}$$

□

By Lemma 4.4, the global optimum of $\Pi^{Col}(a_h, a_l, w_h, w_l)$ may occur either at a case boundary or within a case. As shown in the proof of Proposition 4.3 below, $\Pi^{Col}(a_h, a_l, w_h, w_l)$ is strictly concave in the relevant variables in each case after substituting the binding constraints, so each case has a unique optimum obtained from the first-order conditions. The two High Quality regions both drive the collector to the same solution, so they yield a single High Quality Only candidate; together with the Partially Utilized and Hybrid candidates, this leaves three, summarized in Table 4.3. The collector's profit is moreover continuous across the Partially Utilized–Hybrid boundary, so the global optimum is attained at one of these three candidates.

Proposition 4.3. *Under Case 1 ($w_h + c_h \leq w_l + c_l$), the collector's optimal acquisition and transfer prices in each of the three candidate regions of the remanufacturer's best*

response are as follows. Each candidate is the collector's optimal solution within its region and is valid when the accompanying feasibility condition holds.

High Quality Only.

$$a_h^{HQ*} = \beta_h + \frac{\alpha - \gamma(\beta_h + c_h)}{2(2\delta_h + \gamma)}, \tag{4.35}$$

$$w_h^{HQ*} = \frac{(\delta_h + \gamma)(\alpha - \gamma c_h) + \delta_h \gamma \beta_h}{\gamma(2\delta_h + \gamma)}, \tag{4.36}$$

$$Q_h^{HQ*} = \frac{\delta_h[\alpha - \gamma(\beta_h + c_h)]}{2(2\delta_h + \gamma)}, \quad Q_l^{HQ*} = 0, \tag{4.37}$$

$$\Pi^{Col,HQ*} = \frac{\delta_h[\alpha - \gamma(\beta_h + c_h)]^2}{4\gamma(2\delta_h + \gamma)}. \tag{4.38}$$

This candidate is feasible whenever the high quality class is viable,

$$\alpha > \gamma(\beta_h + c_h),$$

which is guaranteed by market viability and holds for all parameter values considered.

Partially Utilized.

$$a_h^{PU*} = \beta_h + \frac{\delta_l(c_l - c_h - \beta_h + \beta_l) + \frac{\alpha - \gamma c_l}{2}}{2(\delta_h + \delta_l)}, \tag{4.39}$$

$$w_l^{PU*} = \frac{\alpha - \gamma c_l}{2\gamma}, \quad w_h^{PU*} = w_l^{PU*} + c_l - c_h, \tag{4.40}$$

$$Q_h^{PU*} = \delta_h(a_h^{PU*} - \beta_h), \tag{4.41}$$

$$Q_l^{PU*} = \frac{\alpha - \gamma c_l}{4} - Q_h^{PU*}, \tag{4.42}$$

$$a_l^{PU*} = \beta_l + \frac{Q_l^{PU*}}{\delta_l}. \tag{4.43}$$

This candidate is feasible when the supplementary low quality quantity is nonnegative, $Q_l^{PU*} \geq 0$*, equivalently*

$$\delta_h\big(a_h^{PU*} - \beta_h\big) \leq \frac{\alpha - \gamma c_l}{4}.$$

Hybrid.

$$a_h^{Hyb*} - \beta_h = \frac{(2\delta_l + \gamma)[\alpha - \gamma(\beta_h + c_h)] - 2\delta_l[\alpha - \gamma(\beta_l + c_l)]}{2\gamma(2\delta_h + 2\delta_l + \gamma)}, \tag{4.44}$$

$$a_l^{Hyb*} - \beta_l = \frac{(2\delta_h + \gamma)[\alpha - \gamma(\beta_l + c_l)] - 2\delta_h[\alpha - \gamma(\beta_h + c_h)]}{2\gamma(2\delta_h + 2\delta_l + \gamma)}, \tag{4.45}$$

$$w_h^{Hyb*} = \frac{\alpha - 2Q^{Hyb*}}{\gamma} - c_h, \quad w_l^{Hyb*} = \frac{\alpha - 2Q^{Hyb*}}{\gamma} - c_l, \tag{4.46}$$

$$Q_h^{Hyb*} = \delta_h(a_h^{Hyb*} - \beta_h), \quad Q_l^{Hyb*} = \delta_l(a_l^{Hyb*} - \beta_l), \tag{4.47}$$

where $Q^{Hyb*} = Q_h^{Hyb*} + Q_l^{Hyb*}$*. The Hybrid candidate is admissible only if it prescribes non-negative supply of both quality classes and a non-negative low quality transfer price:*

$$a_h^{Hyb*} \geq \beta_h, \qquad a_l^{Hyb*} \geq \beta_l, \qquad w_l^{Hyb*} \geq 0. \tag{4.48}$$

Substituting (4.45)*, the second condition is*

$$\alpha + 2\delta_h(\beta_h - \beta_l + c_h - c_l) - \gamma(\beta_l + c_l) \;\geq\; 0,$$

and it is this condition, rather than $w_l^{Hyb*} \geq 0$*, that binds first as* c_l *increases or as* δ_h *increases.*

These three candidates are exhaustive: the remanufacturer's High Quality Unconstrained and High Quality Constrained regions drive the collector to the same solution ($w_h = \bar{w}_h(a_h)$*,* $a_l = \beta_l$*), giving the single High Quality Only candidate, while the Partially Utilized and Hybrid regions give the remaining two. The collector's optimum is therefore one of the three. Proposition 4.4 shows that the Partially Utilized candidate is never selected, so the equilibrium is always the Hybrid or the High Quality Only candidate.*

Proof. We derive each candidate in turn, verifying feasibility after each derivation. Throughout Case 1, $w_h + c_h \leq w_l + c_l$.

We begin with the High Quality Only candidate. In the High Quality Unconstrained case, $Q_h^*(w_h) = Q_h^{unc}(w_h)$ and $Q_l^* = 0$, so the collector's profit is:

$$\Pi^{Col}(a_h, a_l, w_h, w_l) = w_h \cdot Q_h^{unc}(w_h) - a_h R_h(a_h) - a_l R_l(a_l).$$

Taking partial derivatives with respect to a_h and a_l:

$$\frac{\partial \Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial a_h} = -\delta_h(2a_h - \beta_h) < 0 \quad \text{for all } a_h > \beta_h,$$
$$\frac{\partial \Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial a_l} = -\delta_l(2a_l - \beta_l) < 0 \quad \text{for all } a_l > \beta_l.$$

Both are strictly negative, so $\Pi^{Col}(a_h, a_l, w_h, w_l)$ is strictly decreasing in a_h and a_l throughout this case. The collector would reduce both acquisition prices, but doing so reduces $R_h(a_h)$ toward zero, violating the High Quality Unconstrained condition $R_h(a_h) \geq Q_h^{unc}(w_h) > 0$. The collector is therefore pushed to the High Quality Unconstrained – High Quality Constrained boundary.

In the High Quality Constrained case, $Q_h^*(a_h) = R_h(a_h)$ and $Q_l^* = 0$, so:

$$\Pi^{Col}(a_h, a_l, w_h, w_l) = R_h(a_h)(w_h - a_h) - a_l R_l(a_l). \tag{4.49}$$

Taking the partial derivative with respect to w_h:

$$\frac{\partial \Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial w_h} = R_h(a_h) > 0 \quad \text{for all } a_h > \beta_h.$$

$\Pi^{Col}(a_h, a_l, w_h, w_l)$ is strictly increasing in w_h, so the collector raises w_h to the upper boundary of this case. This boundary is defined by $R_h(a_h) = Q_h^{unc}(w_h)$, that is, the point at which high quality supply exactly meets unconstrained demand. We denote the transfer price at this boundary by $\bar{w}_h(a_h)$. Substituting $R_h(a_h) = \delta_h(a_h - \beta_h)$ and $Q_h^{unc}(w_h) = (\alpha - \gamma(w_h + c_h))/2$ into the boundary condition:

$$\delta_h(a_h - \beta_h) = \frac{\alpha - \gamma(\bar{w}_h(a_h) + c_h)}{2}.$$

Solving for $\bar{w}_h(a_h)$:

$$\bar{w}_h(a_h) = \frac{\alpha - 2\delta_h(a_h - \beta_h)}{\gamma} - c_h. \tag{4.50}$$

Taking the partial derivative with respect to a_l:

$$\frac{\partial \Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial a_l} = -\delta_l(2a_l - \beta_l) < 0 \quad \text{for all } a_l > \beta_l,$$

so the collector sets $a_l = \beta_l$. Both the High Quality Unconstrained and High Quality Constrained cases therefore push the collector to the same point: $w_h = \bar{w}_h(a_h)$ and $a_l = \beta_l$.

At this boundary, substituting $w_h = \bar{w}_h(a_h)$, $Q_h^*(a_h) = \delta_h(a_h - \beta_h)$, $Q_l^* = 0$, and $a_l = \beta_l$ into (4.49):

$$\begin{aligned}\Pi^{Col}(a_h, \beta_l, \bar{w}_h(a_h), w_l) &= \delta_h(a_h - \beta_h)\Big(\frac{\alpha - 2\delta_h(a_h - \beta_h)}{\gamma} - c_h - a_h\Big) \\ &= \frac{\delta_h(a_h - \beta_h)}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) \\ &\quad - (2\delta_h + \gamma)(a_h - \beta_h)\big]. \end{aligned} \tag{4.51}$$

The second derivative of (4.51) with respect to a_h is:

$$\frac{d^2\Pi^{Col}(a_h, \beta_l, \bar{w}_h(a_h), w_l)}{da_h^2} = -\frac{2\delta_h(2\delta_h + \gamma)}{\gamma} < 0,$$

so (4.51) is strictly concave in a_h. Setting the first derivative to zero:

$$\frac{d\Pi^{Col}(a_h, \beta_l, \bar{w}_h(a_h), w_l)}{da_h} = \frac{\delta_h}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) - 2(2\delta_h + \gamma)(a_h - \beta_h)\big] = 0,$$

which gives (4.35). Substituting a_h^{HQ*} into $Q_h^*(a_h^{HQ*}) = \delta_h(a_h^{HQ*} - \beta_h)$ gives (4.37). Substituting into $\bar{w}_h(a_h^{HQ*})$ from (4.50) gives (4.36). Substituting into (4.51) gives (4.38).

We verify feasibility of the High Quality Only candidate. $a_h^{HQ*} \geq \beta_h$: from (4.35), $a_h^{HQ*} - \beta_h = [\alpha - \gamma(\beta_h + c_h)]/[2(2\delta_h + \gamma)] > 0$ since $\alpha > \gamma(\beta_h + c_h)$ by market viability and $2(2\delta_h + \gamma) > 0$. $a_l^{HQ*} = \beta_l \geq \beta_l$ holds with equality by construction. $w_h^{HQ*} \geq 0$: from (4.36), the numerator $(\delta_h + \gamma)(\alpha - \gamma c_h) + \delta_h \gamma \beta_h > 0$ since $\alpha > \gamma c_h$ by market viability and both remaining terms are positive. $w_l^{HQ*} \geq 0$: since $Q_l^{HQ*} = 0$, the transfer price w_l does not enter the collector's profit at this candidate and any $w_l \geq 0$ is feasible without further restriction.

We now derive the Partially Utilized candidate. In the Partially Utilized case, $Q_h^*(a_h) = R_h(a_h)$ and $Q_l^*(w_l, a_h) = Q_l^{unc}(w_l) - R_h(a_h)$, so:

$$\begin{aligned}\Pi^{Col}(a_h, a_l, w_h, w_l) &= w_l \cdot \frac{\alpha - \gamma(w_l + c_l)}{2} + (w_h - w_l)R_h(a_h) \\ &\quad - a_h R_h(a_h) - a_l R_l(a_l). \end{aligned} \tag{4.52}$$

Taking the partial derivative with respect to w_h:

$$\frac{\partial\Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial w_h} = R_h(a_h) > 0 \quad \text{for all } a_h > \beta_h.$$

$\Pi^{Col}(a_h, a_l, w_h, w_l)$ is strictly increasing in w_h, so the collector raises w_h to its upper bound. Under Case 1, the ordering constraint $w_h + c_h \leq w_l + c_l$ binds, giving $w_h = w_l + c_l - c_h$. Substituting into (4.52):

$$\Pi^{Col}(a_h, a_l, w_h, w_l) = w_l \cdot \frac{\alpha - \gamma(w_l + c_l)}{2} + (c_l - c_h)R_h(a_h) - a_h R_h(a_h) - a_l R_l(a_l). \tag{4.53}$$

Taking the partial derivative with respect to w_l:

$$\frac{\partial \Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial w_l} = \frac{\alpha - \gamma(2w_l + c_l)}{2}. \tag{4.54}$$

Setting (4.54) to zero:

$$\alpha - \gamma(2w_l^* + c_l) = 0 \;\Rightarrow\; w_l^{PU*} = \frac{\alpha - \gamma c_l}{2\gamma},$$

which is (4.40). The second derivative $\partial^2 \Pi^{Col}(a_h, a_l, w_h, w_l)/\partial w_l^2 = -\gamma < 0$ confirms this is a maximum. Substituting w_l^{PU*} and $w_h^{PU*} = w_l^{PU*} + c_l - c_h$ into (4.53):

$$\Pi^{Col}(a_h, a_l, w_h^{PU*}, w_l^{PU*}) = \frac{(\alpha - \gamma c_l)^2}{8\gamma} + (c_l - c_h)R_h(a_h) - a_h R_h(a_h) - a_l R_l(a_l). \tag{4.55}$$

Taking the partial derivative with respect to a_l:

$$\frac{\partial \Pi^{Col}(a_h, a_l, w_h^{PU*}, w_l^{PU*})}{\partial a_l} = -\delta_l(2a_l - \beta_l) < 0 \quad \text{for all } a_l > \beta_l.$$

$\Pi^{Col}(a_h, a_l, w_h^{PU*}, w_l^{PU*})$ is strictly decreasing in a_l, so the collector reduces a_l to the lowest feasible value consistent with the Partially Utilized condition $Q_l^*(w_l^{PU*}, a_h) > 0$. The binding constraint gives $R_l(a_l) = Q_l^{unc}(w_l^{PU*}) - R_h(a_h)$, so:

$$\delta_l(a_l - \beta_l) = \frac{\alpha - \gamma c_l}{4} - \delta_h(a_h - \beta_h), \tag{4.56}$$

giving $Q_l^{PU*}(a_h)$ as in (4.42) and $a_l^{PU*}(a_h)$ as in (4.43).

Since a_l is now determined by a_h through (4.56), we substitute to express the profit as a function of a_h alone. Writing $a_l^{PU*}(a_h) = \beta_l + ((\alpha - \gamma c_l)/4 - \delta_h(a_h - \beta_h))/\delta_l$ and substituting into (4.55):

$$\begin{aligned}\Pi^{Col}(a_h, a_l^{PU*}(a_h), w_h^{PU*}, w_l^{PU*}) = {} & \frac{(\alpha - \gamma c_l)^2}{8\gamma} + (c_l - c_h)\delta_h(a_h - \beta_h) \\ & - a_h \delta_h(a_h - \beta_h) \\ & - \beta_l\Big(\frac{\alpha - \gamma c_l}{4} - \delta_h(a_h - \beta_h)\Big) \\ & - \frac{1}{\delta_l}\Big(\frac{\alpha - \gamma c_l}{4} - \delta_h(a_h - \beta_h)\Big)^2.\end{aligned} \tag{4.57}$$

The second derivative of (4.57) with respect to a_h is:

$$\frac{d^2\Pi^{Col}(a_h, a_l^{PU*}(a_h), w_h^{PU*}, w_l^{PU*})}{da_h^2} = -\frac{2\delta_h(\delta_h + \delta_l)}{\delta_l} < 0,$$

so (4.57) is strictly concave in a_h. Taking the first derivative and setting to zero:

$$\frac{d\Pi^{Col}(a_h, a_l^{PU*}(a_h), w_h^{PU*}, w_l^{PU*})}{da_h} = \delta_h\Big[c_l - c_h - a_h + \beta_l - 2(a_h - \beta_h) + \frac{2}{\delta_l}\Big(\frac{\alpha - \gamma c_l}{4} - \delta_h(a_h - \beta_h)\Big)\Big] = 0.$$

Collecting terms in a_h:

$$c_l - c_h - \beta_h + \beta_l + \frac{\alpha - \gamma c_l}{2\delta_l} = \frac{2(\delta_h + \delta_l)}{\delta_l}(a_h - \beta_h),$$

which gives (4.39).

We verify feasibility of the Partially Utilized candidate. $a_h^{PU*} \geq \beta_h$: from (4.39), $a_h^{PU*} - \beta_h = [\delta_l(c_l - c_h - \beta_h + \beta_l) + (\alpha - \gamma c_l)/2]/[2(\delta_h + \delta_l)]$. By market viability, $\alpha > \gamma(\beta_l + c_l) > \gamma c_l$, so $(\alpha - \gamma c_l)/2 > 0$. By quality ordering, $\beta_h + c_h \leq \beta_l + c_l$, which rearranges to $c_l - c_h + \beta_l \geq \beta_h$, giving $\delta_l(c_l - c_h - \beta_h + \beta_l) \geq 0$. Both terms are non-negative and the second is strictly positive, so $a_h^{PU*} > \beta_h$. $a_l^{PU*} \geq \beta_l$: from (4.43), $a_l^{PU*} - \beta_l = Q_l^{PU*}(a_h^{PU*})/\delta_l \geq 0$ whenever $Q_l^{PU*}(a_h^{PU*}) \geq 0$, which is precisely the Partially Utilized feasibility condition $\delta_h(a_h^{PU*} - \beta_h) \leq (\alpha - \gamma c_l)/4$. $w_l^{PU*} \geq 0$: from (4.40), $w_l^{PU*} = (\alpha - \gamma c_l)/(2\gamma) > 0$ by market viability. $w_h^{PU*} \geq 0$: from (4.40), $w_h^{PU*} = w_l^{PU*} + c_l - c_h > w_l^{PU*} > 0$ since $c_l > c_h$ by quality ordering.

We now derive the Hybrid candidate. In the Hybrid case, $Q_h^*(a_h) = R_h(a_h)$ and $Q_l^*(a_l) = R_l(a_l)$, so:

$$\Pi^{Col}(a_h, a_l, w_h, w_l) = R_h(a_h)(w_h - a_h) + R_l(a_l)(w_l - a_l). \tag{4.58}$$

Taking partial derivatives with respect to w_h and w_l:

$$\frac{\partial\Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial w_h} = R_h(a_h) > 0,$$
$$\frac{\partial\Pi^{Col}(a_h, a_l, w_h, w_l)}{\partial w_l} = R_l(a_l) > 0.$$

$\Pi^{Col}(a_h, a_l, w_h, w_l)$ is strictly increasing in both w_h and w_l. The collector raises w_l to its upper bound within the Hybrid case. Raising w_l beyond $\bar{w}_l(a_h, a_l)$, defined by $R_h(a_h) + R_l(a_l) = Q_l^{unc}(w_l)$, would move the remanufacturer into the Partially

Utilized case, where $Q_l^*(w_l, a_h) < R_l(a_l)$, reducing the collector's revenue from low quality. The collector therefore sets:

$$w_l = \bar{w}_l(a_h, a_l) = \frac{\alpha - 2[R_h(a_h) + R_l(a_l)]}{\gamma} - c_l, \tag{4.59}$$

$$w_h = \bar{w}_l(a_h, a_l) + c_l - c_h = \frac{\alpha - 2[R_h(a_h) + R_l(a_l)]}{\gamma} - c_h. \tag{4.60}$$

Substituting (4.59) and (4.60) into (4.58), and writing $R_h(a_h) = \delta_h(a_h - \beta_h)$ and $R_l(a_l) = \delta_l(a_l - \beta_l)$:

$$\begin{aligned}\Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l)) = {} & \frac{\delta_h(a_h - \beta_h)}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) \\ & - (2\delta_h + \gamma)(a_h - \beta_h) \\ & - 2\delta_l(a_l - \beta_l)\big] \\ & + \frac{\delta_l(a_l - \beta_l)}{\gamma}\big[\alpha - \gamma(\beta_l + c_l) \\ & - 2\delta_h(a_h - \beta_h) \\ & - (2\delta_l + \gamma)(a_l - \beta_l)\big].\end{aligned} \tag{4.61}$$

The second-order partial derivatives of (4.61) are:

$$\begin{aligned}\frac{\partial^2 \Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l))}{\partial a_h^2} &= -\frac{2\delta_h(2\delta_h + \gamma)}{\gamma} < 0, \\ \frac{\partial^2 \Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l))}{\partial a_l^2} &= -\frac{2\delta_l(2\delta_l + \gamma)}{\gamma} < 0, \\ \frac{\partial^2 \Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l))}{\partial a_h \partial a_l} &= -\frac{4\delta_h\delta_l}{\gamma}.\end{aligned}$$

The Hessian is:

$$H = -\frac{2}{\gamma}\begin{pmatrix} \delta_h(2\delta_h + \gamma) & 2\delta_h\delta_l \\ 2\delta_h\delta_l & \delta_l(2\delta_l + \gamma) \end{pmatrix}.$$

The leading principal minor is $-2\delta_h(2\delta_h + \gamma)/\gamma < 0$ and the determinant is:

$$\begin{aligned}\det(H) &= \frac{4}{\gamma^2}\big[\delta_h(2\delta_h + \gamma)\delta_l(2\delta_l + \gamma) - 4\delta_h^2\delta_l^2\big] \\ &= \frac{4\delta_h\delta_l\gamma(2\delta_h + 2\delta_l + \gamma)}{\gamma^2} > 0.\end{aligned}$$

Since the leading principal minor is negative and the determinant is positive, H is negative definite and (4.61) is strictly jointly concave in (a_h, a_l). Setting both first-

order conditions to zero:

$$\frac{\partial \Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l))}{\partial a_h} = \frac{\delta_h}{\gamma}\big[\alpha - \gamma(\beta_h + c_h) - (4\delta_h + 2\gamma)(a_h - \beta_h) - 4\delta_l(a_l - \beta_l)\big] = 0, \tag{4.62}$$

$$\frac{\partial \Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l))}{\partial a_l} = \frac{\delta_l}{\gamma}\big[\alpha - \gamma(\beta_l + c_l) - 4\delta_h(a_h - \beta_h) - (4\delta_l + 2\gamma)(a_l - \beta_l)\big] = 0. \tag{4.63}$$

This is a 2×2 linear system in $(a_h - \beta_h, a_l - \beta_l)$. To eliminate $(a_l - \beta_l)$, multiply (4.62) by $(4\delta_l + 2\gamma)$ and (4.63) by $4\delta_l$, then subtract:

$$(4\delta_l + 2\gamma)[\alpha - \gamma(\beta_h + c_h)] - 4\delta_l[\alpha - \gamma(\beta_l + c_l)] - \big[(4\delta_h + 2\gamma)(4\delta_l + 2\gamma) - 16\delta_h\delta_l\big](a_h - \beta_h) = 0.$$

The bracketed coefficient simplifies:

$$(4\delta_h + 2\gamma)(4\delta_l + 2\gamma) - 16\delta_h\delta_l = 8\delta_h\gamma + 8\delta_l\gamma + 4\gamma^2 = 4\gamma(2\delta_h + 2\delta_l + \gamma).$$

Solving for $a_h - \beta_h$ gives (4.44). By the same steps, eliminating $(a_h - \beta_h)$ instead gives (4.45). Substituting a_h^{Hyb*} and a_l^{Hyb*} into (4.59) and (4.60) gives (4.46), and $Q_h^{Hyb*} = \delta_h(a_h^{Hyb*} - \beta_h)$ and $Q_l^{Hyb*} = \delta_l(a_l^{Hyb*} - \beta_l)$ give (4.47).

We verify feasibility of the Hybrid candidate. $a_h^{Hyb*} \geq \beta_h$: expanding the numerator of (4.44):

$$(2\delta_l + \gamma)[\alpha - \gamma(\beta_h + c_h)] - 2\delta_l[\alpha - \gamma(\beta_l + c_l)] = \gamma[\alpha - \gamma(\beta_h + c_h)] + 2\delta_l\gamma[(\beta_l + c_l) - (\beta_h + c_h)].$$

The first term is strictly positive by market viability and the second is non-negative by quality ordering, so $a_h^{Hyb*} > \beta_h$. $a_l^{Hyb*} \geq \beta_l$: expanding the numerator of (4.45),

$$(2\delta_h+\gamma)[\alpha-\gamma(\beta_l+c_l)]-2\delta_h[\alpha-\gamma(\beta_h+c_h)] = \gamma[\alpha-\gamma(\beta_l+c_l)]-2\delta_h\gamma\big[(\beta_l+c_l)-(\beta_h+c_h)\big].$$

The first term is strictly positive by market viability, but the second is non-positive by quality ordering, so the sum is not signed in general. Dividing by $\gamma > 0$, non-negativity of this numerator is equivalent to

$$\alpha + 2\delta_h(\beta_h - \beta_l + c_h - c_l) - \gamma(\beta_l + c_l) \;\geq\; 0,$$

which is the second condition of (4.48). Unlike $a_h^{Hyb*} > \beta_h$, which holds for all admissible parameters, $a_l^{Hyb*} \geq \beta_l$ is a genuine restriction: quality ordering favours the high quality class, so only the low quality class's participation can fail.

It remains to verify that the transfer prices are non-negative. Since $w_h^{Hyb*} = w_l^{Hyb*} + c_l - c_h > w_l^{Hyb*}$ as $c_l > c_h$ by quality ordering, it suffices to verify $w_l^{Hyb*} \geq 0$. From (4.46):

$$w_l^{Hyb*} = \frac{\alpha - 2Q^{Hyb*}}{\gamma} - c_l,$$

where $Q^{Hyb*} = \delta_h(a_h^{Hyb*} - \beta_h) + \delta_l(a_l^{Hyb*} - \beta_l)$. Adding (4.44) multiplied by δ_h and (4.45) multiplied by δ_l:

$$\begin{aligned} Q^{Hyb*} = &\frac{\delta_h[(2\delta_l + \gamma)(\alpha - \gamma(\beta_h + c_h)) - 2\delta_l(\alpha - \gamma(\beta_l + c_l))]}{2\gamma(2\delta_h + 2\delta_l + \gamma)} \\ &+ \frac{\delta_l[(2\delta_h + \gamma)(\alpha - \gamma(\beta_l + c_l)) - 2\delta_h(\alpha - \gamma(\beta_h + c_h))]}{2\gamma(2\delta_h + 2\delta_l + \gamma)}. \end{aligned}$$

In the numerator, the coefficient of $[\alpha - \gamma(\beta_h + c_h)]$ is $\delta_h(2\delta_l + \gamma) - 2\delta_h\delta_l = \delta_h\gamma$, and the coefficient of $[\alpha - \gamma(\beta_l + c_l)]$ is $\delta_l(2\delta_h + \gamma) - 2\delta_h\delta_l = \delta_l\gamma$. Therefore:

$$Q^{Hyb*} = \frac{\delta_h[\alpha - \gamma(\beta_h + c_h)] + \delta_l[\alpha - \gamma(\beta_l + c_l)]}{2(2\delta_h + 2\delta_l + \gamma)}.$$

Substituting into $w_l^{Hyb*} \geq 0$ and multiplying through by $\gamma > 0$:

$$\alpha - \frac{\delta_h[\alpha - \gamma(\beta_h + c_h)] + \delta_l[\alpha - \gamma(\beta_l + c_l)]}{2\delta_h + 2\delta_l + \gamma} - \gamma c_l \geq 0,$$

which is the third condition of (4.48). The first condition, $a_h^{Hyb*} \geq \beta_h$, holds automatically as shown above, and the second is equivalent to the non-negativity of the numerator of (4.45).

When Case 2 ($w_l + c_l \leq w_h + c_h$) holds, the same analysis applies with h and l interchanged throughout. □

Table 4.3 summarizes the three candidates and their feasibility conditions. The Hybrid candidate dominates both the Partially Utilized and High Quality Only candidates whenever it is feasible, and the Partially Utilized candidate is never optimal even when the Hybrid candidate is infeasible. The decentralized equilibrium is therefore always either the Hybrid or the High Quality Only candidate, as established in Proposition 4.4 below.

Table 4.3: Summary of collector's equilibrium candidates under Case 1.

Candidate	Q_h^*	Q_l^*	**Feasibility**
High Quality Only	$R_h(a_h^{HQ*})$	0	Always feasible under market viability (4.35)
Partially Utilized	$R_h(a_h^{PU*})$	$\in (0,\ R_l(a_l^{PU*}))$	$\delta_h(a_h^{PU*} - \beta_h) \leq (\alpha - \gamma c_l)/4$ (4.42)
Hybrid	$R_h(a_h^{Hyb*})$	$R_l(a_l^{Hyb*})$	Condition (4.48): $a_h^{Hyb*} \geq \beta_h$, $a_l^{Hyb*} \geq \beta_l$, $w_l^{Hyb*} \geq 0$

Proposition 4.4. *The Partially Utilized candidate is never the decentralized equilibrium. The decentralized equilibrium is the Hybrid candidate when* (4.48) *holds, and the High Quality Only candidate otherwise.*

Proof. Within the Hybrid case the collector sets the transfer prices at their upper bounds (4.59) and (4.60), so its profit is the function $\Pi^{Col}(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l))$ of (4.61), and its problem is

$$\max_{a_h,\, a_l}\ \Pi^{Col}\big(a_h, a_l, \bar{w}_h(a_h, a_l), \bar{w}_l(a_h, a_l)\big) \quad \text{where} \quad a_h \geq \beta_h,\ \ a_l \geq \beta_l,\ \ \bar{w}_l(a_h, a_l) \geq 0. \tag{4.64}$$

The objective is strictly concave in (a_h, a_l), since its Hessian is negative definite as shown above, and the three constraints are linear in (a_h, a_l), so (4.64) is a strictly concave program over a polyhedron and has a unique maximizer. We characterize it, and then show that the Partially Utilized candidate cannot attain it.

The maximizer of (4.64). If the unconstrained stationary point (a_h^{Hyb*}, a_l^{Hyb*}) of Proposition 4.3 satisfies (4.48), it is the maximizer, and the equilibrium is the Hybrid candidate.

Suppose instead $a_l^{Hyb*} < \beta_l$. The maximizer then lies on the face $a_l = \beta_l$, on which $R_l(\beta_l) = 0$ and the objective reduces to

$$\Pi^{Col}(a_h, \beta_l, \cdot) = R_h(a_h)\left[\frac{\alpha - 2R_h(a_h)}{\gamma} - c_h - a_h\right],$$

which is precisely the High Quality Only problem, whose maximizer is a_h^{HQ*} with profit $\Pi^{Col,HQ*}$ given in (4.38).

Suppose instead $w_l^{Hyb*} < 0$. The maximizer then lies on the face $\bar{w}_l(a_h, a_l) = 0$, on which the collector receives no transfer revenue for low quality cores while paying $a_l \geq \beta_l > 0$ for each one collected. Its profit is therefore strictly decreasing in a_l on that face, so it sets $a_l = \beta_l$, again yielding $R_l(\beta_l) = 0$ and the High Quality Only candidate.

In every case the maximizer of (4.64) is either the Hybrid candidate or the High Quality Only candidate.

The Partially Utilized candidate is feasible for (4.64). By Proposition 4.3, $a_h^{PU*} > \beta_h$. The collector reduces a_l to the lowest value consistent with $Q_l^{PU*} > 0$, so $R_l(a_l^{PU*}) = Q_l^{PU*} > 0$ and hence $a_l^{PU*} > \beta_l$. Because $R_l(a_l^{PU*}) = Q_l^{PU*}$, by (4.42) the total supply satisfies $R_h(a_h^{PU*}) + R_l(a_l^{PU*}) = (\alpha - \gamma c_l)/4$, and substituting this into (4.59) gives

$$\bar{w}_l\big(a_h^{PU*}, a_l^{PU*}\big) = \frac{\alpha - 2\big[R_h(a_h^{PU*}) + R_l(a_l^{PU*})\big]}{\gamma} - c_l = \frac{\alpha - \gamma c_l}{2\gamma} = w_l^{PU*} > 0,$$

by market viability. The Partially Utilized candidate therefore satisfies all three constraints of (4.64) and attains the value $\Pi^{Col,PU*}$ there.

Since the Partially Utilized candidate is feasible for (4.64) but is not its maximizer, and since that maximizer is either the Hybrid candidate or the High Quality Only candidate,

$$\Pi^{Col,PU*} \leq \max\big\{\Pi^{Col,Hyb*},\ \Pi^{Col,HQ*}\big\}.$$

The Partially Utilized candidate is therefore never strictly optimal, and the decentralized equilibrium is the Hybrid candidate when (4.48) holds and the High Quality Only candidate otherwise. □

Proposition 4.5 (Decentralized inefficiency)**.** *The decentralized equilibrium produces strictly less total quantity and strictly lower total profit than the centralized optimum:*

$$Q^{D*} < Q^{C*} \qquad \textit{and} \qquad \Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*}) < \Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}).$$

Proof. In the High Quality Only equilibrium the system reduces to the single quality class model of Chapter 3, and the claim follows from Proposition 3.3. We therefore treat the Hybrid equilibrium, in which both quality classes are acquired. Throughout,

$\alpha - \gamma(\beta_h + c_h) \geq \alpha - \gamma(\beta_l + c_l)$ by the quality ordering, and $\alpha - \gamma(\beta_l + c_l) > 0$ because the low quality class is acquired at a positive quantity: by (4.29), $a_l^{C*} > \beta_l$ requires $(\delta_h + \gamma)\,[\alpha - \gamma(\beta_l + c_l)] > \delta_h\,[\alpha - \gamma(\beta_h + c_h)] \geq 0$.

Because the transfer prices are internal payments between the collector and the remanufacturer, and because all collected cores are remanufactured in the Hybrid equilibrium ($Q_k^{Hyb*} = R_k(a_k^{Hyb*})$ for $k \in \{h, l\}$), the total decentralized profit equals the centralized objective evaluated at the equilibrium acquisition prices:

$$\Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*}) = \Pi^C(a_h^{Hyb*}, a_l^{Hyb*}),$$

where $\Pi^C(a_h, a_l)$ is the centralized objective with the market-clearing price substituted, as in Lemma 4.2. By Lemma 4.2 this function is strictly jointly concave, with unique maximizer (a_h^{C*}, a_l^{C*}) given in Proposition 4.1. It therefore suffices to show that the equilibrium acquires strictly less of each class, $a_h^{Hyb*} < a_h^{C*}$ and $a_l^{Hyb*} < a_l^{C*}$.

Comparing the equilibrium acquisition prices (4.63)–(4.64) with the centralized ones (4.28)–(4.29), the inequality $a_h^{Hyb*} < a_h^{C*}$ is equivalent, after clearing the positive denominators $2\gamma(2\delta_h + 2\delta_l + \gamma)$ and $2\gamma(\delta_h + \delta_l + \gamma)$, to

$$\begin{aligned}(\delta_h + \delta_l + \gamma)\big[(2\delta_l + \gamma)(\alpha - \gamma(\beta_h + c_h)) - 2\delta_l(\alpha - \gamma(\beta_l + c_l))\big] \\ < (2\delta_h + 2\delta_l + \gamma)\big[(\delta_l + \gamma)(\alpha - \gamma(\beta_h + c_h)) - \delta_l(\alpha - \gamma(\beta_l + c_l))\big].\end{aligned}$$

The two sides differ by $-\gamma\big[\delta_h(\alpha - \gamma(\beta_h + c_h)) + \delta_l(\alpha - \gamma(\beta_l + c_l))\big]$, so the inequality holds if and only if

$$\delta_h(\alpha - \gamma(\beta_h + c_h)) + \delta_l(\alpha - \gamma(\beta_l + c_l)) > 0,$$

which is immediate since both terms are positive. The inequality $a_l^{Hyb*} < a_l^{C*}$ is equivalent, in the same way, to

$$\begin{aligned}(\delta_h + \delta_l + \gamma)\big[(2\delta_h + \gamma)(\alpha - \gamma(\beta_l + c_l)) - 2\delta_h(\alpha - \gamma(\beta_h + c_h))\big] \\ < (2\delta_h + 2\delta_l + \gamma)\big[(\delta_h + \gamma)(\alpha - \gamma(\beta_l + c_l)) - \delta_h(\alpha - \gamma(\beta_h + c_h))\big],\end{aligned}$$

whose two sides differ by the same quantity and which reduces to the same condition. Both acquisition prices therefore fall short of the centralized level by the same positive amount,

$$a_k^{C*} - a_k^{Hyb*} = \frac{\delta_h(\alpha - \gamma(\beta_h + c_h)) + \delta_l(\alpha - \gamma(\beta_l + c_l))}{2(\delta_h + \delta_l + \gamma)(2\delta_h + 2\delta_l + \gamma)} > 0, \qquad k \in \{h, l\}.$$

Since $Q_k = \delta_k(a_k - \beta_k)$ is strictly increasing in a_k,

$$Q^{D*} = \delta_h(a_h^{Hyb*} - \beta_h) + \delta_l(a_l^{Hyb*} - \beta_l) < \delta_h(a_h^{C*} - \beta_h) + \delta_l(a_l^{C*} - \beta_l) = Q^{C*}.$$

Moreover $(a_h^{Hyb*}, a_l^{Hyb*}) \neq (a_h^{C*}, a_l^{C*})$, so strict concavity of Π^C at its unique maximizer gives

$$\Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*}) = \Pi^C(a_h^{Hyb*}, a_l^{Hyb*}) < \Pi^C(a_h^{C*}, a_l^{C*}) = \Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}).$$

The source of the inefficiency is the same double-marginalization channel identified in Chapter 3: acting as an intermediary, the collector sets each transfer price above its per-unit acquisition cost, which raises the remanufacturer's effective marginal cost for both classes and leads the decentralized chain to acquire strictly less of each than the integrated benchmark. □

4.3 Coordination

A contract *coordinates* the two-quality supply chain if it induces the decentralized system to achieve the centralized quantities Q_h^{C*} and Q_l^{C*} and total profit $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$. A contract satisfies *individual rationality* if both the collector and the remanufacturer earn at least their decentralized equilibrium profits, $\Pi^{Col}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})$ and $\Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*})$ respectively, so that both parties voluntarily participate. Since $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) > \Pi^D(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})$, the coordination surplus is always available to satisfy individual rationality. We analyze two coordination contracts: a two-part tariff and a revenue sharing contract.

4.3.1 Two-Part Tariff

Under a two-part tariff contract, the collector charges per-unit transfer prices w_h and w_l for high-quality and low-quality cores respectively, and a fixed fee $F \geq 0$ paid by the remanufacturer to the collector regardless of quantity. The fixed fee does not affect the remanufacturer's marginal costs and therefore does not distort the quantity decisions. It serves purely as a profit allocation mechanism. The profit functions

under the two-part tariff are:

$$\Pi^R_{TPT}(Q_h, Q_l) = \frac{\alpha - Q_h - Q_l}{\gamma}(Q_h + Q_l) - (w_h + c_h)Q_h - (w_l + c_l)Q_l - F,$$

$$\Pi^{Col}_{TPT}(a_h, a_l, w_h, w_l, F) = w_h R_h(a_h) + w_l R_l(a_l) - a_h R_h(a_h) - a_l R_l(a_l) + F.$$

We characterize the equilibrium under the two-part tariff by backward induction, as in Section 4.2.

4.3.1.1 Remanufacturer's Problem

For given (a_h, a_l, w_h, w_l, F) set by the collector, the remanufacturer chooses Q_h and Q_l to maximize $\Pi^R_{TPT}(Q_h, Q_l)$:

$$\begin{aligned} \max_{Q_h, Q_l} \quad & \Pi^R_{TPT}(Q_h, Q_l) \\ \text{where} \quad & 0 \le Q_h \le R_h(a_h) \\ & 0 \le Q_l \le R_l(a_l). \end{aligned}$$

Since F does not appear in the first-order conditions with respect to Q_h and Q_l, the remanufacturer's problem is structurally identical to the decentralized Stage 2 problem of Section 4.2.1. By Lemma 4.3, the remanufacturer's optimal quantities are given by the same piecewise best response (4.34), with $w_h + c_h$ and $w_l + c_l$ as the effective costs.

4.3.1.2 Collector's Problem

The collector chooses (a_h, a_l, w_h, w_l, F) to maximize $\Pi^{Col}_{TPT}(a_h, a_l, w_h, w_l, F)$:

$$\begin{aligned} \max_{a_h, a_l, w_h, w_l, F} \quad & \Pi^{Col}_{TPT}(a_h, a_l, w_h, w_l, F) \\ \text{where} \quad & a_h \ge \beta_h && (4.65) \\ & a_l \ge \beta_l && (4.66) \\ & w_h \ge 0 && (4.67) \\ & w_l \ge 0 && (4.68) \\ & F \ge 0 && (4.69) \\ & \Pi^R_{TPT}(Q^*_h, Q^*_l) \ge \Pi^R(Q^{Hyb*}_h, Q^{Hyb*}_l). && (4.70) \end{aligned}$$

Constraints (4.65)–(4.66) ensure acquisition prices attract supply. Constraints (4.67)–(4.68) ensure transfer prices are non-negative. Constraint (4.69) ensures the fixed fee is non-negative. Constraint (4.70) is the individual rationality constraint for the remanufacturer: it must earn at least its decentralized equilibrium profit, or it would prefer the decentralized contract instead.

Since Π^{Col}_{TPT} is strictly increasing in F, the collector sets F as large as possible subject to constraint (4.70) binding. Let $\Pi^{R}_{gross}(Q_h, Q_l)$ denote the remanufacturer's profit before the fixed fee:

$$\Pi^{R}_{gross}(Q_h, Q_l) = \Pi^{R}_{TPT}(Q_h, Q_l) + F.$$

Setting constraint (4.70) to equality gives $F = \Pi^{R}_{gross}(Q_h^{C*}, Q_l^{C*}) - \Pi^{R}(Q_h^{Hyb*}, Q_l^{Hyb*})$. With F determined this way, maximizing Π^{Col}_{TPT} over (a_h, a_l, w_h, w_l) is equivalent to maximizing total system profit. The collector therefore effectively maximizes $\Pi^{C}(a_h, a_l, p)$ and sets $a_h = a_h^{C*}$, $a_l = a_l^{C*}$.

Theorem 4.1. *The two-part tariff contract with acquisition prices $a_h = a_h^{C*}$, $a_l = a_l^{C*}$, transfer prices*

$$w_h^{TPT} = \frac{\alpha - 2Q^{C*}}{\gamma} - c_h = 2a_h^{C*} - \beta_h, \tag{4.71}$$

$$w_l^{TPT} = \frac{\alpha - 2Q^{C*}}{\gamma} - c_l = 2a_l^{C*} - \beta_l, \tag{4.72}$$

and fixed fee

$$F = \frac{(Q^{C*})^2}{\gamma} - \Pi^{R}(Q_h^{Hyb*}, Q_l^{Hyb*}), \tag{4.73}$$

coordinates the two-quality supply chain. The remanufacturer earns exactly its decentralized equilibrium profit, and the collector earns strictly more than its decentralized equilibrium profit.

Proof. Since the collector effectively maximizes total system profit, it sets $a_h = a_h^{C*}$ and $a_l = a_l^{C*}$, so that $R_h(a_h^{C*}) = Q_h^{C*}$ and $R_l(a_l^{C*}) = Q_l^{C*}$ by Lemma 4.1.

For the remanufacturer to choose (Q_h^{C*}, Q_l^{C*}), both supply constraints must bind and the remanufacturer must be willing to order the centralized mix. For this, both effective costs must be equal:

$$w_h^{TPT} + c_h = w_l^{TPT} + c_l.$$

This common effective cost must equal the remanufacturer's unconstrained optimum at Q^{C*}. Setting the first-order condition of $\Pi^R_{gross}(Q_h, Q_l)$ with respect to total quantity $Q = Q_h + Q_l$ equal to zero at $Q = Q^{C*}$:

$$\left.\frac{\partial \Pi^R_{gross}(Q_h, Q_l)}{\partial Q}\right|_{Q=Q^{C*}} = \frac{\alpha - 2Q^{C*}}{\gamma} - (w_i^{TPT} + c_i) = 0, \quad i \in \{h, l\}.$$

Solving for the common effective cost:

$$w_i^{TPT} + c_i = \frac{\alpha - 2Q^{C*}}{\gamma}, \quad i \in \{h, l\},$$

which gives (4.71) and (4.72). The second equality in each, $w_h^{TPT} = 2a_h^{C*} - \beta_h$ and $w_l^{TPT} = 2a_l^{C*} - \beta_l$, follows from the centralized first-order condition (4.28), which gives $\frac{\alpha - 2Q^{C*}}{\gamma} = 2a_i^{C*} - \beta_i + c_i$ for each $i \in \{h, l\}$.

Since both effective costs are equal, the remanufacturer is willing to order the centralized mix (Q_h^{C*}, Q_l^{C*}) with $R_h(a_h^{C*}) = Q_h^{C*}$ and $R_l(a_l^{C*}) = Q_l^{C*}$ from (4.34). Total system profit equals $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$.

For the fixed fee, the remanufacturer's gross profit at (Q_h^{C*}, Q_l^{C*}) is:

$$\Pi^R_{gross}(Q_h^{C*}, Q_l^{C*}) = \left(p^{C*} - w_h^{TPT} - c_h\right) Q_h^{C*} + \left(p^{C*} - w_l^{TPT} - c_l\right) Q_l^{C*}.$$

From (4.71) and (4.72), $p^{C*} - w_i^{TPT} - c_i = p^{C*} - \frac{\alpha - 2Q^{C*}}{\gamma} = \frac{Q^{C*}}{\gamma}$ for both $i \in \{h, l\}$. Therefore:

$$\Pi^R_{gross}(Q_h^{C*}, Q_l^{C*}) = \frac{Q^{C*}}{\gamma}(Q_h^{C*} + Q_l^{C*}) = \frac{(Q^{C*})^2}{\gamma}.$$

Setting constraint (4.70) to equality gives (4.73). The remanufacturer earns $\Pi^R_{gross}(Q_h^{C*}, Q_l^{C*}) - F = \Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*})$ by construction. The collector earns $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) - \Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*})$, which strictly exceeds $\Pi^{Col}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})$ since $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) > \Pi^D(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})$. □

We verify feasibility of the contract.

$w_h^{TPT} \geq 0$: from (4.71), $w_h^{TPT} = 2a_h^{C*} - \beta_h > \beta_h > 0$ since $a_h^{C*} > \beta_h$ from Section 4.1.

$w_l^{TPT} \geq 0$: by the same argument, $w_l^{TPT} = 2a_l^{C*} - \beta_l > 0$.

$F \geq 0$: we need $\Pi^R_{gross}(Q_h^{C*}, Q_l^{C*}) \geq \Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*})$. Since the remanufacturer earns a strictly positive margin $\frac{Q^{C*}}{\gamma} > 0$ per unit at the coordinating contract, and

$Q^{C*} > Q^{Hyb*}$ (by an analogous inefficiency argument to Chapter 3), the gross profit strictly exceeds the decentralized profit, so $F > 0$.

Theorem 4.2. *A feasible fixed fee interval $F \in [\underline{F}, \bar{F}]$ satisfying both individual rationality constraints always exists, where:*

$$\bar{F} = \frac{(Q^{C*})^2}{\gamma} - \Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*}),$$
$$\underline{F} = \Pi^{Col}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*}) - \left[\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) - \frac{(Q^{C*})^2}{\gamma}\right].$$

Proof. $\bar{F} - \underline{F} = \Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) - \Pi^D(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*}) > 0$ by Proposition 4.5, so the interval $[\underline{F}, \bar{F}]$ is non-empty. □

4.3.2 Revenue Sharing

Under a revenue sharing contract, the remanufacturer retains a fraction $\phi \in (0, 1)$ of sales revenue and pays per-unit transfer prices $w_{rs,h}$ and $w_{rs,l}$ to the collector for high-quality and low-quality cores respectively. The collector receives the remaining fraction $1 - \phi$ of revenue plus the transfer payments. Following Cachon [5], we treat $(\phi, w_{rs,h}, w_{rs,l})$ as a triple agreed jointly in advance, fixed before the game is played.

4.3.2.1 Remanufacturer's Problem

For given $(a_h, a_l, w_{rs,h}, w_{rs,l})$ and agreed ϕ, the remanufacturer chooses Q_h and Q_l to maximize its profit:

$$\Pi_{RS}^R(Q_h, Q_l) = \phi \cdot \frac{\alpha - Q_h - Q_l}{\gamma}(Q_h + Q_l) - (w_{rs,h} + c_h)Q_h - (w_{rs,l} + c_l)Q_l. \quad (4.74)$$

The remanufacturer's problem is:

$$\begin{aligned} \max_{Q_h, Q_l} \quad & \Pi_{RS}^R(Q_h, Q_l) \\ \text{where} \quad & 0 \le Q_h \le R_h(a_h) \\ & 0 \le Q_l \le R_l(a_l). \end{aligned}$$

Lemma 4.5. *The remanufacturer's optimal quantities under the revenue sharing contract follow the same piecewise structure as Lemma 4.3, with unconstrained optima:*

$$Q_{rs,h}^{unc}(w_{rs,h}) = \frac{\phi\alpha - \gamma(w_{rs,h} + c_h)}{2\phi}, \tag{4.75}$$

$$Q_{rs,l}^{unc}(w_{rs,l}) = \frac{\phi\alpha - \gamma(w_{rs,l} + c_l)}{2\phi}. \tag{4.76}$$

Proof. The second-order derivative of (4.74) with respect to total quantity $Q = Q_h + Q_l$ is $-2\phi/\gamma < 0$, so Π_{RS}^R is strictly concave in Q. The remanufacturer uses the cheaper quality first, as in Lemma 4.3. Setting the first-order derivative of (4.74) with respect to Q_h, holding $Q_l = 0$, equal to zero:

$$\frac{\partial \Pi_{RS}^R(Q_h, 0)}{\partial Q_h} = \phi \cdot \frac{\alpha - 2Q_h}{\gamma} - (w_{rs,h} + c_h) = 0.$$

Solving for Q_h gives (4.75). By the same steps with $Q_h = 0$:

$$\frac{\partial \Pi_{RS}^R(0, Q_l)}{\partial Q_l} = \phi \cdot \frac{\alpha - 2Q_l}{\gamma} - (w_{rs,l} + c_l) = 0,$$

giving (4.76). The full piecewise best response follows by the same argument as Lemma 4.3, replacing Q_h^{unc} and Q_l^{unc} with $Q_{rs,h}^{unc}$ and $Q_{rs,l}^{unc}$. □

4.3.2.2 Coordination via an Implementable Contract

Coordination requires the quantities actually transacted to equal Q_h^{C*} and Q_l^{C*}. When the collector freely optimizes $w_{rs,h}$ and $w_{rs,l}$ as Stackelberg leader for a fixed, exogenously agreed ϕ, the resulting equilibrium quantities fall strictly short of Q_h^{C*} and Q_l^{C*} for every $\phi \in (0, 1)$. The full derivation of this result, and the proof that no $\phi \in (0, 1)$ can coordinate the system through the Stackelberg mechanism, are given in Appendix B.

Proposition 4.6. *For every $\phi \in (0, 1)$, the freely-optimizing revenue sharing contract cannot coordinate the two-quality supply chain. This is why $(\phi, w_{rs,h}, w_{rs,l})$ are instead fixed jointly in advance, as below.*

We therefore treat $(\phi, w_{rs,h}, w_{rs,l})$ as jointly agreed in advance. Coordination requires both $R_h(a_h) = Q_h^{C*}$ and $R_l(a_l) = Q_l^{C*}$, so the collector must set $a_h = a_h^{C*}$ and

$a_l = a_l^{C*}$. Additionally, the remanufacturer's unconstrained demands must equal Q_h^{C*} and Q_l^{C*}:

$$Q_{rs,h}^{unc}(w_{rs,h}) = Q_h^{C*}, \qquad Q_{rs,l}^{unc}(w_{rs,l}) = Q_l^{C*}.$$

Theorem 4.3. *For any $\phi \in (0,1)$, the revenue sharing contract with acquisition prices $a_h = a_h^{C*}$, $a_l = a_l^{C*}$, and quality-specific transfer prices*

$$w_{rs,h}^{coord} = \phi\left(a_h^{C*} + c_h\right) - c_h, \tag{4.77}$$

$$w_{rs,l}^{coord} = \phi\left(a_l^{C*} + c_l\right) - c_l, \tag{4.78}$$

coordinates the two quality class supply chain, achieving (Q_h^{C*}, Q_l^{C*}) and total profit $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$, and allocates shares ϕ and $1-\phi$ of system profit to the remanufacturer and the collector respectively.

Proof. Setting $a_q = a_q^{C*}$ gives $R_q(a_q^{C*}) = Q_q^{C*}$ for $q \in \{h, l\}$, so the supply available of each quality class is exactly the centralized quantity.

By (4.77)–(4.78), the remanufacturer's effective unit cost of class q is $w_{rs,q}^{coord} + c_q = \phi\big(a_q^{C*} + c_q\big)$. Facing revenue share ϕ, the remanufacturer solves

$$\max_{0 \le Q_q \le R_q(a_q^{C*})} \; \phi \cdot \frac{\alpha - Q_h - Q_l}{\gamma}\,(Q_h + Q_l) - \sum_{q \in \{h,l\}} \phi\big(a_q^{C*} + c_q\big) Q_q,$$

which is strictly concave in (Q_h, Q_l). The remanufacturer acquires both classes up to their supply limits provided its marginal revenue at the total centralized quantity is at least the effective cost of each class, that is, provided

$$\phi\big(a_q^{C*} + c_q\big) \;\le\; \phi \cdot \frac{\alpha - 2Q^{C*}}{\gamma}, \qquad q \in \{h, l\}. \tag{4.79}$$

From the centralized first-order conditions, $\dfrac{\alpha - 2Q^{C*}}{\gamma} = 2a_q^{C*} - \beta_q + c_q$ for each $q \in \{h, l\}$. Since $a_q^{C*} > \beta_q$,

$$\phi\big(a_q^{C*} + c_q\big) \;<\; \phi\big(2a_q^{C*} - \beta_q + c_q\big) = \phi \cdot \frac{\alpha - 2Q^{C*}}{\gamma}, \qquad q \in \{h, l\},$$

so (4.79) holds strictly for both classes. The remanufacturer therefore purchases the entire available supply of each class: $Q_q = R_q(a_q^{C*}) = Q_q^{C*}$, and the transacted total is Q^{C*}. Total system profit is $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$.

For the profit split, the transacted quantities are (Q_h^{C*}, Q_l^{C*}) and the market price is p^{C*}, so

$$\Pi_{RS}^R = \phi\, p^{C*}Q^{C*} - \sum_{q\in\{h,l\}} \big(w_{rs,q}^{coord} + c_q\big)Q_q^{C*} = \phi\, p^{C*}Q^{C*} - \phi \sum_{q\in\{h,l\}} \big(a_q^{C*} + c_q\big)Q_q^{C*}$$
$$= \phi\Big[p^{C*}Q^{C*} - \sum_{q\in\{h,l\}} c_q Q_q^{C*} - \sum_{q\in\{h,l\}} a_q^{C*} R_q(a_q^{C*})\Big] = \phi \cdot \Pi^C\big(a_h^{C*}, a_l^{C*}, p^{C*}\big),$$

using $R_q(a_q^{C*}) = Q_q^{C*}$. The collector receives the residual, $(1-\phi)\,\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$. □

Remark 4.2. Setting $w_{rs,q}^{coord} = 0$ identifies, for each quality class, the revenue share at which the per-unit transfer payment for that class vanishes:

$$0 = \phi\big(a_q^{C*} + c_q\big) - c_q \quad \text{gives} \quad \phi = \frac{c_q}{a_q^{C*} + c_q}, \qquad q \in \{h, l\}.$$

Only the larger of the two is attainable: at the smaller value the other class would require a negative transfer price, which is infeasible. The larger value therefore coincides with the feasibility lower bound ϕ_{min} defined in (4.80) below, at which one class is acquired with no per-unit transfer payment and the collector recovers its acquisition cost for that class solely through the revenue share $1 - \phi$.

We require $w_{rs,h}^{coord} \geq 0$ and $w_{rs,l}^{coord} \geq 0$. By (4.77)–(4.78), these hold if and only if

$$\phi \;\geq\; \phi_{min} \;\equiv\; \max\left\{\frac{c_h}{a_h^{C*} + c_h},\; \frac{c_l}{a_l^{C*} + c_l}\right\}. \tag{4.80}$$

Neither class binds uniformly: although $c_l > c_h$ raises the low quality threshold, the centralized acquisition prices satisfy $a_l^{C*} < a_h^{C*}$ in general, and the ordering of the two ratios depends on the parameters. Since $a_q^{C*} > \beta_q \geq 0$ and $c_q > 0$, each ratio lies in $(0, 1)$, so $\phi_{min} \in (0, 1)$.

Under the coordinating contract, the profits are:

$$\Pi_{RS}^R(\phi) = \phi \cdot \Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}),$$
$$\Pi_{RS}^{Col}(\phi) = (1-\phi) \cdot \Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}).$$

Individual rationality requires both parties to earn at least their decentralized equilibrium profits. For the remanufacturer, $\Pi_{RS}^R(\phi) \geq \Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*})$ requires:

$$\phi \geq \frac{\Pi^R(Q_h^{Hyb*}, Q_l^{Hyb*})}{\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})} \equiv \underline{\phi}.$$

For the collector, $\Pi^{Col}_{RS}(\phi) \geq \Pi^{Col}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})$ requires:

$$\phi \leq 1 - \frac{\Pi^{Col}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})}{\Pi^{C}(a_h^{C*}, a_l^{C*}, p^{C*})} \equiv \bar{\phi}.$$

Theorem 4.4. *A feasible range of coordinating revenue shares $\phi \in [\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$ satisfying both individual rationality constraints always exists.*

Proof. $\bar{\phi} - \underline{\phi} = \frac{\Pi^{C}(a_h^{C*}, a_l^{C*}, p^{C*}) - \Pi^{D}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*})}{\Pi^{C}(a_h^{C*}, a_l^{C*}, p^{C*})} > 0$ by the inefficiency result, so $[\underline{\phi}, \bar{\phi}]$ is non-empty. Whether $\phi_{min} \leq \bar{\phi}$ depends on parameter values and is verified computationally in Chapter 5. □

Both the two-part tariff and the revenue sharing contract coordinate the two-quality supply chain. The two-part tariff coordinates through quality-specific transfer prices (w_h^{TPT}, w_l^{TPT}) and a feasible fixed fee interval $F \in [\underline{F}, \bar{F}]$. The revenue sharing contract coordinates through a family of implementable contracts $(\phi, w_{rs,h}^{coord}, w_{rs,l}^{coord})$ parametrized by $\phi \in [\max(\phi_{min}, \underline{\phi}), \bar{\phi}]$, where for each ϕ both transfer prices are uniquely determined by (4.77)–(4.78), and the acquisition prices are fixed at a_h^{C*} and a_l^{C*}. Higher ϕ allocates more profit to the remanufacturer, and lower ϕ allocates more to the collector. Both contracts exploit the coordination surplus $\Pi^{C}(a_h^{C*}, a_l^{C*}, p^{C*}) - \Pi^{D}(a_h^{Hyb*}, a_l^{Hyb*}, w_h^{Hyb*}, w_l^{Hyb*}) > 0$ to satisfy individual rationality.

CHAPTER 5

COMPUTATIONAL STUDY

Although the single-quality model has closed-form solutions, a computational study is still useful for two reasons. First, it provides numerical benchmarks for direct comparison with the two-quality results. Second, it uncovers a notable property of the efficiency loss (it depends only on supply and demand sensitivity parameters, not on remanufacturing cost or market size), which is not immediately apparent from the formulas alone.

The two-quality model involves a piecewise equilibrium structure with two candidates, parameter-dependent transitions between the Hybrid and High Quality Only equilibria, and coordination conditions whose feasibility cannot be determined analytically in general. A computational study is therefore both necessary and informative for the two-quality model.

This chapter presents a computational study of both the single-quality and two-quality class models developed in Chapters 3 and 4 respectively, organized around the following three research questions:

1. How do equilibrium quantities, system profits, efficiency loss, and consumer surplus respond to changes in cost and market parameters?

2. How does the feasible fixed-fee interval $[\underline{F},\ \bar{F}]$ of the two-part tariff vary with parameters?

3. Under what parameter conditions is the revenue sharing implementable contract feasible, that is, for what parameter values does the feasible range $[\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$ remain non-empty?

The chapter is organized as follows. Section 5.1 presents the computational study of the single-quality class model, including Base Case results, sensitivity analysis of profits and efficiency loss, and sensitivity analysis of coordination contracts. Section 5.2 presents the same analysis for the two-quality model. Section 5.3 compares the key findings across the two models.

All numerical results in this chapter are computed using MATLAB. For each parameter, $n = 200$ equally spaced grid points are used over the experimental range, giving the step sizes reported in the experimental setting tables. The step sizes are chosen small enough that the discrete curves are indistinguishable from continuous functions in the figures.

5.1 Single Quality Class of Returns

5.1.1 Experimental Setting

Table 5.1 reports the Base Case parameter values and experimental ranges. Each parameter is varied individually over its experimental range while all others are held at their Base Case values. The Base Case parameter values are selected to be consistent with the empirical setting of Guide et al. [7], who study ReCellular, a remanufacturer of used cell phones. The remanufacturing cost $c = 5$ corresponds to the best quality grade (Class 1) in their classification. The market size $\alpha = 100$ and demand sensitivity $\gamma = 2$ are chosen to ensure a sufficiently large and profitable market while maintaining market viability $\alpha > \gamma(\beta + c) = 14$. The supply sensitivity $\delta = 2$ reflects a moderately elastic supply of cores, and the minimum acquisition price $\beta = 2$ reflects a positive reservation price for core holders. The experimental range for $\delta \in [0.5,\ 15]$ covers a broad spectrum from inelastic to highly elastic supply, allowing us to examine the approach to the theoretical upper bound of 25% on the efficiency loss. The range for $\gamma \in [1,\ 10]$ is bounded below at 1 to avoid extreme profit levels at very low demand sensitivity, and bounded above at 10 to maintain market viability $\alpha > \gamma(\beta + c) = 7\gamma$. The range for $c \in [0.1,\ 45]$ spans from near-zero remanufacturing cost to the worst quality grade in Guide et al.'s classification, and the range for $\alpha \in [15,\ 200]$ covers markets from barely viable to large and profitable. The

minimum acquisition price β is held fixed throughout the analysis. Every equilibrium quantity, price, and profit in the model depends on β and c only through their sum: the centralized quantity is $Q^{C*} = \delta[\alpha - \gamma(\beta + c)]/[2(\delta + \gamma)]$ and the centralized profit is $\Pi^C(a^{C*}, p^{C*}) = \delta[\alpha - \gamma(\beta + c)]^2/[4\gamma(\delta + \gamma)]$, and the same holds for the decentralized equilibrium. Varying β is therefore equivalent to varying c by the same amount, and the sensitivity analysis of c already covers it. The only quantities that depend on β separately are the acquisition prices a^{C*} and a^{D*}, which shift by exactly β. The relative efficiency loss, $\delta^2/(2\delta+\gamma)^2$ by Corollary 3.1, is invariant to α, β, and c altogether.

Table 5.1: Base Case parameters and experimental ranges for the single quality class model.

Par.	**Description**	**Base Case**	**Range**	**Step**
α	Market size	100	$[15,\ 200]$	0.9296
γ	Demand sensitivity	2	$[1,\ 10]$	0.0452
δ	Supply sensitivity	2	$[0.5,\ 15]$	0.0729
β	Min. acquisition price	2	—	—
c	Remanufacturing cost	5	$[0.1,\ 45]$	0.2256

The performance measures used throughout are defined and computed as follows:

- The centralized total profit $\Pi^C(a^{C*}, p^{C*})$ is the system-level profit under the centralized optimal solution:

$$\Pi^C(a^{C*}, p^{C*}) = p^{C*}(\alpha - \gamma p^{C*}) - (a^{C*} + c)\delta(a^{C*} - \beta).$$

- The decentralized total profit $\Pi^D(a^{D*}, w^{D*})$ is the sum of the collector's and remanufacturer's profits at the decentralized equilibrium:

$$\Pi^D(a^{D*}, w^{D*}) = \Pi^{Col}(a^{D*}, w^{D*}) + \Pi^R(Q^{D*}),$$

  where:

$$\Pi^{Col}(a^{D*}, w^{D*}) = (w^{D*} - a^{D*})Q^{D*},$$
$$\Pi^R(Q^{D*}) = (p^{D*} - w^{D*} - c)Q^{D*}.$$

- The absolute efficiency loss $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})$ measures the total profit sacrificed due to double marginalization.

- The relative efficiency loss $[\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})]/\Pi^C(a^{C*}, p^{C*})$ expresses the efficiency loss as a percentage of the centralized profit.

- Consumer surplus measures the welfare captured by consumers beyond the market price. Under the linear inverse demand $p(Q) = (\alpha - Q)/\gamma$, consumer surplus at equilibrium quantity Q^* is:

$$CS(Q^*) = \int_0^{Q^*} \frac{\alpha - q}{\gamma}\, dq - p^* \cdot Q^* = \frac{(Q^*)^2}{2\gamma}.$$

  Consumer surplus therefore increases in total quantity and decreases in demand sensitivity γ. We report $CS(Q^{C*}) = (Q^{C*})^2/(2\gamma)$ and $CS(Q^{D*}) = (Q^{D*})^2/(2\gamma)$ for the centralized and decentralized solutions respectively. The CS relative loss $[CS(Q^{C*}) - CS(Q^{D*})]/CS(Q^{C*})$ expresses the consumer welfare loss as a percentage of the centralized consumer surplus.

5.1.2 Base Case Results

Table 5.2 reports the Base Case numerical results. Market viability $\alpha = 100 > \gamma(\beta + c) = 14$ is satisfied.

The decentralized equilibrium produces $Q^{D*} = 14.33$ units compared to $Q^{C*} = 21.50$ at the centralized optimum, a reduction of 33% in quantity due to double marginalization. The transfer price markup $w^{D*} - a^{D*} = 21.50$ inflates the remanufacturer's effective marginal cost from $a^{D*} + c = 14.17$ to $w^{D*} + c = 35.67$, causing it to order strictly less than the system optimum. The resulting relative efficiency loss of 11.11% equals $\delta^2/(2\delta + \gamma)^2 = 4/36$ exactly, consistent with Corollary 3.1. Consumers bear a disproportionately larger welfare loss of 55.56%, since consumer surplus is proportional to $(Q^*)^2$ and the quantity reduction is amplified when squared.

For the coordinating contracts, the two-part tariff is always feasible with fee interval $[\underline{F},\ \bar{F}] = [77.04,\ 128.40]$ of width 51.36, equal to the coordination surplus. The revenue sharing implementable contract is feasible for $\phi \in [0.2817,\ 0.3333]$, with $\phi_{min} = 0.2817$ exceeding $\underline{\phi} = 0.2222$, so the binding constraint is the feasibility

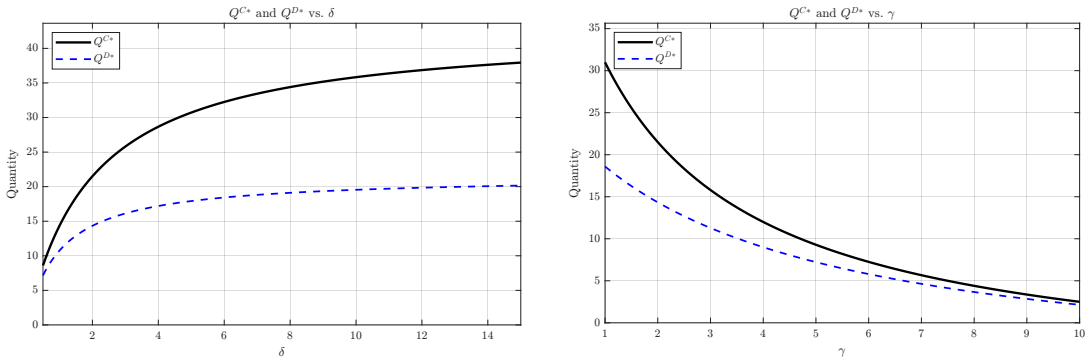
Table 5.2: Base Case results for the single quality class model ($\alpha = 100$, $\gamma = 2$, $\delta = 2$, $\beta = 2$, $c = 5$).

Measure	**Value**	**Description**
Centralized solution		
a^{C*}	12.75	Acquisition price
Q^{C*}	21.50	Quantity
p^{C*}	39.25	Selling price
$\Pi^C(a^{C*}, p^{C*})$	462.25	Total profit
$CS(Q^{C*})$	115.56	Consumer surplus
Decentralized equilibrium		
a^{D*}	9.17	Acquisition price
w^{D*}	30.67	Transfer price
Q^{D*}	14.33	Quantity
p^{D*}	42.83	Selling price
$\Pi^{Col}(a^{D*}, w^{D*})$	308.17	Collector profit
$\Pi^R(Q^{D*})$	102.72	Remanufacturer profit
$\Pi^D(a^{D*}, w^{D*})$	410.89	Total profit
$CS(Q^{D*})$	51.36	Consumer surplus
Efficiency analysis		
Absolute efficiency loss	51.36	Total profit loss due to double marginalization
Relative efficiency loss	11.11%	As a percentage of centralized profit
CS relative loss	55.56%	Consumer surplus loss as a percentage of $CS(Q^{C*})$
Coordinating contracts		
w^{TPT}	23.50	TPT transfer price
$[\underline{F},\ \bar{F}]$	$[77.04,\ 128.40]$	TPT fee interval
$\bar{F} - \underline{F}$	51.36	Fee interval width
ϕ_{min}	0.2817	RS feasibility lower bound
$\underline{\phi}$	0.2222	RS remanufacturer IR bound
$\bar{\phi}$	0.3333	RS collector IR bound
$[\phi_{min},\ \bar{\phi}]$	$[0.2817,\ 0.3333]$	RS feasible range
$\bar{\phi} - \underline{\phi}$	0.1111	RS IR interval width

condition $w_{rs}^{coord} \geq 0$ rather than the remanufacturer's individual rationality condition. The IR interval width $\bar{\phi} - \underline{\phi} = 0.1111 = \delta^2/(2\delta + \gamma)^2$ equals the relative efficiency loss, consistent with Theorem 3.3.

5.1.3 Sensitivity Analysis

Figure 5.1 shows the sensitivity of the equilibrium quantities Q^{C*} and Q^{D*} with respect to changing values of supply sensitivity δ and demand sensitivity γ. As δ increases, both quantities increase monotonically since greater supply elasticity allows the collector to source larger volumes at lower marginal acquisition cost. The gap $Q^{C*} - Q^{D*}$ widens as δ increases, reflecting the amplified double marginalization effect when supply is elastic: the collector's markup on w^{D*} diverts an increasing volume of trade away from the centralized optimum. As γ increases, both quantities decrease monotonically since higher price sensitivity reduces the market price for any given quantity, making production less profitable.



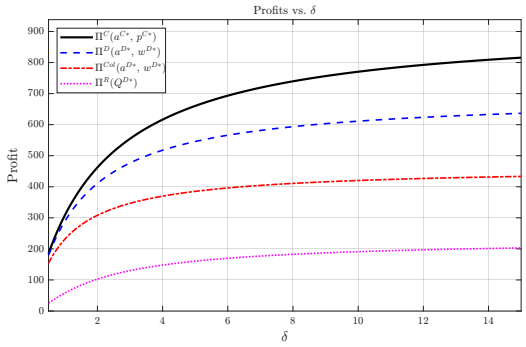
(a) Q^{C*} and Q^{D*} vs. δ when $\gamma = 2$, $\beta = 2$, $c = 5$.

(b) Q^{C*} and Q^{D*} vs. γ when $\delta = 2$, $\beta = 2$, $c = 5$.

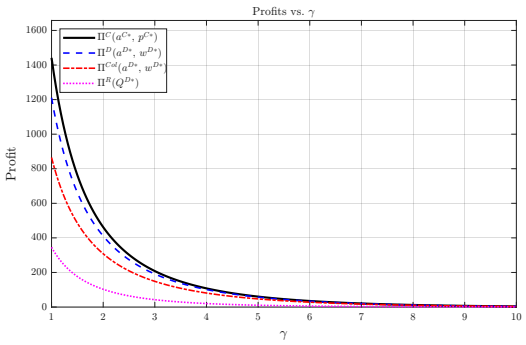
Figure 5.1: Sensitivity of equilibrium quantities Q^{C*} and Q^{D*} to δ and γ when $\alpha = 100$.

Figure 5.2 shows the sensitivity of profits to δ and γ. As δ increases, all four profit functions $\Pi^C(a^{C*}, p^{C*})$, $\Pi^D(a^{D*}, w^{D*})$, $\Pi^{Col}(a^{D*}, w^{D*})$, and $\Pi^R(Q^{D*})$ increase monotonically, consistent with the quantity results above. The gap between $\Pi^C(a^{C*}, p^{C*})$ and $\Pi^D(a^{D*}, w^{D*})$ widens as δ increases. As γ increases, all profit functions decrease monotonically since the shrinking market reduces the revenue base for both systems.

Consumer surplus is defined as $CS(Q^*) = (Q^*)^2/(2\gamma)$, representing the welfare captured by consumers beyond the market price (see Section 5.1.1). Since $CS(Q^*)$ is proportional to $(Q^*)^2$, it follows the same directional pattern as the equilibrium



(a) Profits vs. δ when $\gamma = 2$, $\beta = 2$, $c = 5$.



(b) Profits vs. γ when $\delta = 2$, $\beta = 2$, $c = 5$.

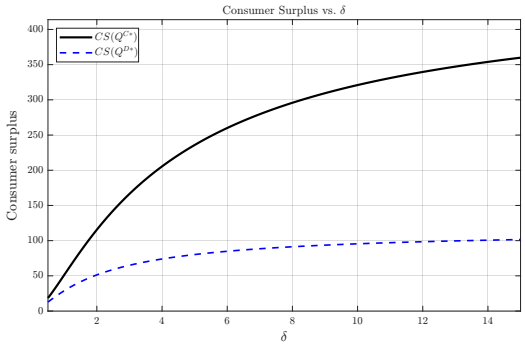
Figure 5.2: Sensitivity of $\Pi^C(a^{C*}, p^{C*})$, $\Pi^D(a^{D*}, w^{D*})$, $\Pi^{Col}(a^{D*}, w^{D*})$, and $\Pi^R(Q^{D*})$ to δ and γ when $\alpha = 100$.

quantities but with amplified differences: the gap between $CS(Q^{C*})$ and $CS(Q^{D*})$ is larger in relative terms than the quantity gap, since squaring amplifies the difference between Q^{C*} and Q^{D*}.

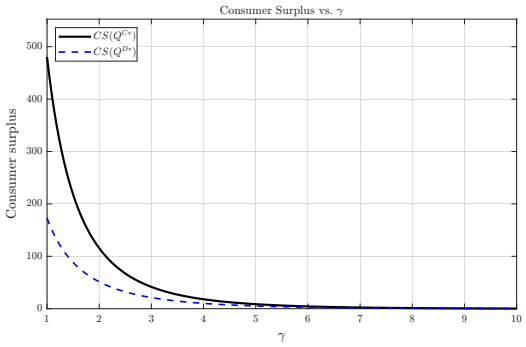
Figure 5.3 shows the sensitivity of consumer surplus to δ and γ. As δ increases, both $CS(Q^{C*})$ and $CS(Q^{D*})$ increase monotonically. The centralized consumer surplus $CS(Q^{C*})$ grows faster than $CS(Q^{D*})$ since the centralized system fully exploits the elastic supply while the decentralized system does not, widening the consumer welfare gap. As γ increases, both $CS(Q^{C*})$ and $CS(Q^{D*})$ decrease monotonically and approach zero, since consumers' willingness to pay above the market price diminishes rapidly when demand is highly price-sensitive.

Figure 5.4 shows the sensitivity of the absolute efficiency loss to δ and γ.

As δ increases, the absolute efficiency loss increases monotonically: greater supply elasticity expands the centralized optimum faster than the decentralized equilibrium, widening the gap $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})$. The relative efficiency loss also increases monotonically, from approximately 3% at $\delta = 0.5$ to approximately 22% at $\delta = 15$, consistent with Corollary 3.1, and approaches the theoretical upper bound of 25% asymptotically. This confirms that more elastic supply amplifies the distortion from double marginalization: when cores are easy to collect, the centralized system fully exploits this while the decentralized system does not, as the collector's markup



(a) $CS(Q^{C*})$ and $CS(Q^{D*})$ vs. δ when $\gamma = 2$, $\beta = 2$, $c = 5$.



(b) $CS(Q^{C*})$ and $CS(Q^{D*})$ vs. γ when $\delta = 2$, $\beta = 2$, $c = 5$.

Figure 5.3: Sensitivity of consumer surplus $CS(Q^{C*}) = (Q^{C*})^2/(2\gamma)$ and $CS(Q^{D*}) = (Q^{D*})^2/(2\gamma)$ to δ and γ when $\alpha = 100$.

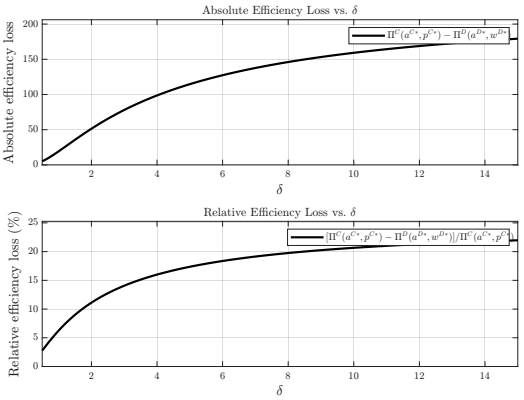
on w^{D*} diverts an increasing volume of trade away from the system optimum.

As γ increases, both absolute and relative efficiency loss decrease monotonically. As the market shrinks, there is less surplus to be lost to double marginalization. The relative efficiency loss decreases from approximately 20% at $\gamma = 1$ to approximately 2% at $\gamma = 10$, consistent with Corollary 3.1. In highly price-sensitive markets, the demand-side friction already limits the equilibrium quantity regardless of the supply-side markup, reducing the structural distortion.

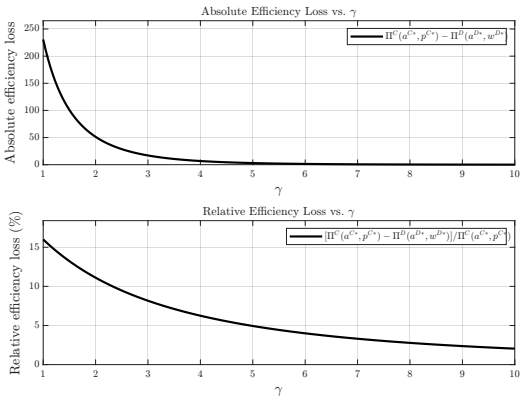
Contrasting Figure 5.4(a) with Figure 5.4(b) reveals an important asymmetry: increasing δ drives the relative efficiency loss toward its upper bound of 25%, while increasing γ drives it toward zero.

The two-part tariff coordinates the supply chain for all parameter values examined. Under the two-part tariff, the remanufacturer pays a per-unit transfer price $w^{TPT} = ((\alpha - 2Q^{C*})/\gamma) - c$ and a fixed fee $F \in [\underline{F},\ \bar{F}]$ to the collector, where the fee interval width $\bar{F} - \underline{F} = \Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})$ equals the coordination surplus (Proposition 3.4).

Figure 5.5 shows the sensitivity of the fee interval to δ, γ, c, and α. As δ increases, both $\bar{F}$ and $\underline{F}$ increase since the coordination surplus grows with supply elasticity. As γ increases, both bounds decrease since the shrinking market reduces the coordination



(a) Absolute and relative efficiency loss vs. δ when $\gamma = 2$, $\beta = 2$, $c = 5$.



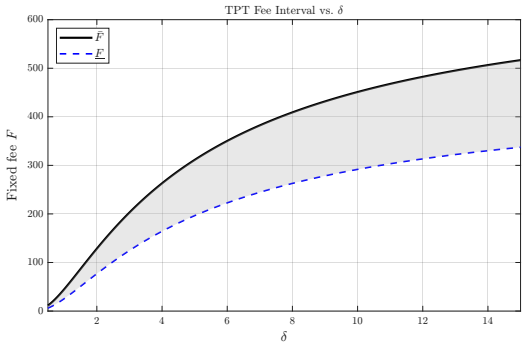
(b) Absolute and relative efficiency loss vs. γ when $\delta = 2$, $\beta = 2$, $c = 5$.

Figure 5.4: Sensitivity of absolute efficiency loss $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})$ (top panel) and relative efficiency loss $[\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})]/\Pi^C(a^{C*}, p^{C*})$ (bottom panel) to δ and γ when $\alpha = 100$.

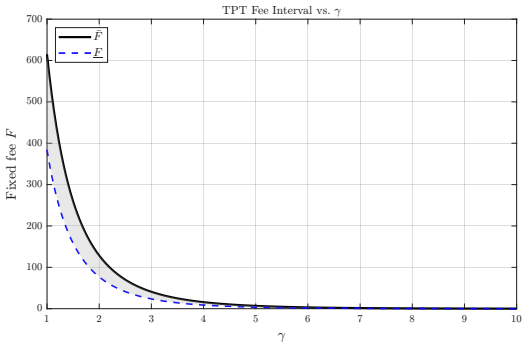
surplus. As c increases, both bounds decrease since higher remanufacturing cost erodes the coordination surplus. As α increases, both bounds increase since a larger market generates more coordination surplus. In all cases the fee interval width equals the absolute efficiency loss, consistent with Proposition 3.4.

The revenue sharing implementable contract is feasible for $\phi \in [\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$, where $\phi_{min} = c/(a^{C*} + c)$ is the feasibility lower bound ensuring $w_{rs}^{coord} \geq 0$ (Remark 3.2), $\underline{\phi} = \Pi^R(Q^{D*})/\Pi^C(a^{C*}, p^{C*})$ is the remanufacturer's individual rationality lower bound, and $\bar{\phi} = 1 - \Pi^{Col}(a^{D*}, w^{D*})/\Pi^C(a^{C*}, p^{C*})$ is the collector's individual rationality upper bound (Theorem 3.3). Note that the width $\bar{\phi} - \underline{\phi} = \delta^2/(2\delta + \gamma)^2$ equals the relative efficiency loss and is invariant to c and α (Corollary 3.1), so these parameters affect feasibility solely through ϕ_{min}. In each figure, the shaded green region indicates the feasible range $[\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$.

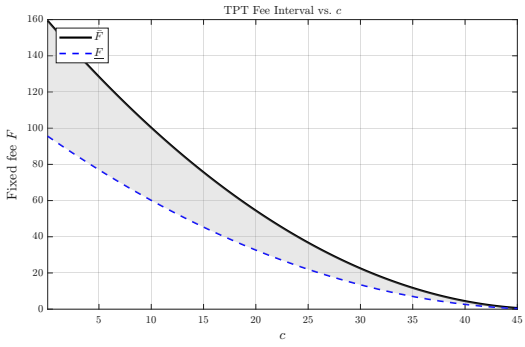
Figure 5.6(a) shows the effect of δ on the implementable contract. The feasibility lower bound ϕ_{min} increases in δ: greater supply elasticity raises Q^{C*} and therefore $a^{C*} = \beta + Q^{C*}/\delta$, increasing $\phi_{min} = c/(a^{C*} + c)$. The upper bound $\bar{\phi}$ also increases in δ but at a slower rate. The contract is feasible for $\delta \in [0.86,\ 8.15]$. For $\delta < 0.86$ and $\delta > 8.15$, ϕ_{min} exceeds $\bar{\phi}$ and the contract becomes infeasible.



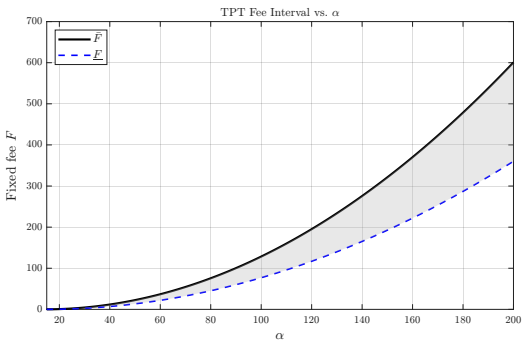
(a) Fee interval vs. δ when $\gamma = 2$, $\beta = 2$, $c = 5$.



(b) Fee interval vs. γ when $\delta = 2$, $\beta = 2$, $c = 5$.



(c) Fee interval vs. c when $\alpha = 100$, $\gamma = 2$, $\delta = 2$, $\beta = 2$.



(d) Fee interval vs. α when $\gamma = 2$, $\delta = 2$, $\beta = 2$, $c = 5$.

Figure 5.5: Sensitivity of the two-part tariff fee interval $[\underline{F},\ \bar{F}]$ to δ, γ, c, and α. Shaded region indicates the feasible interval $[\underline{F},\ \bar{F}]$.

Figure 5.6(b) shows the effect of γ on the implementable contract. As γ increases, a^{C*} decreases, raising $\phi_{min} = c/(a^{C*} + c)$, while $\bar{\phi}$ decreases as the coordination surplus shrinks. The contract is feasible for $\gamma \in [1.00,\ 2.45]$. For $\gamma > 2.45$, ϕ_{min} exceeds $\bar{\phi}$ and the contract becomes infeasible. At $\gamma = 1$, $\underline{\phi}$ exceeds ϕ_{min} and the binding constraint switches to the remanufacturer's individual rationality condition.

Figure 5.6(c) shows the effect of c on the implementable contract. The IR bounds $\bar{\phi}$ and $\underline{\phi}$ are invariant to c since $\bar{\phi} - \underline{\phi} = \delta^2/(2\delta + \gamma)^2$ depends only on δ and γ. However, $\phi_{min} = c/(a^{C*} + c)$ increases in c: higher remanufacturing cost requires a larger revenue share to keep $w_{rs}^{coord} = \phi(a^{C*} + c) - c \geq 0$. The contract is feasible for $c \in [0.10,\ 6.19]$. For $c > 6.19$, ϕ_{min} exceeds $\bar{\phi}$ and the contract becomes infeasible. In industries with high remanufacturing costs, the two-part tariff is therefore the only

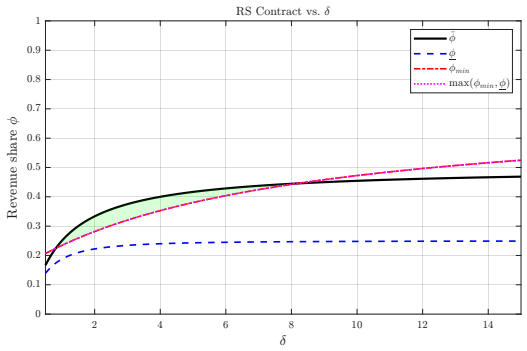
Table 5.3: Sensitivity analysis summary for the single quality class model. Equilibrium quantities Q^{C*} and Q^{D*} follow the same directional pattern as profits and are therefore omitted. $CS(Q^{D*})$ follows the same directional pattern as $CS(Q^{C*})$ in all cases, only $CS(Q^{C*})$ is reported. The two-part tariff is feasible throughout all parameter ranges and is therefore omitted.

Par.	Π^{C*}	Π^{D*}	$CS(Q^{C*})$	**Rel. eff. loss**	**RS feasible range**
δ	Increases	Increases	Increases	Increases toward 25%	$[0.86,\ 8.15]$
γ	Decreases	Decreases	Decreases	Decreases toward 0	$[1.00,\ 2.45]$
c	Decreases	Decreases	Decreases	Invariant	$[0.10,\ 6.19]$
α	Increases	Increases	Increases	Invariant	$[78.22,\ 200]$

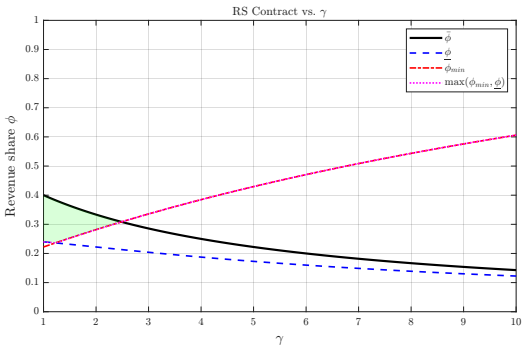
available coordinating mechanism.

Figure 5.6(d) shows the effect of α on the implementable contract. As α increases, Q^{C*} and a^{C*} increase, lowering $\phi_{min} = c/(a^{C*} + c)$, while $\bar{\phi}$ and $\underline{\phi}$ remain invariant. The contract is feasible for $\alpha \in [78.22,\ 200]$. For $\alpha < 78.22$, the market is too small for a^{C*} to bring ϕ_{min} below $\bar{\phi}$ and the contract becomes infeasible.

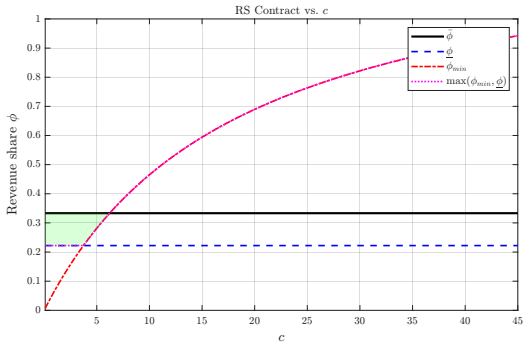
Table 5.3 summarizes the sensitivity findings for the single quality class model. The two-part tariff is feasible throughout all parameter ranges examined. The revenue sharing implementable contract is more sensitive: it becomes infeasible at high remanufacturing cost ($c > 6.19$), small market size ($\alpha < 78.22$), very low or very high supply elasticity ($\delta \notin [0.86,\ 8.15]$), and low demand sensitivity ($\gamma > 2.45$), in all cases because ϕ_{min} exceeds $\bar{\phi}$.



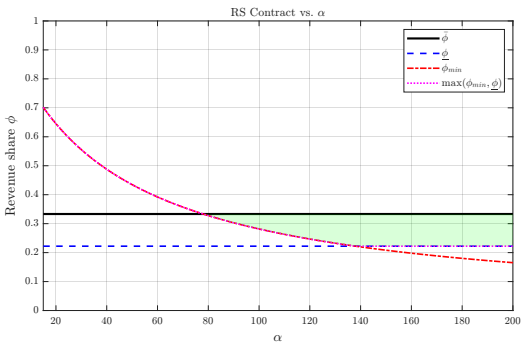
(a) RS contract vs. δ when $\gamma = 2$, $\beta = 2$, $c = 5$.



(b) RS contract vs. γ when $\delta = 2$, $\beta = 2$, $c = 5$.



(c) RS contract vs. c when $\alpha = 100$, $\gamma = 2$, $\delta = 2$, $\beta = 2$.



(d) RS contract vs. α when $\gamma = 2$, $\delta = 2$, $\beta = 2$, $c = 5$.

Figure 5.6: Sensitivity of the revenue sharing implementable contract feasible range $[\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$ to δ, γ, c, and α. Shaded green region indicates the feasible range.

5.2 Two Quality Classes of Returns

All results are reported under Case 1, in which the high quality class is the more economical one for the remanufacturer. Case 2 follows by interchanging the indices (Remark 4.1).

5.2.1 Experimental Setting

Table 5.4 reports the shared parameter values, the two Base Case settings, and the experimental ranges. The Base Case parameter values are motivated by the empirical setting of Guide et al. [7], who study ReCellular, a remanufacturer of used cell phones, and identify six quality grades with remanufacturing costs ranging from \$5 to \$45. The high quality remanufacturing cost $c_h = 5$ corresponds to the best quality grade (Class 1). Base Case A uses the second-best grade ($c_l = 20$, Class 2), representing a setting in which both quality classes are acquired at positive quantities in equilibrium. Base Case B uses the worst grade ($c_l = 45$, Class 6), representing a setting in which the low quality class is unprofitable and the model reduces to the single quality class case. All other parameters are shared across both Base Cases and are chosen to match the single quality class Base Case of Section 5.1.1 for direct comparability.

The minimum acquisition prices β_h and β_l are excluded from the sensitivity analysis. Every equilibrium quantity, profit, and efficiency measure in the model depends on the minimum acquisition price and the remanufacturing cost of a quality class only through their sum. The centralized quantities, for instance, are

$$Q_h^{C*} = \frac{\delta_h\big[\alpha - \gamma(\beta_h + c_h) - \delta_l\big((\beta_h + c_h) - (\beta_l + c_l)\big)\big]}{2(\delta_h + \delta_l + \gamma)},$$

and the same is true of Q_l^{C*}, of the decentralized equilibrium, and of the efficiency loss. Varying β_l is therefore equivalent to varying c_l by the same amount, and the analysis of c_l below already covers it; the same argument applies to β_h and c_h. The only quantities that depend on β_h and β_l separately are the acquisition prices themselves, which shift by exactly β_h and β_l. The sensitivity analysis is carried out over δ_h, δ_l, γ, α, and c_l.

The performance measures are the same as in Section 5.1.1, extended to two quality

Table 5.4: Shared parameters, Base Case settings, and experimental ranges for the two quality class model.

Par.	Description	Base A	Base B	Range	Step
α	Market size	100	100	$[15,\ 200]$	0.9296
γ	Demand sensitivity	2	2	$[1,\ 10]$	0.0452
δ_h	HQ supply sensitivity	2	2	$[0.5,\ 15]$	0.0729
δ_l	LQ supply sensitivity	1	1	$[0.1,\ 10]$	0.0497
β_h	HQ min. acq. price	2	2	—	—
β_l	LQ min. acq. price	1	1	—	—
c_h	HQ remfg. cost	5	5	—	—
c_l	LQ remfg. cost	20	45	$[5,\ 47]$	0.2111

classes. The centralized profit is $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$ and the decentralized profit is $\Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$, with collector profit $\Pi^{Col}(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$ and remanufacturer profit $\Pi^R(Q_h^{D*}, Q_l^{D*})$. Consumer surplus is $CS(Q^{C*}) = (Q^{C*})^2/(2\gamma)$ and $CS(Q^{D*}) = (Q^{D*})^2/(2\gamma)$, where $Q^{C*} = Q_h^{C*} + Q_l^{C*}$ and $Q^{D*} = Q_h^{D*} + Q_l^{D*}$ are total quantities.

5.2.2 Base Case Results

Table 5.5 report the Base Case numerical results for both settings.

The two part tariff coordinates both Base Cases. In each case the fee interval width equals the absolute efficiency loss exactly, $\bar{F} - \underline{F} = \Pi^C - \Pi^D$, which is 61.99 for Base Case A and 51.36 for Base Case B: the fixed fee can redistribute the coordination surplus but cannot create or destroy it.

The revenue sharing implementable contract behaves very differently across the two settings. For Base Case B it is feasible over $\phi \in [0.2817,\ 0.3333]$. For Base Case A it is infeasible: the feasibility lower bound $\phi_{min} = 0.8333$ far exceeds the collector's individual rationality ceiling $\bar{\phi} = 0.3491$. By (4.80), ϕ_{min} is the larger of $c_h/(a_h^{C*} + c_h) = 5/17 = 0.2941$ and $c_l/(a_l^{C*} + c_l) = 20/24 = 0.8333$. It is the low quality class that binds: because $a_l^{C*} = 4$ is small relative to $c_l = 20$, keeping $w_{rs,l}^{coord}$ non-

Table 5.5: Base Case results for the two quality class model, coordinating contracts.

Measure	**Base A**	**Base B**	**Description**
Centralized reference			
a_h^{C*}	12.00	12.75	HQ centralized acquisition price
a_l^{C*}	4.00	—	LQ centralized acquisition price
$\Pi^C - \Pi^D$	61.99	51.36	Coordination surplus
Decentralized profit split			
$\Pi^{Col}(a^{D*}, w^{D*})$	308.19	308.17	Collector equilibrium profit
$\Pi^R(Q^{D*})$	103.32	102.72	Remanufacturer equilibrium profit
Π^{Col}/Π^R	2.98	3.00	Profit ratio $= (2\delta_* + \gamma)/\delta_*$
Two part tariff			
w_h^{TPT}	22.00	23.50	HQ transfer price
w_l^{TPT}	7.00	—	LQ transfer price
$\underline{F}$	99.19	77.04	Minimum fixed fee
$\bar{F}$	161.18	128.40	Maximum fixed fee
$\bar{F} - \underline{F}$	61.99	51.36	Fee interval width
Revenue sharing implementable contract			
$c_h/(a_h^{C*} + c_h)$	0.2941	0.2817	HQ feasibility bound
$c_l/(a_l^{C*} + c_l)$	0.8333	—	LQ feasibility bound
ϕ_{min}	0.8333	0.2817	Binding feasibility bound
$\underline{\phi}$	0.2182	0.2222	Remanufacturer IR bound
$\bar{\phi}$	0.3491	0.3333	Collector IR bound
Feasible range	None	$[0.2817,\ 0.3333]$	$[\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$

negative requires the remanufacturer to retain more than five sixths of revenue, which the collector will not accept. This observation drives the coordination analysis that follows.

5.2.3 Profit Distribution Between the Collector and the Remanufacturer

Table 5.5 reports the total chain profit in the centralized benchmark, the decentralized equilibrium, and under each contract; it is equally instructive to examine how that profit is divided between the collector and the remanufacturer, how the division

changes across the two contracts, and whether either party fares systematically better under one contract than the other. In the decentralized equilibrium the collector, as the Stackelberg leader, sets the acquisition and transfer prices first and leaves the remanufacturer only enough margin to participate. Their profit ratio is

$$\frac{\Pi^{Col}(a^{D*}, w^{D*})}{\Pi^{R}(Q^{D*})} \;=\; \frac{2\delta + \gamma}{\delta} \tag{5.1}$$

in the single class case, independent of α, β, and c. In Base Case B, which is acquired as high quality only and therefore reduces to the single class model, this ratio equals 3: the collector earns 308.17 against the remanufacturer's 102.72, a $75\%/25\%$ division of the decentralized profit 410.89. In Base Case A the same structural advantage produces a near-identical 75% collector share. The advantage arises from the order of moves rather than from any cost or market asymmetry, and it grows with the supply elasticity and shrinks with the demand sensitivity, mirroring the efficiency loss itself. Table 5.6 reports the resulting division under each regime, at both ends of each contract's feasible range.

Table 5.6: Profit division between the collector and the remanufacturer, for both base cases of the two quality class model. Each contract is shown at both ends of its feasible range: for the two-part tariff the fixed fee runs from $\bar{F}$, where the remanufacturer retains exactly its decentralized profit, to $\underline{F}$, where the collector does; for revenue sharing the share runs from $\underline{\phi}$ to $\bar{\phi}$. Revenue sharing is infeasible in Base Case A.

	Base Case A		**Base Case B**	
Regime	**Collector**	**Remanuf.**	**Collector**	**Remanuf.**
Centralized (total)	473.50		462.25	
Decentralized	308.19	103.32	308.17	102.72
share	74.9%	25.1%	75.0%	25.0%
Two-part tariff, $F = \bar{F}$	370.18	103.32	359.53	102.72
Two-part tariff, $F = \underline{F}$	318.81	154.69	308.17	154.08
Revenue sharing, $\phi = \underline{\phi}$	infeasible		332.03	130.22
Revenue sharing, $\phi = \bar{\phi}$	infeasible		308.18	154.07

Both coordinating contracts restore the full centralized profit, so coordination is not a question of whether the chain is better off but of how the recovered surplus $\Pi^C - \Pi^D$,

equal to 51.36 in Base Case B and 61.99 in Base Case A, is divided, and the two contracts distribute it through different instruments. Under the two-part tariff the fixed fee F is a pure transfer between the firms and does not affect the transacted quantity; any $F \in [\underline{F}, \bar{F}]$ is feasible, so at the upper bound $\bar{F}$ the remanufacturer earns exactly its decentralized profit and the collector captures the entire surplus, while at the lower bound $\underline{F}$ the roles reverse. The fee thus slides the division continuously across the whole surplus. Under revenue sharing the split is $(1-\phi, \phi)$ by construction, the remanufacturer receiving a share ϕ of the centralized profit and the collector the complement, so the share ϕ plays the same allocative role as the fee F.

Within each feasible range the division changes linearly. Under revenue sharing the remanufacturer's profit rises from 130.22 to 154.07 as ϕ increases across $[\underline{\phi}, \bar{\phi}]$, and the collector's falls by an equal amount, since the two always sum to Π^C. Under the two-part tariff the same linear trade-off is driven by the fee, the remanufacturer's profit rising from 102.72 to 154.08 as F falls from $\bar{F}$ to $\underline{F}$. In both contracts one party's gain is exactly the other's loss, because the total is fixed at the centralized profit; the contracts differ only in which instrument slides the division and over what floor.

Under both contracts each firm earns at least its decentralized profit, which is precisely the individual rationality requirement: no firm would sign a contract leaving it worse off than the decentralized outcome. The feasible interval is non-empty precisely because coordination generates the surplus $\Pi^C - \Pi^D > 0$ in which both floors can be accommodated. A direct consequence is that neither firm earns systematically more under one contract than the other: both restore the same total and both admit a range of divisions, so a remanufacturer that obtains a given share of chain profit under revenue sharing can obtain the same share under a two-part tariff with the corresponding fee. This is why, in Base Case B, the two contracts are interchangeable on profit grounds. The two ranges are not identical, however. Both reach the same ceiling, where the collector's individual rationality binds, but their floors differ: the two-part tariff can drive the remanufacturer down to its decentralized profit of 102.72, whereas revenue sharing cannot fall below 130.22, the share $\underline{\phi} = \phi_{min}$ at which the transfer price reaches zero. The very constraint that renders revenue sharing infeasible in Base Case A thus works in the remanufacturer's favour in Base Case B, guaranteeing

it a strictly higher minimum. What ultimately distinguishes the two contracts is not the profit either party earns but their robustness: the two-part tariff always admits a feasible fee interval, whereas the revenue sharing interval may be empty. Base Case A is exactly such an instance, and Table 5.5 shows it directly: both contracts would restore the same total, but the revenue sharing feasible range is empty because the binding bound $\phi_{min} = 0.8333$ exceeds the collector's ceiling $\bar{\phi} = 0.3491$, while the two-part tariff coordinates as before. The profit question and the feasibility result are thus two views of the same phenomenon.

Finally, the decentralized $75\%/25\%$ split does not carry over to the coordinated outcome. That split is a product of the Stackelberg timing, where the collector commits to prices before the remanufacturer responds. A coordinating contract is implementable: its terms are fixed jointly in advance, before the game is played, so the sequential timing, and with it the collector's first-mover advantage, no longer applies. Because the trade-off along each feasible range is one-for-one, where the division falls within $[\underline{F}, \bar{F}]$, or within $[\underline{\phi}, \bar{\phi}]$, is a purely distributional question that the timing does not settle; both firms must consent, so the outcome reflects their relative bargaining power. Resolving that point would require an explicit bargaining model over the coordination surplus and lies outside the scope of this thesis, which characterizes the feasible set of divisions consistent with coordination and individual rationality rather than the particular division bargaining would select.

5.2.4 Sensitivity Analysis

In each figure, thick lines correspond to Base Case A ($c_l = 20$, Hybrid equilibrium) and thin lines to Base Case B ($c_l = 45$, High Quality Only). The vertical dashed line marks the transition between the Hybrid and High Quality Only equilibria for Base Case A only. Base Case B remains in the High Quality Only equilibrium throughout all parameter ranges examined and has no transition.

Two transition thresholds recur throughout this section and are worth stating once. For Base Case A, the decentralized equilibrium leaves the Hybrid case when the collector's low quality acquisition price a_l^{Hyb*} falls to its lower bound β_l, so that $R_l(a_l^{Hyb*}) = 0$; the centralized solution leaves it when Q_l^{C*} reaches zero. The two

thresholds do not coincide, and the decentralized system is always the one that acquires the low quality class over the narrower parameter range. Along δ_h the thresholds are 2.07 (decentralized) and 4.14 (centralized); along c_l they are 20.33 and 27.50; along α they are 98 and 70, the order reversed because a larger market makes low quality acquisition more attractive rather than less.

Figure 5.7 shows the sensitivity of equilibrium quantities to δ_h, δ_l, γ, α, and c_l.

As δ_h increases, Q_h^{C*} and Q_h^{D*} increase for both Base Cases, while Q_l^{C*} and Q_l^{D*} decrease for Base Case A: a more elastic high quality supply allows the collector to source the cheaper class in greater volume, displacing low quality acquisition. At $\delta_h \approx 2.07$ the collector's low quality acquisition price a_l^{Hyb*} falls to β_l and the decentralized equilibrium transitions to High Quality Only. At $\delta_h = 58/14 \approx 4.14$, Q_l^{C*} reaches zero and the centralized optimum follows. Above that point Base Case A coincides with Base Case B.

As δ_l increases, Q_l^{C*} and Q_l^{D*} increase for Base Case A since greater low quality supply elasticity allows the collector to source more low quality cores at lower marginal cost. Q_h^{C*} declines steadily as the low quality class displaces it in the centralized solution, falling from 21.32 to 16.14 across the range, while Q_h^{D*} remains approximately flat: the decentralized system acquires only a small volume of low quality cores, so the displacement effect is negligible there. Base Case B quantities are unaffected since low quality is never acquired at $c_l = 45$.

As γ increases, all quantities decrease for both Base Cases since higher price sensitivity reduces the profitable market quantity. Base Case A becomes infeasible at $\gamma \approx 4.76$, where market viability $\alpha > \gamma(\beta_l + c_l)$ fails.

As α increases, all quantities increase for both Base Cases. For Base Case A the low quality class enters in two stages. At $\alpha = 70$ the centralized solution begins acquiring low quality cores, so Q_l^{C*} becomes positive while Q_l^{D*} remains zero. At $\alpha = 98$ the decentralized equilibrium follows, a_l^{Hyb*} rises above β_l, and Q_l^{D*} becomes positive as well. Between these two thresholds the centralized system acquires low quality cores while the decentralized system does not.

As c_l increases, Q_l^{C*} and Q_l^{D*} decrease for Base Case A since higher remanufacturing

cost erodes the profitability of low quality acquisition. At $c_l = 20.33$ the collector's low quality acquisition price a_l^{Hyb*} falls to its lower bound β_l, so Q_l^{D*} reaches zero and the decentralized equilibrium transitions to High Quality Only. At $c_l = 27.50$, Q_l^{C*} also reaches zero and the centralized optimum follows. Throughout, Q_h^{C*} and Q_h^{D*} increase as the high quality class absorbs the volume the low quality class gives up. Base Case B quantities are flat throughout.

Figure 5.8 shows the sensitivity of profits to the same five parameters. All profit functions increase in δ_h, δ_l, and α for Base Case A and decrease in γ and c_l, consistent with the quantity results above. Base Case B profits are unaffected by δ_l since low quality is never acquired, and respond to δ_h, γ, α, and c_l only through the high quality class. The gap between $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$ and $\Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$ for Base Case A exceeds that of Base Case B throughout the Hybrid region, reflecting the additional distortion from double marginalization on two supply classes.

Figure 5.9 shows the sensitivity of consumer surplus. Since $CS(Q^*) = (Q^*)^2/(2\gamma)$ is proportional to total quantity squared, consumer surplus follows the same directional pattern as total quantity, but with amplified differences between the centralized and decentralized cases.

As δ_h increases, both $CS(Q^{C*})$ and $CS(Q^{D*})$ increase for both Base Cases, since a more elastic high quality supply raises total quantity. Above $\delta_h \approx 4.14$ the Base Case A curves coincide with those of Base Case B.

As δ_l increases, $CS(Q^{C*})$ increases for Base Case A since the centralized system acquires more low quality cores, increasing total quantity. $CS(Q^{D*})$ remains approximately flat, since the decentralized total quantity is far less responsive to δ_l: the collector acquires only a small volume of low quality cores in equilibrium. Base Case B consumer surplus is unaffected throughout, since low quality is never acquired at $c_l = 45$.

As γ increases, both $CS(Q^{C*})$ and $CS(Q^{D*})$ decrease for both Base Cases, since higher price sensitivity reduces total quantity. Base Case A becomes infeasible at $\gamma \approx 4.76$, where market viability fails for the low quality class.

As α increases, both consumer surplus measures increase for both Base Cases. The

centralized transition at $\alpha = 70$ for Base Case A produces a visible change in slope in $CS(Q^{C*})$, since the centralized system begins acquiring low quality cores and total quantity increases more steeply. A second, smaller change in slope appears in $CS(Q^{D*})$ at $\alpha = 98$, where the decentralized equilibrium follows.

As c_l increases, $CS(Q^{C*})$ decreases for Base Case A as total centralized quantity falls. At $c_l = 27.50$ it converges to the Base Case B value and remains flat thereafter. $CS(Q^{D*})$ for Base Case A converges to Base Case B earlier, at $c_l = 20.33$, where the decentralized equilibrium transitions to High Quality Only.

Figures 5.10 and 5.11 show the sensitivity of the absolute and relative efficiency loss.

As δ_l increases, the absolute efficiency loss increases for Base Case A: greater low quality supply elasticity expands the centralized optimum faster than the decentralized equilibrium, widening the gap $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) - \Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$. The relative efficiency loss also increases in δ_l, reflecting the growing contribution of the low quality class to total inefficiency. Base Case B is flat throughout since low quality is never acquired.

As γ increases, both absolute and relative efficiency loss decrease for both Base Cases. Base Case A has consistently higher values than Base Case B throughout the feasible range, since operating two quality classes amplifies double marginalization. Base Case A becomes infeasible at $\gamma \approx 4.76$.

As α increases, the absolute efficiency loss increases for both Base Cases, since a larger market generates more coordination surplus to be lost. The relative efficiency loss for Base Case A shows a visible change in slope at $\alpha = 98$: below this threshold the decentralized equilibrium is in the High Quality Only case and, once α is also below 70, so is the centralized solution, in which case the relative loss equals 11.11%, identical to Base Case B. Above $\alpha = 98$ both systems acquire low quality cores and the relative loss rises to approximately 13%, reflecting the additional distortion introduced by the low quality class. Base Case B remains flat at 11.11% throughout, since it never leaves the High Quality Only case. This change in slope identifies the threshold market size above which the two quality class model generates strictly greater inefficiency than the single quality class model.

The c_l analysis reveals the most informative structural feature of the model. Two distinct transitions are visible for Base Case A. At $c_l = 20.33$ the decentralized equilibrium leaves the Hybrid case: a_l^{Hyb*} falls to its lower bound β_l and the collector ceases low quality acquisition. At $c_l = 27.50$ the centralized optimum follows, as Q_l^{C*} reaches zero. Between these two thresholds the centralized system still acquires low quality cores while the decentralized system does not, and the efficiency loss decreases more steeply. For $c_l > 27.50$ both systems are in the High Quality Only case and the relative efficiency loss converges to 11.11%, confirming structural continuity with the single quality class model. Base Case B is flat throughout at 51.36 and 11.11%.

The relative efficiency loss under δ_h deserves separate comment. It increases throughout, as it does in the single quality class model, but the two models do not agree until the low quality class leaves the centralized solution. Above $\delta_h \approx 4.14$, Base Case A is in the High Quality Only case, the model is the single quality class model of Chapter 3, and the relative efficiency loss is exactly $\delta_h^2/(2\delta_h + \gamma)^2$. Below that threshold both quality classes are acquired and the loss is strictly larger than the single quality class value at the same δ_h, as Table 5.7 shows.

Table 5.7: Relative efficiency loss under δ_h, Base Case A, compared with the single quality class model at the same δ_h.

δ_h	Two quality class	Single quality class	Centralized case
0.5	8.43%	2.78%	Hybrid
1.0	10.33%	6.25%	Hybrid
2.0	13.09%	11.11%	Hybrid
3.0	14.39%	14.06%	Hybrid
4.5	16.74%	16.74%	High Quality Only
8.0	19.75%	19.75%	High Quality Only

The gap is largest where high quality supply is least responsive: at $\delta_h = 0.5$ the two quality class loss is more than three times the single quality class value. Quality differentiation therefore does not simply add a second class to the model, it amplifies the inefficiency of decentralization, and it does so most severely precisely where the high quality class is hardest to acquire. Once δ_h is large enough that both the collector

and the centralized system abandon the low quality class, the amplification disappears and the two models coincide.

The two part tariff coordinates the supply chain for both Base Cases throughout all parameter ranges examined. Under the two part tariff, the remanufacturer pays per unit transfer prices $w_h^{TPT} = (\alpha - 2Q^{C*})/\gamma - c_h$ and $w_l^{TPT} = (\alpha - 2Q^{C*})/\gamma - c_l$ and a fixed fee $F \in [\underline{F},\ \bar{F}]$ to the collector, where the fee interval width equals the coordination surplus (Theorem 4.2).

Figure 5.12 shows the sensitivity of the fee interval. As δ_h increases, both bounds increase for both Base Cases as the coordination surplus grows with the volume of high quality trade; above $\delta_h \approx 4.14$ the two Base Cases coincide. As δ_l increases, both bounds increase for Base Case A since the coordination surplus grows as the low quality class contributes more to total surplus, while Base Case B is flat.

At low γ, the fee interval for Base Case A substantially exceeds that of Base Case B, since the large market supports profitable low quality acquisition and the coordination surplus from both classes is substantial. As γ increases, the two Base Cases converge as the shrinking market makes low quality acquisition less profitable. Base Case A terminates at $\gamma \approx 4.76$, where market viability $\alpha > \gamma(\beta_l + c_l)$ fails, while Base Case B remains feasible throughout since $\alpha > \gamma(\beta_h + c_h)$ holds for all $\gamma \in [1,\ 10]$.

As α increases, both bounds increase for both Base Cases. For small α, Base Case A is in the High Quality Only case and its fee interval coincides with Base Case B, since the two models are structurally identical there. At $\alpha = 98$, Base Case A transitions to the Hybrid case and the two curves diverge, with the low quality class beginning to contribute to the coordination surplus.

As c_l increases, both bounds decrease for Base Case A since higher remanufacturing cost reduces the coordination surplus from the low quality class. At $c_l = 20.33$ the rate of decline steepens. This is the decentralized transition, where a_l^{Hyb*} falls to β_l and the collector ceases low quality acquisition, so $\Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$ becomes invariant to c_l while $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$ continues to fall. The fee interval width, which equals the difference between the two profits, therefore narrows more rapidly beyond that point rather than dropping discontinuously: across the transition itself it falls only

from 61.78 to 61.44. At $c_l = 27.50$ the centralized optimum also transitions to High Quality Only, both profits become invariant to c_l, and Base Case A converges exactly to Base Case B, with the interval width constant at 51.36 thereafter.

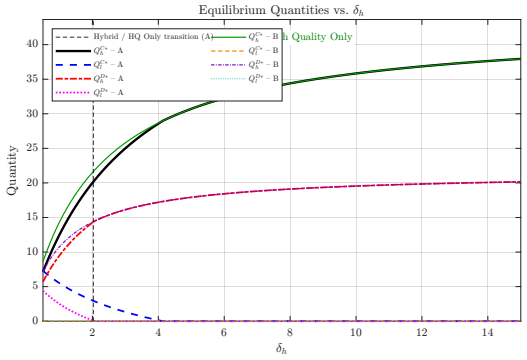
The revenue sharing implementable contract requires $\phi \geq \phi_{min} = \max_q \; c_q/(a_q^{C*} + c_q)$, the maximum taken over the quality classes acquired in the centralized solution ((4.80)). When only the high quality class is acquired this reduces to $c_h/(a_h^{C*} + c_h)$, the single quality class formula of Chapter 3. When both classes are acquired, the low quality class adds a second bound, $c_l/(a_l^{C*} + c_l)$, and it is this bound that governs the results below.

Figure 5.13 shows the sensitivity of the feasible revenue share interval. The contract is feasible for Base Case B over most of the ranges examined. For Base Case A it is feasible only over $\delta_h \in [4.14,\ 8.15]$ and $c_l \in [27.58,\ 47.00]$, and over no part of the δ_l, γ, or α ranges.

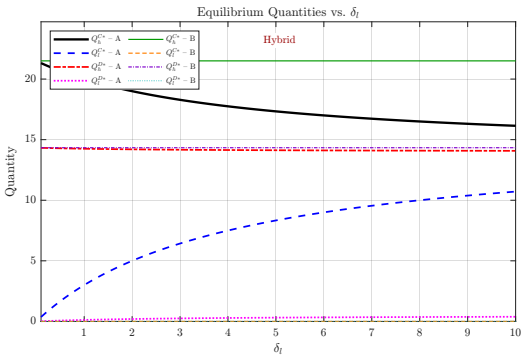
These two intervals are not independent findings. The centralized solution stops acquiring low quality cores at $\delta_h \approx 4.14$ and at $c_l = 27.50$, and the revenue sharing contract becomes feasible at exactly those two points. The mechanism is the low quality bound $c_l/(a_l^{C*} + c_l)$. While the centralized solution acquires both classes, a_l^{C*} is small relative to c_l, so this bound is large: at the Base Case A parameter values it equals 0.8333, against a collector individual rationality ceiling of $\bar{\phi} = 0.3491$. The moment Q_l^{C*} reaches zero the bound disappears, ϕ_{min} falls to $c_h/(a_h^{C*} + c_h) \approx 0.28$, and a feasible interval opens. Above those thresholds Base Case A coincides numerically with Base Case B and inherits its feasibility.

The individual parameter effects follow from this. As δ_l increases, Q_l^{C*} rises and the low quality bound tightens further; the contract is infeasible throughout $\delta_l \in [0.1,\ 10]$, while Base Case B, which never acquires low quality, is feasible throughout. As γ increases, ϕ_{min} for Base Case A rises steeply, exceeding 1 before Base Case A becomes infeasible at $\gamma \approx 4.76$, so the contract is infeasible over the whole range; for Base Case B it is feasible over $\gamma \in [1.00,\ 2.13]$. As α increases, the centralized solution of Base Case A begins acquiring low quality cores at $\alpha = 70$ and the contract becomes infeasible from that point on, while Base Case B is feasible over $\alpha \in [92.16,\ 169.32]$.

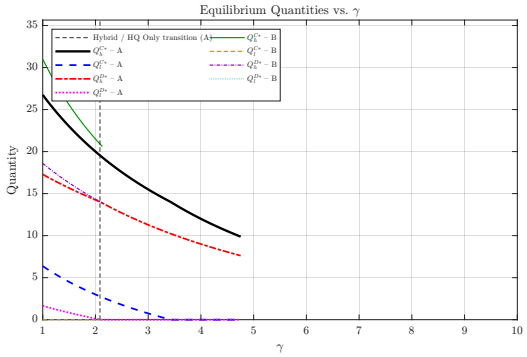
Revenue sharing therefore coordinates the two quality class supply chain only when the chain has ceased to be a two quality class chain. Wherever both quality classes are acquired at positive quantities, the low quality class demands a revenue share that the collector will not accept. The two part tariff, by contrast, remains feasible over every parameter range examined in both Base Cases: its quality specific per unit transfer prices can price the two classes independently, whereas a single revenue share cannot.



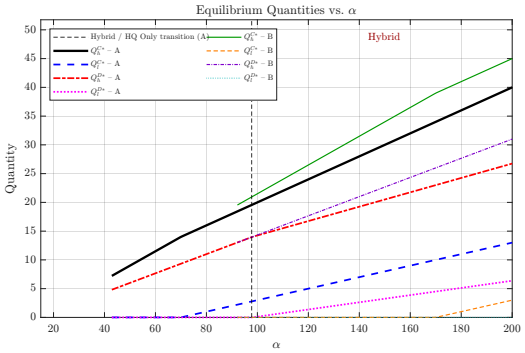
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



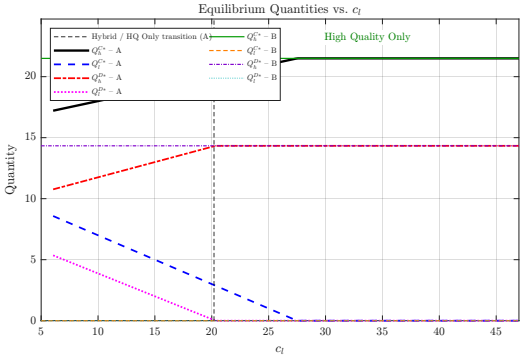
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

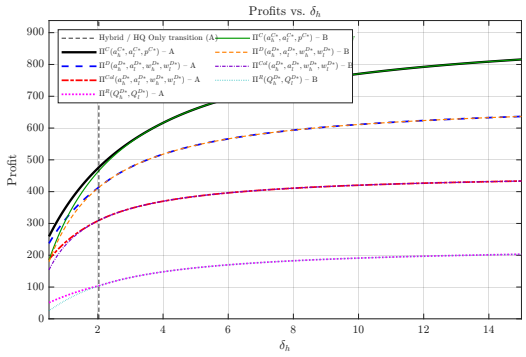


(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

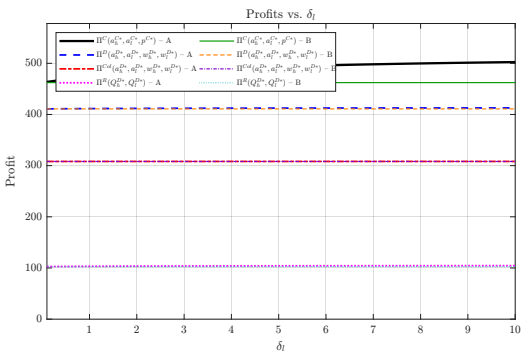


(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

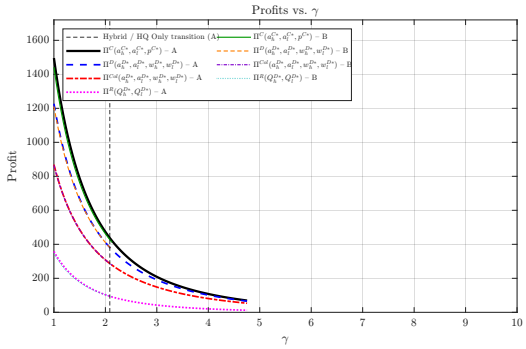
Figure 5.7: Sensitivity of equilibrium quantities Q_h^{C*}, Q_l^{C*}, Q_h^{D*}, and Q_l^{D*} to δ_h, δ_l, γ, α, and c_l. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.



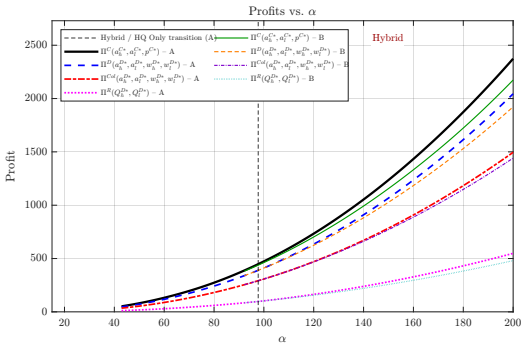
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



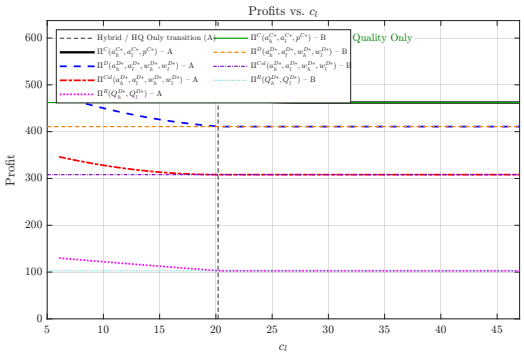
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

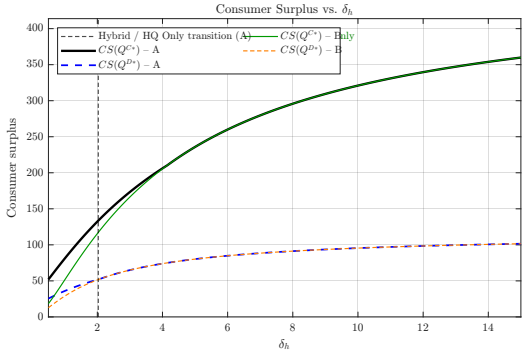


(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

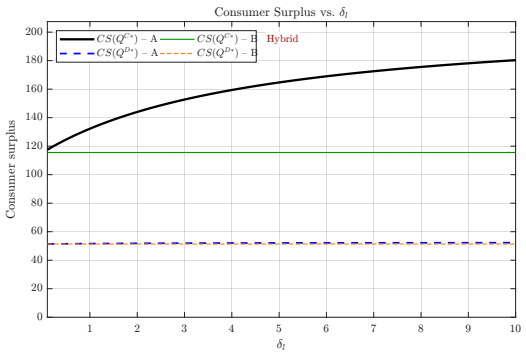


(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

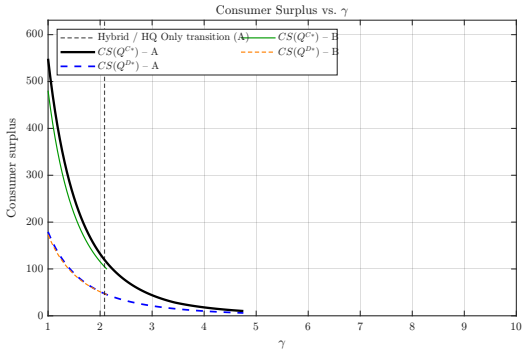
Figure 5.8: Sensitivity of $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*})$, $\Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$, $\Pi^{Col}(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$, and $\Pi^R(Q_h^{D*}, Q_l^{D*})$ to δ_h, δ_l, γ, α, and c_l. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.



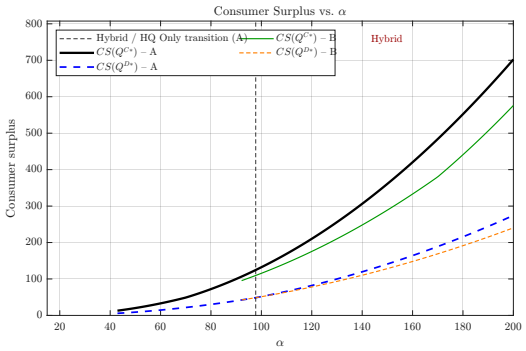
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



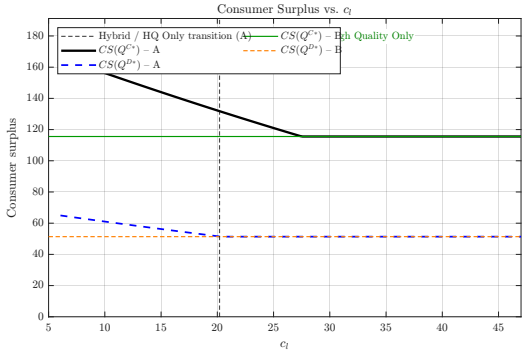
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

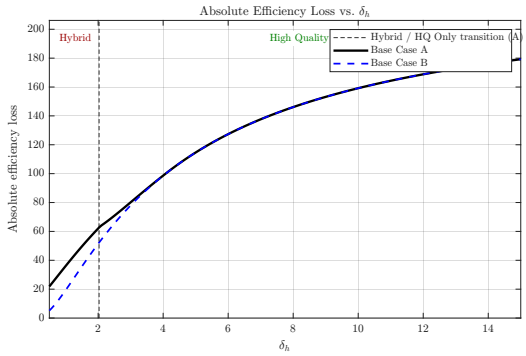


(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

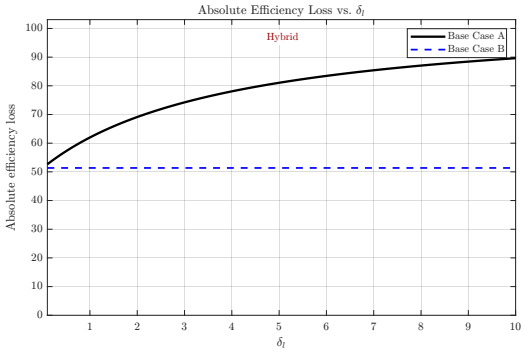


(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

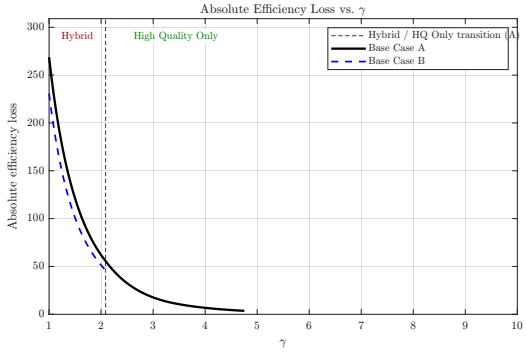
Figure 5.9: Sensitivity of consumer surplus $CS(Q^{C*}) = (Q^{C*})^2/(2\gamma)$ and $CS(Q^{D*}) = (Q^{D*})^2/(2\gamma)$ to δ_h, δ_l, γ, α, and c_l. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.



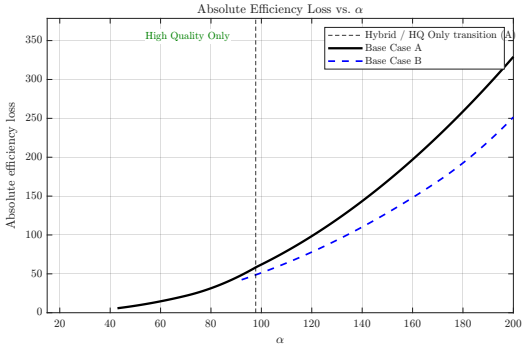
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



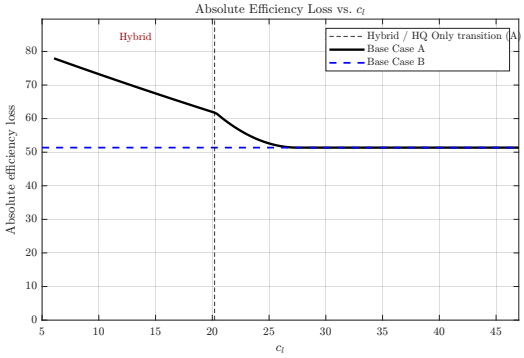
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

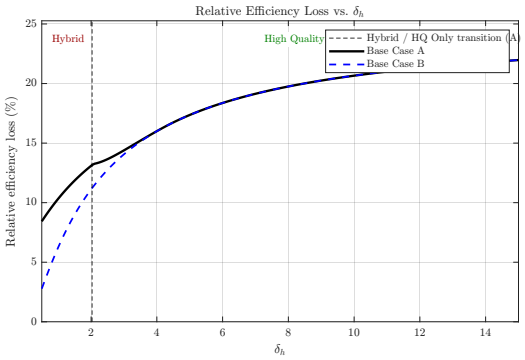


(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

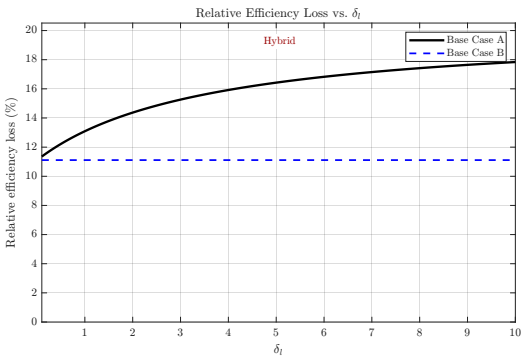


(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

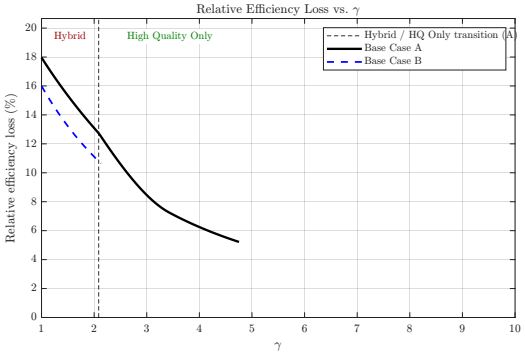
Figure 5.10: Sensitivity of the absolute efficiency loss $\Pi^C(a_h^{C*}, a_l^{C*}, p^{C*}) - \Pi^D(a_h^{D*}, a_l^{D*}, w_h^{D*}, w_l^{D*})$ to δ_h, δ_l, γ, α, and c_l. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.



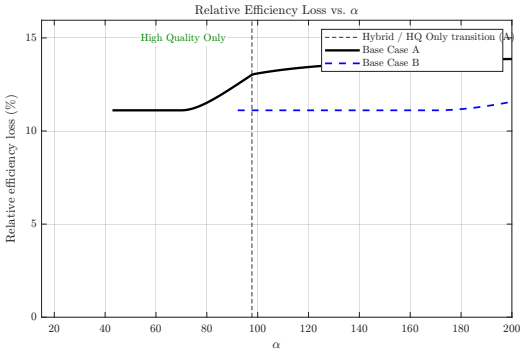
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



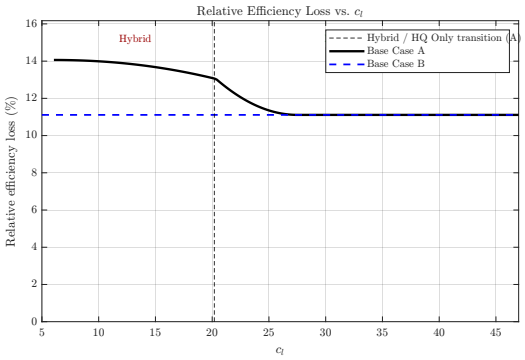
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

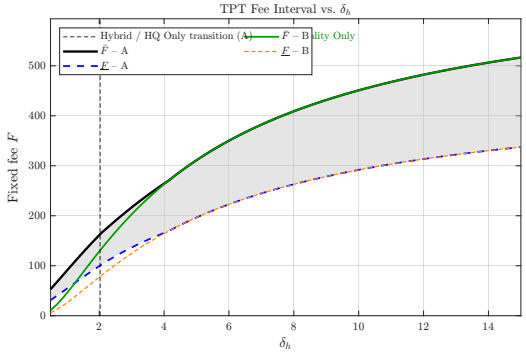


(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

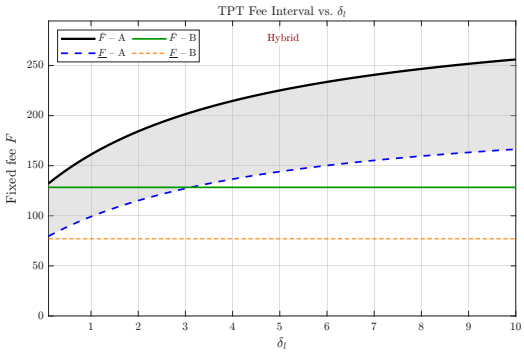


(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

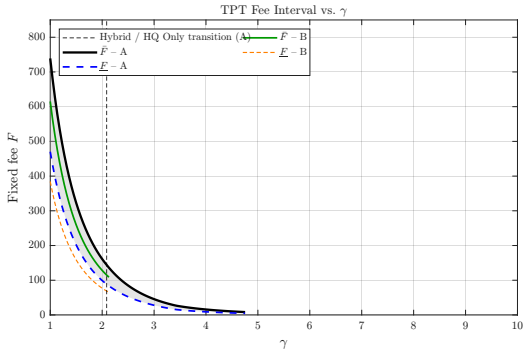
Figure 5.11: Sensitivity of the relative efficiency loss to δ_h, δ_l, γ, α, and c_l. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.



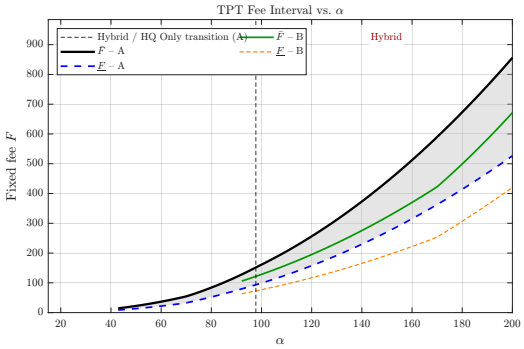
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



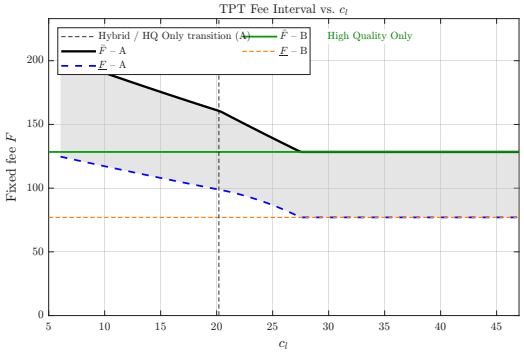
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

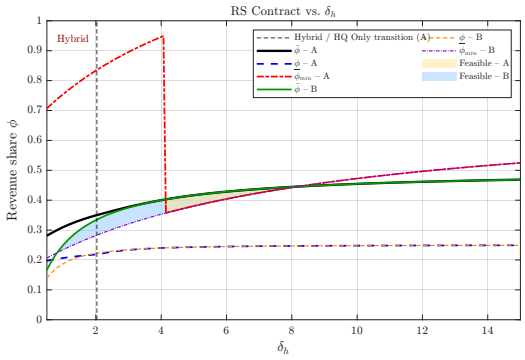


(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

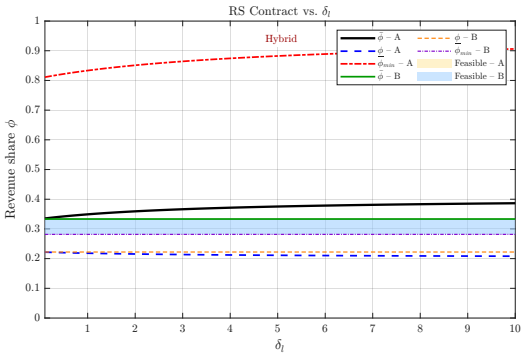


(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

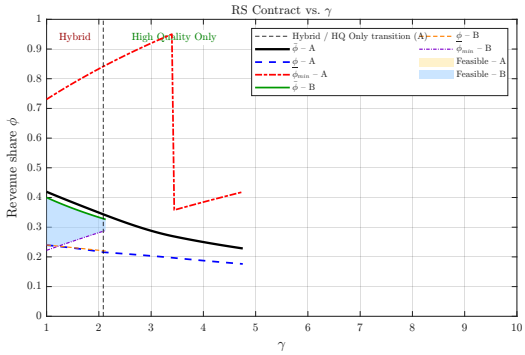
Figure 5.12: Sensitivity of the two part tariff fee interval $[\underline{F},\ \bar{F}]$ to δ_h, δ_l, γ, α, and c_l. Shaded grey region indicates the feasible fee interval for Base Case A. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.



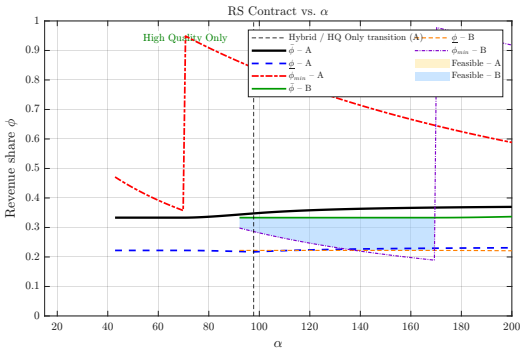
(a) vs. δ_h when $\alpha = 100$, $\gamma = 2$, $\delta_l = 1$, $c_h = 5$.



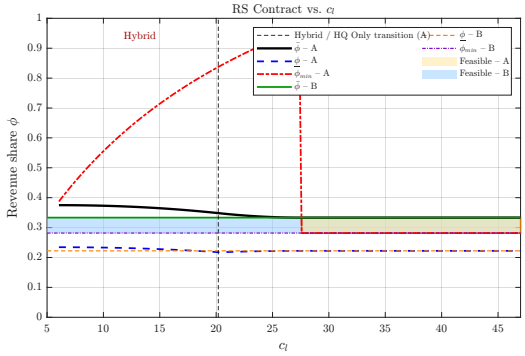
(b) vs. δ_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $c_h = 5$.



(c) vs. γ when $\alpha = 100$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.



(d) vs. α when $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.



(e) vs. c_l when $\alpha = 100$, $\gamma = 2$, $\delta_h = 2$, $\delta_l = 1$, $c_h = 5$.

Figure 5.13: Sensitivity of the revenue sharing implementable contract feasible range $[\max(\phi_{min}, \underline{\phi}),\ \bar{\phi}]$ to δ_h, δ_l, γ, α, and c_l. Orange shading: feasible range for Base Case A. Blue shading: feasible range for Base Case B. Thick lines: Base Case A ($c_l = 20$). Thin lines: Base Case B ($c_l = 45$). Dashed vertical line marks the Hybrid to High Quality Only transition for Base Case A.

Table 5.8 summarizes the sensitivity results.

Table 5.8: Sensitivity analysis summary for the two quality class model. A/B denotes Base Case A / Base Case B.

Par.	Π^{C*}	$CS(Q^{C*})$	**Rel. eff. loss**	**RS feasible range**
δ_h	A: Increases	A: Increases	A: Increases	A: $[4.14,\ 8.15]$
	B: Increases	B: Increases	B: Increases	B: $[0.86,\ 8.15]$
δ_l	A: Increases	A: Increases	A: Increases	A: Never feasible
	B: Invariant	B: Invariant	B: Invariant	B: Always feasible
γ	A: Decreases	A: Decreases	A: Decreases	A: Never feasible
	B: Decreases	B: Decreases	B: Decreases	B: $[1.00,\ 2.13]$
α	A: Increases	A: Increases	A: Increases then	A: Never feasible
	B: Increases	B: Increases	invariant	B: $[92.16,\ 169.32]$
			B: Invariant	
c_l	A: Decreases	A: Decreases	A: Decreases then	A: $[27.58,\ 47.00]$
	B: Invariant	B: Invariant	invariant	B: Always feasible
			B: Invariant	

The most important finding is the contrast between the two Base Cases in the feasibility of the revenue sharing implementable contract. Base Case B, which reduces to the single quality class model, admits a feasible contract over meaningful ranges of every parameter. Base Case A, which operates in the Hybrid equilibrium with $c_l = 20$, admits one only over $\delta_h \in [4.14,\ 8.15]$ and $c_l \in [27.58,\ 47.00]$, and over no part of the δ_l, γ, or α ranges.

These are not two separate windows of feasibility. Both begin at the point where the centralized solution stops acquiring low quality cores: at $\delta_h = 58/14 \approx 4.14$ and at $c_l = 27.50$. The reason is the feasibility bound (4.80), which requires the revenue share to be at least $c_h/(a_h^{C*} + c_h)$ for the high quality class and at least $c_l/(a_l^{C*} + c_l)$ for the low quality class.

Each of these ratios has a direct interpretation: it is the remanufacturer's share of the unit cost of a core of that quality class. The collector pays the acquisition price to obtain the core and the remanufacturer pays the remanufacturing cost to process it,

so the condition $\phi \geq \phi_{min}$ requires the remanufacturer's share of profit to cover its share of cost. The two quality classes divide this cost in opposite proportions. A high quality core is expensive to acquire and cheap to process, so the collector bears most of its unit cost; a low quality core is cheap to acquire and expensive to process, so the remanufacturer bears most of its unit cost. The low quality class is therefore remanufacturer intensive, and the revenue share it demands exceeds the collector's individual rationality bound. No single revenue share satisfies both quality classes.

While the centralized solution acquires both quality classes, the low quality bound $c_l/(a_l^{C*} + c_l)$ is therefore the binding one: at Base Case A it equals 0.8333, against a collector individual rationality ceiling of $\bar{\phi} = 0.3491$. The moment Q_l^{C*} reaches zero that bound disappears, ϕ_{min} falls to $c_h/(a_h^{C*} + c_h) \approx 0.28$, and a feasible interval opens.

Over the parameter ranges examined, the revenue sharing implementable contract becomes feasible for Base Case A only where the centralized solution has ceased to acquire low quality cores. The mechanism behind this is not the number of quality classes as such, but the way the unit cost of a core is divided between the two firms. The bound $c_l/(a_l^{C*} + c_l)$ is the remanufacturer's share of the unit cost of a low quality core: the collector pays a_l^{C*} to acquire it and the remanufacturer pays c_l to process it. The feasibility condition requires the remanufacturer's share of profit to cover its share of cost. Under the empirically motivated configuration of Guide et al. [7], a low quality core is cheap to acquire and expensive to process, so this share is large: it is 0.8333 at Base Case A, against a high quality share of 0.2941. The collector, as Stackelberg leader, will concede at most $\bar{\phi} = 0.3491$ of the coordinated profit, and the two shares cannot be reconciled. Revenue sharing therefore fails whenever the low quality class is remanufacturer intensive in this sense. The two part tariff, by contrast, remains feasible over every parameter range examined in both Base Cases: its quality specific per unit transfer prices act on the cost side and can price the two classes independently, whereas a single revenue share acts on the revenue side and pays the collector the same amount for every unit sold, regardless of which class of core it came from.

5.3 Comparison of the Single and Two Quality Class Models

This section summarizes the key differences between the single quality class and two quality class models across the three research questions. Because the two quality class model is studied under two Base Cases, Base Case A ($c_l = 20$, Hybrid equilibrium, in which both quality classes are acquired at positive quantities) and Base Case B ($c_l = 45$, High Quality Only equilibrium, in which only the high quality class is acquired), the comparison distinguishes between them throughout. Base Case B reduces exactly to the single quality class model and therefore serves as a structural continuity check, whereas Base Case A isolates the effect of also acquiring the low quality class at a positive quantity.

The first research question concerns how equilibrium quantities, profits, efficiency loss, and consumer surplus respond to cost and market parameters. The main points of comparison are:

- In both models the decentralized equilibrium produces strictly less quantity than the centralized optimum. In the single quality class model, $Q^{C*} = 21.50$ falls to $Q^{D*} = 14.33$, a reduction of 33%, and Base Case B reproduces these values exactly. Under Base Case A the centralized benchmark acquires both quality classes and reaches a larger total quantity $Q^{C*} = 23.00$, while the decentralized quantity $Q^{D*} = 14.38$ is only marginally above the single quality class level, so the low quality class widens the gap between the centralized and decentralized quantities rather than closing it.

- Profits decrease with remanufacturing cost and demand sensitivity, and increase with supply sensitivity and market size, in both models.

- The single quality class relative efficiency loss depends only on δ and γ and is invariant to c, α, and β, as established analytically in Chapter 3. In the two quality class model this invariance does not survive: under Base Case A the relative efficiency loss depends on every parameter of the model. It increases in δ_h and δ_l, decreases and then flattens in c_l, and increases and then stabilizes in α, whereas under Base Case B it collapses to the invariant single quality class pattern. At the Base Case parameter values, the relative efficiency loss

is 11.11% in the single quality class model, 13.09% under Base Case A, and 11.11% under Base Case B.

- The amplification is not a fixed offset. Along δ_h, the two quality class relative efficiency loss exceeds the single quality class value at the same δ_h by a factor that grows as the high quality supply becomes less elastic: at $\delta_h = 2$ the two values are 13.09% and 11.11%, but at $\delta_h = 0.5$ they are 8.43% and 2.78%, a threefold difference. Above $\delta_h = 58/14 \approx 4.14$ the centralized solution stops acquiring low quality cores and the two models agree exactly. Quality differentiation therefore amplifies the inefficiency of decentralization, and does so most severely precisely where the high quality class is hardest to acquire.

- Decentralization reduces consumer surplus substantially in both models. In the single quality class model it falls from 115.56 to 51.36, a loss of 55.56%. Under Base Case A it falls from 132.25 to 51.66, a loss of 60.94%, the larger centralized value reflecting the greater total quantity obtained when both quality classes are acquired at positive quantities. Under Base Case B the loss is 55.56%, identical to the single quality class model.

The second research question concerns the feasible fixed fee interval of the two part tariff:

- In both models the two part tariff always coordinates the supply chain, and the feasible fee interval $[\underline{F},\ \bar{F}]$ has width equal to the absolute efficiency loss $\Pi^C(a^{C*}, p^{C*}) - \Pi^D(a^{D*}, w^{D*})$. The fixed fee can redistribute the coordination surplus but cannot create or destroy it.

- The sensitivity patterns are similar across both models: the fee interval widens with increasing supply sensitivity and market size, and narrows with increasing remanufacturing cost and demand sensitivity.

- At the Base Case parameter values, the single quality class fee interval width is 51.36, compared with 61.99 under Base Case A and 51.36 under Base Case B. The larger interval width under Base Case A reflects the greater inefficiency introduced by the low quality class, which the two part tariff recovers in full.

The third research question concerns the feasibility of the revenue sharing implementable contract, and this is where the two models differ most sharply:

- In the single quality class model the contract is feasible over meaningful ranges of every parameter, and at the Base Case parameter values the feasible range is $\phi \in [0.2817,\ 0.3333]$.
- Base Case B, which reduces to the single quality class model, admits the same feasible range $\phi \in [0.2817,\ 0.3333]$.
- Base Case A, in which both quality classes are acquired at positive quantities, is infeasible: the minimum revenue share required to keep the low quality transfer price non-negative, $\phi_{min} = c_l/(a_l^{C*} + c_l) = 0.8333$, far exceeds the upper bound $\bar{\phi} = 0.3491$ set by the collector's individual rationality constraint, so no coordinating share exists.

Table 5.9 summarizes the feasibility comparison.

Table 5.9: Comparison of revenue sharing implementable contract feasibility between the single quality class and two quality class models. Entries are the feasible ranges of ϕ over each parameter; "Never feasible" and "Always feasible" denote infeasibility and feasibility over the entire range examined.

		Two quality class model	
Condition	Single quality class	Base Case A (Hybrid)	Base Case B (HQ Only)
δ (δ_h) increasing	$[0.86,\ 8.15]$	$[4.14,\ 8.15]$	$[0.86,\ 8.15]$
δ_l increasing	—	Never feasible	Always feasible
γ increasing	$[1.00,\ 2.45]$	Never feasible	$[1.00,\ 2.13]$
α increasing	$[78.22,\ 200]$	Never feasible	$[92.16,\ 169.32]$
c (c_l) increasing	$[0.10,\ 6.19]$	$[27.58,\ 47.00]$	Always feasible
Base Case feasibility	$[0.2817,\ 0.3333]$	Infeasible	$[0.2817,\ 0.3333]$

The mechanism behind this difference is that a single revenue share ϕ must simultaneously satisfy the feasibility condition of every quality class acquired in the centralized solution. By Theorem 4.3, the coordinating transfer price for class q is

$w_{rs,q}^{coord} = \phi(a_q^{C*} + c_q) - c_q$, so the contract requires $\phi \geq \max_q \, c_q/(a_q^{C*} + c_q)$. The low quality class is the binding one at Base Case A: because $a_l^{C*} = 4$ is small relative to $c_l = 20$, keeping $w_{rs,l}^{coord}$ non-negative requires the remanufacturer to retain more than five sixths of revenue, which the collector will not accept.

The two intervals over which Base Case A is feasible, $\delta_h \in [4.14, \ 8.15]$ and $c_l \in [27.58, \ 47.00]$, are therefore not independent findings. The centralized solution stops acquiring low quality cores at $\delta_h \approx 4.14$ and at $c_l = 27.50$, and the contract becomes feasible at exactly those two points: the low quality bound disappears, ϕ_{min} falls to $c_h/(a_h^{C*} + c_h) \approx 0.28$, and a feasible interval opens. Revenue sharing coordinates the two quality class supply chain only when the chain has ceased to be a two quality class chain. Wherever both quality classes are acquired at positive quantities, the low quality class demands a revenue share the collector will not accept. In the single quality class model only one feasibility condition applies, which is why the contract remains robust across parameters there.

Overall, acquiring the low quality class at a positive quantity, as under Base Case A, distinguishes the two quality class model from the single quality class model in three notable respects. First, the relative efficiency loss depends on every cost and market parameter, rather than only on δ and γ as in the single quality class model, and it is strictly larger than the single quality class value at the same supply and demand sensitivities. Second, the model switches between the Hybrid and High Quality Only cases as c_l, δ_h, or α varies, a transition that has no counterpart in the single quality class model, and the decentralized system always acquires the low quality class over the narrower parameter range than the centralized system does. Third, the revenue sharing implementable contract fails to coordinate whenever the low quality class is remanufacturer intensive, that is, whenever its cores are cheap to acquire relative to the cost of processing them. This holds under Base Case A and across the parameter ranges examined, whereas in the single quality class model the contract coordinates over a feasible range throughout. Base Case B demonstrates the complementary point: when only the high quality class is acquired, the two quality class model collapses to the single quality class model across every performance measure and coordination result. Across both models the two part tariff remains the more robust coordination mechanism, since it always coordinates and its feasible fee interval

is well defined for all parameter values. Its quality specific per unit transfer prices can price the two classes independently, whereas a single revenue share cannot. This leaves the two part tariff as the only reliable coordinating mechanism when heterogeneous quality classes with distinct remanufacturing costs are jointly acquired.

CHAPTER 6

CONCLUSION

This thesis studied closed-loop supply chain coordination under acquisition pricing, analyzing both a single quality class and a two quality class model in which an independent collector and remanufacturer interact through a Stackelberg game.

For the single quality class model, the centralized benchmark was characterized in closed form using a joint KKT argument to establish that the market-clearing constraint binds at the optimum. The decentralized Stackelberg equilibrium was derived analytically and shown to be strictly inefficient relative to the centralized benchmark due to double marginalization. The relative efficiency loss depends only on the supply and demand sensitivity parameters and is invariant to remanufacturing cost, market size, and minimum acquisition price. It is bounded above by twenty-five percent, approached asymptotically as supply becomes infinitely elastic.

Two coordination contracts were analyzed. The two part tariff always coordinates the supply chain through a single transfer price and a fixed fee, with the fixed fee interval width equal to the coordination surplus. The revenue sharing implementable contract coordinates through a family of contracts parametrized by the revenue share. Coordination alone does not determine the transfer price uniquely; what selects it is the requirement that the revenue share translate into a profit share of the same size, which pins the transfer price to the centralized acquisition price and the remanufacturing cost. The feasible range of revenue shares is bounded below by a feasibility condition ensuring a non-negative transfer price, and above by the collector's individual rationality constraint.

For the two quality class model, the analysis was extended to a setting with two

quality classes, each with its own acquisition price, supply sensitivity, and remanufacturing cost. The centralized solution was characterized in closed form for two cases: the Hybrid case, in which both quality classes are acquired at the optimum, and the High Quality Only case, in which only the high quality class is acquired. In the decentralized setting, the collector faces three candidate equilibria: High Quality Only, Partially Utilized, and Hybrid.

The Partially Utilized candidate was shown never to be optimal. The argument is structural rather than computational: the collector's problem within the Hybrid case is a strictly concave program over a polyhedron, whose maximizer lies either strictly inside the feasible set, giving the Hybrid candidate, or on the boundary at which the low quality acquisition price reaches its lower bound, which is precisely the High Quality Only candidate. The Partially Utilized candidate is a feasible point of that same program and therefore can never attain its maximum. The decentralized equilibrium is consequently the Hybrid candidate whenever that candidate is admissible, meaning that it prescribes non-negative supply of both quality classes and a non-negative low quality transfer price, and the High Quality Only candidate otherwise. Across the parameter ranges examined it is the requirement of non-negative low quality supply, rather than the non-negativity of the transfer price, that determines the transition.

The two part tariff coordinates the two quality class supply chain through quality specific transfer prices that equalize the remanufacturer's effective unit costs across both quality classes, together with a fixed fee interval of width equal to the coordination surplus. The revenue sharing implementable contract coordinates through a family of contracts parametrized by the revenue share, where each quality specific transfer price is determined by the revenue share together with the centralized acquisition price and the remanufacturing cost of that class.

The computational study confirmed the analytical findings and revealed additional structure. For the single quality class model, the relative efficiency loss was verified to be invariant to remanufacturing cost, market size, and minimum acquisition price, and increasing in supply elasticity, consistent with the closed-form expression. The revenue sharing implementable contract was found to be feasible over meaningful parameter ranges, becoming infeasible at very low or very high supply elasticity, at

high demand sensitivity, at high remanufacturing cost, and at small market size.

For the two quality class model, two findings stand out. First, quality differentiation amplifies the inefficiency of decentralization. Whenever both quality classes are acquired, the relative efficiency loss strictly exceeds the single quality class value at the same supply and demand sensitivities, and the gap is largest where the high quality class is hardest to acquire: at a high quality supply sensitivity of one half it is more than threefold. Once the system ceases to acquire the low quality class, the two models agree exactly.

Second, the revenue sharing implementable contract is substantially more fragile in the two quality class setting. A single revenue share must satisfy the feasibility condition of every quality class acquired in the centralized solution, and the low quality class is the binding one: because its centralized acquisition price is small relative to its remanufacturing cost, keeping its transfer price non-negative requires a revenue share that the collector's individual rationality constraint will not admit. The binding bound is the remanufacturer's share of the unit cost of a low quality core, and revenue sharing fails whenever that share is large, that is, whenever low quality cores are cheap to acquire and expensive to process. This is the empirically motivated configuration. Over the parameter ranges examined, the contract becomes feasible only at the points where the centralized solution stops acquiring low quality cores, and not before. The two part tariff, whose quality specific transfer prices can price the two classes independently, remains the only robust coordinating mechanism across all parameter ranges examined.

Several modeling assumptions limit the generality of the present analysis and suggest directions for future research. First, the model assumes deterministic linear demand and supply functions; extending the analysis to stochastic settings would capture important real-world complexities where demand uncertainty or uncertain return rates affect acquisition pricing decisions. Second, the model assumes that quality classes are perfectly observable and that returns are correctly sorted; incorporating inspection costs, misclassification, and asymmetric information about core quality would add realism. Third, the two quality class model is a simplification of the multi-class framework of Guide et al. [7]; extending the analytical results to N quality classes

would generalize the present findings. Since the relative efficiency loss is strictly larger whenever a second quality class is acquired, the inefficiency of decentralization may grow further with the number of classes acquired. Finally, other contract forms such as quantity flexibility contracts, buyback contracts, or option contracts may be better suited to remanufacturing settings where supply uncertainty and quality heterogeneity play a larger role, and remain an open direction for future research.

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APPENDIX A

REVENUE SHARING CONTRACT UNDER FREE STACKELBERG OPTIMIZATION

This appendix presents the full derivation showing that a revenue sharing contract cannot coordinate the supply chain when the collector freely optimizes the transfer price w_{rs} as Stackelberg leader for a fixed, exogenously agreed revenue share $\phi \in (0, 1)$.

For given (a, w_{rs}) and agreed ϕ, the remanufacturer's optimal quantity is $Q^*_{RS}(a, w_{rs}) = \min\{Q^{unc}_{RS}(w_{rs}), R(a)\}$, where $Q^{unc}_{RS}(w_{rs}) = [\phi\alpha - \gamma(w_{rs} + c)]/(2\phi)$, by the same argument as Lemma 3.3. As in Section 3.2, the collector's problem depends on whether the remanufacturer's unconstrained demand exceeds the available supply. Setting $R(a) = Q^{unc}_{RS}(w_{rs})$ and solving for w_{rs} gives the threshold transfer price:

$$\bar{w}_{RS}(a) = \frac{\phi[\alpha - 2\delta(a - \beta)]}{\gamma} - c. \tag{A.1}$$

When $w_{rs} \leq \bar{w}_{RS}(a)$, the remanufacturer is supply-constrained: $Q^*_{RS} = R(a)$. We refer to this case as Region 1. When $w_{rs} > \bar{w}_{RS}(a)$, the remanufacturer is demand-constrained: $Q^*_{RS} = Q^{unc}_{RS}(w_{rs})$. We refer to this case as Region 2. The collector's profit is:

$$\Pi^{Col,1}_{RS}(a, w_{rs}) = (1 - \phi)\frac{\alpha - \delta(a - \beta)}{\gamma}\delta(a - \beta) + w_{rs}\,\delta(a - \beta) - a\,\delta(a - \beta), \tag{A.2}$$

$$\Pi^{Col,2}_{RS}(a, w_{rs}) = (1 - \phi)\frac{\alpha - Q^{unc}_{RS}(w_{rs})}{\gamma}Q^{unc}_{RS}(w_{rs}) + w_{rs}\,Q^{unc}_{RS}(w_{rs}) - a\,\delta(a - \beta). \tag{A.3}$$

Region 1. The coefficient of w_{rs} in (A.2) is $\delta(a - \beta) > 0$, so $\Pi^{Col,1}_{RS}(a, w_{rs})$ is strictly increasing in w_{rs} throughout Region 1, and the optimal w_{rs} lies at the upper boundary

$w_{rs} = \bar{w}_{RS}(a)$. Substituting (A.1) into (A.2) and simplifying, since $(1-\phi)[\alpha-\delta(a-\beta)] + \phi[\alpha - 2\delta(a-\beta)] = \alpha - (1+\phi)\delta(a-\beta)$:

$$\Pi_{RS}^{Col,1}(a) = \frac{\delta(a-\beta)}{\gamma}\big[\alpha - (1+\phi)\delta(a-\beta) - \gamma c - \gamma a\big]. \tag{A.4}$$

The second-order derivative of (A.4) with respect to a is $-2\delta[(1+\phi)\delta+\gamma]/\gamma < 0$, so $\Pi_{RS}^{Col,1}(a)$ is strictly concave in a. Setting the first-order derivative equal to zero and solving for a gives:

$$a_{RS}^{1*} = \frac{\alpha + 2(1+\phi)\delta\beta + \gamma\beta - \gamma c}{2[(1+\phi)\delta+\gamma]}, \tag{A.5}$$

$$Q_{RS}^{1*} = \frac{\delta[\alpha - \gamma(\beta+c)]}{2[(1+\phi)\delta+\gamma]}. \tag{A.6}$$

Region 2. The partial derivative of (A.3) with respect to a is $-\delta(2a-\beta) < 0$ for all $a > \beta$, so $\Pi_{RS}^{Col,2}(a, w_{rs})$ is strictly decreasing in a throughout Region 2, and the optimal a lies at the lower boundary $a = \bar{a}_{RS}(w_{rs})$, obtained by solving (A.1) for a. Substituting this boundary into (A.3) and solving the resulting first-order condition in w_{rs} gives:

$$Q_{RS}^{2*} = \frac{\delta[\alpha - \gamma(\beta+c)]}{2[(1+\phi)\delta+\gamma]}. \tag{A.7}$$

Comparing (A.6) and (A.7) shows that both regions yield the same equilibrium quantity:

$$Q_{RS,eq}^{*}(\phi) = \frac{\delta[\alpha - \gamma(\beta+c)]}{2[(1+\phi)\delta+\gamma]}. \tag{A.8}$$

Proposition A.1. *For every $\phi \in (0,1)$, $Q_{RS,eq}^{*}(\phi) < Q^{C*}$. The freely-optimizing revenue sharing contract cannot coordinate the supply chain for any $\phi \in (0,1)$.*

Proof. Setting (A.8) equal to $Q^{C*} = \delta[\alpha-\gamma(\beta+c)]/[2(\delta+\gamma)]$ requires $(1+\phi)\delta+\gamma = \delta+\gamma$, that is, $\phi\delta = 0$. Since $\delta > 0$, this requires $\phi = 0$, which lies outside $(0,1)$. For $\phi \in (0,1)$, $(1+\phi)\delta+\gamma > \delta+\gamma$, so $Q_{RS,eq}^{*}(\phi) < Q^{C*}$ strictly. □

The source of this failure mirrors the inefficiency of the plain decentralized equilibrium (Proposition 3.3): even after the revenue share ϕ reduces the remanufacturer's effective margin, the collector, optimizing w_{rs} freely, still marks up the transfer price above acquisition cost, reproducing a form of double marginalization that no choice of ϕ alone can undo.

APPENDIX B

REVENUE SHARING CONTRACT UNDER FREE STACKELBERG OPTIMIZATION: TWO-QUALITY MODEL

This appendix derives the Stackelberg equilibrium under the revenue sharing contract for the two quality class model, when the collector freely optimizes the transfer prices $w_{rs,h}$ and $w_{rs,l}$ as Stackelberg leader for a fixed, exogenously agreed $\phi \in (0,1)$, rather than fixing them jointly in advance as in the implementable contract. We show that the resulting equilibrium quantity satisfies $Q^{RS*} < Q^{C*}$ for every $\phi \in (0,1)$, establishing Proposition B.1.

The analysis parallels Appendix A for the single quality class model. Because coordination targets the centralized solution in which both quality classes are acquired at positive quantities, we consider the Hybrid candidate of the Stackelberg equilibrium, in which both supply constraints bind: $Q_h^* = R_h(a_h)$ and $Q_l^* = R_l(a_l)$.

Stage 2: Remanufacturer's Best Response

By Lemma 4.5, the remanufacturer's unconstrained optimal quantities are $Q_{rs,h}^{unc}(w_{rs,h})$ and $Q_{rs,l}^{unc}(w_{rs,l})$, given by (4.75)–(4.76). In the Hybrid candidate both supply constraints bind, so

$$Q_h^* = R_h(a_h) = \delta_h(a_h - \beta_h), \qquad Q_l^* = R_l(a_l) = \delta_l(a_l - \beta_l).$$

Stage 1: Collector's Problem

The collector's profit under the revenue sharing contract, when both supply constraints bind, is

$$\begin{aligned}\Pi_{RS}^{Col}(a_h, a_l, w_{rs,h}, w_{rs,l}) = (1-\phi)\frac{\alpha - R_h(a_h) - R_l(a_l)}{\gamma}\big[R_h(a_h) + R_l(a_l)\big] \\ + w_{rs,h}R_h(a_h) + w_{rs,l}R_l(a_l) \\ - a_h R_h(a_h) - a_l R_l(a_l). \end{aligned} \tag{B.1}$$

The partial derivatives with respect to the transfer prices are

$$\frac{\partial \Pi_{RS}^{Col}}{\partial w_{rs,h}} = R_h(a_h) > 0, \qquad \frac{\partial \Pi_{RS}^{Col}}{\partial w_{rs,l}} = R_l(a_l) > 0,$$

so the collector raises both transfer prices to the upper boundary of the Hybrid candidate, at which the remanufacturer's unconstrained demands exactly equal the available supplies:

$$w_{rs,h}^{RS*} = \frac{\phi\big[\alpha - 2\big(R_h(a_h) + R_l(a_l)\big)\big]}{\gamma} - c_h, \tag{B.2}$$

$$w_{rs,l}^{RS*} = \frac{\phi\big[\alpha - 2\big(R_h(a_h) + R_l(a_l)\big)\big]}{\gamma} - c_l. \tag{B.3}$$

Substituting (B.2) and (B.3) into (B.1), the transfer payments contribute $\phi\big[\alpha - 2(R_h + R_l)\big](R_h + R_l)/\gamma - c_h R_h - c_l R_l$, and adding the collector's own revenue share $(1-\phi)\big[\alpha - (R_h + R_l)\big](R_h + R_l)/\gamma$ gives

$$\begin{aligned}\Pi_{RS}^{Col}(a_h, a_l) = \frac{R_h(a_h) + R_l(a_l)}{\gamma}\Big[\alpha - (1+\phi)\big(R_h(a_h) + R_l(a_l)\big)\Big] \\ - R_h(a_h)\big(c_h + a_h\big) - R_l(a_l)\big(c_l + a_l\big), \end{aligned} \tag{B.4}$$

which is the two quality class analogue of (A.4).

First-Order Conditions and Equilibrium

The Hessian of (B.4) with respect to (a_h, a_l) has determinant $4\delta_h\delta_l\big[(1+\phi)(\delta_h + \delta_l) + \gamma\big]/\gamma > 0$ and leading entry $-2\delta_h\big[(1+\phi)\delta_h + \gamma\big]/\gamma < 0$, so $\Pi_{RS}^{Col}(a_h, a_l)$ is strictly

concave and the first-order conditions characterize its unique maximizer. Solving them jointly:

$$a_h^{RS*} = \beta_h + \frac{\alpha - \gamma(\beta_h + c_h) - (1+\phi)\delta_l\big[(\beta_h + c_h) - (\beta_l + c_l)\big]}{2\big[(1+\phi)(\delta_h + \delta_l) + \gamma\big]}, \tag{B.5}$$

$$a_l^{RS*} = \beta_l + \frac{\alpha - \gamma(\beta_l + c_l) - (1+\phi)\delta_h\big[(\beta_l + c_l) - (\beta_h + c_h)\big]}{2\big[(1+\phi)(\delta_h + \delta_l) + \gamma\big]}. \tag{B.6}$$

The corresponding quantities are $Q_q^{RS*} = R_q(a_q^{RS*})$ for $q \in \{h, l\}$, and the cross terms in $(\beta_h + c_h) - (\beta_l + c_l)$ cancel on summation, giving the total

$$Q^{RS*}(\phi) = \frac{\delta_h\big[\alpha - \gamma(\beta_h + c_h)\big] + \delta_l\big[\alpha - \gamma(\beta_l + c_l)\big]}{2\big[(1+\phi)(\delta_h + \delta_l) + \gamma\big]}. \tag{B.7}$$

Proposition B.1. *For every $\phi \in (0,1)$, $Q^{RS*}(\phi) < Q^{C*}$. The freely optimizing revenue sharing contract cannot coordinate the two quality class supply chain for any $\phi \in (0,1)$.*

Proof. By Proposition 4.1, the centralized quantity is

$$Q^{C*} = \frac{\delta_h\big[\alpha - \gamma(\beta_h + c_h)\big] + \delta_l\big[\alpha - \gamma(\beta_l + c_l)\big]}{2(\delta_h + \delta_l + \gamma)},$$

which shares the numerator of (B.7). That numerator is strictly positive by market viability, so

$$\frac{Q^{RS*}(\phi)}{Q^{C*}} = \frac{\delta_h + \delta_l + \gamma}{(1+\phi)(\delta_h + \delta_l) + \gamma}. \tag{B.8}$$

Equality with one requires $(1+\phi)(\delta_h + \delta_l) = \delta_h + \delta_l$, that is $\phi(\delta_h + \delta_l) = 0$, which is impossible for $\phi \in (0,1)$ since $\delta_h, \delta_l > 0$. For $\phi \in (0,1)$ the denominator of (B.8) strictly exceeds its numerator, so $Q^{RS*}(\phi) < Q^{C*}$ strictly. □

Comparing (B.8) with the single quality class ratio implied by (A.8), the two expressions are identical once δ is replaced by $\delta_h + \delta_l$. The failure of the freely optimizing revenue sharing contract is therefore not caused by quality differentiation: it is the same double marginalization present in the single quality class model, and the aggregate supply elasticity is all that governs its severity. Even after the revenue share ϕ reduces the remanufacturer's effective margin, the collector, optimizing the transfer prices freely, still marks them up above acquisition cost, and no choice of ϕ alone can undo this.

APPENDIX C

PROOF OF HYBRID REGION DOMINANCE

This appendix provides the full derivation of the two profit differences stated in Proposition 4.4.

Hybrid Collector Profit

From Proposition 4.3, $\Pi^{Col,Hyb*} = R_h(a_h^{Hyb*})\big(w_h^{Hyb*} - a_h^{Hyb*}\big) + R_l(a_l^{Hyb*})\big(w_l^{Hyb*} - a_l^{Hyb*}\big)$. Substituting (4.44)–(4.46) and grouping terms:

$$\begin{aligned}
&\Pi^{Col,Hyb*} \\
&\quad = \frac{1}{\gamma(2\delta_h + 2\delta_l + \gamma)}\Big(\frac{(\delta_h + \delta_l)\alpha^2}{4} \\
&\qquad - \frac{\gamma\alpha}{2}\big[\delta_h(\beta_h + c_h) + \delta_l(\beta_l + c_l)\big] \\
&\qquad + \frac{\delta_h\gamma^2(\beta_h + c_h)^2}{4} + \frac{\delta_l\gamma^2(\beta_l + c_l)^2}{4} \\
&\qquad + \frac{\delta_h\delta_l\gamma}{2}\big[(\beta_h + c_h) - (\beta_l + c_l)\big]^2\Big).
\end{aligned} \tag{C.1}$$

Difference: Hybrid Minus High Quality Only

From Proposition 4.3:

$$\Pi^{Col,HQ*} = \frac{\delta_h\big[\alpha - \gamma(\beta_h + c_h)\big]^2}{4\gamma(2\delta_h + \gamma)}. \tag{C.2}$$

Subtracting (C.2) from (C.1), bringing both over the common denominator $4\gamma(2\delta_h + \gamma)(2\delta_h + 2\delta_l + \gamma)$, expanding the numerator, and factoring:

$$\Pi^{Col,Hyb*} - \Pi^{Col,HQ*} = \frac{\delta_l\big[\alpha + 2\delta_h(\beta_h - \beta_l + c_h - c_l) - \gamma(\beta_l + c_l)\big]^2}{4(2\delta_h + \gamma)(2\delta_h + 2\delta_l + \gamma)}. \tag{C.3}$$

Since $\delta_l > 0$ and the denominator is strictly positive, (C.3) is non-negative for all parameter values. Equality holds if and only if $a_l^{Hyb*} = \beta_l$, that is, when the Hybrid candidate degenerates to the High Quality Only candidate.Equality holds if and only if $a_l^{Hyb*} = \beta_l$, that is, when the Hybrid optimum degenerates to the High Quality Only boundary.

Difference: Hybrid Minus Partially Utilized

At the Partially Utilized optimum the collector reduces a_l until the low quality supply exactly meets the remanufacturer's residual demand, so that $R_l(a_l^{PU*}) = Q_l^{PU*}$ and hence, by (4.42),

$$R_h(a_h^{PU*}) + R_l(a_l^{PU*}) = \frac{\alpha - \gamma c_l}{4}. \tag{C.4}$$

Substituting (C.4) into the Hybrid transfer price rule (4.59) gives

$$\bar{w}_l\big(a_h^{PU*}, a_l^{PU*}\big) = \frac{\alpha - 2\big[R_h(a_h^{PU*}) + R_l(a_l^{PU*})\big]}{\gamma} - c_l = \frac{\alpha - \gamma c_l}{2\gamma} = w_l^{PU*},$$

so the Partially Utilized candidate satisfies the Hybrid transfer price rule. It is therefore a feasible point of the collector's problem over the Hybrid case, and the inequality $\Pi^{Col,Hyb*} \geq \Pi^{Col,PU*}$ follows directly from the optimality of (a_h^{Hyb*}, a_l^{Hyb*}). The remainder of this section computes the magnitude of the gap.

From Proposition 4.3, $\Pi^{Col,PU*} = w_h^{PU*}R_h(a_h^{PU*}) + w_l^{PU*}Q_l^{PU*} - a_h^{PU*}R_h(a_h^{PU*}) - a_l^{PU*}R_l(a_l^{PU*})$. Substituting (4.39)–(4.42), subtracting from (C.1) over the common denominator $16(\delta_h + \delta_l)\gamma(2\delta_h + 2\delta_l + \gamma)$, expanding the numerator, and factoring:

$$\Pi^{Col,Hyb*} - \Pi^{Col,PU*} = \frac{\gamma\big[\alpha + 2\delta_h(\beta_h + c_h - c_l) + 2\delta_l\beta_l - \gamma c_l\big]^2}{16(\delta_h + \delta_l)(2\delta_h + 2\delta_l + \gamma)}. \tag{C.5}$$

Since $\gamma > 0$ and the denominator is strictly positive, (C.5) is non-negative for all parameter values. The gap is a perfect square because, by the argument above, the Partially Utilized candidate is an interior feasible point of the same concave program that the Hybrid candidate maximizes.

Equations (C.3) and (C.5) give the two dominance gaps referred to in Proposition 4.4.