STABILITY OF THE ZERO SOLUTION OF IMPULSIVE DIFFERENTIAL EQUATIONS BY LYAPUNOV SECOND METHOD

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis, we firstly introduce some basic concepts and several well-known criteria for stability, asymptotic stability and instability of the zero solution of impulsive differential equations (IDEs). Next, in the light of the known results, we used Lyapunov second method as a tool in obtaining new stability criterion for the zero solution of an IDE with variable moments.

Keywords: Differential equation, impulse effect, stability, instability, Lyapunov second method, variable moment .

ÖZ

LYAPUNOV'UN İKİNCİ YÖNTEMİ İLE İMPULSİVE DİFERENSİYEL DENKLEMLERİN SIFIR CÖZÜMÜNÜN KARARLILIĞI

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Bu tezde, ilk olarak, değişken zamanlı "impulsive" diferensiyel denklemler üzerindeki temel bazı kavramlar ve bilinen kararlılık kriterleri verilmiş ve sonra, bilinen sonuçlar ışığında Lyapunov'un ikinci yöntemi kullanılarak sıfır çözümünün kararlılığı ile ilgili yeni bir sonuç elde edilmiştir.

Anahtar Kelimeler: Diferensiyel denklem, "impulse" etkisi, kararlılık, Lyapunov'un ikinci yöntemi, değişken zaman.

To my lovely parents

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CHAPTER 1

INTRODUCTION

1.1 Impulsive Differential Equations (IDEs)

The impulsive differential equations are adequate mathematical models of numerous real processes and phenomena studied at physics, biology, population dynamics, bio-technologies, control theory, industrial robotics, etc. In spite of great possibilities of applications, the theory of these equations has been developed rather slowly. Recently there have been intensive studies on the theory of impulsive differential equations, especially on the stability of their solutions.

Moreover, the theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. For instance, inital value problems of such equations may not, in general, possess any solution at all even when the functions involved are smooth enough. Furthermore, fundamental properties, such as continuous dependance relative to initial data, may be violated and qualitative properties like stability may need a convenient interpretation. A simple IDE may exhibit several new phenomenon such as rhythmical beating, merging of solutions, and noncontinuability of solutions. Consequently, the theory of IDEs is interesting in itself and will assume greater consideration because of the increase in the application of the theory onto various fields. Although there are limited monographs related to this subject [4, 10, 16], the study of impulsive systems has been the subject of intensive investigations for the last 20 years [1, 2, 3, 5, 6, 10, 12, 13, 15, 17, 18, 19, 20]. It seems that this topic will call close attention of many scientists for years.

One of the important issues in the study of IDEs is to determine whether or not a solution of an IDE is stable. In general, this can not be carried over to stability of the zero solution of an IDE. That is why we have the additional concern of determining if system inhibits the stability of the zero solution.

In this thesis, by using Lyapunov second method, we investigate the stability, asymptotic stability and instability of the zero solution of an impulsive differential equation of the form

$$\begin{aligned} &\frac{dx}{dt} = f(t,x), \quad t \neq \theta_i(x) \\ &\Delta x|_{t=\theta_i(x)} = J_i(x), \quad i \in N = \{1, 2, \ldots\}, \end{aligned} \tag{1.1}$$

where $\Delta x|_{t=\theta} := x(\theta+) - x(\theta)$, $x(\theta+) = \lim_{t \to \theta+} x(t)$.

After presenting some theorems and examples, we discuss the qualitative properties of (1.1) and derive a new stability criterion for its trivial solution in the light of some known results obtained for such IDE. In particular, sufficient conditions are obtained under which stability and asymptotically stability of the zero solution of (1.1) are ensured.

Let us start with introducing some preliminaries [3]. For our purpose, we first introduce the following notations :

$$G = \{(t,x) : \ t \geq 0, \ \ x \in S_\rho\},$$

$$S_\rho = \{x \in R^n : \ \|x\| < \rho\},$$

where $\rho > 0$ is a fixed real number and $\|x\|$ denotes the euclidean norm of $x \in R^n$. Next, we assume the following conditions :

(a) $f(t,x) \ : \ G \to R^n$ is piecewise continuous with discontinuities of the first kind at $t = \theta_i(x)$, where it is left continuous with respect to t; $\sup_{(t,x) \in G} \|f(t,x)\| = M < \infty$, and $f(t,0) = 0$ for all $t \geq t_0$.

(b) $\theta_i(x) : S_\rho \to R_+$ are continuous; $\theta_i(x) < \theta_{i+1}(x)$ for all $x \in S_\rho$ and $i \in N$; $\lim_{i \to \infty} \theta_i(0) = \infty$.

(c) $J_i(x) : S_\rho \to R^n$; $J(0) = 0$.

By a solution of (1.1), we mean a function $\varphi(t)$ defined on I such that

(i) $\varphi(t)$ is piecewise continuous on I having discontinuities of the first kind at τ_i where $\tau_i = \theta_i(x(\tau_i))$ and it is continuous from the left, i.e.,

$$\varphi(\tau_i) = \lim_{t \to \tau_i^-} \varphi(t),$$

(ii) $\varphi(t)$ is differentiable for $t \neq \tau_i$, $t \in I$, and

$$\begin{aligned} \frac{d\varphi(t)}{dt} &= f(t, \varphi(t)), \qquad t \neq \tau_i, \\ \varphi(\tau_i+) - \varphi(\tau_i) &= J_i(\varphi(\tau_i)). \end{aligned}$$

Letting $P(t) = (t, x(t))$, we can describe the trajectory of the integral curve as follows: The point $P(t)$, starting at the point (t_0, x_0), traverses along the curve $\{t, x(t)\}$ defined by the solution $\varphi(t) = x(t; t_0, x_0)$ of system $\dot{x} = f(t, x)$ until $t = \tau_1$. At the moment $t = \tau_1$, the point $P(t)$ is affected by the impulse effect which causes the solution to *momentarily* jump from the point $(\tau_1, x(\tau_1))$ to the point $(\tau_1, x(\tau_1) + \Delta x|_{t=\tau_1})$. Then the point $P(t)$, beginning at this point, moves along the solution curve until $t = \tau_2$. All in all, this process keeps going on provided that the solution exists.

An initial value problem (IVP) related to system (1.1) is the problem of finding a solution $x(t)$ of (1.1) which satisfies the initial condition $x(t_0) = x_0$. It can be shown that the IVP is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s, x_0))ds + \sum_{t_0 < \tau_i < t} J_i(x(\tau_i, x_0)). \tag{1.2}$$

The following basic existence and uniqueness theorem is taken from [16].

Theorem 1.1.1 *In addition to $f \in C(G)$ and $J_i \in C(S_\rho)$, assume that f and J_i satisfy Lipschitz conditions with respect to x in G and S_ρ respectively. Then for each $(t_0, x_0) \in G$, IVP (1.2) has a unique solution $x(t)$ defined on $[t_0, t_0 + \alpha]$ for some $\alpha > 0$.*

In the stability theory solutions are required to exist on an infinite interval of the form $[t_*, \infty)$ for some t_*. The following theorem provides conditions for such solution to exist on infinite interval, see [10] for details.

Theorem 1.1.2 *Assume that the hypothesis of Theorem 1.1.1 hold. Suppose further that*

$$\|f(t,x)\| \leq g(t, \|x\|), \qquad (t,x) \in G,$$
$$\|x + J_i(x)\| \leq \|x\|, \qquad x \in S_\rho,$$

where $g(t,u) \in C[R_+ \times R_+, R_+]$ *is nondecreasing with respect to* u *for each* $t \in R_+$. *If the maximal solution of*

$$u' = g(t,u), \qquad u(t_0) = u_0 \geq 0,$$

exists on $[t_0, \infty)$*, then the interval of existence of solution* $x(t) = x(t, t_0, x_0)$ *of (1.1) such that* $\|x_0\| \leq u_0$ *is* $[t_0, \infty)$.

The proof of this theorem is similar to one in differential equations without impulse effect. We should only mention that the condition $\|x + J_i(x)\| \leq \|x\|$, $x \in S_\rho$, is sufficient for solutions $x(t)$ not to exceed $u(t)$ at the impulse points. Furthermore, in order for a solution $x(t)$ to continue on the infinite interval, we need to have some conditions for absence of beating. In the following section we give some lemmas providing no pulsation.

1.2 Pulse Phenomena

Although solutions of system (1.1) are (left) piecewise continuous, the points of discontinuity, depending on the solution, make the study of such systems considerably more difficult. One of difficulties arises from the pulsations of the solution at the surfaces $t = \theta_i(x)$ which often hampers the solution to be on the domain of impulsive system. To illustrate this let us look at the following example.

Example 1.2.1 Let $\lambda \in R_+$.

$$\frac{dx}{dt} = -x, \quad t \neq \arctan x + i\pi$$
$$\Delta x|_{t=\arctan x + i\pi} = \lambda x, \quad i \in \{0, 1, 2, \ldots\}, \tag{1.3}$$

Keeping in mind that $\lambda > 0$, we see that the integral curve of an arbitrary solution of this system, $\varphi(t) = \varphi(0)e^{\lambda t}$ with $\varphi(0) \leq 0$, intersect the hypersurface $t = \arctan x$ only once, and the integral curve of the solution $\varphi(t) = \varphi(0)e^{\lambda t}$ with $\varphi(0) > 0$ meets the surface of discontinuity $t = \arctan x$ countably many times, that is, pulsations occur on this surface.

As is seen from the example, the solutions of the (1.1) may experience the pulse phenomena, namely the solution may hit the same surface finite or infinite number of times causing rhtmical beating. Consequently, it is desirable to find conditions that guarantee the absence or presence of pulse phenomena. In this section we shall deal with this problem. Accordingly assuming that conditions (a)-(c) are satisfied, we state some lemmas for such system to have no pulsations.

Lemma 1.2.1 *[16] Let the functions $f(t,x)$, $J_i(x)$, and $\theta_i(x)$ be continuous for $(t,x) \in G$, $\theta_i(x)$ be continuously differentiable with respect to x, and*

$$\max_{(t,x)\in G} \left\| \frac{\partial \theta_i(x)}{\partial x} \right\| \leq N. \tag{1.4}$$

Moreover, suppose that the inequality

$$\max_{0\leq s\leq 1} \left\langle \frac{\partial \theta_i(x + sJ_i(x))}{\partial x}, J_i(x) \right\rangle \leq 0 \tag{1.5}$$

holds for all $(t,x) \in G$. Then there is a number N_0 such that for all $N \leq N_0$, the integral curve of any solution of system (1.2), $x(t)$, for $t_0 < t \leq t_0 + \lambda$ ($\lambda \geq 0$), intersects each hypersurface $t = \theta_i(x)$ only once.

Proof. Proving the lemma amounts to proving that, for sufficiently small values of N, any solution $x(t)$ of system (1.1), starting at $x_0 + J_i(x_0)$ and lying in the domain G, does not hit the surface $t = \theta_i(x)$ for $\theta_i(x_0) < t < t_i^m$, where $t_i^m = \max_{(t,x)\in G} \theta_i(x)$.

Suppose, on the contrary, that a solution $x(t)$ starting at the point $x_0 + J_i(x_0)$ intersects the hypersurface at the point $(t_i^\star, x^\star)$, (i.e, $t_i^\star = \theta_i(x^\star)$), for $t_i^\star > \theta_i(x_0)$. Clearly $x(t)$ is continuous in the interval $(\theta_i(x_0), t_i^m)$. Moreover, we know that

$$x^\star = x_0 + J_i(x_0) + \int_{\theta_i(x_0)}^{t_i^\star} f(\tau, x(\tau))d\tau. \tag{1.6}$$

Letting $\int_{\theta_i(x_0)}^{t_i^\star} f(\tau, x(\tau))d\tau = \eta$ and $\phi(t) = \theta_i(x + J_i(x) + t\eta)$ we observe that

$$\begin{aligned}\theta_i(x^\star) - \theta_i(x_0 + J_i(x_0)) &= \phi(1) - \phi(0) \\ &= \int_0^1 \phi'(\tau)d\tau \\ &= \int_0^1 \left\langle \frac{\partial\theta_i(x + J_i(x) + \eta\tau)}{\partial x}, \eta \right\rangle d\tau \end{aligned} \tag{1.7}$$

Likewise,

$$\theta_i(x_0 + J_i(x_0)) - \theta_i(x_0) = \int_0^1 \left\langle \frac{\partial\theta_i(x + \tau J_i(x))}{\partial x}, J_i(x) \right\rangle d\tau \tag{1.8}$$

Therefore, we see that

$$\begin{aligned} t_i^\star - \theta_i(x_0) &= \theta_i(x^\star) - \theta_i(x_0) \\ &= \theta_i(x^\star) - \theta_i(x_0 + J_i(x_0)) + \theta_i(x_0 + J_i(x_0)) - \theta_i(x_0) \\ &= \int_0^1 \left\langle \frac{\partial\theta_i(x + J_i(x) + \eta\tau)}{\partial x}, \eta \right\rangle d\tau + \\ &+ \int_0^1 \left\langle \frac{\partial\theta_i(x + \tau J_i(x))}{\partial x}, J_i(x) \right\rangle d\tau \end{aligned} \tag{1.9}$$

It follows from $\sup\limits_{(t,x)\in G} \|f(t,x)\| \leq M$ that the first of these integrals in the right hand side of (1.9) admits the estimate

$$\int_0^1 \left\langle \frac{\partial\theta_i(x + J_i(x) + \eta\tau)}{\partial x}, \eta \right\rangle d\tau \leq MN(t_i^\star - \theta_i(x_0)),$$

and thus,

$$(1 - MN)(t_i^\star - \theta_i(x_0)) \leq \int_0^1 \left\langle \frac{\partial\theta_i(x + \tau J_i(x))}{\partial x}, J_i(x) \right\rangle d\tau \tag{1.10}$$

Now it suffices to choose N_0 such that $MN_0 < 1$ because, in this case, inequality (1.10) can not hold due to the condition (1.5).

Let us next illustrate lemma 1.2.1 by an example.

Example 1.2.2 [10] Consider the impulsive differential equation

$$\frac{dx}{dt} = \cos t, \quad t \neq \theta_i(x), x(0) = 0$$
$$\Delta x|_{t=\theta_i(x)} = 1, \quad i \in N = \{1, 2, \ldots\}, \tag{1.11}$$

where $\theta_i(x) = -(x+1) + (2\pi + 1)i$. Since

$$\max_{(t,x)\in G} \left\| \frac{\partial \theta_i(x)}{\partial x} \right\| \leq 1 \text{ and } \max_{0\leq s\leq 1} \left\langle \frac{\partial \theta_i(x + sJ_i(x))}{\partial x}, J_i(x) \right\rangle = -1 \leq 0$$

then every solution of (1.11) meets any given surface $t = -(x+1) + (2\pi + 1)i$ only once.

Now let us introduce the following lemma:

Lemma 1.2.2 *[16] Let, in system (1.1), the functions $f(t,x)$ and $J_i(x)$ satisfy the conditions of the the lemma 1.2.1 and the functions $\theta_i(x)$ satisfy the Lipschitz condition*

$$|\theta_i(x_1) - \theta_i(x_2)| \leq N\|x_1 - x_2\|, \quad x_1, x_2 \in G \tag{1.12}$$

and the inequality

$$\theta_i(x) \geq \theta_i(x + J_i(x)) \tag{1.13}$$

for all $(t,x) \in G$ instead of (1.5). Then there is a positive number N_0 such that for all $N \leq N_0$, the integral curve of any solution $x(t)$ of system (1.1), which belongs to the domain G for $t_0 < t \leq t_0 + \lambda$ $(\lambda \geq 0)$, intersects each hypersurface $t = \theta_i(x)$ only once.

Proof. The proof of this lemma is similar to the proof of lemma 1.2.1. In this case, instead of (1.7), we invoke the Lipschitz condition (1.12) to infer that

$$\begin{aligned} |\theta_i(x^\star) - \theta_i(x_0 + J_i(x_0))| &\leq N \int_{\theta_i(x_0)}^{t_i^\star} \|f(\tau, x(\tau))\| d\tau \\ &\leq NM(t_i^\star - \theta_i(x_0)). \end{aligned} \tag{1.14}$$

Accordingly,

$$\begin{aligned} t_i^\star - \theta_i(x_0) &= \theta_i(x^\star) - \theta_i(x_0) \\ &= \theta_i(x^\star) - \theta_i(x_0 + J_i(x_0)) + \theta_i(x_0 + J_i(x_0)) - \theta_i(x_0) \\ &\leq NM(t_i^\star - \theta_i(x_0)) + \theta_i(x_0 + J_i(x_0)) - \theta_i(x_0) \end{aligned} \tag{1.15}$$

That is to say,

$$(1 - MN)(t_i^\star - \theta_i(x_0)) \leq \theta_i(x_0 + J_i(x_0)) - \theta_i(x_0) \tag{1.16}$$

It follows that if N is so small that $1 - MN > 0$, then inequality (1.16) can not hold due to inequality (1.13). This completes the proof.

The proof the following lemma is quite similar to that of the previous one, and hence is disregarded.

Lemma 1.2.3 *[16] Let the function $f(t,x)$ be continuous with respect to t and x (piecewise continuous with respect to t) for $t \geq t_0$, and ϵ be sufficiently small number. If the functions $J_i(x)$, $i = 1, 2, \ldots$, are continuous for $\|x\| \leq \epsilon$, and if, for $\|x_1\| \leq \epsilon$, $\|x_2\| \leq \epsilon$, and all $i = 1, 2, \ldots$, the functions $\theta_i(x)$ satisfy the Lipschitz condition*

$$|\theta_i(x_1) - \theta_i(x_2)| \leq N\|x_1 - x_2\|$$

and the inequality

$$\theta_i(x) \geq \theta_i(x + J_i(x)).$$

Then, for $t \geq t_0$, the integral curve of any solution $x(t)$ with $\|x(t_0)\| \leq \epsilon$ of the system (1.1) intersects each surface $t = \theta_i(x)$ only once.

1.3 Continuous Dependence of Solutions

Now let us examine the following simple but important example:

Example 1.3.1 [16] Consider the system

$$\begin{aligned} &\frac{dx}{dt} = 0, \quad t \neq 2i - x \\ &\Delta x|_{t=2i-x} = \lambda \quad i \in N = \{1, 2, \ldots\}, \end{aligned} \tag{1.17}$$

where $\lambda \in R_+$. We see with just an overview that $\phi(t) = 1$ and $\psi(t) = 1+\epsilon$ are two solutions of this system starting at $t_0 = 0$ where ϵ is an arbitrarily small number. These solutions on the interval [0,2] do not depend continuously on x_0 because $|\phi(t) - \psi(t)| = \lambda + \epsilon$ on the interval [1-ϵ,1] if $\epsilon > 0$ (or [1,1-ϵ] if $\epsilon < 0$) no matter how small ϵ may be. But, outside this interval, the difference can be made arbitrarily small by making ϵ sufficently small.

However, if we exclude from the interval $[t_0, t_0+\lambda]$ sufficiently small neigborhoods of the points where the integral curve intersects the surfaces $t = \theta_i(x)$, then the solutions will depend uniformly on the remaining values of independent variable.

Suppose that the functions $f(t, x)$, $J_i(x)$, and $\theta_i(x)$ in (1.1) are continuous for $(t, x) \in G$ and the inequalities

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|, \qquad \|J_i(x_1) - J_i(x_2)\| \leq L\|x_1 - x_2\|,$$

$$|\theta_i(x_1) - \theta_i(x_2)| \leq N\|x_1 - x_2\| \tag{1.18}$$

hold for all $t \in I, x, x_1, x_2 \in D$.

Let $x(t, x_0)$ and $x(t, y_0)$ be two solutions of the system (1.1), which belong to the domain G for all $t \in [t_0, t_0 + \lambda]$. Suppose that each of these solutions intersects every hypersurface $t = \theta_i(x)$ only once, and denote by $\theta_i^{x_0}, \theta_i^{y_0}$ the corresponding times when these solutions intersect the surfaces $t = \theta_i(x)$.

Lemma 1.3.1 *[16] If the stated above conditions are satisfied and $MN < 1$, then*

$$\|x(t, x_0) - x(t, y_0)\| \leq \left(1 + \frac{L}{1 - MN}\right)^p e^{L\lambda}\|x_0 - y_0\| \tag{1.19}$$

for all $t \in [\theta_i^m, \theta_{i+1}^M]$, where $\theta_i^m = \min(\theta_i^{x_0}, \theta_i^{y_0}), \theta_i^M = \max(\theta_i^{x_0}, \theta_i^{y_0})$ and p is the number of points θ_i^m (or θ_i^M) in the interval $[t_0, t_0 + \lambda]$.

Now we consider the following theorem.

Theorem 1.3.2 *[16] Let the functions $f(t, x)$, $J_i(x)$, and $\theta_i(x)$ in (1.1) satisfy inequalities (1.18), the inequality (1.19) hold, and $MN < 1$. If a solution of (1.1), $x(t, x_0)$, is defined for $t \in [t_0, t_0 + \lambda]$, then this solution depends continuously on the initial condition x_0 in the following sense: for arbirary $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for any other solution $x(t, y_0)$ of (1.1), the inequality $\|x_0 - y_0\| < \delta$ implies*

$$\|x(t, x_0) - x(t, y_0)\| < \epsilon \tag{1.20}$$

for all $t \in [t_0, t_0+\lambda]$, satisfying $|t-\theta_i^{x_0}| > \epsilon$, where $\theta_i^{x_0}$ are the times, at which the integral curve of the solution $x(t, x_0)$ intersects the hypersurfaces $t = \theta_i(x)$.

Proof. The conditions of the theorem imply that there are no pulsations of the solutions of the system (1.1) at the surface $t = \theta_i(x)$ and that the conditions of lemma 1.3.1 hold. By lemma 1.3.1, two solutions of the system (1.1) satisfy the estimate (1.19). Fix an arbitrary $\epsilon > 0$ and choose $\delta_1 > 0$ so small that

$$\max_{\|x_0-y_0\|<\delta_1} |\theta_i^{x_0} - \theta_i^{y_0}| < \epsilon$$

for all i such that $\theta_i^{x_0} \in [t_0, t_0+\lambda]$. As of the choice of δ, we take

$$\delta = \min\left(\delta_1, \epsilon\left(1+\frac{L}{1-MN}\right)^{-p} e^{-L\lambda}\right).$$

If y_0 is such that $\|x_0 - y_0\| < \delta$, then, by (1.19), the solutions $x(t, x_0), x(t, y_0)$ satisfy the inequality (1.20) for all $t \in [t_0, t_0+\lambda]$ with $|t-\theta_i^{x_0}| > \epsilon$, which completes the proof.

Let $x(t, x_0), y(t, x_0)$ be two different solutions of the system (1.1) on $[t_0, t_0+\lambda]$. Suppose now that there exists t_i^* such that $x(t, x_0)$ and $y(t, x_0)$ differ in size with value ϵ at $t = t_i^*$ for the first time . Namely, $\|x(t_i^*, x_0) - y(t_i^*, x_0)\| = \epsilon$. But if $|t_i^* - \theta_i^{x_0}| > \epsilon$, then, by theorem 1.3.2, $\|x(t_i^*, x_0) - y(t_i^*, x_0)\| < \epsilon$. Hence $|t_i^* - \theta_i^{x_0}| \le \epsilon$. But for $0 < |t_i^* - \theta_i^{x_0}| < \epsilon$, there exists ϵ_1, $\epsilon_1 < \epsilon$, such that $|t_i^* - \theta_i^{x_0}| > \epsilon_1$. In this case, again by theorem 1.3.2, $\|x(t_i^*, x_0) - y(t_i^*, x_0)\| < \epsilon_1$. At last, for if $t_i^* = \theta_i^{x_0}$, then since both $x(t, x_0)$ and $y(t, x_0)$ undertake the same pulsation at $t = t_i^*$, it is not possible to obtain different values of $x(t, x_0)$ and $y(t, x_0)$ on interval $[t_0, t_0+\lambda]$. Therfore we have $x(t, x_0) = y(t, x_0)$. This means that continuous dependance of solutions implies the uniqueness of the solution.

1.4 Some Definitions

The following definitions are extracted from [14].

Definition 1.4.1 *The zero solution of (1.1) is called stable if for any given $\epsilon > 0$ and $t_0 \in R_+$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \epsilon$ for all $t \ge t_0$.*

Definition 1.4.2 *The zero solution of (1.1) is called asymptotically stable if it is stable and there exists* $\tilde{\delta} > 0$ *such that any solution* $x(t, t_0, x_0)$ *with* $\|x_0\| < \tilde{\delta}$ *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Definition 1.4.3 *A continuous function* $\psi : R_+ \to R_+$ *is said to belong to class* $\mathcal{K}$, *if* ψ *strictly increasing and* $\psi(0) = 0$.

Definition 1.4.4 *A scalar-valued function* $V(t, x)$ *is called positive definite on* G, *if there exists* $\psi \in \mathcal{K}$ *such that* $V(t, x) \geq \psi(\|x\|)$ *for all* $(t, x) \in G$; *it is called positive semidefinite on* G, *if* $V(t, x) \geq 0$ *for all* $(t, x) \in G$. *The function* $V(t, x)$ *is called negative definite (negative semidefinite) on* G, *if* $-V(t, x)$ *is positive definite (positive semidefinite) on* G.

Definition 1.4.5 *A scalar-valued continuous function* $V(t, x) : R_+ \times R^n \to R$ *is said to be decrescent, if there exists a positive definite function* $w : R^n \to R$ *such that*

$$|V(t, x)| \leq w(x) \;\; for\ all \;\; t \geq t_0 \;\; and\ for\ all \;\; x \in S_\rho \;\; for\ some \;\; \rho > 0.$$

Now we are ready to see some results obtained by employing Lyapunov method.

CHAPTER 2

SOME KNOWN RESULTS ON THE STABILITY OF THE ZERO SOLUTION

2.1 Introduction

In this chapter, we state certain known stability criteria for impulsive differential equations with variable moment.

As is mentioned in the preceding chapter, there are various sets of sufficient conditions for the absence of pulse phenomena. Therefore, it is essential to impose some conditions that the beating of solutions of (1.1) on each surface of discontinuity be absent. To this end, namely for the system (1.1) to have no pulsation, in this chapter, in addition to conditions (a)-(c) we assume that there exists $L > 0$ with $ML < 1$ such that

$$|\theta_i(x) - \theta_i(y)| \leq L\|x - y\| \qquad \text{and} \qquad \theta_i(x + J_i(x)) \leq \theta_i(x) \tag{2.1}$$

for all $x, y \in S_\rho$ and $i \in N$. By the way, it is straightforward to see that these conditions are not necessary [16].

It is also well-known that the solution curve of the sytem (1.1) may intersect a surface of discontinuity at certain time and stay there for a while [4]. Therefore some caution should be taken, when such a system is under investigation. In this chapter, we have only two kinds of surfaces of discontinuities, namely $t = \theta_i^0$ and $t = \theta_i(x)$. It can be shown that no solution curve of (1.1) can stay on the surface $t = \theta_i^0$ for a period of time. Even in the case of $t = \theta_i(x)$, this behavior is still not possible, since each $t = \theta_i(x)$ is also a surface of impulse points.

2.2 Lyapunov Second Method for the Stability of Zero Solution

It is natural to ask whether it is convenient, in some situations, to utilize several Lyapunov function. As we shall see, the answer is positive and this approach offers a more flexible mechanism to investigate stability of the zero solution in a unified way by using a single Lyapunov function and the theory of impulsive differential inequalities. Lyapunov's second method has its origin in simple theorems that form the core of what he himself called his second method for dealing with question of stability. The method is an indispensible tool not only in the theory of stability but also in the study of other qualitative properties of solutions of differential equations. Each Lyapunov function needs to satisfy some requirements which will be described in the theorems.

Let $V(t,x)$ be a continuous real valued Lyapunov function defined on G with $V(t,0) = 0$ for $t \geq t_0$. We assume that $V(t,x)$ is locally Lipschitz in x, and denote

$$D^{+}V(t,x) = \limsup_{h \to 0^{+}} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h}.$$

Notice that if V is differentiable then

$$D^{+}V(t,x) = \dot{V}(t,x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x).$$

Note that, in what follows, we denote by $\mathcal{A}$ the set of all continuous functions $\psi : R \to R$ such that $\psi(0) = 0$ and $\psi(s) > 0$ for $s > 0$.

In view of the assumption (2.1), we shall take into account the qualitative behaviour of the zero solution of the system (1.1) and consider the method of Lyapunov by indicating its fruitfulness.

2.3 Theorems on the Stability of IDE

In what follows, firstly some basic theorems with their proofs will be examined and secondly the foregoing results combined with comparison theorem will be derived.

Theorem 2.3.1 *[16] If there is a positive definite differentiable function $V(t,x)$ satisfying in G the inequalities*

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \leq 0$$
$$V(\theta_i(x), x + J_i(x)) \leq V(\theta_i(x), x), \tag{2.2}$$

then the trivial solution of the system (1.1) is stable. If, instead of the second inequality of (2.2), we have

$$V(\theta_i(x), x + J_i(x)) - V(\theta_i(x), x) \leq -\psi(V(\theta_i(x), x)) \tag{2.3}$$

for all $i = 1, 2, \ldots,$ and $\psi \in \mathcal{A}$, then the zero solution of (1.1) is asymptotically stable.

Proof. Let $\epsilon > 0$ be fixed and $\delta > 0$ be sufficiently small such that

$$l = \inf_{t \geq t_0, \epsilon \leq \|x\| < \rho} V(t,x), \qquad m = \sup_{\|x\| < \delta} V(t_0, x). \tag{2.4}$$

and $m < l$. Take an arbitrary solution $x(t)$ with $x(t_0) = x_0$ of the system (1.1) such that $x_0 \in \mathcal{B}_\delta$, which is a ball centered at x_0 with radius δ, and consider the function $v(t) = V(t,x)$. If we assume that, at a moment t^*, $\|x(t^*\| = \epsilon$ for $\|x_0\| < \delta$, then $v(t^*) = V(t^*, x(t^*)) \geq l$. Observe that for $\theta_i = \theta_i(x(\theta_i))$, we have

$$\begin{aligned} v(\theta_i+) &= V(\theta_i+, x(\theta_i+)) \\ &= V(\theta_i+, x(\theta_i) + J_i(x(\theta_i))) \\ &\leq V(\theta_i, x(\theta_i)) \\ &= v(\theta_i) \end{aligned} \tag{2.5}$$

Besides, the inequalities (2.2) imply that the function $v(t)$ is nonincreasing along any solution of (1.1) that lies in the region G, so that $v(t^*) \leq v(t_0) = V(t_0, x(t_0)) \leq m < l$. This contradiction proves the first part of the theorem. As of the proof of second part, we now suppose that, instead of the second inequality of (2.2), the inequality (2.3) holds. Our aim is to prove that the zero solution is asymtotically stable. To this end, it suffices to show that $\lim_{t\to\infty} v(t) = 0$. The first of inequality of (2.2) and inequality (2.3) imply that the fuction that $v(t)$ is nonincreasing and, since it is bounded from below, the limit $\lim_{t\to\infty} v(t) = \alpha$ exists. Suppose, for the sake of contradiction, that $\alpha > 0$. Let $\tau = \min_{\alpha \le s \le v(t_0)} \psi(s)$. If the function $x(t)$ intersects the surfaces $t = \theta_i(x)$ at the points $(\theta_i(x_i), x_i)$, then by (2.3), we have

$$v(\theta_i(x_i)+) - v(\theta_i(x_i)) \le -\psi(v(\theta_i(x_i))$$

for all $i = 1, 2, \ldots$. Since $\alpha < v(\theta_i(x_i)) \le v(t_0)$, we see that $-\psi(v(\theta_i(x_i)) \le -\tau$, and accordingly,

$$v(\theta_i(x_i)+) - v(\theta_i(x_i)) \le -\tau.$$

Inasmuch as, by the first inequality of (2.2), the function $v(t)$ is nonincreasing on every interval where it is continuous, we obtain from $v(\theta_i(x_i)+) \ge v(\theta_{i+1}(x_{i+1}))$ that, for any natural k,

$$\begin{aligned} v(\theta_k(x_k)+) &\le v(\theta_k(x_k)+) + \sum_{i=1}^{k-1}[v(\theta_i(x_i)+) - v(\theta_{i+1}(x_{i+1}))] \\ &= v(t_0) + \sum_{i=1}^{k}[v(\theta_i(x_i)+) - v(\theta_i(x_i))] \\ &\le v(t_0) - k\tau. \end{aligned}$$

That the right-hand side of this inequality becomes negative for large values of k contradicts the fact that the function $V(t,x)$ is positive definite, yielding the result that $\alpha > 0$ is not possible. Proof ends up with this contradiction.

Example 2.3.1 [16] We shall consider stability of the lower equilibrium point of a pendulum subject to an impulsive effect with the following motion equations

$$\begin{aligned} &\dot{x} = y, \qquad \dot{y} = -\sin x, \qquad t \ne \theta_i(x,y), \\ &\Delta x|_{t=\theta_i(x,\dot{x})} = -x + \arccos\left(-\frac{y^2}{2} + \cos x\right), \\ &\Delta y|_{t=\theta_i(x,\dot{x})} = -y, \qquad i \in N = \{1, 2, \ldots\}. \end{aligned}$$

Let us study stability of the zero solution of the system. Take the total energy of an unperturbed pendulum

$$V(x,y) = 1 - \cos x + \frac{y^2}{2}.$$

to be a Lyapunov function. We easily find that

$$\frac{dV}{dt} = y\sin x - y\sin x = 0,$$

$$\begin{aligned} V(x+\Delta x, y+\Delta y) &= 1 - \cos\left(\arccos\left(\cos x - \frac{y^2}{2}\right)\right) \\ &= 1 - \cos x + \frac{y^2}{2} = V(x,y). \end{aligned}$$

Since the conditions of theorem 2.3.1 are satisfied, the zero solution of this system is stable.

Theorem 2.3.2 *[16] Suppose there exists a positive definite funtion $V(t,x)$, which satisfies in G the inequalities*

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \leq -\varphi(V(t,x)), \tag{2.6}$$

$$V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x)), \tag{2.7}$$

where $\varphi, \psi \in \mathcal{A}$ and let

$$\sup_i(\min_{\|x\|\leq\rho} \theta_{i+1}(x) - \max_{\|x\|\leq\rho} \theta_i(x)) = \theta > 0. \tag{2.8}$$

Then, if the functons $\varphi(s)$ and $\psi(s)$ are such that

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta \tag{2.9}$$

for some a_0 and all $a \in (0, a_0]$, then the zero solution of the system (1.1) is stable. Furthermore, if, instead of inequality (2.9), we have

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta - \gamma \tag{2.10}$$

for some $\gamma > 0$, then the zero solution of the system (1.1) will be asymptotically stable.

Proof. Let us fix an arbitrary sufficently small $\epsilon > 0$. For this ϵ choose $\delta > 0$ to be so small that the inequalies (2.4) hold. Take a solution $x(t_0) = x_0$ of the system (1.1) such that $x_0 \in \mathcal{B}_\delta$, which is a ball centered at x_0 with radius δ. We will show that $x(t)$ never leaves the ball $\mathcal{B}_\epsilon$. Consider the function $v(t) = V(t, x)$. To prove the theorem it suffices to show that $v(t) < l$ for all $t \geq t_0$. An assumption that $x(t)$ leaves $\mathcal{B}_\epsilon$ without reaching the surface $t = \theta_1(x)$ at a moment t^* leads to a contradiction, because on one hand $v(t^*) = V(t^*, x(t^*)) \geq l$, and on the other hand, the function $v(t)$ is not increasing and $v(t^*) \leq v(t_0) = V(t_0, x(t_0)) \leq m < l$. So, $x(t)$ intersects the surface $t = \theta_1(x)$, for example at a point $(\theta_1(x_1), x_1)$. By using (2.7), $t_0 \leq \theta_1(x_1)$ and $v'(t) \leq -\varphi(v(t)$ we have

$$-\int_{t_0}^{\theta_1(x_1)} \frac{v'(t)dt}{\varphi(v(t))} \geq \theta_1(x_1) - t_0. \tag{2.11}$$

By setting in (2.11) $v(t) = s$ and using (2.8), we get

$$\int_{v(\theta_1(x_1))}^{v(t_0)} \frac{ds}{\varphi(s)} \geq \theta_1(x_1) - t_0 \geq \theta. \tag{2.12}$$

Now by replacing a by $v(\theta_1(x_1), x_1)$ in the inequality (2.9) and using (2.7), we obtain

$$\int_{v(\theta_1(x_1))}^{v(\theta_1(x_1)+)} \frac{ds}{\varphi(s)} \leq \int_{v(\theta_1(x_1))}^{\psi(v(\theta_1(x_1)))} \frac{ds}{\varphi(s)} \leq \theta. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\int_{v(\theta_1(x_1)+)}^{v(t_0)} \frac{ds}{\varphi(s)} = \int_{v(\theta_1(x_1))}^{v(t_0)} \frac{ds}{\varphi(s)} - \int_{v(\theta_1(x_1))}^{v(\theta_1(x_1)+)} \frac{ds}{\varphi(s)} \geq 0$$

which clearly implies that $v(\theta_1(x_1)+) \leq v(t_0)$. To end the proof of the first part of the thorem what we need to do is to apply the induction to get $v(\theta_i(x_i)+) \leq v(t_0)$ for all $i = 1, 2, \ldots$.

Now suppose that, instead of (2.9), inequality (2.10) holds and the solution $x(t)$ intersect the surfaces $t = \theta_i(x)$ at the points $(\theta_i(x_i), x_i)$. By inequality (2.7) and $\theta_i(x_i)+ \geq \theta_{i+1}(x_{i+1})$, we have

$$-\int_{\theta_{i+1}(x_{i+1})}^{\theta_i(x_i)+} dt \geq -\int_{\theta_i(x_i)}^{\theta_{i+1}(x_{i+1})} \frac{v'(t)dt}{\varphi(v(t))} \geq \theta_{i+1}(x_{i+1}) - \theta_i(x_i) \geq \theta.$$

Setting $a = v(\theta_{i+1}(x_{i+1}))$ in (2.10) and using (2.7), we see that

$$\int_{v(\theta_{i+1}(x_{i+1}))}^{v(\theta_{i+1}(x_{i+1})+)} \frac{ds}{\varphi(s)} \leq \int_{v(\theta_{i+1}(x_{i+1}))}^{\psi(v(\theta_{i+1}(x_{i+1})))} \frac{ds}{\varphi(s)} \leq \theta - \gamma. \tag{2.14}$$

In view of (2.14), we see that

$$\int_{a_{i+1}^+}^{a_i^+} \frac{ds}{\varphi(s)} = \int_{a_{i+1}}^{a_i^+} \frac{ds}{\varphi(s)} - \int_{a_{i+1}}^{a_{i+1}^+} \frac{ds}{\varphi(s)} \geq \gamma,$$

where $a_i^+ = v(\theta_i(x_i)+)$. Since the sequence a_i^+ is decreasing for all i, the following inequality is satisfied

$$\int_{a_{i+1}^+}^{a_i} \frac{ds}{\varphi(s)} \geq \gamma. \tag{2.15}$$

Now let us show that $\lim_{i\to\infty} v(\theta_i(x_i)+) = 0$. Suppose the converse, i.e. assume that $\lim_{i\to\infty} v(\theta_i(x_i)+) = \alpha > 0$. Let $\tau = \min_{\alpha \leq s \leq v(t_0)} \varphi(s)$. From (2.15) we get

$$\gamma \leq \int_{a_{i+1}^+}^{a_i^+} \frac{ds}{\varphi(s)} \leq \frac{1}{\tau}[v(\theta_i(x_i)+) - v(\theta_{i+1}(x_{i+1})+)],$$

That is to say, $v(\theta_i(x_i)+) - v(\theta_{i+1}(x_{i+1})+) \geq \gamma\tau = constant$, which contradicts the convergence of the sequence $v(\theta_i(x_i)+)$. Thus $v(\theta_i(x_i)+) \to 0$ for $i \to \infty$. To end the proof, recall that by (2.7), $v(t)$ is decreasing on every interval of continuity $(\theta_i(x_i), \theta_{i+1}(x_{i+1})]$, and hence $\sup_{\theta_i(x_i)<t<\theta_{i+1}(x_{i+1})} v(t) = v(\theta_i(x_i)+)$, which, together with the inequality $v(\theta_i(x_i)+) > v(\theta_{i+1}(x_{i+1})+)$ that holds for all i, leads to the inequality $v(t) < v(\theta_i(x_i)+)$ for all $t > \theta_i(x_i)$. Thus, it follows that $v(\theta_i(x_i)+) \to 0$ for all $i \to \infty$ that $\lim_{t\to\infty} v(t) = 0$, and so $\lim_{t\to\infty} \|x(t)\| = 0$.

For the following analogous theorem, additionally we assume that the function $\theta_i(x)$ are such that, for some $\theta_1 > 0$

$$\max_{\|x\|\leq\rho} \theta_i(x) - \min_{\|x\|\leq\rho} \theta_{i-1}(x) \leq \theta_1. \tag{2.16}$$

for all $i = 1, 2, \ldots$.

Theorem 2.3.3 *[16] Let there exists a positive definite function $V(t,x)$ satisfying in the domain G the conditions*

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \leq \varphi(V(t,x)), \tag{2.17}$$

$$V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x)), \tag{2.18}$$

where $\varphi, \psi \in \mathcal{A}$. If the functons $\varphi(s)$ and $\psi(s)$ are such that

$$\int_{\psi(a)}^{a} \frac{ds}{\varphi(s)} \geq \theta_1 \tag{2.19}$$

for some a_0 and all $a \in (0, a_0]$, then the zero solution of (1.1) is stable. Furthermore, if, instead of inequality (2.18), we have

$$\int_{\psi(a)}^{a} \frac{ds}{\varphi(s)} \geq \theta_1 + \gamma \tag{2.20}$$

for some $\gamma > 0$, then the zero solution of (1.1) is asymptotically stable.

We skip the proof of this theorem, because it can be similarly done by adopting previously straightforward approach. The principal purpose of the preceding discussion is to point out that stability of the zero solution can be provided under some circumstances. Before embarking on the results for the theory of instability of solutions, let us look at the following examples on the stability of solutions for illustration.

Example 2.3.2 [16] Consider the system

$$\begin{aligned}
&\dot{x} = y, \qquad \dot{y} = -\sin x, \qquad t \neq \theta_i(x, y),\\
&\Delta x|_{t=\theta_i(x,\dot{x})} = \alpha x + \beta y,\\
&\Delta y|_{t=\theta_i(x,\dot{x})} = -\beta x + \alpha y. \qquad i \in N = \{1, 2, \ldots\}.
\end{aligned}$$

As in the example 2.3.1, we take the function $V(t, x)$ to be

$$V(x, y) = 1 - \cos x + \frac{y^2}{2}.$$

The derivative of this function along the solution is identically equal to zero. Moreover, we observe that

$$\begin{aligned}
V(x + \Delta x, y + \Delta y) - V(x, y) &= \frac{1}{2}(\alpha^2 + 2\alpha + \beta^2)(x^2 + y^2) + O(x^2 + y^2).\\
&= \frac{1}{2}((\alpha + 1)^2 + \beta^2 - 1)(x^2 + y^2) + \gamma(x^2 + y^2),
\end{aligned}$$

where $\gamma(x^2 + y^2) \to 0$ for $x^2 + y^2 \to 0$.
Let $\frac{1}{2}((\alpha + 1)^2 + \beta^2 - 1) = l < 0$, i.e. $((\alpha + 1)^2 + \beta^2 < 1$. Then there exists $\lambda > 0$ such that $|\gamma(x^2 + y^2)| \leq \epsilon < -l$ as long as $x^2 + y^2 \leq \lambda^2$, and we have

$$V(x + \Delta x, y + \Delta y) - V(x, y) \leq (\epsilon + l)(x^2 + y^2).$$

Consequently, if $((\alpha+1)^2+\beta^2<1$, then the zero solution of the system is asymptotically stable.

Example 2.3.3 [16] Consider stability of the zero solution of the system

$$\begin{aligned}
&\dot{x}=-y+x^3, \qquad \dot{y}=x+y^3, \qquad t\neq\theta_i(x,y),\\
&\Delta x|_{t=\theta_i(x,\dot{x})}=-\alpha x^3+\beta y^3,\\
&\Delta y|_{t=\theta_i(x,\dot{x})}=\beta x^3-\alpha y^3, \qquad i\in N=\{1,2,\ldots\}.
\end{aligned}$$

where $\alpha>0,\beta>0$, and $\theta_i(x,y)=i+x^2+y^2$. It can be checked that solutions of this system satisfy the conditions of lemma (1.2.3) in a sufficiently small neigborhood of the origin and so there are no beating at the surfaces $t=\theta_i(x)$. Set $V(x,y)=x^2+y^2$. Then

$$\frac{dV}{dt}=2x^4+2y^4\leq 2V^2(x,y),$$

$$\begin{aligned}
V(x+\Delta x,y+\Delta y) &= x^2+y^2-2\alpha(x^4+y^4)+2(x^2+y^2)xy+\\
&+ (\alpha^2+\beta^2)(x^6+y^6)-4\alpha\beta x^3y^3\\
&\leq V(x,y)-(\alpha-\beta)V^2(x)+(\alpha^2+\beta^2)V^3(x,y).
\end{aligned}$$

Thus, it is better to take $\varphi(s)=2s^2$ and $\psi(s)=s-(\alpha-\beta)s^2+(\alpha^2+\beta^2)s^3$. By using the fact that

$$\max_{x^2+y^2\leq\rho}\theta_i(x,y)-\min_{x^2+y^2\leq\rho}\theta_{i-1}(x,y)\leq 1+\rho,$$

$$\int_{\psi(a)}^{a}\frac{ds}{2s^2}=\frac{\alpha-\beta-(\alpha^2-\beta^2)a}{2(1-(\alpha-\beta)a+(\alpha^2+\beta^2)a^2)},$$

we see that, for the zero solution of the system under consideration to be asymptotically stable, it is sufficient to impose the condition $\alpha-\beta>2$.

We are now in position to find out the sufficient conditions for the zero solution of the (1.1) to be unstable. For such theorems that give these conditions, we first require that the function $V(t,x)$ exist and have the following properties [16]:

i) The intersection of the region $\Omega = \{(t,x) \in G | V(t,x) > 0\}$, where a plane t =constant, is a nonempty open set adherent to the origin for any $t \geq t_0$;

ii) The function $V(t,x)$ is bounded in Ω

Theorem 2.3.4 *[16] If there there exists a function $V(t,x)$ having properties i) and ii) and satisfying in the region Ω the conditions*

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \geq 0, \tag{2.21}$$

$$V(\theta_i(x), x + J_i(x)) - V(\theta_i(x), x) \geq \psi(V(\theta_i(x), x)), \tag{2.22}$$

where $\psi \in \mathcal{A}$, then the zero solution of system (1.1) is unstable.

Proof. By the conditions of the theorem, in any neighborhood of the point $x = 0$ there exists such a point x_0 that $V(t_0, x_0) > 0$. We will prove that the solution $x(t)$ that starts at the point x_0 will eventually leave the ball $\mathcal{B}_\rho$. Suppose, on the contrary, that $x(t) \in \mathcal{B}_\rho$ for all $t \geq t_0$. Let $x(t)$ intersect the surfaces $t = \theta_i(x)$ at points $(\theta_i(x_i), x_i)$. Consider the function $v(t) = V(t, x(t))$. By the inequalities (2.21) and (2.22), $v(t)$ is nondecreasing function, and hence $v(t) \geq v(t_0) > 0$ for all $t \geq t_0$. This means that $(t, x(t)) \in \Omega$ for all $t \geq t_0$. Let $\tau = \min\limits_{v(t_0) \leq s \leq \zeta} \psi(s)$, where $\zeta = \sup\limits_{(t,x) \in \Omega} V(t,x)$. It is clear that $\tau > 0$ and $v(\theta_i(x_i)+) - v(\theta_i(x_i)) \geq \tau, i = 1, 2, \ldots$. So, for any natural k, we have that

$$\begin{aligned} v(\theta_k(x_k)+) &\geq v(\theta_k(x_k)+) + \sum_{i=1}^{k-1} [v(\theta_i(x_i)+) - v(\theta_{i+1}(x_{i+1}))] \\ &= v(t_0) + \sum_{i=1}^{k} [v(\theta_i(x_i)+) - v(\theta_i(x_i))] \\ &\geq v(t_0) + k\tau. \end{aligned}$$

The right-hand side of the last inequality becomes unbounded as $k \to \infty$, which contradicts the assumption *ii*). This completes the proof.

Theorem 2.3.5 *[16] If there there exists a function $V(t,x)$ having properties i) and ii) and satisfying in the region Ω the inequalities*

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \geq -\varphi(V(t,x)), \tag{2.23}$$

$$V(\theta_i(x), x + J_i(x)) \;\; \geq \;\; \psi(V(\theta_i(x), x)), \tag{2.24}$$

where $\psi, \varphi \in \mathcal{A}$*. Also, assume that the functions* $\theta_i(x)$ *satisfy the condition (2.15). If the functions* $\psi(s)$ *and* $\varphi(s)$ *are such that, for some* $\gamma > 0$*,*

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \geq \theta_1 + \gamma \tag{2.25}$$

for all $a \in (0, a_0]$*, then the zero solution of the system (1.1) is unstable.*

Proof. In every arbitrarily small neighborhood of the point $x = 0$ there exists such a point x_0 that $V(t_0, x_0) > 0$. We will prove that the solution that starts at this point will eventually leave the ball $\mathcal{B}_\rho$. Suppose, conversely, that $x(t) \in \mathcal{B}_\rho$ does not not intersect the surfaces $t = \theta_i(x)$ at the points $(\theta_i(x_i), x_i)$ for all $t \geq t_0$. Let us assume that, from the assumption $x(t) \in \mathcal{B}_\rho$, it will follow that $(t, x(t)) \in \Omega$ for all $t \geq t_0$. Indeed, it is impossible that $(\theta_i(x_i), x_i) \in \Omega$ and $(\theta_i(x_i), x(\theta_i(x_i)+)) \notin \Omega$, because it follows from (2.24) that

$$V(\theta_i(x_i), x(\theta_i(x_i)+)) = V(\theta_i(x_i), J_i(x_i)) \geq \psi(V(\theta_i(x_i), x_i)) > 0.$$

If we assume that the phase point $(t, x(t))$ leaves the region Ω, then it necessarily intersects its boundary. Let t^* be the smallest moment when this occurs. If $\theta_k(x_k) < t \leq \theta_{k+1}(x_{k+1})$, then, denoting $v(t) = V(t, x(t))$, we have that $v(t^*) = 0$ and $v(\theta_k(x_k)+) > 0$. It follows from (2.23) that

$$-\int_{\theta_k(x_k)}^{t^*} \frac{v'(t)dt}{\varphi(v(t))} \leq t^* - \theta_k(x_k).$$

and, by using (2.15),

$$\int_{v(t^*)}^{v(\theta_k(x_k)+)} \frac{ds}{\varphi(s)} \leq t^* - \theta_k(x_k) \leq \theta_1. \tag{2.26}$$

Fix $a' > 0$ such that $0 < \psi(a') \leq v(\theta_k(x_k)+)$. By using inequalities (2.24)-(2.26) we get a contradictory chain of the inequalities

$$\theta_1 + \gamma \leq \int_{a'}^{\psi(a')} \frac{ds}{\varphi(s)} \leq \int_0^{v(\theta_k(x_k)+)} \frac{ds}{\varphi(s)} = \int_{v(t^*)}^{v(\theta_k(x_k)+)} \frac{ds}{\varphi(s)} \leq \theta_1.$$

Hence, if $x(t) \in \mathcal{B}_\rho$, then $(t, x(t)) \in \Omega$ for $t \geq t_0$. So it follows from (2.23) that

$$\int_{a_{i+1}}^{a_i^+} \frac{ds}{\varphi(s)} \leq \theta_1, \qquad i = 0, 1, 2, \ldots$$

where $a_i = v(\theta_i(x_i))$ and $a_i^+ = v(\theta_i(x_i)+)$. By substracting this inequality from(2.25) with $a = a_{i+1} = v(\theta_{i+1}(x_{i+1}))$ and using (2.24), we obtain

$$\gamma \le \int_{a_{i+1}}^{\psi(a_{i+1})} \frac{ds}{\varphi(s)} - \int_{a_{i+1}}^{\psi(a_i^+)} \frac{ds}{\varphi(s)} \le \int_{a_{i+1}}^{\psi(a_{i+1}^+)} \frac{ds}{\varphi(s)} - \int_{a_{i+1}}^{a_i^+} \frac{ds}{\varphi(s)}$$

or equivalently

$$\int_{a_i^+}^{a_{i+1}^+} \frac{ds}{\varphi(s)} \ge \gamma, \qquad i = 0, 1, 2, \dots. \tag{2.27}$$

This shows that $\{a_i\}$ is an increasing sequence and it is bounded by a_0 due to $(t, x(t)) \in \Omega$. Let $\tau = \min_{v(t_0) \le s \le a_0} \varphi(s) > 0$. From (2.27) we get

$$\gamma \le \frac{1}{\tau} \int_{a_i^+}^{a_{i+1}^+} ds = \frac{1}{\tau} \left[v(\theta_{i+1}(x_{i+1})+) - v(\theta_i(x_i)+) \right],$$

that is, $v(\theta_{i+1}(x_{i+1})+) - v(\theta_i(x_i)+) \ge \gamma\tau$, and hence, for any natural k, $v(\theta_i(x_i)+) \ge \gamma\tau k + v(t_0)$, which is a contradiction since the sequence $\{v(\theta_i(x_i)+)\}$ is bounded.

Now we state one more analogous theorem that can be proved similarly.

Theorem 2.3.6 *[16] If there there exists a function $V(t, x)$ with properties i) and ii) and satisfying in the region Ω the inequalities*

$$\begin{aligned} \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x) &\ge \varphi(V(t, x)), & (2.28)\\ V(\theta_i(x), x + J_i(x)) &\ge \psi(V(\theta_i(x), x)), & (2.29) \end{aligned}$$

where $\psi, \varphi \in \mathcal{A}$. Also assume that the functions $\theta_i(x)$ satisfy the condition (2.8). If the functions $\psi(s)$ and $\varphi(s)$ are such that, for some $\gamma > 0$,

$$\int_{\psi(a)}^{a} \frac{ds}{\varphi(s)} \ge \theta - \gamma \tag{2.30}$$

for all $a \in (0, a_0]$, then the zero solution of the system (1.1) is unstable.

The following basic comparison results, extracted from [11], are to be used in the remaining part of the thesis.

Lemma 2.3.7 *Let $V(t,x)$ be as above, and*

$$D^{+}V(t,x) \leq g(t,V(t,x)), \quad (t,x) \in G, \tag{2.31}$$

where $g \in C[R_{+} \times R_{+}, R]$. Suppose that $u(t) = u(t,t_0,u_0)$ is the maximal solution of the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0,$$

which exists to the right of t_0. If $x(t) = x(t,t_0,x_0)$ is any solution of (2.31) such that $V(t_0,x_0) \leq u_0$, then $V(t,x(t)) \leq u(t)$ for $t \geq t_0$.

Proof. First of all, let us start by refreshing our memory with the definition of the maximal solution. To do this suppose that $u(t)$ is a solution of the system

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0 \tag{2.32}$$

and, for sufficiently small $\epsilon > 0$, $u(t,\epsilon)$ is a solution of

$$u' = g(t,u) + \epsilon, \quad u(t_0) = u_0 + \epsilon. \tag{2.33}$$

$u_m(t) = u_m(t,t_0,u_0)$ is called maximal solution of the system (2.32), if there exists a sequence $\{\xi_n\}$ with $\xi_{n+1} < \xi_n$ and $\lim_{n\to\infty} \xi_n = 0$ such that $\lim_{n\to\infty} u(t,\xi_n) = u_m(t)$. To prove the lemma, suppose that $u_m(t)$ is a maximal solution of (2.32). If $x(t)$ is any solution of (2.31) and if $V(t_0,x_0) \leq u_0$, then we see that $V(t_0,x_0) \leq u_0 < u_0 + \epsilon = u(t_0,\epsilon)$, which means that $V(t_0,x_0) < u(t_0,\epsilon)$. Suppose on the contrary that $V(t,x(t)) \leq u_m(t)$ fails to hold for all $t \geq t_0$. This means that there exists $t^* > t_0$, the smallest moment, such that $V(t^*,x(t^*)) = u(t^*,\epsilon)$ with $V(t,x(t)) < u(t,\epsilon)$ for $t \in [t_0,t^*)$ and $V(t,x(t)) > u(t,\epsilon)$ for some $t > t^*$. Now we observe that, for arbitrary sequence $\xi_n > 0$ with $\xi_n \to 0$ as $n \to \infty$,

$$\frac{V(t^*+\xi_n,x) - V(t^*,x)}{\xi_n} = \frac{V(t^*+\xi_n,x) - u(t^*,\epsilon)}{\xi_n} > \frac{u(t^*+\xi_n,\epsilon) - u(t^*,\epsilon)}{\xi_n}$$

Since ξ_n is arbitrary positive sequence converging to zero, it follows that

$$D^{+}V(t^*,x(t^*)) > u'(t^*,\epsilon)$$

This implies, for $\epsilon > 0$, that

$$\begin{aligned} g(t^*, V(t^*, x(t^*))) &\geq D^+V(t^*, x(t^*)) \\ &> u'(t^*, \epsilon) \\ &= g(t^*, u(t^*, \epsilon)) + \epsilon \\ &> g(t^*, u(t^*, \epsilon)) \\ &= g(t^*, V(t^*, x(t^*))). \end{aligned}$$

This contradiction completes the proof of the lemma.

One can similarly prove the following analogous comparison result:

Lemma 2.3.8 *Let $V(t,x)$ be as above, and*

$$D^+V(t,x) \geq g(t, V(t,x)), \quad (t,x) \in G, \tag{2.34}$$

where $g \in C[R_+ \times R_+, R]$. Suppose that $u(t) = u(t, t_0, u_0)$ is the minimal solution of the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0,$$

which exists to the right of t_0. If $x(t) = x(t, t_0, x_0)$ is any solution of (2.34) such that $V(t_0, x_0) \geq u_0$, then $V(t, x(t)) \geq u(t)$ for $t \geq t_0$.

Note that by taking $g(t,u) \equiv 0$, one can easily verify that if $D^+V(t,x) \geq 0$ then $V(t,x(t))$ is nondecreasing, and if $D^+V(t,x) \leq 0$ then $V(t,x(t))$ is nonincreasing.

We note that until now the change of Lyapunov function in the interval of continuity is compared with its change at the moments of discontinuity. In the following theorems, the changes of a Lyapunov function in the vicinity of the moments, where solutions undertake an impulse effect are comparied. In fact, the following results play an important role in our study as we will try to improve them.

In the sequel, we denote

$$G_i = \{(t,x) \in D : \ t \text{ is between } \theta_i^0 \text{ and } \theta_i(x)\}.$$

Theorem 2.3.9 *[3] Assume that the following conditions are fulfilled.*

(i) $V(t,x)$ *is positive definite and decrescent on* G.

(ii) $D^+V(t,x)$ *is negative semidefinite on* G.

(iii) $D^+V(t,x) \leq -\varphi(V(t,x))$ *for some* $\varphi \in \mathcal{A}$ *and for all* $(t,x) \in \cup_{i\in N} G_i$.

(iv) $V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x)$ *for some* $\psi \in \mathcal{A}$ *and for all* $x \in S_\rho$ *and* $i \in N$.

(v) There exists a real number $L > 0$ *such that* $|\theta_i(x) - \theta_i(y)| \leq L\|x - y\|$ *for all* $x, y \in S_\rho$ *and* $i \in N$.

(vi) There exists a real number $L_1 > 0$ *such that* $|\theta_i(x) - \theta_i^0| \geq L_1\|x\|$ *for all* $x \in S_\rho$ *and* $i \in N$.

(vii) There exists a real number $\gamma \geq 0$ *such that*

$$\int_{V(\theta_i(x),x)}^{\psi(V(\theta_i(x),x))} \frac{ds}{\varphi(s)} \leq (L_1 - \gamma)\|x\|$$

for all $x \in S_\rho$ *and* $i \in N$.

Then, the zero solution of (1.1) is stable if $\gamma = 0$, *and is asymptotically stable if* $\gamma > 0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be a solution of (1.1) having discontinuities at $t = \tau_i$ for $i \in N$. It follow that $v(t) := V(t, x(t)$ is nonincreasing on each of the intervals $[t_0, \tau_1)$ and (τ_i, τ_{i+1}), $i \in N$. To obtain more information on the behavior of $v(t)$, we need to investigate its change in G_i. For each fixed $i \in N$, there are two possible cases:

Case 1. $\tau_i \geq \theta_i^0$. We let $u(t)$ be the maximal solution of $u' = -\varphi(u)$ on $[\theta_i^0, \tau_i]$ such that $u(\theta_i^0) = v(\theta_i^0)$. In view of Lemma 2.3.7 and (vi), we have

$$L_1\|x(\tau_i)\| \leq \tau_i - \theta_i^0 = \int_{u(\tau_i)}^{u(\theta_i^0)} \frac{ds}{\varphi(s)} \leq \int_{v(\tau_i)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)}. \tag{2.35}$$

Using (iv) and (vii) we also have

$$(\gamma - L_1)\|x(\tau_i)\| \leq -\int_{v(\tau_i)}^{\psi(v(\tau_i))} \frac{ds}{\varphi(s)} \leq -\int_{v(\tau_i)}^{v(\tau_i+)} \frac{ds}{\varphi(s)}. \tag{2.36}$$

Summing (2.35) and (2.36) leads to

$$\int_{v(\tau_i+)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)} \geq \gamma \|x(\tau_i)\| \tag{2.37}$$

and hence we obtain that

$$v(\theta_i^0) \geq v(\tau_i+). \tag{2.38}$$

Case 2. $\tau_i < \theta_i^0$. We let $u(t)$ be the maximal solution of $u' = -\varphi(u)$ on $[\tau_i, \theta_i^0]$ such that $u(\tau_i) = v(\tau_i+)$. It follows that

$$L_1\|x(\tau_i)\| \leq \theta_i^0 - \tau_i = \int_{u(\theta_i^0)}^{u(\tau_i)} \frac{ds}{\varphi(s)} \leq \int_{v(\theta_i^0)}^{v(\tau_i+)} \frac{ds}{\varphi(s)}. \tag{2.39}$$

From (2.36) and (2.39) we get

$$\int_{v(\theta_i^0)}^{v(\tau_i)} \frac{ds}{\varphi(s)} \geq \gamma \|x(\tau_i)\| \tag{2.40}$$

and so

$$v(\tau_i) \geq v(\theta_i^0). \tag{2.41}$$

Define $\Gamma := \cup_{i=1}^{\infty}(\xi_i, \zeta_i]$ and $\Lambda := [t_0, \infty) \setminus \Gamma$, where $\xi_i = \tau_i$ and $\zeta_i = \theta_i^0$ if $\tau_i \leq \theta_i^0$, $\xi_i = \theta_i^0$ and $\zeta_i = \tau_i$ if $\tau_i \geq \theta_i^0$. From (2.38) and (2.41) we may write $v(\zeta_i+) \leq v(\xi_i)$, and hence conclude that $v(t)$ is nonincreasing on Λ.
Let $0 < \epsilon < \rho$ and t_0 be given. Without loss of generality, we may assume that $t_0 \in \Lambda$. It is now clear that $v(t) \leq v(t_0)$ for all $t \in \Lambda$. Set

$$\epsilon_1 = \frac{\epsilon}{2(1+ML)} \quad \text{and} \quad \eta = \inf_{t \geq t_0,\, \|x\| \geq \epsilon_1} V(t,x).$$

Since $V(t,x)$ is continuous and $V(t,0) = 0$, it is possible to find a positive real number δ such that $\delta < \epsilon_1$ and

$$\sup_{\|x\| \leq \delta} V(t_0, x) < \eta.$$

We first claim that if $\|x_0\| < \delta$, then $\|x(t, t_0, x_0)\| < \epsilon_1$ for all $t \in \Lambda$. Suppose on the contrary that this is not true. Then, there would exist a $t^* \in \Lambda$ such that $\|x(t^*, t_0, x_0)\| \geq \epsilon_1$. But this leads us to the contradiction that $\eta \leq v(t^*) \leq$

$v(t_0) < \eta$.

Next suppose that $t \in (\xi_i, \zeta_i]$ for some i. Clearly,

$$x(t) = x(\xi_i) + \int_{\xi_i}^{t} f(s, x(s))ds \ \text{ if } \tau_i > \theta_i^0, \tag{2.42}$$

and

$$x(t) = x(\zeta_i) + \int_{\zeta_i}^{t} f(s, x(s))ds \ \text{ if } \tau_i < \theta_i^0. \tag{2.43}$$

In view of (v), we easily obtain from both (2.42) and (2.43) that

$$x(t) \leq \epsilon_1(1 + ML) < \epsilon.$$

Therefore, the zero solution of (1.1) is stable. We shall now show that if $\gamma > 0$ then $\lim_{t\to\infty} x(t) = 0$. We first observe that since $v(t)$ is positive and nonincreasing on Λ, there is a nonnegative real number μ such that

$$\lim_{t\to\infty} v(t) = \mu, \quad t \in \Lambda. \tag{2.44}$$

We claim that $\mu = 0$. Suppose on the contrary that $\mu > 0$. Because of (2.44) and (i), there exists a positive real number μ_1 such that

$$\|x(t)\| \geq \mu_1 \ \text{ for all } t \in \Lambda. \tag{2.45}$$

If $\theta_i^0 > \tau_i$ then, since $\tau_i \in \Lambda$, (2.45) implies that

$$\|x(\tau_i)\| \geq \mu_1. \tag{2.46}$$

Suppose that $\theta_i^0 < \tau_i$. In this case, $\theta_i^0 \in \Lambda$, and therefore by (2.45) we have

$$\|x(\theta_i^0)\| \geq \mu_1. \tag{2.47}$$

Using (v), we also have

$$|\theta_i(x(\tau_i)) - \theta_i^0| \leq L\|x(\tau_i)\|. \tag{2.48}$$

In view of (2.47) and (2.48), we easily obtain from

$$x(\tau_i) = x(\theta_i^0) + \int_{\theta_i^0}^{\tau_i} f(s, x(s))\, ds$$

that

$$\|x(\tau_i)\| \geq \mu_1 - ML\|x(\tau_i)\|$$

and hence

$$\|x(\tau_i)\| \geq \mu_1/(1+ML) \tag{2.49}$$

Thus, we see from (2.46) and (2.49) that

$$\|x(\tau_i)\| \geq \mu_2 \text{ for all } i \in N, \tag{2.50}$$

where $\mu_2 = \mu_1/(1+ML)$.
On the other hand, since $(\zeta_i, \xi_{i+1}] \in \Lambda$ for all $i \in N$, $\lim_{i\to\infty} v(\zeta_i+) = \lim_{i\to\infty} v(\xi_i) = \mu$ and $v(\xi_i) \geq v(\zeta_i+) \geq \mu$. Letting $m = \min_{\mu \leq s \leq v(t_0)} \varphi(s)$, it follows from (2.37), (2.40), and (2.50) that

$$v(\xi_i) - v(\zeta_i+) \geq \gamma m \mu_2 \text{ for all } i \in N. \tag{2.51}$$

Using $v(\xi_{i+1}) \leq v(\zeta_i+)$ in (2.51) and then summing the resulting inequality over i from 1 to k we get

$$v(\xi_1) - v(\xi_{k+1}) \geq (\gamma m \mu_1)k \text{ for all } k \in N. \tag{2.52}$$

It is clear from (2.52) that if k is sufficiently large, then the function v takes on negative values. But this contradicts the fact that v is positive definite. Thus, we must have $\mu = 0$. As in the classical case, it follows that $\lim_{t\to\infty} x(t) = 0$, and hence we may conclude that the zero solution is asymptotically stable.

Corollary 2.3.10 *[3] Let all conditions of Theorem 3.2.1 except (v) are satisfied. In addition, suppose that the family $\{\theta_i(x)\}$ is equicontinuous at $x = 0$, and $\theta_i^0 \geq \theta_i(x)$ for all $x \in S_\rho$. Then the conclusion of Theorem 2.3.9 remains valid.*

Proof. We proceed as in the proof of Theorem 2.3.9, until ϵ_1 is picked. Now since the family $\{\theta_i(x)\}$ is equicontinuous at $x = 0$ and $\theta_i^0 \geq \theta_i(x)$ for all $x \in S_\rho$, given any ϵ_2, $0 < \epsilon_2 < \epsilon/M$, we can find $\epsilon_3 > 0$ such that $\theta_i^0 - \theta_i(x) < \epsilon_2$ for all $\|x\| < \epsilon_3$ and $i \in N$. Fix $\epsilon_1 > 0$ such that $\epsilon_1 < \min\{\epsilon_3, \epsilon - M\epsilon_2\}$. Then, it follows from (2.42) and (2.43) that

$$\|x(t)\| \leq \epsilon_1 + M\epsilon_2 < \epsilon.$$

Clearly, (2.44), (2.45), and (2.46) hold and by our assumption the case $\theta_i^0 < \tau_i$ does not exist. Thus (2.50) is satisfied with $\mu_2 = \mu_1$. The rest of the proof is the same as that of Theorem 2.3.9.

Example 2.3.4 Let $\theta_i(x) = i - \sqrt{x_1^2 + x_2^2}$ so that $G_i = \{(t,x) \in G : i - \sqrt{x_1^2 + x_2^2} < t \leq i\}$. We define $S = \bigcup_{i=1}^{\infty} G_i$ and consider the impulsive system

$$\dot{x}_1 = \begin{cases} -x_2, & (t,x) \notin S, \\ -x_1, & (t,x) \in S \end{cases}$$

$$\dot{x}_2 = \begin{cases} x_1, & (t,x) \notin S, \\ -x_2, & (t,x) \in S \end{cases}$$

$$\Delta x_1|_{t=\theta_i(x)} = -\alpha x_1 + \beta x_2,$$

$$\Delta x_2|_{t=\theta_i(x)} = \beta x_1 - \alpha x_2.$$

We choose $V(x) = x_1^2 + x_2^2$ and make the following observations:

(a) $\dot{V}(x) = 0$ if $(t,x) \notin S$ and $\dot{V}(x) = -2V(x)$ if $(t,x) \in S$. Since $V(x + \Delta x) = ((1-\alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1x_2$, we have

$$V(x + \Delta x) \leq \ell(\alpha, \beta)V(x),$$

where $\ell(\alpha,\beta) = (|\beta| + |1-\alpha|)^2$.

(b) $\|x + \Delta x\|^2 = ((1-\alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha-1)x_1x_2$ and so $\|x + \Delta x\|^2 \geq (|1-\alpha|^2 - |\beta|^2)^2\|x\|^2$. It follows that if $|\,|1-\alpha| - |\beta|\,| \geq 1$, then

$$\theta_i(x + \Delta x) \leq \theta_i(x).$$

(c) $|\theta_i(x) - \theta_i^0| = \sqrt{x_1^2 + x_2^2} = \|x\|.$

(d) Let $\varphi(s) = 2s$ and $\psi(s) = \ell(\alpha\,\beta)s$, and fix a positive number $\gamma < 1$. If $\ell(\alpha, \beta) \leq 1$, then $\ln \ell(\alpha, \beta) \leq 2(1-\gamma)\|x\|$ and hence

$$\int_V^{\ell V} \frac{ds}{2s} \leq (1-\gamma)\ \|x\|.$$

In view of Theorem 2.3.9 we deduce that the zero solution of (1.1) is asymptotically stable, if

$$|\,|1-\alpha| - |\beta|\,| \geq 1 \ \text{ and } |1+\alpha| + |\beta| \leq 1.$$

If we take $\theta_i(x) = i - \sqrt[4]{x_1^2 + x_2^2}$, then it is easy to verify condition (v) is not satisfied and therefore Theorem 2.3.9 does not apply. However, since the additional conditions stated in Corollary 2.3.10 are true, we may conclude that the above conclusion is valid. We note that the beating is still absent in this case, since $\frac{d}{dt}\|x(t)\| = 0$ for all $(t, x) \notin S$ and (b) holds.

In the next theorem we do not require that $D^+V(t, x)$ be negative semidefinite on $\cup_{i \in N} G_i$.

Theorem 2.3.11 *[3] Assume that the following conditions are fulfilled.*

(i) $V(t, x)$ *is positive definite on* G.

(ii) $D^+V(t, x)$ *is negative semidefinite on* $G \setminus \cup_{i \in N} G_i$.

(iii) $D^+V(t, x) \leq \varphi(V(t, x))$ *for some* $\varphi \in \mathcal{A}$ *and for all* $(t, x) \in \cup_{i \in N} G_i$.

(iv) $V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x)$ *for some* $\psi \in \mathcal{A}$ *and for all* $x \in S_\rho$ *and* $i \in N$.

(v) There exists a real number $L > 0$ *such that* $|\theta_i(x) - \theta_i(y)| \leq L\|x - y\|$ *for all* $x, y \in S_\rho$ *and* $i \in N$.

(vi) There exists $\gamma \geq 0$ *such that*

$$\int_{V(\theta_i(x),x)}^{\psi(V(\theta_i(x),x))} \frac{ds}{\varphi(s)} \geq (L+\gamma)\|x\|$$

for all $x \in S_\rho$ *and* $i \in N$.

Then, the zero solution of (1.1) is stable if $\gamma = 0$, and is asymptotically stable if $\gamma > 0$.

Proof. Let $u(t)$ be the maximal solution of $u' = \varphi(u)$ on $[\xi_i, \zeta_i]$ such that $u(\xi_i) = v(\xi_i+)$, where ξ_i and ζ_i are as defined in the proof of Theorem 2.3.9. Proceeding as in the proof of Theorem 2.3.9 we easily obtain

$$\int_{v(\theta_i^0)}^{v(\tau_i)} \frac{ds}{\varphi(s)} \le L\|x(\tau_i)\| \text{ for } \tau_i \ge \theta_i^0 \tag{2.53}$$

and

$$\int_{v(\tau_i+)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)} \le L\|x(\tau_i)\| \text{ for } \tau_i < \theta_i^0 \tag{2.54}$$

Using (iv) and (vi) we also have

$$\int_{v(\tau_i)}^{v(\tau_i+)} \frac{ds}{\varphi(s)} \le -(L+\gamma)\|x(\tau_i)\|. \tag{2.55}$$

It follows from (2.53), (2.54), and (2.55) that $v(\zeta_i+) \le v(\xi_i)$ for all $i \in N$. The remainder of the proof is similar to that of Theorem 2.3.9 and hence is omitted.

Example 2.3.5 Let $\theta_i(x) = i + x_1^2 + x_2^2$. Clearly $G_i = \{(t,x) : \ i < t \le i + x_1^2 + x_2^2\}$. Define $S = \bigcup_{i=1}^{\infty} G_i$ and consider the impulsive system

$$\dot{x}_1 = \begin{cases} -x_2, & (t,x) \notin S, \\ -x_2 + x_1^3, & (t,x) \in S \end{cases}$$

$$\dot{x}_2 = \begin{cases} x_1, & (t,x) \notin S, \\ x_1 + x_2^3, & (t,x) \in S, \end{cases}$$

$$\Delta x_1|_{t=\theta_i(x)} = -\alpha x_1 + \beta x_2,$$

$$\Delta x_2|_{t=\theta_i(x)} = \beta x_1 - \alpha x_2.$$

We choose $V(x) = x_1^2 + x_2^2$ and make the following observations:

(a) $\dot{V}(x) = 0$ if $(t,x) \notin S$ and $\dot{V}(x) \le 2V^2(x)$ if $(t,x) \in S$. Since $V(x + \Delta x) = ((1-\alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1x_2$, we have

$$V(x + \Delta x) \le \ell(\alpha, \beta)V(x),$$

where $\ell(\alpha, \beta) = (|\beta| + |1 - \alpha|)^2$.

(b) Let $x, y \in R^n$ such that $||x|| \leq h$ and $||y|| \leq h$, where $h > 0$ is some real number. It follows that

$$|\theta_i(x) - \theta_i(y)| \leq 2h||x - y||.$$

(c) If $\ell(\alpha, \beta) < 1$ then $\theta_i(x + \Delta x) \leq i + \ell(\alpha, \beta)||x|| < \theta_i(x)$.

(d) Let $g = (-x_2 + x_1^3, x_1 + x_2^3)$ and $m(h) = \max_{||x|| \leq h} ||g||$. Clearly $m(h) \to 0$ as $h \to 0$ and so there exists h_0 such that $2hm(h) < 1$ for all $h \leq h_0$.

(e) Let $\varphi(s) = 2s^2$ and $\psi(s) = \ell(\alpha, \beta)s$, and fix a positive real number γ. Choose $||x|| \leq \min\{h_0, \sqrt[3]{(1-l)/(2l(2h_0 + \gamma))}\}$. It follows that $1 - \ell \geq 2\ell(2h + \gamma)||x||^3$ and so

$$\int_{\ell V}^{V} \frac{ds}{2s^2} \geq (2h + \gamma)||x||.$$

By Theorem 2.3.11, the zero solution of (1.1) is asymptotically stable if

$$|\beta| + |\alpha - 1| < 1.$$

In this case, $2hM < 1$ is sufficient for the absence of beating.

The following result is a Chetaev's type instability theorem [14], for the zero solution of (1.1).

Theorem 2.3.12 *[3] Assume that the following conditions are fulfilled.*

(i) For every $\epsilon > 0$ and for every $t \geq t_0$ there exists points $\bar{x} \in S_\epsilon$ such that $V(t, \bar{x}) > 0$. The set B of all points (t, x) such that $\bar{x} \in S_\rho$ and such that $v(t, \bar{x}) > 0$ is called the "domain $v > 0$." The set B is bounded by the hypersurfaces $||x|| = \rho$ and by $v(t, x) = 0$. We assume that v is bounded from above in B and $0 \in \partial B$ for all $t \geq t_0$.

(ii) $D^+V(t, x)$ is positive semidefinite on $B \setminus \cup_{i \in N}(G_i \cap B)$.

(iii) $D^+V(t, x) \geq -\varphi(V(t, x))$ for some $\varphi \in \mathcal{A}$ and for all $(t, x) \in \cup_{i \in N}(G_i \cap B)$;

(iv) $V(\theta_i(x), x + J_i(x)) \geq \psi(V(\theta_i(x), x)$ *for some* $\psi \in \mathcal{A}$ *and for all* $(t, x) \in \cup_{i \in N}(\Gamma_i \cap B)$, *where* $\Gamma_i = \{(t, x) : t = \theta_i(x)\}$.

(v) There exists a positive real number L *such that* $|\theta(x) - \theta_i^0| \leq L\|x\|$ *for* $x \in S_\rho$ *and* $i \in N$.

(vi) There exists a positive real number γ *such that*

$$\int_{V(\theta_i(x),x)}^{\psi(V(\theta_i(x),x))} \frac{ds}{\varphi(s)} \geq (L + \gamma)\|x\|$$

for all $x \in S_\rho$ *and* $i \in N$.

Then the zero solution of (1.1) is unstable.

Proof. Fix $\epsilon > 0$ and t_0, $(t_0, x_0) \in B$, and let $x(t) = x(t, t_0, x_0)$ be a solution of (1.1) having discontinuities at $t = \tau_i$ for $i \in N$. We shall show that $x(t)$ must leave the ball S_ϵ in finite time. In view of (ii), we see that $v(t)$ is nondecreasing on each interval of its continuity in Λ We need to prove that $v(t)$ is nondecreasing for all $t \in \Lambda$. So we let $u(t)$ be the minimal solution of $u' = -\varphi(u)$ on $[\xi_i, \zeta_i]$ such that $u(\xi_i) = v(\xi_i+)$, where ξ_i and ζ_i are as defined in the proof of Theorem 3.2.1. By using Lemma 2 and (vi) we see that

$$\int_{v(\theta_i^0)}^{v(\tau_i+)} \frac{ds}{\varphi(s)} \geq \gamma\|x(\tau_i)\| \text{ if } \tau_i > \theta_i^0 \tag{2.56}$$

and

$$\int_{v(\tau_i)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)} \geq \gamma\|x(\tau_i)\| \text{ if } \tau_i < \theta_i^0. \tag{2.57}$$

From (2.56) and (2.57) we may deduce that $v(\xi_i) \leq v(\zeta_i+)$. Therefore, $v(t) \geq v(t_0)$ for all $t \in \Lambda$, implying that $(t, x(t)) \in B \setminus \cup_{i \in N} G_i$ for all $t \in \Lambda$.

Let $M > 0$ be real number such that $V(t, x) \leq M$ for all $(t, x) \in B$, which is possible by (i). Since $v(t) \geq v(t_0)$, there is a $\mu_1 > 0$ such that $\|x(t)\| \geq \mu_1$ for all $t \in \Lambda$. If we now define $m = \min_{v(t_0) \leq s \leq M} \varphi(s)$, then it follows from (2.56) and (2.57) that

$$v(\zeta_i+) - v(\xi_i) \geq \gamma m \mu_1. \tag{2.58}$$

Since $v(\xi_{i+1}) \geq v(\zeta_i+)$ we get

$$v(\xi_{i+1}) - v(\xi_i) \geq \gamma m \mu_1 \text{ for all } i \in N \tag{2.59}$$

Summing (2.59) over i from 1 to k we see that

$$v(\xi_{k+1}) - v(\xi_1) \geq (\gamma m \mu_1)k \text{ for all } k \in N. \tag{2.60}$$

But (2.60) leads to a contradiction that $v(t)$ is unbounded in B. This completes the proof.

CHAPTER 3

A NEW STABILITY THEOREM OF THE ZERO SOLUTION

3.1 Introduction

In this chapter, we shall try to improve the theorem 2.3.9. We define

$$\tilde{G}_i = \{(t,x) \in G : \ t \text{ is between } t = w_i(x) \text{ and } t = \theta_i(x)\},$$

where $t = w_i(x)$ is any given surface like $t = \theta_i(x)$ such that $w_i(0) = \theta_i(0)$. We note that $w_i(x) = \theta_i(0)$ in [3].

As is mentioned before, the solutions of differential equations with variable moments of impulse effect may experience pulse phenomena. That is to say, they may hit given surfaces of discontinuity finite or infinite number times causing rhythmical beating [10, 16]. One of the things we came to realise was that this results in additional complications in studying such systems and therefore in most cases it is necessary to find conditions that guarantee the absence of beating and the stability of zero solution. In this chapter, we assume that $\theta_i(x)$ is Lipshitzian and satisfies the inequality $\theta_i(x + J_i(x)) \leq \theta_i(x)$.

It is worthwhile noting that the arguments developed in [5, 6] were based on a comparison method. Specifically, the change of a Lyapunov function in the interval of continuity was compared with its change at the moments of discontinuity. Our technique is also based on a comparison, but it is somewhat different. We compare the changes of a Lyapunov function in the vicinity of the moments where solutions meet a surface of discontinuity. Moreover, there is no restriction on the distance between moments of impulses. As a result, the following result seem to be very useful for stabilization and controllability of impulsive systems [9].

3.2 The Main Result

Theorem 3.2.1 *Assume that the following conditions are fulfilled.*

(i) $V(t,x)$ *is positive definite on* G.

(ii) $D^+V(t,x)$ *is negative semidefinite on* G.

(iii) $D^+V(t,x) \leq -\varphi(V(t,x))$ *for some* $\varphi \in \mathcal{A}$ *and for all* $(t,x) \in \cup_{i\in N}\tilde{G}_i$.

(iv) $V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x)$ *for some* $\psi \in \mathcal{A}$ *and for all* $x \in S_\rho$ *and* $i \in N$.

(v) There exist positive real numbers L_1, L_2 *such that* $|\theta_i(x) - \theta_i(y)| \leq L_1\|x - y\|$ *and* $|w_i(x) - w_i(y)| \leq L_2\|x - y\|$ *for all* $x, y \in S_\rho$ *and* $i \in N$.

(vi) There exists a real number $L_3 > 0$ *such that* $|\theta_i(x) - w_i(x)| \geq L_3\|x\|$ *for all* $x \in S_\rho$ *and* $i \in N$.

(vii) There exists a real number $\gamma \geq 0$ *such that*

$$\int_{V(\theta_i(x),x)}^{\psi(V(\theta_i(x),x))} \frac{ds}{\varphi(s)} \leq \left(\frac{L_3}{1 + ML_2} - \gamma\right)\|x\|$$

for all $(t,x) \in \tilde{G}_i$ *and* $i \in N$.

If $\gamma = 0$*, then the zero solution of (1.1) is stable. If* $\gamma > 0$ *and* $V(t,x)$ *is decrescent on* G*, then the zero of (1.1) is asymptotically stable.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be a solution of (1.1) that has discontinuities at $t = \tau_i$ for which $\tau_i = \theta_i(x(\tau_i))$. Let $t = \eta_i$ be points satisfying $\eta_i = w_i(x(\eta_i))$ for all $i \in N$. It follows from *(ii)* and *(iii)* that $v(t) := V(t, x(t))$ is nonincreasing on G except the points of discontinuities of the solution $x(t)$ of (1.1). To determine the change of $v(t)$ we look at its behaviour in $\tilde{G}_i$. For each fixed $i \in N$, there are two possible cases: either $\eta_i > \tau_i$ or $\tau_i > \eta_i$.

We assume without loss of generality that $\tau_i < \eta_i$. Before embarking on utilizing the conditions of the theorem, we observe that

$$x(\eta_i) = x(\tau_i) + \int_{\tau_i}^{\eta_i} f(s, x(s))ds$$

and

$$\begin{aligned}
\eta_i - \tau_i &= w_i(x(\eta_i)) - \theta_i(x(\tau_i)) \\
&\geq |w_i(x(\eta_i)) - \theta_i(x(\eta_i))| - |\theta_i(x(\eta_i)) - \theta_i(x(\tau_i))| \\
&\geq L_3\|x(\eta_i)\| - ML_2(\eta_i - \tau_i)
\end{aligned}$$

equivalently

$$\eta_i - \tau_i \geq \frac{L_3}{1 + ML_2}\|x(\eta_i)\|. \tag{3.1}$$

Let $u(t)$ be the maximal solution of $u' = -\varphi(u)$ on $[\tau_i, \eta_i]$ such that $u(\tau_i) = v(\tau_i)$. We invoke Lemma 2.3.7 and (vi) to see that

$$\frac{L_3}{1 + ML_2}\|x(\eta_i)\| \leq \eta_i - \tau_i = \int_{u(\eta_i)}^{u(\tau_i)} \frac{ds}{\varphi(s)} \leq \int_{v(\eta_i)}^{v(\tau_i)} \frac{ds}{\varphi(s)}. \tag{3.2}$$

Employing conditions of theorem (iv) and (vii), we also have

$$(\gamma - \frac{L_3}{1 + ML_2})\|x(\eta_i)\| \leq -\int_{v(\eta_i)}^{\psi(v(\eta_i))} \frac{ds}{\varphi(s)} \leq -\int_{v(\eta_i)}^{v(\eta_i+)} \frac{ds}{\varphi(s)}. \tag{3.3}$$

Adding (3.2) and (3.3) yields that

$$\int_{v(\eta_i+)}^{v(\tau_i)} \frac{ds}{\varphi(s)} \geq \gamma\|x(\eta_i)\|. \tag{3.4}$$

Thus we get

$$v(\tau_i) \geq v(\eta_i+). \tag{3.5}$$

Similarly, we can show that $v(\tau_i+) \leq v(\eta_i)$ if $\tau_i > \eta_i$. Accordingly we define $\Omega := \cup_{i=1}^{\infty}(\xi_i, \zeta_i]$ and $\Lambda := [t_0, \infty) \setminus \Omega$, where $\xi_i = \tau_i$ and $\zeta_i = \eta_i$ if $\tau_i < \eta_i$, $\xi_i = \eta_i$ and $\zeta_i = \tau_i$ if $\tau_i > \eta_i$. It follows from (3.5) that $v(\zeta_i+) \leq v(\xi_i)$, and that $v(t)$ is nonincreasing on Λ.

Let $0 < \epsilon < \rho$ and t_0 be given. Without loss of generality we may assume that

$t_0 \in \Lambda$. It is now clear that $v(t) \le v(t_0)$ for all $t \in \Lambda$.

Suppose that $\gamma = 0$. Set

$$\epsilon_1 = \frac{\epsilon}{2(1 + M(L_1 + L_2))}. \tag{3.6}$$

According to this ϵ_1, we can find $\lambda = \inf_{t \ge t_0,\, \|x\| \ge \epsilon_1} V(t,x)$, because $V(t,x)$ is positive definite. Since $V(t,x)$ is continuous and $V(t,0) = 0$, it is also possible to find a positive real number δ such that $\delta < \epsilon_1$ and

$$\kappa = \sup_{\|x\| \le \delta} V(t_0, x) < \lambda.$$

We first claim that if $\|x_0\| < \delta$, then

$$\|x(t, t_0, x_0)\| < \epsilon_1 \tag{3.7}$$

for all $t \in \Lambda$. Suppose on the contrary that this is not true. Then, there would exist a $t^* \in \Lambda$ such that $\|x(t^*, t_0, x_0)\| \ge \epsilon_1$. But this yields the contradiction that $\lambda \le v(t^*) \le v(t_0) \le \kappa < \lambda$. This means that (3.7) is valid. Now we estimate $\|x(t)\|$ in Ω. For $t \in [w_i(0), \eta_i]$ we write

$$x(t) = x(\eta_i) + \int_{\eta_i}^{t} f(s, x(s))ds \tag{3.8}$$

which yields

$$\|x(t)\| \le \epsilon_1(1 + ML_2) \quad \text{for all} \quad t \in [w_i(0), \eta_i]. \tag{3.9}$$

In particular, we have $\|x(w_i(0))\| \le \epsilon_1(1 + ML_2)$. Using $w_i(0) = \theta_i(0)$, we can write

$$x(t) = x(\theta_i(0)) + \int_{\theta_i(0)}^{t} f(s, x(s))ds \;\text{ for all }\; t \in [\tau_i, \theta_i(0)], \tag{3.10}$$

which implies that

$$\|x(t)\| \le \|x(\theta_i(0))\| + M|t - \theta_i(0)| \le \epsilon_1(1 + M(L_1 + L_2)) \tag{3.11}$$

for all $t \in [\tau_i, \theta_i(0)]$. In view of (v), we easily obtain from both (3.9) and (3.11) that

$$x(t) \le \epsilon_1(1 + M(L_1 + L_2)) = \frac{\epsilon}{2} < \epsilon$$

for all $t \in \Omega$, and hence, we deduce from (3.7) and (3.10) that the zero solution of (1.1) is stable.

We shall now show that if $\gamma > 0$, then $\lim_{t\to\infty} x(t) = 0$. We first observe that since $v(t)$ is positive and nonincreasing on Λ, there is a nonnegative real number α such that

$$\lim_{t\to\infty} v(t) = \alpha, \quad t \in \Lambda. \tag{3.12}$$

We claim that $\alpha = 0$. Note that $v(t)$ is decrescent, that is to say, there exists a positive definite function $W : S_\rho \to R$ such that $|v(t)| \le W(x)$ for all $t \ge t_0$. Now suppose on the contrary that $\alpha > 0$. In this case, we can always find $\mu > 0$ such that

$$0 < \sigma = \sup_{\|x\|<\mu} W(x) < \alpha$$

and hence,

$$\alpha \le v(t) \le W(x(t)) \le \sigma < \alpha.$$

That is, $\alpha < \alpha$. This conradiction yields that there exists a positive real number α_1 such that

$$\|x(t)\| \ge \alpha_1 \;\text{ for all } t \in \Lambda. \tag{3.13}$$

On the other hand, since $(\zeta_i, \xi_{i+1}] \in \Lambda$ for all $i \in N$, we have $\lim_{i\to\infty} v(\zeta_i+) = \lim_{i\to\infty} v(\xi_i) = \alpha$. Therefore from (ii) and (3.5) we have

$$v(\xi_i) \ge v(\zeta_i+) \ge \alpha \quad \text{for all } i \in N. \tag{3.14}$$

Let $\beta = \min_{\alpha\le s\le v(t_0)} \varphi(s)$. It follows from (ii), (3.4) and (3.13) that

$$v(\xi_i) - v(\zeta_i+) \ge \alpha_1\beta\gamma \;\text{ for all } i \in N \tag{3.15}$$

Since $v(t)$ is nonincreasing for $t \ge t_0$, we have

$$v(\xi_{i+1}) \le v(\zeta_i+) \quad \text{for all } i \in N. \tag{3.16}$$

Substituting (3.16) into (3.15), we obtain

$$v(\xi_i) - v(\xi_{i+1}) \ge \alpha_1\beta\gamma \;\text{ for all } i \in N. \tag{3.17}$$

Summing the resulting inequalities (3.17) over i from 1 to k, we obtain

$$\sum_{i=1}^{k} v(\xi_i) - v(\xi_{i+1}) \geq \sum_{i=1}^{k} \alpha_1 \beta \gamma \quad \text{for all } k \in N. \tag{3.18}$$

Correspondingly, we have

$$v(\xi_1) - v(\xi_{k+1}) \geq (\alpha_1 \beta \gamma) k \ \text{ for all } k \in N. \tag{3.19}$$

That is,

$$v(\xi_{k+1}) \leq v(\xi_1) - (\alpha_1 \beta \gamma) k \ \text{ for all } k \in N. \tag{3.20}$$

It is easy to see from (3.20) that if k is sufficiently large, then the function $v(t)$ takes on negative values. But this contradicts the fact that $v(t)$ is positive definite. Thus, we must have $\alpha = 0$. Namely, we have

$$\lim_{t \to \infty} v(t) = 0. \tag{3.21}$$

As in the classical case, it follows that

$$\lim_{t \to \infty} \|x(t)\| = 0 \ , \tag{3.22}$$

and hence we can infer that the zero solution is asymtotically stable.

Let us put Theorem 3.2.1 to a concrete test:

Example 3.2.1 Let $\theta_i(x) = i - \sqrt{x_1^2 + x_2^2}$ and $w_i(x) = i - \frac{1}{2}\sqrt{x_1^2 + x_2^2}$ so that $\tilde{G}_i = \{(t,x) \in G : \ i - \sqrt{x_1^2 + x_2^2} < t \leq i - \frac{1}{2}\sqrt{x_1^2 + x_2^2}\}$. We define $S = \bigcup_{i=1}^{\infty} \tilde{G}_i$ and consider the impulsive system

$$\dot{x}_1 = \begin{cases} -x_2, & (t,x) \notin S, \\ -x_1, & (t,x) \in S \end{cases}$$

$$\dot{x}_2 = \begin{cases} x_1, & (t,x) \notin S, \\ -x_2, & (t,x) \in S \end{cases}$$

$$\Delta x_1|_{t=\theta_i(x)} = -\alpha x_1 + \beta x_2,$$

$$\Delta x_2|_{t=\theta_i(x)} = \beta x_1 - \alpha x_2.$$

We take $V(x) = x_1^2 + x_2^2$ and make the following observations :

(a) $\dot{V}(x) = 0$ if $(t,x) \notin S$ and $\dot{V}(x) = -2V(x)$ if $(t,x) \in S$. Since

$$V(x + \Delta x) = ((1-\alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1x_2$$

we have

$$V(x + \Delta x) \le k(\alpha,\beta)V(x),$$

where $k(\alpha,\beta) = (|\beta| + |1-\alpha|)^2$.

(b) $\|x+\Delta x\|^2 = ((1-\alpha)^2+\beta^2)(x_1^2+x_2^2) - 4\beta(\alpha-1)x_1x_2$ and so $\|x+\Delta x\|^2 \ge (|1-\alpha|^2 - |\beta|^2)^2\|x\|^2$. It follows that if $|\,|1-\alpha| - |\beta|\,| \ge 1$, then

$$\theta_i(x + \Delta x) \le \theta_i(x).$$

(c) $|\theta_i(x) - w_i(x)| = |i - \sqrt{x_1^2 + x_2^2} - (i - \frac{1}{2}\sqrt{x_1^2 + x_2^2})| = \frac{1}{2}\sqrt{x_1^2 + x_2^2} = \frac{1}{2}\|x\|$.

(d) $|\theta_i(x) - \theta_i(y)| = |i - \sqrt{x_1^2 + x_2^2} - (i - \sqrt{y_1^2 + y_2^2})| = |\|y\| - \|x\|| \le \|x - y\|$.

(e) $|w_i(x) - w_i(y)| = |i - \frac{1}{2}\sqrt{x_1^2 + x_2^2} - (i - \frac{1}{2}\sqrt{y_1^2 + y_2^2})| = \frac{1}{2}|\|y\| - \|x\|| \le \frac{1}{2}\|x - y\|$.

(f) Let $\varphi(s) = 2s$ and $\psi(s) = k(\alpha,\beta)s$, and fix a positive number $\gamma < \frac{1}{3}$. Note that $\|f(t,x)\| = \|x\| \le 1$ for all $(t,x) \in S$. If $k(\alpha,\beta) \le 1$, then $\ln k(\alpha,\beta) \le 2(\frac{1}{3} - \gamma)\|x\|$ and hence

$$\int_V^{kV} \frac{ds}{2s} \le (\frac{1}{3} - \gamma)\ \|x\|.$$

In view of theorem 3.2.1, we deduce that the zero solution of (1.1) is asymptotically stable if

$$|\,|1-\alpha| - |\beta|\,| \ge 1 \text{ and } |1+\alpha| + |\beta| \le 1.$$

If we compare this example with example 2.3.4, we infer that we have opportunity to make use of suitable $w_i(x)$ which is closer to $\theta_i(x)$ than θ_i^0. In other words, we estimate the difference $|\theta_i(x) - w_i(x)|$ over a region $\tilde{G}_i$, which is smaller than G_i, to ensure that the zero solution is stable or asymptotically stable. In particular, we can chose $w_i(x)$ to be θ_i^0 and the only thing that remains is to find γ satisfying the condition (vii).

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