# APPROXIMATING SMOOTH MAPS BY REAL ALGEBRAIC MORPHISMS

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# ABSTRACT

## APPROXIMATION OF SMOOTH MAPS BY REAL ALGEBRAIC MORPHISMS

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Given two nonsingular real algebraic varieties, we can consider them as smooth manifolds and view regular maps between them as a subset of the topological space of smooth mappings between them. Thus we can ask when can a smooth map be approximated by algebraic ones. In this thesis, we deal with sufficient and necessary conditions for the set of regular maps to be dense in the smooth mappings, based on two main results of J. Bochnak and W. Kucharz.

Keywords: Real Algebraic Varieties, Regular Maps, Weierstrass Approximation, Algebraic Vector Bundles, Abelian Varieties.

# ÖZ

## DÜZGÜN DÖNÜŞÜMLERİN REEL CEBİRSEL DÖNÜSÜMLERLE YAKLAŞTIRILMASI

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İki cebirsel varyete verildiğinde, bunları düzgün manifoldlar olarak, aralarındaki regüler dönüşümleri de sözkonusu düzgün manifoldlar arasındaki düzgün dönüşümlerden müteşekkil topolojik uzayın bir alt kümesi olarak ele alabiliriz. Böylelikle, bu uzayda hangi düzgün dönüşümlerin cebirsel olanlarca yaklaşık ifade edilebileceği sorusunu sorabiliriz. Bu tezde, cebirsel dönüşümler alt kümesinin düzgün dönüşümler uzayında yoğun olması için gerek ve yeter şartları, J. Bochnak ve W. Kucharz'ın iki esas sonucunu temel alarak inceleyeceğiz.

Anahtar Kelimeler: Reel Cebirsel Varyeteler, Regüler Dönüşümler, Weierstrass Yaklaştırması, Cebirsel Vektör Demetleri, Abel Varyeteleri.

Nemit'e

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# CHAPTER 1

# INTRODUCTION AND PRELIMINARIES

## 1.1 Basic concepts and motivation

An *affine real algebraic variety* $X$ is an irreducible Zariski closed subset of $\mathbb{R}^n$, for some $n$. Thus $X$ is the common zero locus of finitely many irreducible polynomials taken from $\mathbb{R}[x_1, \dots, x_n]$. Morphisms of real algebraic varieties are called *regular maps.* If $f$ is a regular map between the real algebraic varieties $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, then it is of the form $f = (f_1/g_1, \dots, f_m/g_m)$, where $f_i, g_i \in \mathbb{R}[x_1, \dots, x_n]$ and $g_i^{-1}(0) \cap X = \emptyset$, for all $i = 1, \dots, n$. If there exist regular maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ satisfying $g \circ f = id_X$ and $f \circ g = id_Y$, then real algebraic varieties $X$ and $Y$ are said to be *isomorphic.* In this context, an affine real algebraic variety $X$ will actually mean the isomorphism class of $X$, unless we mention the ambient space. We define a *real algebraic variety* as a topological space that has a finite open cover consisting of affine real algebraic varieties. Note that every projective real algebraic variety is actually affine ([3], Theorem 3.4.4., Proposition 3.2.10).

Every real algebraic variety is endowed with the Zariski topology, but also carries the usual Euclidean topology. The latter topology is much more finer than the former. Here and throughout this thesis, all topological notions related to real algebraic varieties will refer to the Euclidean topology, unless explicitly stated otherwise.

Given two real algebraic varieties $X$ and $Y$, let $\mathcal{R}(X,Y)$ denote the set of all regular maps from $X$ to $Y$. Let $\mathcal{C}(X,Y)$ be the space of all continuous maps from $X$ to $Y$, endowed with the compact-open topology. It is clear that we can consider $\mathcal{R}(X,Y)$ as a subset of this function space, and endow it with the relative topology. If $X$ and $Y$ are nonsingular, then it is well-known that $X$ and $Y$ are smooth manifolds. In this case, let $\mathcal{C}^\infty(X,Y)$ be the space of all smooth maps from $X$ to $Y$, endowed with the $C^\infty$ compact-open topology. Then $\mathcal{R}(X,Y)$ is a subset of this space and is endowed with the relative $C^\infty$ compact-open topology this time. We ask the following two questions:

**Problem(1)** When $\overline{\mathcal{R}(X,Y)} = \mathcal{C}(X,Y)$?

**Problem(2)** When $\overline{\mathcal{R}(X,Y)} = \mathcal{C}^\infty(X,Y)$?

Here the closures are in the topologies of $\mathcal{C}(X,Y)$ and $\mathcal{C}^\infty(X,Y)$, respectively. One can weaken these questions and ask, in general, when $f : X \overset{continuous}{\longrightarrow} Y$ in $\overline{\mathcal{R}(X,Y)}$ is or when $f : X \overset{smooth}{\longrightarrow} Y$ in $\overline{\mathcal{R}(X,Y)}$ is. However, we are going to concentrate on the stronger formulations given above. Note that $\mathcal{C}^\infty(X,Y)$ is dense in $\mathcal{C}(X,Y)$ and $C^\infty$ compact-open topology is coarser, so (2) $\Rightarrow$ (1).

Mainly, there are two motivations for these problems. The first one is the classical Weierstrass Approximation Theorem. Together with the improvement of M. H. Stone in 1937, this classical result answers our questions positively, when the target variety is a real affine space ([13], Theorem 3.2.21.). That is; any smooth map from a compact smooth manifold embedded into some $\mathbb{R}^n$ to $\mathbb{R}^m$, for any $m$, can be approximated by $n$-variable real polynomials defined on this compact manifold. On the other hand, J.

Nash showed in 1952 that any closed smooth manifold is diffeomorphic to a component of a nonsingular real algebraic variety ([26]), and later in 1973, A. Tognoli proved that any closed smooth manifold is diffeomorphic to a nonsingular real algebraic variety ([32]). According to these, one can always pick a nonsingular real algebraic variety $X$ from the diffeomorphism class of a smooth manifold $M$. Such an $X$ is called an *algebraic model* of $M$. What naturally accompanies this process of making smooth objects algebraic is approximating smooth maps by algebraic ones.

## 1.2 Concise history

In general, very little is known about regular maps between real algebraic varieties. The situation is the same to a large extent for the aforementioned density problem. Most results are obtained for maps between spheres, whereas the others are for products of spheres, flag varieties, Grassmannians, and rational surfaces, which are all examples of rational varieties (see Section 1.3). In the related studies, general results mostly occur in low-dimensions, actually when the dimension of the domain or the target variety is 1 or 2. All the results are obtained by making use of topological/algebraic (co)homology, topological/algebraic vector bundles and algebraic K-theory, together with some analytic methods and some classical methods in algebraic geometry.

Chronologically, we may accept Jean-Louis Loday's studies on regular mappings to be the avant-garde of studies in this topic ([22]). It was Jacek Bochnak and Wojciech Kucharz who advanced the studies on the subject. Together with M. Coste, M.F. Roy, D.Y. Suh and Y. Ozan, they have extensively studied mappings into spheres in [3], [5], [6], [8], [9], [10], [12], [27], and [31]. The results that Bochnak and Kucharz obtained in 1987 were the initiators for these works (see [5]). Among several results for maps between spheres, they showed that $\overline{\mathcal{R}(S^n, S^k)} = \mathcal{C}^\infty(S^n, S^k)$ when $n \geq 1$ and $k = 1, 2$, or 4.

Kucharz studied mappings into flag varieties in [19]. Suh, Kucharz and Rusek obtained results for the case in which the target variety is a Grassmann variety in [21] and [30]. In particular, Kucharz and Rusek showed in 1997 that $\overline{\mathcal{R}(X, \mathbb{G}_\mathbb{R}(n,k))} = \mathcal{C}^\infty(X, \mathbb{G}_\mathbb{R}(n,k))$, provided that $X$ is a compact nonsingular curve (see [21]). Finally, Bochnak put forth results for mappings into rational surfaces in [20].

## 1.3 Main theorems

**Definition 1.3.1.** Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be real algebraic varieties. If $f : X \longrightarrow Y$ is of the form $f = (f_1/g_1, \dots, f_m/g_m)$, where $f_i, g_i \in \mathbb{R}[x_1, \dots, x_n]$ and $g_i^{-1}(0) \cap X \neq X$, for all $i = 1, \dots, n$, then $f$ is said to be a *rational map* between $X$ and $Y$. If there exist rational maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ satisfying $g \circ f = id_X$ and $f \circ g = id_Y$, then $X$ and $Y$ are said to be *birational.*

**Definition 1.3.2.** An $n$-dimensional real algebraic variety $X$ is called *rational* if it is birationally equivalent to $\mathbb{RP}^n$.

**Example 1.3.3.** Affine spaces and spheres of any dimension are rational. An important type of rational varieties is the *Grassmann variety,* which is defined to be the space of all $k$-dimensional vector subspaces of $\mathbb{R}^n$, and denoted by $\mathbb{G}_\mathbb{R}(n,k)$. A *flag variety* is the set of all possible linear cell decompositions of $\mathbb{RP}^n$, for some $n$. Also flag varieties are rational.

In this thesis, we focus on the results that Bochnak and Kucharz obtained in [11], in 1999. There are two main theorems in consideration. The first theorem generalizes the ideas in the works we mentioned in the previous section, for the case when the domain is a compact curve, as all varieties appearing there are rational. The second theorem provides a necessary condition for the density of regular maps.

**Main Theorem S (Bochnak-Kucharz).** *Let $X$ be a compact nonsingular real algebraic curve and let $Y$ be a nonsingular real algebraic variety. If $Y$ is rational, then $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$.*

**Main Theorem N (Bochnak-Kucharz).** *Let $X$ be a real algebraic variety of positive dimension and let $Y$ be a compact nonsingular real algebraic variety. If $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}(X, Y)$, then $b_1(Y, \mathbb{C}) = 0$.*

Note that Theorem S proposes a sufficient condition also for Problem(2) and Theorem N proposes a necessary condition also for Problem(1). We discuss and prove Theorem S in Chapter II, and Theorem N in Chapter III. All the required preliminaries but some general concepts are given in the related chapters. The remaining sections in this chapter constitute a preliminary part for both Chapter II and Chapter III.

## 1.4 Blowing-up

In this section we present blowing-up for both smooth real manifolds and for real algebraic varieties. For details and proofs in this section, we refer to [2] and [3]. Although we will mostly deal with algebraic blowing-up, we begin with the blowing-up in the smooth category, as it provides a visualization for what goes on. Note that everything described below can be also translated into the analytic category and each proposition mentioned here has an analytic analogue. Proofs in the analytic case are the same to the letter.

Let $M$ be a smooth real manifold and let $N$ be a proper smooth real submanifold of $M$. We will construct a new manifold $\mathcal{B}(M,N)$, called the *blow-up of* $M$ *along* $N$, and a proper map $\pi(M,N) : \mathcal{B}(M,N) \to M$, instead of which we mostly use $\pi$, in short. We call $N$ the *blow-up center* and call $\pi$ the *blow-up projection.*

Let $\rho : E \longrightarrow N$ be the projective normal bundle of $N$ in $M$, that is, $E$ is the space of lines in the normal bundle of $N$. So, if $N$ has codimension $n$, $\rho$ will be a smooth fiber bundle with fiber $\mathbb{RP}^{n-1}$. As a point set, $\mathcal{B}(M,N)$ is $M \setminus N$ union $E$. We then put a natural manifold structure on this space. The map $\pi(M,N)$ is the identity on $M \setminus N$ and is the bundle projection to $N$ on $E$.

There is a global description of the smooth structure on $\mathcal{B}(M,N)$. Let $\mu : T \longrightarrow N$ be the normal bundle of $N$ and let $\eta : L \longrightarrow E$ be the canonical line bundle over its projectivization $E$, so

$$L = \{(x, \lambda) \in T \times E \,|\, x \,\text{is a point in the line}\, \lambda\}.$$

Identify $E$ and $N$ with their 0-sections in $L$ and $T$, respectively. Then there is a canonical isomorphism $\nu : L \setminus E \longrightarrow T \setminus N$ induced by projection to the first factor. Let $\psi : T \longrightarrow M$ be a tubular neighborhood of $N$. Then $\mathcal{B}(M,N)$ is the manifold obtained by gluing $L$ and $M \setminus N$ together via the embedding $\psi \circ \nu|_{L \setminus E} : L \setminus E \longrightarrow M \setminus N$. We have a natural smooth projection $\pi(M,N) : \mathcal{B}(M,N) \longrightarrow M$ which is the identity on $M \setminus N$ and is $\rho$ on $E$.

**Example 1.4.1.** Let us determine $\mathcal{B}(\mathbb{R}^3, 0)$. $E$ is just $\mathbb{RP}^2$ and we can identify $L$ with $\mathbb{RP}^3 - [1,0,0,0]$. The maps $\eta$ and $\nu$ are given by $\eta[w,x,y,z] = [x,y,z]$ and $\nu[w,x,y,z] = (x,y,z)w/(x^2+y^2+z^2)$. In this case, $\psi$ can be onto, so we just get $\mathcal{B}(\mathbb{R}^3,0) = L$ and $\pi(\mathbb{R}^3,0) = \psi \circ \nu$.

Another description of $\mathcal{B}(\mathbb{R}^3,0)$ is obtained by describing its charts. The manifold $E = \mathbb{RP}^2$ has three charts $[1,y,z]$, $[x,1,z]$, $[x,y,1]$, and of course the bundle $L$ is trivial over each chart. This gives us three charts for $L = \mathcal{B}(\mathbb{R}^3,0)$ and the map $\pi(\mathbb{R}^3,0)$ is given by $(x,xy,xz)$, $(xy,y,yz)$ and $(xz,yz,z)$ respectively on the three charts.

The concrete description of the blow-up in the example is useful, so we state it generally:

Let $\xi_{in} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ denote the map whose $j$-th coordinate is :

$$\xi_{in}(x_1, x_2, \cdots, x_m)_j = \begin{cases} x_j x_i & \text{if } i \neq j \leq n \\ x_j & \text{if } j = i \text{ or } j > n \end{cases}$$

Then the following describes exactly the local topology of a blow-up:

**Proposition 1.4.1.** *Let $N$ be a proper submanifold of the real smooth manifold $M$. Let $\theta : U \to M$ be an embedding onto an open set $V$ in $M$ where $U \subset \mathbb{R}^m$ is open and $\theta^{-1}(N) = \{(x_1, x_2, \cdots, x_m) \in U \mid x_i = 0 \text{ for all } i \leq n\}$. Then $\pi(M,N)^{-1}(V)$ is covered by $n$ charts $\varphi_i : U_i \longrightarrow \mathcal{B}(M,N)$ where $U_i = \xi_{in}^{-1}(U)$ and*

$$\theta^{-1} \circ \pi(M,N) \circ \varphi_i = \xi_{in}|_{U_i}.$$

**Remark 1.4.2.** If $N \subset M$ has codimension one, then $\mathcal{B}(M,N) = M$ and $\pi(M,N) = id_M$. Thus if $\operatorname{codim} N < 2$, then blow-up does not change anything. If $N$ is a point in $M$, then $\mathcal{B}(M,N)$ is diffeomorphic to $M \# \mathbb{RP}^m$, connected sum of $M$ with $\mathbb{RP}^m$.

Now we define the notion of blowing-up in the algebraic category in such a way that it coincides with our previous notion in the case of nonsingular real algebraic varieties:

**Definition 1.4.3.** Let $X$ be an affine real algebraic variety and $Y$ a Zariski closed subset of $X$ with $I(Y) = (f_1, \cdots, f_m)$. Define

$$Z = \{(x, [f_1(x), \cdots, f_m(x)]) \in X \times \mathbb{RP}^{m-1} | \, x \in X \setminus Y\}.$$

Denote by $\mathcal{B}(X,Y)$ the Zariski closure of $Z$ in $X \times \mathbb{RP}^{m-1}$, ($\mathcal{B}(X,Y)$ is thus an affine real algebraic variety) and $\pi(X,Y) : \mathcal{B}(X,Y) \longrightarrow X$, the projection mapping. The variety $\mathcal{B}(X,Y)$ is called the *blowing-up of* $X$ *with center* $Y$, and $\pi$ is called the *blow-up projection.*

**Remark 1.4.4.** The algebraic variety $\mathcal{B}(X,Y)$ does not depend on the choice of generators of $I(Y)$, up to isomorphism compatible with $\pi(X,Y)$.

**Proposition 1.4.2.** *If* $X$ *is a real algebraic variety and* $Y$ *is a Zariski closed subvariety of* $X$*, then*

$$\pi(M,Y)|_{\mathcal{B}(M,Y)\setminus \pi(M,Y)^{-1}(Y)} : \mathcal{B}(M,Y) \setminus \pi(M,Y)^{-1}(Y) \longrightarrow M \setminus Y$$

*is a birational isomorphism.*

Lastly we present an important proposition on lifting maps between real algebraic varieties to blow-up. There we use the following concept:

**Definition 1.4.5.** We say a smooth map $f : N \longrightarrow M$ hits a submanifold $L \subset M$ *cleanly* if $f^{-1}(L)$ is a submanifold and *df* injects the normal bundle of $f^{-1}(L)$ at each point into the normal bundle of $L$.

**Proposition 1.4.3.** *Suppose $M$ and $N$ are smooth manifolds or nonsingular real algebraic varieties, $L$ is a smooth submanifold or nonsingular Zariski closed subset of $M$ and the smooth map (or regular function) $f: N \to M$ hits $L$ cleanly. Then there is a unique smooth map $\tilde{f}: \mathcal{B}(N, f^{-1}(L)) \longrightarrow \mathcal{B}(M, L)$ compatible with projections. The map $\tilde{f}$ is a regular function in the algebraic case. In other words, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{B}(N, f^{-1}(L)) & \xrightarrow{\tilde{f}} & \mathcal{B}(M, L) \\ \downarrow_{\pi(N, f^{-1}(L))} & & \downarrow_{\pi(M,L)} \\ N & \xrightarrow{f} & M \end{array}$$

## 1.5 Resolution of singularities

The possibility of resolution of singularities in characteristic zero was established by Hironaka ([17]). Let $X$ be a real or complex algebraic variety, then $Sing(X)$ is a proper closed subvariety of $X$. By Hironaka's theorem, we can desingularize $X$ by blowing-up Zariski closed nonsingular subvarieties contained in the $Sing(X_i)$. That is, we have a finite sequence of blowing-ups on Zariski closed nonsingular subvarieties:

$$\tilde{X} = X_k \xrightarrow{\pi_k} X_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

such that $B_{k-1}, \ldots, B_1, B_0$ are the corresponding blow-up centers, all contained in $Sing(X)$. Then $\pi = \pi_1 \circ \cdots \circ \pi_k$ is a surjective, proper map and its restriction is an isomorphism between $\tilde{X} \setminus \pi^{-1}(X)$ and $X$. In fact, what Hironaka presented was the choice of such admissible centers, despite that his work does not give an explicit resolution algorithm.

We are going to use the following proposition quite often:

**Proposition 1.5.1.** *Every nonsingular real algebraic variety $X$ is isomorphic to a Zariski open subvariety of a nonsingular projective real algebraic variety $Y$ with $dimY = dimX$.*

*Proof.* Let $X \subseteq \mathbb{R}^n$ be a nonsingular real algebraic variety. Projectivizing this pair we have $\mathcal{P}(X) \subseteq \mathcal{P}(\mathbb{R}^n) = \mathbb{RP}^n$. The projective closure $\mathcal{P}(X)$ of $X$ is possibly singular. As $X \subseteq \mathcal{P}(X)$ is nonsingular, $Sing(\mathcal{P}(X))$ is away from $X$. Thus, without spoiling $X$, we can resolve the singular locus of $\mathcal{P}(X)$ and obtain a nonsingular projective real algebraic variety $Y$. So $Y = \widetilde{\mathcal{P}(X)}$ in the notation we used above. Let $\pi : Y \longrightarrow X$ be the composition of blowing-up projections in this desingularization process. Since we can regard $\mathcal{P}(X)$ as a Zariski open subvariety of $Y$ and regard $X$ as a Zariski open subvariety of $\mathcal{P}(X)$, the conclusion follows. □

**Remark 1.5.1.** Any projective real algebraic variety is isomorphic to a compact affine real algebraic variety. So the previous proposition states that any nonsingular real algebraic variety $X$ can be regarded as a Zariski open subvariety of a compact nonsingular real algebraic variety $Y$. If $\dim X = 1$, then this $Y$ is uniquely determined, up to isomorphism (see [16]).

## 1.6 Maps between real algebraic varieties

Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ be real algebraic varieties and let $f = (f_1/g_1, \cdots, f_m/g_m)$ be a rational map between them. $Z(g_1 \cdot g_2 \cdots g_m) \cap X$ is the indeterminacy set of $f$, so, by definition, it is a proper Zariski closed subset of $X$. Denote this closed set by $\Delta$. Thus $f$ is regular on the Zariski open set $X \setminus \Delta$.

We implicitly make use of the following basic fact in many proofs:

**Proposition 1.6.1.** *If $f : X \longrightarrow Y$ is a rational map from a real algebraic variety to a projective real algebraic variety, then the indeterminacy set $\Delta$ of $f$ has codim$\geq 2$.*

*Proof.* Let $f$ be of the form $f = [f_1, \dots, f_m]$, so $\Delta = Z(f_1, \dots, f_m)$ Suppose that codim$\Delta = 1$. Then the subvariety $\Delta = Z(g)$, where $g$ is a nonconstant irreducible polynomial. Write each component function of $f$ as $f_i = g^{r_i} \tilde{f}_i$ such that $g$ does not divide $\tilde{f}_i$. Putting $r = \min\{r_1, \dots, r_m\}$, we have $f = [g_1^{r_1 - r} \tilde{f}_1, \dots, g_m^{r_m - r} \tilde{f}_m]$. The right hand side is defined on the whole $X$. □

**Theorem 1.6.2 (Elimination of indeterminacy).** *Let $f : X \to Y \subseteq \mathbb{RP}^m$ be a rational map from a nonsingular real algebraic variety to a projective nonsingular real algebraic variety. Then there exists a nonsingular real algebraic variety $\tilde{X}$ and a surjective, proper map $\pi : \tilde{X} \longrightarrow Y$ such that $f \circ \pi : \tilde{X} \longrightarrow Y$ is regular.*

*Proof.* Let $\Delta \subsetneq X$ be the indeterminacy set of $f$. By Hironaka's theorem, we can make the subvariety $\Delta$ a union of codimension one nonsingular subvarieties with normal crossings by blowing up Zariski closed nonsingular subvarieties contained in $Sing(\Delta)$. Denote this union by $\tilde{\Delta}$. Let $B$ be one of the nonsingular closed subvarieties contained in $\tilde{\Delta}$, and let $\pi : \tilde{X} \longrightarrow X$ be the blow-up of $X$ along $B$. Since $B$ is a Zariski closed nonsingular subvariety of codimension 1, the rational map $f \circ \pi : \tilde{X} \longrightarrow \mathbb{RP}^m$ extends over $B$. Continuing this process, we extend $f$ over $\tilde{\Delta}$. Hence we obtain a regular map on $\tilde{X}$. □

**Example 1.6.1.** Let $f : \mathbb{RP}^2 \to \mathbb{RP}^1$ be a rational map defined by $f([x, y, z]) = [x, y]$. The indeterminacy set of $f$ is $\Delta = \{[0, 0, 1]\}$. We blow-up $\mathbb{RP}^2$ at this point. Then we have

$$\mathcal{B}(\mathbb{RP}^2, [0,0,1]) = \{([x,y,z],\ell) \in \mathbb{RP}^2 \times \mathbb{RP}^1 \mid [x,y,z] \in \ell, [0,0,1] \in \ell\}$$

and

$$\tilde{f} = f \circ \pi([x,y,z],\ell) = \begin{cases} [x,y] & \text{if } [x,y,z] \neq [0,0,1] \\ \ell \cap \{z=0\} & \text{otherwise} \end{cases}$$

where $\{z=0\} \cong \mathbb{RP}^1$. Hence $\tilde{f}$ is regular on $\tilde{X} = \mathcal{B}(\mathbb{RP}^2, [0,0,1])$.

# CHAPTER 2

# A SUFFICIENT CONDITION

## 2.1 Statement of Theorem S

In this chapter we are going to discuss and prove the following theorem:

**Theorem 2.1.1 (Theorem S).** *Let $X$ be a compact nonsingular real algebraic curve and let $Y$ be a nonsingular real algebraic variety. If $Y$ is rational, then $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$*

**Remark 2.1.1.** If $X$ is non-compact, then the theorem immediately fails, even when $Y$ is a Euclidean space. To illustrate this, take $X = Y = \mathbb{R}$ and consider the smooth map $f(x) = \sin x$ between them. Clearly it is impossible to approximate $\sin x$ by polynomials.

**Remark 2.1.2.** $X$ can not be replaced by a higher dimensional real algebraic variety, even if $Y = \mathbb{RP}^1$, the simplest rational variety. One can see this as follows: Bochnak and Kucharz showed that for a compact connected orientable smooth manifold $M$, the following are equivalent: $(i)$ For each algebraic model $X$ of $M$, the set of regular mappings $\mathcal{R}(X, S^1)$ is dense in $\mathcal{C}^\infty(X, S^1)$ and $(ii)$ $b_1(M) = 0$ or $\dim M = 1$. ([7], Corollary 1.7.) Thus for any compact connected orientable smooth manifold $M$ with dim$\geq 2$ and nonzero first betti number, there exists an algebraic model $X$ of $M$ such that $\mathcal{R}(X, S^1)$ is not dense in $\mathcal{C}^\infty(X, S^1)$. Noting that $\mathbb{RP}^1$ is real isomorphic to $S^1$, we have the conclusion. For example take $M$ as a sphere with $g > 0$ handles.

In order to prove Theorem S, we are going to prove a more general statement. Before stating this general form, we need the following two definitions:

**Definition 2.1.3.** Given a smooth map $f : M \longrightarrow N$ between smooth manifolds, a point $a \in M$, and a nonnegative integer $r$, let $a_0, a_1, \dots, a_r$ be the first $r+1$ terms of Taylor expansion of $f$ at $a$. We define the *r-jet of $f$ at $x$* as the $(r+1)$-tuple $(a_0, a_1, \dots, a_r)$, and denote it by $j^r f(a)$.

**Definition 2.1.4 (Property(X)).** Let $X$ be a compact nonsingular real algebraic curve. A nonsingular real algebraic variety $Y$ is said to have *Property*$(X)$ if for every map $f \in \mathcal{C}^\infty(X, Y)$, every neighborhood $U$ of $f$ in $\mathcal{C}^\infty(X, Y)$, every finite subset $A$ of $X$, and every nonnegative integer $s$, there exists a map $g \in \mathcal{R}(X, Y)$ such that $g \in U$ and $j^s g(a) = j^s f(a)$ for all $a \in A$.

**Theorem 2.1.2 (Theorem S$'$).** *Every rational real algebraic variety $Y$ has Property$(X)$ for every compact nonsingular real algebraic curve $X$.*

$Y$ has *Property*$(X)$ implies, in particular, that $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$. Thus Theorem S is a corollary of Theorem S$'$. Although the latter is more general, the former states the core.

To facilitate, reader may view *Property*$(X)$ as the indicated density condition plus 'some r-jet condition'. Actually, the latter is used only in the proof of Lemma 2.5.1 and in a very technical manner. This 'r-jet condition' is easily transferred everywhere else it appears, so reader may omit those lines without loosing the main idea.

Now we can sketch the outline of the proof of Theorem S$'$: First we prove that $\mathbb{RP}^n$ has *Property*$(X)$ for any compact nonsingular real algebraic curve $X$. Secondly, we show that *Property*$(X)$ can be lifted to blow-up. There we use an intermediate category; namely the real analytic category, introduced in the next section. Thus in this second step, supplementary job will be showing that an analytic map can be lifted to blow-up, too. Finally, we combine these results to complete the proof. These steps correspond to Section 2.4, Section 2.5, Section 2.6 in order.

## 2.2 Real analytic manifolds and maps

We say that a function $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is *real analytic* if it has a convergent power series expansion with positive radius, at each $x \in D$. The analyticity of a real-valued function with domain in $\mathbb{R}^n$ is then defined by multi-index and rearrangement. Finally, for a vector-valued function analyticity can be defined by the analyticity of each component function.

**Definition 2.2.1.** A manifold $M$ is called a *real analytic manifold* if it has an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that each chart transform $\varphi_j \circ \varphi_k^{-1}$ is a real analytic function for $j, k \in I$. A map $f : M \longrightarrow N$ between real analytic manifolds is called *analytic* if for each $x \in X$ the composition $\psi \circ f \circ \varphi^{-1}$ is analytic where $\varphi$ and $\psi$ are analytic charts at $x$ and $f(x)$, respectively.

**Remark 2.2.2.** Zeros of an analytic function are isolated. If an analytic function is nonconstant then it is not constant on any nonempty open set.

**Remark 2.2.3.** Every nonsingular real algebraic variety is a real analytic manifold and every real analytic manifold is a smooth manifold. Every regular map is real analytic and every real analytic map is smooth.

It is convenient to improve the above picture. Chow's theorem remarkably states that in $\mathbb{P}^n$, any analytic variety is algebraic. Indeed, algebraic category and analytic category are locally similar, whereas smooth category is so far away from these. One bridge between these categories is that an analytic map is locally an infinite sum of polynomial terms. On the other hand, analytic maps are dense in smooth maps with respect to $\mathcal{C}^\infty$ compact-open topology, which contributes another bridge. Thus, analytic category appears to be an intermediate step that Bochnak and Kucharz use to approximate smooth maps by algebraic morphisms. The following two propositions are used throughout the proof of Theorem S and its Lemmas:

**Proposition 2.2.1.** *Let $A$ be a finite subset of a compact nonsingular algebraic subset $X$ of $\mathbb{R}^n$. Then for any $f \in \mathcal{C}^\infty(X)$ and any neighborhood $V$ of $f$ in $\mathcal{C}^\infty(X)$, there exists a polynomial $g \in \mathcal{R}(X)$ such that $g \in V$ and $j^s_a f = j^s_a g$ for all $a \in A$.*

*Proof.* See [4], Corollary 1. □

**Proposition 2.2.2.** *Let $A$ be a discrete subset of a compact analytic manifold $X$, $\{s_a\}_{a\in A}$ a sequence of nonnegative integers, $f : X \longrightarrow Y$ a smooth map and $V$ a neighborhood of $f$ in $\mathcal{C}^\infty(X,Y)$. Then there exists an analytic map $h : X \longrightarrow Y$ such that $h \in V$ and $j^{s_a} f(a) = j^{s_a} h(a)$ for all $a \in A$.*

*Proof.* See [4], Corollary 2. Note that the "very strong Whitney $\mathcal{C}^\infty$ topology" mentioned there coincides with our $\mathcal{C}^\infty$ compact-open topology as $X$ is taken to be compact. □

## 2.3 Algebraic vector bundles

Here we present a plenty of definitions and propositions about algebraic vector bundles. All these are used in the next section to prove that $\mathbb{RP}^n$ satisfies $Property(X)$. We start with the basic definitions:

**Definition 2.3.1.** A *pre-algebraic* $\mathbb{R}$ *vector bundle* is a triple $\xi = (E, \pi, X)$, where:

(i) $E$ is a real algebraic variety (not necessarily affine), and $\pi : E \longrightarrow X$ is a regular mapping,

(ii) for each $x \in X$, the fiber $\pi^{-1}(x)$ is a finite dimensional $\mathbb{R}$-vector space,

(iii) there exists a finite covering $\{U_i\}_{i \in I}$ of $X$ by Zariski open sets and, for each $i \in I$, an integer $n$ and a biregular isomorphism $\varphi_i : U_i \times \mathbb{R}^n \longrightarrow \pi^{-1}(U_i)$ such that $\pi \circ \varphi_i$ is the canonical projection of $U_i \times \mathbb{R}^n$ onto $U_i$ and, for every $x \in U_i$, the restriction $x \times \mathbb{R}^n \longrightarrow \pi^{-1}(x)$ of $\varphi_i$ is an $\mathbb{R}$-linear isomorphism.

An *algebraic section* of $\xi$ is a regular mapping $s : X \longrightarrow E$ such that $\pi \circ s = id_X$. The variety $X$ is called the *base space of* $\xi$ and $E$ is called the *total space of* $\xi$.

**Remark 2.3.2.** The *rank* of an $\mathbb{R}$-vector bundle $\xi = (E, \pi, X)$ is the function from $X$ to $\mathbb{N}$ which assigns to $x \in X$ the dimension of the $\mathbb{R}$-vector space $\pi^{-1}(x)$. By (iii), the rank is locally constant for the Zariski topology. Hence if $X$ is connected in the Zariski topology, the rank of a pre-algebraic $\mathbb{R}$-vector bundle over $X$ is constant.

**Definition 2.3.3.** Given two pre-algebraic $\mathbb{R}$-vector bundles $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X)$ over $X$, an *algebraic morphism* $\psi : \xi \longrightarrow \xi'$ is a regular mapping $\psi : E \longrightarrow E'$ such that $\pi' \circ \psi = \pi$ and, for every $x \in X$, $\psi_x : \pi^{-1} \longrightarrow (\pi')^{-1}(x)$ is $\mathbb{R}$-linear. The bundles $\xi$ and $\xi'$ are *algebraically* *isomorphic* if there exist algebraic morphisms $\psi : \xi \longrightarrow \xi'$ and $\varphi : \xi' \longrightarrow \xi$ such that $\varphi \circ \psi = id_\xi$ and $\psi \circ \varphi = id_{\xi'}$.

**Definition 2.3.4.** Let $\epsilon_X^n$ denote the vector bundle $(X \times \mathbb{R}^n, \pi, X)$ where $\pi$ is the canonical projection. A pre-algebraic vector bundle over $X$ is said to be *algebraically trivial* if it is algebraically isomorphic to $\epsilon_X^n$ for some $n$.

**Definition 2.3.5.** Let $f : Y \longrightarrow X$ be a regular mapping between real algebraic varieties and let $\xi = (E, \pi, X)$ be a pre-algebraic vector bundle. The *induced vector bundle or pull-back bundle* is $f^*(\xi) = (E', \pi', Y)$ where $E' = \{(v, y) \in E \times Y \mid \pi(v) = f(y)\}$ and $\pi'(v, y) = y$ is equipped with a canonical structure of pre-algebraic vector bundle. If $Y$ is an algebraic subvariety of $X$ and $f$ is the inclusion map $i : Y \hookrightarrow X$, the bundle $f^*(E)$ is called the *restriction of the bundle $\xi$ to $Y$* and is denoted by $\xi|_Y$.

It is time to notice that the total space $E$ in the Definition 2.3.1 is not necessarrily affine. Indeed, this makes a difference between pre-algebraic vector bundles and the bundles we are going to define:

**Definition 2.3.6.** A pre-algebraic vector bundle $\xi$ over $X$ is said to be an *algebraic vector bundle* if there exists an injective algebraic morphism from $\xi$ to a trivial bundle $\epsilon_X^n$, that is, if $\xi$ is algebraically isomorphic to a pre-algebraic vector subbundle of a trivial bundle.

**Example 2.3.7.** Let $X$ be a nonsingular real algebraic variety. The tangent bundle $TX$ and the cotangent bundle $T^*X$ are algebraic vector bundles over $X$. If $Z$ is a nonsingular subvariety of $X$, then the normal bundle $NZ$ is an algebraic vector bundle over $Z$.

Let us note two facts to complete the picture: It is quite straightforward to show that the total space of an algebraic vector bundle over an affine real algebraic variety is itself an affine real algebraic variety. The converse is also valid: A pre-algebraic vector bundle with affine total space is algebraic (see [18]). Secondly, the category of algebraic $k$-bundles is equivalent to the category of projective modules of finite type over the ring $\mathcal{R}(X)$ of regular functions on $X$ (see [3]). The latter fact may provide an inspiration for the next propositions:

**Proposition 2.3.1.** *Let $\xi = (E, \pi, X)$ be a pre-algebraic vector bundle of rank $k$ over $X$. Then $\xi$ is algebraic if and only if there exists a pre-algebraic vector bundle $\xi'$ over $X$ such that $\xi \oplus \xi'$ is algebraically isomorphic to a trivial bundle $\epsilon_X^n$ for some $n$.*

*Proof.* See [3], Theorem 12.1.7. □

**Proposition 2.3.2.** *If $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X)$ are algebraic vector bundles over $X$, then $\xi \oplus \xi'$, $\xi \otimes \xi'$, $\xi^*$ and $Hom(\xi, \xi')$ are algebraic vector bundles.*

*Proof.* Put $\xi \oplus \xi' = (E \oplus E', (\pi, \pi'), X)$ and $\xi \otimes \xi' = (E \otimes E', (\pi, \pi'), X)$ and the proof is straightforward. Since $\xi$ is algebraic, it is a pre-algebraic direct factor of a trivial bundle, say $\epsilon_X^n$. Then the projection map induces a surjective algebraic morphism $\epsilon_X^n \longrightarrow \xi$. Dualizing this morphism we obtain an injective algebraic morphism $\xi^* \longrightarrow (\epsilon_X^n)^*$. Dual of the trivial bundle is algebraically isomorphic to itself, so $\xi^*$ is an algebraic vector bundle by definition. Lastly, the algebraic isomorphism $Hom(\xi, \xi') \cong \xi^* \otimes \xi'$ together with our previous results imply that $Hom(\xi, \xi')$ is an algebraic vector bundle over $X$ as well. □

We finish this section with the following theorem:

**Theorem 2.3.3.** *Let $X$ be a compact nonsingular affine real algebraic curve. Every smooth vector bundle of constant rank over $X$ is $C^\infty$ isomorphic to an algebraic vector bundle.*

*Proof.* See [3], Theorem 12.5.1. □

## 2.4 Approximation for $\mathbb{RP}^n$

Now we prove that for any compact nonsingular real algebraic curve $X$, $\mathbb{RP}^n$ has $Property(X)$:

Let $X$ be a compact nonsingular real algebraic curve $X$ and $f : X \to \mathbb{RP}^n$ be smooth. Let $\gamma = (E, \pi, \mathbb{RP}^n)$ be the universal line bundle on $\mathbb{RP}^n$ with the total space $E = \{(\ell, e) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid e \in \ell\}$. Here, $\mathbb{RP}^n$ is regarded as the space of one dimensional vector subspaces of $\mathbb{R}^{n+1}$.

Consider the pull-back bundle $f^*(\gamma)$ on $X$. Since $\gamma$ is a smooth subbundle of the trivial bundle $\epsilon^{n+1}_{\mathbb{RP}^n}$, then $f^*(\gamma)$ is a smooth subbundle of $f^*(\epsilon^{n+1}_{\mathbb{RP}^n}) \cong \epsilon^{n+1}_X$. By Theorem 2.3.3, there exists an algebraic vector bundle $\xi$ and a smooth bundle isomorphism $\varphi : \xi \longrightarrow f^*(\gamma)$. We can define a smooth section

$$u : X \longrightarrow Hom(\xi, \epsilon^{n+1}_X)$$
$$u(x)(e) = \varphi(e) \quad \text{for all } x \in X \text{ and } e \in \xi_x.$$

Since $\xi$ and $\epsilon^{n+1}_X$ are both algebraic vector bundles, so is $Hom(\xi, \epsilon^{n+1}_X)$ by Proposition 2.3.2. Then as we have mentioned in the previous section, there exists a pre-algebraic vector bundle $\epsilon$ over $X$ such that $Hom(\xi, \epsilon^{n+1}_X) \oplus \zeta$ is isomorphic to a trivial bundle $\epsilon^N_X$ for some $N$. So $Hom(\xi, \epsilon^{n+1}_X) \oplus \zeta \cong \epsilon^N_X$ with $\pi : \epsilon^N_X \longrightarrow Hom(\xi, \epsilon^{n+1}_X)$, the projection vector bundle morphism -which is regular. Thus, we can define

$$\tilde{u} : X \longrightarrow \epsilon_X^N$$
$$\pi \circ \tilde{u}(x) = u(x) \quad \text{for all } x \in X.$$

Now, we begin to construct the approximation. By Proposition 2.2.1, there exists a regular section $\tilde{v} : X \longrightarrow \epsilon_X^N$, arbitrarily close in the $C^\infty$ compact-open topology to $\tilde{u}$. As $\pi$ is regular, it follows that the map

$$v : X \longrightarrow Hom(\xi, \epsilon_X^{n+1})$$
$$v(x) = \pi \circ \tilde{v}(x) \quad \text{for all } x \in X$$

is a regular section. Here $v$ is arbitrarily close to $u$. Clearly $j^s v(a) = j^s u(a)$ for all $a \in A$.

It follows from the definition of $u$ that the linear map $u(x) : \xi_x \longrightarrow (\epsilon_X^{n+1})_x = \{x\} \times \mathbb{R}^{n+1}$ is injective for all $x \in X$. Since $X$ is compact, then for a sufficiently close $v$ to $u$ the linear map $v(x) : \xi_x \longrightarrow \{x\} \times \mathbb{R}^{n+1}$ is injective, for all $x \in X$, as well.

Let $\rho : X \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the canonical projection and $q : \mathbb{R}^{n+1} \to \mathbb{RP}^n$ be the quotient map taking each nonzero element $y \in \mathbb{R}^{n+1}$ to $[y] \in \mathbb{RP}^n$. We define

$$g : X \longrightarrow \mathbb{RP}^n$$
$$g(x) = q \circ \rho(v(x)(\xi_x)) \quad \text{for all } x \in X.$$

If $g(x) = 0$ then $v(x)(\xi_x) = \{x\} \times 0$ and then $v$ cannot be an injection. So $g$ is nowhere zero which implies that it is well-defined. Moreover, $g$ is regular because all $v$, $\rho$ and $q$ are regular. Lastly, $g$ is arbitrarily close to $f$ because

$f(x) = q \circ \rho(f(x) \times \mathbb{R}^{n+1}) = q \circ \rho(f^*(\gamma)_x) = q \circ \rho(\varphi(\xi_x)) = q \circ \varphi(u(x)(\xi_x))$ and smoothness of all these functions induces the closeness of $v$ and $u$ to $g$ and $f$, respectively. Trivially $j^s g(a) = j^s f(a)$ for all $a \in A$. This completes the proof.

## 2.5 Lemmas on approximation and blow-ups

Given real analytic manifolds $X$ and $Y$, we denote by $\mathcal{O}(X,Y)$ the space of all analytic maps from $X$ into $Y$, endowed with the topology induced from the ambient space $\mathcal{C}^\infty(X,Y)$. If $Z$ is a subset of $Y$, let $\mathcal{O}(X,Y)_Z$ be the subset of $\mathcal{O}(X,Y)$ consisting of all maps $f$ for which $f^{-1}(Z)$ is a finite set.

Let $X$ be a compact real analytic curve and $Z$ be a closed analytic submanifold of the real analytic manifold $Y$. Let $\pi : \tilde{Y} \longrightarrow Y$ be the blow-up of $Y$ along $Z$. Then there exists a unique $\tilde{f} \in \mathcal{O}(X,\tilde{Y})$ such that $\pi \circ \tilde{f} = f$ (see Proposition 1.4.3).

To prove that one can lift *Property*$(X)$ to blow-up, we make use of the following lemma:

**Lemma 2.5.1.** *Let $f \in \mathcal{O}(X,Y)_Z$ and let $\tilde{N}$ be a neighborhood of $\tilde{f}$ in $\mathcal{O}(X,\tilde{Y})$. Let $A$ be a finite subset of $X$ and let $s$ be a nonnegative integer. Then there exists a neighborhood $N$ of $f$ in $\mathcal{O}(X,Y)$ and a positive integer $r$ such that $\mathcal{O}(X,Y) \subseteq \mathcal{O}(X,Y)_Z$ and for every $g \in N$ satisfying $j^r g(a) = j^r f(a)$ for all $a \in A \cup f^{-1}(Z)$, we have $\tilde{g} \in \tilde{N}$ and $j^s\tilde{g}(a) = j^s\tilde{f}(a)$ for all $a \in A$.*

*Proof.* First of all observe that $\mathcal{O}(X,Y)_Z$ is open in $\mathcal{O}(X,Y)$. So any neighborhood in $\mathcal{O}(X,Y)_Z$ is a neighborhood in $\mathcal{O}(X,Y)$. Let $n = \dim Y$ and $k = \dim Z$.

The proof will be done in local coordinates. Choose a point $x_0 \in f^{-1}(Z)$. There is a 'preferred chart', say $(V|_Z, \psi)$, at $f(x_0)$; that is, $\psi : V \longrightarrow \mathbb{R}$ is an analytic diffeomorphism with $\psi(V \cap Z) = \mathbb{R}^k \times \{0\}$. Choose a chart $(U, \varphi)$ at $x_0$ such that $f(U) \subseteq V$ and the analytic diffeomorphism $\varphi$ maps $x_0$ to 0. Consider the composite map:

$$\begin{aligned} \psi \circ f \circ \varphi^{-1} : \mathbb{R} &\longrightarrow \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \\ t &\longmapsto (f_1(t), \dots, f_k(t), \dots, f_n(t)) \end{aligned}$$

Here $\psi \circ f \circ \varphi^{-1}(0) = \psi \circ f(x_0) \in \mathbb{R}^k \times \{0\} = \psi(V \cap Z)$ so $f_{k+1}(0) = \cdots = f_n(0) = 0$. On the other hand at least one of $f_k, \dots, f_n$ is not identically zero. Otherwise $\psi \circ f \circ \varphi^{-1}(\mathbb{R}) \subseteq \mathbb{R}^k \times \{0\} = \psi(V \cap Z)$ which means that $f(U) \subset Z$. It follows that $f^{-1}(Z)$ contains the open set $U$ and therefore it is infinite, contradicting to that $f \in \mathcal{O}(X, Y)_Z$.

We may select $(V, \psi)$ in such a way that $f_n$ has order $p$ at 0 and $f_{k+1}, \dots, f_{n-1}$ all has order at least $p$. This can be done by putting $p = \min\{\text{order of } f_i \text{ at } 0\}_{i=k+1}^n$ (which is well-defined as at least one $f_j$ is not identically zero) and rearranging the coordinates. Furthermore, we may choose $(U, \varphi)$ so that $f_n^{-1}(0) = \{0\}$. Since $f_n$ is analytic and $X$ is compact, then $f_n^{-1}(0)$ is a finite set, say, $f_n^{-1}(0) = \{p_1, \dots, p_r, 0\}$. Let $0 < \epsilon < \min\{|p_1|, \dots, |p_r|\}$, $\phi : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ be an analytic diffeomorphism, and replace $(U, \varphi)$ by $(\varphi^{-1}((-\epsilon, \epsilon)), \phi \circ \varphi)$, without changing the denotation. Now by the construction the real meromorphic functions $f_{k+1}/f_n, \dots, f_{n-1}/f_n$ are analytic on $\mathbb{R}$.

By Proposition 1.4.1, there exists a chart $(\tilde{V}, \tilde{\psi})$ around $\tilde{f}(x_0)$ in $\tilde{Y}$ such that $\tilde{\psi} : \tilde{V} \longrightarrow \mathbb{R}^n$ is an analytic diffeomorphism, $\tilde{f}(U) \subseteq \tilde{V}$ and

$$\begin{aligned}\tilde{\psi} \circ \tilde{f} \circ \varphi^{-1} : \mathbb{R} \longrightarrow & \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \\ t \longmapsto & (f_1(t), \dots, f_k(t), \frac{f_{k+1}(t)}{f_n(t)}, \dots, \frac{f_{n-1}(t)}{f_n(t)}, f_n(t)).\end{aligned}$$

Note that $\tilde{f}(x_0)$ is in this chart because n-th entry of $\psi \circ f \circ \varphi^{-1}(0)$ is nonzero, due to our construction.

Let $I$ be a bounded open interval in $\mathbb{R}$ containing 0. Let $N$ be a small neighborhood of $f$ in $\mathcal{O}(X,Y)_Z$ such that for any $g \in N$, $g(\varphi^{-1}(I)) \subseteq V$. Consider the map

$$\begin{aligned}\psi \circ g \circ \varphi^{-1} : I \longrightarrow & \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \\ t \longmapsto & (g_1(t), \dots, g_k(t), \dots, g_n(t)).\end{aligned}$$

If $N$ is sufficiently small and if $r$ is a sufficiently large integer (in particular, $r \geq \max\{p+1, s\}$), and if $j^r g(x_0) = j^r f(x_0)$, then also $g_n^{-1}(0) = \{0\}$, the meromorphic functions $g_{k+1}/g_n, \dots, g_{n-1}/g_n$ on $I$ are analytic, and $j^s(g_i/g_n)(0) = j^s(f_i/f_n)(0)$ for $i = k+1, \dots, n-1$. Thus $\tilde{g}(x_0) = \tilde{f}(x_0)$, so we can use the same chart and define

$$\begin{aligned}\tilde{\psi} \circ \tilde{g} \circ \varphi^{-1} : \mathbb{R} \longrightarrow & \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \\ t \longmapsto & (g_1(t), \dots, g_k(t), \frac{g_{k+1}(t)}{g_n(t)}, \dots, \frac{g_{n-1}(t)}{g_n(t)}, g_n(t)).\end{aligned}$$

It follows that, if $N$ is small enough, then $\tilde{\psi} \circ \tilde{g} \circ \varphi^{-1} = \tilde{\psi} \circ \tilde{g} \circ \varphi^{-1}|_I$ is close to $\tilde{\psi} \circ \tilde{f} \circ \varphi^{-1}|_I$ and $j^s\tilde{g}(a) = j^s\tilde{f}(a)$ for all $a \in \pi^{-1}(X, f^{-1}(Z)(A \cup f^{-1}(Z)) = A$. Now the conclusion follows. $\square$

**Lemma 2.5.2.** *Let $Y$ be a nonsingular real algebraic variety and let $Z$ be a Zariski closed nonsingular subvariety of $Y$. Let $\pi : \tilde{Y} \longrightarrow Y$ be the blowing-up of $Y$ along $Z$. Let $X$ be a compact nonsingular real algebraic curve. If $Y$ has Property$(X)$, then so does $\tilde{Y}$.*

*Proof.* Let $\varphi \in \mathcal{C}^\infty(X, \tilde{Y})$ and let $V$ be a neighborhood of $\varphi$. Let $A$ be a finite subset of $X$ and let $s$ be a nonnegative integer. Since $Z$ is a proper closed subvariety of $Y$, then $\operatorname{codim} \pi^{-1}(Z) \geq 1$. So there exists a transverse $\psi \in V$ with $j^s\psi(a) = j^s\varphi(a)$ for all $a \in A$. As $\psi \pitchfork \varphi$, $\psi(C)$ is not contained in $\pi^{-1}(Z)$ for any connected component $C$ of $X$.

By Proposition 2.2.2, we can find an analytic map $F \in \mathcal{O}(X, \tilde{Y})$ close to $\psi$ in the $\mathcal{C}^\infty$ compact-open topology, satisfying $j^sF(a) = j^s\psi(a)$ for all $a \in A$. Since transversality is an open condition, by taking $F$ sufficiently close to $\psi$ in $V$, we ensure that $F$ is also transversal to $\pi^{-1}(Z)$. On the other hand, being an analytic submanifold of $X$, $F^{-1}(\pi^{-1}(Z))$ is either a finite set of points or a connected component $C$ of $X$. The latter contradicts with that $F(C)$ is not contained in $\pi^{-1}(Z)$ for any connected component $C$ of $X$, so $F^{-1}(\pi^{-1}(Z))$ is necessarily finite. Hence, $f = \pi \circ F \in \mathcal{O}(X, Y)_Z$ and, in the notation we used in the previous lemma, $\tilde{f} = F$.

Now we can apply the previous lemma. There exists a neighborhood $N$ of $f$ in $\mathcal{O}(X, Y)$ and a positive integer $r$ such that $N \subseteq \mathcal{O}(X, Y)_Z$ and for every map $g \in N$ satisfying $j^rg(a) = j^rf(a)$ for all $a \in A \cup f^{-1}(Z)$, the map $\tilde{g} \in V$ and $j^s\tilde{g}(a) = j^s\tilde{f}(a)$ for all $a \in A$. Since $Y$ has *Property*$(X)$, we can choose $g \in \mathcal{R}(X, Y)$ as above. Then we have a $\tilde{g} \in \mathcal{R}(X, \tilde{Y})$ such that $\tilde{g} \in V$ and $j^s\tilde{g}(a) = j^s\psi(a) = j^s\varphi(a)$ for all $a \in A$. It shows that $\tilde{Y}$ has *Property*$(X)$. □

Lastly we show that an analytic map between two projective varieties can be lifted to blow-up:

**Lemma 2.5.3.** *Let $\varphi : Z \longrightarrow Y$ be a regular map between projective nonsingular real algebraic varieties. Let $X$ be an analytic curve and let $h : X \longrightarrow Y$ be an analytic map. If $\varphi$ is a birational map, then there exists an analytic map $H : X \longrightarrow Z$ such that $\varphi \circ H = h$.*

*Proof.* By applying elimination of indeterminacy theorem to the rational map $\varphi^{-1} : Y \longrightarrow Z$, we obtain a sequence

$$\tilde{Y} = Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = Y$$

and a regular map $\psi : \tilde{Y} \longrightarrow Z$ such that each $\pi_i$ is the blowing-up of $Y_{i-1}$ along a Zariski closed nonsingular subvariety $B_{i-1}$ of $Y_{i-1}$ for $1 \leq i \leq k$, and $\varphi \circ \psi = \pi_1 \circ \cdots \circ \pi_k$.

We now successively construct analytic maps $H_i : X \longrightarrow Y_i$ satisfying $H_0 = h$ and $\pi_{i+1} \circ H_{i+1} = H_i$ for $0 \leq i \leq k-1$. Without loss of generality, we may assume that $X$ is connected, because once we construct maps for connected components then they can be combined to gather the analytic map on the whole $X$. Since each $B_i$ is a closed analytic variety and each $H_i$ is an analytic map, then $H_i^{-1}(B_i)$ is an analytic subvariety of $X$. As $X$ is compact, $H_i^{-1}(B_i)$ is either a finite set of points or the whole $X$.

If $H_i^{-1}(B_i)$ is a finite set, then $H_{i+1}$ is uniquely determined. Suppose that $H_i^{-1}(B_i) = X$. So $H_i(X) \subseteq B_i$. Let $\nu$ be the smooth normal bundle of $B_i$ in $Y$, and consider the pull-back bundle $H_i^*(\nu)$ over $X$. Identifying $B_i$ with the 0-section, let $T$ be a tubular neighborhood of $B_i$ in $Y$ and $T'$ be the tubular neighborhood of the 0-section in $\nu$ such that $T$ is diffeomorphic to $T'$. By Remark 1.4.2 we know that if $\text{codim} B_i < 2$, then the blow-up process had to be stopped. So we assume that codim$B \geq 2$ in $Y$ and therefore rank$H_i^*(\nu)$ = codim$B_i \geq 2$. However any smooth vector bundle over $X$, which is diffeomorphic to $S^1$, has a nowhere zero section, provided that $k \geq 2$. Clearly we can choose such a section whose image is contained in $T'$. Thus $H_i^*(\nu)$ has a nowhere vanishing smooth section $\sigma : X \longrightarrow T' \subset H_i^*(V)$ so that $\sigma(X) \cap H_i^*(B_i) = \emptyset$. By Theorem 2.3.3, $H_i^*(B_i)$ is $\mathcal{C}^\infty$ isomorphic to an algebraic vector bundle $\xi$ over $X$. As $\xi$ is algebraic, it is a direct summand of a trivial bundle, say $\epsilon_X^n$. Consider

$$\tilde{\sigma} : X \longrightarrow \epsilon_X^N$$
$$pr \circ \tilde{\sigma}(x) = \sigma(x) \quad \text{for all } x \in X,$$

where $pr : \epsilon_X^N \longrightarrow \xi$ is the projection vector bundle morphism. By Proposition 2.2.1, there exists a regular section $\tilde{g} : X \longrightarrow \epsilon_X^N$, arbitrarily close in the $\mathcal{C}^\infty$ compact-open topology to $\tilde{\sigma}$. As $pr$ is regular, it follows that the map

$$g : X \longrightarrow \xi$$
$$g(x) = pr \circ \tilde{g}(x) \quad \text{for all } x \in X$$

is a regular section arbitrarily close to $\sigma$. Taking $g$ close enough, we ensure that $g(X) \cap H_i^*(B_i) = \emptyset$. Thus $g$ is away from the blow-up center and $\pi_i \circ g = H_i$. We finish the proof by putting $H_{i+1} = g$. □

## 2.6 Proof of Theorem S$'$

Now we prove Theorem S$'$: By Proposition 1.5.1 $Y$ is isomorphic to a Zariski open subvariety of a projective nonsingular real algebraic variety $V$. Since

$X$ is compact, so is $f(X) \subset Y$. Therefore if $\{p_k\}_{k=1}^{\infty}$ is a sequence of regular maps converging to the smooth map $f : X \longrightarrow V$ with $f(X) \subset Y$, then there is a subsequence $\{p_{i_k}\}_{k=1}^{\infty}$ with images in $Y$ and converging to $f$. So, if we can approximate the maps from $X$ into $V$, we will be approximating maps into $Y$ in particular. Because of this, we will assume that $Y$ itself is projective.

As $Y$ is rational, there exist rational maps $\phi : Y \to \mathbb{RP}^n$ and $\psi : \mathbb{RP}^n \to Y$ such that $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_{\mathbb{RP}^n}$. By elimination of indeterminacy theorem, there exists a sequence

$$\tilde{Y} = Y_k \xrightarrow{\pi_k} Y_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = \mathbb{RP}^n$$

and a regular map $\varphi : \tilde{Y} \longrightarrow Y$ such that each $\pi_i$ is the blowing-up of $Y_{i-1}$ along a Zariski closed nonsingular subvariety $B_{i-1}$ of $Y_{i-1}$, for $1 \leq i \leq k$, and $\varphi$ is birational. It follows from Lemma 2.4 and Lemma 2.5.2 that each $Y_i$ has $Property(X)$ and thus $\tilde{Y}$ has $Property(X)$.

Let $A$ be a finite subset of $X$ and $s$ be a nonnegative integer. By Proposition 2.2.2, for any given $f \in C^{\infty}(X, Y)$, we can find an analytic map $h \in \mathcal{O}(X, Y)$, arbitrarily close to $f$, with $j^s h(a) = j^s f(a)$ for all $a \in A$. By Lemma 2.5.3, there exists $H \in \mathcal{O}(X, \tilde{Y})$ satisfying $\varphi \circ H = h$. Since $\tilde{Y}$ has $Property(X)$, we can choose $G \in \mathcal{R}(X, \tilde{Y})$ such that $G$ is arbitrarily close to $H$ and $j^s G(a) = j^s H(a)$ for all $a \in A$. Then $g = \varphi \circ G \in \mathcal{R}(X, Y)$ is close to $h = \varphi \circ H$ and $j^s g(a) = j^s h(a)$ for all $a \in A$. Hence $Y$ has $Property(X)$, the proof is complete.

## 2.7 Some corollaries

The following corollary is deduced from the above proof but has its own significance:

**Corollary 2.7.1.** *Let $X$ be a compact nonsingular real algebraic curve, $Y$ and $Y'$ be nonsingular real algebraic varieties which are birational. If $Y'$ has Property$(X)$, then so does $Y$. In particular, if $\mathcal{R}(X,Y')$ is dense in $\mathcal{C}^\infty(X,Y')$ then so is $\mathcal{R}(X,Y)$ in $\mathcal{C}^\infty(X,Y)$.*

*Proof.* In the proof of Theorem S' just replace $\mathbb{RP}^n$ with $Y'$. □

There are examples of varieties satisfying *Property*$(X)$ which are not rational, namely stably-rational varieties. Although they are defined in a purely algebraic setting (in relation to transcendental field extensions), here we use the following geometric characterization:

**Definition 2.7.1.** An algebraic variety $V$ is *stably rational* if $V \times \mathbb{P}^k$ is rational for some $k$.

Whether stably rational implies rational had been asked by Zariski. This is not true for dim $\geq 3$, there are examples of stably rational but not rational varieties (see [1]). Thus the set of rational varieties is a proper subset of stably rationals. These varieties occur quite often in moduli problems. Here we present the statement of Theorem S$'$ renewed for stably rational varieties.

**Corollary 2.7.2.** *Every stably rational nonsingular real algebraic variety $Y$ has Property$(X)$ for every compact nonsingular real algebraic curve $X$.*

*Proof.* Let $Y$ be a stably rational variety and $f \in \mathcal{C}^\infty(X,Y)$. Let $V$ be a neighborhood of $f$ in $\mathcal{C}^\infty(X,Y)$, $A$ be a finite subset of $X$, and $s$ be a nonnegative integer. There exists $k$ such that $Y \times \mathbb{RP}^k$ is birational to $\mathbb{RP}^m$ where $m = \dim Y + k$. Consider the smooth map

$$F : X \longrightarrow Y \times \mathbb{RP}^k$$
$$x \longmapsto (f(x), c)$$

where $c$ is a point in $\mathbb{RP}^k$. By Lemma 2.4 and Proposition 2.7.1, $Y \times \mathbb{RP}^k$ has $Property(X)$ so we can find $G \in \mathcal{R}(X, Y \times \mathbb{RP}^k)$ such that $G \in V \times W$ ($W$ is a neighborhood of the constant function) and $j^sG(a) = j^sF(a)$ for all $a \in A$. It follows that $g = pr_1 \circ G \in \mathbb{R}(X, Y)$ is in $V$ and $j^sg(a) = j^sf(a)$ for all $a \in A$, where $pr_1$ is the projection map $Y \times \mathbb{RP}^k \longrightarrow Y$. □

# CHAPTER 3

# A NECESSARY CONDITION

## 3.1 Real parts of complex varieties

It is convenient to consider real algebraic varieties as real parts of complex algebraic varieties so that one can use the most of the knowledge accumulated in complex algebraic geometry. For this purpose we deal with a certain class of complex varieties:

**Definition 3.1.1.** Let $V$ be a complex algebraic variety. If there is an embedding $i : V \hookrightarrow \mathbb{C}^N$ such that the defining equations of $i(V) \subseteq \mathbb{C}^N$ can be given by real polynomials, then $V$ is said to be *defined over* $\mathbb{R}$. Then *the real part of* $V$, denoted by $\mathbb{R}V$ is the real algebraic variety $X \subseteq \mathbb{R}^N$ defined by these polynomials. Let $f : V \longrightarrow V'$ be a morphism between complex algebraic varieties and let $V$ and $V'$ be defined over $\mathbb{R}$, then we also say $f$ *is defined over* $\mathbb{R}$ if it can be defined by means of real polynomials.

**Remark 3.1.2.** Equivalently, a complex algebraic variety $V$ is said to be *defined over* $\mathbb{R}$ if there is an embedding (indeed the same embedding as above) $i : V \hookrightarrow \mathbb{C}^N$ such that $i(V)$ is preserved under the complex conjugation ($i(V) = \overline{i(V)}$). Then $\mathbb{R}V$ is the fixed point set of this conjugation, i.e

$\mathbb{R}V = i(V) \cap \mathbb{R}^N$. In a general setting, we can think complex algebraic varieties defined over $\mathbb{R}$ as complex algebraic varieties with an antiholomorphic involution. In this case, the real part is the fixed point set of this involution. Finally, a morphism $f : V \to V'$ between two complex algebraic varieties defined over $\mathbb{R}$ is a morphism which commutes with the involutions on $V$ and $V'$.

**Remark 3.1.3.** If $V$ is a nonsingular complex algebraic variety with complex dimension $n$, then $\mathbb{R}V$ is either empty or a nonsingular real algebraic variety of dimension $n$ (see [29], Section I.1).

All these have a natural counterpart:

**Definition 3.1.4.** Given a real algebraic variety $X \subseteq \mathbb{R}^N$, complexification of $X$ is the complexification of the pair $X \subseteq \mathbb{R}^N$ and denoted by $X_{\mathbb{C}}$. That is, if $X = Z(I)$, where $I$ is an ideal of $\mathbb{R}[x_1, \ldots, x_N]$, then $X_{\mathbb{C}} \subseteq \mathbb{C}^N$ is obtained by considering $I$ as an ideal of $\mathbb{C}[x_1, \ldots, x_N]$. In other words $X_{\mathbb{C}}$ is the smallest complex algebraic variety (in $\mathbb{C}^N$) containing $X$.

**Remark 3.1.5.** $X_{\mathbb{C}}$ is a complex variety defined over $\mathbb{R}$. Indeed, $X_{\mathbb{C}} = \overline{X_{\mathbb{C}}}$ and $\mathbb{R}X_{\mathbb{C}} = X$ (see [33]).

Although the complexification depends on the ambient space, we will talk about complexification of a real algebraic variety $X$ (without any specified embedding). This will mean the complexification of $X$ obtained after 'an' embedding. Of course this definition makes sense only when the considered properties of the complexification is independent from the choice. We'll see an example of this in the following context.

## 3.2 Statement of Theorem N

**Definition 3.2.1.** Given a compact nonsingular real algebraic variety $Y$, $b_1(Y, \mathbb{C})$ is defined to be the first Betti number of a nonsingular projective complexification $V$ of $Y$. In the next section we show that $b_1(Y, \mathbb{C})$ is well-defined for any nonsingular real algebraic variety.

We are going to give a complete proof of the following theorem:

**Theorem 3.2.1 (Theorem N).** *Let $X$ be a real algebraic variety of positive dimension and let $Y$ be a compact nonsingular real algebraic variety. If $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}(X, Y)$, then $b_1(Y, \mathbb{C}) = 0$.*

## 3.3 $b_1(Y, \mathbb{C})$ is well-defined

Let $V$ be a complexification of the compact nonsingular real algebraic variety $Y$. By resolution of singularities theorem we can obtain a nonsingular complex algebraic variety $\tilde{V}$ from $V$ by a sequence of blowing-ups on Zariski closed subvarieties; say, $\tilde{V} = V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0 = V$ such that $B_{k-1}, \ldots, B_1, B_0$ are the corresponding blow-up centers, all contained in $Sing(V)$. Since $Sing(V)$ is defined over $\mathbb{R}$ then each $B_i$ and $V_i$ are defined over $\mathbb{R}$. As $Y = \mathbb{R}V$ is nonsingular, it follows that this blowing-up process does not effect $Y$. Therefore $Y = \mathbb{R}V_i$ for all $i = 1, 2, \ldots, k$ and $\tilde{V}$ is a nonsingular complexification of $Y$. By Proposition 1.5.1 we have a nonsingular projective variety $\tilde{\tilde{V}}$ obtained from $\tilde{V}$ and by the same reasons we have written above, $\tilde{\tilde{V}}$ is a complexification of $Y$ as well.

Let us simply denote this nonsingular projective complexification of $Y$ by $V$ and let $V'$ be such an other complexification. As $\mathbb{R}V \cong \mathbb{R}V'$, this isomorphism extends to -at least- a birational isomorphism between $V$ and $V'$.

If these are isomorphic then we are done. Suppose that they are birational but not isomorphic. Then by Proposition 1.6.1 there are closed subvarieties $\Delta \subset V$ and $\Delta' \subset V'$, each having codim$\geq 2$, such that $V \setminus \Delta \cong V' \setminus \Delta'$. Considering all these as real manifolds, we have $V_{\mathbb{R}} \setminus \Delta_{\mathbb{R}} \cong V'_{\mathbb{R}} \setminus \Delta'_{\mathbb{R}}$ where codim$\Delta_{\mathbb{R}}$, codim$\Delta'_{\mathbb{R}} \geq 4 > 2$. For any real topological manifold $M$, if $N$ is a submanifold of $M$ with codim$> 2$ then $\pi_1(M) = \pi_1(M \setminus N)$. Therefore $\pi_1(V, \mathbb{C}) = \pi_1(V_{\mathbb{R}}) = \pi_1(V'_{\mathbb{R}}) = \pi_1(V', \mathbb{C})$. Finally, $b_1(Y, \mathbb{C})$ is well-defined because of the isomorphism $H_1(M) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]$ for any compact topological manifold $M$.

**Remark 3.3.1.** If $Y$ is rational then $b_1(Y, \mathbb{C}) = 0$. To see this let $Y$ be in $\mathbb{R}^m$. Complexifying the ambient space $\mathbb{R}^m$ we obtain a complexification $V$ of $Y$. As $Y$ is rational, by definition it is birational to $\mathbb{RP}^n$ for some $n$. Then same maps define a birational isomorphism between $V$ and $\mathbb{CP}^n$, where the latter is simply connected. Thus $\pi_1(Y, \mathbb{C}) = \pi_1(\mathbb{CP}^n) = 0$, implying that $H_1(Y, \mathbb{C})$ is trivial.

## 3.4 Abelian varieties

This is a preliminary section for the proof of Theorem N. Here we introduce some definitions and properties concerning Abelian varieties.

### 3.4.1 General definitions

**Definition 3.4.1.** A lattice $\Lambda$ in $\mathbb{C}^n$ is by definition a discrete subgroup of maximal rank in $\mathbb{C}^n$. The quotient $X = \mathbb{C}^n/\Lambda$ is called a *complex torus.* A complex Abelian variety (shortly, "Abelian variety" in this context) is a complex torus admitting a positive definite line bundle, where positive definite means that the first Chern class of the bundle is a positive definite Hermitian form. Let $X$ and $X'$ be two Abelian varieties with positive definite line bundles $L$ and $L'$. A morphism $f : X \longrightarrow X'$ is *a morphism of Abelian varieties* if $f^*(L') = L$.

**Remark 3.4.2.** By definition, $\Lambda$ is a free Abelian group of rank $2n$ and a complex torus is an $n$-dimensional smooth complex manifold. A complex torus inherits the structure of a complex Lie group from $\mathbb{C}^n$ (see [23]). It turns to be an Abelian variety if it satisfies certain conditions, called Riemann conditions. (see [15]). Equivalently a complex torus $X$ is an Abelian variety if it can be embedded to a complex projective space (i.e, if $X$ is projective).

As the above definition emphasizes the geometric structure of an Abelian variety, we give an other -equivalent- definition which emphasizes the group structure of it. These two definitions in balance will be used substitutively:

**Definition 3.4.3.** A group variety over $\mathbb{C}$ is a complex algebraic variety with morphisms

$$M : V \times V \longrightarrow V \quad (\textit{multiplication})$$
$$inv : V \longrightarrow V \quad (\textit{inverse})$$

and an identity element $e \in V$ with respect to $m$ and $inv$. A complete group variety over $\mathbb{C}$ (i.e its center and outer automorphism groups are both trivial) is called a complex Abelian variety. An algebraic morphism $f : X \longrightarrow X'$ is *a morphism of Abelian varieties* if it is also a group homomorphism.

**Remark 3.4.4.** A group variety is nonsingular. The group structure on an Abelian variety is commutative (see [24]).

**Definition 3.4.5.** A nonzero Abelian variety $A$ is said to be *simple* if it does not contain an Abelian subvariety different from $\{0\}$ and $A$.

**Definition 3.4.6.** An *isogeny* $f : X \longrightarrow X'$ is a morphism of Abelian groups with finite kernel. In other words, $f$ is an isogeny if and only if it is surjective and $dimX = dimX'$.

**Theorem 3.4.1 (Poincaré's Complete Reducibility Theorem).** *Given an Abelian variety $X$, there is an isogeny*

$$X \longrightarrow X_1^{n_1} \times \cdots , X_r^{n_r}$$

*with simple Abelian varieties $X_1, \ldots , X_r$ which are mutually nonisogenous. Moreover, Abelian varieties $X_1, \ldots , X_r$ and the integers $n_1, \ldots , n_r$ are uniquely determined up to isogenies and permutations.*

*Proof.* See [23], Chapter V Theorem 3.7, or [24], Section 12. □

### 3.4.2 Jacobian of a curve and Albanese varieties

To any compact nonsingular complex algebraic variety one can associate a canonical Abelian variety, so called the Albanese variety:

**Definition 3.4.7.** Let $V$ be a compact nonsingular complex algebraic variety. Then we define *the Albanese variety of $M$* as

$$Alb(V) = \frac{H^0(V, \Omega^1)^*}{H_1(V, \mathbb{Z})/Tor}$$

and denote it by $Alb(V)$. Choosing a base point $p_0 \in V$ and a basis

$\omega_1, \dots, \omega_q \in H^0(V, \Omega^1)$ the map

$$\begin{aligned} \mu : V &\longrightarrow Alb(V) \\ p &\longmapsto \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_q\right) \end{aligned}$$

is called the *canonical Albanese map corresponding to* $p_0$. In particular, if $V$ is a compact nonsingular complex algebraic curve, then $Alb(V)$ is called *the Jacobian of* $V$ and denoted by $J_V$.

**Remark 3.4.8.** Albanese map is a well-defined morphism of complex algebraic varieties. Indeed Albanese varieties are generalizations of Jacobians (see [14] for a detailed treatment). Note that another useful characterization can be given by means of Hodge theory, where we obtain $Alb(V) \cong H^{n-1,n}(V)/H^{2n-1}(V, \mathbb{Z})$. Albanese varieties are Abelian. For the proofs, see [15], Section II.6.

**Definition 3.4.9.** Given a set $S$, an Abelian group $A$, a positive integer $n$, and a map $f : S \longrightarrow A$, we define

$$\begin{aligned} f^n : S^n &\longrightarrow A \\ (x_1, \dots, x_n) &\longmapsto f(x_1) + \dots + f(x_n) \end{aligned}$$

for all $(x_1, \dots, x_n) \in S^n$. Let $S$ be a nonsingular complex algebraic curve $C$ and let $\Sigma_n$ denote the permutation group on $\{1, \dots, n\}$. Then each $\sigma \in \Sigma_n$ can be considered as a morphism $\sigma : C^n \longrightarrow C^n$, mapping each $(p_1, \dots, p_n)$ to $(p_{\sigma(1)}, \dots, p_{\sigma(n)})$, and hence $\Sigma_n$ can be regarded as a subgroup of the automorphism group of $C^n$. We define $f^{(n)} : C^{(n)} \longrightarrow A$ as $f^{(n)}(x) = f^n(x)$ where $C^{(n)} = C^n/\Sigma_n$, the symmetric product of $C$. In a different point of view, $C^{(n)}$ is the set of all effective divisors of degree $n$ of $C$.

We are going to make use of the following theorem:

**Theorem 3.4.2 (Strong form of the Jacobi inversion theorem).** *Let $C$ be a compact nonsingular complex algebraic curve and $p_0 \in C$. Let $g$ be the genus of $C$ and $\omega_1, \ldots, \omega_q$ be a basis of $H^0(V, \Omega^1)$. Then the restriction of Abel-Jacobi map corresponding to $p_0$:*

$$u : C^{(g)} \longrightarrow J(C)$$
$$\sum_{i=1}^{g} p_i \longmapsto \left(\sum_{i=1}^{g} \int_{p_0}^{p_i} \omega_1, \ldots, \sum_{i=1}^{g} \int_{p_0}^{p_i} \omega_g\right)$$

*is surjective and 'generically' (i.e. those objects in the family not satisfying this property are parametrized by a subvariety of strictly lower dimension) injective.*

*Proof.* See [25], or for a detailed and clear proof see [14] together with [15]. □

Lastly we present the following important property of Albanese varieties. Note that this property can also be used to characterize $Alb(V)$ for a given compact nonsingular complex algebraic variety $V$.

**Theorem 3.4.3 (Universal property of Alb(V)).** *Let $V$ be a compact nonsingular complex algebraic variety and $\mu : V \longrightarrow Alb(V)$ be the canonical Albanese map corresponding to $p_0 \in V$. Suppose that $A$ is an Abelian variety and $\varphi : V \longrightarrow A$ is a morphism. Then there exists a unique morphism $\tilde{\varphi} : Alb(V) \longrightarrow A$ of Abelian varieties such that $\varphi = \tilde{\varphi} \circ \mu$.*

*Proof.* See [23], where a clear proof is given in the case of Jacobian variety. We refer to [28] for the general case. □

Up to here, we haven't touch on the real structure of these varieties and morphisms. The crucial point is that exactly all these definitions and propositions can well be carried out for the complex algebraic varieties defined over $\mathbb{R}$; one can just find and replace all "complex algebraic varieties" by "complex algebraic varieties defined over $\mathbb{R}$" and "morphisms" by "morphisms defined over $\mathbb{R}$". Proofs of theorems 3.4.1, 3.4.2, and 3.4.3 that we have mentioned above are the same as in the general case; except one must notice that the real structure on the isogeny and decomposition in 3.4.1 are induced by the real structure of $V$ throughout the proof. Yet, the following is not so straightforward: If $V$ is a compact nonsingular complex algebraic variety defined over $\mathbb{R}$ then $Alb(V)$ and the Albanese map corresponding to any $p_0 \in \mathbb{R}V$ are also defined over $\mathbb{R}$. We sketch the proof below. Meanwhile we show that the Abel-Jacobi map is defined over $R$, when the base point of the complex variety defined over $\mathbb{R}$ is chosen from the real part.

Let $V$ be a compact nonsingular complex algebraic variety defined over $\mathbb{R}$ with an anti-holomorphic involution $\sigma$. We can always choose charts $(\phi_1, \dots, \phi_n)$ on $V$ such that $\phi_i^\sigma(x) = (j \circ \phi_i \circ \sigma)(x)$ equals to $\phi_i(\sigma(x))$, where $j$ is the complex conjugation ([29], Section I.1). Let $\omega = \sum f_i dz_i$ be a holomorphic 1-form, written in these local coordinates and define $\omega^\sigma = \sum j \circ f_i \circ \sigma dz_i$. Then $\omega_1 = (\omega + \omega^\sigma)/2$ and $\omega_2 = i(\omega - \omega^\sigma)/2$ are holomorphic 1-forms on $V$ with coefficient functions defined over $\mathbb{R}$. A basis for $\Omega^1$ can be formed among these. Fix this basis and consider the canonical Albanese map $\mu$ corresponding to $p_0 \in \mathbb{R}V$. Any component function of $\mu$ is of the form $\int_{p_0}^{p} \omega$, where $\omega = \sum f_i dz_i$ is such a basis element. Then $(j \circ \mu)(\sigma(p)) = \int_{p_0}^{\sigma(p)} \omega$ is locally $j \circ \int_{x_0}^{\sigma(x)} f_i(x) dz_i$, where we compute the integration on the curve $\gamma : [x_0, x] \longrightarrow V$. So we have $j \circ \int_{x_0}^{x} f_i(\sigma(x)) dz_i = \int_{x_0}^{x} j \circ f_i(\sigma(x))\ dz_i = \int_{x_0}^{x} f_i(x) dz_i = \mu(x)$. Hence $\mu$ and $Alb(V)$ are defined over $R$.

## 3.5 Lemmas on real parts of Abelian varieties

In this section we are going to state and prove some results on real parts of Abelian varieties defined over $\mathbb{R}$. Lemmas we state here are going to be used in the proof of Theorem N.

**Proposition 3.5.1.** *Let $A$ be a complex Abelian variety defined over $\mathbb{R}$ and let $\mathbb{R}A_0$ be the connected component of $\mathbb{R}A$ containing 0. Then $\mathbb{R}A_0$ is a real analytic Lie manifold isomorphic to the usual torus.*

*Proof.* Pick any $a, b \in \mathbb{R}A_0$ and let $\alpha : [0,1] \longrightarrow \mathbb{R}A_0$, $\beta : [0,1] \longrightarrow \mathbb{R}A_0$ be paths connecting 0 and $a$, 0 and $b$, respectively. Define $\alpha - \beta : [0,1] \longrightarrow \mathbb{R}A$ which maps each $x \in [0,1]$ to $\alpha(x) - \beta(x)$. Since this map is continuous, it follows that $(\alpha - \beta)(0) = 0$ and $(\alpha - \beta)(1) = a - b$ are in the same connected component; i.e. $a - b \in \mathbb{R}A_0$. Hence $\mathbb{R}A_0$ is a subgroup of $\mathbb{R}A$ -which is clearly a subgroup of $A$. Having induced the analytic structure of $A$, $\mathbb{R}A_0$ is a real analytic Lie manifold. Besides; characterizing $A$ by $\mathbb{C}^n/\Lambda$ (where $n = dimA$ and $\Lambda$ is a full lattice), we have $\mathbb{R}A_0 = \mathbb{R}(\mathbb{C}^n/\Lambda)_0$ real isomorphic to $\mathbb{R}(\mathbb{C}^n/\mathbb{Z}^{2n})_0 = \mathbb{R}^n/\mathbb{Z}^n$, because there always exists $B \in GL(2n, \mathbb{R})$ satisfying $B\Lambda = \mathbb{Z}^{2n}$. That is; $\mathbb{R}A_0 \cong \mathbb{R}^n/\mathbb{Z}^n$, the real torus. □

Now we can prove the following:

**Lemma 3.5.2.** *Let $f : A \longrightarrow B$ be a morphism of complex Abelian varieties defined over $\mathbb{R}$. If $f(A) = B$ then $f(\mathbb{R}A_0) = \mathbb{R}B_0$. (Note that also $f$ is assumed to be defined over $\mathbb{R}$.)*

*Proof.* First we show that $a + \bar{a} \in \mathbb{R}A_0$ for all $a \in A$. Let $\gamma : [0,1] \longrightarrow A$ be a path connecting 0 and $a$, and define $\delta : [0,1] \longrightarrow A$ which maps each $x \in [0,1]$ to $\gamma(x) + \overline{\gamma(x)}$. As it is a continuous map into $\mathbb{R}A$, it follows that

$\gamma(0) = 0$ and $\gamma(1) = a + \bar{a}$ are in the same connected component of $\mathbb{R}A$. Thus $a + \bar{a} \in \mathbb{R}A_0$ for all $a \in A$.

Let $b \in \mathbb{R}B_0$. Since $\mathbb{R}B_0$ is algebraically a torus, we can consider $b$ in $S^1 \times \cdots \times S^1 = T^n$. So $b = (b_1, \ldots, b_n)$ with $b_j = (\cos\theta_j, \sin\theta_j)$ where $0 \leq \theta_j \leq 2\pi$ for all $j = 1, \ldots, n$. Choose $b' = (b'_1, \ldots, b'_n) \in T^n$ with $b'_j = (\cos(\frac{\theta_j}{2}), \sin(\frac{\theta_j}{2}))$ for all $j = 1, \ldots, n$. Then $b' \in \mathbb{R}B_0$ is such that $2b' = b$. Choose $a \in A$ such that $f(a) = b'$. Then $f(a+\bar{a}) = f(a) + f(\bar{a}) = f(a) + \bar{f(a)}$ (as $f$ is defined over $\mathbb{R}$). Thus $f(a + \bar{a}) = b' + \bar{b'} = b' + b' = b$ and hence the conclusion follows from the fact that $a + \bar{a} \in \mathbb{R}A_0$. □

**Lemma 3.5.3.** *Let $C$ be a nonsingular projective complex algebraic curve defined over $\mathbb{R}$, $c_0 \in \mathbb{R}C$, and $\mu : C \longrightarrow J_C$ be the canonical morphism corresponding to $c_0$. For every connected neighborhood $N \subseteq \mathbb{R}C$ of $c_0$ there exists $n_{C,N} \in \mathbb{Z}^+$ such that $\mu^n(N^n) = (\mathbb{R}J_C)_0$ for every integer $n \geq n_{C,N}$.*

*Proof.* First of all, since $\mu$ is defined over $\mathbb{R}$, then $\mu^n(\mathbb{R}C) \subseteq \mathbb{R}J_C$. Moreover, $\mu(c_0) = 0$, so any connected neighborhood of $c_0$ is mapped into $(\mathbb{R}J_C)_0$.

Let $g$ be the genus of $C$. If $g = 0$ then $J_C = 0$ and the conclusion is vacously satisfied. Assume that $g \geq 1$. Then by Theorem 3.4.2, $\mu^{(g)}$, which equals to restriction of Abel-Jacobi map on $C^{(g)}$, is surjective. The same holds for $\mu^g$. Then $\mu^g(C^g) = J_C$. Since the set $N^g$ is Zariski dense in $C^g$ and $\mu^g$ is Zariski continuous, then there exists a point in $N^g$ at which $\mu^g$ has rank $g$. For, the jacobian matrix of a variety is given by polynomials and it follows that if $rank(\mu^g) < g$ on the dense set $N^g$ then it is also strictly less than $g$ on the Zariski closure $C^g$. However, this contradicts that $\mu^g(C^g) = J_C$, so there exists such a point. Now by the Inverse Function Theorem there are connected neighborhoods $U$ of $x_0$ in $N^g$ and $V$ of $\mu^g(x_0)$ in $(\mathbb{R}J_C)_0$, diffeomorphic to each other.

As $(\mathbb{R}J_C)_0$ is algebraically a torus, we can consider $V$ in $S^1\times\cdots\times S^1 = T^g$. So $V$ can be taken of the form $V_1 \times \cdots \times V_g \subseteq T^g$, where each $V_i$ is a nonempty open subset of $S^1$. For each $V_i$ we have $k_i$ such that $k_iV_i = S^1$. Put $k = \max\{k_1, \ldots, k_g\}$, then $kV = (\mathbb{R}J_C)_0$. Hence for $n_{C,N} = kg$ if $n \geq n_{C,N}$ then $\mu^n(U) \supseteq \mu^{kg}(U) \supseteq k\mu^g(U) = kV = (\mathbb{R}J_C)_0$. Since $\mu^n(U) \subseteq \mu^n(N^g) \subseteq (\mathbb{R}J_C)_0$, the conclusion is satisfied. □

**Lemma 3.5.4.** *Let $C$,$N$ and $n_{C,N}$ be as in the previous lemma. Let $A$ be a simple complex Abelian variety defined over $\mathbb{R}$, let $U$ be a Zariski neighborhood of $c_0$ in $\mathbb{R}C$ and $f : U \longrightarrow \mathbb{R}A$ be a regular map satisfying $f(c_0) = 0$. Let $N$ be contained in $U$. Then for every integer $n \geq n_{C,N}$, either $f^n(N^n) = 0$ or $f^n(N^n) = (\mathbb{R}A)_0$.*

*Proof.* $\mathbb{R}C \setminus U$ has codim$\leq 1$ so $f$ can be extended to a regular map $g$ on $C$ by Proposition 1.6.1. Theorem 3.4.3 implies that there exists a morphism $\varphi : J_C \longrightarrow A$ of Abelian varieties defined over $\mathbb{R}$, such that $\varphi \circ \mu = g$

Since $A$ is simple, the complex Abelian group $\varphi(J_C)$ defined over $\mathbb{R}$ is either $\{0\}$ or $A$. Therefore either $\varphi((\mathbb{R}J_C)_0) = \{0\}$ or $\varphi((\mathbb{R}J_C)_0) = \mathbb{R}A_0$, implied by Lemma 3.5.2. If the former, then $\varphi \circ \mu(c_0) = \{0\} \subseteq \varphi \circ \mu(N) \subseteq \varphi \circ \mu((\mathbb{R}J_C)_0) = \{0\}$ and therefore $g^n(N^n) = ng(N) = n(\varphi \circ \mu)(N) = \{0\}$. If the latter, then $\mathbb{R}A_0 = \varphi((\mathbb{R}J_C)_0) = \varphi(\mu^n(N^n))$ by Lemma 3.5.3. It follows that $\mathbb{R}A_0 = \varphi \circ \mu^n(N^n) = (\varphi \circ \mu)^n(N^n) = g^n(N^n)$. Recall that $g|_U \equiv f$. So $g|_N \equiv f$ and we have either $f^n(N^n) = \{0\}$ or $f^n(N^n) = \mathbb{R}A_0$ for all $n \geq n_{C,N}$. □

## 3.6 Proof of Theorem N

We are now ready to prove Theorem N. Idea is simple: We are going to construct a smooth map whose restriction to a fixed curve on $X$, when composed with a nonconstant map from $Y$ to real part of a simple Abelian variety $A$ (we'll see that this morphism is defined naturally) have an image far from being $\{0\}$ or $\mathbb{R}A_0$. Whereas restriction of any regular map from $X$ to $Y$ on the same curve, composed with the same morphism from $Y$ to $A$, will necessarily have an image equal to either $\{0\}$ or $\mathbb{R}A_0$, owing to the results in the previous section.

*Proof.* We may assume that $Y$ is of the form $\mathbb{R}V$ where $V$ is a nonsingular projective complex algebraic variety defined over $\mathbb{R}$. Assume that $b_1(Y, \mathbb{C})$ is not zero. With this assumption, we will be able to define a nonconstant morphism from $V$ to a simple Abelian variety due to the fact that $Alb(V)$ is nontrivial.

As $Y$ is a compact nonsingular complex algebraic variety, it is a Kähler manifold. Thus by Hodge decomposition

$$H^r(M, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(M)$$

$$H^{p,q}(M) \cong \overline{H^{p,q}(M)}$$

and $H^{p,q}(M) \cong H^q(M, \Omega^p)$ (which is derived from the first two). In particular, $H^1(M, \mathbb{C}) \cong H^{1,0}(M) \bigoplus H^{0,1}(M)$, $H^{1,0}(M) = \overline{H^{0,1}(M)}$ and $H^{1,0}(M) \cong H^0(M, \Omega^1)$. Thus $b_1(Y, \mathbb{C}) \neq 0$ implies that $dim(H^{1,0}(M)) = dim(H^0(M, \Omega^1)) = dim(H^0(M, \Omega^1)^*$ is positive. Hence, nontriviality of $Alb(M)$ arise from its definition.

Choose a point $y_0 \in V(\mathbb{R})$ and let $\alpha : V \longrightarrow Alb(V)$ be the corresponding Albanese morphism (in particular $\alpha(y_0) = 0$). As we mentioned in section 3.4 both $Alb(V)$ and $\alpha$ are defined over $\mathbb{R}$. Also by Theorem 3.4.1 there exists an isogeny $\eta : Alb(V) \longrightarrow A_1 \times \cdots A_r$ defined over $\mathbb{R}$. Let $\pi : A_1 \times \cdots A_r \longrightarrow A_1$

be the canonical projection on the first component and let $\varphi = \pi \circ \eta \circ \alpha$, i.e

$$\varphi : V \xrightarrow{\alpha} Alb(V) \xrightarrow{\eta} A_1 \times \cdots A_r \xrightarrow{\pi} A_1.$$

Let us simply use $A$ to denote $A_1$. As we have noted in the section on Albanese varieties, the group generated by $\alpha(V)$ is an Abelian variety and the universal property of $Alb(V)$ implies that it is the smallest Abelian variety that $\alpha(V)$ generates as a group. Therefore these two coincides; $<\alpha(V)>= Alb(V)$. It follows that $<\varphi(V)>= A$, so $\varphi : V \longrightarrow A$ is nonconstant.

Choose a nonsingular Zariski locally closed real algebraic curve $Z$ in $X$. Choose a point $x_0$ in $Z$ and let $G$ be a connected neighborhood of $x_0$ in $Z$, whose closure is compact. By Remark 1.5.1 there is a compact real algebraic curve $C'$ with a Zariski open subvariety $D$ isomorphic to $Z$. As we have discussed in Section 3.3, we can find a nonsingular projective complexification $C$ of $C'$. $D \cong Z$ is Zariski open in $C' \cong \mathbb{R}C$, so we can consider $Z$ as a Zariski open subvariety of the real part of a nonsingular projective complex algebraic curve. If $\psi : Z \longrightarrow D$ is the isomorphism then put $c_0 = \psi(x_0)$ and $N = \psi(G)$. Lastly, fix $n \in \mathbb{Z}^+$ as in Lemma 3.5.3 (and 3.5.4).

We can choose a small neighborhood $U$ of $y_0$ in $V(\mathbb{R})$ such that the set $\mathbb{R}A_0 \setminus \varphi^n(U^n)$ contains a nonempty open subset of $\mathbb{R}A_0$. Any smooth real algebraic variety is locally defined by means of analytic charts so shrinking $U$ if necessary we can find a real analytic diffeomorphism $\sigma : U \longrightarrow \mathbb{R}^m$ with $\sigma(y_0) = 0$, where $m = \dim V$. We know that $\varphi$ is nonconstant on $V$. However, $U$ is Zariski dense in $V$ and $\varphi$ is continuous in the Zariski topology so it follows that $\varphi$ is also nonconstant on $U$. Choose a point $y_1 \in U$ with $\varphi(y_1) \neq 0$ and consider its image in $\mathbb{R}^m$ under $\sigma$. As $\sigma$ is one-to-one, $\sigma(y_0) = 0$ and $\sigma(y_1)$ defines a line $L$ in $\mathbb{R}^m$ passing through the origin. Let $M$ be the one dimensional real analytic submanifold of $U$ with $\sigma(M) = L$. Thus it is diffeomorphic to $\mathbb{R}$, say, via $\phi$ and suppose that $\phi(y_0) = 0$. Furthermore, both $y_0$ and $y_1$ lie in $M$ so $\varphi$ is nonconstant on $M$.

Now we can construct a 'perverse' map $f : X \longrightarrow Y$. For instance we can take the following map: Let $i : X \hookrightarrow \mathbb{R}^N$ be an embedding of $X$. Since $X$ has positive dimension, there exist $x_j \in 1, \ldots, N$ such that the projection map onto $i$-th component is nonconstant around $x_0$. Let $p_j : \mathbb{R}^N \longrightarrow \mathbb{R}$ be this projection and define $h : \mathbb{R}^N \longrightarrow \mathbb{R}$ as $p_j(x) - p_j(x_0)$ for all $x \in X$. Put $f = \phi^{-1} \circ h \circ i$ but ensure that $y_1 \in f(M)$ (if not, use a self-diffeomorphism of $\mathbb{R}$ in the composition of $f$).

Thus we can define define a continuous map $f : X \longrightarrow Y = \mathbb{R}V$ with $f(X) \subseteq M$, $f(x_0) = y_0$ and $f$ is nonconstant on any neighborhood of $x_0$ in $Z$. By construction, $f$ is nonconstant on $G$ and $\varphi$ is nonconstant on $f(G) \subseteq M$, therefore $\varphi \circ f$ is nonconstant on $G$ and $(\varphi \circ f)^n$ is nonconstant on $G^n$. On the other hand since $f(X) \subseteq M$ then $\mathbb{R}A_0 \setminus (\varphi \circ f)^n(X^n)$ contains a nonempty open subset of $\mathbb{R}A_0$. Hence $(\varphi \circ f)^n(G^n)$ is far from being equal to $\{0\}$ or $\mathbb{R}A_0$. However by Lemma 3.5.4 for any regular map $g : X \longrightarrow Y$ the image of $(\varphi \circ g)^n(G^n)$ is either $\{0\}$ or $\mathbb{R}A_0$. This implies that $(\varphi \circ g)^n$ cannot be arbitrarily close to $(\varphi \circ f)^n$, implying that $(\varphi \circ g)$ cannot be arbitrarily close to $(\varphi \circ f)$. □

## 3.7 Insufficiency of the theorem

It is clear that the Theorem N not only provides a necessary condition for the approximation problem but also provides a practicable one. Because we do two concrete jobs: complexification and computing the first Betti number. Yet, this condition is too weak to insure validity of Theorem S. In this section we illustrate that.

We begin with the following observation: Let $X$ be a compact nonsingular real algebraic curve and let $Y$ be a compact nonsingular real algebraic variety. Let $H_1^{alg}(Y)$ be the subgroup of $H_1(Y, \mathbb{Z}_2)$ generated by the homology classes represented by the Zariski closed real algebraic curves in $Y$. Suppose that $\mathcal{R}(X, Y)$ is dense in $\mathcal{C}^\infty(X, Y)$. As $X$ is topologically disjoint union of circles, then for any 1-cycle $\alpha$ generating $H_1(Y, \mathbb{Z}_2)$ we can find a smooth map $f : X \longrightarrow Y$ with $Imf = \alpha$. Since we can approximate $f$ by regular functions then there is a close enough regular map $g$ homotopic to it. Because we can consider a tubular neighborhood of $Imf$ and if $Img$ is in it then we can define a linear homotopy inside this neighborhood. Thus $Img$ is homotopic to $\alpha$ and therefore $[\alpha] = g_*([X]) \in H_1^{alg}(Y)$, by definition. Hence the subgroup $H_1^{alg}(Y)$ is equal to $H_1(Y, \mathbb{Z}_2)$ when $\overline{\mathcal{R}(X, Y)} = \mathcal{C}^\infty(X, Y)$.

Now let $Y$ be a nonsingular surface in $\mathbb{RP}^3$ of $deg \geq 4$ which is not homeomorphic to $S^2$ or $\mathbb{RP}^2$. Then it is a complete intersection and hence $b_1(Y, \mathbb{C}) = 0$ by virtue of Lefschetz Theorem.

On the other hand Bochnak and Kucharz showed that ([7]) any 'general surface' (see [15], section on Kodaira number and the classification of surfaces) $Y \subset \mathbb{RP}^3$ of $deg \geq 4$ has $dim_{\mathbb{Z}_2} H_1^{alg}(Y, \mathbb{Z}_2) \leq 1$. By our assumption, $Y$ is homeomorphic to either a sphere with $g$ handles or to a connected sum of $k\mathbb{RP}^2$s due to classification of compact surfaces, where $g \geq 1$ and $k > 1$. But then $dimH^1(Y, \mathbb{Z}_2)$ is either $2g$ or $k$ and it is strictly greater than 1. Hence the observation we have done above and the conclusion of Bochnak and Kucharz imply that $\mathcal{R}(X, Y)$ can not be dense in $\mathcal{C}^\infty(X, Y)$.

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