# SOME REMARKS ON MASSEY PRODUCTS,
# TIED CLASSES
# AND
# THE LUSTERNIK-SCHNIRELMAN CATEGORY

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MEHMETCİK PAMUK



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Prof. Dr. Tayfur ÖZTÜRK
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Ersan AKYILDIZ
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Yıldıray OZAN
Supervisor

Examining Committee Members

Prof. Dr. Turgut ÖNDER

Assoc. Prof. Yıldıray OZAN (Supervisor)

Assist. Prof. Feza ARSLAN

Assist. Prof. Ergün YALÇIN

Instructor Dr. Muhiddin UĞUZ

# ABSTRACT

## SOME REMARKS ON MASSEY PRODUCTS, TIED CLASSES AND THE LUSTERNIK-SCHNIRELMAN CATEGORY

Mehmetcik Pamuk
M.Sc., Department of Mathematics
Supervisor: Assoc. Prof. Yıldıray Ozan

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In this thesis, we present the work of Claude Viterbo on the problem of determining the minimal number of critical points of a function on a Hilbert manifold. In accordance to C. Viterbo, the notion of tied cohomology classes and tie length are introduced, and a better lower bound for the number of critical points of a function is found.

Keywords: Hilbert Manifold, Tied Cohomology Classes, Tie Length.

# ÖZ

## MASSEY ÇARPIMLARI, BAĞLI SINIFLAR VE LUSTERNIK-SCHNIRELMAN KATEGORİSİ

Mehmetcik Pamuk
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Assoc. Prof. Yıldıray Ozan

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Bu tezde, Claude Viterbo'nun, bir Hilbert manifoldu üzerinde tanımlı bir fonksiyonun kritik noktalarının minimum sayısını belirleme problemi üzerindeki çalışmaları incelendi. Bağlı kohomoloji sınıfları ve bağlılık uzunluğu kavramları tanımlandı ve bir fonksiyonun kritik noktalarının sayısı için daha iyi bir alt sınır bulundu.

Anahtar Kelimeler: Hilbert Manifold, Bağlı Kohomoloji Sınıfları, Bağlılık Uzunluğu.

To My Family

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# CHAPTER 1

# Introduction and Preliminaries

Let $f : X \to \mathbb{R}$ denote a smooth real valued function on a smooth finite or infinite dimensional Hilbert manifold $X$. We will make three basic assumptions about $X$ and $f$:

(a) (Completeness) $X$ is a complete Riemannian manifold

(b) (Boundedness from below) The function $f$ is bounded below on $X$. We will let $B$ denote the greatest lower bound of $f$, so our assumption is that $B > -\infty$

(c) (Palais-Smale condition)(PS) Any sequence $(x_n)$ such that $f'(x_n) \to 0$ and $f(x_n)$ is bounded has a converging subsequence, $(x_{n_k}) \to p$ (by continuity, $f'(p) = 0$, so that $p$ is a critical point of $f$ )

Of course, if $X$ is compact then with any choice of Riemannian metric for $X$ all three conditions are automatically satisfied.

Let $X$ be a Hilbert manifold, and $f : X \to \mathbb{R}$ be a smooth function. We denote by $crit(f)$ the number of critical points of $f$. The minimal value of the cardinality of the set $crit(f)$ for fixed $X$ and variable $f$ is called the F-category of $X$ and it is denoted by $F(X)$.

**Remark 1.1.** Let $f$ be a smooth real valued function on a finite dimensional manifold $X$. A point $p \in X$ is called a critical point of $f$ if the induced map $f_* : TX_p \to T\mathbb{R}_{f(p)}$ is zero. If we choose a local coordinate system $(x^1, \dots, x^n)$ in a neighbourhood $U$ of $p$, this means that $\frac{\partial f}{\partial x^1}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0$. The number $f(p)$ is called a critical value of $f$.

We denote by $X^a$ the set of all points $p \in X$ such that $f(p) \leq a$, i.e., $X^a = \{p \in X \mid f(p) \leq a\}$. If $a$ is not a critical value of $f$ then it follows from implicit function theorem that $X^a$ is smooth manifold with boundary. The boundary $f^{-1}(a)$ is a smooth submanifold of $X$.

A critical point $p$ is called non-degenerate if the matrix $(\frac{\partial^2 f}{\partial x^i \partial x^j}(p))$ is non-singular. It can be checked directly that non-degeneracy does not depend on the coordinate system (see [11] ).

Note that for $-\infty < a \leq b < +\infty$, $(PS)$ implies that the set of critical points in the closure of $X^b - X^a$ is compact, and the set of critical values is closed (in fact; if we denote by $K$ the set of critical points of $f$ and $K_c$ the set of critical points at level $c$, then the restriction of $f$ to $K$ is proper. In particular, for any $c \in \mathbb{R}$, $K_c$ is compact: To see this, we must show that $f^{-1}([a, b]) \cap K$ is compact, i.e. if $(x_n)$ is a sequence of critical points with $a \leq f(x_n) \leq b$ then $(x_n)$ has a convergent subsequence. However, since $f'(x_n) = 0$ this is immediate from (PS). Since proper maps are closed the set $f(K)$ of critical values of $f$ is a closed subset of $\mathbb{R}$).

Let $F(X)$ be the minimal number of critical points for a function bounded from below and satisfying $(PS)$ on $X$. (If $X$ has a boundary $\partial$X, we assume that $\lim_{x \to \partial X}$ f(x)= $+\infty$). More generally, for a submanifold $A$ (with boundary) of $X$, $F(X, A)$ is defined as the minimal number of critical points in $X - A$ for a non negative function on $X$, satisfying $(PS)$ and vanishing on $A$. Changing $f$ to $1/f$, we see that it is also the number of critical points for a function $f$ on $X$ - $A$ such that $\lim_{x \to A}$f(x)= $+\infty$.

A lower bound for $F(X)$ was given by Lusternik-Schnirelman, denoted by $cat(X)$, is the minimal number of open subsets, contractible in $X$, needed to cover $X$. Note that the definition of $cat(X)$; is valid on any topological space, not just a manifold.

$cat(X) = 1$ if and only if $X$ is contractible. Also $cat(X) \leq 2$ if $X$ is a suspension, in particular spheres have category 2.

Recall that $f : X \to Y$ and $g : Y \to X$ be maps such that $fog \simeq 1$. Then $f$ is a homotopy left inverse of $g$ and $g$ is a homotopy right inverse of $f$.

**Definition 1.1.** The space $X$ dominates the space $Y$ if there exists a map $Y \to X$ which admits a homotopy left inverse.

$cat(X)$ is an invariant of homotopy type. This follows at once from

**Proposition 1.1.** If $X$ dominates $Y$ then $cat(X) \geq cat(Y)$.

*Proof.* Let $f : X \to Y$ and $g : Y \to X$ be maps such that $fog \simeq 1$. If $U \subset X$ is open then $V = g^{-1}U \subset Y$ is open. If $U$ is contractible in $X$ then $f|U$ is nullhomotopic, hence $fog|V$ is nullhomotopic, hence $V$ is contractible in $Y$, since $fog \simeq 1$. Thus any categorical open covering of $X$ pulls back to a categorical open covering of $Y$. □

It might seem more natural, in some ways, to define category using subsets of $X$ which are contractible in themselves, instead of contractible in $X$. The number thus defined is known as the *strong category*, and written $CatX$. However R. H. Fox has constructed examples in [5], showing that $CatX$ is not a homotopy invariant.

T. Ganea, in [6] has studied the minimum value of $CatX$, for all spaces of the same homotopy type of $X$, and has shown that this minimum is equal to $catX - 1$ or $catX$, for a large class of spaces $X$, and examples of both possibilities are known(see [7]).

The relation between $cat(X)$ and $F(X)$ has two special cases, see for example [17] :

1. Let $X$ be a closed n-manifold; does $cat(X) = 2$ imply $F(X) = 2$? This is equivalent with the Poincaré conjecture, because $cat(X)$ is 2 if and only if

$X$ is a homotopy sphere and $F(X)$ is 2 if and only if $X$ is homeomorphic with an n-sphere. The answer is affirmative for $n \geq 5$ by Smale ([13,14]).

**Theorem 1.1.** (Reeb) If $X$ is a compact manifold and $f$ is a differentiable function on $X$ with only two critical points, both of which is non-degenerate, then $X$ is homeomorphic to a sphere.

**Remark 1.2.** The theorem remains true even if the critical points are degenerate. However, the proof is more difficult.

2. Let $X$ be a compact manifold with boundary. Does $cat(X) = 1$ imply $F(X) = 1$? This problem is very close to the following problem :
Is every contractible compact manifold a disc? This is related with the Poincaré conjecture and (partially) solved together with the Poincaré conjecture. The answer is affirmative if $\dim(X) \geq 6$ and $(X, \partial X)$ is 2-connected (again by Smale).

A well-known consequence of Lusternik and Schnirelman's work is that, the cuplength of $X$ (denoted by $cl(X)$), defined by

$$cl(X) = \max \{k : \exists\, \beta_1, \ldots, \beta_{k-1} \in H^*(X) - H^0(X),\ \beta_1 \ldots \beta_{k-1} \neq 0\ \}$$

satisfies

$$cat(X) \geq cl(X)$$

To see this, let $X = U_1 \cup U_2 \cup \ldots \cup U_l$, where each $U_i$ is contractible in $X$ then $l \geq cat(X)$. Now let $a_i \in H^*(X) \cong H^*(X, pt) \cong H^*(X, U_i)$ so $a_i \cup a_2 \cup \ldots \cup a_l \in H^*(X, U_1 \cup U_2 \cup \ldots \cup U_l) = H^*(X, X) = 0$. Hence, $l \geq cl(X)$ which implies $cat(X) \geq cl(X)$.

Note that $cl(X)$ depends on the choice of the coefficient ring (but inequality is valid for any coefficient ring).

In the case of a manifold, the number of charts in a chart structure is an upper bound for the category (indeed the strong category), [9].

**Example 1.1.** Consider $\mathbb{RP}^n$. There is a standard chart structure consisting of the subsets $V_i = \{[x_0 : x_1 : \ldots : x_i : \ldots : x_n] \mid x_i \neq 0\}$ for $i = 0, \ldots, n$. Hence $cat(\mathbb{RP}^n) \leq n+1$. The mod 2 cohomology ring is a truncated polynomial ring of height $n$, $cat(\mathbb{RP}^n) \geq n+1$. We conclude that $cat(\mathbb{RP}^n) = n+1$.

Now we need another definition.

**Definition 1.2.** Let $\alpha \in H^*(X^b, X^a) - \{0\}$, and set

$$c(\alpha, f) = \inf\{\lambda : \alpha \neq 0 \text{ in } H^*(X^\lambda, X^a)\}$$

Here, note that $c(\alpha, f)$ is indeed independent of $a$, if $a$ is near -$\infty$. We then have the following theorem.

**Theorem 1.2.** The number $c(\alpha, f)$ is a critical value of f. If $\beta \in H^*(X)$, and $\alpha.\beta \neq 0$ in $H^*(X)$, then

$$c(\alpha \cdot \beta, f) \geq c(\alpha, f) \qquad (*)$$

Moreover, in case $(*)$ is an equality, $K_c$ which is the set of critical points of $f$ at level $c = c(\alpha, f) = c(\alpha.\beta, f)$; then $\beta$ is nonzero on $H^*(K_c)$. As a result, $\dim(K_c) \geq \deg(\beta)$, and in particular if $\deg(\beta) \neq 0$, *then*
$K_c$ is uncountable (hence infinite).

From the theorem, it immediately follows that $F(X) \geq cl(X)$. To see this let $cl(X) = k$, that is $\exists\, \beta_1, ...., \beta_{k-1} \in H^*(X) - H^0(X)$ such that $\beta_1 \cdots \beta_{k-1} \neq 0$, then $c(\beta_1, f) \leq c(\beta_1 \cup \beta_2, f) \leq \cdots \leq c(\beta_1 \cup \cdots \cup \beta_{k-1}, f)$. If there is any equality then the set of critical points is infinite and if all of them are strict inequalities we have $k$ distinct critical values so at least $k$ distinct critical points.

**Remark 1.3.** Usually the theorem is stated as follows: If $(*)$ is an equality, then $K_c$ is infinite.

**Remark 1.4.** If $K_c$ is a manifold, the statement clearly implies in case of equality that $\dim(K_c) \geq \deg(\beta)$.

**Remark 1.5.** We may extend the definition of $c(\alpha, f)$ to a set of cohomology classes as follows: Let $E$ be a subset of $H^*(X)$, then

$$c(E, f) = \sup\{c(u, f) : u \in E\}.$$

Our theorem may then be generalized to $c(E \cdot F, f) \geq c(E, f)$ with equality only if for all $\beta \in F$, $\beta$ is nonzero on $K_c$. Here $E \cdot F$ is the set of all products of an element in $E$ and one in $F$.

The proof of above and subsequent theorem is based on the following lemma.

**Lemma 1.3 (Main Lemma).** Let $K_c$ be the set of critical points at level $c$. Then for any neighbourhood $V$ of $K_c$, there exists $\epsilon > 0$ and an isotopy $\phi_t : X^{c+\epsilon} \longrightarrow X^{c+\epsilon}$ such that $\phi_0 = Id$ and $\phi_1 : X^{c+\epsilon} \longrightarrow X^{c-\epsilon} \cup V$.

*Proof.* Let $V$ be a neighbourhood of $K_c$. According to the Palais-Smale condition, there exist $\eta > 0$ such that on ( $X^{c+\epsilon_0} - X^{c-\epsilon_0}) \cap CV$, where $CV$ denotes the complement of $V$, we have $|df(x)| \geq \eta$ (suppose not, then for each positive integer $n$ we could choose an $x_n$ in ( $X^{c+\frac{1}{n}}$ - $X^{c-\frac{1}{n}}) \cap CV$ such that $|df(x_n)| < \eta$. By (PS), a subsequence of $(x_n)$ would converge to a critical point $p$ of $f$ with $f(p) = c$, so $p \in K_c$ and eventually the subsequence must get inside the neighborhood $V$ of $K_c$, a contradiction since $K_c$ is compact). Let $\phi_t$ be the flow of $Y = -\nabla f(x)$. We want to compute the length of a trajectory of $\phi_s$ contained in ( $X^{c+\epsilon_0} - X^{c-\epsilon_0}) \cap CV$. It is given by

$$\begin{aligned} \int_{s_0}^{s_1} |Y(\phi_s(x))| ds &\leq \frac{-1}{\eta} \int_{s_0}^{s_1} df(\phi_s(x)) Y(\phi_s(x)) ds \\ &= \frac{-1}{\eta} [f(\phi_{s_1}(x)) - f(\phi_{s_0}(x))] \leq \frac{2\epsilon}{\eta} \end{aligned}$$

Thus, if we assume $\epsilon < \frac{\eta}{2}\,\delta$, then this length is less than $\delta$. Now let $U$ be a neighbourhood of $K_c$, and $V$ another neighbourhood with

$$K_c \subset V \subset \overline{V} \subset U$$

Since $K_c$ is compact, $\overline{V}$ may be assumed to be compact, and then $\delta = d(V, CU) > 0$.

If we add to $V$ the portion of the trajectories of $Y$, which go through $V$ and are contained in ( $X^{c+\epsilon_0} - X^{c-\epsilon_0}$), we get a neighbourhood $W$ of $K_c$, such that $W \subset U$, and $W \cap (X^{c+\epsilon_0} - X^{c-\epsilon_0})$ is saturated for $Y$ (i.e., if $z \in W \cap (X^{c+\epsilon_0} - X^{c-\epsilon_0})$ and $\phi_s(z) \in (X^{c+\epsilon_0} - X^{c-\epsilon_0})$, then $\phi_s(z) \in W$).

In other words, since $\frac{d}{ds} f(\phi_s(x)) = -|\nabla f(\phi_s(x))|^2 \leq -\eta^2$ outside $W$, we have for $z \in X^{c+\epsilon_0} - X^{c-\epsilon_0})$ and $s_0 = \frac{2\epsilon}{\eta^2}$ that either :

(*i*) $\forall\ s \geq s_0\ \phi_s(z) \in X^{c-\epsilon}$ if the trajectory of $z$ does not meet $W$,

or

(*ii*) $\forall\ s \geq s_0\ \phi_s(z) \in W \cup X^{c-\epsilon}$ if the trajectory of $z$ meets $W$.

□

The Lemma can be restated as follows ( [12] is a good reference for the arguments below) :

**Theorem 1.4.** (First Deformation Theorem) Let $V$ be any neighbourhood of $K_c$ in $X$. Then for $\epsilon > 0$ sufficiently small $\phi_1(X^{c+\epsilon} \setminus V) \subseteq X^{c-\epsilon}$

**Corollary 1.1.** If $c$ is a regular value of $f$ then, for some $\epsilon > 0$, $\phi_1(X^{c+\epsilon}) \subseteq X^{c-\epsilon}$

*Proof.* Since $K_c = \phi$ we can take $V = \phi$. □

Let $\mathcal{F}$ denote a nonempty family of non compact subsets of $X$. We define $c_m(f)$ minimax $(f, \mathcal{F})$, the minimax of $f$ over the family $\mathcal{F}$, to be the infimum over all $F$ in $\mathcal{F}$ of the maximum of $f$ on $F$. Now the maximum value of $f$ on $F$ is just the smallest $c$ such that $F \subseteq X^c$. So $c_m(f) =$ minimax $(f, \mathcal{F})$ is the smallest $c$ such that, for any positive $\epsilon$, we can find an $F$ in $\mathcal{F}$ with $F \subseteq X^{c+\epsilon}$. Since the family $\mathcal{F}$ is said to be invariant under the time flow of $-\nabla f$ if whenever $F \in \mathcal{F}$ and $t > 0$ it follows that $\phi_t(F) \in \mathcal{F}$

**Proposition 1.2.** (Minimax Principle) If $\mathcal{F}$ is a family of compact subsets of $X$ invariant under the positive time flow of $-\nabla f$ then minimax $(f, \mathcal{F})$ is a critical value of $f$.

*Proof.* By definition of minimax we can find an $F$ in $\mathcal{F}$ with $F \subseteq X^{c+\epsilon}$. If $c$ were a regular value of $f$, then by corollary above, $\phi_1(X^{c+\epsilon}) \subseteq X^{c-\epsilon}$ and $\phi_1(F) \subseteq X^{c-\epsilon}$. However, since $\mathcal{F}$ is invariant under the positive time flow of $-\nabla f$, $\phi_1(F)$ is also in the family $\mathcal{F}$ and it follows that minimax $(f, \mathcal{F}) \leq c - \epsilon$, a contradiction. □

Suppose that $cat(X) \geq m$ and define for a subset $A$ of $X$ $cat(A, X)$ to be minimal number of open subsets, contractible in $X$ needed to cover $A$. ($cat(X) = cat(X, X)$). Let $\mathcal{F}_m = \{X^a \mid cat(X^a, X) \geq m\}$. Note that $\mathcal{F}_m$ contains $X$ itself so it is nonempty. Now let

$$\widetilde{\mathcal{F}_m} = \{\phi_t(X^a) \mid cat(X^a, X) \geq m\}$$

Since minimax$(f, \mathcal{F}_m) =$ minimax$(f, \widetilde{\mathcal{F}_m})$ and by The Minimax Principle minimax$(f, \widetilde{\mathcal{F}_m})$ is a critical value of $f$ so is minimax$(f, \mathcal{F}_m)$.(By the monotonicity of $cat(\quad, X)$, we can equally well define $c_m(f)$ by the formula $c_m(f) = \inf\{a \in \mathbb{R} \mid cat(X^a, X) \geq m\}$.) Thus for $m = 0, 1, ..., cat(X)$, $c_m(f) =$ minimax$(f, \mathcal{F}_m)$ is a critical value of $f$.

Now $\mathcal{F}_{m+1}$ is clearly a subset of $\mathcal{F}_m$, so $c_m(f) \leq c_{m+1}(f)$. Of course equality can occur, for example when $f$ is constant. However, as the next result shows, this will be compensated for by having more critical points at this level.

**Theorem 1.5.** (Lusternik-Schnirelman Multiplicity Theorem) If $c_{n+1}(f) = c_{n+2}(f) = \cdots = c_{n+k}(f) = c$ , then there are $k$ critical points at level $c$. Hence if $1 \leq m \leq cat(X)$ then $f$ has at least $m$ critical points at or below level $c_m(f)$. In particular every smooth function $f : X \to \mathbb{R}$ has at least $cat(X)$ critical points altogether.

*Proof.* Suppose that there are only a finite number $r$ critical points $x_1, ..., x_r$ at the level $c$ and choose open neighborhoods $U_i$ of the $x_i$ whose closures are disjoint closed disks (hence in particular contractible). Putting $O = O_1 \cap ... \cap O_r$ , clearly $cat(O, X) \leq r$ . By the First Deformation Theorem, for some $\epsilon > 0 X^{c+\epsilon} - O$ can be deformed into $X^{c-\epsilon}$. Since $c - \epsilon < c = c_{n+1}$, $cat(X^{c-\epsilon}, X) < n+1$, and so $cat(X^{c+\epsilon} - O) \leq n$. Thus by the subadditivity and monotonicity of $cat$, $cat(X^{c+\epsilon}, X) \leq cat((X^{c+\epsilon} - O) \cap O, X) \leq n + r$ ,and hence $c < c + \epsilon < \inf \{a \in \mathbb{R} : cat(X^a, X) > n + r + 1\} = c_{n+r+1}(f)$. Since on the other hand $c = c_{n+k}(f)$, (and $c_m(f) \leq c_{m+1}(f)$) it follows that $n + r + 1 > n + k$, so $r \geq k$. □

**Remark 1.6.** Another motivation for studying the F-category of a manifold is the Arnold conjecture.

**Arnold Conjecture:** Let $\text{Arn}(X, \omega)$ be the minimum number of critical fixed points for any Hamiltonian symplectomorphism of $M$. Then, the following inequality holds for every closed symplectic manifold $(X, \omega)$ :

$$Arn(X, w) \geq F(X)$$

Although, the Arnold Conjecture was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Futaya-Ono, Lin-Tian using Floer homology; Yuli B. Rudyak and John Oprea, using the Lusternik-Schnirelman category, proved that (in 1998) for any closed symplectic manifold whose symplectic form vanishes on the image of the Hurewicz map, the required inequality holds.

In practice, one is interested in good lower bounds for $F(X)$, as this number is actually very hard to compute. As far as we know, the only general lower bound for $F(X)$ or *cat(X)* was given by *cl(X)* (at least if $X$ is simply connected ), until the paper by Fadell and Husseini [1]. They proved that if $M$ is a closed manifold, $\bigwedge M$ its free loop space, i.e., $\bigwedge M = C^0(S^1, M)$, then $F(M) = +\infty$, while $cl(\bigwedge M)$ may be finite (this is the case if $M = S^2$ or $\mathbb{CP}^n$). Their argument is based on a clever use of the fibration $\bigwedge M \to M$ given by $u \to u(1)$ using the fact that fiber $\Omega M = \{u \in \bigwedge M : u(1) = x_0\}$ has infinite cup length.

**Remark 1.7.** It is a standard result that, if $X$ is a simply connected compact manifold and $\Omega(X) = \Omega(X, x_0)$ is the space of based loops on $X$ (based at $x_0$), then the Lusternik-Schnirelman category $cat(\Omega(X)) = \infty$. This follows from the classical result that the real (or rational ) cohomology of $\Omega(X)$ has nontrivial cup products of arbitrary high length. An inspection of the proof convinces us that compactness is not required for the proof of this result. All that is required is that the real (or rational) cohomology $H^*(X)$ be finitely generated and for some $i > 0$ , $H^i(X) \neq 0$. However, for the free loop space $\bigwedge(X) = \{\alpha \in X^I : \alpha(0) = \alpha(1)\}$, it is not necessary that the cohomology of $\bigwedge(X)$ has nontrivial cup products.

Our aim was then to see whether, at least for simple spaces, the cohomology of $\bigwedge M$ could explain this phenomenon. We were led to use Massey's triple product to find more critical levels for the energy function on $\bigwedge M$. Eventually, we were led to define a new invariant, the *tie length*, which gives a lower bound for the $F$-category. It is not clear whether this new invariant is always infinite for a free loop space, but it is at least so for complex projective spaces and even dimensional spheres and many other examples are easy to construct in view of Chapter 3. On the other hand, this invariant has certain stability properties explained in Chapter 2, which make it more widely applicable than the category. In particular this is usually the case for applications to symplectic topology. We first recall the definition of Massey product.

**Definition 1.3.** Let $\alpha$, $\gamma \in H^*(X)$, $\beta \in H^*(X, A)$ be such that $\alpha \cdot \beta =$

$\gamma \cdot \beta = 0$. (This is just what happens when the cup product will not give critical points anymore). Let $a, b, c$ be cochain representatives of $\alpha, \beta, \gamma$; we may thus write $a \cdot b = d\lambda$, $b \cdot c = d\mu$ for some cochains $\lambda, \mu$ vanishing on $A$. Then $a \cdot \mu - (-1)^{deg(a)} \lambda \cdot c$ is closed (its differential is, up to sign , $a(bc) - (ab)c$), and its cohomology class is well defined in $H^*(X, A)/[\alpha \cdot H^*(X, A) + \gamma \cdot H^*(X, A)]$.

We denote by $< \alpha, \beta, \gamma >$ the class we defined in the quotient of $H^*(X, A)$; it is called Massey's triple product (of $\alpha, \beta, \gamma$ ). Similarly, we may define $< \alpha, \beta, \gamma >$ for $\alpha$ ,$\gamma \in H^*(X, A)$ and $\beta \in H^*(X)$.

**Remark 1.8.** Massey products are functorial in the sense that if $f\colon X \to Y$ is a map such that $f(A) \subset B$, then the induced map $f^*\colon H^*(Y, B) \to H^*(X, A)$ sends $< \alpha, \beta, \gamma >$, an element of $H^*(Y, B)/[\alpha.H^*(Y, B) + \gamma.H^*(Y, B)]$ to $< f^*\alpha, f^*\beta, f^*\gamma >$, an element of $H^*(X, A)/[f^*\alpha.H^*(X, A) + f^*\gamma.H^*(X, A)]$. This means that

$$f^*(< \alpha, \beta, \gamma >) \subset < f^*\alpha, f^*\beta, f^*\gamma >.$$

**Remark 1.9.** There are also higher-order Massey products, see [10]. The product $< u_1, ..., u_k >$ is defined only if $< u_1, ..., u_{k-1} > = < u_2, ..., u_k > = 0$ (but this is condition is not sufficient). We shall not deal with them here, but our results and proofs extend to these higher-order products.

**Remark 1.10.** It is known that the following relation holds : For $p$ odd prime $u \in H^{2m+1}(X; \mathbb{Z}_p)$, then $\beta P^m(u) \in < u, ..., u >$. Here $\beta$ is the mod $p$ Bockstein, and $P^m$ is the mod $p$ Steenrod $m^{th}$ power. It is then natural to ask whether the following hold.

(1) If $\beta P^m(u) \neq 0$, is it true that the critical levels associated to $u$ and $\beta P^m(u)$ are distinct unless the common critical level contains infinitely many critical points? In other words, if $0 \in < u, ..., u >$ then our method (see section 2) does not apply; however, $\beta P^m(u)$ may still be nonzero. Is

that enough to get distinct critical points? (Fadell and Husseini give an answer to this question in a positive way in [2]).

(2) Same question with $\beta P^j(u)$ for $j \neq m$. However, this cannot be expected for all $j$, since the Steenrod operations are stable operations (in a different sense than Massey products!). As a result, we have, for instance, that the Bockstein operation is nonzero on the lens spaces $L(1,p)$. Hence it is also nonzero on its suspension. However, the category of a suspension is at most two, while this would give us that it is at least three. Thus if there is any hope of a positive answer to our question, it seems that $j$ has to be a function of $m$.

**Remark 1.11.** In a space with category at most two, not only all cup products, but also all Massey products should vanish (see [9]).

**Remark 1.12.** Fadell and Husseini proved that if $k$ is a field with suitable (nonzero) characteristic, $H^*(\Lambda X, k)$ has infinite cup length for $X = S^{2n}, \mathbb{CP}^n$, etc. However, cohomology computations in nonzero characteristic are usually very hard, since one cannot make use of Sullivan's minimal model. We will see that on the contrary, Massey products in rational cohomology are easily detected on the minimal model.

# CHAPTER 2

# TIED CLASSES

In this section we shall try to partially answer the following two questions [18]:

(i) Given a function $f$ on the compact manifold $X$, and a class $x$ in $H^*(X)$, which classes $y$ satisfy $c\ (y, f) \geq c\ (x, f)$ for all f ? We shall then say that $y$ is *weakly tied* to $x$.

(ii) What are the classes $y$, weakly tied to $x$, such that equality cannot hold unless $H^j(K_c) \neq 0$ for some $j > 0$? We shall then say $y$ is *strongly tied* to $x$.

Let us recall the definition of Thom isomorphism. Let $E \to X$ be an orientable vector bundle of dimension $n$. Choose a metric on $E$. Let $D(E) \subset E$ be the unit disk subbundle with respect to this metric, and let $S(E)$ be the unit sphere subbundle. We have a relative fibration $(D(E), S(E)) \to X$ whose fiber is the pair $(D^n, S^{n-1})$. There is a class $u \in H^n(D(E), S(E))$ so that $H^*(X) \cong H^*(D(E)) \to H^*(D(E), S(E))$ (cupping with $u$) is an isomorphism.

There are stable versions of the definitions above: If $x, y$ are cohomology classes on $X$, we may consider an orientable vector bundle $E$ on $X$, and the image $u \cup x$, $u \cup y$ of $x, y$ by the Thom isomorphism in $H^*(D(E), S(E))$, where $D(E), S(E)$ are the canonical disk and sphere bundle associated to $E$.

If for any orientable bundle $E$, $u \cup y$ is strongly(resp., weakly) tied to $u \cup x$, we shall say that $x$ and $y$ are *stably tied*(resp., *weakly stably tied*).

Note that $y$ is weakly tied to $x$, in particular if the following property holds: For any map $h : P \to X$, we have $h^*(x) = 0 \Longrightarrow h^*(y) = 0$.

We may similarly define the notion of tied sets of cohomology classes. We may thus set the next definition.

**Definition 2.1.** We define $Tl(M)$ to be the maximal integer $k$ such that there is a sequence of cohomology classes $\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \beta_{k-1}, \alpha_k$ such that $\beta_j$ is weakly tied to $\alpha_j$ and $\alpha_{j+1}$ is strongly tied to $\beta_j$. The number $tl(M)$ is defined similarly, with strongly tied replaced by stably tied and weakly tied by stably weakly tied (see [18]).

**Remark 2.1.** Most of the time, we only need the case $\beta_j = \alpha_j$.

**Remark 2.2.** One of the motivations for introducing the stability property comes from symplectic topology (see, for example, [19] ). There one deals with functions on a vector bundle which are quadratic at infinity. This means that there is a function $Q(x, \xi)$ ($x$ is the base variable, $\xi$ the fiber variable ) which for fixed $x$ is a nondegenerate quadratic form, such that $f(x, \xi) - Q(x, \xi)$ has compact support.

The simplest example is given, of course, by the following corollary.

**Corollary 2.1.** The classes $x$ and $y = x \cup z$ are strongly tied, provided of course $y \neq 0$, and $\deg(z) > 0$.

This is nothing else than a rephrasing of Theorem 1.1.

Clearly, $cl(X) \leq tl(X) \leq Tl(X)$.

Another example is given through cohomology operations. A cohomology operation is a natural map, $\Psi : H^*(X, R_1) \to H^*(X, R_2)$ where $R_1, R_2$ are rings, satisfying certain naturality assumptions. The only fact we need is that if $h^*(x) = 0$, we have $h^*(\Psi(x)) = \Psi(h^*(x)) = 0$ (this is the naturality assumption). Applying this to the inclusion $i_\lambda : X^\lambda \to X$, we easily get $c(\Psi(x), f) \geq c(x, f)$. This is the next proposition.

**Proposition 2.1.** Given a cohomology operation, $\Psi$, $\Psi(x)$ is always weakly tied to $x$.

Let us define the stable $F-$ *category* of $M$, $SF(M)$, to be the minimal number of critical points for a function $f$ defined on a vector bundle over $M$, coinciding at infinity with a nondegenerate quadratic form in the fibers (see[18]).

Returning to the notion of tie length, an obvious result is the next proposition.

**Proposition 2.2.** We have the inequality $F(M) \geq Tl(M)$ and $SF(M) \geq tl(M)$.

The rest of the chapter is devoted to giving examples of tied cohomology classes.

**Proposition 2.3.** Let $x, y \in H^*(M)$ with $\deg(y)$ odd and $x \cdot y = d\eta$. Then $y \cdot \eta$ is closed and defines a cohomology class strongly tied to $y$. To be precise, we have if $c\ (y, f) = c\ (y \cdot \eta, f) = c$ that $H^j(K_c)$ is nonzero for some $j \in \{\deg(x), \deg(x+y)-1\}$.

*Proof.* We set $c = c(y, f)$ and shall prove, assuming $H^j(K_c)$ vanishes for $j$ in the above set, that $c(y \cdot \eta, f) > c + \epsilon$. According to our Main Lemma, we identify $M^{c+\epsilon}$ with $M^{c-\epsilon} \cup V$, where $V$ is a neighbourhood of $K_c$. From the definition of $c$, we get a cochain $\psi$ that we may assume to be defined on all of $M$, such that $d\psi = y$ on $M^{c-\epsilon}$. Now change $y$ to $y' = y - d\psi$, and $\eta$ to $\eta' = \eta - (-1)^{\deg(x)} x \cdot \psi$. Then we have that $y \cdot \eta$ and $y' \cdot \eta'$ are cohomologous.

Indeed,

$$(y-d\psi)\cdot(\eta-(-1)^{\deg(x)}x\cdot\psi) = y\cdot\eta-d\psi\cdot\eta-(-1)^{\deg(x)}y\cdot x\cdot\psi+(-1)^{\deg(x)}d\psi\cdot x\cdot\psi$$
$$= y\cdot\eta - d(\psi\cdot\eta) + (-1)\deg(\psi)\psi\cdot d\eta - (-1)^{\deg(x)}y\cdot x\cdot\psi + (-1)^{\deg(x)}d\psi\cdot x\cdot\psi$$
$$= y\cdot\eta - d(\psi\cdot\eta) + \psi\cdot x\cdot y - x\cdot y\cdot\psi + (-1)^{\deg(x)}d\psi\cdot x\cdot\psi$$
$$= y\cdot\eta - d(\psi\cdot\eta) + 1/2(-1)^{\deg(x)}d(x\cdot\psi\cdot\psi)$$

Similarly, if we change $x$ to $x' = x + d\alpha$ and $\eta$ to $\eta' = \eta + \alpha\cdot y$ the cohomology classes represented by $y\cdot\eta$ and $y\cdot\eta'$ are same (note that we have $x'\cdot y = x\cdot y + d(\alpha\cdot y)$
$= d(\eta+\alpha\cdot y))$. The same is true again if, without changing $x$ or $y$, we add to $\eta$ an exact cochain.

Now we may choose $y'$ to be zero in $M^{c-\epsilon}$ and $x'$ to be zero in $V$; we have $d\eta' = x'y' = 0$ in $M^{c-\epsilon}\cup V$. Now $\eta'$ is closed in $V$. Then, because $H^{deg\eta}(K_c) = 0$, we have for $V'$ a subset of $V$ containing $K_c$ but small enough, $\eta'$ vanishes in $H^{deg(\eta)}(V')$=0, so we may write $\eta' = d\rho$ in $V'$. Then $y'(\eta'-d\rho)$ is cohomologous to zero $y'\eta'$. However, it is clear that $y'(\eta'-d\rho)$ vanishes on $M^{c-\epsilon}\cup V'$; thus $y\cdot\eta$ is cohomologous to zero in $H^(M^{c+\epsilon})$, and finally get $c(y\cdot\eta, f)\geq c+\epsilon$.

□

**Remark 2.3.** The class of $y\cdot\eta$ is nothing else than the class of $< y, x, y >$. The only difference is that in principle $< y, x, y >$ is a family of cohomology class (differing by an element in $y\cdot H^*(M)$), while we proved that the class of $y\cdot\eta$ is independent of any choice. In fact, a more careful study of Massey products shows that in many cases where some class is repeated in the product, the indeterminacy on the resulting Massey product may be reduced. In particular, this is the case for $< x, x, ..., x >$ (see [10]).

**Proposition 2.4.** Let $\alpha,\gamma$ in $H^*(M^b, M^a)$ and $\beta\in H^*(M)$. Then

$$c\ (<\alpha,\beta,\gamma>,f) \geq \ \min\ (c\ (\alpha,f), c\ (\gamma,f)).$$

Equality implies that $H^j(K_c) \neq 0$ for
$j \in \{\deg(\beta), \deg(\alpha)+\deg(\beta)-1, \deg(\beta)+\deg(\gamma)-1\}$.

*Proof.* Assume $a, c$ are representatives of $\alpha, \gamma$, vanishing on $M^{c-\epsilon}$ (with $c = \min\ (c\ (\alpha,f), c\ (\beta,f)))$ and assume $b=0$ on $V$, then

$$ab = d\lambda \text{ vanishes on } M^{c-\epsilon} \cup V,$$
$$bc = d\mu \text{ vanishes on } M^{c-\epsilon} \cup V.$$

Now assume the cohomology classes of $\gamma$ and $\mu$ are zero on $V$. We may assume that these cochains actually vanish on $V$, hence $a\mu \pm c\mu = 0$ on $M^{c-\epsilon} \cup V$. We thus get an element in $<\alpha,\beta,\gamma>$ that vanishes in $H^*(M^{c+\epsilon})$, so $c\ (<\alpha,\beta,\gamma>,f) = sup\{c\ (u,f) : u \in <\alpha,\beta,\gamma>\} > c$.

Otherwise, one of the cohomology classes of $b, \lambda$, or $\mu$ is nonzero in $H^*(V)$, and since we may choose for $V$ arbitrarily small neighbourhood of $K_c$, our proof is complete.

□

**Corollary 2.2.** Assume in the above proposition that, moreover, $\alpha$ is weakly tied to $\gamma$. Then the right-hand side of the above inequality is equal to $c\ (\gamma, f)$; thus $<\alpha,\beta,\gamma>$ is strongly tied to $\gamma$.

Note that in particular, we allow the case $\alpha = \gamma$. We now compare $c\ (<\alpha,\beta,\gamma>,f)$ to $c\ (\beta,f)$ , where $\beta \in H^*(M^b, M^a)$ and $\alpha,\gamma \in H^*(M)$.

**Proposition 2.5.** Assume $<\alpha,\beta,\gamma>$ to be nonzero. Then, for $c = c\ (\beta,f)$, there is a critical level strictly above $c$, unless $H^j(K_c) \neq 0$ for some $j \in \{\deg(\alpha), \deg(\gamma)\ \}$.

**Remark 2.4.** We cannot claim in general that $<\alpha,\beta,\gamma>$ is even weakly tied to $\beta$.

*Proof.* Again, we may assume $b = 0$ on $M^{c-\epsilon}$ and $a = c = 0$ on $V$. Then $ab = d\lambda$ and $bc = d\mu$ vanish on $M^{c-\epsilon} \cup V$, this last set being identified to $M^{c+\epsilon}$. We may consider $\widetilde{\beta}$ the class of $b$ in $H^*(M, M^{c-\epsilon})$. We shall distinguish two cases. We may choose $\lambda$ and $\mu$ such that their cohomology classes on $M^{c+\epsilon}$ vanish. Then we have $\alpha \cdot \widetilde{\beta} = \widetilde{\beta} \cdot \gamma = 0$ in $H^*(M, M^{c+\epsilon})$. Thus $< \alpha, \widetilde{\beta}, \gamma >$ is well defined in $H^*(M, M^{c+\epsilon})$ and its image in $H^*(M)$ is $< \alpha, \beta, \gamma >$, which does not contain zero by assumption. In this case, since $c\ (< \alpha, \widetilde{\beta}, \gamma >, f)$ is nonzero in $H^*(M, M^{c+\epsilon})$ by definition $c\ (< \alpha, \widetilde{\beta}, \gamma >, f) \geq c + \epsilon$. Otherwise, either $\alpha \cdot \widetilde{\beta}$ or $\widetilde{\beta} \cdot \gamma$ is nonzero in $H^*(M, M^{c+\epsilon})$ and $c\ (\alpha \cdot \widetilde{\beta}, f) \geq c + \epsilon$. □

Since we do not know in which case we end up, we may not infer that $c\ (< \alpha, \beta, \gamma >, f) \geq c + \epsilon$.

**Remark 2.5.** One of the unpleasant facts about this result is that we do not know exactly how the new critical level has been obtained. In particular, this is unfit for iterating the procedure.

We now wish to prove that, in the above propositions, we may replace strongly tied by stably tied (provided in Corollary 2.2 we replace weakly tied by stably weakly tied).

Our main tool will be the following lemma.

**Lemma 2.1.** (Stability of Massey Products) Let $p : E \to M$ be an orientable vector bundle. Let $T : H^*(M) \to H^*(D(E), S(E))$ be the Thom isomorphism. Then for $\alpha, \gamma \in H^*(M), \beta \in H^*(M)$, we have (up to sign)

$$< T\alpha, p^*\beta, p^*\gamma > = < p^*\alpha, T\beta, p^*\gamma > = < p^*\alpha, p^*\beta, T\beta > = \ T \ < \alpha, \beta, \gamma >.$$

*Proof.* The Thom isomorphism is given by taking the product with the Thom class $U \in \ H^n(D(E), S(E))$. It is easy to check from the definition of Massey product that

$$< p^*\alpha, T\beta, p^*\gamma > = U \cdot p^* < \alpha, \beta, \gamma >$$

□

Hence, we have for $x, y$ in $H^*(M)$ such that deg($y$) is odd and $x \cdot y = d\eta$ that $Ty$ and $T(y \cdot \eta)$ are tied. Thus the classes which were found to be tied according to Propositions 2.3 and 2.4 are in fact stably tied.

From Lemma 2.1. and Proposition 2.4., one can easily conclude the following proposition:

**Proposition 2.6.** Let $E$ be an orientable vector bundle over $M$, and $f$ a function on $D(E)$ going to infinity on $S(E)$. Then if $T$ is the Thom isomorphism, we have

$$c\ (T < \alpha, \beta, \gamma >, f) \geq \min\ (c(T\alpha, f), c\ (T\gamma, f))$$

We then have the following easy consequence of the above results.

**Corollary 2.3.** Let $E$ be an orientable vector bundle over $M$, and $f$ a function on $E$, quadratic at infinity. If $T$ is the Thom isomorphism associated to the negative bundle of the quadratic form, then

$$c\ (T < \alpha, \beta, \gamma >, f) \geq \min(c\ (T\alpha, f), c\ (T\gamma, f))$$

Similarly, we have for $x, y$ in $H^*(M)$ such that deg($y$) is odd and $x \cdot y = d\eta$, that $Ty$ and $T(y \cdot \eta)$ are tied.

**Remark 2.6.** We could define a third invariant, the product length, denoted by $pl(X)$, defined in the same way as the tie length, except that we now assume that $\alpha_{j+1}$ is tied to $\beta_j$ through a cohomology operation, cup product, or Massey product (i.e. $\alpha_{j+1}$ is related to $\beta_j$ either because $\alpha_{j+1} = \psi(\beta_j)$, where $\psi$ is a cohomology operation, or $\alpha_{j+1} = \beta_j \cup \gamma_j$, or $\alpha_{j+1} =< \beta_j, \gamma_j, \alpha_i >$ with $i \leq j$, or ... ). Note that the analogue of our results for higher order

Massey products may also be used. Let us point out that Massey products are sets of cohomology classes, so that the indeterminacy increases at each step. It is thus important to specify in which order the operations are done, and this order actually has an effect on the length of the chain (because even though $< a, bd, c > \subset < a, b, c > d$ equality does not hold in general). Clearly, $pl$ is a stable invariant and yields a lower bound for $tl$.

# CHAPTER 3

# EXAMPLES

**Example 3.1 (Circle Bundles Over Manifolds).** Let$N$ be a principal $S^1$ bundle over the two torus $T^2$ with nonzero chern class. According to Chern-Weil theory, there is a connection on this bundle, denoted by $\eta$, such that $d\eta = x \cdot y$, where $x, y$ are generators of the 1-dimensional de-Rham cohomology of $T^2$. (For a cochain on $T^2$, we denote by the same letter its pullback on $N$). As a result, we saw that in the sequence of cochains $1, x, x \wedge \eta, x \wedge y \wedge \eta$, each cochain is strongly tied to previous one. Therefore, the stable F-category of $N$ is 4. We may generalize this as the following proposition.

**Proposition 3.1.** Let $P$ be a circle bundle over a manifold $M$. Then, if $c_1(P)$ is the Chern class of the bundle, and there are classes in
$H^*(M) - H^0(M)$ with $x_1 \cdot x_2 \cdots x_q \neq 0$ and $x_1 \cdot x_2 \cdots x_q \cdot c_1 = 0$, we have that $Tl(P) \geq q$. If moreover, $p^*(x_1) \neq 0$, we have $Tl(P) \geq q + 1$

*Proof.* Using the Gysin exact sequence associated to the circle fibration, we see that in the cohomology class of $P$ the classes $p^*(u) \cdot \eta$ (where $p$ is the projection $p : P \to N$ and $\eta$ the connection cochain of $P$) when closed are always nonzero. The closedness is equivalent to $u \cdot c_1 = 0$. On the other hand, $p^*(u)$ is automatically closed and exact if and only if $u = v \cdot c_1$. Thus the cochains $p^*(x_1 \cdot x_2 \cdots x_j).\eta$ are closed by assumption and are nonzero in cohomology. This shows that $Tl(P) \geq q$. If moreover $p^*(x_1) \neq 0$, we may consider the sequence $p^*(x_1), p^*(x_1) \cdot \eta, p^*(x_1 \cdot x_2 \cdots x_j) \cdot \eta$ , to show the second inequality. □

Note in particular that if $x_1$ has degree 1, $p^*(x_1) \neq 0$, since $x_1$ cannot be of the form $c_1 \cdot u$, and if $x_1$ has degree 2 and is not proportional to $c_1$, $p^*(x_1) \neq 0$ for the same reason.

**Example 3.2 (The Complement of the Borromean Rings).** Let $B \subset S^3$ be the Borromean rings of Figure1. Note that $B$ is the union of three pairwise unlinked circles that, however, may not be taken apart by an isotopy of $S^3$ as shown below:

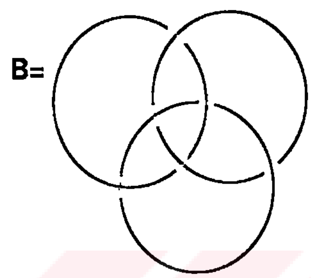


Figure 3.1: Borromean rings

First we will prove that $S^3 - B$ is not homeomorphic to the complement of three standard circles, using Massey products as follows:

There is a geometric form of Poincaré duality. The basis for the duality is the following. Let $M^k \subset N^n$ be an embedded submanifold. Suppose that $M^k$ and $N^n$ are both closed and oriented. By the tubular neighborhood theorem there is a neighborhood $\nu(M \subset N)$ which is diffeomorphic to a disk bundle over $M$:

Let ${D_0}^{n-k}$ be the fiber over a point $m_0 \in M$. Since $M$ and $N$ are oriented, ${D_0}^{n-k}$ admits an orientation. By the Thom isomorphism theorem there is a unique class $u_M \in H^{n-k}(\nu(M \subset N), \partial\nu(M \subset N); \mathbb{Z})$ so that $\int_{D_0} u_M = 1$ (Here, we assume that $M$ is connected). There is a $C^\infty$ differential form representing $U$, which vanishes identically near $\partial\nu(M \subset N)$. If we extend by 0 to $N - \nu(M \subset N)$, then we have a closed $C^\infty$ – form, $\widetilde{U_M}$ on all of $N$. Its cohomology class is the Poincaré dual of $[M] \in H_k(N)$. We will write $U_M$ instead of $\widetilde{U_M}$.

There is a form of this duality for manifolds with boundary. If $(M, \partial M) \subset (N, \partial N)$ with $M$ meeting $\partial N$ normally in $\partial M$, then the same construction yields a class $U_M \in H^{n-k}(N)$ which is the Lefschetz dual of the class $[M, \partial M] \in H_k(N, \partial N)$.

Under the correspondence $M \to U_M$, transverse intersection of manifolds corresponds to wedge product of forms. Thus, if ${M_0}^k$ and ${M_1}^l$ are transverse in $N^n$ with intersection ${M_{0,1}}^{k+l-n}$, and if $U_0$ and $U_1$ are Thom forms for $M_0$ and $M_1$ supported in sufficiently small tubes, then $U_0 \Lambda U_1$ is a Thom form for $M_{0,1}$. This means that $U_0 \Lambda U_1$ is a closed form supported in a tube about $M_{0,1}$ and integrating to 1 over each fiber.

The problem of finding a solution for $d\eta = U_M$, given $M$ and $U_M$, corresponds to finding a submanifold of $N$ whose boundary is $M$. Thus, if $M^k = \partial L^{k+1}$ then there is a form $U_L$ supported in a tube about $L$, closed outside the tube about $M$, integrating to 1 over fibers of $\nu(L \subset N)$ which are outside the tube about $M$, and so that $dU_L = U_M$. For the details see [8].

The first cohomology of $S^3 - B$ has rank 3 and is generated by classes which are dual to 2-disks spanning the components. We choose these disks as pictured in Figure 3.2.

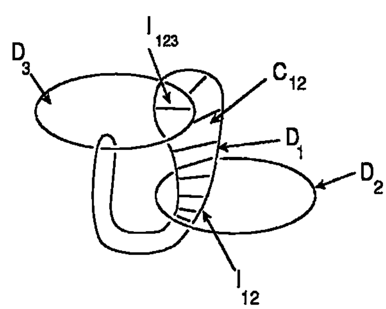


Figure 3.2: Disks in $S^3 - B$

Let $U_1, U_2, \text{and} U_3$ be the dual Thom classes in $H^1(S^3 - B)$.

The geometric fact the linking number of any pair is zero in $S^3 - B$ means that $U_i \cup U_j \neq 0$ for all $i \neq j$. Clearly $U_2 \wedge U_3 = 0$ as a form since $D_2 \cap D_3 = \phi$. The form $U_1 \cup U_2$ is the Thom form for the interval $I_{12}$. To solve the equation $d\xi = U_1 \cup U_2$ we must find a proper 2-dimensional submanifold whose boundary is $I_{12}$. We can take this to be the part of $D_1$ cut off by $I_{12}$ which lies above $D_2, C_{12}$ (see Figure 3.2). To form the Massey product $< U_1, U_2, U_3 >$ we must take $n_{12} \wedge U_3 + U_1 \wedge n_{23}$, where $dn_{ij} = U_i \wedge U_j$. In this case $n_{23} = 0$ and $n_{12}$ is supported near $C_{12}$. Thus $n_{12} \wedge U_3 + U_1 \wedge n_{23}$ is represented by the Thom form of $C_{12} \cap D_3$. The intersection is $I_{123}$. Since $I_{123}$ is an arc with end points on different components of $B$, $[I_{123}] \in H_1(S^3 - B)$ is nonzero. Thus, $< U_1, U_2, U_3 > \neq 0$. If we do similar calculations for $S^3 - B'$, where $B'$ is three unlinked circles, then all Massey products $< U_{i_1}, U_{i_2}, U_{i_3} >$ are trivial.

We may thus conclude the following:

**Proposition 3.2.** We have $F(S^3 - B) = 3$

*Proof.* The inequality $F(S^3 - B) \geq 3$ follows from the following argument. Consider $c(U_1, f)$ and $c(U_3, f)$ and the following two possibilities:

(1) Either $c(U_1, f) \neq c(U_3, f)$, and since $c(1, f) < c(U_1, f)$ ,$c(1, f) < c(U_3, f)$, we get in this case three distinct critical levels , unless there are infinitely many critical points.

or,

(2) $c(U_1, f) = c(U_3, f)$. Then since $< U_1, U_2, U_3 >$ is nonzero, we have $c(< U_1, U_2, U_3 >, f) > \min\{(c(U_1, f), c(U_3, f))\} = c(U_1, f) = c(U_3, f)$ unless $H^1(K_c)$ is nonzero. So we have again that if $f$ has only finitely many critical points, there are three critical levels :
$c(1, f), c(U_1, f), c(< U_1, U_2, U_3 >, f) > c(U_1, f)$.

Because $S^3 - B$ is open, we have $F(S^3 - B) \leq \dim(S^3 - B) = 3$ (see Remark 3.1 below). This concludes our proof.

□

**Remark 3.1.** If $X$ is closed, i.e. $\partial X = \phi$, then $F(X) \leq \dim(X) + 1$. If $\partial X \neq \phi$, then $F(X) \leq \dim(X)$ (see [17] )

**Example 3.3 (Tied Classes, Massey Products and Minimal Models).** In this section all cohomologies will be with rational coefficients. Massey products are then easily computed from the minimal model of a space.

A differential graded algebra (or differential algebra for short), $A$, is a vector space over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$,

$$A = \bigoplus_{p \geq 0} A^p$$

having

(i) a differentiation $d : A^p \to A^{p+1}$ with $d^2 = 0$

(ii) a product $A^p \bigotimes A^q \to A^{p+q}$ satisfying $\alpha\beta = (-1)^{pq}\beta\alpha$ (The algebra is then said to be commutative)

(iii) $d(\alpha\beta) = d(\alpha)\beta + (-1)^p \alpha d(\beta)$ (Leibnitz Rule)

We use the notation DGA for a differential graded algebra.

Given a DGA, $A$, we denote by $H^*(A)$ the cohomology algebra. It is again a DGA with $d = 0$. We assume throughout that $H^0(A)$ is the ground field and that $H^1(A) = 0$. Thus, $A$ is so to speak connected and simply connected.

**Definition 3.1.** A DGA $A$ is said to be minimal if

(i) $A$ is free as a graded-commutative algebra

(ii) $A^1 = 0$, and

(iii) $dA \subset A^+ \bigwedge A^+$, where $A^+ = \bigoplus_{k>0} A^k$

Condition (i) means that $A$ is a tensor product of polynomial algebras on generators of even degrees and exterior algebras on generators of odd degrees. Condition (iii) says that d is decomposable.For the details see [8].

Given a DGA, $A$, we wish to construct a minimal model $MA$, for $A$. By definition this means that $MA$ is a minimal DGA and there is a map $\rho : MA \to A$ of DGA's inducing an isomorphism on cohomology.

**Remark 3.2.** It is known that every simply connected DGA has a minimal model, [8].

Now if $M$ is a manifold, $\Omega^* M$ the DGA of exterior forms on $M$, and $d$ the exterior differential, a theorem of Sullivan [14] tells us that there is a minimal model $M\Omega$ for $\Omega^* M$.

Results of Sullivan, Vigue-Poirrier [16] tell us how to compute the minimal model of $\bigwedge M$, the free loop space of $M$, starting from $M\Omega$ . To compute Massey products in $H^*(M, \mathbb{R})$ from the knowledge of $M\Omega$, one proceeds as if $(M\Omega, d)$ was the de Rham complex of $M$.

If $H^*(M) = \{1, x, x^2, ..., x^n\}$ with $x^{n+1} = 0$ and $\deg(x) = 2k$, then the minimal model of $\bigwedge(M)$ is $S(x, y') \bigotimes E(x', y)$ where

(1) S(x, y') is the polynomial algebra with one generator $x$ of degree $2k$ and one $y'$ of degree $2(k(n+1)-1)$

(2) E(x', y) is the exterior algebra on the generators $x', y$ of degrees $2k-1$ and $2k(n+1)-1$ respectively.

Moreover , $dx = dx' = 0$, $dy = x^{n+1}$ and $dy' = (n+1)x^n x'$. Also in the cohomology ring of $\bigwedge(M)$ every $(n+1)$ fold product is zero.(see [16] )

To illustrate this, let us give some examples. In what follows, subscripts will always indicate the degree of an element. We consider the space $\bigwedge(\mathbb{CP}^n)$. Its minimal model is $S(x_2, y_{2n}) \bigotimes E(x_1, y_{2n+1})$

Moreover $dx_2 = 0, dx_1 = 0,\ dy_{2n} = {x_2}^n x_1,\ dy_{2n+1} = {x_2}^{n+1}$.

We claim that $< x_1, {x_2}^n, {y_{2n}}^q x_1 > = {y_{2n}}^{q+1} x_1$ (up to some nonzero constant factor).

First, observe that $d{y_{2n}}^q x_1 = 0$. Indeed, ${y_{2n}}^q x_1 . {x_2}^n = dy_{2n}^{q+1}$ and $dy_{2n} = {x_2}^n x_1$, so that $< x_1, {x_2}^n, {y_{2n}}^q x_1 >$ is represented by ${y_{2n}}^{q+1} x_1 + y_{2n} . {y_{2n}}^q x_1$.

As a result, $c({y_{2n}}^{q+1} x_1, f) \geq \min(c(x_1, f), c({y_{2n}}^q x_1, f))$. But using this result for $q = 0$ we see that ${y_{2n}}^q x_1$ is strongly tied to $x_1$. Now, by Proposition 2.3 we have $y_{2n}^{2q+1} x_1$ is strongly tied to $y_{2n}^{q+1} x_1$ (Use ${x_2}^{2n}$ and $y_{2n}^{q+1} x_1$ for $x$ and $y$ respectively, then $\eta = y_{2n}^{q+1}$)

Thus, we have a sequence of cohomology classes $x_1, y_{2n} x_1, y_{2n}^3 x_1, ..., y_{2n}^{2k-1} x_1, ...$ where each term is strongly tied to the previous one.

In particular $Tl(\bigwedge \mathbb{CP}^n) = F(\bigwedge \mathbb{CP}^n) = \infty$, while $cl(\bigwedge \mathbb{CP}^n) = n + 2$. This is still true for any space $X$ with the rational homotopy type of $\mathbb{CP}^n$

Similar computations show that minimal model of $\bigwedge S^{2n}$ is $S(x_{2n}, y_{4n-2}) \bigotimes E(x_{2n-1}, y_{4n-1})$ with $dx_{2n} = dx_{2n-1} = 0$ , $dy_{4n-1} = {x_{2n}}^2$ , $dy_{4n-2} = x_{2n} \cdot x_{2n-1}$. Again, $x_{2n-1} \cdot {y_{4n-2}}^{q+1} =< x_{2n-1}, x_{2n}, {y_{4n-2}}^q . x_{2n-1} >$ is strongly tied to $x_{2n-1}$ for $q = 0$ and $x_{2n-1} \cdot {y_{4n-2}}^{2q+1}$ is strongly tied to $x_{2n-1} \cdot {y_{4n-2}}^q$. As a result $Tl(\bigwedge S^{2n}) = \infty$, while $cl(\bigwedge S^{2n}) = 3$.

We now deal with an example by Felix and Halperin [3], concerning the category of the space $X = \mathbb{CP}^2 \bigvee S^2 \bigcup_w e^7$, the seven-cell $e^7$ is attached by $w = [\alpha, \beta] \in \pi_6$ $(\mathbb{CP}^2 \bigvee S^2)$, where $\alpha \in \pi_5(\mathbb{CP}^2)$, $\beta \in \pi_2(S^2)$ are the obvious basis elements and $w$ is the Whitehead Product of $\alpha$ and $\beta$. They proved that this space has category 4. Now a minimal model for this space is given by $x_2, x'_2, x_7, y_3, y'_3, y_5, y_8, y'_8, v_4, v_6, ....$ satisfying $dx_2 = dx'_2 = dx_7 = 0$
$dy_3 = x_2 x'_2, \quad dy'_3 = {x'_2}^2, \quad dy_5 = {x_2}^3, \quad dv_4 = y_3 x'_2 - y'_3 x_2,$
$dv_6 = y_5 x'_2 - y_3 {x_2}^2 - x_7$ (There are more generators and relations , but they do not occur in our computations).

Now we will show that $Tl(X) \geq 4$. First, we see that using the sequence $1, x_2, {x_2}^2 \quad Tl(X) \geq 3$ (${x_2}^2$ is strongly tied to $x_2$ by Corollary 2.1). Now, observe that $< x_2, {x_2}^2, x'_2 > = {x_2}^2 y_3 - x'_2 y_5$, using the equalities

$d(y_3x_2) = x'_2{x_2}^2, \quad dy_5 = x_2{x_2}^2$. We know that

$$< x_2, {x_2}^2, x'_2 > \in H^7(X)/[x_2.H^5(X) + {x'_2}^2.H^5(X)].$$

However, since $H^5(X) = 0$ $(H^*(X) \cong H^*(\mathbb{CP}^2 \bigvee S^2 \bigvee S^7))$, ${x_2}^2y_3 - x'_2y_5$ is the unique element in $< x_2, {x_2}^2, x'_2 >$, which is nonzero (there exists no $z_6$ such that $dz_6 = {x_2}^2y_3 - x'_2y_5$ ). Therefore, we found a new critical level by Proposition 2.4.

**Remark 3.3.** One may wonder how much our results extend from $F(X)$ to $cat(X)$. The answer is everything, i.e all results hold with $F(X)$ replaced by $cat(X)$. This is because all our results rest on the Main Lemma (i.e. Lemma 1.3). Indeed, given a covering $V$ of $X$ by open sets $V_1, ..., V_q$ , we set for $\alpha$ a cohomology class in $X$:

$$n(\alpha, V) = \inf\{j : \alpha \neq 0 \text{ in } H^*(V_1 \cup V_2 \cup ... \cup V_j)\}$$

We see that all the results hold when $c(\alpha, f)$ is replaced by $n(\alpha, V)$. Of course, in Chapter 2, the concept of tied classes must be adapted in an obvious way, i.e. we say $\alpha$ is weakly tied to $\beta$ if $n(\alpha, V) \geq n(\beta, V)$ for any covering $V$. Moreover, the examples of tied classes we constructed work also in this modified setting. For example, Theorem 1.1 may be restated as:

**Theorem 3.1.** If $\beta \in H^*(X)$, and $\alpha \cdot \beta \neq 0$ *in* $H^*(X)$, then

$$n(\alpha \cdot \beta, V) \geq n(\alpha, V) \qquad (*)$$

Moreover, in case (*) is an equality, set $n = n(\alpha, V) = n(\alpha \cdot \beta, V)$; then $\beta$ is nonzero in $H^*(V_n)$.

As a result, if $V$ is a covering by subsets that are contractible in $X$, this last case may not happen, thus we get the inequality $cat(X) \geq cl(X)$.

Similarly, Proposition 2.4 would also yield the following :

**Proposition 3.3.** Let $\alpha, \beta, \gamma$ in $H^*(X)$. Then

$$n(<\alpha,\beta,\gamma>, V) \geq \min(n(\alpha, V), n(\gamma, V))$$

Equality implies that for $n$ the common value of both sides, we have $H^j(V_n) \neq 0$ for some $j \in \{\deg(\alpha), \deg(\alpha)+\deg(\beta)-1, \deg(\beta)+\deg(\gamma)-1\}$.

One may follow the examples to extend these notions to $cat(X)$, and find a new bound for $cat(X)$.

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