STRONGLY RESIDUALLY FINITE FC-GROUPS

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ABSTRACT

STRONGLY RESIDUALLY FINITE FC-GROUPS

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It is well known that the homomorphic image of a residually finite group is not necessarily residually finite. A group G is called *a strongly residually finite group* if every homomorphic image of G is residually finite. Recall that a group G is called an FC-group if the conjugacy class of every element of G is a finite subset of G. This thesis is a survey characterization of strongly residually finite FC-groups depending on an unpublished article of L.A. Kurdachenko and J. Otal. They characterized locally nilpotent and locally soluble strongly residually finite FC-groups. In 1959, P. Hall proved that a periodic residually finite countable FC-group can be embedded in a direct product of finite groups. P. Hall gave examples of uncountable periodic FC-groups which cannot be embedded into a direct product of finite groups. Therefore the question, " What are the sufficient conditions for periodic FC-groups to be embedded into a direct product of finite groups? " is interesting. Observe that subgroups of a direct product of finite groups are natural examples of periodic residually finite FC-groups. In particular, the authors proved that strongly residual finiteness is a sufficient condition for periodic FC-groups to be embedded into a direct product of finite groups.

Keywords: FC-groups, Residually finite groups, Strongly residually finite groups.

ÖZ

GÜÇLÜ ARTIKSAL SONLU FC-GRUPLAR

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Bilindiği gibi artıksal sonlu grupların homomorfik görüntüsü artıksal sonlu olmak zorunda değildir. Eğer herhangi bir G grubunun her homomorfik görüntüsü artıksal sonlu ise bu G grubuna *güçlü artıksal sonlu grup* denir. Her elemanının eşlenik sınıfı o grubun sonlu bir alt kümesi olan grubların FC-grup olarak adlandırıldığını anımsayalım. Bu tez, L.A. Kurdachenko ve J. Otal'ın henüz yayınlanmamış bir makalesine dayanan, güçlü artıksal sonlu grupların bazı sınıflandırmaları üzerine yapılan bir derlemedir. L.A. Kurdachenko ve J. Otal, lokal nilpotent ve lokal çözünür güçlü artıksal sonlu FC-grupları sınıflandırmışlardır. 1959'da P. Hall, periyodik artıksal sonlu sayılabilir FC-grupların, sonlu grupların direkt çarpımına gömülebileceğini ispatlamıştır. P. Hall, sonlu grupların direkt çarpımının içine gömülemeyecek sayılamaz periyodik FC-grupların örneklerini vermiştir. Bu sebepten, "Periyodik FC-grupların, sonlu grupların direkt çarpımının içine gömülebilmesi için yeterli koşullar nelerdir?" sorusu ilginçtir. Sonlu grupların direkt çarpımının alt grupları periyodik artıksal sonlu FC-grupların doğal örnekleri olarak gözlemlenebilir. Yazarlar bu çalışmada, periyodik FC-gruplarda güçlü artıksal sonluluk özelliği sağlanırsa, bu grupların sonlu grupların direkt çarpımının içine gömülebileceğini göstermişlerdir.

Anahtar Kelimeler: FC-gruplar, Artıksal sonlu gruplar, Güçlü artıksal sonlu gruplar.

To my family

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CHAPTER 1

INTRODUCTION

A group G is called an FC-group (finite conjugacy) if conjugacy class of every element of G is a finite subset of G. Residually finite groups, whose subgroups with finite index have trivial intersection, play an important role in the theory of FC-groups. The connection between residually finite groups and FC-groups is that the factor group of an FC-group by its center is residually finite (see 2.3.10). The most popular study on residually finite FC-groups is its relation to the class of groups which can be embedded into a direct product of finite groups. Clearly, the class SDF (subgroup of direct product of finite groups) is contained in the class of residually finite periodic FC-groups. The converse of this statement was proved for countable periodic residually finite FC-groups by P. Hall [4] in 1959. Moreover, P. Hall has shown that, there exists uncountable residually finite FC-groups which can not be embedded in a direct product of finite groups (see example 2.3.15). L.A. Kurdachenko and J. Otal, among other characterizations, by assuming that homomorphic image of a residually finite group is residually finite, proved that every such periodic FC-group is in SDF. Let's recall the definition of direct variety.

A class of groups $\mathfrak{X}$ is called a direct variety if it satisfies the following conditions:

(i) If $G \in \mathfrak{X}$ and H is a subgroup of G, then $H \in \mathfrak{X}$.

(ii) If $G \in \mathfrak{X}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{X}$.

(iii) If $G_i \in \mathfrak{X}$ for all $i \in I$, then $Dr_{i \in I} G_i \in \mathfrak{X}$.

Since homomorphic image of a residually finite group is not necessarily residually finite (i.e. the class of residually finite groups is not Q-closed), the class of residually finite groups is not a direct variety.

Let us denote by QSDF the class of quotient subgroups of the groups in SDF. Note that QSDF is a direct variety (see proposition 2.1.3). In 1971, Yu. M. Gorchakov proved that any (not necessarily countable) periodic residually finite FC-group belongs to the class QSDF (see [3], Theorem 2). But P. Hall constructed an example of an uncountable periodic FC-group P that satisfies; P has exponent 4 such that $|P| = 2^{\aleph_0}$ and $|P'| = 2$ which is not a homomorphic image of a subgroup of any direct product of finite groups (see Theorem 2.6 in [4]). In particular, QSDF does not coincide with the class of all periodic FC-groups.

By *a variety of groups* we mean an equationally defined class of groups. Since the intersection of varieties is also a variety, we can talk about the variety generated by a certain set of groups. In 1969, Olsanskii proved that all groups of a variety $\mathfrak{B}$ are residually finite if and only if $\mathfrak{B}$ is generated by a finite group all of whose Sylow subgroups are abelian (see [11] Theorem 1). Note that all the groups in a variety which is generated by a finite group are locally finite (see [10] Theorem 15.71). Hence the variety in the theorem of Olsanskii mentioned above consists of residually and locally finite groups. Therefore, in connection with this theorem, it is an interesting study to analyse the structure of residually finite periodic FC-groups which form a residually finite variety. Hence the structure of periodic FC-groups whose every factor is residually finite appears to be interesting.

A group G is called *a strongly residually finite (SRF) group* if every homomorphic image of G is residually finite. Observe that if G is an abelian SRF-group, then every subgroup of G is SRF (see proposition 2.2.14). For non-abelian groups, subgroups of SRF-groups may not be SRF-groups (see example 2.2.15). Therefore, the class of SRF-groups does not form a variety. Because, every variety is closed with respect to forming subgroups and homomorphic images (see [13] Theorem 2.3.4).

The objective of this thesis is to give a detailed and clarified proof of the results of L.A. Kurdachenko and J. Otal on characterization of strongly residually finite FC-groups.

The following is a brief description of the thesis.

Chapter 1 is the introduction of the thesis. Chapter 2 contains some well-known results that will be necessary in the subsequent chapter. Chapter 3 consists of six sections. Each of which is a part of characterization of SRF-groups. In section 3.1 we study torsion-free abelian SRF-groups. Since every FC-group can be embedded in the direct product of a periodic FC-group and a torsion-free abelian group (see [14] Theorem 1.7), in order to understand the structure of strongly residually finite FC-groups, it is important to study torsion-free strongly residually finite abelian groups. Note that if G is an FC-group and T is the periodic part of G, then G/T is a torsion-free abelian group. Here we clarified the following theorem which is used in the proof of the main theorem of section 3.2.

Theorem 3.1.10 Let G be an FC-group and let T be the periodic part of G. Suppose that $r_0(G/T)$ is finite and $Sp(G/T) = \emptyset$. If T is an SRF-group, then G is an SRF-group.

In Section 3.2 we give the necessary and sufficient condition for a locally nilpotent FC-group to be an SRF-group. Here we give the definition of ZN-p-factor which is a powerful tool for our characterization. It is shown in detail that there exists no SRF-group which contains a ZN-p-factor. Finally the proof of the following theorem is given.

Theorem 3.2.7 Let G be a locally nilpotent FC-group. Then G is a strongly residually finite group if and only if $Z(G)$ includes a finitely generated torsion-free subgroup V such that $G/V = Dr_{p\in\pi(G)}Z_p$, where Z_p is a bounded central-by-finite p-group.

The subgroup generated by all the minimal normal subgroups of a group G is called *the socle of* G, denoted by $Soc(G)$. We denote the non-central part of the $Soc(G)$ by $Socnc(G)$. In Section 3.3, the structure of the socle of a group is determined and the following subgroup is considered. Let G be a group. Take $M_0 = 1$, $M_1 = Socnc(G)$ and construct a series of G by taking $M_{\alpha+1}/M_\alpha = Socnc(G/M_\alpha)$ for every $\alpha < \gamma$ where $Socnc(G/M_\gamma) = 1$. The last term M_γ, denoted by $Z^*(G)$, is called *the non-central hypersocle of the group* G. The following theorem shows that $Z^*(G)$ has an important role in our study.

Theorem 3.3.4 Let G be an FC-group. Then G is an SRF-group iff $G/Z^*(G)$ is an SRF-group.

The subject of section 3.4 and section 3.5 is locally soluble strongly residually finite FC-groups. In section 3.4 we deal with the groups in which the set of prime divisors of the orders of their elements is finite. In the latter one, we consider the groups which are metahypercentral. In particular, in these sections the proofs of the following theorems are clarified.

Theorem 3.4.4 Let G be a locally soluble FC-group with $\pi(G)$ is finite. Then G is an SRF-group iff $G/Z^*(G) \leq T \times A$, where T is bounded central-by-finite group and A is a torsion-free abelian group of finite rank and empty spectrum.

Theorem 3.5.10 Let G be a metahypercentral FC-group. Then G is an SRF-group if and only if $G/Z^*(G) \leq T \times A$, where

(i) A is a torsion-free abelian group of finite rank and empty spectrum,

(ii) T includes a normal subgroup $L = Dr_{p\in\pi(L)}L_p$, L_p a finite p-group.

(iii) $T/L = Dr_{p\in\pi(T/L)}\bar{Q}_p$, $\bar{Q}_p$ is a bounded central-by-finite group.

(iv) the Sylow p-subgroups of T are bounded central-by-finite groups.

Finally, in section 3.6, we characterize periodic strongly residually finite FC-groups. As we mention above, P. Hall has shown that countable periodic residually finite FC-groups are contained in the class SDF. We see that; if we replace residually finiteness by strongly residually finiteness we may ignore the countability. In particular, we clarified the following theorem.

Theorem 3.6.1 Let G be a periodic strongly residually finite FC-group. Then G can be embedded in a direct product of finite groups.

CHAPTER 2

PRELIMINARIES

In this chapter we give the basic definitions and primary results that play an important role through other chapters.

2.1 Subgroup of Direct Product of Finite Groups and Quotients

We will denote the class of subgroups of direct product of finite groups by SDF and the class of quotient subgroups of the groups in SDF by QSDF.

Note that the class of direct product of finite groups is not closed with respect to forming subgroups as in the following example shows.

Example 2.1.1. For each integer n, let G_n be a dihedral group of order 8. That is $G_n = \langle a_n, b_n : a_n^2 = b_n^2 = 1, [a_n, b_n] = c_n$, where c_n is a central element of order $2\rangle$. Let $G = \langle g_{2n-1}, g_{2n} : g_{2n-1} = a_{2n-1}a_{2n},\ g_{2n} = b_{2n}b_{2n+1}$, where $-\infty \leq n \leq \infty\rangle$ be a subgroup of $Dr_{n=-\infty}^{\infty} G_n$. Consider $[g_{2n-1}, g_{2n}] = [a_{2n-1}a_{2n}, b_{2n}b_{2n+1}]$. Since the elements in a direct product with different indices commute with each other, $[g_{2n-1}, g_{2n}] = [a_{2n}, b_{2n}] = c_{2n}$. Similarly, $[g_{2n}, g_{2n+1}] = [b_{2n+1}, a_{2n+1}] = c_{2n+1}$. So, each c_n is in G'. Then $G' = Z(G) = Dr_{n=-\infty}^{\infty}\langle c_n\rangle$. Each non-central element g of G can be written uniquely in the form $g_{i(1)} \ldots g_{i(r)} z$ where $z \in Z(G)$ and $i(1) < \cdots < i(r)$. Call this form as the standard form of g and say that

$g_{i(1)}$ is the initial component, $g_{i(r)}$ is the final component of g. Assume that $G = H \times K$ and that h, k are non-central elements of H and K respectively. Then $h = g_{i(1)} \dots g_{i(r_1)} z_1$ and $k = g_{j(1)} \dots g_{j(r_2)} z_2$ where $z_1, z_2 \in Z(G)$, $i(1) < \dots < i(r_1)$ and $j(1) < \dots < j(r_2)$. Assume that h and k have the same initial component. Then $c_{i(1)} = [g_{i(1)-1}, g_{i(1)}] = [g_{i(1)-1}, g_{j(1)}] = [g_{i(1)-1}, h] \in H$ and $c_{i(1)} = [g_{i(1)-1}, g_{i(1)}] = [g_{i(1)-1}, k] \in K$. Then $c_{i(1)} \in H \cap K = 1$ which is a contradiction. So, $g_{i(1)} \neq g_{j(1)}$. Similarly assume that $g_{i(r_1)} = g_{j(r_2)}$ then $c_{i(r)} = [g_{i(r)-1}, g_{i(r)}] \in H \cap K = 1$. Hence the final components are not the same. It follows that when hk is expressed in the standard form, we must have $r \geq 2$. Note that g_n is a non-central element of length 1. So g_n is not a product of two non-central elements. Therefore $g_n \in HZ(G)$ or $g_n \in KZ(G)$. Hence, for all n there exists z_n in $Z(G)$ such that $g_n z_n \in H$ or $g_n z_n \in K$. Since $[g_n z_n, g_{n+1} z_{n+1}] = [g_n, g_{n+1}] = c_{n+1} \neq 1$, $g_n z_n$ and $g_{n+1} z_{n+1}$ belong the same factor and so $G = HZ(G)$ or $G = KZ(G)$. If $g_n z_n, g_{n+1} z_{n+1} \in H$ then $c_{n+1} = [g_n z_n, g_{n+1} z_{n+1}] \in H$. Hence $Z(G) \leq H$ and $G = HZ(G) = H$. Similarly, if $g_n z_n, g_{n+1} z_{n+1} \in K$ then $c_{n+1} \in K$ and so $Z(G) \leq K$. Thus $G = KZ(G) = K$. Therefore G has no direct decomposition.

Definition 2.1.2. A class of groups $\mathfrak{X}$ is called a direct variety if it satisfies the following conditions:

(i) If $G \in \mathfrak{X}$ and H is a subgroup of G, then $H \in \mathfrak{X}$.

(ii) If $G \in \mathfrak{X}$ and N is a normal subgroup of G, then $G/N \in \mathfrak{X}$.

(iii) If $G_i \in \mathfrak{X}$ for all $i \in I$, then $Dr_{i \in I} G_i \in \mathfrak{X}$.

Proposition 2.1.3. *QSDF is a direct variety.*

Proof. Let G/K be an element of QSDF. Then $G \leq Dr_{i \in I} F_i$ where F_i is a finite group. Let H/K be a subgroup of G/K. Then $H \leq G \leq Dr_{i \in I} F_i$. Hence, H/K

is in QSDF. Take a normal subgroup N/K of G/K. Since $(G/K)/(N/K) \cong G/N$ and $G \leq Dr_{i\in I}F_i$, it is seen that $(G/K)/(N/K)$ is an element of QSDF. To check the last condition to be a direct variety, let G_i/K_i be an element of QSDF for all $i \in I$.

Define a map $\alpha : Dr_{i\in I}G_i \longrightarrow Dr_{i\in I}G_i/K_i$ such that

$$\alpha(g_{i_1}, \ldots, g_{i_n}) = (g_{i_1}K_{i_1}, \ldots, g_{i_n}K_{i_n}) \text{ where } i_j \in I.$$

It is trivial that α is well defined epimorphism.

$$\begin{aligned} Ker\alpha &= \{(g_{i_1}, \ldots, g_{i_n}) \in Dr_{i\in I}G_i : (g_{i_1}K_{i_1}, \ldots, g_{i_n}K_{i_n}) = (K_{i_1}, \ldots, K_{i_n})\} \\ &= Dr_{i\in I}K_i. \end{aligned}$$

By the first isomorphism theorem, $Dr_{i\in I}G_i/K_i \cong (Dr_{i\in I}G_i)/(Dr_{i\in I}K_i)$. Also we have that $Dr_{i\in I}G_i$ is in SDF, therefore $Dr_{i\in I}G_i/K_i$ is in QSDF. □

2.2 Residually Finite Groups and Strongly Residually Finite Groups

Definition 2.2.1. A group G is said to be *residually finite* if the intersection of all subgroups having finite index in G is trivial.

Equivalently, a group G is said to be *residually finite* if given $1 \neq g \in G$ there exists a normal subgroup N of G such that $g \notin N$ and $|G : N| < \infty$.

Lemma 2.2.2. *Every subgroup of a residually finite group is residually finite.*

Proof. Let G be a residually finite group and H be a subgroup of G. Since G is residually finite, for all $h \neq 1$ in H there exists a normal subgroup N_h of G such that $h \notin N_h$ and $|G : N_h| < \infty$. Since N_h is normal in G, $N_h \cap H$ is normal in $G \cap H = H$. We need to show that $|H : N_h \cap H| < \infty$. Note that

$H/(N_h \cap H) \cong (HN_h)/N_h \leq G/N_h$. But G/N_h is finite. Then $H/(N_h \cap H)$ is finite. Hence H is residually finite. $\square$

Lemma 2.2.3. *The direct product of two residually finite groups is residually finite.*

Proof. Let G and H be residually finite groups. Take an element $(g,h) \in G \times H$ then there exists $N \trianglelefteq G$ such that $g \notin N$, $|G : N| < \infty$ and $M \trianglelefteq H$ such that $h \notin M$, $|H : M| < \infty$. Thus, $N \times M \trianglelefteq G \times H$ and $(g,h) \notin N \times M$ Now, it is enough to say that $|(G \times H)/(N \times M)| < \infty$. Since $(G \times H)/(N \times M) \cong (G/N) \times (H/M)$, we have $|(G \times H)/(N \times M)| = |G/N||H/M| < \infty$. $\square$

Lemma 2.2.4. *A restricted direct product of residually finite groups is residually finite.*

Proof. (a) Finite case. Let $G = Dr_{i=1}^{n} G_i$ where G_i is a residually finite group for all $i \in \{1, 2, \ldots, n\}$. Proof is done by induction on n.

(i) Let $G = G_1 \times G_2$ where G_1 and G_2 are residually finite groups. By above lemma G is residually finite.

(ii) Assume that it is true for $n = k$ for some $k \in \mathbb{N}$. Let $G = Dr_{i=1}^{k+1} G_i$ where G_i is a residually finite group for all $i \in \{1, 2, \ldots, k+1\}$. Set $H = Dr_{i=1}^{k} G_i$ then, by assumption, H is residually finite. Then $G = H \times G_{k+1}$ where H and G_{k+1} are residually finite groups. Hence, by (i), G is a residually finite group.

(b) Infinite case. Let $G = Dr_{i \in \lambda} G_i$ where each G_i is a residually finite group and λ is an infinite index set. Take any $g \in G$ then $g = g_{i_1} g_{i_2} \cdots g_{i_n}$ where $g_{i_j} \in G_{i_j}, n \in \mathbb{N}$. Set $H = Dr_{j=1}^{n} G_{i_j}$ then $G \cong H \times Dr_{i \notin \{i_1, i_2, \ldots i_n\}} G_i$ and $g \in H$. Since H is residually finite, there exists $M \trianglelefteq H$ such that $g \notin M$

and $|H : M| < \infty$. Set $M_1 = M \times Dr_{i \notin \{i_1, i_2, \dots i_n\}} G_i$. Since any element of M commutes with every element in $Dr_{i \notin \{i_1, i_2, \dots i_n\}} G_i$ and normal in H, M_1 is normal in G. Moreover $g \notin M_1$ and $|G : M_1| = |H \times Dr_{i \notin \{i_1, i_2, \dots i_n\}} G_i : M \times Dr_{i \notin \{i_1, i_2, \dots i_n\}} G_i| = |H/M \times Dr_{i \notin \{i_1, i_2, \dots i_n\}} G_i / Dr_{i \notin \{i_1, i_2, \dots i_n\}} G_i| = |H/M| < \infty$.

□

Remark 2.2.5. A homomorphic image of a residually finite group is not necessarily residually finite.

Example 2.2.6. Let G be a free abelian group of infinite rank with a basis $\{x_i : i \in \mathbb{N}\}$, that is $G = Dr_{i \in \mathbb{N}} \langle x_i \rangle$.
Consider the subgroup $J = \langle x_i x_{i+1}^{-p} : i \geq 0$ and let $x_0 = 1 \rangle$ where p is a prime. $G/J = \langle x_i J : i \geq 0 \rangle$. So we have the following

$$x_1^p J = J$$
$$x_2^p J = x_1 J$$
$$x_3^p J = x_2 J$$
$$\vdots$$

i.e. $x_i^{p^i} J = J$. Thus $G/J \cong \mathbb{C}_{p^\infty}$ which is not residually finite. Hence G/J is not residually finite.

A similar example can be given for periodic groups as follows.

Example 2.2.7. Let $G = Dr_{i \in \mathbb{N}} \langle g_i \rangle$ such that $o(g_i) = p^i$ for some prime p. Set $J = \langle g_i g_{i+1}^{-p} : i \geq 0$ and let $g_0 = 1 \rangle$. Then $G/J = \langle g_i J : i \geq 0 \rangle$ and so $G/J \cong C_{p^\infty}$ which is not residually finite.

Lemma 2.2.8. *Let G be a residually finite group with $|G'| < \infty$. Then G is abelian by finite.*

Proof. Let $x_1, \ldots, x_n$ be the nontrivial elements of G'. Since G is residually finite for each i, there exists $N_i \trianglelefteq G$ such that $|G/N_i| < \infty$ and $x_i \notin N_i$. Set $N = \bigcap_{i=1}^{n} N_i$. For any x,y in N, $[x, y] \in G' \cap N = 1$. Hence N is abelian. $|G/N_i| < \infty$ for all i implies $|G/N| < \infty$. So G is abelian by finite. □

Definition 2.2.9. A group G is called *a strongly residually finite (SRF) group* if every homomorphic image of G is residually finite.

Example 2.2.10. Periodic FC-groups with finite Sylow subgroups are SRF-groups.

Indeed, see 2.4.4.

Example 2.2.11. Direct product of finite simple non-abelian groups are SRF-groups.

To see this, let $G = Dr_{i \in I} G_i$ where each G_i is a finite simple non-abelian group. Let N be a normal subgroup of G. Then, by Theorem 3.3.16 in [13], $N = Dr_{j \in J} G_j$ where $J \subseteq I$. Hence $G/N \cong Dr_{i \notin J, i \in I} G_i$ which is residually finite.

Example 2.2.12. Bounded abelian groups are SRF-groups.

Bounded abelian groups are residually finite and homomorphic image of a bounded abelian group is bounded abelian. In order to observe this, let G be a bounded abelian group and let N be a subgroup of G. Then G/N is a bounded abelian group. Then G/N is a direct sum of cyclic groups with boundedly finite orders (see [13], Theorem 4.3.5) and so G/N is residually finite.

Example 2.2.13. Periodic FC-groups with abelian Sylow subgroups are SRF-groups.

To see this, let G be a periodic FC-group with abelian Sylow subgroups and let N be a normal subgroup of G. Since G is a periodic FC-group, every Sylow p-subgroup of G/N is in the form SN/N where S is a Sylow p-subgroup of G (see [14] Theorem 5.4). So, every Sylow p-subgroup of G/N is abelian. Hence G/N is also a periodic FC-group with abelian Sylow subgroups. Then, by [11] Theorem 1 and [10] Theorem 15.71, G/N is residually finite.

Proposition 2.2.14. *If G is an abelian SRF-group, then every subgroup of G is SRF.*

Proof. Let G be an abelian SRF-group and H be a subgroup of G. If N is normal in H then we need to show that H/N is residually finite. Since G is abelian, every subgroup is normal. Hence $N \trianglelefteq G$. Since G is an SRF-group, G/N is residually finite. $H/N \leq G/N$ and every subgroup of residually finite group is residually finite. Hence H/N is residually finite. It follows that H is an SRF-group. □

Indeed, subgroup H of an SRF-group G is an SRF-group if every normal subgroup of H is normal in G.

But for non-abelian groups the above proposition is not true.

Example 2.2.15. Let $Alt(p^i)$ denote the alternating group of a set of p^i elements where p is an odd prime, $i \in \mathbb{N}$. Let $G = Dr_{i=1}^{\infty} Alt(p^i)$. Since G is a direct product of finite simple non-abelian groups, it is an SRF-group. Since p is odd, $Alt(p^i)$ contains p^i-cycles so there exists elements of order p^i in $Alt(p^i)$. Let $H = Dr_{i=1}^{\infty} C_{p^i}$ where C_{p^i} is the cyclic subgroup generated by the cycle x_i of length p^i of order p^i in $Alt(p^i)$. Now, for H as in the example 2.2.7 there exists subgroup $J = \langle x_i x_{i+1}^{-p} \mid i = 1, 2, 3, \ldots \rangle$ such that H/J is isomorphic to C_{p^∞}. Hence H is not an SRF-group.

Lemma 2.2.16. *Every factor group of an SRF group is an SRF group.*

Proof. Let G be an SRF group. Then every factor group G/N of G is a residually finite group. Take a factor group $(G/N)/(H/N)$ of G/N. By third isomorphism theorem, $(G/N)/(H/N) \cong G/H$ which is residually finite. $\square$

Lemma 2.2.17. *If a group G is central by finite and bounded, then G is an SRF-group.*

Proof. Take any factor group G/N of a group G. If G is bounded, then G/N is bounded. $|(G/N) : (Z(G/N))| \leq |(G/N) : (Z(G)N/N)| = |G : (Z(G)N)| \leq |G : Z(G)| < \infty$. Hence if G is central by finite, then G/N is also central by finite. Thus, it is enough to show that a bounded central by finite group G is residually finite.

Let Z be the center of G. Then G/Z is finite and Z is a bounded abelian group. So Z is a direct sum of cyclic subgroups of finite order (see [13], Theorem 4.3.5.). Hence Z is residually finite. Take any $y \in G$. It is enough to find a normal subgroup N of G such that $y \notin N$ and $|G : N| < \infty$. If $y \notin Z$ then choose $N = Z$. Since G is central by finite, $|G : N| = |G : Z| < \infty$. If $y \in Z$ then there exists $N \trianglelefteq Z$ such that $y \notin N$ and $|Z : N| < \infty$. Then $|G : N| = |G : Z||Z : N| < \infty$. Note that $N \leq Z$ so every element of N commutes with every element of G. Then N is a normal subgroup of G. Hence G is residually finite. $\square$

2.3 Finite Conjugacy (FC) Groups

Definition 2.3.1. A group G is said to be an *FC-group* if for each $g \in G$, $|G : C_G(g)|$ is finite.

Example 2.3.2. Every finite group is an *FC-group.*

Example 2.3.3. Every abelian group is an *FC-group.*

Example 2.3.4. Let G be a restricted direct product of finite groups, then G is an *FC-group.*

Proof. Let $G = Dr_{i\in\lambda}G_i$ where $|G_i| < \infty$. Take any $g \in G$ then $g = g_{i_1}g_{i_2}\dots g_{i_n}$ where $g_{i_j} \in G_{i_j}, n \in \mathbb{N}$. Note that $C_G(g) \geq Dr_{j\notin\{i_1,i_2,\dots i_n\}}G_j$. Then

$$|G : C_G(g)| \leq |G : Dr_{j\notin\{i_1,i_2,\dots i_n\}}G_j| = |G_{i_1}||G_{i_2}|\dots|G_{i_n}| < \infty.$$ □

Lemma 2.3.5. *A subgroup of an FC-group is an FC-group.*

Proof. Let G be an FC-group and $H \leq G$. Take $h \in H$ then $|G : C_G(h)| < \infty$. Since $H = G \cap H$ and $C_H(h) = C_G(h) \cap H$ we have $|H : C_H(h)| = |G \cap H : C_G(h) \cap H| < \infty$. So H is an FC-group. □

Lemma 2.3.6. *A homomorphic image of an FC-group is an FC-group.*

Proof. Let G be an FC-group and $N \trianglelefteq G$. Take an element $xN \in G/N$. It is easily seen that $C_G(x)N/N \leq C_{G/N}(xN)$. Therefore, $|G/N : C_{G/N}(xN)| \leq |G/N : C_G(x)N/N| \leq |G : C_G(x)| < \infty$. □

Lemma 2.3.7. *A restricted direct product of FC-groups is an FC-group.*

Proof. Let us show first for direct product of finitely many FC-groups. i.e. If $G = Dr_{i=1}^{n}G_i$ where G_i is an FC-group for all $i \in \{1, 2, \dots, n\}$, then G is an FC-group.

Proof is done by induction on n.

(i) Let $G = G_1 \times G_2$ where G_1 and G_2 are FC-groups. Take $(g_1, g_2) \in G_1 \times G_2$.
$C_{G_1\times G_2}((g_1, g_2)) = \{(x, y) \in G_1 \times G_2 : (g_1, g_2)(x, y) = (x, y)(g_1, g_2)\}$
$= \{(x, y) \in G_1 \times G_2 : g_1x = xg_1 \text{ and } g_2y = yg_2\}$
$= C_{G_1}(g_1) \times C_{G_2}(g_2)$

$$|G_1 \times G_2 : C_{G_1\times G_2}((g_1, g_2))| = |G_1 \times G_2 : C_{G_1}(g_1) \times C_{G_2}(g_2)|$$
$$= |G_1 \times G_2 : C_{G_1}(g_1) \times G_2||C_{G_1}(g_1) \times G_2 : C_{G_1}(g_1) \times C_{G_2}(g_2)|$$
$$= |G_1 : C_{G_1}(g_1)||G_2 : C_{G_2}(g_2)| < \infty$$

(ii) Assume that it is true for $n = k$ for some $k \in \mathbb{N}$. Let $G = Dr_{i=1}^{k+1} G_i$ where G_i is an FC-group for all $i \in \{1, 2, \ldots, k+1\}$. Set $H = Dr_{i=1}^{k} G_i$ then, by assumption, H is an FC-group. Then $G = H \times G_{k+1}$ where H and G_{k+1} are FC-groups. Hence, by (i), G is an FC-group.

For the infinite case, let $G = Dr_{i\in\lambda} G_i$ where each G_i is an FC-group and λ is an infinite index set. Take any $g \in G$ then $g = g_{i_1} g_{i_2} \cdots g_{i_n}$ where $g_{i_j} \in G_{i_j}$ and $n \in \mathbb{N}$. Set $H = Dr_{j=1}^{n} G_{i_j}$ then $G \cong H \times Dr_{i\notin\{i_1,i_2,\ldots i_n\}} G_i$ and $C_G(g) = C_H(g) \times Dr_{i\notin\{i_1,i_2,\ldots i_n\}} G_i$. Therefore $|G : C_G(g)| = |H \times Dr_{i\notin\{i_1,i_2,\ldots i_n\}} G_i : C_H(g) \times Dr_{i\notin\{i_1,i_2,\ldots i_n\}} G_i| = |H : C_H(g)| < \infty$. Thus G is an FC-group. □

Lemma 2.3.8. *(Dicman's Lemma) Let $x_1, \ldots, x_n$ be elements of the group G having finite order and each having finitely many conjugates in G. Then there exists a finite normal subgroup N of G containing $x_1, \ldots, x_n$.*

Proof. Let $C_i = C_G(x_i)$ for all $i \in \{1, \ldots, n\}$. Then by our assumption $|G : C_i| < \infty$. Let $K_i = \bigcap_{g\in G} C_i^g$ then $K_i \trianglelefteq G$ and G/K_i is finite. Let $K = K_1 \cap \cdots \cap K_n$ and $N = \langle x_1, \ldots, x_n \rangle^G$. Since K is an intersection of finitely many K_i and for each i, G/K_i is finite, we get G/K is finite. Note that $K \cap N \leq Z(N)$. Then $|N : Z(N)| \leq |N : K \cap N|$ and $N/(K \cap N) \cong NK/K \leq G/K$ so $N/Z(N)$ is finite. Then N' is finite (see [14] Theorem 1.2). Also N is finitely generated and so N/N' is a finitely generated abelian group. Let T/N' be the torsion part of N/N'. Now $(N/N')/(T/N') \cong N/T$ is a torsion-free abelian group so all conjugates of x_i must lie in T. Since T is a subgroup of N and N is the smallest normal subgroup containing x_i, we get $N = T$. Since $T/N' = N/N'$ is a finitely generated torsion abelian group, N/N' is finite. Since both N/N' and N' are finite, N is finite. □

Corollary 2.3.9. *A periodic FC-group is locally finite and locally normal.*

Lemma 2.3.10. *The factor group of an FC-group by its center is residually finite.*

Proof. Let G be an FC-group. We need to show that $G/Z(G)$ is residually finite. Take any $xZ(G) \in G/Z(G)$ such that $xZ(G) \neq Z(G)$. So, $x \notin Z(G)$. Then there exists $y \in G$ such that $x \notin C_G(y)$ this implies $xZ(G) \notin C_G(y)/Z(G)$. Since $|(G/Z(G)) : (C_G(y)/Z(G))| = |G : C_G(y)| < \infty$, there exists a normal subgroup $N/Z(G)$ of $G/Z(G)$ such that $N/Z(G) \leq C_G(y)/Z(G)$ and $|(G/Z(G)) : (N/Z(G))| < \infty$. Thus $G/Z(G)$ is residually finite. □

Lemma 2.3.11. *Every abelian by finite FC-group is a central by finite group.*

Proof. Let G be an abelian by finite FC-group and A be an abelian subgroup of G such that $|G : A| < \infty$. Let S be the set of left cosets of A in G. Then $S = \{x_1A, \ldots, x_nA\}$ for some $n \in \mathbb{N}$ and $x_i \in G$, $i \in \{1, 2, \ldots, n\}$. Since G is an FC-group, $|G : C_G(x_i)| < \infty$ for all i. Set $T = \bigcap_{i=1}^{n} C_G(x_i)$. Then $|G : T| < \infty$. We have $|G : A| < \infty$ and $|G : T| < \infty$ so, $|G : A \cap T| = |G : T||T : A \cap T| = |G : T||G \cap T : A \cap T| < \infty$.

Therefore it is enough to show that $A \cap T$ is central. Take $z \in A \cap T$ and $g \in G$. Since $G = \bigcup_{i=1}^{n} x_iA$, $g = x_ia$ for some $a \in A$ and $i \in \{1, \ldots, n\}$. Note that $z \in T = \bigcap_{i=1}^{n} C_G(x_i)$ so z commutes with x_i and since $z, a \in A$, z commutes with a. Then we have the following:

$$zg = zx_ia = x_iza = x_iaz = gz$$ □

Definition 2.3.12. A subgroup H of an abelian group G is called *pure* if $nG \cap H = nH$ for all integers $n \geq 0$.

Definition 2.3.13. Let G be an abelian torsion group. A subgroup B of G is called a *basic subgroup* if it is pure in G, it is a direct sum of cyclic groups, and G/B is divisible.

Lemma 2.3.14. *Every abelian torsion group G has a basic subgroup.*

Proof. See [13], Theorem 4.3.4. □

In the following example we will show that there exist uncountable residually finite FC-groups which can not be embedded into a direct product of finite groups.

Example 2.3.15. Let G be the periodic subgroup of the cartesian sum $Cr_{i=1}^{\infty} C_{p^i}$ where C_{p^i} is a cyclic group of order p^i for a fixed prime p. Since G is abelian, G is an FC-group. By Lemma 2.3.14, G has a basic subgroup.

Let $B = Dr_{i=1}^{\infty} C_{p^i}$ and let m be a positive integer. If $x \in p^m G \cap B$, then $x = p^m(x_1, x_2, x_3, \dots) = (y_1, \dots, y_k, 0, 0, \dots)$, for some $x_i, y_i \in C_{p^i}$ and $k \geq 0$. Hence $y_i = p^m x_i$ and $x = (p^m x_1, \dots, p^m x_k, 0, 0, \dots) = p^m(x_1, \dots, x_k, 0, 0, \dots) \in p^m B$. Therefore, B is pure in G. Next let $x = (x_1, x_2, \dots)$ in G has order p^m. Then $p^m x_i = 0$ for all components x_i of x. Since $|C_{p^i}| = p^i$, if $i > m$ we have $x_i \in pC_{p^i}$. Hence, $x \in B + pG$ and so $G/B = p(G/B)$ which implies that G/B is divisible. Thus, B is a basic subgroup of G.

Set $G_j = Cr_{i=1, i\neq j}^{\infty} C_{p^i}$. Then G/G_j is finite for all j and $\bigcap_{j=1}^{\infty} G_j = 1$. Hence $Cr_{i=1}^{\infty} C_{p^i}$ is residually finite. Since every subgroup of a residually finite group is residually finite, G is residually finite. Note that $G \geq Cr_{i=1}^{\infty} C_p$. Therefore, G is uncountable. Moreover G consists of elements $x = (x_1, x_2, x_3, \dots)$ such that each component of x has bounded order. Assume that G is contained in a direct product $D = Dr_{i \in I} F_i$ where each F_i is finite. Then B would be contained in a countable direct factor X of D. Since $GX/X \cong G/G \cap X \cong (G/B)/(G \cap X/B)$

and every homomorphic image of a divisible abelian group is divisible, GX/X is a divisible group. Recall that an abelian group is divisible if and only if it is a direct sum of isomorphic copies of $\mathbb{Q}$ and of $\mathbb{C}_{p^\infty}$ (see [13], Theorem 4.1.5). Since GX/X is a p-group, it is a direct sum of isomorphic copies of $\mathbb{C}_{p^\infty}$. Since X is a direct factor of D, D/X is a subgroup of direct product of finite groups. Since such groups are residually finite, we get D/X is a residually finite group and so GX/X is residually finite. This is a contradiction.

2.4 Thin Groups

Definition 2.4.1. A group G is said to be a *thin group* if every Sylow subgroup of G is finite.

Lemma 2.4.2. *A subgroup of a thin group is a thin group.*

Proof. Let G be a thin group and $H \leq G$. If S_p^H is a Sylow p-subgroup of H where p is a prime then $S_p^H \leq S_p^G$ for some Sylow p-subgroup S_p^G of G which is finite. □

Lemma 2.4.3. *A homomorphic image of a thin periodic FC-group is also a thin periodic FC-group.*

Proof. Let G be a thin periodic FC-group and H be a normal subgroup of G. Since G is a periodic FC-group, every Sylow p-subgroup of G/H is in the form SH/H where S is a Sylow p-subgroup of G (see [14], Theorem 5.4). Since S is finite, SH/H is finite. Hence G/H is a thin periodic FC-group. □

A group satisfies *min-p* for the prime p if each of its p-subgroups satisfies minimal condition on their subgroups.

If π is a set of primes then a π-group is a periodic group in which the order of each element is a π-number. The maximal normal π-subgroup of a group G is denoted by $O_\pi(G)$. If $\pi = \{p\}$, we write $O_p(G)$. Denote the set of all primes distinct from p by p'. Then $O_{p'p}(G)$ is defined by $O_p(G/O_{p'}(G)) = O_{p'p}(G)/O_{p'}(G)$.

Lemma 2.4.4. *Thin periodic FC-groups are SRF-groups.*

Proof. Since every homomorphic image of a thin periodic FC-group is also a thin periodic FC-group, it is enough to show that every thin periodic FC-group is residually finite.

Let G be a thin periodic FC-group. Then G is a locally finite group satisfying min-p for all primes in $\pi(G)$. Fix a prime p in $\pi(G)$. Assume that G involves an infinite simple group S. Then $S \cong H/K$ where $K \trianglelefteq H \leq G$. Since G is a periodic FC-group, H/K is an infinite periodic FC-group. Take a non-trivial element xK in H/K. Then, by Dicman's Lemma, $\langle xK \rangle^G$ is a finite normal subgroup of H/K which contradicts to H/K is infinite simple. Hence, G does not involve an infinite simple group. Then the index $|G : O_{p'p}(G)|$ is finite (see [6], Theorem 3.17). By Lemma 2.4.3, $G/O_{p'}(G)$ is a thin group. Since, by definition, $O_{p'p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$, $O_{p'p}(G)/O_{p'}(G)$ is finite. Therefore, $G/O_{p'}(G)$ is finite for all prime p in $\pi(G)$. Note that $\bigcap_{p\in\pi(G)} O_{p'}(G) = 1$. Hence the intersection of all subgroups that has finite index in G is trivial. That is G is residually finite. $\square$

2.5 Rank of an Abelian Group

Definition 2.5.1. Let G be an abelian group and S be a nonempty subset of G. S is called *independent* if $0 \notin S$ and, given distinct elements $s_1, s_2, \ldots, s_r$ of

S and integers $m_1, m_2, \ldots, m_r$ the relation $m_1s_1+m_2s_2+\cdots+m_rs_r = 0$ implies that $m_is_i = 0$ for all i.

Let $\sum$ be the set of all independent subsets of G. Every subset consisting of only one nonzero element is linearly independent. So $\sum$ is non-empty. $\sum$ is a partially ordered set by set inclusion. Take an ascending chain $\{S_i\}_{i\in I}$ of elements of $\sum$. Set $S = \bigcup_{i\in I} S_i$. S is a nonempty subset of G. Take distinct elements $s_1, s_2, \ldots, s_r$ of S and integers $m_1, m_2, \ldots, m_r$ such that $m_1s_1+m_2s_2+\cdots+m_rs_r = 0$. There exist j in I such that $s_1, s_2, \ldots, s_r \in S_j$. Since S_j is an independent subset of G, $m_is_i = 0$ for all i. So $S \in \sum$. Hence by Zorn's Lemma $\sum$ has a maximal element. By using the above argument one can show that every independent subset of G is contained in a maximal independent subset of G. In the same way, we can say that every independent subset consisting of elements of G of infinite order (respectively, p-power order for some fixed prime) is contained in a maximal independent subset consisting of elements of G of infinite order (respectively, p-power order).

Definition 2.5.2. Let G be an abelian group.

(i) The cardinality of a maximal independent subset of elements of infinite order is called the *0-rank of G* or the *torsion-free rank of G* and denoted by $r_0(G)$.

(ii) The cardinality of a maximal independent subset of elements of p-power order is called the *p-rank of G* and denoted by $r_p(G)$.

Lemma 2.5.3. *Let G be an abelian group and H be a subgroup of G. Then the following are valid.*

(i) $r_0(G) = r_0(H) + r_0(G/H)$ *when* $r_0(G)$, $r_0(H)$ *and* $r_0(G/H)$ *are finite.*

(ii) $r_p(G) \leq r_p(H) + r_p(G/H)$ *when* $r_p(G)$, $r_p(H)$ *and* $r_p(G/H)$ *are finite.*

Proof. (i) Let $S_1 = \{s_1, s_2 \ldots, s_t\}$ be a maximal independent set of elements of H of infinite order and $S = \{s_1, s_2 \ldots, s_t, s_{t+1}, s_{t+2}, \ldots, s_k\}$ be a maximal independent set of elements of G of infinite order containing S_1. So $|S| = |S_1| + (k-t)$. Consider $\{s_{t+1}, s_{t+2}, \ldots, s_k\}$. Since S_1 is maximal, $s_j + H \neq H$ for all j in $\{t+1, \ldots, k\}$. Assume that $s_i + H = s_j + H$ for some $i, j \in \{t+1, \ldots, k\}$. Then $s_i - s_j \in H$. Since S_1 is maximal torsion-free independent, $\{s_i - s_j\} \cup S_1$ is dependent, i.e. there is a relation $m\{s_i - s_j\} + m_1 s_1 + \cdots + m_t s_t = 0$ where $m_i, m \in \mathbb{Z}$ and $m\{s_i - s_j\} \neq 0$.

$0 = m\{s_i - s_j\} + m_1 s_1 + \cdots + m_t s_t = m s_i + (-m) s_j + m_1 s_1 + \cdots + m_t s_t$ then $m s_i = 0 = (-m) s_j$ since S is independent. This is a contradiction to $m\{s_i - s_j\} \neq 0$. So for all $i, j \in \{t+1, \ldots, k\}$, $s_i + H \neq s_j + H$. Now, assume that $m_{t+1}(s_{t+1} + H) + m_{t+2}(s_{t+2} + H) + \cdots + m_k(s_k + H) = 0 + H$. Then $m_{t+1}s_{t+1} + m_{t+2}s_{t+2} + \cdots + m_k s_k \in H$. So $\{m_{t+1}s_{t+1} + m_{t+2}s_{t+2} + \cdots + m_k s_k\} \cup S_1$ is dependent. Then $n(m_{t+1}s_{t+1} + m_{t+2}s_{t+2} + \cdots + m_k s_k) + n_1 s_1 + \cdots + n_t s_t = 0$ for some $n, n_i \in \mathbb{Z}$ and $n(m_{t+1}s_{t+1} + m_{t+2}s_{t+2} + \cdots + m_k s_k) \neq 0$. Since S is an independent set, $n m_i s_i = 0$ for all $i \in \{t+1, \ldots, k\}$ which is a contradiction to $n(m_{t+1}s_{t+1} + m_{t+2}s_{t+2} + \cdots + m_k s_k) \neq 0$. Hence $\{s_{t+1} + H, \ldots, s_k + H\}$ is an independent set. Therefore $r_0(G/H) \geq k - t$. Thus $r_0(G) \leq r_0(H) + r_0(G/H)$.

For the converse, assume that $r_0(H) = n$, $r_0(G/H) = k$ and $S_H = \{h_1, \ldots, h_n\}$, $S_{G/H} = \{g_1 + H, \ldots, g_k + H\}$ are maximal independent set of H and G/H respectively. Suppose that $m_1 h_1 + \cdots + m_n h_n + m_{n+1} g_1 + \cdots + m_{n+k} g_k = 0$ for some $m_i \in \mathbb{Z}$. Then $0 + H = (m_1 h_1 + \cdots + m_n h_n + m_{n+1} g_1 + \cdots + m_{n+k} g_k) + H = (m_{n+1} g_1 + H) + \cdots + (m_{n+k} g_k + H)$. Since $S_{G/H}$is independent, $m_{n+i} g_i \in H$ for all $i \in \{1, 2, \ldots, k\}$. Since each

$g_i + H$ has infinite order, $m_{n+i} = 0$ for all $i \in \{1, 2, \ldots, k\}$. Therefore $m_1h_1 + \cdots + m_nh_n = 0$. Since S_H is independent, $m_jh_j = 0$ for all $j \in \{1, 2, \ldots, n\}$. So, $S_H \cup \{g_1, g_2, \ldots, g_k\}$ is an independent set of elements of G. Thus, $r_0(G) \geq r_0(H) + r_0(G/H)$.

(ii) The proof is *mutatis mutandis* as in the first paragraph.

□

Lemma 2.5.4. *If A is a torsion-free abelian group of rank n and K is a finitely generated subgroup of A such that A/K is periodic, then for some fixed prime p, the p-rank of A/K is $\leq n$.*

Proof. We will do the proof in two parts.

Assume that $n = 1$ and there exists two subgroups B/K and C/K of A/K such that $|B/K| = |C/K| = p$ and $B/K \cap C/K = 1$. i.e. p-rank is ≥ 2. Let bK and cK be the generators of B/K and C/K respectively. Then $pb = k_1$ and $pc = k_2$ for some $k_1, k_2 \in K$. Since K is abelian of rank 1, k_1 and k_2 are dependent so there exists t and s in $\mathbb{Z}$ such that $tk_1 - sk_2 = tpb - spc = 0$. Then $p(tb - sc) = 0$. Since A is torsion-free we get $tb = sc$. Using the same technique we can reduce t and s to the case $(t, s) = 1$.

If $p \mid t$, then $t = pt_1$ for some t_1 in $\mathbb{Z}$. Then $scK = tbK = t_1pbK = t_1k_1K = K$ but $p \nmid s$ since $(t, s) = 1$ contradiction to cK has order p. So $(t, p) = 1$ and there exists x, y in $\mathbb{Z}$ such that $xp + yt = 1$. Then $bK = (xp + yt)bK = xk_1K + ytbK = yscK$ but $(B/K) \cap (C/K) = 1$ so we get a contradiction. Hence p-rank of A/K is 1.

Assume that $n > 1$ and p-rank of A/K is n+1. So there exists $n + 1$ subgroups of order p, say $B_1/K, \ldots, B_{n+1}/K$ of A/K such that $B_i/K \cap \langle \bigcup_{j \neq i, j=1,\ldots,n+1} B_j/K \rangle = 1$. Let b_iK be the generator of B_i/K. Then there exists $k_i \in K$ such that $pb_i = k_i$ for all i. Since the rank of K is not greater than

n, there exists $t_1, \dots, t_{n+1}$ in $\mathbb{Z}$ not all zero such that $t_1k_1 - \dots - t_{n+1}k_{n+1} = 0$. This implies $t_1pb_1 - \dots - t_{n+1}pb_{n+1} = 0$. Since A is torsion-free we get $t_1b_1 - \dots - t_{n+1}b_{n+1} = 0$ and $t_1, \dots, t_{n+1}$ are relatively prime. So there exist at least one t_i such that $xp + yt_i = 1$ for some x, y in $\mathbb{Z}$. Then

$b_iK = (xp + yt_i)b_iK = xpb_iK + yt_ib_iK = xk_iK + y(t_1b_1 + \dots + t_{i-1}b_{i-1} + t_{i+1}b_{i+1} + \dots + t_{n+1}b_{n+1})K$

which is a contradiction to $B_i/K \cap \langle \bigcup_{j \neq i, j=1,\dots,n+1} B_j/K \rangle = 1$. Thus p-rank of A/K is n. □

2.6 Nilpotent Groups,Hypercentral Groups and Metahypercentral Groups

Definition 2.6.1. A group G is *nilpotent* if it has a central series, that is, a normal series $1 = Z_0 \leq Z_1 \leq \dots \leq Z_n = G$ such that Z_{i+1}/Z_i is contained in the center of G/Z_i for all i.

Lemma 2.6.2. *A finite p-group is nilpotent.*

Proof. See [13], Theorem 5.1.3. □

Lemma 2.6.3. *Let G be a finite nilpotent group. Then G is the direct product of its Sylow p-subgroups.*

Proof. See [5], Theorem 17.1.4. □

Definition 2.6.4. Let G be a group. If every finitely generated subgroup of G is nilpotent then G is called *a locally nilpotent group.*

Lemma 2.6.5. *Subgroups and factor groups of a locally nilpotent group are locally nilpotent.*

Proof. See [5], Exercise 18.1.1. □

Lemma 2.6.6. *Each minimal normal subgroup of a locally nilpotent group is central.*

Proof. See [13], Theorem 12.1.6 □

Lemma 2.6.7. *Let G be a locally finite locally nilpotent group. Then G is the direct product of its Sylow p-subgroups.*

Proof. Let G be a locally finite locally nilpotent group and let $\sum$ be a local system of G consisting of finite subgroups. It can be easily seen by Zorn's Lemma that Sylow p-subgroups exist for G. Take S_i in $\sum$ then S_i is a finite nilpotent group. So by Lemma 2.6.3, $S_i = P_{i_1} \times \cdots \times P_{i_k}$ where P_{i_j} is a Sylow p_j-subgroup of S_i. That is Sylow p-subgroups of S_i are normal and so unique. In particular, every element s in S can be written uniquely as $s_{p_1} \ldots s_{p_k}$ where s_{p_j} in P_{i_j}. Let p be a fixed prime and $P_n \in Syl_p(S_n)$. Since $S_n \leq S_{n+1} \in \sum$, P_n is contained in a Sylow p-subgroup of S_{n+1}. But there exists unique Sylow p-subgroup of S_{n+1} namely P_{n+1}. So $P_n \leq P_{n+1}$. Set $P = \bigcup_{n \in I} P_n$ where I is the index set of $\sum$. Take an element x in P and g in G then there exists n and m in I such that $x \in P_n$ and $g \in S_m$. If $m \leq n$ then $g \in S_n$ and so $x^g \in P_n \leq P$. If $m > n$ then $x^g \in P_m \leq P$. Therefore P is normal in G. Moreover P is a maximal p-subgroup of G. Since this is true for all primes, G is a direct product of its Sylow p-subgroups. □

The property of locally nilpotent groups given above is partially true for locally soluble groups. Let us deal in detail.

Lemma 2.6.8. *Let G be a locally finite locally soluble group. Then $G = S_\pi S_{\pi'}$ where π is a subset of $\pi(G)$ and S_π is a Sylow π-subgroup, $S_{\pi'}$ is a Sylow π'-subgroup of G.*

Proof. Let G be a locally finite locally soluble group. Let π be a subset of $\pi(G)$ and let $\sum$ be a local system of G consisting of finite subgroups of G. Take S_i in $\sum$ then S_i is a finite soluble group and so, by Theorem 20.1.1 in [5], $S_i = S_{i_\pi} S_{i_{\pi'}}$ where S_{i_π} is a Sylow π-subgroup and $S_{i_{\pi'}}$ is a Sylow π'-subgroup of S_i. Also, by Hall's Theorem, any two Hall π-subgroup are conjugate and every π-subgroup is contained in a Hall π-subgroup. So there exists a Hall π-subgroup S_{i+1_π} of S_{i+1} such that $S_{i_\pi} \leq S_{i+1_\pi}$ and similarly $S_{i_{\pi'}} \leq S_{i+1_{\pi'}}$ and $S_i = S_{i_\pi} S_{i_{\pi'}}$. Set $S_\pi = \bigcup_{i\in I} S_{i_\pi}$ and $S_{\pi'} = \bigcup_{i\in I} S_{i_{\pi'}}$ where I is the index set of $\sum$. Then $G = S_\pi S_{\pi'}$. Take a π element g in G. Then $g \in S_j$ for some $S_j \in \sum$. Then g is an element of S_π. Hence S_π is a maximal π-subgroup of G. Similarly, $S_{\pi'}$ is a maximal π'-subgroup of G. □

Definition 2.6.9. A group G is called *hypercentral*, if it coincides with its *hypercenter*, which is the terminal member of its (transfinite) central series.

Lemma 2.6.10. *Every subgroup of a hypercentral group is hypercentral.*

Proof. Let G be a hypercentral group and let H be a subgroup of G. Since G is hypercentral, there exists an ascending series $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_\gamma = G$ such that $Z_\beta \trianglelefteq G$ and $Z_{\beta+1}/Z_\beta \leq Z(G/Z_\beta)$ for every $\beta < \gamma$. Then $Z_\beta \cap H \trianglelefteq G \cap H = H$ and $(Z_{\beta+1} \cap H)/(Z_\beta \cap H) \leq Z(H/Z_\beta \cap H)$ for every $\beta < \gamma$ as $[H, Z_{\beta+1} \cap H] \leq Z_\beta \cap H$. So, $1 = Z_0 \cap H \leq Z_1 \cap H \leq \cdots \leq Z_\gamma \cap H = H$ is the required central series. Thus H is hypercentral. □

Lemma 2.6.11. *If G is hypercentral, then every homomorphic image of G is hypercentral.*

Proof. Let G be a hypercentral group and N be a normal subgroup of G. Then there exists a central series of G which terminates at G say $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_\gamma = G$. Consider the series $Z_0N/N \leq Z_1N/N \leq \cdots \leq Z_\gamma N/N = G/N$. Since both Z_β and N are normal in G, $Z_\beta N/N$ is normal in G/N for all $\beta < \gamma$. Since $[G, Z_{\beta+1}] \leq Z_\beta$, $[G/N, Z_{\beta+1}N/N] \leq Z_\beta N/N$. Then $(Z_{\beta+1}N/N)/(Z_\beta N/N) \leq Z((G/N)/(Z_\beta N/N))$ for all $\beta < \gamma$. Thus G/N is hypercentral. □

Lemma 2.6.12. *Let G be an FC-group. G is hypercentral if and only if G is locally nilpotent.*

Proof. Indeed, every hypercentral group is locally nilpotent. Let G be a hypercentral group and H be a subgroup of G generated by $\{x_1, \dots, x_n\}$. Then H is hypercentral. Let w be the first infinite ordinal and let $Z_\beta(H)$ denotes the elements of central series of H. Then $Z_w(H) = \bigcup_{i=1}^{\infty} Z_i(H)$. Let $a \in Z_{w+1}(H)$ then $[a, x_j] \in Z_w(H)$. So, for each $j \in \{1, \dots, n\}$ there exists i_j such that $[a, x_j] \in Z_{i_j}(H)$. Since the series is linearly ordered, there exists $m = max\{i_1, \dots, i_n\}$ such that for all $j \in \{1, \dots, n\}$, $[a, x_j]$ is in $Z_m(H)$. Then $aZ_m(H)$ commutes with $x_j Z_m(H)$ for all j. But $\langle x_1, \dots, x_n \rangle = H$ hence $aZ_m(H)$ commutes with $hZ_m(H)$ for all h in H. Then $aZ_m(H) \in Z(H/Z_m(H)) = Z_{m+1}(H)/Z_m H$. So $a \in Z_w(H)$. Therefore each generator x_j is in $Z_{j_k}(H)$ where $1 \leq k \leq n$. Let $t = max\{j_1, \dots, j_n\}$ then $H = Z_t(H)$. Thus H is nilpotent as required.

Conversely, let G be a locally nilpotent FC-group. Then $G/Z(G)$ is a periodic FC-group (see [14], Theorem 1.4) and so it has a socle series $1 = S_0/Z(G) \leq S_1/Z(G) \leq \cdots \leq \bigcup_{i=1}^{\infty} S_i/Z(G) = G/Z(G)$ (see [14],Lemma 1.14). Now consider the series $1 \leq Z(G) \leq S_1 \leq \cdots \leq \bigcup_{i=1}^{\infty} S_i = G$. Since the socle of a group is characteristic, each S_i is normal in G. $S_1/Z(G) = Soc(G/Z(G))$ which is a direct product of some minimal normal subgroups of $G/Z(G)$. By Lemma 2.6.6, $S_1/Z(G) \leq Z(G/Z(G))$. Similarly since $(G/Z(G))/(S_i/Z(G))$ is locally nilpotent for all i, see Lemma 2.6.5, $(S_{i+1}/Z(G))/(S_i/Z(G)) \leq Z((G/Z(G))/S_i/Z(G))$. Thus $S_{i+1}/S_i \leq Z(G/S_i)$. Therefore G is hypercentral. □

Lemma 2.6.13. *Every periodic hypercentral group is locally finite.*

Proof. Let G pe a periodic hypercentral group. By Lemma 2.6.12, G is locally nilpotent and so locally soluble. Note that finitely generated periodic soluble group is finite as shown below.

Let H be a finitely generated subgroup of G. Then H is a periodic soluble group. Assume that the derived length of H is 1. Then $H^{'} = 1$. So H is a finitely generated periodic abelian group. Hence H is finite.

Assume that all finitely generated subgroups of G of derived length $\leq n$ are finite for some $n \in \mathbb{N}$. Let K be a finitely generated subgroup of G of derived length $n+1$. Since $K/K^{'}$ is a finitely generated periodic abelian group, it is finite. Since $K/K^{'}$ is finite and K is finitely generated, $K^{'}$ is finitely generated (see [1] Lemma 1.1.3). Hence, by assumption, $K^{'}$ is finite. Since $K/K^{'}$ and $K^{'}$ are finite, K is finite.

Hence G is locally finite. □

Lemma 2.6.14. *If a hypercentral group G contains a p-element, where p is a prime, then $Z(G)$ contains a p-element.*

Proof. First we show that if $1 \neq N \trianglelefteq G$, then $N \cap Z(G) \neq 1$. Let G be a hypercentral group and N be a normal subgroup of G. Let $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_\gamma = G$ be the upper central series of G. Then there exists a smallest β, which is not a limit ordinal, such that $N \cap Z_\beta(G) \neq 1$. Then $[G, N \cap Z_\beta(G)] \leq N \cap Z_{\beta-1}(G) = 1$. Hence $1 \neq N \cap Z_\beta(G) \leq Z(G)$ and so $N \cap Z(G) \neq 1$.

Since G is hypercentral, the periodic part T of G is a subgroup of G. T is a periodic hypercentral group so by Lemma 2.6.13 and Lemma 2.6.12, it is a locally finite, locally nilpotent group. By Lemma 2.6.7, $T = Dr_{p\in\pi(T)}T_p$ where T_p is the Sylow p-subgroup of T. Since T_p is hypercentral, $Z(T_p) \neq 1$. Note that $Z(T_p)$ char $Z(T)$ char T char G and so $Z(T_p) \trianglelefteq G$. Then, by above paragraph, $Z(T_p) \cap Z(G) \neq 1$. Hence $Z(G)$ contains a p-element. □

Definition 2.6.15. If a group G has a normal series $1 \lhd N \lhd G$ where N and G/N are hypercentral, then G is called *metahypercentral group.*

Lemma 2.6.16. *Every homomorphic image of a metahypercentral group is metahypercentral.*

Proof. Assume that G is metahypercentral and $1 \lhd N \lhd G$ is a series of G such that N and G/N are hypercentral. Let M be a normal subgroup of G. Since N is hypercentral, by Lemma 2.6.11, $NM/M \cong N/(N \cap M)$ is hypercentral. Similarly, $(G/M)/(NM/M) \cong (G/N)/(NM/N)$ is hypercentral. Hence $M/M \lhd NM/M \lhd G/M$ is the required series. $\square$

Lemma 2.6.17. *Every periodic metahypercentral FC-group is locally soluble.*

Proof. Let G be a metahypercentral group. Then G has a series $1 \lhd N \lhd G$ where N and G/N are hypercentral. Then N and G/N are locally soluble. Since G is a periodic FC-group, by Dicman's Lemma, G is locally finite. So, N and G/N are locally finite. Let $\langle x_1, \ldots, x_n \rangle$ be a finitely generated subgroup of G. Then $\langle x_1, \ldots, x_n \rangle N/N \leq G/N$ is a finite soluble group. Since $\langle x_1, \ldots, x_n \rangle/(\langle x_1, \ldots, x_n \rangle \cap N) \cong \langle x_1, \ldots, x_n \rangle N/N$ is finite and $\langle x_1, \ldots, x_n \rangle$ is finitely generated, $\langle x_1, \ldots, x_n \rangle \cap N$ is a finitely generated subgroup of N and so it is soluble. Recall that extension of a soluble group by a soluble group is soluble (see [13], Theorem 5.1.1). Therefore, $\langle x_1, \ldots, x_n \rangle$ is a soluble group. Hence, G is locally soluble. $\square$

CHAPTER 3

STRONGLY RESIDUALLY FINITE FC-GROUPS

3.1 Reduction to Periodic Groups

An FC-group can be embedded into the direct product of a periodic FC-group and a torsion-free abelian group (see [14], Theorem 1.7 or [13], Theorem 14.5.10). Therefore in order to understand the structure of strongly residually finite FC-groups, it is helpful to consider the structure of torsion-free abelian groups which can be SRF-groups. For torsion-free abelian groups of infinite rank we give an example [2.2.6] that shows they are not SRF. So from the point of view of SRF-groups it is enough to consider torsion-free abelian groups of finite rank.

Definition 3.1.1. Let G be a torsion-free abelian group of finite rank and F be a finitely generated subgroup of G such that G/F is periodic. Set $Sp(G)$=\{p prime : the Sylow p-subgroups of G/F are infinite\}. Then $Sp(G)$ is called *the spectrum* of G.

Example 3.1.2. Let $G = \mathbb{Z} \times \mathbb{Z}$ and $F =< (p^n, q^m) >$ where $n, m \in \mathbb{N}$ and p, q are primes. F is a finitely generated subgroup of G such that $G/F \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/q^m\mathbb{Z} \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$ is periodic. Sylow p-subgroup of G/F is isomorphic to $\mathbb{Z}_{p^n}$ and Sylow q-subgroup of G/F is isomorphic to $\mathbb{Z}_{q^m}$. Hence $Sp(G) = \emptyset$.

Example 3.1.3. $\mathbb{Z}$ is a finitely generated subgroup of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z} \cong Dr_{p\in\pi}\mathbb{C}_{p^\infty}$ where π is the set of all primes in $\mathbb{Z}$. Hence $Sp(\mathbb{Q}) = \pi$.

Lemma 3.1.4. *$Sp(G)$ is independent of the choice of F.*

Proof. Let F and K be two finitely generated subgroups of G such that G/F and G/K are periodic (i.e $r_0(G/F) = 0$ and $r_0(G/K) = 0$). $K/(K\cap F) \cong KF/F \leq G/F$ then $K/(K\cap F)$ is also periodic. Since K is finitely generated and abelian $K/(K\cap F)$ is a finitely generated, periodic abelian group. Thus $K/(K\cap F)$ is finite. Similarly, $F/(F\cap K)$ is finite. We have

$$|G:K||K:F\cap K|=|G:F\cap K|=|G:F||F:F\cap K|$$

where $K/(K\cap F)$ and $F/(F\cap K)$ are finite. Therefore, Sylow p-subgroups of G/F are infinite iff Sylow p-subgroups of G/K are infinite. □

Lemma 3.1.5. *A torsion-free abelian group G is an SRF-group iff $r_0(G)$ is finite and $Sp(G) = \emptyset$.*

Proof. First we will show that for a torsion-free abelian SRF-group G, it is necessary that $r_0(G)$ is finite and $Sp(G) = \emptyset$.

Let G be a torsion-free (i.e. $r_p(G) = 0$ for all prime p) abelian SRF-group. Assume that $r_0(G)$ is infinite. So we can take a set $S_0 = \{x_n : n \geq 1\}$ of linearly independent elements of infinite order. Define $S = Dr_{x_n\in S_0}\langle x_n\rangle$. Given a prime p, define $J = \langle x_n x_{n+1}^{-p} : n \geq 0 \text{ and } x_0 = 1\rangle$. So we have $S/J \cong \mathbb{C}_{p^\infty}$ [Example 2.1.4]. That is $S/J \leq G/J$ is not residually finite. Then, by Lemma 2.2.2, G/J is not residually finite. But G is an SRF-group. Hence $r_0(G)$ can not be infinite.

Let F be a finitely generated subgroup of G such that G/F is periodic and let S/F be a Sylow p-subgroup of G/F. Every finitely generated subgroup of S/F is a finitely generated periodic abelian group and so they are finite. Therefore S/F is a locally finite group. Since $r_0(G)$ is finite, $r_0(S)$ is also finite. Let $S_0 = \{s_1 + F, \ldots, s_n + F\}$ be an independent set of S/F. (i.e. if $(l_1 s_1 + F) + \cdots + (l_n s_n + F) = F$ for some integer $l_i, i \in \{1, \ldots, n\}$ then $l_i s_i + F = F$, for all i.) Say $m_1 s_1 + m_2 s_2 + \cdots + m_n s_n = 0$ for some $m_i \in \mathbb{Z}$. Then $(m_1 s_1 + F) + (m_2 s_2 + F) + \cdots + (m_n s_n + F) = 0 + F = F$. By our assumption, $m_i s_i + F = F$, for all i. Since each $s_i + F$ is nontrivial and S/F is a p-group, for each s_i there exists $k_i \neq 0$ in $\mathbb{N}$ such that p^{k_i} is the order of $s_i + F$. So p^{k_i} divides m_i for all i. Without loss of generality let k_1 be the smallest k_i. Then $0 = m_1 s_1 + m_2 s_2 + \cdots + m_n s_n = p^{k_1}(t_1 s_1 + t_2 p^{k_2 - k_1} s_2 + \cdots + t_n p^{k_n - k_1} s_n)$ where p and t_i are relatively prime for all i. Since S is torsion-free, $t_1 s_1 + t_2 p^{k_2 - k_1} s_2 + \cdots + t_n p^{k_n - k_1} s_n = 0$. Then $t_1 s_1 + F = F$ and so $t_1 s_1$ is in F. Since p and t_1 are coprime and S/F is a p-group, $t_1 = 0$. Then $m_1 s_1 = 0$. Repeating this process for each s_i we obtain $m_i s_i = 0$ for all i. Hence $S_1 = \{s_1, \ldots, s_n\}$ is an independent set of S. That is $r_p(S/F) = r_0(S)$. Therefore $r_p(S/F)$ is finite which means that S/F is a direct sum of finitely many cyclic and quasicyclic groups (see [13], Theorem 4.3.13). But, since G is an abelian SRF-group, S/F is residually finite and so it has no quasicyclic subgroup. Hence S/F is finite. Thus $Sp(G) = \emptyset$.

For the converse, assume that G is a torsion-free abelian group such that $r_0(G)$ is finite and $Sp(G) = \emptyset$. Let F be a finitely generated subgroup of G such that G/F is periodic. Since $Sp(G) = \emptyset$, G/F has no infinite Sylow subgroups. Let $\varphi : G \to A$ be an epimorphism and let $\varphi(F) = B$. Then $A/B = \varphi(G)/\varphi(F)$ is a periodic abelian group with finite Sylow subgroups. Since every abelian group is an FC-group, A/B is an FC-group. Therefore, by Lemma 2.4.4, A/B is an SRF-group. Since F is a finitely generated abelian group, B is a finitely generated abelian group. Therefore, B is isomorphic to a direct sum of finitely many $\mathbb{Z}$ and a finite group. Hence B is residually finite.

We need to show that A is residually finite. Take an element x in A. If $x \notin B$, since A/B is residually finite, there exists $D/B \trianglelefteq A/B$ such that $xB \notin D/B$ and $|A/B : D/B| < \infty$. In particular, $x \notin D$ and $|A : D| < \infty$. If $x \in B$, since B is residually finite, there exists $N \trianglelefteq B$ such that $x \notin N$ and $|B : N| < \infty$. Since A/B is a periodic group with finite Sylow subgroups and B/N is finite, A/N is a periodic group with finite Sylow subgroups. Moreover, since A is an abelian group, A/N is an abelian group and so it is an FC-group. So, by Lemma 2.4.4, A/N is an SRF-group. Hence there exists $C/N \trianglelefteq A/N$ such that $xN \notin C/N$ and $|A/N : C/N| < \infty$. In particular, $x \notin C$ and $|A : C| < \infty$. □

Corollary 3.1.6. *To apply Lemma 3.1.5 on G/T it is sufficient to say that G is a strongly residually finite FC-group.*

Proof. Let G be an FC-group and T be the periodic part of G. Then G/T is a torsion-free FC-group. Since G is an FC-group, $G' \leq T$. Therefore G/T is abelian. If G is an SRF-group then, by Lemma 2.2.16, G/T is an SRF-group. □

Lemma 3.1.7. *Let G be a periodic FC-group and N be a normal subgroup of G such that G/N is thin. If N is residually finite and $1 \neq x \in N$, then there exists a G-invariant subgroup $U = U(x) \leq N$ such that $x \notin U$ and $|N : U|$ is finite.*

Proof. If N is residually finite, then there exists a normal subgroup H of N such that $x \notin H$ and $|N : H| < \infty$. Set $K = Core_G(H) = \bigcap_{g \in G} H^g$. Then N/K is residually finite central-by-finite and bounded (see [8], Lemma 3). $x \notin H \geq K$ and so $K \neq xK \in N/K$. G/K is a periodic FC-group and N/K is a normal residually finite subgroup of G/K where $(G/K)/(N/K) \cong G/N$ is thin. So we may replace G by G/K to assume that N is a bounded, central by finite group.

Put $C = Z(N)$. If $x \notin C$ take U as C. Since $Z(N)$ char $N \trianglelefteq G$, $C \trianglelefteq G$. Hence we are done.

So assume that $x \in C$. Since G is a periodic FC-group, by Dicman's Lemma, $X = \langle x^g : g \in G \rangle$ is a finite normal subgroup of G. Since G is a periodic group and every cyclic group is abelian, $\langle x \rangle$ is a finite abelian group. Hence $\langle x \rangle$ can be written as a direct product of its finite Sylow subgroups. Then $x = x_1 \dots x_n$ where each x_i is a p_i-element. We claim that if for each x_i there exists a G-invariant subgroup U_i such that $x_i \notin U_i$ and $|N : U_i| < \infty$ then there exists a G-invariant subgroup U such that $x \notin U$ and $|N : U|$ is finite. Set $U = \bigcap_{i=1}^{n} U_i$ then U is G-invariant and $|N : U|$ is finite. We need to show that $x \notin U$. Let us show this by induction on n. If $n = 2$ then $x = x_1 x_2$, $o(x_1) = p_1^i$ and $o(x_2) = p_2^j$ are relatively prime where $i, j \in \mathbb{N}$. If $x \in U = U_1 \cap U_2$ then $x_1 x_2 U_1 \cap U_2 = U_1 \cap U_2$. Then $({x_2}^{-1})^{p_1^i} U_1 \cap U_2 = x_1^{p_1^i} U_1 \cap U_2 = U_1 \cap U_2$ which is a contradiction. Hence $x \notin U_1 \cap U_2$. Assume that $n = k$ and the assumption is true for $n = k - 1$. Then $x = x_1 \dots x_{k-1} x_k$ and $x_k \notin U_k$, $x_1 \dots x_{k-1} \notin \bigcap_{i=1}^{k-1} U_i$. Hence as we show above $x \notin \bigcap_{i=1}^{k} U_k$. So, we may assume that x is a p-element. Hence X is a p-group.

Let $\sum$ be the set of G invariant subgroups B of C all of which has trivial intersection with X. $1 \in \sum \neq \emptyset$. $\sum$ is partially ordered with respect to set inclusion. Let $\{B_i : i \in I\}$ be a chain consisting of elements of $\sum$. Set $B = \bigcup_{i \in I} B_i$ then B is a G-invariant subgroup of C and $B \cap X = 1$. Hence, by Zorn's Lemma, $\sum$ has a maximal element V. Consider G/V. Since G is a periodic FC-group, so is G/V. Since N is a bounded normal subgroup of G, $N/V \trianglelefteq G/V$ and is bounded. $|N/V : Z(N/V)| \leq |N/V : Z(N)V/V|$. Since $V \leq Z(N)$, $|N/V : Z(N)V/V| = |N/V : Z(N)/V| = |N : Z(N)|$ and since N is central by finite, $|N/V : Z(N/V)| < \infty$. Then, by Lemma 2.2.17, N/V is residually finite and so, all the conditions of the theorem is satisfied. Therefore, there is no loss if we assume that $V = 1$. Since C is a periodic abelian group, C can be written by its p_i-primary components (see [5], Exercise 10.1.2). For all $p_i \neq p$, $X \cap C_{p_i} = 1$. Since each C_{p_i} is normal and $V = 1$ is the maximal of such groups, $C_{p_i} = 1$ for all $p_i \neq p$. Therefore C is a p-group.

Assume that C is infinite. Since C is a bounded periodic abelian p-group, it is an infinite direct sum of its cyclic p-subgroups (see [13] Theorem 4.3.5). So, C contains a subgroup $Dr_{i=1}^{\infty} C_p$ such that each C_p is a cyclic p-group of order p. Then $\Omega_1(C) = \{c \in C : c^p = 1\}$ is infinite.

Let α be an automorphism of C. Take any $\alpha(a) \in \alpha(\Omega_1(C))$. Then $\alpha(a)^p = \alpha(a^p) = \alpha(1) = 1$. So, $\alpha(\Omega_1(C)) \subseteq \Omega_1(C)$. Therefore $\Omega_1(C)$ char $C \trianglelefteq G$. Set $A = \Omega_1(C)$ then A is an infinite G-invariant elementary abelian p-subgroup of G. Put $Y = X \cap A$. Since A is a G-invariant elementary abelian p-subgroup of G contained in C and X is a G-invariant finite subgroup of G, Y is a G-invariant finite elementary abelian p-subgroup of C.

G/N is a thin periodic FC-group. Then, by Lemma 2.4.3, $(G/N)/O_{p'}(G/N)$ is a thin periodic FC-group. Since $O_p((G/N)/O_{p'}(G/N))$ is a p-subgroup of $(G/N)/O_{p'}(G/N)$, we get $O_{p'p}(G/N)/O_{p'}(G/N) = O_p((G/N)/O_{p'}(G/N))$ is finite. Assume that G/N involves an infinite simple group S/N. Then $S/N \cong (H/N)/(K/N)$ where $K/N \trianglelefteq H/N \leq G/N$. Since G is a periodic FC-group, $H/K \cong (H/N)/(K/N)$ is an infinite periodic FC-group. Take a non-trivial element xK in H/K. Then, by Dicman's Lemma, $\langle xK \rangle^G$ is a finite normal subgroup of H/K which contradicts to H/K is infinite simple. Hence, G/N does not involve an infinite simple group. Since G/N is a locally finite group satisfying min-p and G/N does not involve an infinite simple group, by ([6] Theorem 3.17) the index $|G/N : O_{p'p}(G/N)|$ is finite. Therefore $|G/N : O_{p'}(G/N)|$

is finite.

Let $L/N = O_{p'}(G/N)$. Then $|G : L| = |G/N : L/N| = |G/N : O_{p'}(G/N)|$ is finite. On the other hand, there exists an L-invariant subgroup $B \leq A$ such that $A = Y \times B$. In particular, $L \leq N_G(B)$ and, since G/L is finite, $|G : N_G(B)|$ is finite. Let $\{y_1, \ldots, y_m\}$ be a transversal of $N_G(B)$ in G and let $D = B^{y_1} \cap \cdots \cap B^{y_m}$. Take any g in G. Then $D^g = B^{y_1 g} \cap \cdots \cap B^{y_m g}$. Since $y_j g \in G = \bigcup_{i=1}^{m} N_G(B) y_i$ for all $j \in \{1, \ldots, n\}$, for each $j \in \{1, \ldots, n\}$ there exists $i \in \{1, \ldots, n\}$ and $n \in N_G(B)$ such that $y_j g = n y_i$. Then $B^{y_j g} = B^{n y_i} = B^{y_i}$. Hence D is normal in G. Since $Y \cong A/B$, $|A : B^{y_i}|$ is finite for all i. Therefore $|A : D|$ is finite. Note that $D \cap X = D \cap A \cap X = D \cap Y \leq B \cap Y = 1$. Then $D = 1$ which is a contradiction to $|A : D|$ is finite. Thus C is finite. Since N/C is also finite, N is finite. □

Corollary 3.1.8. *Let G be a periodic FC-group and suppose that N is a normal subgroup of G such that G/N is thin. Then N is residually finite iff G is residually finite.*

Proof. If G is residually finite then, by Lemma 2.2.2, N is residually finite.

For the converse, assume that N is residually finite. Since G is a periodic FC-group G/N is a thin periodic FC-group so by Lemma 2.4.4 G/N is residually finite. Given $x \in G$. If $x \notin N$ then there exists a normal subgroup M/N of G/N such that $xN \notin M/N$ and $|G/N : M/N| < \infty$. Then $M \trianglelefteq G$ such that $x \notin M$ and $|G : M| < \infty$. So we are done for the case $x \notin N$.

Assume $x \in N$ then by Lemma 3.1.7 there exists a G-invariant subgroup U of N such that $x \notin U$ and $|N : U|$ is finite. $U \leq N \leq G$, $|N : U| < \infty$ and G/N is thin so G/U is thin. Thus G/U is a thin periodic FC-group hence it is residually finite. Since $x \notin U$, $xU \neq U$. Since G/U is residually finite, there exists $Y/U \trianglelefteq G/U$ (then $Y \trianglelefteq G$) such that $xU \notin Y/U$ (then $x \notin Y$) and

$|G/U : Y/U| < \infty$ (then $|G : Y| < \infty$). Hence G is residually finite. $\square$

Corollary 3.1.9. *Let G be an FC-group and let T be the periodic part of G. Suppose that $r_0(G/T)$ is finite and $Sp(G/T) = \emptyset$. If T is residually finite, then G is residually finite.*

Proof. Let G be an FC-group and let T be the periodic part of G. Suppose that $r_0(G/T)$ is finite, $Sp(G/T) = \emptyset$ and T is residually finite. Since G is an FC-group, $G/Z(G)$ is periodic (see [14] Theorem 1.4) and so $G/Z(G)T$ is periodic. So $r_0(G/Z(G)T) = 0$. Since $G/Z(G)T \cong (G/T)/(Z(G)T/T)$, by Lemma 2.5.3, $r_0(G/T) = r_0(Z(G)T/T)$. Similarly, since $Z(G)T/T \cong Z(G)/(T \cap Z(G))$ and $r_0(Z(G) \cap T) = 0$, $r_0(Z(G)T/T) = r_0(Z(G))$. Hence $r_0(Z(G)) = r_0(G/T)$.

Set $V_i = \langle x_i \rangle$. Since each x_i has infinite order, each V_i is an infinite cyclic group. Put $V = Dr_{i=1}^{s} V_i$. Then V is a torsion free finitely generated subgroup of $Z(G)$ such that $r_0(V) = r_0(G/T)$. Let $1 \neq x \in G$ then there exists $n \geq 1$ such that $x \notin nV$. This can be shown as follows: If $x \notin V$ then take $n = 1$. If $x \in V$ then $x = k_1x_1 + \cdots + k_sx_s$ where $k_i \in \mathbb{N}$. Let p be a prime such that $p > max\{k_1, \ldots, k_s\}$ then $p \nmid k_i$ for all i. Take $n = p$. If $x \in pV$ then there exists $g = l_1x_1 + \cdots + l_sx_s \in V$ such that $x = pg$. $k_1x_1 + \cdots + k_sx_s = p(l_1x_1 + \cdots + l_sx_s) = pl_1x_1 + \cdots + pl_sx_s$ contradiction to $p \nmid k_i$ for all i. Let $U = nV$ such that $n \geq 1$ and $x \notin U$. Since U is in the center of G, $U \trianglelefteq G$. Take $v + U \in V/U$. $n(v + U) = nv + U = U$ then every element of V/U has order $\leq n$. Hence V/U is periodic. Therefore $r_0(U) = r_0(V) = r_0(G/T)$. Since T is the torsion part of G and U is torsion free so $T \cap U = 1$. Then $TU/T \cong U/T \cap U \cong U$. Thus TU/T is finitely generated and $r_0(TU/T) = r_0(G/T)$. The latter property brings us that $G/TU \cong (G/T)/(TU/T)$ is periodic.

Since $Sp(G/T) = \emptyset$, for all finitely generated subgroup H/T with $(G/T)/(H/T)$ is periodic, $(G/T)/(H/T)$ has no infinite Sylow subgroup. So, G/TU is a thin group. In particular $(G/U)/(TU/U)$ is a thin group. Since $TU/U \cong T/T \cap U \cong T$, TU/U is residually finite. Since $(G/U)/(TU/U) \cong G/TU$ and extension of a periodic group by a periodic group is periodic, G/U is periodic. Then by Corollary 3.1.8, G/U is residually finite. So for any $x + U \in G/U$ (respectively for $x \in G$), there exists $N/U \trianglelefteq G/U$ (then $N \trianglelefteq G$) such that $|G/U : N/U| < \infty$ (then $|G : N| < \infty$) and $x + U \notin N/U$ (then $x \notin N$). Hence G is residually finite. □

Theorem 3.1.10. *Let G be an FC-group and let T be the periodic part of G. Suppose that $r_0(G/T)$ is finite and $Sp(G/T) = \emptyset$. If T is an SRF-group, then G is an SRF-group.*

Proof. Let G be an FC-group and let T be the periodic part of G. Suppose that $r_0(G/T)$ is finite, $Sp(G/T) = \emptyset$ and T is an SRF-group. Let H be a normal subgroup of G. Since $TH/H \cong T/T \cap H$ and T is an SRF-group, TH/H is residually finite. Let P/H be the periodic part of G/H. Since G is an FC-group, G/H is an FC-group and so P/H is a periodic FC-group. Since T is periodic, TH/H is periodic. So $TH/H \leq P/H$. Moreover, since $T \trianglelefteq G$ and $H \trianglelefteq G$, $TH/H \trianglelefteq P/H$. Note that $P/TH \cong (P/H)/(TH/H)$ is a thin group. Then by Corollary 3.1.8, P/H is residually finite.

Since G/T is a torsion-free abelian group with $r_0(G/T) < \infty$ and $Sp(G/T) = \emptyset$, by Lemma 3.1.5, G/T is an SRF-group. Then $(G/H)/(P/H) \cong (G/T)/(P/T)$ is a torsion-free abelian SRF-group. So,

$$r_0((G/H)/(P/H)) < \infty \text{ and } Sp((G/H)/(P/H)) = \emptyset.$$

Then, by Corollary 3.1.9, G/H is residually finite. □

3.2 Locally Nilpotent Strongly Residually Finite FC-Groups

By using the same technique in the paper [9], we can prove the following generalization of the Lemma 3.1 in [9]. Recall that a periodic abelian group can be written $G = Dr_{p\in\pi(G)}G_p$ where G_p is a Sylow p-subgroup of G. G_p is called the p-component of G.

Lemma 3.2.1. *Let G be an SRF-group and let H and K be normal subgroups of G such that $H \leq K$, $[G,K] \leq H$. If K/H is periodic then each p-component of K/H is bounded.*

Proof. Since G is an SRF-group, G/H is an SRF-group. Then K/H is residually finite. $[G,K] \leq H$ iff $[G/H, K/H] = 1$ iff $K/H \leq Z(G/H)$. So K/H is abelian. Every periodic abelian group can be written as a direct sum of its maximal p-subgroups. Let U be a maximal p-subgroup of K/H. Assume that U is unbounded. Then there exists an infinite chain $n_1 < n_2 < n_3 < \dots$ such that $o(u_i) = p^{n_i}$ for some $u_i \in U$. Set $U_1 = \langle u_i : i = 1, 2, 3, \dots \rangle$. Since U_1 is generated by countably many infinite elements, it is an unbounded, countable subgroup of U. Let $\sum$ be the set of unbounded countable subgroups of U. Then $\sum$ is a nonempty set. Take $V \in \sum$. Then $V \leq U$ is residually finite. Assume V has an element v of infinite height, that is there exist $v_i \in V$, $i \in \mathbb{N}$ such that $v = pv_1 = p^2v_2 = p^3v_3 = \dots$. But U is a p-group so $o(v) = p^m$ for some $m \in \mathbb{N}$ and $1 = p^m v = p^{m+1}v_1 = p^{m+2}v_2 = p^{m+3}v_3 = \dots$ Set $V_1 = \langle v, v_i : i \in \mathbb{N} \rangle$. Then $p^{m-1}V_1 = \langle p^{m-1}v, p^{m-1}v_1, p^{m-1}v_2, \dots \rangle \cong \mathbb{C}_{p^\infty}$ contradiction to V is residually finite. Hence every element in V has finite height. Then V is a direct product of cyclic p-groups (see [2] Theorem 17.3). Since V is unbounded, V includes a subgroup $W = Dr_{n\geq 1}\langle w_n \rangle$ where $o(w_n) = p^n$. If we take $Y = \langle w_n w_{n+1}^{-p} : n \geq 0$

and $w_0 = 1\rangle$ then $W/Y \cong \mathbb{C}_{p^\infty}$.

$Y \leq W \leq V \leq U \in Z(G/H)$. Then $Y = L/H$ for some $L \leq K$. Since $L/H \leq Z(G/H)$, $L/H \trianglelefteq G/H$ and so $L \trianglelefteq G$.

Denote W as W_1/H where $W_1 \leq K$. Then $\mathbb{C}_{p^\infty} \cong W/Y = (W_1/H)/(L/H) \cong W_1/L \leq G/L$ which contradicts to G/L is residually finite. Therefore U is bounded. □

Corollary 3.2.2. *Let G be an SRF-group and let H and K be normal subgroups of G such that $H \leq K$, $[G,K] \leq H$ and K/H is a p-group, for certain prime p. Then K/H is bounded.*

Corollary 3.2.3. *Let G be an SRF-group and let H and K be normal subgroups of G such that $H \leq K$, $[G,K] \leq H$ and K/H is a periodic group order of which is divided by only finitely many primes. Then K/H is bounded.*

Proof. Recall that K/H is a direct sum of its maximal p-subgroups. Since the order of K/H is divided by finitely many primes, this direct sum is finite. Each maximal p-subgroup of K/H is bounded and hence K/H is bounded. □

Definition 3.2.4. Let G be a group and let p be a prime. Suppose that U and H are normal subgroups of G satisfying the following conditions:

(i) There are G-invariant subgroups U_n and H_n, $n \geq 1$ such that $H = Dr_{n\geq 1}H_n$ and $U = Dr_{n\geq 1}U_n$.

(ii) $U_n \leq H_n$ for all $n \geq 1$.

(iii) $H_n/U_n = \langle g_nU_n, a_nU_n : [g_nU_n, a_nU_n] = c_nU_n \neq U_n, c_n^p \in U_n\rangle$ where $c_nU_n \in Z(G/U_n)$ for every $n \geq 1$.

Then the factor H/U is said to be ZN-p-factor of G. If $U = 1$, then H is called a ZN-p-subgroup of G.

Lemma 3.2.5. *Let G be a strongly residually finite group then G has no ZN-p-factor.*

Proof. Assume that H/U is a ZN-p-factor of G. Define $J/U = \langle (c_n c_{n+1}^{-1})U : n \geq 1 \rangle$. Take $c_n U_n \in Z(G/U_n)$ then $c_n^{-1} g^{-1} c_n g \in U_n \subseteq U$. So $c_n U \in Z(G/U)$ for all $n \geq 1$. Then $c_n c_{n+1}^{-1} U \in Z(G/U)$ for all $n \geq 1$. Then $J/U \leq Z(G/U)$. Hence J is normal in G.

Consider $[H/J : H/J] = \langle [xJ, yJ] : xJ, yJ \in H/J \rangle$. Note that $H/J = \langle g_n J, a_n J : n \geq 1 \rangle$. Since $c_n U_n \in Z(G/U_n)$, we get $[g_n J, a_n J] = [g_n, a_n]J = c_n J \in Z(G/J)$. So, if $n = m$, then $[g_n J, a_m J] = c_n J \in Z(H/J)$ and if $n \neq m$, then $[g_n J, a_m J] = J$. Therefore $[H/J, H/J] = \langle c_n J : n \geq 1 \rangle$.

Take an element $g \in [H/J, H/J]$. Then $g = [g_{i_1} J, a_{i_1} J]^{k_{i_1}} \ldots [g_{i_l} J, a_{i_l} J]^{k_{i_l}} = c_{i_1}^{k_{i_1}} \ldots c_{i_l}^{k_{i_l}} J$ for some $l, i_j \in \mathbb{N}$, $k_{i_j} \in \mathbb{Z}$. Since $c_n J = c_{n+1} J$ for all $n \geq 1$, $g = c_{i_1}^{lk_{i_1}} J$. Note that $c_n^p \in U \leq J$ for all $n \geq 1$. If $p > mk_{i_1}$ then $mk_{i_1} = pq + r$ for some q,r$\in \mathbb{N}$ where $0 \leq r < p$. Then $c_{i_1}^{mk_{i_1}} J = (c_{i_1}^p)^q J c_{i_1}^r J = c_{i_1}^r J$. Therefore $[H/J, H/J]$ has order at most p. Since $[H/J, H/J]$ is finite, by [[14], Theorem 1.1], H/J is an FC-group. Moreover, since G is an SRF-group, G/J is residually finite. Then H/J is residually finite. So, by Lemma 2.2.8, H/J is abelian by finite. Hence, by Lemma 2.3.11, H/J is central by finite group. However we can show that $|H/J : Z(H/J)|$ is infinite. As shown above, $J/U \leq Z(G/U)$. Since $[g_n U, a_n U] = c_n U \neq U$, $g_n U \notin Z(G/U)$ so $g_n U \notin J/U$. Then $g_n J \neq J$. $[g_n J, a_n J] = c_n J \neq J$ then $g_n J \notin Z(H/J)$. Since we have infinitely many such g_n, $|H/J : Z(H/J)|$ is infinite. Hence G has no ZN-p-factor. □

Lemma 3.2.6. *Let G be a strongly residually finite periodic FC-group. For every prime p, let C_p, Z_p be the Sylow p-subgroups of $Z(G)$ and hypercenter(Z) of G respectively. Then Z_p/C_p is finite.*

Proof. Assume that C_p is finite. Since G is residually finite, Z is a residually finite group. Then there exists $M \trianglelefteq Z$ such that $M \cap C_p = 1$ and Z/M is finite. Since M is hypercentral, by Lemma 2.6.14, if M contains a p-element then $Z(M)$ contains a p-element. But $Z(M) \cap C_p \leq M \cap C_p = 1$ so M has no p-element. Then $M \cap Z_p = 1$ and $Z_p \cong Z_p/M \cap Z_p \cong Z_pM/M \leq Z/M$ is finite. Then Z_p/C_p is finite. So we may assume that C_p is infinite.

Put $E_1 = C_p$ and let E_2/E_1 be the Sylow p-subgroup of $Z(G/E_1)$. If $E_2 = E_1$, then $Z_p = C_p$ and Z_p/C_p is trivially finite. If $E_2 \neq E_1$ take an element $a_1 \in E_2 \backslash E_1$ such that $a_1^p \in E_1$. Define $\phi_1 : G \to G$ by $\phi_1(g) = [g, a_1]$. $a_1E_1 \in Z(G/E_1)$ so $[g, a_1] \in E_1$. Take $x, y \in G$, $\phi_1(xy) = [xy, a_1] = [x, a_1]^y[y, a_1]$. Since $[x, a_1] \in E_1 \leq Z(G)$, $\phi_1(xy) = [x, a_1][y, a_1] = \phi_1(x)\phi_1(y)$. ϕ_1 is an endomorphism of G such that $Ker\phi_1 = \{g \in G \mid [g, a_1] = 1\} = C_G(a_1)$ and $Im\phi_1 = \{[g, a_1] | g \in G\} = [G, a_1]$. If $a_1 \in Z(G)$ then $a_1 \in E_1$ which is a contradiction so $Im\phi_1 \neq 1$. Hence there exists $g_1, c_1 \in G$ such that $[g_1, a_1] = c_1 \neq 1$. Since $a_1^p \in Z(G)$ and $[g_1, a_1] \in Z(G)$, (Recall the formula $[x, yz] = [x, z][x, y]^z$. When $y = z$ and $[x, y] \in Z(G)$ we get $[x, y^2] = [x, y]^2$. By induction one can see that $[x, y^n] = [x, y]^n$ for all $n \in \mathbb{N}$.) $c_1^p = [g_1, a_1]^p = [g_1, a_1^p] = 1$.

Define $F_1 = \langle g_1, a_1 \rangle^G$ then, by Dicman's Lemma, F_1 is finite. Since G is residually finite, for each element of F_1 there exists a normal subgroup of G of finite index such that it does not contain that element. Let H_1 be the intersections of these normal subgroups, then $H_1 \trianglelefteq G$ of finite index and $H_1 \cap F_1 = 1$.

Put $E_3 = E_1 \cap H_1$ and let E_4/E_3 be the Sylow p-subgroup of $Z(H_1/E_3)$. Similar to first paragraph, since G is SRF and $E_3 = E_1 \cap H_1 \trianglelefteq G$, G/E_3 is residually finite. So H_1/E_3 is residually finite. If the Sylow p-subgroup of the upper hypercenter of H_1/E_3 is finite then Z_p/C_p is finite. So we may assume that E_4/E_3 is infinite. Given any element $e \in E_4 \backslash E_3$, then the mapping $\psi_e : H_1 \to$

H_1 by $\psi_e(h) = [h, e]$ is an endomorphism of H_1. If $Im\psi_e = [H_1, e] = 1$ for all e in E_4 then $E_4 \leq Z(H_1)$. So $C_G(E_4) \geq H_1$ and, since G/H_1 is finite, $G/C_G(E_4)$ is finite. Then $(G \cap E_4)/(C_G(E_4) \cap E_4)$ is finite. Note that $C_G(E_4) \cap E_4 = Z(G) \cap E_4$ hence $E_4/(E_4 \cap Z(G))$ is finite. On the other hand, since E_4/E_3 and E_3 are p-groups, E_4 is a p-group. Hence $E_4 \cap Z(G) = E_4 \cap C_p$. Since $E_4 \leq H_1$, $E_4 \cap Z(G) = E_4 \cap H_1 \cap C_p = E_3$ contradiction to E_4/E_3 is infinite.

Therefore, there are $a_2 \in E_4$, $g_2 \in H_1$ and $c_2 \in E_3$ such that $a_2^p \in E_3$ and $[g_2, a_2] = c_2 \neq 1$. Moreover $c_2^p = [g_2, a_2]^p = [g_2, a_2^p] = 1$.

Similar to F_1, define $F_2 = \langle g_2, a_2 \rangle^G$. Since g_2 and a_2 in H_1, $F_1 \cap F_2 = 1$. F_1F_2 is finite so we can choose a normal subgroup H_2 of G of finite index such that $H_2 \cap F_1F_2 = 1$.

If we iterate this process, it stops in finitely many steps. Otherwise $Dr_{n \geq 1} F_n$ is a ZN-p-subgroup of G but this is impossible by 3.2.5. Therefore there is some $t \geq 1$ such that E_{2t}/E_{2t-1} is finite. Hence Z_p/C_p is finite. □

Theorem 3.2.7. *Let G be a locally nilpotent FC-group. Then G is a strongly residually finite group if and only if $Z(G)$ includes a finitely generated torsion-free subgroup V such that $G/V = Dr_{p \in \pi(G)} Z_p$ where, Z_p is a bounded central-by-finite p-group.*

Proof. Assume that G is an SRF group and let T be the periodic part of G then G/T is a torsion free abelian SRF group. Then, by Lemma 3.1.5, $r_0(G/T)$ is finite. Since G is an FC-group, $G/Z(G)$ is locally finite (see [12], Theorem 4.32) so $(G/T)/(Z(G)T/T) \cong G/Z(G)T$ is periodic. Hence $r_0(G/T) = r_0(Z(G)T/T) = r_0(Z(G))$. Recall that every infinite abelian group contains a maximal torsion-free subgroup. Let V be the subgroup generated by F which is a maximal independent subset of elements of $Z(G)$ of infinite order. Then V is a finitely generated torsion-free subgroup such that $r_0(V) = r_0(Z(G))$. If there exists z in $Z(G)$ such that $z^n V \neq V$ for all n in $\mathbb{N}$ then $V \cap \langle z \rangle = 1$ contradiction to the maximality of F. Hence $Z(G)/V$ is periodic. Since $G/Z(G)$ and $Z(G)/V$ are periodic, G/V is periodic. G/V is a periodic locally nilpotent SRF, FC-group hence we may assume that $V = 1$ so that G is a periodic group. Since periodic FC-groups are locally finite and every locally nilpotent locally finite group is a direct product of its Sylow p-subgroups, $G = Dr_{p\in\pi(G)} Z_p$ where Z_p is the Sylow p-subgroup of G. $Z_p \trianglelefteq G$ and $[G, Z_p] \leq Z_p$ then by Lemma 3.2.1, Z_p is bounded. Every locally nilpotent FC-group is hypercentral (see Lemma 2.6.12) so Z_p is hypercentral. Moreover $Z_p \cong G/Dr_{q\in\pi(G), q\neq p} Z_q$ is an SRF-group. So by Lemma 3.2.6, Z_p is central by finite.

Conversely, assume that $Z(G)$ includes a finitely generated torsion-free subgroup V such that $G/V = Dr_{p\in\pi(G)} Z_p$, where Z_p is a bounded central-by-finite group. TV/V is a periodic locally nilpotent FC-group hence $TV/V = Dr_{p\in\pi(G)} S_p$ where S_p is the Sylow p-subgroup of TV/V. $TV/V \leq G/V$ so $S_p \leq Z_p$ hence bounded. $Z_p/Z(Z_p)$ is finite so $S_p/Z(S_p) \leq (Z_p \cap S_p)/(Z(Z_p) \cap S_p)$ is finite. By Lemma 2.2.17, S_p is an SRF-group. Since direct product of residually finite groups is residually finite, TV/V is residually finite. By Lemma 2.3.6 and Lemma 2.6.5, every homomorphic image of TV/V is a periodic locally nilpotent FC-group. Since TV/V has bounded central-by-finite Sylow subgroups, every homomorphic image of TV/V has bounded central-by-finite Sylow subgroups. Therefore every homomorphic image of TV/V is also residually finite. Hence $T \cong TV/V$ is an SRF-group. $(G/T)/(VT/T) \cong G/VT$. Since VT/T is finitely generated (so $r_0(VT/V)$ is finite) and G/VT is periodic (so $r_0(G/VT) = 0$), $r_0(G/T)$ is finite and for the spectrum of G/T it is enough to consider the Sylow p-subgroups of G/VT. Since G/V has bounded Sylow p-subgroups and G/VT is abelian (since $G' \in T$), G/VT has bounded abelian

Sylow p-subgroups L_p. So each L_p is a direct sum of cyclic subgroups with boundedly finite orders (see [13], Theorem 4.3.5). Since G/VT has finite p-rank by Lemma 2.5.4, this direct sum is finite and so L_p is finite. We conclude that $Sp(G/T) = \emptyset$. Then by Theorem 3.1.10, G is an SRF-group. □

3.3 The Non-central Series in Strongly Residually Finite Groups

In this section we determine the structure of the socle of a group and we see that the non-central hypersocle of a group has an important role in the characterization of strongly residually finite FC-groups.

Definition 3.3.1. The subgroup generated by all the minimal normal subgroups of a group G is called *the socle of* G, represented by $Soc(G)$.

If a group G does not have a minimal normal subgroup then we may assume $Soc(G) = 1$.

Example 3.3.2. If G is a simple group, then $Soc(G) = 1$.

Remark 3.3.3. The socle of a group is a direct product of minimal normal subgroups (see [13], page 87).

Let M be a minimal normal subgroup of a group G. Since $[M, G]$ is a G-invariant subgroup of M and M is minimal, either $[M, G] = 1$ or $[M, G] = M$. In the first case, $M \leq Z(G)$, that is M is central in G. In the second case, $C_G(M) \neq G$, that is M is non-central in G.

Say $Soc(G)=Dr_{i\in I}M_i$ and let $Z=\{i \in I \mid M_i$ is central in $G\}$, $E=\{j \in I \mid M_j$ is non-central in $G\}$. Then $Soc(G) = S_1 \times S_2$ where $S_1 = Dr_{i\in Z}M_i$ and

$S_2 = Dr_{j \in E} M_j$. To see this, take any minimal normal subgroup M of G and assume that M is central in G (respectively, noncentral in G). Since $S_1 \cap M \trianglelefteq G$; if S_1 does not contain M (respectively, S_2 does not contain M), then $S_1 \cap M = 1$ (respectively, $S_2 \cap M = 1$. Hence $S_1M/S_1 \cong M/(S_1 \cap M) \cong M$ (respectively, $S_2M/S_2 \cong M/(S_2 \cap M) \cong M$). In particular $C_G(M) \cong C_G(S_1M/S_1)$ (respectively, $C_G(M) = C_G(S_2M/S_2)$). However $S_1M/S_1 \leq Soc(G)/S_1 \cong S_2$ (respectively, $S_2M/S_2 \leq Soc(G)/S_2 \cong S_1$). That is $S_1M/S_1 \cong M_j$ for some j in E (respectively, $S_2M/S_2 \cong M_i$ for some i in Z). But $C_G(M_j) \neq G$ for j in E (respectively, $C_G(M_i) \neq G$ for i in Z) which is a contradiction. So S_1 and S_2 are independent of the choice of the decomposition of $Soc(G)$. Moreover every central minimal normal subgroup is contained in S_1 and non-central minimal normal subgroup is contained in S_2.

The subgroups $S_1 = Socc(G)$ and $S_2 = Socnc(G)$ are called *the central and the non-central socle of the group G* respectively.

Take $M_0 = 1$, $M_1 = Socnc(G)$ and construct a series of G by taking $M_{\alpha+1}/M_\alpha = Socnc(G/M_\alpha)$ for every $\alpha < \gamma$ where $Socnc(G/M_\gamma) = 1$. This series is called *the upper non-central socular series of G* and the last term $M\gamma$, denoted by $Z^*(G)$, is called *the non-central hypersocle of the group G.*

Theorem 3.3.4. *Let G be an FC-group then G is an SRF-group iff $G/Z^*(G)$ is an SRF-group.*

Proof. Since every factor group of an SRF-group is SRF, it is enough to show the converse. Set $U = Z^*(G)$ and assume that G/U is an SRF-group. Let H be a normal subgroup of G then we need to show that G/H is residually finite. Let $S/H = Soc(G/H) \cap (UH/H)$. Then $S/H = Dr_{i \in I} M_i/H$ where M_i/H is a minimal normal subgroup of G/H(see [13], Theorem 3.3.12). Since there exists a G-chief factor $L_i/(H \cap U)$ such that $M_i/H \cong L_i/(H \cap U)$, M_i/H is a non-central chief factor of G. That is $M_i/H \cap Z(G/H) = 1$ for all i in I. Hence $S/H \cap Z(G/H) = 1$ and so $UH/H \cap Z(G/H) = 1$. Therefore,

$$G/H \hookrightarrow (G/H)/Z(G/H) \times (G/H)/(UH/H)$$

By Lemma 2.3.10, $(G/H)/Z(G/H)$ is residually finite. Note that

$$(G/H)/(UH/H) \cong (G/U)/(UH/U)$$

which is a homomorphic image of G/U. Hence $(G/H)/(UH/H)$ is also residually finite. Since direct product of residually finite groups and subgroup of a residually finite group is residually finite, G/H is residually finite. □

3.4 Some Locally Soluble Strongly Residually Finite FC-Groups

In this section we deal with the strongly residually finite locally soluble FC-groups in which the set of prime divisors of the orders of their elements is finite.

Lemma 3.4.1. *Let G be a locally soluble FC-group and let E be a p-subgroup of G for a prime p and $C \leq E$. If $C \leq Z(G)$ and E/C is a non-central G chief factor, then either E contains a G invariant subgroup B such that $E = B \times C$ or there are $g \in G$, $a \in E$ and $c \in C$ such that $[g, a] = c \neq 1$ and $c^p = 1$.*

Proof. Since G is an FC-group and E/C is a chief factor of G, E/C is finite (see [14], Theorem 1.13). Let T be the periodic part of G. Then $E \leq T$ and E/C is a chief factor of T. Since G is locally soluble, E/C is locally soluble and since T is a periodic and FC-group, T is locally finite so E/C is a finite elementary abelian p-group (see [6], Corollary 1.B.4).

If E is nonabelian, then there exists $a, b \in E$ such that $[a,b] = c \neq 1$. Since E/C is elementary abelian p-group, $[a,b]C = [aC, bC] = C$ (that is $c \in C$) and $a^p \in C \leq Z(G)$. Then $c^p = [a,b]^p = [a^p, b] = 1$. Hence we are done for the case E is non abelian.

Assume that E is abelian. Let $H = C_G(E/C) = \{g \in G \mid [g,e] \in C$ for all e in $E\}$. Let $g \in G$ and $x \in H$, $g^{-1}xg \in H$ iff $(g^{-1}xg)e(g^{-1}xg)^{-1}e^{-1} \in C$ for all e in E. $(g^{-1}xg)e(g^{-1}xg)^{-1}e^{-1} = g^{-1}xe^{g^{-1}}x^{-1}ge^{-1}(g^{-1}g) = x(e^{g^{-1}})x^{-1}(e^{g^{-1}})^{-1} \in C$. Therefore H is a normal subgroup of G. G acts on the finite group E/C by conjugation. Define $\phi : G \to Sym(E/C)$ by $\phi(g) = \phi_g$ such that $\phi_g : E/C \to E/C$ with $\phi_g(eC) = e^gC$. $Ker\phi = \{g \in G \mid \phi_g = 1\} = \{g \in G \mid e^gC = C$ for all e in E $\}$ $=H$. Note that since E/C is finite, $Sym(E/C)$ is finite. Hence $G/H = G/Ker\phi$ is finite. Suppose that $Z(H)$ does not contain E. Then there exists $g \in H$ and $a \in E$ such that $[g,a] = c \neq 1$. As we mention above, $a^p \in C$ and $c \in C$, then $c^p = 1$. Again all is proved.

Therefore we may assume in the remaining case $E \leq Z(H)$. If $E \leq Z(H)$, then $H \leq C_G(E)$. Let $g \in C_G(E)$. Then $[g,e] = 1 \in C$ for all $e \in E$. That is $C_G(E) \leq H$. Hence $H = C_G(E)$. Since G/H is finite and G is locally soluble, G/H is a soluble group hence it contains a maximal normal abelian subgroup L/H.Since G/H is finite, $|G:L| = |G/H : L/H|$ is finite. Let $S = \{x_1, \ldots x_n\}$ be a transversal of L in G. Since E/C is noncentral, $C_G(E/C) \neq G$ so G/H is nontrivial. In particular $L/H \neq 1$. Chose an L-chief factor U/C of E/C. C, E and L are normal subgroups of G so U^{x_i}/C is an L-chief factor of E/C for all x_i in S. Set $T/C = \langle U^{x_i}/C \mid x_i \in S\rangle$ and take $g \in G = \cup_{x_i \in S} Lx_i$, $uC \in T/C$ then $g = lx_i$ for some l in L, x_i in S and $uC = u_{1_1}^{x_1} \ldots u_{n_1}^{x_n}C$ where $u_{i_1}^{x_i}$ in U^{x_i}.

$$(uC)^g = (u_{1_1}^{x_1} \ldots u_{n_1}^{x_n}C)^{lx_i} = ((u_{1_1}^{x_1})^l \ldots (u_{n_1}^{x_n})^l C)^{x_i}$$

Since each U^{x_i}/C is L-invariant, $(uC)^g = (u_{1_2}^{x_1} \ldots u_{n_2}^{x_n}C)^{x_i}$ where $u_{i_2}^{x_i} \in U^{x_i}$. For all x_i, x_j in S, $x_ix_j \in G$ then $x_ix_j = l_kx_{i_k}$ where $l_k \in L$ and $x_{i_k} \in S$. Then $(uC)^g = u_{1_2}^{x_1x_i} \dots u_{n_2}^{x_nx_i}C = u_{1_2}^{l_1x_{i_1}} \dots u_{n_2}^{l_nx_{i_n}}C = u_{1_3}^{x_{i_1}} \dots u_{n_3}^{x_{i_n}}C$ where $u_{i_3}^{x_{i_k}} \in U^{x_{i_k}}$. Hence $(uC)^g \in T/C$. Therefore T/C is G-invariant . Since each generator U^{x_i}/C of T/C is in E/C but E/C is a chief factor of G, $T/C = E/C$.

Let $\{U^{x_{i_1}}/C, \dots, U^{x_{i_k}}/C\}$ be the smallest generator set of E/C. Then for all $U^{x_{i_j}}/C$ in $\{U^{x_{i_1}}/C, \dots, U^{x_{i_k}}/C\}$, we get

$$(U^{x_{i_j}}/C) \bigcap (U^{x_{i_1}}/C) \dots (U^{x_{i_{j-1}}}/C)(U^{x_{i_{j+1}}}/C) \dots (U^{x_{i_k}}/C) = C.$$

Since E/C is abelian, every element of $\{U^{x_{i_1}}/C, \dots, U^{x_{i_k}}/C\}$ commutes with each other. Therefore $E/C = U^{x_{i_1}}/C \times \cdots \times U^{x_{i_k}}/C$.

Assume that U/C is central in L. Then $C_G(U/C) \geq L$ and so $(C_G(U/C))^x \geq L^x = L$ for all x in G. Take g in $C_G(U^x/C)$ where x in G.

$$g \in C_G(U^x/C) \text{ iff } g^{-1}(u^x)^{-1}gu^x = (g^{x^{-1}})^{-1}u^{-1}(g^{x^{-1}})u \in C \text{ for all } u \text{ in } U.$$
$$\text{And } (g^{x^{-1}})^{-1}u^{-1}(g^{x^{-1}})u \in C \text{ iff } g^{x^{-1}} \in C_G(U/C).$$

So, $C_G(U^x/C) = (C_G(U/C))^x$. $H = C_G(E/C) = C_G(U^{x_{i_1}}/C \times \cdots \times U^{x_{i_k}}/C) = C_G(U^{x_{i_1}}/C) \times \cdots \times C_G(U^{x_{i_k}}/C) = (C_G(U/C))^{x_{i_1}} \times \cdots \times (C_G(U/C))^{x_{i_k}} \geq Dr_{i=1}^{k}L \geq L$ contradiction to $H < L$. Therefore U/C is non-central in L. Since $U/C \cong U^{x_i}/C$, we get U^{x_i}/C is non-central for all x_i in S. For simple notation, let us use U_i/C instead of U^{x_i}/C for every $1 \leq i \leq k$.

Given $a \in L \backslash H$, consider the mapping $\theta_a : E \to E$ by $\theta_a(e) = [e, a]$. Note that $a \notin H = C_G(E)$ hence $[E, a] \neq 1$. Take e_1, e_2 in E. $\theta_a(e_1e_2) = [e_1e_2, a] = [e_1, a]^{e_2}[e_2, a]$. Since E is abelian, $[e_1, a]^{e_2}[e_2, a] = \theta_a(e_1)\theta_a(e_2)$. So, θ_a is an endomorphism of E.

Since $E \trianglelefteq G$ and $L \leq G$, L acts on E by conjugation. Take $l \in L$ and $e \in E$; $\theta_a(le) = \theta_a(e^l) = [e^l, a] = (e^l)^{-1}a^{-1}(e^l)a = (e^{-1}a^{l^{-1}}ea^l)^l = [e, lal^{-1}]^l = [e, a[a, l^{-1}]]^l = ([e, [a, l^{-1}]][e, a]^{[a, l^{-1}]})^l$. Since L/H is abelian, $[a, l^{-1}] \in H$ which centralizes E. So $([e,[a,l^{-1}]][e,a]^{[a,l^{-1}]})^l = [e,a]^l = \theta_a(e)^l = l\theta_a(e)$.

$\theta_a(1e) = \theta_a(e^1) = \theta_a(e) = \theta_a{}^1(e) = 1\theta_a(e)$. Assume that $\theta_a((n-1)e) = (n-1)\theta_a(e)$. $\theta_a(ne) = \theta_a(e^n) = [e^n, a] = [ee^{n-1}, a] = [e,a]^{e^{n-1}}[e^{n-1}, a] = [e,a][e^{n-1}, a] = [e,a][e,a]^{n-1} = [e,a]^n = n\theta_a(e)$.

So θ_a is a $\mathbb{Z}L$-endomorphism of E.

Recall that $U_i/C \trianglelefteq J/C$ (so $U_i \trianglelefteq J$) and $U_i/C \leq E/C$ (so $U_i \leq E$) for all $1 \leq i \leq k$. Take $\theta_a(u_1) \in \theta_a(U_1)$. $\theta_a(u_1) = [u_1, a] = u_1^{-1}u_1 \in U_1$. Hence $\theta_a|_{U_1} : U_1 \to U_1$ is a $\mathbb{Z}L$-endomorphism. $[a, U_1] = \theta_a(U_1) = Im(\theta_a|_{U_1}) \cong U_1/Ker(\theta_a|_{U_1}) = U_1/(Ker\theta_a \cap U_1)$. $Ker\theta_a = \{e \in E \mid [e,a] = 1\}$. Since $[E,a] \neq 1$ and θ_a is a homomorphism, $Ker\theta_a \triangleleft E$. Since $C \leq Z(G)$, $C \leq Ker\theta_a$. Then $C = C \cap U_1 \trianglelefteq Ker\theta_a \cap U_1 \triangleleft E \cap U_1 = U_1$. Then $(Ker\theta_a \cap U_1)/C \triangleleft U_1/C$. Take $e \in Ker\theta_a$ and $l \in L$. $\theta_a(e^l) = \theta_a(e)^l = 1^l = 1$. Hence $Ker\theta_a \trianglelefteq L$. Then $(Ker\theta_a \cap U_1)/C \trianglelefteq L/C$. Recall that U_1/C is an L-chief factor. Then $Ker\theta_a \cap U_1 = C$. Hence $\theta_a(U_1) \cong U_1/C$. So $\theta_a(U_1)$ is a minimal L-invariant subgroup of E such that $C_L(\theta_a(U_1)) = C_L(U_1/C) \neq L$. In particular, $U_1 = \theta_a(U_1) \times C$.

The argument can be applied on $U_2U_1/\theta_a(U_1)$ and similar quotients. After finitely many steps, we conclude that there exists an L-invariant subgroup B such that $E = B \times C$. Since $B \cong E/C$, B has a finite series of L-invariant subgroups $\langle 1 \rangle = B_0 \leq B_1 \leq \cdots \leq B_n = B$ every factor of which is L-chief and it is not L-central. Repeating this on any conjugate B^g of B, $g \in G$, and on the product BB^g, we find that the latter has a finite series every factor of which is L-chief and it is not L-central. It follows that $BB^g \cap C = 1$ and hence $B = B^g$. Therefore B is a normal subgroup of G. □

Definition 3.4.2. Let H be a normal subgroup of a group G. H is called *essential normal subgroup*, if whenever $N \trianglelefteq G$ and $N \cap H = 1$, then $N = 1$.

Lemma 3.4.3. *Let G be a periodic strongly residually finite FC-group. If $Z(G)$ is an essential normal subgroup of G, then $O_p(G/Z)$ is finite for every prime p, where Z is the upper hypercenter of G.*

Proof. Z is a periodic locally nilpotent group and $Z(G)$ is a periodic abelian group so we can decompose these groups to their Sylow p-subgroups. Say $Z = Dr_{p\in\pi(Z)}Z_p$ and $Z(G) = Dr_{p\in\pi(Z)}C_p$ where Z_p and C_p are Sylow p-subgroups of Z and $Z(G)$ respectively. By Lemma 3.2.6, Z_p/C_p is finite for all prime p. But C_p char $Z(G)$ and Z_p char Z so Z_p has a finite central G-series. (i.e. we have the series $C_p \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_n = Z_p$ where $H_{i+1}/H_i \leq Z(G/H_i)$, $n \in \mathbb{N}$)

Let q be a prime such that $O_q(G/Z) = Q/Z$ is infinite. Let M/Z be a minimal G-invariant subgroup of Q/Z. $C_G(M/Z) = G$ implies that $M/Z \leq Z(G/Z)$. Then M is contained in the hypercenter of G contradiction to M/Z is non-trivial. So $C_G(M/Z) \neq G$. Let $Y = Dr_{p\neq q}Z_p$ so Y is a q'-group. We need to prove the following statement.

M/Y contains no G-invariant subgroup L/Y such that $L/Y \cap Z/Y = 1$.

Assume that there exists such L/Y. Note that $M/Z \cong (M/Y)/(Z/Y)$. Since M/Z and Z/Y are q-group, M/Y is a q-group. So, L/Y is a q-group. Then there exists $X \leq L$ such that $L = YX$ and $X \cap Y = 1$ (see [14], Theorem 5.25). X is a Sylow q-subgroup of L and so X char $L \trianglelefteq G$. Since $L/Y \cap Z/Y = 1$, $Y = L \cap Z$. Then $Y = YX \cap Z = Y(X \cap Z)$. So $X \cap Z \leq Y \cap X = 1$. But then $X \cap Z(G) = 1$ which is a contradiction to $Z(G)$ is an essential normal subgroup of G. Hence there exists no such L/Y.

Since $Z/Y = DrZ_p/Dr_{p\neq q}Z_p \cong Z_q$ and Z_p/C_p is finite for all prime p, there exists a finite central G-series $Y = V_0 \leq \cdots \leq V_k = Z$. Note that for all j in $\{0, \ldots, k-1\}$, G/V_j is a soluble FC-group and M/V_j is a q-subgroup of G/V_j containing the central subgroup Z/V_j. $(M/V_j)/(Z/V_j) \cong M/Z$ is a non-central

G/V_j chief factor. Then by Lemma 3.4.1, either there are elements $gV_j \in G/V_j$, $aV_j \in M/V_j$ and $cV_j \in Z/V_j$ such that $[gV_j, aV_j] = cV_j \neq V_j$ and $c^q \in V_j$ or M/V_j includes a G-invariant subgroup L_1/V_j such that $M/V_j = L_1/V_j \times Z/V_j$. In the above paragraph we showed that the latter case does not hold for $j = 0$. So there exist an index j and elements $g_1, a_1, c_1 \in G$ such that

$$[g_1V_j, a_1V_j] = c_1V_j \neq V_j,\ c_1^q \in V_j \text{ and } c_1V_j \in Z(G/V_j).$$

Set $E_1 = \langle g_1, a_1 \rangle^G$ and $F_1 = V_j \cap E_1$. Since G is a periodic FC-group, by Dicman's Lemma, E_1 is finite. Since G is residually finite group and E_1 is a finite subgroup of G, there exists a normal subgroup H_1 of G such that $|G : H_1| < \infty$ and $E_1 \cap H_1 = 1$. Application of the previous argument to $H_1 \cap Z$ and $H_1 \cap Q$ gives the existence of a G-invariant subgroup W and elements $g_2, a_2, c_2 \in G$ such that

$$[g_2W, a_2W] = c_2W \neq W,\ c_2^q \in W \text{ and } c_2W \in Z(G/W).$$

Set $E_2 = \langle g_2, a_2 \rangle^G$ and $F_2 = W \cap E_2$. If we continue this process infinitely many times, we can construct the G-invariant subgroups $E = Dr_{n\geq 1}E_n$ and $F = Dr_{n\geq 1}F_n$. But E/C is a ZN-q-factor of G which contradicts to Lemma 3.2.5. □

Theorem 3.4.4. *Let G be a locally soluble FC-group with $\pi(G)$ is finite. Then G is an SRF-group iff $G/Z^*(G) \leq T \times A$, where T is a bounded central-by-finite group and A is a torsion-free abelian group of finite rank and empty spectrum.*

Proof. Assume that G is an SRF-group then, by Theorem 3.3.4, $G/Z^*(G)$ is an SRF-group. Recall that $Z^*(G) = M\gamma = Dr_{j\in E}M_j$ for some index set E, where M_j is a noncentral minimal normal subgroup of $G/M_{\gamma-1}$. Since $G/M_{\gamma-1}$ is an FC-group, M_j is finite (see [14], Theorem 1.13) and so $Z^*(G)$ is in $T(G)$. Take a torsion element $yZ^*(G)$ in $G/Z^*(G)$. If $y^n \in Z^*(G)$ then there exists m such that $y^{nm} = 1$. Therefore the prime factors of nm are in $\pi(G)$ and so $\pi(G/Z^*(G))$ is also finite. So $G/Z^*(G)$ is a locally solvable strongly residually finite FC-group with $\pi(G/Z^*(G))$ is finite. Hence we may assume that $Z^*(G) = 1$ and show that $G \hookrightarrow T \times A$.

Let U be a maximal torsion-free subgroup of $Z(G)$ (By Zorn's Lemma such U exists). Then $Z(G)/U$ is a periodic group. By [14], Theorem 1.4, $G/Z(G)$ is also periodic. Recall that extension of a periodic group by a periodic group is periodic. So G/U is a periodic FC-group. Let $C/U = Z(G/U)$ and denote $\pi(G)$ by $\sum$. Since every periodic abelian group is the direct product of its primary components, $C/U = C_1/U \times C_2/U$ where C_1/U (respectively C_2/U) is a maximal $\sum$-subgroup (respectively $\sum'$-subgroup) of C/U. Assume that there exists a torsion element c_2 in C_2 then $c_2 \notin U$. Since C_2/U is a periodic $\sum'$-group, there exists an integer k whose prime factors are in $\sum$ such that the torsion element c_2^k in U. Hence $c_2^k = 1$ which contradicts to $c_2 \in G$. Therefore $C_2 \cap T(G) = 1$. Since $C/U = Z(G/U)$ char G/U, C_1/U and C_2/U are normal in G/U and so C_1 and C_2 are normal in G. Note that there exists no torsion-free normal subgroup $N \gneqq U$ of G. Otherwise $[G, N] \leq G' \cap N \leq T(G) \cap N = 1$. That is $N \leq Z(G)$. But this is a contradiction to U is maximal. Therefore, $C_2 = U$. That is, $\pi(C/U) \subseteq \sum$ which is finite.

We claim that $Z^*(G/U) = 1$. Assume that the claim is not true and let M/U is a noncentral minimal normal subgroup of G/U. Let V be the periodic part of M. Note that $V \neq 1$, otherwise $M \gneqq U$ is a torsion-free subgroup of G which is not possible. Since V is periodic and U is torsion-free, $V \cap U = 1$ and $V \cong VU/U$. V char $M \trianglelefteq G$ and $U \trianglelefteq G$ so $VU/U \trianglelefteq G/U$. Since M/U is a minimal normal subgroup of G/U, we have $VU/U = M/U$. So V is a chief factor of G. G is locally soluble hence V is abelian. Since $C_G(M/U) \neq G$ and

V is abelian, $C_G(V) \neq G$. That is V is a noncentral minimal normal subgroup of G which contradicts to $Z^*(G) = 1$. So, $Z^*(G/U) = 1$.

Let Z/U be the upper hypercenter of the periodic group G/U and let $S/Z = Soc(G/Z)$. Since G/Z is locally soluble, locally finite and S/Z is direct product of minimal normal subgroups, S/Z is a periodic abelian group. Moreover, for every prime p, by Lemma 3.4.3, the Sylow p-subgroup of S/Z is finite. Our next claim is that $\pi(S/Z)$ is finite. Since S/Z is a periodic abelian group we can write $S/Z = S_1/Z \times S_2/Z$ where S_1/Z is a $\sum'$-group and S_2/Z is a $\sum$-group. Since $\sum$ is finite, if $\pi(S/Z)$ is infinite, then the Sylow $\sum'$-subgroup S_1/Z is infinite. In particular, S_1/Z is non-trivial. Since $\pi(C/U) \subseteq \sum$, $\pi(Z/U) \subseteq \sum$. Therefore Z/U is a normal $\sum$-subgroup of S_1/U. But S_1/U is a periodic FC-group where $(S_1/U)/(Z/U)$ is a $\sum'$ group then there exists a $\sum'$ subgroup R/U such that $S_1/U = (Z/U)(R/U)$ (see [14], Theorem 5.25). Now $R/U \cong (S_1/U)/(Z/U) \cong S_1/Z \leq S/Z$. Hence R/U is abelian. Since Z/U is upper hypercenter of G/U and S_1/U is abelian, S_1/U is hypercentral. So, by Lemma 2.6.12, S_1/U is a locally nilpotent periodic group hence, it can be written as a direct product of its Sylow p-subgroups. Then every Sylow-$\sum'$ subgroup of S_1/U is characteristic in S_1/U so R/U is normal in G/U. Similar to C_2, R is a torsion-free group but $R \trianglelefteq G$ so $R = G$. Hence $\pi(S/Z)$ is finite. G/U is an SRF-group, $\pi(C/U)$ is finite and $[G/U, C/U] \leq U$ hence by Lemma 3.2.3, C/U is bounded. G/Z and Z/C are finite so $(G/U)/(C/U)$ is finite. Moreover C/U is bounded hence G/U is bounded. Take $T = G/U$ so T is central by finite and bounded. Set $A = G/T(G)$ then A is a torsion-free abelian SRF-group hence it is of finite rank and empty spectrum. Since $U \cap T(G) = 1$, $G \hookrightarrow T \times A$.

Conversely, assume that $G/Z^*(G) \leq T \times A$ and that all the conditions of the statement are satisfied. Let $Y = T(G)$. Since G/Y is torsion-free and

$Z^*(G) \leq Y$, G/Y isomorphic to a subgroup of A. Then by proposition 2.2.14, $(G/Z^*(G))/(Y/Z^*(G)) \cong G/Y$ is an SRF-group. Hence, by Lemma 3.1.5,

$$r_0((G/Z^*(G))/(Y/Z^*(G))) < \infty \text{ and } Sp((G/Z^*(G))/(Y/Z^*(G))) = \emptyset.$$

Similarly, $Y/Z^*(G)$ is isomorphic to a subgroup of T. So $Y/Z^*(G)$ is a bounded central-by-finite group. Then by Lemma 2.2.17, $Y/Z^*(G)$ is an SRF-group. Note that $Y/Z^*(G)$ is the torsion part of $G/Z^*(G)$. Then, by Theorem 3.1.10, $G/Z^*(G)$ is an SRF-group. Hence, by Theorem 3.3.4, G is an SRF-group. □

3.5 Metahypercentral Strongly Residually Finite FC-Groups

In this section we will deal with the characterization of metahypercentral strongly residually finite FC-groups. But first we need some preliminary work.

Definition 3.5.1. A group G is called *radical group* or *hyper-(locally nilpotent) group* if its radical series reaches G itself. In other words, there exists an ordinal γ such that we have an ascending series $1 = L_0 \leq L_1 \leq \cdots \leq L_\alpha \leq L_{\alpha+1} \leq \cdots \leq L_\gamma = G$ where, for each $\alpha < \gamma$, $L_{\alpha+1}/L_\alpha$ is the locally nilpotent radical (that is the maximal normal locally nilpotent subgroup) of G/L_α. In addition, the ordinal γ is said to be *the locally nilpotent length* of G.

Definition 3.5.2. A group G is called *monolithic* if the intersection M of all its non-trivial normal subgroups is non-trivial. The intersection M is called the *monolith* of G.

Example 3.5.3. Let G be a cyclic group of order p^n where $n \in \mathbb{N}$. G has a unique subgroup of order p. So G is monolithic.

Example 3.5.4. All nontrivial normal sugroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ where $n \in \mathbb{N}$. Since $\bigcap_{n\in\mathbb{N}} n\mathbb{Z} = 1$, $\mathbb{Z}$ is not monolithic.

Lemma 3.5.5. *Let G be a locally soluble FC-group with bounded central-by-finite Sylow subgroups. If G is monolithic, then G is finite.*

Proof. We claim that G is periodic. Suppose not. Let a be a torsion-free element in G. Since G is an FC-group, $H = \langle a^x \mid x \in G \rangle$ is a finitely generated G-invariant FC-group. Then $H/Z(H)$ is finite. Hence $Z(H)$ is a finitely generated abelian group. Then $Z(H) = T \oplus F$ where T is periodic and F torsion-free (see [13], 4.2.10). Since T is a finitely generated periodic FC-group, T is finite of order n for some $n \in \mathbb{N}$. Since F is a finitely generated torsion-free abelian group, F is isomorphic to direct sum of finitely many $\mathbb{Z}$. Hence $\bigcap_{l\in\mathbb{N}} F^l = 1$. Note that $Z(H)^{nl}$ char $Z(H)$ char $H \trianglelefteq G$ for all l in $\mathbb{N}$ and $Z(H)^n \leq F$. Let M be the monolith of G. Then $M \leq \bigcap_{l\in\mathbb{N}} Z(H)^{nl} \leq \bigcap_{l\in\mathbb{N}} F^l = 1$. Contradiction. So, $F = 1$ and hence G is periodic as we claimed.

If $Z(G) = 1$ then, by Lemma 2.3.10, G is residually finite. Then for any element x in G, there exists $N_x \trianglelefteq G$ such that $|G : N_x|$ is finite and $\bigcap_{x\in G} N_x = 1$. But G is monolithic. So for each x in G, $N_x = 1$. Then G is finite and we are done.

If $Z(G) \neq 1$ then M is a subgroup of $Z(G)$. So M does not have any proper subgroup. Therefore M is a cyclic abelian p-group for some prime p.

Let $P = O_p(G)$. Then $M \leq P$. Our next claim is that P is finite. Suppose that P is infinite. Set $C = Z(P)$ and $C_1 = \Omega_1(C)$. Then $M \leq C_1$. Since P is a bounded central-by-finite group, $Z(P)$ is an infinite bounded abelian group so direct sum of cyclic groups of bounded order(see [5], Theorem 10.1.5). Hence C_1 is infinite. Since G is a periodic locally soluble group, by Lemma 2.6.8,

$G = S_pS_{p'}$ where S_p is a Sylow p-subgroup of G and $S_{p'}$ is a Sylow p'-subgroup of G. Set $H = C \times S_{p'}$. H is a periodic FC-group, C is an abelian subgroup of H and $\pi(C) \cap \pi(S_{p'}) = \emptyset$. Further, C_1 is an elementary abelian p-subgroup of C and since C_1 characteristic subgroup of C, C_1 is normal in H. M is a finite subgroup of C_1 which is normal in H. Then C_1 contains a subgroup U which is normal in H such that $C_1 = M \times U$ (see [7], Lemma 2). Trivially U is $S_{p'}$-invariant. Since $Z(S_p)$ normalizes U and $|S_p : Z(S_p)|$ is finite, $|S_p : N_{S_p}(U)|$ is finite. Choose a transversal $\{x_1, \ldots, x_n\}$ for $N_{S_p}(U)$ in S_p. Since C_1 is a normal subgroup of G, each U^{x_i} is in C_1 and since $|C_1 : U|$ is finite, $|C_1 : U^{x_i}|$ is finite for all i. Let $V = U^{x_1} \cap \cdots \cap U^{x_n}$. Then $|C_1 : V|$ is finite. Since C_1 is infinite and $|C_1 : V|$ is finite, we get $V \neq 1$. By construction V is S_p invariant and, since every subgroup of C commutes with $S_{p'}$, it is also $S_{p'}$-invariant. Hence V is a non-trivial normal subgroup of G. But $V \cap M \leq U \cap M = 1$, which is a contradiction as M is a monolith. Hence P is finite.

If there exists an element g of prime order $q \neq p$ in $Z(G)$, then $\langle g \rangle \trianglelefteq G$ and so $M \leq \langle g \rangle$. But this is impossible. Hence $Z(G) \leq P$. In particular, $Z(G)$ is finite. We have $O_p(G/Z(G)) = O_p(G)/Z(G) = P/Z(G)$ and so $O_p(G/Z(G))$ is finite.

Assume that $G/Z(G)$ is infinite. Take an element $gZ(G)$ in $G/Z(G)$. Then $H_1 = \langle gZ(H) \rangle^{G/Z(G)}$ is a minimal normal subgroup of $G/Z(G)$. Since $G/Z(G)$ is infinite and residually finite, there exists an infinite normal subgroup N_1 of $G/Z(G)$ such that $H_1 \cap N_1 = 1$. Now take an element in H_1 and apply the same argument. Since this process will not terminate in finitely many steps, $G/Z(G)$ has infinitely many minimal normal subgroup. Hence $J/Z(G) = Soc(G/Z(G))$ is infinite. Since $G/Z(G)$ is locally soluble, each minimal normal subgroup of $G/Z(G)$ is abelian. Therefore $J/Z(G)$ is a periodic abelian group. Hence $J/Z(G) = J_1/Z(G) \times J_2/Z(G)$ where $J_1/Z(G)$ and $J_2/Z(G)$ are Sylow p and p'-subgroups of $J/Z(G)$ respectively. Then $J_1/Z(G) \leq O_p(G/Z(G))$ is finite. Hence $J_2/Z(G)$ is infinite. Consider the lower central series of J_2. Since $J_2/Z(G)$ is abelian, $\gamma_1(J_2) = [J_2, J_2] \leq Z(G)$. $\gamma_2(J_2) \leq [Z(G), J_2] = 1$. So J_2 is a periodic nilpotent group and so it can be written as a direct product of p-subgroups. Then $J_2 = Z(G) \times E$, where E is the Sylow p'-subgroup of J_2. E char J_2 and $J_2/Z(G)$ char $J/Z(G) \trianglelefteq G/Z(G)$. So, $E \trianglelefteq G$. But $E \cap M \leq E \cap Z(G) = 1$ which is a contradiction. Therefore $G/Z(G)$ is finite. Since $Z(G)$ is also finite, G is finite. □

Theorem 3.5.6. *Let G be a periodic locally soluble FC-group. If the Sylow subgroups of $G/Z^*(G)$ are bounded central-by-finite groups, then G is residually finite.*

Proof. Assume that $Z^*(G) = 1$. Take an element $1 \neq x \in G$ and let H be the maximal of normal subgroups which does not contain x. It is easily seen that each non-trivial normal subgroup of G/H contains xH. Therefore G/H is a monolithic group whit monolith $\langle xH \rangle^{G/H}$. Since the Sylow subgroups of G are bounded and central-by-finite, so are the Sylow subgroups of G/H (see [14], Theorem 5.4). So by Lemma 3.5.5, G/H is finite. Hence G is residually finite.

Now assume that $Z^*(G) \neq 1$. $G/Z^*(G)$ is a locally soluble FC-group. Since $Z^*(G/Z^*(G)) = 1$ and the Sylow subgroups of $(G/Z^*(G))/Z^*(G/Z^*(G))$ are bounded central-by-finite , by first paragraph, $G/Z^*(G)$ is residually finite. So, if $x \notin Z^*(G)$ then there exists $M_x/Z^*(G) \trianglelefteq G/Z^*(G)$ such that $xZ^*(G) \notin M_x/Z^*(G)$ and $|G/M_x|$ is finite. Hence G is residually finite. So assume that $x \in Z^*(G)$. Let $1 = R_0 \leq R_1 \leq \cdots \leq R_\gamma = Z^*(G)$ be an ascending G-chief series of $Z^*(G)$. There exists some $\beta < \gamma$ such that $x \in R_{\beta+1} \backslash R_\beta$. Since $R_{\beta+1}/R_\beta$ is a noncentral minimal normal subgroup of G/R_β, $(R_{\beta+1}/R_\beta) \cap Z(G/R_\beta) = 1$. Hence $xR_\beta \notin Z(G/R_\beta)$. Since G/R_β is an FC-group, $(G/R_\beta)/Z(G/R_\beta)$ is residually finite. Then there exists a normal subgroup M_x/R_β of G/R_β of finite index such that $xR_\beta \notin M_x/R_\beta$. Hence M_x is the required subgroup. Thus G is residually finite. □

Corollary 3.5.7. *Let G be a periodic locally soluble FC-group. If the Sylow subgroups of G are bounded central-by-finite groups, then G is residually finite.*

Proof. It is shown in the first paragraph of the proof of above theorem. □

Corollary 3.5.8. *Let G be a periodic locally soluble FC-group. If the Sylow subgroups of $G/Z^*(G)$ are bounded central-by-finite groups, then G is strongly residually finite.*

Proof. By Theorem 3.3.4, it is enough to show that $G/Z^*(G)$ is strongly residually finite. Since $Z^*(G/Z^*(G)) = 1$, we may assume that $Z^*(G) = 1$. Let H be a normal subgroup of G. Since G is a periodic FC-group, each Sylow p-subgroup of G/H is of the form S_pH/H, where S_p is a Sylow p-subgroup of G and p is a prime (see [14], Theorem 5.4). In particular the Sylow subgroups of G/H are bounded central by finite groups. Therefore, by Corollary 3.5.7, G/H is residually finite. □

Lemma 3.5.9. *Let G be a periodic metahypercentral FC-group and suppose that $Z^*(G) = 1$. If G is an SRF-group, then*

(i) G includes a normal subgroup $L = Dr_{p\in\pi(L)}L_p$, where L_p is a finite p-group for every $p \in \pi(L)$,

(ii) $G/L = Dr_{p\in\pi(G/L)}Q_p/L$, where Q_p/L is a bounded central-by-finite p-group for every $p \in \pi(G/L)$,

(iii) the Sylow p-subgroups of G are bounded central-by-finite groups for every $p \in \pi(G)$.

Proof. Let R be the locally nilpotent radical of the group G and Z be the hypercenter of G, $\pi = \pi(G)$. Since R (respectively, Z) is a periodic locally nilpotent group, $R = Dr_{p\in\pi} R_p$ and $Z = Dr_{p\in\pi} Z_p$ where R_p and Z_p are the Sylow p-subgroups of R and Z respectively. Since $Z^*(G) = 1$, every minimal normal subgroup of G is central. Take any normal subgroup N of G. Let $x \in N$. Since N is normal and G is a periodic FC-group, $\langle x\rangle^G$ is a finite G-invariant subgroup of N. So, $\langle x\rangle^G$ contains a minimal normal subgroup M of G. Hence $M \leq Z(G)$ and so $N \cap Z(G) \neq 1$. Therefore $Z(G)$ is essential in G. Then by Lemma 3.4.3, $O_p(G/Z)$ is finite for every prime p. Then $R_p/Z_p = R_p/(R_p \cap Z) \cong R_pZ/Z \leq O_p(G/Z)$ is finite. Choose a coset representation $\{x_1, \ldots, x_n\}$ of Z_p in R_p. Let $L_p = \langle x_1, \ldots, x_n\rangle^G$. R_p char $R \trianglelefteq G$ hence $L_p \leq R_p$. Note that L_p is a finite normal subgroup of G. Set $L = Dr_{p\in\pi} L_p$. Then L is a normal subgroup of G. So we have shown the first property.

Since L_p contains all coset representations of Z_p in R_p, $R_p = Z_pL_p$. $R/L = (Dr_{p\in pi} R_p)/L = (Dr_{p\in\pi} Z_pL_p)/L = (Dr_{p\in\pi} Z_pL_p)L/L = (Dr_{p\in\pi} Z_p)L/L = ZL/L$. Using the same method applied in the proof of Lemma 2.6.11, we can say that the hypercenter of G/L includes ZL/L. So there exists a series $L/L \leq L_1/L \leq L_2/L \leq \cdots \leq L_\alpha/L = R/L$ such that $Z(G/L_\beta) = L_{\beta+1}/L_\beta$ for all $\beta < \alpha$. Since G is metahypercentral, there exists a series $1 \triangleleft M \triangleleft G$ such that M and G/M is hypercentral. Since R is the maximal locally nilpotent normal subgroup of G, M is contained in R. $G/R \cong (G/M)/(R/M)$ hence by Lemma 2.6.11, G/R is hypercentral. So there exists a series $R/L \leq Z_1/L \leq Z_2/L \leq \cdots \leq Z_\gamma/L = G/L$ such that $Z(G/R) = Z_1/R$ and $Z(G/Z_\beta) = Z_{\beta+1}/Z_\beta$ for all $\beta < \gamma$. Hence G/L is hypercentral group and so it is a locally nilpotent, strongly residually finite, FC-group. Then by Theorem 3.2.7, $G/L = Dr_{p\in\pi(G/L)}Q_p/L$, where Q_p/L is a bounded central-by-finite p-group for every $p \in \pi(G/L)$.

For the last property, let S_p be a Sylow p-subgroup of G and set $J = Dr_{q\neq p}L_q$. Then $L/J \cong L_p$ is finite. Since G is strongly residually finite, G/J is residually finite. So, for each xJ in L/J there exists H_x/J such that $xJ \notin H_x/J$ and $(G/J)/(H_x/J)$ is finite. Set $H/J = \bigcap_{xJ\in L/J} H_x/J$. Then G/H is finite and $H/J \cap L/J = 1$. Then

$$H/J \cong (H/J)/(H/J \cap L/J) \cong [(H/J)(L/J)]/(L/J) = (HL/J)/(L/J) \cong HL/L$$

Therefore H/J is a periodic hypercentral group. Hence $H/J = Dr_{p\in\pi H/J}U_p/J$ where U_p/J is a Sylow p-subgroup of H/J. Since each Q_p/L is a bounded central-by-finite p-group, each U_p/J is bounded and central-by-finite. Then $S_pJ/J \cap H/J \leq U_p/J$ is bounded and central-by-finite. Since $|G:H|$ is finite, SJ/J is also bounded and central-by-finite. Since J is a $p' - group$, $J \cap S = 1$. Then $S \cong S/(J\cap S) \cong SJ/J$. So S is bounded and central-by-finite. □

Theorem 3.5.10. *Let G be a metahypercentral FC-group. Then G is an SRF-group if and only if $G/Z^*(G) \leq T \times A$, where*

(i) A is a torsion-free abelian group of finite rank and empty spectrum,

(ii) T includes a normal subgroup $L = Dr_{p\in\pi(L)}L_p$, L_p is a finite p-group.

(iii) $T/L = Dr_{p\in\pi(T/L)}\bar{Q}_p$, $\bar{Q}_p$ is a bounded central-by-finite group.

(iv) the Sylow p-subgroups of T are bounded central-by-finite groups.

Proof. Suppose that G is an SRF-group. Let $U/Z^*(G)$ be a maximal torsion-free subgroup of $Z(G/Z^*(G))$. Set $T = G/U \cong (G/Z^*(G))/(U/Z^*(G))$ and $A = (G/Z^*(G))/T(G/Z^*(G))$. We claim that $G/Z^*(G) \hookrightarrow T \times A$ where T and

A as in the theorem. Since $U/Z^*(G) \cap T(G/Z^*(G)) = 1$, we have $G/Z^*(G) \cong (G/Z^*(G))/[U/Z^*(G) \cap T(G/Z^*(G))] \hookrightarrow [(G/Z^*(G))/(U/Z^*(G))] \times [(G/Z^*(G))/(T(G/Z^*(G)))] \cong T \times A$.

Consider $T = G/U$. Since G is a strongly residually finite FC-group, T is a periodic strongly residually finite FC-group. Since G is metahypercentral, by Lemma 2.6.16, T is metahypercentral. Therefore, to apply Lemma 3.5.9 on T, it is enough to show that $Z^*(T) = 1$. Since $Z^*(G/Z^*(G)) = 1$, we may assume that $Z^*(G) = 1$. Let M/U be a non-central minimal normal subgroup of T and let V be the periodic part of M. If $V = 1$, then M is a torsion-free group. Since G is an FC-group, G' is periodic (see [14] Theorem 1.6) and so $M \cap G' = 1$. Therefore $[G, M] \leq M \cap G' = 1$. That is $M \in Z(G)$ which is a contradiction to M/U is non-central. Hence $V \neq 1$. Since the periodic part of an FC-group is characteristic, $V \cong V/(U \cap V) \cong VU/U$ char $M/U \trianglelefteq G/U$. Since M/U is a minimal normal subgroup of G/U, $V \cong M/U$ and so V is a chief factor of G/U. So, V appears in a chief series of G. Note that $C_G(V) = C_G(M/U) \neq G$. So $V \in Z^*G = 1$ which is a contradiction. So, $Z^*(T) = Z^*(G/U) = 1$. Hence by Lemma 3.5.9, T safisfies the properties (ii) and (iii) of the theorem.

Now consider the group A. Since G is a strongly residually finite FC-group, A is also a strongly residually finite FC-group. Moreover, since A is a torsion-free FC-group, it is abelian. Hence, by Lemma 3.1.5, A is a torsion-free abelian group of finite rank and empty spectrum.

Conversely, set $Y = T(G/Z^*(G))$. Since G is metahypercentral, Y is also metahypercentral. So Y is a periodic metahypercentral FC-group. Then by Lemma 2.6.17, it is locally soluble. Since Y is periodic, it embeds in T. So, Sylow p-subgroups of Y are bounded and central by finite. Then Sylow p-subgroups of $Y/Z^*(G)$ are bounded and central by finite. Hence, by Corollary

3.5.8, Y is an SRF-group. Since $(G/Z^*(G))/T(G/Z^*(G))$ is torsion-free, it embeds in A. By Lemma 3.1.5, A is an SRF-group. Since A is an abelian SRF-group, by Lemma 2.2.14, $(G/Z^*(G))/T(G/Z^*(G))$ is also an SRF-group. Hence $(G/Z^*(G))/T(G/Z^*(G))$ has finite rank and empty spectrum. Then, by Theorem 3.1.10, $G/Z^*(G)$ is an SRF-group. Then, by Theorem 3.3.4, G is an SRF-group. □

3.6 Strongly Residually Finite FC-Groups and SDF-Groups

In this section we consider periodic strongly residually finite FC-groups. As we mention above, P. Hall has shown that countable periodic residually finite FC-groups are contained in the class SDF. We see here that countability is not a necessary condition for periodic residually finite FC-groups to be in the class SDF.

Theorem 3.6.1. *Let G be a periodic strongly residually finite FC-group. Then G can be embedded in a direct product of finite groups.*

Proof. Let G be a periodic strongly residually finite FC-group and G' be the commutator subgroup of G. Since G is an SRF-group, G/G' is also an SRF-group. Let S/G' be a Sylow p-subgroup of G/G' for some prime p. Since $G' \leq S$, S is a normal subgoup of G. Therefore $S/G' \trianglelefteq G/G'$. Note that $[G/G', S/G'] \leq G'$. Hence, by Lemma 3.2.1, S/G' is bounded.

By Theorem 2.24 in [14], G is isomorphic to a subgroup of a centrally restricted product of finite groups. So, we may express G as follows:

$$G \leq (Dr_{i\in I}F_i)(T(Z(\textstyle\prod_{i\in I} F_i))) \text{ where each } F_i \text{ is finite.}$$

Set $D = Dr_{i\in I}F_i$ and $Z = T(Z(\prod_{i\in I} F_i))$. Take $x \in n(Z\cap GD)\bigcap(Z\cap G)$ where $n \in \mathbb{N}$. Since $x \in n(Z \cap GD)$, $x = n(x_1, x_2, x_3, \dots)$ and since $x \in (Z \cap D)$, $x = (y_1, \dots, y_k, 0, 0, \dots)$ where $x_i, y_i \in F_i$. Then $y_i = nx_i$ for all i. So, $x = (nx_1, \dots, nx_k, 0, 0, \dots) = n(x_1, \dots, x_k, 0, 0, \dots) \in n(Z \cap D)$. Hence $Z \cap D$ is a pure subgroup of $Z \cap GD$. Since $GD = ZD \cap GD = (Z \cap GD)D$, we get $GD/D \cong (Z \cap GD)D/D \cong (Z \cap GD)/(Z \cap GD \cap D) = (Z \cap GD)/(Z \cap D)$. So, $(Z \cap GD)/(Z \cap D)$ has bounded Sylow subgroups. Thus, by [2] Theorem 27.5, $Z \cap D$ is a direct factor of $Z \cap GD$. Let $B = Z \cap GD$ and $A = Z \cap D$. Then the Sylow p-subgroup A_p of A is a direct factor of the Sylow p-subgroup of B_p of B. That is $B_p = E_p \times A_p$ for every prime p. Therefore, $B = E \times A$ where $E = Dr_{p\in\pi(B)}E_p$. Hence $Z \cap GD = E \times (Z \cap D)$. So we get $G \leq GD = (Z \cap GD)D = (E \times (Z \cap D))D = E \times D$. □

Corollary 3.6.2. *Let G be a periodic FC-group. Then G is an SRF-group if and only if every factor group of G can be embedded into a direct product of finite groups.*

Proof. Let G be a periodic FC-group. If G is an SRF-group, for any normal subgroup H of G, G/H is a periodic strongly residually finite FC-group. Then, by Theorem 3.6.1, G/H can be embedded into a direct product of finite groups.

Conversely, since any direct product of finite groups is residually finite and every subgroup of a residually finite group is residually finite, every factor group of G is residually finite. Hence G is an SRF-group. □

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