SEMI-DYNAMIC MODELING OF TACTICAL LEVEL LAND COMBAT

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ABSTRACT

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In this thesis, we present a modeling framework for tactical level land combat involving heterogeneous forces. We propose an integrated system, consisting of three models interacting with each other. These models are: (1) an optimization model for force allocation, (2) an attrition simulation model, including a discrete-time stochastic model, that validates allocation results, and (3) a weapon effectiveness index update model. Our approach is based on decomposing the overall combat into stages and mini-battles. Mini-battles of one stage are solved using the three models, and the results are passed to the subsequent stage, creating a semi-dynamic environment. We also incorporate in our modeling approach the influence of events such as synergistic effects resulting from force combination and division, engagement, noncombat loss, and reinforcement.

Keywords: Heterogeneous Combat, Force Allocation, Discrete-Time Stochastic Model, Lanchester Model, Weapon Effectiveness

ÖZ

TAKTİK KARA MUHAREBELERİNİN YARI-DİNAMİK OLARAK MODELLENMESİ

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Bu çalışmada, heterojen kuvvetleri içeren taktik seviyedeki kara muharebesi için bir model çatısı sunuyoruz. Burada birbiri ile etkileşim içerisinde olan üç modeli içeren entegre bir sistem önermekteyiz. Bu modeller: (1) kuvvet tahsisi için bir optimizasyon modeli, (2) kesikli-zamanlı bir stokastik model içeren ve kuvvet tahsisi sonuçlarını geçerleyen zayiat simülasyon modeli, ve (3) silah etkinlik indeksi güncelleme modelidir. Bizim yaklaşımımız, muharebeyi safhalara ve alt-muharebelere ayrıştırmaya dayanmaktadır. Herhangi bir safhadaki alt-muharebeler bu üç modeli kullanarak çözümlendikten sonra, sonuçlar bir sonraki safhaya aktarılmakta ve böylece yarı-dinamik bir ortam oluşturulmaktadır. Ayrıca, kuvvetlerin birleşiminden ve bölünmesinden kaynaklanan sinerjik etkiler ile angajman, muharebe dışı kayıp, ve takviye gibi olayların etkileri de modele yansıtılmıştır.

Anahtar Kelimeler: Heterojen Muharebe, Kuvvet tahsisi, Kesikli-Zamanlı Stokastik Model, Lanchester Modeli, Silah Etkinliği

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LIST OF ABBREVIATIONS

ALLM	Allocation model
ARC	Attrition rate coefficient
ASM	Attrition simulation model
DLI	Deterministic local improvement
DSM	Discrete-time stochastic model
GA	Genetic algorithm
HS	Heuristic solution
IP	Integer programming
LM	Lanchester model
LP	Linear programming
OS	Optimal solution
PD	Percent deviation
SLI	Stochastic local improvement
SLM	Stochastic Lanchester model
SSKP	Single shot kill probability
WEI	Weapon effectiveness index

CHAPTER 1

Introduction

Defense expenditures constitute a significant portion of the overall budget of a country and governments are therefore trying to reduce the funds allocated for military activities. Attempts towards the development of cost efficient weapon systems and investigations for effective use of available weapons and ammunition are the main efforts in this direction. The Turkish Army has been planning and allocating the munitions according to the principles described in the Directive of Logistic Factors (KKY 54-5). However, latest progresses in weapon and ammunition technology have resulted in a considerable increase in the effectiveness of munitions in terms of higher probability of target detection and destruction. These developments make us question the current methods since some of them may have lost their validity. Hence, our motivation in this study is to develop a modeling framework that will eventually generate the input required for the weapon and ammunition planning system for Turkish Land Forces.

Perhaps the first step towards a weapon and ammunition planning system would be to estimate the amount of munition required to win the combat. To this end, we propose a modeling approach for heterogeneous combat where different types of military units might be involved. However, considering the complexity of planning and allocation of munitions, we restrict the scope of our study to the tactical level land combat involving units from squad to brigade.

As will be revealed in literature survey, two major classes of combat models are the force allocation models and the combat simulation models. The force allocation models in general aim at optimization, but they have a high level of abstraction and are static in nature. The class of simulation models, on the other hand, is more detailed (realistic) and dynamic, but they can only be used for prediction and do not support optimal planning decisions. Our modeling framework is an attempt to bridge the gap between these two classes, bringing the advantages of both together.

Instead of modeling combat as a whole we decompose it into stages and mini-battles with the purpose of combining optimization and simulation within a semi-dynamic modeling scheme. We keep the problem parameters constant within a stage but allow them to change as we move from one stage to the next, thereby creating a semi-dynamic environment.

Combat modeling requires combined use of several models within an overall framework where implementation of these models in a logical order and synchronization among them become crucial. Hence, we propose an integrated system for analyzing the combat situation by focusing on a mini-battle that takes place in a combat stage. This system consists of three models interacting with each other, and deals with land combat problems involving heterogeneous forces. These models are the allocation model (ALLM), the attrition simulation model (ASM) and the weapon effectiveness index update model (WEI). Each model focuses on different aspect, which is inherent in any combat problem. Having modeled mini-battles of one stage with these three models, we transfer the results to subsequent stages for handling the overall combat.

For each mini-battle in a stage, we begin by solving a discrete optimization model, ALLM, which aims at allocating allied forces (blue units) to enemy forces (red units). ALLM minimizes the weapon effectiveness (value or cost) of blue units used to win the battle, subject to the constraints for satisfying attrition goals set for red units. ALLM facilitates allocation of blue units with some prespecified fractions. Hence, a blue unit can be divided among multiple red units, and multiple blue units can be combined against a red unit. Synergistic effects due to force division and combination are also taken into consideration in ALLM.

A blue unit allocated by ALLM can be heterogeneous, meaning that it consists of a number of subunits of different types. A subunit, however, is homogeneous in itself. For example, an infantry battalion is a heterogeneous unit consisting of homogeneous subunits such as infantry companies, antitank and mortar platoons.

After blue units are allocated, we run ASM to simulate the combat for both sides based on the subunits which are hierarchically one level lower than the units considered in ALLM. Lanchester Model (LM), which is one of the two modules of ASM, is used to simulate the engagements between large forces where the attrition process of each unit is traditionally modeled by a system of differential equations (Taylor 1983). LM results give information about the general behavior of forces. However, randomness plays an important role and must be taken into consideration in combats involving small force sizes. Therefore, we propose a discrete-time stochastic model (DSM) which is the second simulation tool of ASM.

We incorporate in DSM the effects of events such as synergistic effects due to force division and combination, engagement, noncombat loss, and reinforcement through stochastic processes. Allowing multiple casualties and overkills in one firing cycle are other features of DSM which makes the model more realistic when it is compared with the classical stochastic approaches.

Our modeling approach permits some parameters, such as desired duration of a stage and target loss fractions for red forces, to be input by the user as combat goals. Hence, it can be used as a decision support system and/or a training tool for analyzing various combat scenarios. A communication environment between the optimization model and the simulation model is created by running ALLM and ASM iteratively, until the two models converge (Özdemirel and Kandiller, 2001). Convergence occurs when ASM indicates that the user specified combat goals can be achieved by the force allocations obtained from ALLM.

The third model, WEI, provides information that can be used in planning the use of blue forces throughout the combat stages (Özdemirel and Kandiller, 2001). It evaluates the values of blue forces in each stage of combat and aims at utilizing the potential of any blue unit as much as possible in the stage where it is of vital importance (or where it will be most effective).

The organization of the thesis is as follows. In Chapter 2, we present a review of the combat modeling literature. After presenting the literature regarding force allocation models and combat simulation models we address other modeling issues and some critiques on combat models. In Chapter 3, following the definition of our problem environment, we present our overall modeling framework based on decomposition of a combat and solving combat problems in a semi-dynamic environment. In Chapter 4, we describe the allocation model (ALLM). Here, after demonstrating our binary programming formulation, we present our approach for solving ALLM and also address previous studies in this field. In Chapter 5, we describe the attrition simulation model (ASM) which consists of two modules: discrete-time stochastic model (DSM) used for small forces and Lanchester model (LM) used for large forces. We describe DSM in detail addressing the main model and its extensions through use of stochastic processes. We demonstrate how to integrate DSM and LM for combat situations where small and large forces engage each other. In Chapter 6, we first explain the interaction between ALLM and ASM. Next, we describe the use of weapon effectiveness update model (WEI). Finally, we define a combat scenario and illustrate the overall decision support system (DSS) procedure in a mini-battle. Finally, in Chapter 7, following a discussion on our work, we point out some possible future research directions.

Some aspects of the modeling framework proposed in this thesis have been developed by others. For completeness, we describe those developments in related chapters when necessary. For the sake of clarity, however, we summarize the contribution of the thesis in Table 1.1.

Table 1.1 A summary of the thesis' contribution

Chapter	Thesis Work	Other Work
3 and 6	- Decomposition of combat into stages and mini-battles - The idea of using ALLM, ASM and WEI in an overall modeling framework	- DSS procedure for integrated use the three models (Özdemirel and Kandiller, 2001)
4	- Binary formulation of ALLM - Decomposition and column generation (and heuristic) for ALLM solution	- Enumeration method for ALLM solution (Özdemirel and Kandiller, 2001) - Genetic algorithm for ALLM solution (Erdem and Özdemirel, 2002)
5	- DSM for attrition simulation - Integrating DSM and LM for attrition simulation	- LM for attrition simulation (Taylor, 1983)
6	- The idea of using dual prices in WEI	- WEI procedure for use of dual prices and marginal gain (Özdemirel and Kandiller, 2001)

CHAPTER 2

Literature Review

The use of scientific methods in military problems was initiated in early twentieth century with the early development of deterministic combat models by Lanchester (1914). He investigated air combat situations during World War I by applying ordinary differential equations to populations of fighter planes. Later, many researchers have enriched these models by incorporating additional situations. These combat models represent the attrition of opposing sides under different types of engagements. Initial studies, which were concerned with deterministic models, were followed by the developments of stochastic models. As a result of these researches now we have a literature on mathematical modeling of combat.

In this chapter we present the summary of the literature we have reviewed to reveal critical issues relevant to combat modeling followed by a discussion of recognized deficiencies and our contribution to this field. Our taxonomy is presented in Figure 2.1. The publications that we have reviewed are given as a table in Appendix A.

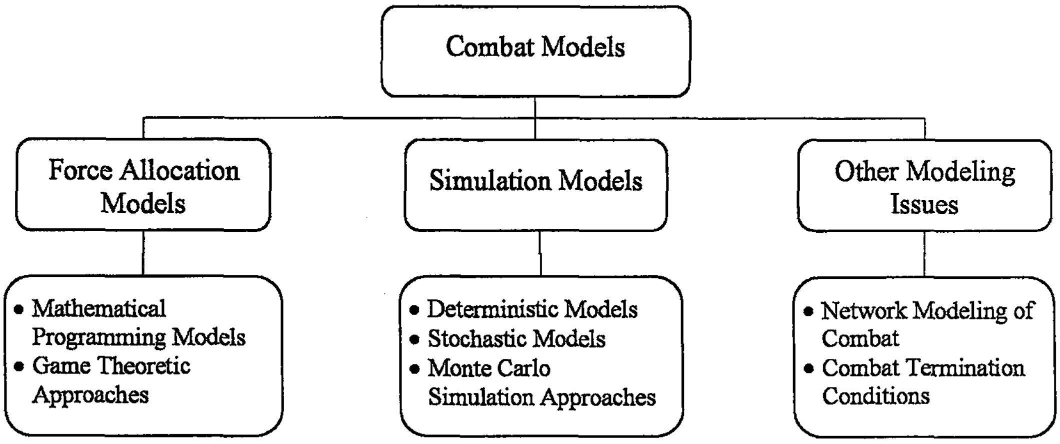


Figure 2.1 A taxonomy of combat models

One may also classify combat models in terms of the resolution. For example, deterministic models are low resolution models where the entities are aggregated. On the other hand, resolution increases in stochastic models monitoring the behavior of entities. Davis (1995) introduced variable-resolution modeling and development of integrated families of models with varied resolutions. After indicating that the term "resolution" can be used for different concepts, he listed the factors that may be used to define high resolution also clarifying the needs for both low-resolution and high-resolution models. Fowler (1999) presented a classification scheme for aggregation in combat models.

2.1 Force Allocation Models

These are the models developed for deciding how to assign forces to opposing units (or distribute firepower over targets) as summarized below.

2.1.1 Mathematical Programming Models

It is possible to use the assignment, and allocation models to formulate force deployment and allocation problems. Kevin and Lam (1995), in their study on force deployment strategies, formulated a nonlinear programming model to find the optimal force level of blue and red forces to be deployed in various phases of combat. They illustrated their model on an example providing sensitivity analysis, and the relationships between objective values and constraints through use of Lagrange multipliers.

Jaiswal (1997) demonstrated four linear programming formulations that can be applied to military decision problems. These are; i) *weapon mix problem* for assigning weapons to aircrafts to protect an important vulnerable area where the objective is to maximize the average number of aircrafts destroyed, ii) *weapon deployment problem* in deploying different types of air defense weapons at some sites, iii) *weapon target allocation problem* to allocate air defense systems to each site in order to maximize the overall protection of the sites, and iv) *sortie allocation problem* for allocating aircraft sorties to attack several targets with the objective of maximizing the average value of targets destroyed.

NATO developed the first version ACROSS (Allied Command Europe Resource Optimization Software System) in 1998, which is an integrated system used at strategic level (corps or higher) for munitions planning under possible threat situations. ACROSS consists of three models, one of which is LEMEM (Land Force Equipment and Munitions Expenditure Model) used in computing the amount of weapon and ammunitions in a cost-effective way. LEMEM is a linear programming formulation to find the optimal fire allocation strategies, weapon and munitions need and overall cost.

Kwon *et al.* (1999) formulated a weapon-target allocation problem as a nonlinear integer programming model, with the objective of minimizing the total firing cost, where various types of weapons are to be allocated to several targets under munition availability constraints. They linearized the nonlinear constraints some of which are relaxed by Lagrangean relaxation and they used a branch-and-bound approach to solve the problem and provided computational results for test problems.

Özdemirel *et al.* (2000), and Özdemirel and Kandiller (2001) presented an integer programming formulation for optimal allocation of blue force to red units and they also incorporated combination and division effects into their model. They presented an enumeration technique which can be applied to small problems. Erdem and Özdemirel (2002) developed a genetic algorithm to deal with large sized allocation problems.

2.1.2 Game Theoretic Approaches

Pugh (1973) modeled the combat as a time sequential two-person zero-sum game. He used a combination of game theory and dynamic programming to determine the allocation of forces to each region and to compute the payoff in each period and also to compute the overall payoff. Pugh and Mayberry (1973) used game theory and utilized the ratio of losses and the weighted difference of losses as valid payoff criteria and indicated that the objective of a strategy should be minimization of one's own losses while maximizing the losses of enemy.

Kawara (1973) modeled and analyzed the problem of allocating support fire between two different targets (enemy primary units and support units) as differential game defining the problem as a two person zero sum game. Later, Taylor (1977) reviewed Kawara's model to determine the class of terminal payoffs that yield force-level-independent optimal strategies to the fire support game.

Maliphant and Smith (1995) used a strategy game, which is named "risk", including an attacker and a defender battling each other to defeat their opponents' forces. They used the game as a means to develop a dynamic programming approach where decisions are made sequentially one at a time.

2.2 Simulation Models

These are the models computing the attritions (and also the number of survivors) in an engagement within a combat duration and thus providing information required for making decisions in remaining part of combat. We classify the simulation models as deterministic, stochastic, and Monte Carlo simulation models and present the review of relevant literature below.

2.2.1 Deterministic Simulation Models

Combat models developed by Lanchester (1914) employ static attrition rate coefficients and explain the combat dynamics describing the changes in the force levels of combatants over time from a deterministic viewpoint. Since then, Lanchester models provide information about the general behavior of units and are applicable to aggregated units involving a large number of combatants. In its original form, combat is assumed to take place in between two homogeneous forces where the effects of each side against the other are reflected through the use of constant attrition rate coefficients (ARCs). These basic models are given as a couple of differential equations where directed fire model is as follows.

$$\frac{dX}{dt} = -aY \tag{2.1}$$

$$\frac{dY}{dt} = -bX \tag{2.2}$$

where $X=X(t)$ and $Y=Y(t)$ denote the number of combatants (or weapons) of blue and red forces at time t whereas a and b are the ARCs representing the effectiveness of one combatant of red and blue respectively. For example, a represents the number of blue combatants killed by one red combatant per unit time. The initial number of blue and red combatants, say X_0 and Y_0, sets the initial conditions for these differential equations. The differential equation system for area fire where the attrition of a force is dependent on both its own size and the size of the opposing force is as follows.

$$\frac{dX}{dt} = -aXY \tag{2.3}$$

$$\frac{dY}{dt} = -bYX \tag{2.4}$$

Third basic model, which is called mixed fire, is given by the following equations where blue is subject to directed fire and red is subject to area fire.

$$\frac{dX}{dt} = -aY \tag{2.5}$$

$$\frac{dY}{dt} = -bYX \tag{2.6}$$

Closed form solutions for time dependent force levels in directed fire and area fire models can be easily obtained solving the relevant differential equations with given initial conditions, when forces are homogeneous and attrition rate coefficients are constant. These solutions are demonstrated by many researchers. It is not so easy to find out closed form solutions for X and Y separately in the mixed fire case; however if a final value of one at time t is given, the force level of the other is easily computed.

The fourth model is called the logarithmic law and formulates the noncombat loss process.

$$\frac{dX}{dt} = -aY - \beta X \tag{2.7}$$

$$\frac{dY}{dt} = bX - \alpha Y \tag{2.8}$$

where β and α are noncombat loss rates of blue and red, respectively. Finally the fifth model, which is named as Helmbold model, is as follows.

$$\frac{dX}{dt} = -a\left(\frac{X}{Y}\right)^{1-\omega} Y \tag{2.9}$$

$$\frac{dY}{dt} = -b\left(\frac{Y}{X}\right)^{1-\theta} X \tag{2.10}$$

Here ω is the fraction of blue unit that can be used effectively against red unit when initial force ratio X_0/Y_0 is too high where $0<\omega<1$. θ is defined similarly for red.

There is a literature on extensions and modifications of basic Lanchester Equations. Unfortunately most of the researches deal with homogeneous forces (consisting of only one type of combatant or weapon) either due to easiness of introducing of a new concept on homogeneous combat or due to computational reasons. Since homogeneous combat models are a well known issue, we addressed the studies handling heterogeneous forces below.

First efforts allowing heterogeneity are the studies investigating the distribution of fire over a number of targets such as the research of Isbell and Marlow (1956), and Taylor (1972).

Howes and Thrall (1973) developed a procedure utilizing Perron-Frobenius theory of eigenvalues and eigenvectors to compute the overall weight (effectiveness) of a heterogeneous force, which is the sum of the weighted averages of individual weapon effectiveness values that are derived from inter-weapon effectiveness matrices, assuming these matrices are given.

Taylor (1974a) used Lanchester equations in combination with optimal control theory to find optimal fire distribution policy where he used range dependent attrition rate coefficients with the assumption that all weapons of a force have the same range capability. He, for instance, defined variable attrition rate coefficients as, $b_i(t)=k_{b_i}\, h(t)$ for blue weapon i where k_{b_i} is the constant portion and $h(t)$ is the variable portion. He assigned values per unit of surviving forces and analyzed a two-on-one combat which is heterogeneous in the sense that different survival values and attrition rate coefficients are assigned for each type of weapon. He presented the formulation for distribution of fire over n targets and then demonstrated his approach on a problem where n=2. Taylor (1974b), in another study, reviewed the common issues related to fire distribution. Starting with a list of main questions regarding fire distribution he indicated the following findings.

- When the targets undergo a square-law (directed fire) attrition process;
  - All fire is always concentrated on one target type (0-1 allocation rule).
  - The allocation is not (directly) dependent upon the force levels.
- When the targets undergo a linear-law (area fire) attrition process;
  - Fire may be divided between target types (policies other than 0-1 rule are possible).
  - The allocation is (directly) dependent upon the force levels.

Finally he reached the following conclusions regarding the dependence of fire distribution on the capabilities of forces.

- When intelligence and command control systems are very efficient, the optimal tactic is to concentrate fire on a specific target type.
- When capability for redirection of fire from destroyed targets is poor, the optimal tactic may be to allocate fire in a proportional fashion over target types.

Taylor (1975), as another extension, handled the fire distribution problem including force-level constraints in time sequential allocation problems and using measures of strategic value of firing at a target indicating that optimal fire distribution policy depends on the force levels and not on time. One of the results he obtained is the motivation to value the targets directly proportional to their fire effectiveness.

Later, Taylor and Brown (1978) made another improvement in the field of fire distribution problems which was demonstrated on a two-on-two heterogeneous combat where they investigated optimal allocation of supporting fires during the approach of attacking infantry to contact enemy defensive positions.

Taylor (1982) defined two types of target acquisition process for Lanchester type combat models. These are: i) serial acquisition where a firer (or a weapon) can not acquire targets while it is engaging another target, and ii) parallel acquisition where it can search targets continuously while engaging other targets.

However, the enrichments to Lanchester models cause the problem become intractable when solving analytically and obtaining a closed form solution is either difficult or impossible. Hence, we should seek procedures to solve these problems numerically. Taylor (1983) illustrated the formulation of Lanchester equations for heterogeneous forces. He suggested the use of simplest methods, such as Euler-Cauchy method, indicating that they work well since Lanchester-type equations are well behaved.

There are some studies including modeling approaches that are inspired from other systems. Protopopescu and Dockery (1989), in an effort of compensating some shortcomings of Lanchester equations, developed a combat model using partial differential equations incorporating the effects of spatial dependence and nonlinear effects. Their formulation includes the following elements, none of which are present in the classical Lanchester systems.

- *Diffusion:* natural tendency of any force to lose its original configuration as it moves, fights, or simply just as time goes by, due to fatigue, loss of concentration, loss of motivation, etc.
- *Advection:* large scale, ordered "flow" of troops on the battlefield.
- *State dependent attrition* of forces, which are closing on one another.

Przemieniecki (1994) addressed how to determine the attrition rate coefficients which are the basic parameter of Lanchester type systems.

Hudges (1995) examined the measure of combat power's mental effect which is the suppression of enemy actions. After reviewing the principles from combat science that are relevant to the proportion of losses of forces and stating that the observable effects of combat power are not only physical but also mental and spiritual, he developed a quantitative approach using Lanchester square law to illustrate the suppression effect of enemy fire.

Bitters (1995) used the concept of elasticity from economics and developed a more meaningful measure of effectiveness called force elasticity,

$$\epsilon_F = (dR/R)/(dB/B) \tag{2.11}$$

which is a dimensionless local measure comparing the percent change in the red and blue forces at a specific time t. He developed the conditions for parity, red win and blue win are shown in terms of force elasticity in a fight to finish battle where one or both sides will annihilate eventually.

Gass (1997) developed a formulation using the continuity equations of fluid dynamics and obtained a model for combat at a micro (small unit) level that may be a part of the overall campaign. The main feature of his model is the inclusion of movement of the battlefront, replenishment of the losses and withdrawal of combat units.

Considering the solution of Lanchester equations for heterogeneous forces, Jaiswall (1997) described a method which depends on the use of eigenvalues and eigenvectors where this method is applicable when some conditions are required. Fowler (1999) introduced two methods for aggregating the heterogeneous quadratic Lanchester system into a homogeneous one.

Özdemirel *et al.* (2000) mentioned a variety of techniques for computing attrition rate coefficients and single shot kill probabilities, which are the main inputs for combat models.

2.2.2 Stochastic Models

Stochasticity plays an important role in engagements between small units where detailed observation of the behavior of each combatant is obviously necessary, such as a 2-on-3 combat situation of tanks.

The researches of Taylor (1983), Gafarian and Ancker (1984), Kress (1987) and, Gafarian and Manion (1989) are the examples of early studies on stochastic combat modeling. They deal with stochastic duels (or small firefights) including small number of combatants. Later, researchers have continued investigating stochastic behavior of combat and contributed by modeling and analyzing different combat characteristics.

Yang and Gafarian (1995) developed an algorithm based on solving a set of exact Kolmogorov equations and approximating the kill rate of one combatant, conditioned on the state of the system, for solving homogeneous stochastic combat models. At the end, they concluded that solving battles greater than 4-on-4 requires huge amount of computation. Speight (1995) used a discrete time Markov model as a stochastic equivalent of deterministic and continuous Lanchester Formulation and compared the results at the mini-battle level.

Jaiswall, Sangeeta, and Gaur (1995) modeled the combat between two opposing forces as a continuous-time discrete-state space Markov process and evaluated some combat characteristics such as the distribution, mean and variance of combat duration, the probability of win of either side, the expected number of survivors at termination, the joint probability distribution of the forces and the ratio of expected losses.

Anderson (1995) described attrition formulas for large scale combat for a variety of problem situations. He handled many-on-many combats which are heterogeneous in the sense that various types of munitions are used and treated directed and area fires separately. Regarding the area fire he assumed that i) targets are uniformly distributed in an area, ii) fires of weapons may overlap in each salvo, and iii) a target is killed in a salvo with a kill probability if it is in the fatal area (otherwise it survives that salvo).

Parkhideh and Gafarian (1996) made a research for developing general solutions to many-on-many heterogeneous stochastic combat. They modeled the combat between two units (each having a number of combatants) as a sequence of aiming and killing events where target selection is at random in the aiming events. Assuming that combat ends when any side reaches its predefined breakpoint, they computed state probabilities by enumerating all possible routes that the combat may go through (defining sequences of aiming and killing events) and then computing probabilities of aiming and killing events that takes the combat to a specific state.

Jaiswall, Sangeeta, and Gaur (1997), in another study, modeled a homogeneous combat as a continuous-time discrete-state space Markov process to analyze the effect of reinforcements made at prespecified force levels on various combat characteristics.

Yildirim (1999), concerning target acquisition process in stochastic combat modeling, addressed the original finite-state Markov chain with states "invisible",

"visible and acquired" and "visible and not acquired" where transition from the "invisible" state to "visible and acquired" state is not allowed. Then he modified the Markov chain such that the above transition is allowed.

McNaught (1999) explored the effects of applying Exponential Stochastic Lanchester (ESL-stochastic version of deterministic square law) to battles which have been split into smaller engagements modeling the combat as a Markov Chain where the forces are homogeneous. McNaught (2001) solved two variants of one-on-one duels modeling as a continuous time Markov Chain where the distribution of inter-firing times follows 2-phase Erlang distribution in the first model, and exponential distribution in the second.

Armstrong (2001) formulated the one-on-one homogeneous combat as a Markov Chain and provided results regarding the computation of expected ammunition consumption and the effect of suppression.

Salim and Hamid (2001) formulated a Bayesian stochastic combat model and introduced the concept of stochastic variable attrition rate coefficients (treating these coefficients as random variables). They chose beta distribution as a prior distribution for survivor probability and provided some results for the expected values of attrition rate coefficients and the distribution of the number of survivors.

2.2.3 Monte Carlo Simulation Models

Although it is not within our scope (and hence, we did not review in detail) we know that there are some researches using Monte Carlo simulation.

JANUS, which is a high resolution simulation system developed in early 1980s for analyzing tactical level combat, uses Monte Carlo in estimating attritions.

Speight (1997) used a Monte Carlo battle simulation as a tool with the purposes of judging whether Lanchester formulation could provide a description of mobile engagement, and adjusting the parameters as the engagement conditions were changed. Şeref (2001) developed a Monte Carlo simulation model for land combats.

2.3 Other Modeling Issues

In this section we review the literature concerning the decomposition of combat into phases and mini-battles and then address the studies regarding combat termination conditions.

2.3.1 Network Modeling of Combat

Speight (1995) made an important contribution to combat modeling by handling an overall combat as network of smaller local engagements. He introduced the "mini-battle" concept to represent each of these small engagements. He stressed that the outcome of a series of mini-battles would be different from that of a single Lanchester battle and the conditions of each mini-battle will have a fundamental effect on the whole process of target acquisition.

Ancker (1995) also suggested that a combat should be analyzed as hierarchical network of firefights (mini battles). Speight (1997) reemphasized mini-battle formation and target acquisition. Another research utilizing the concept of mini-battles is the study of McNaught (1999) in which he split the combat into smaller engagements and applied Exponential Stochastic Lanchester.

Kevin and Lam (1995), after breaking the battle into phases of equal lengths, used constraints analogous to inventory balance equations to describe the losses for each stage of the battle and to relate force levels through stages.

2.3.2 Combat Termination Conditions

Termination conditions which finalize the simulation process are also investigated by some researchers. Taylor (1972) proposed two scenarios for tactical allocation problems. They are the "prescribed duration battle" where the battle ends at the end of a fixed duration, and "terminal control battle" where the battle terminates when certain terminal states (like complete destruction of one side in a fight to finish battle) are reached.

Taylor (1974a), Taylor and Brown (1978) used prescribed duration approach where the combat ends after a fixed duration in their studies on optimal fire allocation policies.

Jaiswal and Nagabhushana (1995) defined four combat termination decision rules depending on the casualties and the ratio of force strengths as follows.

- *Absolute decision rule* (*A*): The combat terminates when the force strength falls below a given threshold value.
- *Proportional decision rule* (*P*): The combat terminates when the force ratio reaches a specified threshold value.
- *AOP rule:* The combat terminates when the force strength curve crosses either the absolute or the proportional threshold lines.
- *AAP rule:* The combat terminates when the force strength curve crosses

both the absolute and the proportional threshold lines.

They examined the effect of these rules on winning condition for blue or red, and on the number of survivors at combat termination using a generalized force strength equation, which satisfies a large class of attrition laws.

Jaiswall, Sangeeta, and Gaur (1995) employed these four termination decision rules in a continuous-time discrete-state space Markov process they developed.

Speight and Rowland (1999) conducted a study on combat degradation stressing that military performance in actual battle tends to be significantly worse than that normally achieved in peacetime operations due to task differences, deterioration of military skills and active contribution to battle. They incorporated the degradation effects into a Monte Carlo simulation using the previously developed mini-battle formation and target acquisition.

2.4 Critiques and Discussion

There were some common critiques in literature indicating the deficiencies of Lanchester Models such as use of constant attrition rate coefficients and use of homogeneous forces only. Other than these, Protopopescu and Dockery (1989) stated the following shortcomings of Lanchester Equations:

- They do not account for the movement of forces on the battlefield.
- Command and Control (C^2) is usually not treated.

As a result of their study they proposed a new combat model using partial differential equations in an effort to incorporate the above issues.

Ancker (1995) reviewed combat theory and reached the conclusion that there are some deficiencies. Proposing two axioms of combats and the combat theorem based on them he emphasized that analysis of a combat as hierarchical network of firefights, where a firefight is a terminating stochastic target attrition process on a discrete state space, is essential to better understand combat and develop models.

During the literature review we identified some deficiencies or drawbacks in combat models and our critics can be summarized as follows.

(1) Force allocation models are static.

(2) Combat simulation models are dynamic but they can not be used in force planning since they yield only prediction.

(3) Models handling different types of units or analyzing different phases of combat are not integrated.

(4) More developments are needed to take into consideration the dynamic nature of combats.

Our contribution based on these is the development of an integrated system bringing force allocation models and combat simulation models together in a semi-dynamic environment. This system consists of three models interacting with each other and allows involvement of heterogeneous forces of any size. Our approach follows from breaking the overall combat into stages and transferring updated information through stages. We also incorporate the effects of possible events such as force combination and division, engagement, noncombat loss, and reinforcement through stochastic models to increase realism. Our system is used in estimating force (or weapon system) requirements and generate information that will lead to weapon and ammunition planning.

CHAPTER 3

Problem Definition and Modeling Approach

Military activities include both peacetime activities (such as training, field exercises and logistics planning) and wartime activities like deployment of forces, combats, and combat support activities. Although we focus on the analysis of wartime operations in our study, our aim is to generate input for weapon and ammunition planning, which is a part of peacetime planning. In this chapter, we first define our specific problem environment and then explain our modeling approach for land combat problems. Finally, we briefly describe the three models, and the overall system.

3.1 Problem Environment

A combat is characterized by the interaction among various force types -land force, air force and navy- where cross interactions such as army vs. air force, navy vs. air force, etc. are also possible. This interaction is mainly characterized by attrition of one side mainly due to fires of the opposing side, and the resulting casualties are called combat losses. However, some additional losses, which depend only on one's own side, take place during a combat as well. These losses are called non-combat losses and occur due to factors other than the enemy fire such as illnesses, accidents and desertions.

Depending on whether the target is visible and can be aimed at, a combatant can use either directed fire (where it fires at a certain target) or area fire (where it fires at a certain area in which targets are distributed). Several models and solution approaches have been developed to handle the attrition process during a combat. These models are mainly classified as deterministic and stochastic models. Deterministic models generally utilize attrition rates whereas stochastic models use single shot kill probabilities to estimate the attrition of forces during a combat.

The organization of a military force consists of hierarchically connected units where the size and structure differs at each level. In terms of planning and operations, there are two main levels: strategic level, which includes the higher level units (corps and higher), and tactical level that includes the lower levels (brigade and lower). Our study is concerned with the engagements occurring in a land combat at tactical level given the higher-level operation plan.

The most significant aspect of a combat is its dynamism. There are many factors yielding this dynamism such as the force size (the number of combatants in a unit), effectiveness of weapons, weather and terrain conditions that change over time. The blue and red force sizes and other basic inputs change during the course of battle in terms of both quantity and quality. The number and effectiveness of combatants, firepower, survivability and mobility are some of the factors subject to change. Combat models should reflect attrition of both sides under these dynamic conditions.

The formulation of combat dynamics for heterogeneous forces is a difficult problem. Unfortunately, most of the studies deal with engagements between homogeneous forces to achieve computational tractability. In this thesis work, we aim modeling heterogeneous combat problems and solving them using a semi-dynamic approach.

3.2 Terminology and definitions

There are two sides in a combat. We use the term **blue** to represent the side for which we will obtain a solution, and we use **red** for the opposing side. The following definitions will be used throughout this study.

Combat: A fight or struggle between two opposing forces to achieve specified objectives. Usually the side that has force superiority and initiative adopts an *offensive* action while the other side *defends*. The offending side generally has the force superiority with a ratio of three to one.

Combat stage: A part of the combat after which some decisions can be made and new plans can be implemented.

Combat objective: It can be capturing or destroying a target, delaying the enemy movement, defending a position, minimizing (maximizing) own (opposing) loss or loss ratio, or other specified objective.

Deployment: The movement of troops to go to a new position or to enter a combat.

Reserve force: The force kept to reinforce or replenish the combating units.

Reinforcement: Deployment of additional troops to strengthen the combating units.

Replenishment: Deployment of additional troops to furnish the combating units by replacing some of them.

Weapon: An instrument or device of any kind used to injure or kill in a combat.

Ammunition: Anything hurled by a weapon or exploded as a weapon, such as bullets, shot, shells, bombs, grenades and rockets.

Munitions: The combat supplies of units consisting of weapons and ammunition.

Homogeneous forces: A force is said to be *homogeneous* if it consists of only one type of combatant (or weapon) such as only infantry or only artillery.

Heterogeneous forces: If a force includes two or more types of combatants (or weapons) such as infantry, artillery, and tanks it is then called *heterogeneous*.

Military unit: Hierarchically connected divisions of the armed forces, having different structures and sizes at each level, such as squad, platoon, company, battalion, and so on. A unit is a heterogeneous formation consisting of subunits of different types and capabilities.

Subunit: Division of a unit which is one level lower in hierarchy like a platoon of a company, or a company of a battalion. We assume that each subunit of a unit is homogeneous, that is all entities (weapons or combatants) within a subunit have the same capabilities.

Weapon effectiveness index (WEI) of a unit: Relative effectiveness of a unit compared with the others which reflects its value and can be defined according to the factors such as firepower, mobility, and survivability. It can also be used to approximate the cost of using each unit.

Attrition rate coefficient (ARC) of a subunit: A measure of effectiveness of the subunit and defines the rate at which it destroys the opposing unit. Equivalently, it can be defined as the number of targets killed by one subunit per unit time.

Single shot kill probability (SSKP): The probability that a combatant kills its target (an opposing combatant) at one shot.

Salvo: A firing cycle of fixed duration within which some weapons fire simultaneously.

Most of the studies assume that ARCs are available where these values are actually difficult to estimate in practice. However, SSKPs can be easily obtained from technical and operational data of weapon systems and from field exercises observing the skills of combatants in aiming and shooting events. Values of ARCs and SSKPs can be found by using one of the available systems such as NATO's TASCFORM illustrated by Regan and Wogt (1988) or OLI developed by Dupuy (1979).

3.3 Overall Modeling Approach

In this section, we first describe our overall modeling framework followed by a brief description of each model. Instead of modeling a combat as a whole, we decompose it into stages where each stage consists of a number of local engagements. Hence, we develop an approach based on three models to analyze a combat stage by stage where these models interact with each other.

An overall combat between two forces can be decomposed into sub-combats or mini-battles interacting with each other to reflect the dynamism (Ancker 1995, Speight 1995). Such a decomposition scheme consisting of a number of mini-battles is shown in Figure 3.1. There are two stages of combat as seen in the figure. The first stage consists of two independent mini-battles taking place simultaneously at two different locations separated by a mountain. The results of the two mini-battles in the first stage determine the starting conditions of the second stage where the remaining forces from the first stage after replenishment and/or reinforcement and noncombat loss become the initial force levels of the second stage.

The basic input for modeling a mini-battle in any stage is obtained from the results of the mini-battles in the previous stage. The inputs required at the beginning of a stage are as follows:

- Current structure of blue and red forces (units and subunits)
- The number of combatants and amount of weapon and ammunition of blue and red units
- Attrition rate coefficients (ARCs) and single shot kill probabilities (SSKPs) of blue and red units

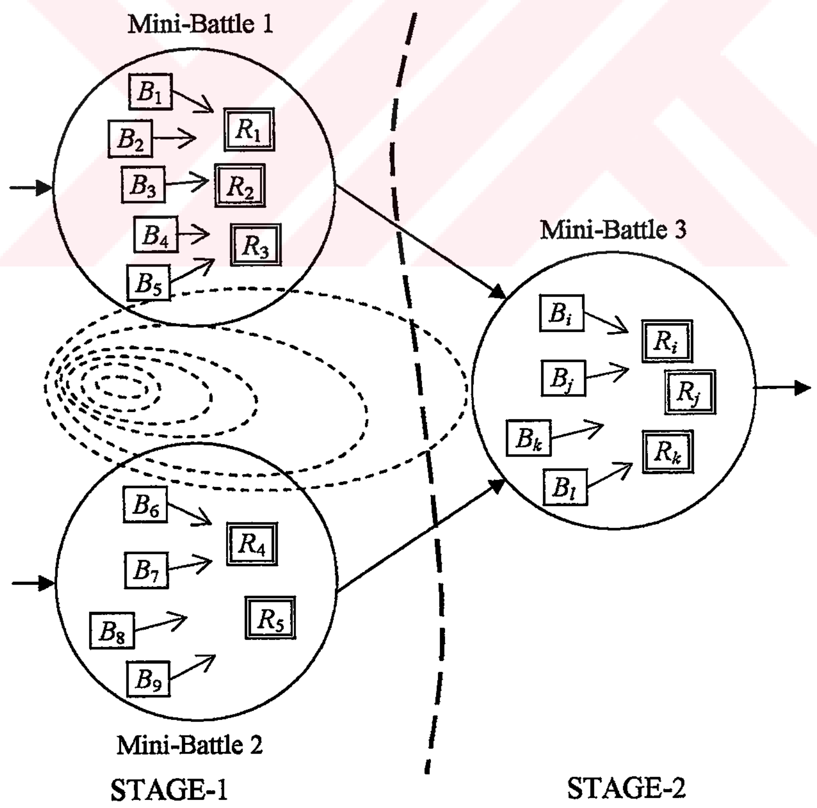


Figure 3.1 Representation of land combats as a system of interacting mini-battles

- Weapon effectiveness indices (WEIs) of blue units to be used in that stage
- Restrictions on allocation of blue forces to red units
- Noncombat loss coefficients of blue and red units
- Weather conditions
- Terrain properties

In our approach, we assume that each mini-battle can be modeled independently and the results of mini-battles at a stage can be used as input (beginning conditions) for the mini-battles in the next stage.

Our proposed approach for modeling a mini-battle in any stage of the land combat is given in Figure 3.2. This approach requires the use of three models that interact with each other. These models are:

a. Allocation Model (ALLM) which allocates blue units to red units.
b. Attrition Simulation Model (ASM) consisting of deterministic Lanchester model (LM) and discrete-time stochastic model (DSM), which simulates the combat under allocations suggested by ALLM.
c. Weapon Effectiveness Index Update Model (WEI) which updates the effectiveness of a military unit at the end of the stage.

3.4 Allocation Model (ALLM)

The first model to be used in modeling a mini-battle is the Allocation Model (ALLM), which is a binary programming model. WEI values, attrition rate coefficients (ARCs), required minimum attrition fractions for red units and restrictions on allocation of blue units are the inputs of ALLM. ALLM determines

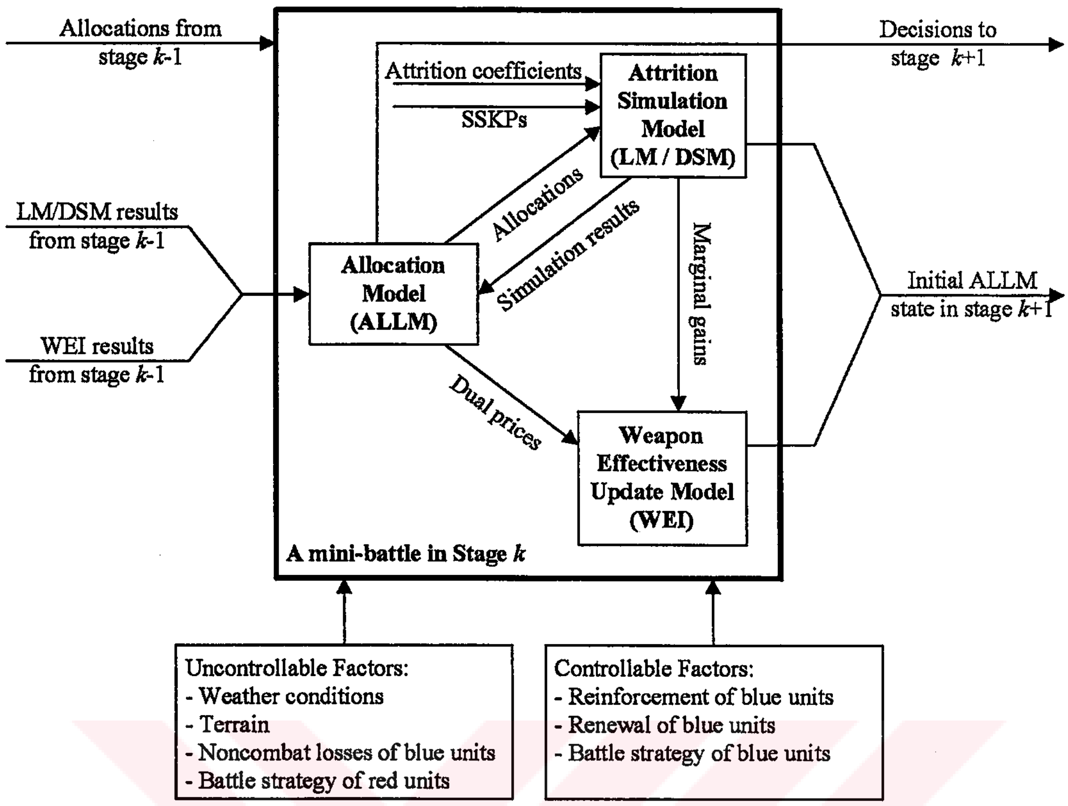


Figure 3.2 The modeling of a mini-battle

the assignment of blue forces to red forces, that is the distribution of blue firepower over red at the tactical level (squad, platoon, company, battalion, or brigade). This model aims to destroy or to wear out the red side using the minimum total blue weapon effectiveness subject to the attrition goal constraints for red units. The details of the formulation and solution approaches of ALLM are given in Chapter 4.

We assume that firepower of a blue unit can be divided among multiple red units by certain fractions. In this case the overall attrition rate caused by the blue unit can be less than it would be if the blue were allocated as a whole to a single red unit. This phenomenon is called the *division effect.* Also, (fractions of) multiple blue units can be allocated to a red unit, in which case the overall attrition effect of these units will be more than the mere sum of the individual attrition effects. This is called the *combination effect*. Incorporating these synergy effects into ALLM increases the realism of the model as well as its complexity.

3.5 Attrition Simulation Model (ASM)

This model is used to estimate attritions of units in a mini-battle, based on the assignments obtained by solving ALLM. Attrition simulation model consists of two simulation tools: deterministic Lanchester model (LM), which is used for simulating the attrition of large unit, and discrete-time stochastic model (DSM) which is used for small units.

ALLM and ASM are implemented at different resolution levels. In ALLM, the basic force element considered for allocation is a blue unit, which can be heterogeneous. In ASM, however, the basic element simulated is the homogeneous subunit. For example, if ALLM allocates a battalion, ASM simulates the infantry soldiers, tanks and anti-aircraft weapons of the battalion as separate subunits.

ALLM is one-sided (allocates only blue units to red units), and does not consider red side's reaction. ASM models the interaction between two sides. Hence, ASM helps us to check if combat objectives can indeed be achieved under ALLM's allocations.

Lanchester model (LM)

LM consists of differential equation systems based on the use of ARCs. Basic differential equations, which were developed by Lanchester (1914) for homogeneous forces under directed and area fire conditions, are adapted in LM for heterogeneous forces. The system of equations for the heterogeneous forces is solved by numerical approximation since closed form solutions are not available. We incorporate division and combination effects used in ALLM also in LM and noncombat loss is included in LM as well.

Deterministic combat problems can be modeled using the LM library. The main characteristic of LM is that it can only be applied to battles involving large number of identical combatants (generally > 20), where the approximate behavior of the whole subunit is simulated. With small force sizes, however, randomness plays an important role and must be taken into consideration. For this reason, we propose DSM, as an alternative to LM, for simulating subunits having a small number of combatants.

Discrete-Time Stochastic Model (DSM)

In our thesis, we have developed a discrete-time stochastic model (DSM) with the purpose of overcoming some drawbacks of LM. DSM is used to model the attrition of units having small number of combatants (generally ≤ 20). The use of single shot kill probabilities (SSKPs), instead of ARCs is also an advantage of DSM since SSKPs are easier to estimate.

In DSM, we treat combat as a stochastic process proceeding in salvos of fixed duration where salvo duration can be easily adjusted according to the fire cycle of a weapon type. Forces on both sizes are heterogeneous each consisting of a number of military units which can be of different types. Both directed fire and area fire are modeled by means of combinatorial analysis where overkills are taken into consideration. Division and combination of units are also allowed in DSM as in LM and ALLM. Furthermore, discrete reinforcements are included whereas engagements and noncombat losses are handled using single dimensional stochastic processes. Finally, using DSM the variance and the mean of remaining force size is found for each military unit at the end of each salvo facilitating the risk analysis.

Use of LM for heterogeneous forces, DSM, and integration of LM and DSM are discussed in detail in Chapter 5.

3.6 Weapon Effectiveness Index Update Model (WEI)

Using the dual prices obtained by solving the LP relaxation of ALLM and the marginal gain results of each military unit from ASM, WEI updates the effectiveness of a blue unit. Marginal gain is the increase in the final force level of a blue unit when we increase its initial force level by adding one subunit. Based on the dual prices WEI computes a new effectiveness for each unit to be used in the next stage of combat. This way, the weapon effectiveness indices are updated semi-dynamically as the combat progresses from one stage to the next.

The purpose of WEI update is to force the model use a blue unit in the stage where it is of vital importance. This is achieved by increasing the WEI value, which is analogous to cost, of this blue unit in other stages except the stage where it should be used. Details of WEI update model are explained by Özdemirel *et al.* (2000, 2001).

3.7 Basic Decision Support System (DSS) Cycle

The three models defined above can be integrated within the framework of a decision support system (DSS) for semi-dynamic modeling of combat. These models, which constitute the model base of the DSS, should be used iteratively in a cyclic manner as explained by Özdemirel *et al.* (2000). The DSS procedure can be summarized as follows.

ALLM and ASM are used iteratively, as shown in Figure 3.2, to obtain a convergence. According to the first solution of ALLM, ASM is run to simulate the battle using the proper tool (LM or DSM) depending on force size, and the results are examined to see the "realized" levels of objective function and attrition goal constraints. If there is a difference between the obtained attrition levels and the preset goals, attrition goals are adjusted. Then ALLM is run again to increase or decrease the associated allocations, followed by another ASM run. When convergence is obtained within a prespecified tolerance limit, then the final solution of ALLM is taken as the optimal solution for the mini-battle.

When a mini-battle is solved by the above ALLM-LM interaction, WEI is used to update the weapon effectiveness values. This process is repeated for each mini-battle of the current stage. At the end of the stage, ASM and WEI results are transferred to the next stage. The implementation of DSS cycle is illustrated on an example in Chapter 6.

CHAPTER 4

Force Allocation Model (ALLM)

Our modeling approach begins with deciding the allocation of blue units to red units in a mini-battle by solving ALLM. In this chapter, we first briefly define our allocation problem and give its formulation. Then, we explain an alternative formulation attempt based on column generation and decomposition, followed by description of a heuristic approach to improve the solution. Finally, we present the experimentation results on test problems.

4.1 Formulation

ALLM, which is actually an integer programming model, allocates blue units to red units to reach the predefined attrition goals for red units where its objective is the minimization of the total weapon effectiveness (or total cost) of blue forces used. Consideration of force division and combination and their effects in allocations is a significant feature of ALLM. Our assumptions concerning force division and combination are summarized below.

A1) A blue unit can divide its force among at most three red units. Considering a military unit typically consists of three subunits, division between two red units can be with fractions (1/3, 2/3) or (1/2, 1/2). In case of division among three red units, the fractions are (1/3, 1/3, 1/3).

A2) When a blue unit divides its force among multiple red units, its total attrition potential will be lower compared to the case when it is allocated to a single red unit. This is called the division effect.

A3) When multiple blue units are allocated to a single red unit, their total attrition potential will be higher than the mere sum of individual attrition potentials. This is called the combination effect.

A4) If a portion of a blue unit is to be used then the entire unit will be allocated. This assumption can be accepted by the decision maker in DSS or relaxed to allow partial allocations. Division and combination effects are assumed to be present with partial allocations as well.

By means of assumptions A2 and A3, combat dynamics can be better represented by considering synergistic effects of force division or combination.

Some allocation combinations may not be meaningful when assigning blue units to red units due to reasons such as terrain conditions and capabilities of blue units. Hence, we give the set of possible allocations, which can be represented by an allocation graph, as input to ALLM. An example of such a graph is given in Figure 4.1 for a mini battle which takes place between three blue and three red infantry units. Blue unit 1 can be divided either with fractions (1/3, 2/3) or (2/3, 1/3) between the first and the second red units. The complete set of allocation alternatives for this unit is then {(1/3, 2/3), (2/3, 1/3), (0, 1), (1, 0), (0, 0)} under assumption A4. If we relax A4, the set also includes the alternatives (1/3, 0), (2/3, 0), (0, 1/3) and (0, 2/3).

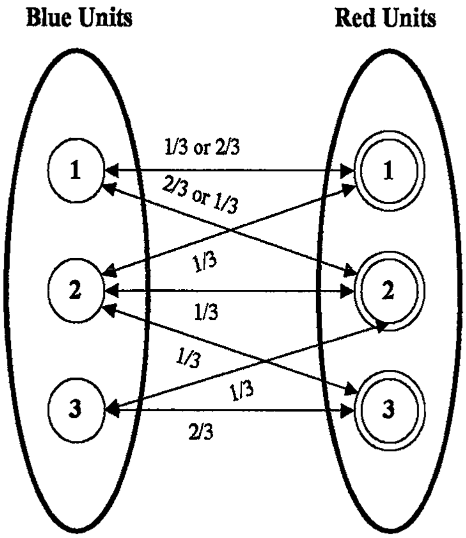


Figure 4.1 Allocation graph

Although mini battles in a stage are modeled independently, we set a common target battle duration T for all them all. Notation used in ALLM and formulation for a mini battle in a stage is given below. Among the parameters, stage targets T and μ_j, which are specified by the decision maker, define combat goals.

i Blue unit index, i=1, …, I.

j Red unit index, j=1, …, J.

x'_{ij} Decision variable indicating fraction of blue unit i assigned to red unit j.

WEI_i Weapon effectiveness index (value or cost) of blue unit i.

T Target battle duration for the stage.

μ_j Target fraction of loss in red unit j to be reached before time T, $0 \leq \mu_j \leq 1$.

α_j Attrition goal for red unit j to be reached before time T. Its initial value is found as $\alpha_j = \mu_j R_{j0}$ where R_{j0} is the initial force level of red unit j.

b'_{ij} Total net attrition given by blue unit i to red unit j after division and combination effects are considered. These parameters are found by running ASM one-to-one for each (i, j) pair. If the remaining force level at time T is $R_j(T)$ then it is found as $b'_{ij} = R_{j0} - R_j(T)$.

A simplified formulation for ALLM where the objective is to minimize the total value or cost of blue forces used subject to the constraints for satisfying red attrition goals is given below.

$$\min \quad z = \sum_{i=1}^{I} \sum_{j=1}^{J} WEI_i x'_{ij}$$

s.t.

$$\sum_{i=1}^{I} b'_{ij} x'_{ij} \geq \alpha_j \qquad j=1, \ldots, J$$

$$\sum_{j=1}^{J} x'_{ij} \leq 1 \qquad i=1, \ldots, I$$

$$x'_{ij} \in \left\{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\right\} \qquad i=1, \ldots, I,\ j=1, \ldots, J$$

A blue unit can be unused, used totally, or used with a fraction as dictated by the allocation graph. In the real formulation, there are a number of binary indicator variables detecting divisions and combinations, hence taking the synergistic effects into consideration. Moreover, there are some technical constraints to regulate unit inclusion/exclusion in combinations and divisions. Below, we define the division and combination effect parameters and illustrate formulation of the complete set of constraints by using two examples.

$\lambda(i, k_i)$ Loss in attrition potential of blue unit i when it is divided among k_i red units, k_i=1, 2, 3, $\lambda(i,1)=0$, $0 \leq \lambda(i,2) \leq \lambda(i,3) \leq 1$.

$\phi(i, j, l_j)$ Gain in attrition potential of blue unit i against red unit j when l_j blue units are combined against red unit j, l_j=1,2, …,n_j, $\phi(i,j,1)=0$, $0 \le \phi(i,j,2) \le \phi(i,j,3) \le 1$, $\phi(i,j,l_j)=\phi(i,j,3)$ for $l_j > 3$.

In the first example, let us consider force combination where three blue units, $i1$, $i2$ and $i3$, can be allocated to red unit j according to the allocation graph. There are three possible cases of combination.

Case 1: Only one of the blue units is assigned to red unit j. In this case, the attrition goal constraint for red unit j takes the simple form

$$b_{i1,j}x_{i1,j} + b_{i2,j}x_{i2,j} + b_{i3,j}x_{i3,j} \ge \alpha_j$$

where b_{ij} is the attrition given by blue unit i to red unit j *before* division and combination effects are considered, and x_{ij} is the binary decision variable indicating whether or not blue unit i is assigned to red unit j.

Case 2: Blue units $i1$ and $i2$ are assigned to red unit j. The attrition goal constraint then takes the nonlinear form

$$b_{i1,j}x_{i1,j} + b_{i2,j}x_{i2,j} + b_{\{i1,i2\},j}x_{i1,j}x_{i2,j} \ge \alpha_j \tag{4.1}$$

where $b_{\{i1,i2\},j} = \phi(i1,j,2)b_{i1,j} + \phi(i2,j,2)b_{i2,j}$ is the two-way combination effect. We can linearize the constraint by defining a new binary decision variable and adding some technical constraints.

$$y_{\{i1,i2\},j} = \begin{cases} 1, & \text{if } x_{i1,j} = x_{i2,j} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$b_{i1,j}x_{i1,j} + b_{i2,j}x_{i2,j} + b_{\{i1,i2\},j}y_{\{i1,i2\},j} \ge \alpha_j \tag{4.2}$$

$$x_{i1,j} + x_{i2,j} \le 1 + y_{\{i1,i2\},j} \tag{4.3}$$

$$y_{\{i1,i2\},j} \le x_{i1,j} \text{ and } y_{\{i1,i2\},j} \le x_{i2,j} \tag{4.4}$$

Note that in constraint (4.2) the new variable replaces $x_{i1,j}x_{i2,j}$ that was used in (4.1).

Constraint set (4.4) indicates that there is no combination if either one of the blue units is not assigned to red unit j. Constraint (4.3) activates the combination effect only when both blue units are assigned.

Case 3: All three blue units are assigned to red unit j. Again we need to define a new decision variable and additional constraints for linearization.

$$y_{\{i1,i2,i3\},j} = \begin{cases} 1, & \text{if } x_{i1,j} = x_{i2,j} = x_{i3,j} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$b_{i1,j}x_{i1,j} + b_{i2,j}x_{i2,j} + b_{i3,j}x_{i3,j} + b_{\{i1,i2,i3\},j}y_{\{i1,i2,i3\},j} \geq \alpha_j \tag{4.5}$$

$$x_{i1,j} + x_{i2,j} + x_{i3,j} \leq 2 + y_{\{i1,i2,i3\},j} \tag{4.6}$$

$$y_{\{i1,i2,i3\},j} \leq x_{i1,j}\text{, } y_{\{i1,i2,i3\},j} \leq x_{i2,j}\text{, and } y_{\{i1,i2,i3\},j} \leq x_{i3,j} \tag{4.7}$$

where $b_{\{i1,i2,i3\},j} = \phi(i1, j,3)b_{i1,j} + \phi(i2, j,3)b_{i2,j} + \phi(i3, j,3)b_{i3,j}$ used in (4.5) is the three-way combination effect.

To take care of all three cases in the same constraint, we need to rearrange the three-way combination effect in order to avoid double counting of two-way effects that are already included in the three-way effect, i.e.

$$\begin{aligned} b_{\{i1,i2,i3\},j} = {} & \phi(i1, j,3)b_{i1,j} + \phi(i2, j,3)b_{i2,j} + \phi(i3, j,3)b_{i3,j} \\ & - \left(b_{\{i1,i2\},j} + b_{\{i1,i3\},j} + b_{\{i2,i3\},j}\right) \end{aligned}$$

The main attrition goal constraint then becomes

$$\begin{aligned} b_{i1,j}x_{i1,j} + b_{i2,j}x_{i2,j} + b_{i3,j}x_{i3,j} + b_{\{i1,i2\},j}y_{\{i1,i2\},j} + b_{\{i1,i3\},j}y_{\{i1,i3\},j} + b_{\{i2,i3\},j}y_{\{i2,i3\},j} \\ + b_{\{i1,i2,i3\},j}y_{\{i1,i2,i3\},j} \geq \alpha_j \end{aligned}$$

To complete the formulation for three-way combination, we should write technical constraints (4.3) and (4.4) for each of the ($i1$, $i2$), ($i1$, $i3$), and ($i2$, $i3$) pairs, and add constraints (4.6) and (4.7).

In the second example, we consider force combination and division together.

Suppose that blue units $i1$, $i2$ and $i3$ can be allocated to red units $j1$ and $j2$ according to the allocation graph. Units $i1$ and $i3$ can only be assigned as a whole either to $j1$ or to $j2$. This restriction can be represented by constraints

$$x_{i1,j1} + x_{i1,j2} \leq 1 \text{ and } x_{i3,j1} + x_{i3,j2} \leq 1.$$

Unit $i2$, on the other hand, can be divided between $j1$ and $j2$ with fractions 1/3 and 2/3. We need other binary variables and technical constraints to facilitate the division.

$$z_{i2}^{1} = \begin{cases} 1, & \text{if } i2 \text{ is divided} \\ 0, & \text{otherwise} \end{cases}$$

$$x_{i2,j1} + x_{i2,j2} \leq 1 + z_{i2}^{1}$$

$$z_{i2}^{1} \leq x_{i2,j1} \text{ and } z_{i2}^{1} \leq x_{i2,j2} \tag{4.8}$$

The superscript used in variable z indicates one of the alternative ways of dividing a blue unit according to the allocation graph.

Case 1: Only one blue unit is assigned to each red unit. The attrition goal constraints for red units $j1$ and $j2$ are then

$$b_{i1,j1}x_{i1,j1} + b_{i2,j1}x_{i2,j1} + b_{i3,j1}x_{i3,j1} - b_{\frac{i2}{3}}z_{i2}^{1} \geq \alpha_{j1} \text{ and}$$

$$b_{i1,j1}x_{i1,j1} + b_{i2,j1}x_{i2,j1} + b_{i3,j1}x_{i3,j1} - b_{\frac{2i2}{3}}z_{i2}^{1} \geq \alpha_{j2}$$

where

$$b_{\frac{i2}{3}} = b_{i2,j1}\left(\frac{2}{3} + \frac{\lambda(i2,2)}{3}\right) \text{ and } b_{\frac{2i2}{3}} = b_{i2,j2}\left(\frac{1}{3} + \frac{2\lambda(i2,2)}{3}\right).$$

Hence, if blue unit $i2$ is divided, its attrition on $j1$ will be $\frac{1-\lambda(i2,2)}{3}b_{i2,j1}$. In this computation, the negative effect of division is shared between the two red units by their division fractions.

Case 2: Blue units $i1$ and $i2$ are assigned to a red unit. In this case, we omit the effect of $i3$ for the sake of simplicity and consider only red unit $j1$. We again need to define a new binary variable, $w^1_{\{i1,i2/3\},j1}$, to simultaneously handle combination of $i1$ and $i2$ as well as division of $i2$.

$$w^1_{\{i1,i2/3\},j1} = \begin{cases} 1, & \text{if } i2 \text{ is divided and combined with } i1 \text{ against } j1 \\ 0, & \text{otherwise} \end{cases}$$

$$z^1_{i2} + y_{\{i1,i2\},j1} \le 1 + w^1_{\{i1,i2/3\},j1}$$

$$w^1_{\{i1,i2/3\},j1} \le z^1_{i2} \text{ and } w^1_{\{i1,i2/3\},j1} \le y_{\{i1,i2\},j1}$$

Then, the main attrition goal constraint for $j1$ takes the form

$$b_{i1,j1}x_{i1,j1} + b_{i2,j1}x_{i2,j1} + b_{\{i1,i2\},j1}y_{\{i1,i2\},j1} - b_{\frac{i2}{3}}z^1_{i2} - b_{\{i1,i2/3\},j1}w^1_{\{i1,i2/3\},j1} \ge \alpha_{j1}$$

where the coefficient of the new variable is found as

$$b_{\{i1,i2/3\},j1} = \phi(i2,j1,2)\left[\lambda(i2,2)b_{i2,j1} + \frac{2(1-\lambda(i2,2))}{3}b_{i2,j1}\right] = \phi(i2,j1,2)b_{i2/3}.$$

Case 3: All three blue units are assigned to a red unit. In this case too, we consider only red unit $j1$ for simplicity. Omitting the technical constraints, we can rewrite the main attrition goal constraint for $j1$ as

$$\begin{aligned} & b_{i1,j1}x_{i1,j1} + b_{i2,j1}x_{i2,j1} + b_{i3,j1}x_{i3,j1} \\ & + b_{\{i1,i2\},j1}y_{\{i1,i2\},j1} + b_{\{i1,i3\},j1}y_{\{i1,i3\},j1} + b_{\{i2,i3\},j1}y_{\{i2,i3\},j1} \\ & - b_{\frac{i2}{3}}z^1_{i2} - b_{\{i1,i2/3\},j1}w^1_{\{i1,i2/3\},j1} - b_{\{i2/3,i3\},j1}w^1_{\{i2/3,i3\},j1} - b_{\{i1,i2/3,i3\},j1}w^1_{\{i1,i2/3,i3\},j1} \ge \alpha_{j1} \end{aligned}.$$

In this example, if blue unit $i2$ can be divided in other alternative ways, e.g. it can be assigned to $j1$ and $j2$ with fractions 2/3 and 1/3 as well as 1/3 and 2/3, then we need to define a second set of variables: z^2_{i1}, $w^2_{\{i1,i2/3\},j1}$, $w^2_{\{i2/3,i3\},j1}$, $w^2_{\{i1,i2/3,i3\},j1}$,

$w^2_{\{i1,i2/3\},j2}$, $w^2_{\{i2/3,i3\},j2}$, $w^2_{\{i1,i2/3,i3\},j2}$. Coefficients of these variables can be computed as illustrated above. To make sure that only one of these divisions takes place, we should rearrange constraint set (4.8) and its counterpart with superscript 2 as

$$z^1_{i2} + z^2_{i2} \le x_{i2,j1} \text{ and } z^1_{i2} + z^2_{i2} \le x_{i2,j2}.$$

In all cases of the second example, the term $-WEI_{i2}\left(z^1_{i2} + z^2_{i2}\right)$ should be added to the objective function to avoid paying the cost of $i2$ twice.

4.2 Column Generation and Decomposition

In an attempt to solve ALLM, we have tried to reformulate ALLM such that we can use column generation and decomposition. Here we expect to obtain optimal or near optimal integer solutions.

Consider the problem where I blue units are to allocated to J red units. Let

$$x^{k_i}_{ij} = \begin{cases} 1 & \text{if } k_i^{th} \text{ portion of blue unit } i, B_i, \text{ is allocated to red unit } j, R_j \quad k_i = 1,\ldots,K_i \\ 0 & \text{otherwise} \end{cases}$$

where K_i is the total number of portions of B_i for a given allocation fraction, e.g. K_i=2 if B_i is divided into two by fractions (1/2, 1/2).

Then we can reformulate the problem as follows.

$$Min\ z = \sum_{i=1}^{I} \sum_{j \in SR_i} \sum_{k_i=1}^{K_i} \frac{WEI_i}{K_i} x_{ij}^{k_i} \tag{4.9}$$

subject to

$$\sum_{i \in SB_j} \sum_{k_i=1}^{K_i} \frac{b_{ij}}{K_i} x_{ij}^{k_i} \geq \alpha_j, \qquad j = 1,\ldots,J \tag{4.10}$$

$$\sum_{j \in SR_i} x_{ij}^{k_i} \leq 1, \qquad i = 1,\ldots,I; k_i = 1,\ldots,K_i \tag{4.11}$$

$$x_{ij}^{k_i} = 0\ or\ 1 \qquad i = 1,\ldots,I; \forall j \in SR_i; k_i = 1,\ldots,K_i$$

where SB_j is the set of blue units that can be allocated to R_j, and SR_i is the set of red units that B_i can be allocated to according to the allocation graph. In this formulation we assume that each blue unit is divided into equal fractions as (1/2, 1/2) or (1/3, 1/3, 1/3) and used WEI_i/K_i and b_{ij}/K_i as coefficients of $x_{ij}^{k_i}$ in the objective function and the attrition goal constraints, respectively. We can use the same formulation when fractions are not equal as in (1/3, 2/3) by defining a decision variable for every 1/3 of blue unit.

In the above formulation, the objective function minimizes the total value of blue units used as before and attrition goal constraints given by Equation 4.10 ensures that the total attrition given to each red unit cannot be less than the attrition goal specified for that unit. Equation 4.11 assures that one portion of a blue unit cannot be allocated to more than one red unit.

The decomposition procedure progresses as follows. The constraints are decomposed into two sets as *hard constraints* (H) and *easy constraints* (E). The attrition goal constraints given by Equation 4.10 represent the overall attrition of a specific red unit, resulting from all allocated blue units. These are treated as hard constraints. However, each of the allocation constraints, given by Equation 4.11, includes variables concerning the use of a portion of one blue unit only and these constraints are referred to as easy constraints. Now we can define the problem as the minimization of objective function subject to hard constraints and easy constraints. Regarding the easy constraints we can define i=1,…,I subproblems:

$$P_i = \{ x_{i*}^{*} \in \mathrm{B}^{K_i|SR_i|} : \sum_{j \in SR_i} x_{ij}^{k_i} \leq 1, \qquad k_i = 1,\ldots,K_i \}$$

where subproblem i indicates the extreme points in a binary space of size $K_i|SR_i|$ and (*) indicates all values corresponding to that index position.

Then the feasible region of the LP relaxation is bounded by a finite number of (say L) extreme points in $K_1|SR_1|+\ldots+K_I|SR_I|$ dimensional space. Let y^1 through y^L denote these extreme points. Then, we can write any point y in the feasible region as a convex combination of y^1, y^2, …, y^L as dictated by Minkowski's Theorem (Nemhauser and Wolsey, 1988). That is, there exist weights β_1, β_2, …, β_L satisfying

$$y = \sum_{l=1}^{L} \beta_l y^l ,$$

$$\beta_1+\beta_2+\ldots+\beta_L = 1 \text{ and } \beta_l \geq 0 \text{ for } l=1,\ldots,L.$$

Now we can rewrite the allocation problem as follows.

$$\begin{aligned} &Min\ z = \sum_i c_i y_i = c^T y \\ &s.t. \qquad A_H y \geq \alpha \\ &\qquad\qquad y \in P_1 \cap P_2 \cap \cdots \cap P_I \end{aligned} \qquad (4.12)$$

Or equivalently

$$
\begin{aligned}
& Min\ z = c^T \sum_{l=1}^{L} \beta_l y^l \\
& s.t. \qquad A_H \left(\sum_{l=1}^{L} \beta_l y^l \right) \geq \alpha \\
& \qquad\qquad \sum_{l=1}^{L} \beta_l = 1 \\
& \qquad\qquad \beta_l \geq 0 \qquad l = 1, \ldots, L
\end{aligned} \tag{4.13}
$$

Note that the weights ($\beta_1, \ldots, \beta_L$) of patterns ($y^1, \ldots, y^L$) are the decision variables of (4.13). Here, c is the vector of weapon effectiveness values with $K_1|SR_1| + \cdots + K_I|SR_I|$ elements where, for example, the values of first $K_1|SR_1|$ entries are equal to WEI_1/K_1. A_H is the coefficient matrix of hard (attrition goal) constraints.

For each hard constraint j of problem (4.13) there is a surplus variable ($s_j, j=1,\ldots,J$) representing the excess and there is an accompanying dual variable (w_j, $j=1,\ldots,J$). Let

$\beta=[\beta_1, \ldots, \beta_L]$,

$q=[q_1, \ldots, q_L]$ where $q_l=c^T y^l$,

$$
D = \left[\begin{array}{ccc} A_H y^1 & \cdots & A_H y^L \\ \hline 1 & \cdots & 1 \end{array} \right]_{(J+1)xL},
$$

$$
S = \left[\begin{array}{ccc} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \\ \hline 0 & \cdots & 0 \end{array} \right]_{(J+1)xL},
$$

$s^T=[s_1, \ \ldots, \ s_J]$, and

$d^T = [\alpha_1, \ \cdots, \ \alpha_J \mid 1]$.

Now we can rewrite problem (4.13) as

$$\begin{aligned} & Min\ z = q^T \beta \\ & s.t. \qquad D\beta + Ss = d \\ & \qquad\qquad \beta \geq 0 \end{aligned} \tag{4.14}$$

Now we can use column generation to solve (4.14) starting with an initial allocation vector y^l and the corresponding initial basis. We check the reduced costs for each solution generated since a solution of the above (minimization) problem is optimal if all reduced costs of nonbasic variables are nonnegative. To determine the optimal allocation, we propose the following column generation procedure.

S0. Find an initial solution y^1 and compute the corresponding initial basis as

$$B = \left[\begin{array}{c|c} A_H y^1 & -I \\ \hline 1 & 0 \cdots 0 \end{array}\right]_{(J+1)x(J+1)}$$

where initial basic variables are β_1, s_1, …, s_J. Compute the values of initial basic variables as $\hat{d} = B^{-1}d$.

S1. Calculate the values of dual variables

$$w^T = q_B^T B^{-1}$$

which is in the following form

$$w = [\,\hat{w}\,|w_{J+1}] = [w_1, \ldots, w_J\,|\,w_{J+1}]$$

where q_B^T is the objective function coefficients of basic variables, $\hat{w}$ is the vector of dual variables corresponding to attrition goal constraints and w_{J+1} is the dual variable for constraint $\beta_1 + \ldots + \beta_L = 1$.

S2. Compute the reduced cost of new (candidate) pattern l

$$\hat{c}_l = q_l - w^T \left[\begin{array}{c} A_H y^l \\ \hline 1 \end{array}\right] = q_l - \left[\hat{w}^T \mid w_{J+1}\right] \left[\begin{array}{c} A_H y^l \\ \hline 1 \end{array}\right]$$

$$= q_l - \hat{w}^T A_H y^l - w_{J+1}$$

$$= c^T y^l - \hat{w}^T A_H y^l - w_{J+1}$$

yielding

$$\hat{c}_l = (c^T - \hat{w}^T A_H) y^l - w_{J+1} \tag{4.15}$$

If we apply Dantzig's rule, which says choose the entering variable whose reduced cost is the minimum, we have the following optimization problem. (If the minimum reduced cost is nonnegative then so are all, and hence optimal solution is secured.)

$$\gamma = \min\ (c^T - \hat{w}^T A_H) y^l - w_{J+1}$$

$$\text{s.t.} \quad y^l \in P_1 \cap P_2 \cap \cdots \cap P_I$$

Discarding w_{J+1} (which is a constant) from the objective function we obtain the following problem

$$\gamma = \min\ (c^T - \hat{w}^T A_H) y^l$$

$$s.t. \qquad y^l \in P_1 \cap P_2 \cap \cdots \cap P_I \tag{4.16}$$

Due to the special structure of the problem (i.e. since each subproblem, P_i; i=1,…,I is independent of the others) instead of solving (4.16) we solve following I independent linear programming problems.

$$\gamma_i = \min\ (c^T - \hat{w}^T A_H)_i y_i^l = \sum_{k_i=1}^{K_i} \sum_{j \in SR_i} (c^T - \hat{w}^T A_H)_{i,k_i} x_{ij}^{k_i} \qquad i=1,\ldots,I \tag{4.17}$$

$$s.t. \qquad y_i^l \in P_i$$

where the subscripts i, k_i indicate the element of a vector corresponding to k_i^{th} portion of B_i. Hence, solving and combining the results of these LPs we get the pattern $y^l = y_1^l + \cdots + y_I^l$ where y_i^l is the vector consisting of the values of decision variables corresponding to all portions of B_i). Here "+" means combining the vectors.

Finally, the reduced cost of new pattern y^l is computed as follows.

$$\hat{c}_l = \gamma_1 + \cdots + \gamma_I - w_{J+1} \tag{4.18}$$

S3. Optimality check

Considering the value of reduced cost of y^l there are two alternatives:

(i) If $\hat{c}_l \geq 0$ then current solution is optimal (entering y^l will not change objective function value if $\hat{c}_l = 0$ and will cause an increase if $\hat{c}_l > 0$) and we STOP.

(ii) If $\hat{c}_l < 0$ then entering y^l will improve the solution (decrease objective function value), hence β_l, which is the weight of y^l, should enter the basis. We then go to S4.

S4. Update basis

- Compute the entering column, $D^l = \begin{bmatrix} A_H y^l \\ \hdashline 1 \end{bmatrix}$
- Compute the updated column of the basis, $\hat{B}_l = B^{-1} D^l = B^{-1} \begin{bmatrix} A_H y^l \\ \hdashline 1 \end{bmatrix}$
- Determine the leaving variable by applying the minimum ratio test.

  Value of the entering variable is found as,

$$\bar{\beta}_l = \min_{1 \leq m \leq J+1} \left\{ \frac{\hat{d}_m}{\hat{b}_{m,l}} : \hat{b}_{m,l} > 0 \right\}.$$

Here,

$\hat{b}_{m,l}$ is m^{th} entry of column $\hat{B}_l$

$\hat{d}_m$ is m^{th} entry of updated right hand side values $\hat{d} = B^{-1} d$

where m^{th} entry corresponds to m^{th} basic variable.

If there is an index k such that $\frac{\hat{d}_k}{\hat{b}_{k,l}}$ is minimum then k^{th} basic variable

leaves the basis.

- Update the basis, B
- Compute values of basic variables, $\hat{d} = B^{-1}d$

We then go to S1.

Now we illustrate the implementation of this process on a problem where two blue units B_1 and B_2 are to be allocated to two red units R_1 and R_2. Problem data are given below.

		Blue, i		
		B_1	B_2	
	$WEI_i \rightarrow$	12	10	
		b_{ij}		attrition goal, α_j
Red, j	R_1	4	3	3
	R_2	7	5	4

Assuming that B_1 is divided into two by fractions [1/2, 1/2] and B_2 is divided into three by fractions [1/3, 1/3, 1/3], objective function is written as follows.

$$Min\ z = \sum_{j=1}^{2}\sum_{k_1=1}^{2}\frac{WEI_1}{2}x_{1j}^{k_1} + \sum_{j=1}^{2}\sum_{k_2=1}^{3}\frac{WEI_2}{3}x_{2j}^{k_2}$$

Then the allocation problem minimizing the overall cost of using blue units while satisfying the attrition goal constraints can be formulated as follows.

$$\text{Min } z = \frac{12}{2}x_{1*}^{*} + \frac{10}{3}x_{2*}^{*}$$

s.t. (1) $\frac{4}{2}(x_{11}^{1} + x_{11}^{2}) + \frac{3}{3}(x_{21}^{1} + x_{21}^{2} + x_{21}^{3}) \geq 3$ (attrition goal of R_1)

(2) $\frac{7}{2}(x_{12}^{1} + x_{12}^{2}) + \frac{5}{3}(x_{22}^{1} + x_{22}^{2} + x_{22}^{3}) \geq 4$ (attrition goal of R_2)

(3) $x_{11}^{1} + x_{12}^{1} \leq 1$ (first portion of B_1)

(4) $x_{11}^{2} + x_{12}^{2} \leq 1$ (second portion of B_1)

(5) $x_{21}^{1} + x_{22}^{1} \leq 1$ (first portion of B_2)

(6) $x_{21}^{2} + x_{22}^{2} \leq 1$ (second portion of B_2)

(7) $x_{21}^{3} + x_{22}^{3} \leq 1$ (third portion of B_2)

$$x_{ij}^{k_i} = 0 \text{ or } 1 \quad (i=1,2;\ k_i=1,\ldots,K_i;\ j=1,2)$$

Attrition goal constraints (1) and (2) are treated as hard constraints, whereas constraints (3) through (7) are referred to as easy constraints. The easy constraints are decomposed into two as E_1 and E_2. E_1 includes constraints (3) and (4) stated for B_1, and E_2 consists of constraints (5) through (7) written for B_2. Now the problem is to minimize

$$z = \frac{12}{2} x_{1*}^{*} + \frac{10}{3} x_{2*}^{*}$$

subject to hard constraints (H), easy constraints (E_1 and E_2), and nonnegativity constraints. Regarding the easy constraints we can define two subproblems:

$$P_1 = \{ x_{1*}^{*} \in \mathrm{B}^4 :\ x_{11}^{1} + x_{12}^{1} \leq 1,\ x_{11}^{2} + x_{12}^{2} \leq 1 \}$$

$$= \operatorname{conv}\left\{ \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \hline 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \hline 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \operatorname{conv}(S_1)$$

where the columns are the vectors $(x_{11}^{1}, x_{11}^{2}, x_{12}^{1}, x_{12}^{2})^{\mathrm{T}}$ indicating the extreme points whose set is S_1. The projection of this set onto a two dimensional space is as follows.

$$T_1 = \left\{ \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hline 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ \hline 1/2 \end{bmatrix} \right\}$$

where each column is the vector $(x_{11}^{*}, x_{12}^{*})^{\mathrm{T}}$. $(1/2 \mid 0)^{\mathrm{T}}$ in T_1 corresponds to $(1\ 0 \mid 0\ 0)^{\mathrm{T}}$ and $(0\ 1 \mid 0\ 0)^{\mathrm{T}}$ in S_1. As we see, the cardinalities of these sets are $|S_1|$=9 and $|T_1|$=6. That is, our formulation in this section requires more binary decision variables compared to the original formulation.

The subproblem for B_2 can be defined in a similar manner.

$$P_2 = \{x_{2*}^{*} \in \mathrm{B}^6 : \ x_{21}^1 + x_{22}^1 \le 1, \ \ x_{21}^2 + x_{22}^2 \le 1, \ \ x_{21}^3 + x_{22}^3 \le 1\}$$

$$= \mathrm{conv}(S_2),$$

where

conv(S_2)=

$$\left\{ \left[\begin{array}{c|c|c|ccc|ccc|cccccc|ccc|ccc|ccc|ccc}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}\right] \right\}$$

Projected form of S_2 in two dimensional space is

$$T_2=\left\{ \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 1 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hline 1/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ \hline 1/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hline 2/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ \hline 2/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ \hline 1/3 \end{bmatrix} \right\}$$

The cardinalities of these sets are $|S_2|$=27 and $|T_2|$=10.

Therefore, the feasible region of the LP is bounded by $L=|S_1|*|S_2|=243$ extreme points. Let y^1 to y^{243} denote these extreme points. Now, any point y in the feasible region can be written as a convex combination of $y^1, y^2, \ldots, y^{243}$ defining weights β_1, $\beta_2, \ldots, \beta_{243}$ as follows.

$$y = \sum_{l=1}^{243} \beta_l y^l ,$$

$$\beta_1+\beta_2+ \ldots + \beta_{243} = 1 \text{ and } \beta_l \ge 0 \text{ for } l=1, \ldots, 243.$$

where y^l, which is the vector of decision variables representing an allocation pattern, is defined as

$$y^l = [x_{11}^1 \quad x_{11}^2 \quad x_{12}^1 \quad x_{12}^2 \quad x_{21}^1 \quad x_{21}^2 \quad x_{21}^3 \quad x_{22}^1 \quad x_{22}^2 \quad x_{22}^3]^T$$

Now we solve the following allocation problem for which the general form is given by (4.7).

$$\begin{aligned} & Min\ z = c^T \sum_{l=1}^{243} \beta_l y^l \\ s.t.\quad & A_H \left(\sum_{l=1}^{243} \beta_l y^l \right) \geq \alpha \\ & \sum_{l=1}^{243} \beta_l = 1 \\ & \beta_l \geq 0 \qquad l = 1, \ldots, 243 \end{aligned}$$

where

$$c = \begin{bmatrix} 12/2 & 12/2 & 12/2 & 12/2 & 10/3 & 10/3 & 10/3 & 10/3 & 10/3 & 10/3 \end{bmatrix}$$

$$A_H = \begin{bmatrix} 4/2 & 4/2 & 0 & 0 & 3/3 & 3/3 & 3/3 & 0 & 0 & 0 \\ 0 & 0 & 7/2 & 7/2 & 0 & 0 & 0 & 5/3 & 5/3 & 5/3 \end{bmatrix}$$

Defining, $\beta = [\beta_1, \ldots, \beta_{243}]$, $q = [q_1, \ldots, q_{243}]$, $s^T = [s_1, s_2]$, $d^T = [3 \quad 4 \mid 1]$ and

$$S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ \hline 0 & 0 \end{bmatrix}$$

we obtain the following decomposition problem

$$\begin{aligned} & Min\ z = q^T \beta \\ s.t.\quad & D\beta + Ss = d \\ & \beta \geq 0 \end{aligned}$$

Let initial solution be the pattern $y^1 = [1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1]^T$, where B_1 is allocated to R_1 and B_2 is allocated to R_2 as a whole. Then,

$$D=\begin{bmatrix} A_H y^1 \\ \hline 1 \end{bmatrix}=\begin{bmatrix} 4 \\ 5 \\ \hline 1 \end{bmatrix}$$

and initial basis found in S0 of the algorithm is

$$B=\left[\begin{array}{c|cc} 4 & -1 & 0 \\ 5 & 0 & -1 \\ \hline 1 & 0 & 0 \end{array}\right] \text{ and } B^{-1}=\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 4 \\ 0 & -1 & 5 \end{bmatrix}.$$

Values of initial basic variables (β_1, s_1, s_2) are computed as $\hat{d}=B^{-1}d$=[1 1 1]^T. Their objective function coefficients are q_B^T=[$c^T y^1$ 0 0] = [22 0 0], yielding an objective function value of $z=q_B^T\hat{d}$=22.

Now we compute the values of dual variables in S1 of the algorithm as follows.

$w^T = q_B^T B^{-1}$= [0 0 | 22] where $\hat{w}^T$=[w_1, w_2] = [0 0] and w_3 = 22.

In S2, we should calculate the reduced cost of the new (candidate) pattern y^2. Using Equation 4.15 we have

$$\hat{c}_2=(c^T-\hat{w}^T A_H)y^2-w_3.$$

To obtain the pattern y^2 and its reduced cost $\hat{c}_2$ we need to solve the following problem.

$$\begin{aligned} \gamma &= \min\ (c^T-\hat{w}^T A_H)y^2 \\ s.t. &\quad y^2 \in P_1 \cap P_2 \end{aligned}$$

where the objective function coefficients are

$$\bar{c}=c^T-\hat{w}^T A_H=c^T=\left[\tfrac{12}{2}\ \ \tfrac{12}{2}\ \ \tfrac{12}{2}\ \ \tfrac{12}{2}\ \ \tfrac{10}{3}\ \ \tfrac{10}{3}\ \ \tfrac{10}{3}\ \ \tfrac{10}{3}\ \ \tfrac{10}{3}\ \ \tfrac{10}{3}\right].$$

We decompose the above minimization problem into two subproblems such that each subproblem solves the allocation of one blue unit independent of the other.

LP1:

$$
\begin{aligned}
\gamma_1 = \min\ & (6x_{11}^1 + 6x_{11}^2 + 6x_{12}^1 + 6x_{12}^2) \\
s.t.\quad & x_{11}^1 + x_{12}^1 \le 1 \\
& x_{11}^2 + x_{12}^2 \le 1 \\
& x_{1j}^{k_1} \ge 0 \quad j = 1,2;\ k_1 = 1,2
\end{aligned}
$$

and

LP2:

$$
\begin{aligned}
\gamma_2 = \min\ & (10/3\, x_{21}^1 + 10/3\, x_{21}^2 + 10/3\, x_{21}^3 + 10/3\, x_{22}^1 + 10/3\, x_{22}^2 + 10/3\, x_{22}^3) \\
s.t.\quad & x_{21}^1 + x_{22}^1 \le 1 \\
& x_{21}^2 + x_{22}^2 \le 1 \\
& x_{21}^3 + x_{22}^3 \le 1 \\
& x_{2j}^{k_2} \ge 0 \quad j = 1,2;\ k_2 = 1,2,3
\end{aligned}
$$

The solutions to LP1 and LP2 are $y_1^2 = [0\ \ 0\ \ 0\ \ 0]^T$ with γ_1=0 and $y_2^2 = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^T$ with γ_2=0 respectively yielding $\hat{c}_2 = \gamma_1 + \gamma_2 - w_3 = 0 + 0 - 22 = -22$ which is the reduced cost. According to S3 of the algorithm, since $\hat{c}_2$ is negative then β_2 for pattern y^2 enters the basis where y^2 is obtained combining y_1^2 and y_2^2 as follows.

$$y^2 = y_1^2 + y_2^2 = [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0].$$

Note that y^2 is infeasible.

In S4, we update the values as follows.

- Entering column: $D^2 = \left[\begin{array}{c} A_H y^2 \\ \hline 1 \end{array}\right]$=$[0\ \ 0\ \ 1]^T$
- Updated column of the basis: $\hat{B}_2 = B^{-1}D^2$=$[1\ \ 4\ \ 5]^T$
- Leaving variable: Value of the entering variable is found as

  β_2= $\min\{1/1, 1/4, 1/5\} = 1/5$.

Since the ratio corresponding to s_2 is minimum, s_2 leaves the basis.

- Updated basis: Now basic variables are (β_1, s_1, β_2) where the basis is updated as

$$B = \begin{bmatrix} 4 & -1 & 0 \\ 5 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 0 & 0.2 & 0 \\ -1 & 0.8 & 0 \\ 0 & -0.2 & 1 \end{bmatrix}.$$

- Updated values of basic variables: $\hat{d} = B^{-1}d$=[0.8 0.2 0.2]^T,

This completes the first iteration giving the following solution, which is the convex combination of two patterns y^1 and y^2.

$$y = \beta_1 y^1 + \beta_2 y^2 = [0.8\ \ 0.8\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.8\ \ 0.8\ \ 0.8]^T.$$

Note that although y^1 and y^2 are binary vectors, y consists of fractional values.

We iterate the algorithm in the same manner, and each time generating new patterns, computing their reduced costs, and checking optimality. The results of the subsequent iterations are summarized below.

<u>Iteration 2:</u>

β_3 enters where y^3 = [0 0 1 1 0 0 0 1 1 1] and s_1 leaves.

<u>Iteration 3:</u>

β_4 enters where y^4 = [0.5 0.5 0.5 0.5 0 0 0 0 0 0] and β_3 leaves.

<u>Iteration 4:</u>

β_5 enters where y^5 = [1 1 0 0 1 1 1 0 0 0] and β_2 leaves.

<u>Iteration 5:</u>

β_6 enters where y^6 = [0 0 1 1 0.5 0.5 0.5 0.5 0.5 0.5] and β_1 leaves.

<u>Iteration 6:</u>

β_7 enters where y^7 = [0 0 1 1 1 1 1 0 0 0] and β_6 leaves.

At the end of iteration 6 we have the optimal solution. Basic variables are β_7, β_4 and β_5 where their values are 0.2857, 0.5714 and 0.1429, respectively. The corresponding patterns are

$$y^7 = [0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0],$$

$$y^4 = [0.5\ 0.5\ 0.5\ 0.5\ 0\ 0\ 0\ 0\ 0\ 0],$$

$$y^5 = [1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0].$$

This yields the following solution which is the weighted sum of the above patterns.

$$y = \beta_4 y^4 + \beta_5 y^5 + \beta_7 y^7$$

$$= [0.4286\ 0.4286\ 0.5714\ 0.5714\ 0.4286\ 0.4286\ 0.4286\ 0\ 0\ 0]^T.$$

According to this result, 0.4286 of B_1 and 0.4286 of B_2 are allocated to R_1 yielding a total attrition of 0.4286(4) +0.4286(3) = 3.00 whereas 0.5714 of B_1 is allocated to R_2 yielding a total attrition of 0.5714(7) = 4.00. Since attrition goals are α_1=3 and α_2=4 the solution is feasible. However, it includes fractional allocations that are not allowable in the allocation problem.

Our incentive in using this approach, which is based on integrated use of decomposition and column generation, is to formulate our allocation problem in an alternative way by grouping the variables and decomposing the constraints for each blue unit. However, as seen in the above example, we obtain continuous allocation fractions in overall solution since the solution is a convex combination of extreme points. We have observed this in solving many other problems.

To overcome this issue, we have developed an approach based on the idea of keeping surplus variables in the basis. Considering the above example we can have the following cases.

(i) No surplus variables are in the basis (i.e. we have only decision variables, say, β_1, β_2 and β_3 as basic variables): In this case we can have an integer solution only if one of them (say β_k) is positive and the others are all zero. Then, we can obtain an integer solution by setting $\beta_k = 1$.

(ii) One surplus variable (either s_1 or s_2) is in the basis: Suppose we have β_1, β_2 and s_1 as basic variables. Then we can obtain an integer solution if either β_1 or β_2 (say β_k) is positive whereas the other is zero. As in case (i), we can obtain an integer solution by setting $\beta_k = 1$.

(iii) Both surplus variables (s_1 and s_2) are in the basis: Since we now have only one decision variable β_k in the basis (other than surplus variables), we can easily obtain an integer solution by setting $\beta_k = 1$.

Using this idea, we have attempted to solve a number of problems where we assign negative values as the objective function coefficients of surplus variables (instead of zeros which are the current coefficients). Our motivation in doing so is to force surplus variables to stay in the basis. This way, we can have only one decision variable (β_k) as the other basic variable, and hence the only pattern remaining in the basis generates the discrete optimal allocation as in case (iii). However, we have observed that we can not keep both s_1 and s_2 in the basis any of the test problems. Only one of them can remain in the basis in some problems, but not always yielding case (ii). When analyzing the problem in detail we have found out that, at each iteration we should generate a feasible pattern as the entering pattern (y^l). Our approach (which is based on decomposition and column generation) does not guarantee this due to the special structure of the problem as shown in Appendix B in detail.

Therefore, we have decided to develop a heuristic approach, which will be applied after column generation solution, to obtain feasible integer solutions to ALLM as described in the following section.

4.3 Heuristic Approach

Given a continuous solution obtained from column generation, the following heuristic can be applied to generate a discrete solution.

S1. Obtain column generation solution. (We solved LP relaxations and checked that LP and column generation give the same solution. Therefore, LP solution can also be used as a starting point.) If all values are discrete fractions then STOP (optimal discrete solution is found). Otherwise go to S2.

S2. Round solution values to nearest discrete fractions. Let $\lceil x_{ij} \rceil$ be the smallest discrete fraction greater than x_{ij} and $\lfloor x_{ij} \rfloor$ be the largest discrete fraction smaller than x_{ij}. If $\lceil x_{ij} \rceil - x_{ij} < x_{ij} - \lfloor x_{ij} \rfloor$ then round x_{ij} up to $\lceil x_{ij} \rceil$, otherwise round x_{ij} down to $\lfloor x_{ij} \rfloor$.

Calculate attrition gain, loss and net gain for each red unit due to rounding. Let g_j (l_j) be the total attrition gain (loss) and G_j be the net attrition gain for R_j. Let S_{RU} and S_{RD} be the set of indices of blue units which are rounded-up and rounded-down respectively. Then, the net gain is calculated as follows.

$$g_j = \sum_{i \in (SB_j \cap S_{RU})} b_{ij} \left(\lceil x_{ij} \rceil - x_{ij} \right), \quad j=1,\dots,J$$

$$l_j = \sum_{i \in (SB_j \cap S_{RD})} b_{ij}\left(x_{ij} - \lfloor x_{ij} \rfloor\right), \quad j=1,\ldots,J$$

$$G_j = g_j - l_j, \qquad j=1,\ldots,J$$

If $G_j<0$ then attrition goal for R_j is undersatisfied (infeasible) whereas if $G_j\geq 0$ then attrition goal constraint for R_j is oversatisfied.

If the solution is feasible go to S5, otherwise go to S3.

S3. Save excess portions of blue units.

For all R_j with $G_j > 0$, check blue units allocated to R_j and save portions of blue units from R_j (if possible) such that feasibility is still maintained in terms of attrition goal for R_j.

S4. Repair infeasibilities for undersatisfied red units.

Compute the objective function coefficient to constraint coefficient ratios for each unused portion of each blue unit as

$$r_{ij} = (WEI_i \,/\, b_{ij}) \,/ K_i$$

where K_i is the total number of portions as defined before. Then,

- Sort infeasible red units in increasing additional attrition requirement (i.e. red unit needing the minimum additional attrition will be in first position).
- Select a red unit (R_j) starting with the first one.
- Sort unused portions of blue units that can be allocated to R_j in increasing r_{ij} values.
- Starting with the first blue unit, allocate necessary portions of blue units to R_j until attrition goal constraint for R_j is satisfied (if possible).
- Repeat this procedure for all infeasible red units.

If the solution becomes feasible, update g_j, l_j and G_j for all red units, and go to S5. Otherwise STOP (a feasible solution cannot be obtained).

S5. Improve solution (save excess portions of blue units).

For all R_j with $G_j > 0$, check blue units allocated to R_j and save portions of blue units from R_j (if possible) such that feasibility is maintained.

Although the above algorithm is described without the combination and division effects for simplicity, we can easily take them into consideration when we compute gains and losses during implementation.

Now we illustrate the implementation of this algorithm on an example assuming division and combination effects are zero. Consider an allocation problem where three blue units B_1, B_2 and B_3 will be allocated to three red units R_1, R_2 and R_3. Assume that B_1 cannot be divided whereas B_2 can be divided into two by fractions [1/2, 1/2] and B_3 can be divided into three by fractions [1/3, 1/3, 1/3]. Furthermore assume that every blue unit can be allocated to every red unit. Problem data are given below.

		Blue, i			
		B_1	B_2	B_3	
	$WEI_i \rightarrow$	14	12	10	
			b_{ij}		attrition goal, α_j
	R_1	7	5	6	6
Red, j	R_2	5	13	6	8
	R_3	5	6	12	4

Defining the allocation pattern as

$$y^l = [x_{11}^1 \quad x_{12}^1 \quad x_{13}^1 \mid x_{21}^1 \quad x_{21}^2 \quad x_{22}^1 \quad x_{22}^2 \quad x_{23}^1 \quad x_{23}^2 \mid x_{31}^1 \quad x_{31}^2 \quad x_{31}^3 \quad x_{32}^1 \quad x_{32}^2 \quad x_{32}^3 \quad x_{33}^1 \quad x_{33}^2 \quad x_{33}^3]^T$$

we obtain the following solution using column generation.

$x_{11}^1=0.2858,\ x_{12}^1=x_{13}^1=0$ (B_1)

$x_{22}^1=x_{22}^2=0.6154,\ x_{21}^1=x_{21}^2=x_{23}^1=x_{23}^2=0,$ (B_2)

$x_{31}^1=x_{31}^2=x_{31}^3=0.6666,\ x_{33}^1=x_{33}^2=x_{33}^3=0.3334,\ x_{32}^1=x_{32}^2=x_{32}^3=0.$ (B_3)

Recall that B_2 and B_3 are divided into two and three equal portions, respectively. Hence, considering the allocations of B_2 and B_3 to a certain red unit, sum of allocation fractions of all portions (which is same for all portions) is also equal to the overall allocation fraction of blue unit. For example, fraction of B_3 allocated to R_1 is $1/3\,x_{31}^1+1/3\,x_{31}^2+1/3\,x_{31}^3=1/3(0.6666)+1/3(0.6666)+1/3(0.6666)=0.6666$.

According to the above solution 0.2858 of B_1 is allocated to R_1, 0.6154 of B_2 is allocated to R_2, and B_3 is allocated to R_1 and R_3 with fractions 0.6666 and 0.3334, respectively yielding an objective function value of 21.39. Hence we rewrite the solution as

$x_{11}=0.2858$, $x_{22}=0.6154$, $x_{31}=0.6666$, and $x_{33}=0.3334$.

Now we apply the heuristic procedure as follows.

Since allocation fractions of B_3 (0.6666 and 0.3334) are discrete whereas those of B_1 and B_2 (0.2858 and 0.6154) are not, we round the values for B_1 and B_2 to nearest discrete fractions as shown below.

We see that $\lceil x_{11}\rceil=\lceil 0.2858\rceil=1$ and $\lfloor x_{11}\rfloor=\lfloor 0.2858\rfloor=0$ for B_1. Since

$1-0.2858=0.7142>0.2858-0=0.2858$ then we set $x_{11}=0$ (round-down).

Similarly, since $\lceil x_{22}\rceil=\lceil 0.6154\rceil=1$, $\lfloor x_{22}\rfloor=\lfloor 0.6154\rfloor=1/2$ and

$1-0.6154=0.3846>0.6154-0.5=0.1154$, then we set $x_{22}=0.5$ (round-down).

At the end of rounding objective function value is reduced to 16. Gains and losses are computed as

$g_1 = g_2 = g_3 = 0$ since there is no round-up whereas,

$l_1 = 0.2858\ (7) = 2$,

$l_2 = 0.1154\ (13) = 1.5$, and

$l_3 = 0$

yielding

$G_1 = g_1 - l_1 = 0 - 2 = -2$,

$G_2 = g_2 - l_2 = 0 - 1.5 = -1.5$,

$G_3 = g_3 - l_3 = 0 - 0 = 0$.

Since G_1 and G_2 are negative the solution is not feasible. Therefore we need to repair these infeasibilities by allocating additional portions of blue units to R_1 and R_2.

Before repairing, we check portions of blue units currently allocated to R_3 to see whether we can save some portions. However we see that only 1/3 of B_3 is allocated to R_3 and hence we cannot save any blue unit (which is also obvious since $G_3 = 0$). Now repairing process is carried out as follows.

Among the two infeasible red units (R_1 and R_2), R_2 has the minimum additional attrition requirement and hence we handle R_2 first. There are two blue units that can be allocated to R_2 where currently we have B_1 (as a whole) and 1/2 of B_2 available for R_2, where r_{i2} values of these blue units are computed as

$r_{12} = 14 / (5(1)) = 2.80$,

$r_{22} = 12 / (13(2)) = 0.46$.

Since r_{22} is the smallest, we choose B_2 first and allocate 1/2 of B_2 to R_2 and observe that R_2 becomes feasible since $1/2(13)=6.5>|G_2 = -1.5|$. As to R_1, since B_2 is used up, there is only B_1 available. Hence we allocate B_1 completely to R_1 where R_1 also becomes feasible since $1(7)=7>|G_1=-2|$. Current objective function value is

14+12+10=36 since all units are fully utilized.

Since the solution is feasible we update net gains as G_1 = –2+7=5 and G_2 = –1.5+6.5=5 where G_3=0 remains unchanged. Now we try to improve the solution by saving some blue units from R_1 and R_2 since they have excess attrition (G_1=G_2=5) whereas no blue units can be saved from R_3 since G_3=0. At this point all of B_1 and 2/3 of B_3 are allocated to R_1. Since we have added B_1 in previous stage (while repairing R_1) we investigate whether we can save some portions of B_3. As saving 2/3 of B_3 from R_1 yields a reduction of 2/3(6)=4 in total attrition given to R_1, and this does not violates feasibility (since G_1=5), we remove 2/3 of B_3 from R_1 yielding a net gain of G_1=5–4=1 for R_1 which is still positive.

We cannot make an improvement regarding R_2 since only B_2 is allocated to R_2 as a whole and removing even 1/2 of B_2 yields a reduction of 1/2(13)=6.5 in total attrition given to R_2, which causes infeasibility (since G_2=5). Hence, the only improvement is the removal of 2/3 of B_3 from R_1 which yields an improved objective function value of 36–2/3(10)=29.33. The solution values obtained through all steps are summarized in Table 4.1. The results obtained from the application of this approach on more complex problems are discussed in the next section.

Table 4.1 Solution values of example problem

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	ΣWEI
Column G.	0.2858				0.6154		0.6666		0.3334	21.39
Rounding	0				1/2		2/3		1/3	16
Saving	0				1/2		2/3		1/3	16
Repair	1				1		2/3		1/3	36
Improve	1				1		0		1/3	29.33

4.4 Experimentation

We first applied the heuristic approach on 20 problems each having same structure as the example problem presented in previous section where we only changed the problem parameters (*WEI*, b_{ij}, and α_j values). Feasible solutions to all problems are obtained where we got the optimal solutions in 18 of them and there were gaps in two solutions (one being 10.00% and the other is 3.44% different from optimal objective values). The optimal solutions of these problems are found solving integer programming models.

Özdemirel and Kandiller (2001), proposed an enumeration procedure for small problems. They enumerated discrete allocation fractions (x'_{ij} values) for each blue unit. The enumeration procedure can be illustrated as follows. Suppose the number of red units that a certain blue unit can engage is three and this blue unit can be divided with fractions 1/3. Then, under assumption A4, the unit has eleven partial solution vectors each of dimesion three as shown below.

$$\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1/3\\2/3\\0\end{bmatrix}\begin{bmatrix}2/3\\1/3\\0\end{bmatrix}\begin{bmatrix}1/3\\0\\2/3\end{bmatrix}\begin{bmatrix}2/3\\0\\1/3\end{bmatrix}\begin{bmatrix}0\\2/3\\1/3\end{bmatrix}\begin{bmatrix}0\\1/3\\2/3\end{bmatrix}\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$$

After these partial solutions are determined for all blue units, candidate solutions are formed by taking Cartesian product of the partial solutions. Finally, these candidate solutions are checked to see whether or not they satisfy red attrition goal constraints, and the best feasible one is chosen as the optimal solution.

Erdem and Özdemirel (2002) developed a genetic algorithm (GA) for solving large ALLM instances. Their approach is summarized as follows. The fractional values in a partial solution (for one blue unit) form a gene, and a candidate solution (for all blue units) constitutes a chromosome. A feasible initial population is generated randomly. The ALLM objective function is used as the fitness function, and this function is improved through evolution of generations. A new generation is created by means of traditional crossover and mutation operators. If newly generated solutions do not satisfy some red attrition goal constraints, they are repaired by gradually switching fractions of blue units from the red units with excess attrition to the red units whose constraints are violated. If this switching is not sufficient to eliminate the infeasibility, unassigned blue units are used. When a large number of generations are produced, i.e. all chromosomes in the population converge to a fitness value, the chromosome with the best fitness value so far is declared as the best solution. They then developed two improvement heuristics which are deterministic local improvement (DLI) and stochastic local improvement (SLI) and experimented their algorithm on a problem set. We used these problems to further experiment our approach as explained below.

Their problem set involves 50 problems (which have more complex structure) where 6 blue units are to be allocated to 6 red units in each. These problems also consider combination and division effects as well as allowable division alternatives for each blue unit according to certain division codes specified. We solved same problems using the approach explained in previous sections where the comparative results are provided in Table 4.2. The second column shows the optimal solution (OS) of the problems obtained from enumeration, the third column displays our heuristic solution (HS) whereas deterministic local improvement (DLI) and stochastic local improvement (SLI) values are given in columns 5 and 7 respectively where all values are in terms of total weapon effectiveness used (i.e. objective function values). Percent differences (PD) from optimal solution are also provided for these approaches right to each column. As seen from the results our approach is not very successful in these problems where infeasible solutions are found for 6 problems. Average percent deviation (from optimal solution) is computed as 8.97 and maximum deviation is 28.39. In only three problems (problems 16, 26 and 39) we got better solutions as compared to GA results. In all solutions our approach requires less CPU time compared to their CPU times reported on PIII-750Mhz processor.

Considering the characteristics of these GA test problems and the features of our column generation and heuristic approach we can explain the reasons for the above results as follows.

(i) Incorporation of combination and division effects: In our column generation process we do not take combination and division effects into consideration and start dealing with them in S2 of the heuristic (when computing gains and losses after rounding). This prevents us from having a better starting solution, in terms of combination and division effects, which eventually may yield infeasible or costly solution. (In some example problems we observed that the solution obtained from column

Table 4.2 Experimentation Results and Their Comparison with GA solution

Problem	OS	HS	PD	DLI	PD	SLI	PD
1	3051.47	I*		3051.46	0.00	3051.46	0.00
2	1947.75	2010.74	3.23	1947.75	0.00	1947.75	0.00
3	2092.68	2141.73	2.34	2102.02	0.45	2102.02	0.45
4	2287.24	2296.64	0.41	2296.64	0.41	2296.64	0.41
5	3133.75	I		3133.75	0.00	3133.75	0.00
6	2587.27	2871.08	10.97	2587.27	0.00	2587.27	0.00
7	2015.44	2195.60	8.94	2024.92	0.47	2024.92	0.47
8	1636.53	2017.48	23.28	1643.51	0.43	1643.51	0.43
9	3247.63	3247.82	0.01	3247.63	0.00	3247.63	0.00
10	2107.23	2222.54	5.47	2107.22	0.00	2107.22	0.00
11	2343.17	2794.25	19.25	2343.17	0.00	2343.17	0.00
12	2155.81	2558.28	18.67	2155.81	0.00	2155.81	0.00
13	1604.07	1624.22	1.26	1612.35	0.52	1612.35	0.52
14	2207.26	2332.98	5.70	2216.63	0.42	2216.63	0.42
15	4100.96	4100.96	0.00	4100.96	0.00	4100.96	0.00
16	2282.67	2291.57	0.39	2335.48	2.31	2290.31	0.33
17	4724.54	I		4724.54	0.00	4724.54	0.00
18	2111.81	2140.89	1.38	2115.23	0.16	2115.23	0.16
19	2770.62	3149.76	13.68	2770.62	0.00	2770.62	0.00
20	2977.93	2986.70	0.29	2986.91	0.30	2986.91	0.30
21	2550.01	3145.21	23.34	2550.01	0.00	2550.01	0.00
22	2828.47	3290.21	16.32	2828.48	0.00	2828.48	0.00
23	1531.21	1786.20	16.65	1539.50	0.54	1539.50	0.54
24	2038.86	I		2055.68	0.82	2055.68	0.82
25	1773.20	1838.71	3.69	1789.42	0.91	1789.42	0.91
26	2396.65	2405.18	0.36	2410.09	0.56	2413.38	0.70
27	1601.54	1654.06	3.28	1607.32	0.36	1607.32	0.36
28	2894.08	3055.86	5.59	2894.08	0.00	2894.08	0.00
29	2597.56	2676.43	3.04	2597.57	0.00	2597.57	0.00
30	2171.89	2634.58	21.30	2171.90	0.00	2171.90	0.00
31	2161.44	2171.06	0.45	2170.97	0.44	2170.97	0.44
32	2469.16	2907.04	17.73	2469.16	0.00	2469.16	0.00
33	2931.45	I		2941.28	0.34	2941.28	0.34
34	2677.36	2688.46	0.41	2686.32	0.33	2686.32	0.33
35	2747.21	3527.16	28.39	2753.61	0.23	2753.61	0.23
36	2212.02	2597.85	17.44	2212.02	0.00	2212.02	0.00
37	2200.96	2399.34	9.01	2200.96	0.00	2200.96	0.00
38	2765.30	3380.53	22.25	2765.31	0.00	2765.31	0.00
39	1635.29	1649.85	0.89	1651.28	0.98	1651.28	0.98
40	1989.02	2150.99	8.14	2077.13	4.43	1998.86	0.49
41	2458.58	2671.99	8.68	2458.58	0.00	2458.58	0.00
42	2525.00	3032.60	20.10	2524.99	0.00	2524.99	0.00
43	2257.45	2278.04	0.91	2266.39	0.40	2266.39	0.40
44	1656.68	1749.07	5.58	1664.64	0.48	1664.64	0.48
45	1839.08	2106.19	14.52	1848.94	0.54	1848.94	0.54
46	2840.98	3023.82	6.44	2850.39	0.33	2850.39	0.33
47	2012.19	2339.75	16.28	2227.72	10.71	2012.19	0.00
48	1832.86	1992.91	8.73	1832.86	0.00	1832.86	0.00
49	2274.83	I		2274.83	0.00	2274.83	0.00
50	1801.85	2120.59	17.69	1808.72	0.38	1808.72	0.38
		ave.	**8.97**				
		max.	*28.39*				

* I: infeasible

CPU time

P-IV (1.6)	P-III (450)
0.19	0.28
0.13	0.34
0.14	0.33
0.14	0.36
0.13	0.37
0.14	0.45
0.11	0.30
0.11	0.26
0.14	0.36
0.11	0.31
0.16	0.40
0.13	0.27
0.13	0.27
0.13	0.31
0.13	0.27
0.14	0.30
0.11	0.28
0.16	0.35
0.13	0.33
0.13	0.30
0.17	0.46
0.13	0.31
0.11	0.25
0.16	0.35
0.13	0.31
0.14	0.32
0.11	0.25
0.13	0.34
0.13	0.31
0.16	0.35
0.11	0.27
0.17	0.44
0.16	0.36
0.13	0.30
0.14	0.33
0.11	0.27
0.13	0.26
0.16	0.42
0.13	0.31
0.11	0.25
0.13	0.31
0.14	0.37
0.13	0.32
0.13	0.26
0.13	0.32
0.16	0.40
0.11	0.24
0.13	0.27
0.09	0.25
0.13	0.26
0.13 ave.	**0.32** ave.

GA (CPU Time)	Init.Pop.	DLI / SLI	total
	0.77	1.04 (DLI)	**1.81**
	0.77	1.05 (SLI)	**1.82**

generation, without combination and division effects, became infeasible when these effects are added.)

(ii) Fixing some fractions at the beginning due to rounding: For instance, a blue unit may be allocated to two red units by fractions (1/2, 1/2) or (1/3, 2/3). Suppose that the column generation solution is (0.4125, 0.0000) for this unit. Then in S2 we round this solution to (1/3, 0) disregarding allocations of the form (1/2, 1/2). Further suppose that 1/2 of this blue unit satisfies the attrition goal for the first red unit whereas 1/3 of it does not. Then we should allocate another blue unit to the undersatisfied red unit in S4 (while repairing infeasibilities). Hence, we obtain a costly solution than the case where 1/2 of the blue unit is allocated initially.

To sum up, use of column generation and decomposition followed by the application of the heuristic approach (as illustrated in this chapter) may give optimal or near optimal solutions to certain types of simplified problems. However, we need to improve further to deal with the problems where combination and division effects should be handled and alternative allocation fractions, such as (1/2, 1/2) or (1/3, 2/3), are possible. Actually, the improvement steps used in the genetic algorithm can be implemented in our heuristic for this purpose. This would provide a fair comparison of the two heuristic approaches.

The group problem is used to obtain integer solutions to linear systems. Hence, it may be considered for rounding continuous solution of an allocation problem to integer values. The group problem is outlined below (Nemhauser and Wolsey, 1988).

Consider the set $S = \{x \in Z_+^n : \sum_{j \in N} a_j x_j = b\}$ where $a_j \in Z^1$ for $j \in N$ and $b \in Z^1$.

Let k be a nonnegative integer. Then we can relax S to

$$S_k = \{x \in Z_+^n : \sum_{j \in N} a_j x_j = b + kw,\ w \in Z^1\}$$

That is, we can subtract multiples of k from each coefficient and the right hand side of the original constraint. Therefore, $x \in S_k$ if and only if $x \in Z_+^n$ and for each $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in Z_+^{n+1}$ there exists a $w' \in Z^1$ such that

$$\sum_{j \in N} (a_j - k\lambda_j) x_j = (b - k\lambda_0) + kw'$$

By choosing $\lambda_j = \lfloor a_j / k \rfloor$ for $j \in N$ and $\lambda_0 = \lfloor b/k \rfloor$, we have that $x \in S_k$ if and only if $x \in Z_+^n$ satisfies

$$\sum_{j \in N} \phi_k(a_j) x_j = \phi_k(b) \qquad (\text{mod } k)$$

where $\phi_k(d) = d - k\lfloor d/k \rfloor$; that is $\phi_k(d) = d(\text{mod } k)$ is the remainder when d is divided by k.

Now we can use the following approach, addressed by Taha (1975), which is based on the idea of relaxing the integer problem and obtaining integer solutions from the relaxed problem.

An integer programming problem can be rewritten as follows

(IP) $\quad \min z = \{cx : Ax = b,\ x \geq 0 \text{ and integer}\}$.

Assume that all coefficients (elements of c and A) and right hand side values (elements of b) are integer. If we relax the integrality condition on x, the optimal solution to the equivalent LP is given by

$$\min z = (c_B B^{-1} b) - (c_B B^{-1} N - c_N)\, x_N$$

subject to

$$x_B = B^{-1} b - B^{-1} N x_N$$

$$x_B,\, x_N \geq 0$$

where x_B and x_N are the (optimal) basic and nonbasic variables, and B is the basis matrix, so that A=[B, N]. Since at the optimal continuous solution $x_N = 0$, then the optimal LP is given by $x_B = B^{-1}b$. The corner polyhedron is obtained from the optimal LP by imposing the integrality condition on x_B and x_N while maintaining the nonnegativity condition on x_N only. This means that the corner polyhedron is given by

$$x_B = B^{-1} b - B^{-1} N x_N$$

x_B integer

$x_N \geq 0$ and integer

Since x_B is unrestricted in sign, the corner polyhedron becomes equivalent to

$$B^{-1} b - B^{-1} N x_N \equiv 0 \;\; (mod\ 1)$$

$x_N \geq 0$ and integer

That is, only the difference $B^{-1} b - B^{-1} N x_N$ must be integer for nonnegative integer x_N. Therefore, adding or subtracting integer values to $B^{-1}b$ and $B^{-1}Nx_N$ the relationship remains unchanged. Hence, the corner polyhedron is given by

$$f(B^{-1} b) - f(B^{-1} N x_N) \equiv 0 \;\; (mod\ 1)$$

$x_N \geq 0$ and integer

where f, which is the fractional remainder in the Gomory's fractional cut, operates on each element of the column vector and is defined as $f(\bullet) = \bullet - \lfloor \bullet \rfloor$, and $\lfloor \bullet \rfloor$ is the largest integer smaller than •. Then the corner polyhedron is defined by Gomory's fractional cuts derived for each of the rows in the continuous optimal tableau.

Therefore, the optimal solution over the corner polyhedron is given by the following relaxed problem

$$\min z_R = (c_N - c_B B^{-1} N)\, x_N$$

subject to

$$f(B^{-1}b) - f(B^{-1}N\, x_N) \equiv 0 \quad (mod\ 1)$$

$$x_N \geq 0 \text{ and integer}$$

Minimization of z_R follows from the minimization of $z = c_B B^{-1} b - z_R$ after neglecting the constant. Also note that from the optimality of LP, $c_B B^{-1} N - c_N \leq 0$ and hence the relaxed problem must have a bounded solution.

CHAPTER 5

Attrition Simulation Model (ASM)

Attrition simulation model (ASM) simulates land combat according to the starting conditions determined by the allocation decisions obtained from ALLM. ASM consists of two alternative modules: deterministic Lanchester model (LM) and discrete-time stochastic model (DSM). ASM employs either or both of them in calculating attritions, depending on sizes and allocations of units involved. When a combat includes units having large number of combatants, it is more appropriate to use LM based on attrition rate coefficients (ARCs). For the case of small units, we can use DSM, which makes use of single shot kill probabilities (SSKPs). The determination of ARCs is difficult whereas obtaining SSKPs is relatively easier.

We propose to use LM for engagements between units each having more than 20 combatants whereas small sized engagements are treated stochastically using DSM. The figure 20 is suggested by Taylor (1983) as a rule of thumb. This follows from the tradeoff between the two issues given below;

i) When the number of combatants is small, such as in an engagement between three blue tanks and four red tanks, the behavior and attrition of each entity is subject to randomness and can be best observed using a stochastic process. Using deterministic LM would not yield accurate results in this case. On the other hand, for combats involving large number of combatants, as in an engagement between two opposing infantry battalions, the general behavior and attrition of units are of interest, and hence use of LM is considered as the proper approach.

ii) The computational time required to analyze a combat stochastically increases to an intractable level as the number of units and combatants increase. This occurs due to; (1) a new unit adds a new dimension and (2) addition of new combatants increases size for existing dimensions. Therefore, this approach is applicable to engagements among a relatively small number of units each having a small number of combatants (20 or less).

In this chapter, we first present our proposed DSM in detail. Then we briefly explain the use of Lanchester Model LM. Finally, we discuss the integration of DSM and LM and illustrate the methodology in an example instance.

5.1 Discrete-Time Stochastic Model (DSM)

In this section, we first present main features of DSM in homogeneous combat. Next, we generalize the model for heterogeneous forces. Finally, we discuss the incorporation of noncombat loss, engagement process, reinforcement, and synergy effects due to force division and combination.

5.1.1 Motivation

Before introducing DSM, we should state the restrictions of stochastic Lanchester model (SLM), which is the classical continuous-time stochastic combat model making use of attrition rate coefficients (ARCs). These restrictions are listed after introducing some concepts.

One can model a combat between two units each having a small number of combatants as a two-dimensional death process. Let the state definition be (t, i, j) where t is the continuous time variable, $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$ are the number of combatants alive in blue and red units at time t. Let $P(t,i,j)$ be the probability of having i blue and j red combatants at time t where the initial condition is $P(0,m,n) = 1$.

In the classical continuous-time treatment of combat, the attrition rates are used to determine the state probabilities where three possible state transitions are no loss, one blue casualty and one red casualty. Let $A(t, i, j)$ and $B(t, i, j)$, defined according to the type of engagement, be the attrition rates of blue and red units at time t when there are i blue and j red combatants. Then,

$$P(t+\Delta t, i, j) = P(t, i, j)P(\text{no loss}) + P(t, i+1, j)P(\text{one blue loss}) + P(t, i, j+1)P(\text{one red loss})$$

where the probability that no loss will occur is

$$\begin{aligned} P(\text{no loss}) &= P(\text{no blue loss})\, P(\text{no red loss}) \\ &= [1-A(t, i, j)][1-B(t, i, j)] \\ &= 1-A(t, i, j) - B(t, i, j) - A(t, i, j)\, B(t, i, j) \end{aligned}$$

The last term is assumed to be zero since no more than one casualty is allowed between t and $t+\Delta t$. Hence,

$$P(t+\Delta t, i, j) = [1-\{A(t, i, j)+B(t, i, j)\}]P(t, i, j)$$

$$+A(t,i+1,j)P(t,i+1,j)+B(t,i,j+1)P(t,i,j+1)$$

Rearranging the above equation and taking the limit yields the Kolmogorov equation.

$$\frac{dP(t,i,j)}{dt} = A(t,i+1,j)P(t,i+1,j) + A(t,i,j+1)P(t,i,j+1)$$

$$-\{A(t,i,j) + B(t,i,j)\}P(t,i,j) \qquad (5.1)$$

Then, selecting a proper infinitesimal time step Δt the state transition probabilities are computed as follows.

$$P(t+\Delta t,i,j) = [1-A(t,i,j)-B(t,i,j)]\Delta t P(t,i,j) + A(t,i,j)\Delta t P(t,i+1,j)$$

$$+B(t,i,j)\Delta t P(t,i,j+1) \qquad (5.2)$$

Assuming that Δt is the salvo length and defining the salvo sequence as 0, Δt, $2\Delta t$, $3\Delta t$, ... we can treat Equation 5.2 as a difference equation and use it to calculate discrete-time state probabilities, time varying expected values, and variances of remaining force sizes.

However, even if the attrition rates are available for SLM, Δt should be determined such that the probability of one blue casualty, $A(t,i,j)\Delta t$, and similarly probability of one red casualty, $B(t,i,j)\Delta t$, does not exceed one. This requirement forces the common Δt value to be very small, resulting in the need for observing the combat dynamics over a large number of salvos.

Hence, our motivation for developing DSM as an alternative to SLM is to relax these restrictions. We treat combat as a stochastic process proceeding in *salvos* during which every military unit engages one or more opposing units. We can summarize the main features of DSM as follows:

- Time advancement mechanism of DSM is salvos of fixed duration, which can be easily adjusted according to the fire cycle of a weapon type. Hence, a battle is modeled as a discrete-time stochastic process based on SSKPs that can be obtained from technical and operational data of weapon systems and from field exercises observing the skills of combatants in aiming and shooting events.

- Engagement and killing events are represented by binomial process.
- Forces on both sizes are heterogeneous, i.e. both blue and red sides consist of a number of military units which can be of different types.
- Both directed fire and area fire are modeled by means of combinatorial analysis. Overkills are taken into consideration.
- Division and combination of units are allowed.
- Stochastic engagement and noncombat loss can be included as well as discrete reinforcements.
- The variance, as well as the mean, of remaining force size is found for each military unit at the end of each salvo, facilitating the risk analysis as in any stochastic analysis.

Therefore, DSM allows us to model combat problems in a more realistic way since i) due to the discrete-time nature of DSM we can compute the effects of several events by means of the binomial process and ii) it allows more than one casualty within a salvo which is quite possible if, for example, we consider the number of casualties of an infantry unit due to a single artillery fire.

In addition, using salvos for time advancement enables us to predict the amount of ammunition used as well as the number of combatants killed, which is the information required for any weapon and ammunition planning system.

Before going into details we summarize basic assumptions of DSM as follows.

(A1) In a salvo, all shots are assumed to take place simultaneously, and the attrition occurs only after the firing is over.

(A2) The maximum number of salvos determines the combat duration.

(A3) A combatant in a unit employs either directed fire against an opposing combatant or area fire towards the opposing unit.

(A4) SSKPs may be different for different units, but remain constant throughout the combat.

5.1.2 DSM for Homogeneous Forces

Let the state space be represented as triplets (t, i, j) where $t = 0, 1, \ldots$ is the salvo counter, $i = m, m-1, \ldots, 1, 0$ represent surviving blue units and $j = n, n-1, \ldots, 1, 0$ represent surviving red units at the end of salvo t. The initial condition is assumed to be $P(0, m, n) = 1$. Considering the discrete-time nature of our model, we focus on binomial processes.

Directed Fire

Directed fire is the situation where a firer detects and aims at a single target and fires. The firer knows the result of the fire (whether the target is destroyed or not) and acts accordingly afterwards. The conflict between opposing tank units approaching towards each other can be thought of as directed fire and the engagement characteristics can be listed as follows:

- Combatants from both sides can concentrate their fires on a single target.
- When a target is destroyed, fire is shifted to a new target.
- The attrition (loss) of each side is determined only by the fires of the opposing force.

If there are i firers and j targets, then we can define a pattern $a=\{a_1, a_2, \ldots, a_j\}$, where a_k (k=1, ..., j) is the number of firers engaging target k. When j targets are distinguishable the number of different arrangements is $\binom{i+j-1}{i}$. However, if the targets are indistinguishable as in our case where each unit is homogeneous, there is no closed-form formula for the number of arrangements. In this case, the equivalent combinatorial problem is the number of different expressions of natural integer i using at most j sums of natural integers.

Suppose we have i identical firers shooting at j=3 identical targets. For simplicity, let the number of firers engaged with each target be A, B and C. Then we have a pattern $a=\{A, B, C\}$ such that $A+B+C=i$. If i=4, for instance, we have the following main patterns where $A \geq B \geq C$.

Pattern $\{A, B, C\}$			Number of repetitions
4	0	0	3
3	1	0	6
2	2	0	3
2	1	1	3
		Total :	15

For example, {2,1,1} means that two firers shoot at one target and each of the remaining two firers engage with each of the remaining targets. Depending on the engagement pattern we may have different number of targets killed. In the above example, with {4,0,0} at most one kill is possible, {3,1,0} and {2,2,0} result in at most two kills, whereas {2,1,1} yields up to three kills. Note that with this pattern definition overkills are possible. These patterns are one of the main tools to be used in conducting the binomial analysis.

The pattern $\{A,B,C\}$ would repeat as (A,B,C), (A,C,B), (B,A,C), (B,C,A), (C,A,B), (C,B,A) if the targets were distinguishable. In general, let there be j elements in pattern a and $n(a)$ be the frequency (total number of repetitions) of a. If we partition this pattern into K parts as $a^1, \ldots, a^K$ such that each one contains m_k identical elements ($k = 1, \ldots, K$), then the frequency of this pattern is calculated as

$$n(a) = \frac{j!}{\prod_{k=1}^{K} m_k!} \tag{5.3}$$

Consider the arrangement a={2,2,2,2,1,1,1,0} which is one of the main patterns of a combat where 11 blue firers are engaging 8 red targets. Then, a^1={2,2,2,2}, a^2={1,1,1} and a^3={0} where m_1=4, m_2=3 and m_3=1. Then the total number of repetitions of pattern a is 8!/(4!3!1!) = 40320/144 = 280.

Hence, for j=3, the total number of repetitions, $n(\{A,B,C\})$, is 3!/(1!1!1!)=6 if A, B, C are all different (e.g. {3,1,0}), 3!/(2!1!)=3 if only two of them are the same (e.g. {2,1,1}), and 3!/(3!)=1 if all three are the same (e.g. {1,1,1} when i=3). For the case of indistinguishable targets, these repetitions are all the same, and the relative frequency, that is the probability of occurrence, of the pattern $\{A,B,C\}$ can be found as

$$P(\{A,B,C\}) = \frac{n(\{A,B,C\})}{\binom{i+3-1}{i}} \tag{5.4}$$

Let p_k be the SSKP of a single firer. If A firers shoot at target τ then the probability that target τ is not killed is

$$P(\tau \text{ is not killed}) = (1-p_k)^A.$$

Then the probability that τ is killed is

$$P(\tau \text{ is killed}) = 1-[(1-p_k)^A].$$

Therefore given a pattern $\{A,B,C\}$ where first, second and third targets are subject to A, B, and C shots respectively, the probability of having l = 0, 1, 2, 3 casualties is calculated considering whether or not each target is killed.

$$P(l \text{ casualties} \mid \{A,B,C\}) = \sum_{\substack{l_1,l_2,l_3:0 \text{ or } 1 \\ l_1+l_2+l_3=l}}$$

$$\left[1-(1-p_k)^A\right]^{l_1}\left[(1-p_k)^A\right]^{1-l_1}\left[1-(1-p_k)^B\right]^{l_2}\left[(1-p_k)^B\right]^{1-l_2}\left[1-(1-p_k)^C\right]^{l_3}\left[(1-p_k)^C\right]^{1-l_3}$$

(5.5)

Hence, the probability of l casualties in the presence of i firers is calculated as

$$P_i(l) = P(l \text{ casualties with } i \text{ firers}) = \sum_{\substack{A \ge B \ge C \\ A+B+C=i}} \frac{n(\{A,B,C\})}{\binom{i+2}{i}} P(l \text{ casualties} \mid \{A,B,C\})$$

(5.6)

Two examples with $j=3$ targets are given in Tables 5.1 and 5.2 where the number of firers are i=4, i=3 and the SSKP values are p_k =0.2, p_k=0.3, respectively. In Table 5.1, for example, the probability of one casualty for each pattern is computed as follows.

$$P(1 \text{ casualty} \mid \{4,0,0\}) = [1-0.8^4] = 0.5904,$$

$$P(1 \text{ casualty} \mid \{3,1,0\}) = [1-0.8^3][0.8] + [0.8^3][0.2] =$$

$$0.3904 + 0.1024 = 0.4928,$$

Table 5.1 Directed fire casualty probabilities for i=4, j=3, p_k =0.2

{A,B,C}	n({A,B,C})		$P_4(0)$	$P_4(1)$	$P_4(2)$	$P_4(3)$	Total
{4,0,0}	3		0.4096	0.5904	0.0000	0.0000	1.0000
{3,1,0}	6		0.4096	0.4928	0.0976	0.0000	1.0000
{2,2,0}	3		0.4096	0.4608	0.1296	0.0000	1.0000
{2,1,1}	3		0.4096	0.4352	0.1408	0.0144	1.0000
Total	15	Ave.	0.4096	0.4944	0.0931	0.0029	1.0000

Table 5.2 Directed fire casualty probabilities for i=3, j=3, p_k =0.3

{A,B,C}	n({A,B,C})		$P_3(0)$	$P_3(1)$	$P_3(2)$	$P_3(3)$	Total
{3,0,0}	3		0.3430	0.6570	0.0000	0.0000	1.0000
{2,1,0}	6		0.3430	0.5040	0.1530	0.0000	1.0000
{1,1,1}	1		0.3430	0.4410	0.1890	0.0270	1.0000
Total	10	Ave.	0.3430	0.5436	0.1107	0.0027	1.0000

Table 5.3 Directed fire casualty probabilities for i=3, j=4, p_k =0.2

{A,B,C}	n({A,B,C})		$P_3(0)$	$P_3(1)$	$P_3(2)$	$P_3(3)$	$P_3(4)$	Total
{3,0,0,0}	4		0.5120	0.4880	0.0000	0.0000	0.0000	1.0000
{2,1,0,0}	12		0.5120	0.4160	0.0720	0.0000	0.0000	1.0000
{1,1,1,0}	4		0.5120	0.3840	0.0960	0.0080	0.0000	1.0000
Total	20	Ave.	0.5120	0.4240	0.0624	0.0016	0.0000	1.0000

$P(1 \text{ casualty} \mid \{2,2,0\}) = [1-0.8^2][0.8^2] + [1-0.8^2][0.8^2] =$

$0.2304 + 0.2304 = 0.4608,$

$P(1 \text{ casualty} \mid \{2,1,1\}) = [1-0.8^2][0.8][0.8] + [0.8^2][0.2][0.8] + [0.8^2][0.8][0.2]=$

$0.2304 + 0.1024 + 0.1024 = 0.4352,$

and the overall probability of one casualty (for i=4 firers) is

$$P_4(1) = (3/15)0.5904 + (6/15)\ 0.4928 + (3/15)0.4608 + (3/15)0.4352 = 0.4944.$$

Table 5.3 illustrates an example where we have j=4 targets, i=3 firers, and the SSKP value is p_k =0.2. The results given in Tables 5.1-5.3 will be used later in this chapter.

Area Fire

In area fire, a firer cannot detect targets individually but knows the general region in which the opposing unit is deployed. Hence, unlike directed fire, firers do not know the result of their fires. The targets are assumed to be uniformly distributed over an area as explained by Anderson (1995). Combat between two opposing artillery units is a good example where both sides engage area fire as explained in detail below.

- Combatants of each side are positioned (uniformly distributed) in a "fixed" area.
- Each side simply directs its fire towards the center of the area where the enemy is located.
- The attrition (loss) of each side is dependent on its own force size as well as the fires of the opposing force.

Consider the scheme shown in Figure 5.1 where red combatants are positioned in a circular region of radius R and are subject to area fire of blue. Let A denote the area of this region. Since area firing weapons such as cannons and mortars generally have circular effective areas, an area shot divides this region into two as fatal area (FA) of radius r and non-fatal area (NFA). Let A_F and A_{NF} be the areas of these regions respectively. We assume that FA with radius r falls completely inside A, that

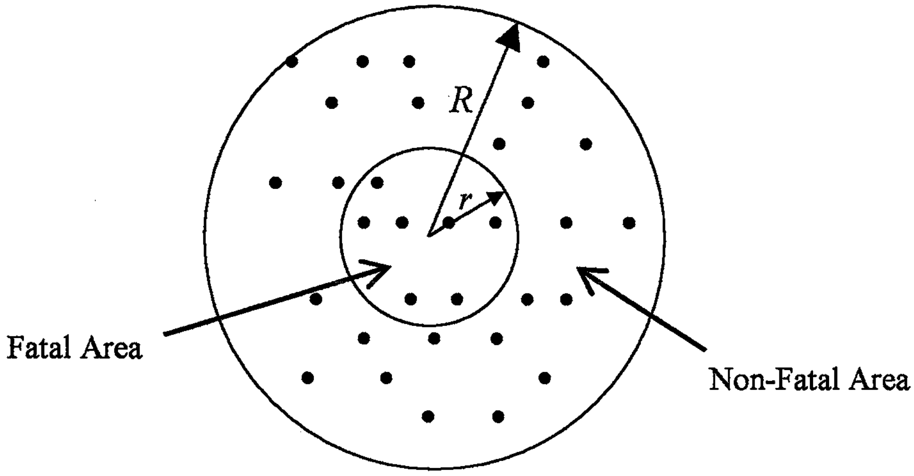


Figure 5.1 Regions of Fatality in Area Fire

is $A = A_{NF}+A_F$. Suppose that a blue unit with SSKP p_k engages a red unit having j combatants.

We assume that SSKP of a blue firer is defined as

$$\text{SSKP} = \begin{cases} 0 & \text{if target is in } NFA \\ p_k & \text{if target is in } FA \end{cases}$$

That is a red target can be killed only if it is located in the fatal area. Suppose that the probability that a red target is in the FA is defined as the ratio of A_F to A and let $\xi=r^2/R^2$ denote this probability.

Consider the case where j=3 targets are subject to area fire of one firer where ξ=0.4 and p_k=0.2. Then the probability of one red casualty with one firer, $P_1(1)$, is computed as follows.

$$P_1(1) = P(1 \text{ casualty} \mid 3 \text{ targets in FA}) + P(1 \text{ casualty} \mid 2 \text{ targets in FA})$$
$$+ P(1 \text{ casualty} \mid 1 \text{ target in FA})$$

$$= \binom{3}{3}0.4^3(0.6)^0 \binom{3}{1}(0.2)^1(0.8)^2 + \binom{3}{2}0.4^2(0.6)\binom{2}{1}(0.2)^1(0.8)^1$$

$$+\binom{3}{1}0.4(0.6)^2\binom{1}{1}(0.2)^1(0.8)^0$$

$$= (1)(0.064)(3)(0.128) + (6)(0.096)(2)(0.16) + (3)(0.144)(1)(0.2) = 0.2953$$

To generalize, when there is only one firer, the probability of l casualties out of j targets depends on the condition that there are f targets in FA, where $l \leq f \leq j$, and l of them are killed. Hence,

$$P_1(l) = P\text{ (}l\text{ casualties with one firer)} = \sum_{f=l}^{j}\binom{j}{f}\xi^f(1-\xi)^{j-f}\binom{f}{l}p_k^l(1-p_k)^{f-l} \quad (5.7)$$

In the case of $i \geq 2$ firers the probability of having l casualties is calculated using i-way convolution. This treatment assumes that FAs of firers fall inside the area A over which the targets are distributed, where the exact locations of these FAs within A are not known. Consider the case where there are three firers. If the targets neutralized by each firer were mutually exclusive (i.e. there were no overkills), then the probability of having l casualties in a salvo would be calculated by the following 3-way convolution:

$$P_3(l) = P(l\text{ casualties with 3 firers}) = \sum_{\substack{l_1, l_2, l_3 \\ l_1+l_2+l_3=l}} P_1(l_1)P_1(l_2)P_1(l_3) \quad (5.8)$$

Let the pattern $(l_1, l_2, l_3 | l_{12}, l_{13}, l_{23} | l_{123})$ denote the number of casualties where l_1 is the number of kills only by the first firer, l_{12} is the number of kills only by the first and the second firer, and so on. Here, we can handle overkills. For instance, the pattern (1,2,0|1,2,0|1) indicates that one target is overkilled by all three firers, another is overkilled by the first and the second together, two targets are overkilled by the first and the third together, one is killed only by the first firer, and two only by the second. If there are i firers, we have patterns of dimension 2^i-1, as

$$\left(\binom{i}{1}\text{entries}\mid\cdots\mid\binom{i}{k}\text{entries}\mid\cdots\mid\binom{i}{i}\text{entry}\right)$$

Since the targets are indistinguishable, the number of repetitions of the pattern is

$$n(l_1, l_2, l_3 \mid l_{12}, l_{13}, l_{23} \mid l_{123}) = \binom{j}{j-l}\binom{l}{l_{123}}\binom{l-l_{123}}{l_{12}}\binom{l-l_{123}-l_{12}}{l_{13}}\cdots\binom{l_1}{l_1}$$

$$=\binom{j}{j-l, l_1, l_2, l_3, l_{12}, l_{13}, l_{23}, l_{123}} = \frac{j!}{(j-l)!\,l_1!\,l_2!\,l_3!\,l_{12}!\,l_{13}!\,l_{23}!\,l_{123}!} \qquad (5.9)$$

where $l= l_1+ l_2+ l_3+ l_{12}+ l_{13}+ l_{23}+ l_{123}$ is the total number of casualties.

Let l_A, l_B, l_C denote the number of kills (including overkills) by the first, second and third firers, i.e., $l_A=l_1+l_{12}+l_{13}+l_{123}$, $l_B= l_2+l_{12}+l_{23}+l_{123}$, $l_C=l_3+l_{13}+l_{23}+l_{123}$. Given l_A, l_B, l_C, we face with some size restrictions. For instance, the two-way overkill value $l_{12}=\max\{0, l_A+l_B-j\}, \ldots, \min\{l_A, l_B\}$. That is, if we have $j=3$ targets, $i=2$ firers, $l_A=3$ and $l_B=1$, then we cannot have the pattern (1,0,0|2,0,0|0) since the number of targets overkilled by the first two firers ($l_{12}=2$) cannot exceed the overall number of casualties due to the second firer ($l_B=1$). The pattern (2,1,0|1,0,0|0) is also impossible when $l_A=3$, $l_B=2$, $j = 3$, since there cannot be more than j casualties. That is, l_{12} should be at least $l_A+l_B-j=2$ in which case $l_{12}=2$, $l_1=1$ and $l_2=0$ should be true. Let (*) denote such restrictions and $n(l_A, l_B, l_C)$ be the number of patterns possible under these restrictions. Then,

$$n(l_A, l_B, l_C) = \sum_{l_1,l_2,l_3,l_{12},l_{13},l_{23},l_{123}\in S_3} n(l_1,l_2,l_3 \mid l_{12},l_{13},l_{23} \mid l_{123}), \qquad (5.10)$$

where

$S_3 = \{(l_1, l_2, l_3, l_{12}, l_{13}, l_{23}, l_{123}) \in Z_+^7$: (*) is satisfied,

$l_A = l_1+l_{12}+l_{13}+l_{123},\ l_B = l_2+l_{12}+l_{23}+l_{123},\ l_C = l_3+l_{13}+l_{23}+l_{123}\ \}$.

Hence, the probability of l casualties is calculated as

$P_3(l) = P(l$ casualties with 3 firers) =

$$\sum_{\substack{l_1,l_2,l_3,l_{12},l_{13},l_{23},l_{123}\in S_3 \\ l_1+l_2+l_3+l_{12}+l_{13}+l_{23}+l_{123}=l \\ l_A \geq l_B \geq l_C}} n(\{l_A,l_B,l_C\}) \frac{n(l_1,l_2,l_3 \mid l_{12},l_{13},l_{23} \mid l_{123})}{n(l_A,l_B,l_C)} P_1(l_A)P_1(l_B)P_1(l_C) \tag{5.11}$$

where $n(\{l_A, l_B, l_C\})$ is the number of doublecounts if we ignore the order $l_A \geq l_B \geq l_C$; or equivalently, it is 3!/(1!1!1!) if l_A, l_B, l_C are all different, 3!/(2!1!) if only two of them are the same, and 3!/3! if all are the same.

An example with j=3 targets and i=2 firers is illustrated in Table 5.4. Suppose that $P_1(l)$ values are found by Equation 5.7 as 0.4096, 0.4944, 0.0931, 0.0029 for l= 0,1,2,3 and they are the same for the two firers. Consider the rows where l_A=2 and l_B=1, generating the patterns {1,0|1} and {2,1|0}. Using Equation 5.9, the number of repetitions is n(1,0|1)=3!/(1!1!0!1!)=6 for the first pattern and n(2,1|0)= 3!/(0!2!1!0!)=3 for the second. Thus, n(2,1)=6+3=9. We have $n(\{l_A,l_B\})$=2! repetitions due to different arrangements of l_A and l_B. Hence, the joint probability $P_1(2)P_1(1)$=0.046 is multiplied by 2! and distributed between $P_2(2)$ and $P_2(3)$ with proportions of 6/9 and 3/9, resulting in the values 0.0614 and 0.0306, respectively. After processing all patterns in this manner, $P_2(2)$ and $P_2(3)$ are found as 0.3035 and 0.0422.

Table 5.4 Area fire casualty probabilities for i=2, j=3 when FA locations are unknown

$\{l_1, l_2 \vert l_{12}\}$	l_A	l_B	$P_1(l_A)$	$P_1(l_B)$	$P_1(l_A)P_1(l_B)$	$n(\{l_A, l_B\})$	$n(l_A, l_B)$	$n(l_1, l_2 \vert l_{12})$	$P_2(0)$	$P_2(1)$	$P_2(2)$	$P_2(3)$
{0,0|3}	3	3	0.0029	0.0029	0.0000	1	1	1				0.0000
{1,0|2}	3	2	0.0029	0.0931	0.0003	2	3	3				0.0006
{2,0|1}	3	1	0.0029	0.4944	0.0014	2	3	3				0.0028
{3,0|0}	3	0	0.0029	0.4096	0.0012	2	1	1				0.0024
{0,0|2}	2	2	0.0931	0.0931	0.0087	1	9	3			0.0029	
{1,1|1}	2	2	0.0931	0.0931	0.0087	1	9	6				0.0058
{1,0|1}	2	1	0.0931	0.4944	0.0460	2	9	6			0.0613	
{2,1|0}	2	1	0.0931	0.4944	0.0460	2	9	3				0.0307
{2,0|0}	2	0	0.0931	0.4096	0.0381	2	3	3			0.0762	
{0,0|1}	1	1	0.4944	0.4944	0.2444	1	9	3		0.0815		
{1,1|0}	1	1	0.4944	0.4944	0.2444	1	9	6			0.1629	
{1,0|0}	1	0	0.4944	0.4096	0.2025	2	3	3		0.4050		
{0,0|0}	0	0	0.4096	0.4096	0.1678	1	1	1	0.1678			
Total									**0.1678**	**0.4865**	**0.3034**	**0.0423**

The above treatment assumes that FAs of the firers fall inside the area A over which the targets are distributed, but the exact locations of these FAs within A are unknown. If the firers are deployed such that locations (and intersections) of their respective FAs are known as shown in Figure 5.2, the total area A is partitioned into regions NFA, FA_1, FA_2, FA_3, $\cdots$. No casualty can occur in NFA, no overkills in FA_1, two-way overkills in FA_2, three-way overkills in FA_3, and so on. Let A_{NF}, A_{F1}, A_{F2}, A_{F3}, ... be the areas of respective regions such that $A = A_{NF}+A_{F1}+A_{F2}+A_{F3}+\cdots$. Hence, the area specific distribution probabilities are

$$\xi_{NF} = \frac{A_{NF}}{A}, \quad \xi_{F1} = \frac{A_{F1}}{A}, \quad \xi_{F2} = \frac{A_{F2}}{A}, \quad \xi_{F3} = \frac{A_{F3}}{A}, \cdots \tag{5.12}$$

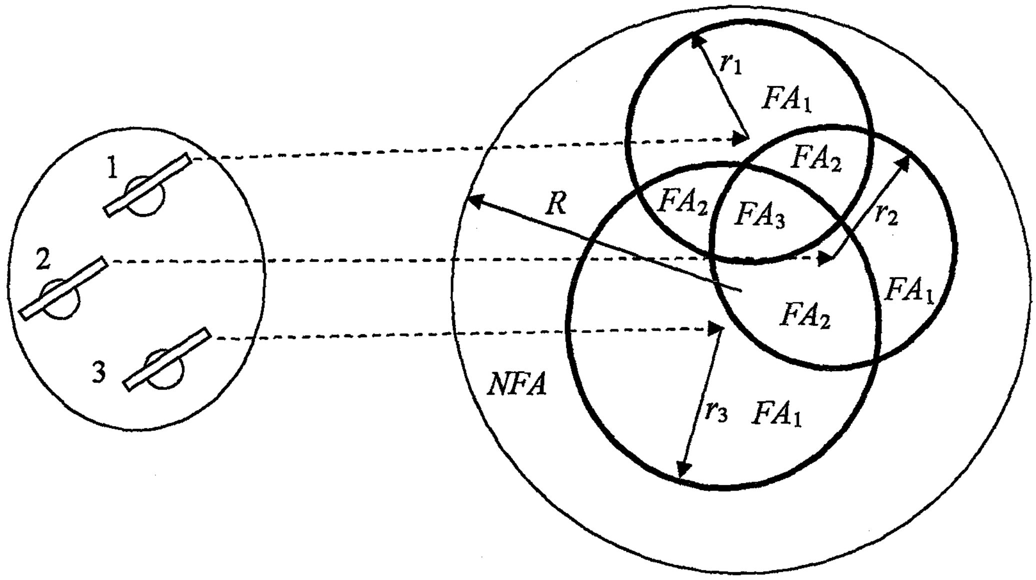


Figure 5.2 Overlapping fatal areas

The area specific SSKPs are computed as follows

$$p_{NF} = 0,\ p_{F1} = p_k,\ p_{F2} = 1-(1-p_k)^2,\ p_{F3} = 1-(1-p_k)^3,\ \ldots \tag{5.13}$$

Let us assume again that we have i=3 firers and j targets. The distribution of targets in four regions such that $j_0 + j_1 + j_2 + j_3 = j$ has the following multinomial probabilities:

$$P(j_0 \text{ in } NFA,\ j_1 \text{ in } FA_1, j_2 \text{ in } FA_2,\ j_3 \text{ in } FA_3) = \binom{j}{j_0, j_1, j_2, j_3} \xi_{NF}^{j_0}\, \xi_{F1}^{j_1}\, \xi_{F2}^{j_2}\, \xi_{F3}^{j_3} \tag{5.14}$$

Then, the probability of having l casualties is calculated as

$$P_3(l) = P(l \text{ casualties with 3 firers}) = \sum_{j_0+j_1+j_2+j_3=j} \binom{j}{j_0, j_1, j_2, j_3} \xi_{NF}^{j_0}\, \xi_{F1}^{j_1}\, \xi_{F2}^{j_2}\, \xi_{F3}^{j_3}$$

$$\sum_{\substack{l_1 \le j_1, l_2 \le j_2, l_3 \le j_3 \\ l_1+l_2+l_3=l}} \binom{j_1}{l_1} p_{F1}^{l_1}(1-p_{F1})^{j_1-l_1} \binom{j_2}{l_2} p_{F2}^{l_2}(1-p_{F2})^{j_2-l_2} \binom{j_3}{l_3} p_{F3}^{l_3}(1-p_{F3})^{j_3-l_3} \tag{5.15}$$

Table 5.5 Area fire casualty probabilities for i=2, j=4, p_k=0.3 when FA locations are known; p_{NF}=0.00, p_{F1}=0.30, p_{F2}=0.51; ξ_{NF}=0.5, ξ_{F1}=0.3, ξ_{F2}=0.2

				$\sum_{\substack{l_1 \le j_1, l_2 \le j_2 \\ l_1 + l_2 = l}} \binom{j_1}{l_1} p_{F1}^{l_1}(1-p_{F1})^{j_1-l_1} \binom{j_2}{l_2} p_{F2}^{l_2}(1-p_{F2})^{j_2-l_2}$					
j_0	j_1	j_2	$P(j_0, j_1, j_2)$	$P_2(0)$	$P_2(1)$	$P_2(2)$	$P_2(3)$	$P_2(4)$	Total
4	0	0	0.0625	1.0000	0.0000	0.0000	0.0000	0. 0000	1.0000
0	4	0	0.0081	0.2401	0.4116	0.2646	0.0756	0. 0081	1.0000
0	0	4	0.0016	0.0576	0.2400	0.3747	0.2600	0. 0677	1.0000
3	1	0	0.1500	0.7000	0.3000	0.0000	0.0000	0. 0000	1.0000
1	3	0	0.0540	0.3430	0.4410	0.1890	0.0270	0. 0000	1.0000
0	1	3	0.0096	0.0824	0.2924	0.3778	0.2076	0 .0398	1.0000
3	0	1	0.1000	0.4900	0.5100	0.0000	0.0000	0. 0000	1.0000
1	0	3	0.0160	0.1176	0.3674	0.3823	0.1327	0. 0000	1.0000
0	3	1	0.0216	0.1681	0.3910	0.3175	0.1096	0.0138	1.0000
2	2	0	0.1350	0.4900	0.4200	0.0900	0.0000	0.0000	1.0000
2	0	2	0.0600	0.2401	0.4998	0.2601	0.0000	0.0000	1.0000
0	2	2	0.0216	0.1176	0.3457	0.3590	0.1542	0.0234	1.0000
2	1	1	0.1800	0.3430	0.5040	0.1530	0.0000	0.0000	1.0000
1	2	1	0.1080	0.2401	0.4557	0.2583	0.0459	0.0000	1.0000
1	1	2	0.0720	0.1681	0.4219	0.3320	0.0780	0.0000	1.0000
Total			1.0000	0.4262	0.4051	0.1444	0.0229	0.0014	1.0000

An example is presented in Table 5.5 where we have i=2 firers and j=4 targets creating three regions with the following SSKPs: p_{NF} = 0, p_{F1} = p_k = 0.3, p_{F2}=1−(1−0.3)^2 = 0.51. Suppose that one half of the total area is nonfatal, and the overkill area is 20 percent of A. The distribution of 4 targets as j_1=l in FA_1 and j_2=2 in FA_2, has the multinomial probability value (4!/(1!1!2!)) 0.5(0.3)0.2^2=0.072. Note that in this case, the probability of having four casualties is zero since we have one target in the NFA. The probability of having two casualties is

P(2 casualties | 1 in NFA, 1 in FA_1, 2 in FA_2) =

$$\binom{1}{0}0.7\binom{2}{2}0.51^2+\binom{1}{1}0.3\binom{2}{1}0.51(0.49)=0.18207 + 14944 = 0.33201.$$

The overall probability of having two targets neutralized is $P_2(2)$=0.1444.

Salvo treatment

State transitions in DSM are calculated using the binomial treatment of engagement and killing phenomena as discussed in the previous two subsections. DSM allows multiple casualties in both forces. Possible states that can be reached from state (t,i,j) in the case of directed fire are as follows:

$(t,i,j) \rightarrow (t+1,i,j), (t+1,i-1,j), (t+1,i,j-1), (t+1,i-1,j-1), (t+1,i-2,j), (t+1,i,j-2),$

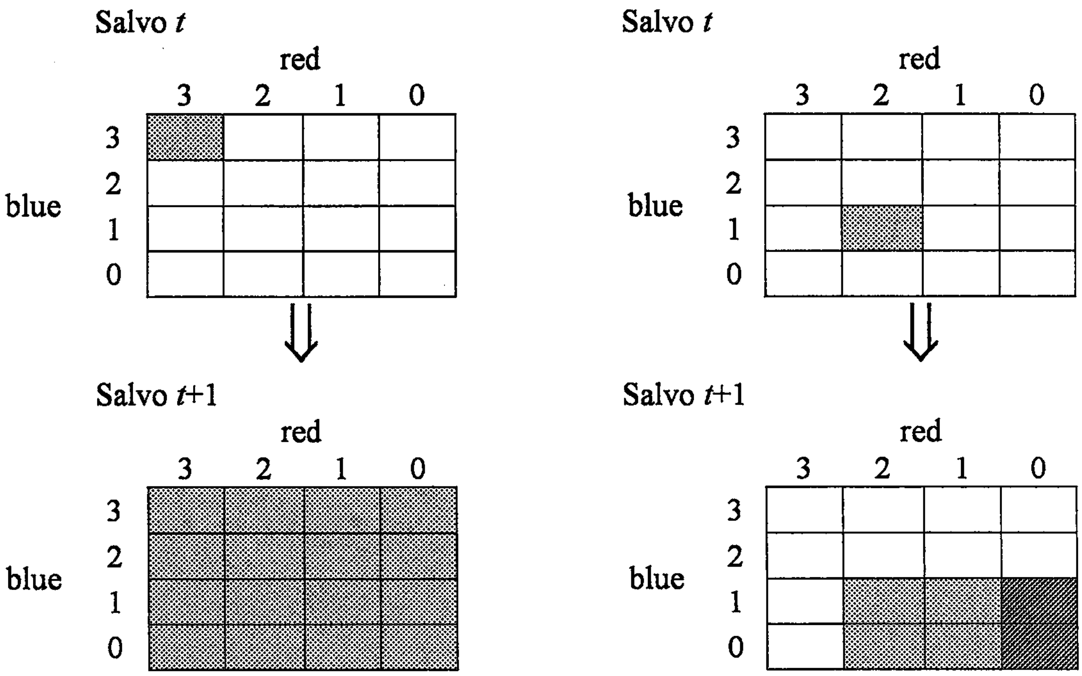
$(t+1,i-2,j-1), (t+1,i-1,j-2), (t+1,i-2,j-2), \ldots, (t+1, \max\{i-j, 0\}, \max\{j-i,0\})$

If area fire is involved, the state $(t+1, 0, 0)$ can also be reached. Possible state transitions for two cases are also shown in Figure 5.3 where the dark shaded cells in (b) are possible only if blue uses area fire.

The state transition $(t,i,j) \rightarrow (t+1, i-\Delta i, j-\Delta j)$ indicates that in salvo t+1 there are Δi blue casualties with probability $P_j^B(\Delta i)$, and Δj red casualties with probability $P_i^R(\Delta j)$ in a duel of i blue versus j red combatants. Regardless of the fire type, the corresponding state transition probability is found as

$$P((t,i,j) \rightarrow (t+1, i-\Delta i, j-\Delta j)) = P_j^B(\Delta i)P_i^R(\Delta j) \qquad (5.16)$$

As an example, Figure 5.4 illustrates all possible state transitions from the state $(t, 4, 3)$ under directed fire. The casualty probabilities for red and blue units are taken from Tables



(a) States from i=3, j=3 (b) States from i=1, j=2

Figure 5.3 Possible states of salvo t+1 resulting from the state of salvo t

5.1 and 5.3. The transition probability of staying in the same state is the probability of no blue or red casualties, 0.5120 (0.4096) = 0.209715. The probability of transition to state (t+1, 2, 2) is the probability of having two blue casualties and one red casualty, which is 0.0624 (0.4944) = 0.030851. Since we have only 3 red combatants, more than 3 casualties in blue is impossible as shown in the last row of Figure 5.4. The state transition probabilities from the state (t, 2, 1) are given in Figure 5.5. Transition to the upper states such as (t+1, 3, 2) is impossible as indicated by the empty cells in the figure.

		Red			
		P(0 cas.)	P(1 cas.)	P(2 cas.)	P(3 cas.)
		0.4096	0.0931	0.4944	0.0029
Blue		$j=3$	$j=2$	$j=1$	$j=0$
P(0 cas.) 0.5120	$i=4$	0.209715	0.253133	0.047677	0.001475
P(1 cas.) 0.4240	$i=3$	0.173670	0.209626	0.039483	0.001221
P(2 cas.) 0.0624	$i=2$	0.025559	0.030851	0.005811	0.000180
P(3 cas.) 0.0016	$i=1$	0.000655	0.000791	0.000149	0.000005
P(4 cas.) 0.0000	$i=0$	0.000000	0.000000	0.000000	0.000000

Figure 5.4 State transition probabilities from the state $(t,4,3)$

				Red	
				P(0 cas.)	P(1 cas.)
				0.6400	0.3600
Blue		$j=3$	$j=2$	$j=1$	$j=0$
	$i=4$				
	$i=3$				
P(0 cas.) 0.7000	$i=2$			0.448000	0.252000
P(1 cas.) 0.3000	$i=1$			0.192000	0.108000
P(2 cas.) 0.0000	$i=0$			0.000000	0.000000

Figure 5.5 State transition probabilities from the state $(t,2,1)$

The state probabilities for the next salvo when both blue and red units are subject to directed fire are calculated as

$$P(t+1,i,j)=\sum_{\Delta i=0}^{\min\{m-i,n\}}\sum_{\Delta j=0}^{\min\{n-j,m\}}\left[P_j^B(\Delta i)P_i^R(\Delta j)\right]P(t,i+\Delta i,j+\Delta j) \quad (5.16)$$

whereas the state probabilities under area fire are

$$P(t+1,i,j)=\sum_{\Delta i=0}^{m-i}\sum_{\Delta j=0}^{n-j}\left[P_j^B(\Delta i)P_i^R(\Delta j)\right]P(t,i+\Delta i,j+\Delta j) \quad (5.17)$$

Because of the initial condition $P(0,m,n)=1$, the state probabilities $P(1,i,j)$, $i=0,1, \ldots, m=4$ and $j=0,1, \ldots, n=3$ of the first salvo are the same as the transition probabilities from the state (0, 4, 3) given in Figure 5.4. The marginal probabilities of having 4, 3, 2, 1 blue combatants alive at the end of the first salvo are 0.5120, 0.4240, 0.0624 and 0.0016, respectively whereas the same probabilities for having 3, 2, 1 red combatants alive at the end of the first salvo are 0.0029, 0.0931, and 0.4944, respectively. Therefore, the expected value and variance of the number of surviving blue combatants at the end of the first salvo are computed as

$$E[i(1)] = \sum_{i=1}^{4} iP(1,i,\bullet) = 1(0.0016)+2(0.0624)+3(0.4240)+4(0.5120) = 3.44640$$

$$\mathrm{Var}[i(1)] = E[i(1)^2] - E[i(1)]^2 = 0.381527$$

where ($\bullet$) indicates summation over corresponding index. For example $P(1,1,\bullet) = P(1,1,0)+P(1,1,1)+P(1,1,2)+P(1,1,3)$. The same values for the red are $E[j(1)]=2.310720$ and $\mathrm{Var}[j(1)]= 0.417693$.

		Red			
		$j=3$	$j=2$	$j=1$	$j=0$
Blue	$i=4$	0.043980	0.103891	0.088842	0.029545
	$i=3$	0.066921	0.169709	0.139746	0.033420
	$i=2$	0.059307	0.115642	0.072698	0.010966
	$i=1$	0.020073	0.028446	0.012143	0.001093
	$i=0$	0.001419	0.001648	0.000488	0.000022

Figure 5.6 State probabilities at the end of the second salvo

The state probabilities $P(2, i, j)$ of the second salvo are given in Figure 5.6. The probability of the transition from state (1,4,3) to (2,3,2) is 0.209626 as in Figure 5.3.

The contribution of this transition to $P(2,3,2)$ is 0.209626x$P(1,4,3)$, which is 0.043962 since $P(1,4,3)$ = 0.209715. Similarly, contributions of transitions to state (2,3,2) from states (1,4,2), (1,3,3) and (1,3,2) are found as 0.047280, 0.025877, and 0.052591. In Figure 5.6, $P(2,3,2)$=0.169709 is found by adding up all four contributions.

The expected number of blue and red combatants, respective variances and 95% confidence intervals around the expected values at the end of the first ten salvos are given in Table 5.6. For example, we can be 95 percent confident that at the end of the first salvo the expected number of blue combatants is in the interval 3.4464 ± 1.21063. As the salvo number increases, the rate of change in the expected number of combatants decreases indicating the convergence. The variance first increases as the diffusion from the initial state takes effect and it starts to decrease as the absorbing states start getting higher probabilities as seen especially in red. The plot for the first ten salvos is given in Figure 5.7.

Table 5.6 Results for the first ten salvos

Salvo	Blue			Red		
	Expected	Variance	95% CI	Expected	Variance	95% CI
0	4.000000	0.000000		3.000000	0.000000	
1	3.446400	0.381527	1.21063	2.310720	0.417693	1.26671
2	2.873403	0.788057	1.73991	1.727690	0.731652	1.67649
3	2.418376	1.140775	2.09338	1.281439	0.891333	1.85041
4	2.097206	1.439102	2.35122	0.985727	0.910048	1.86973
5	1.891530	1.655602	2.52189	0.806947	0.886740	1.84564
6	1.767951	1.801352	2.63055	0.704308	0.865432	1.82333
7	1.696407	1.895509	2.69843	0.646954	0.853520	1.81073
8	1.655798	1.954531	2.74012	0.615304	0.848191	1.80507
9	1.632962	1.990653	2.76532	0.597912	0.846198	1.80295
10	1.620166	2.012350	2.78035	0.588355	0.845623	1.80234

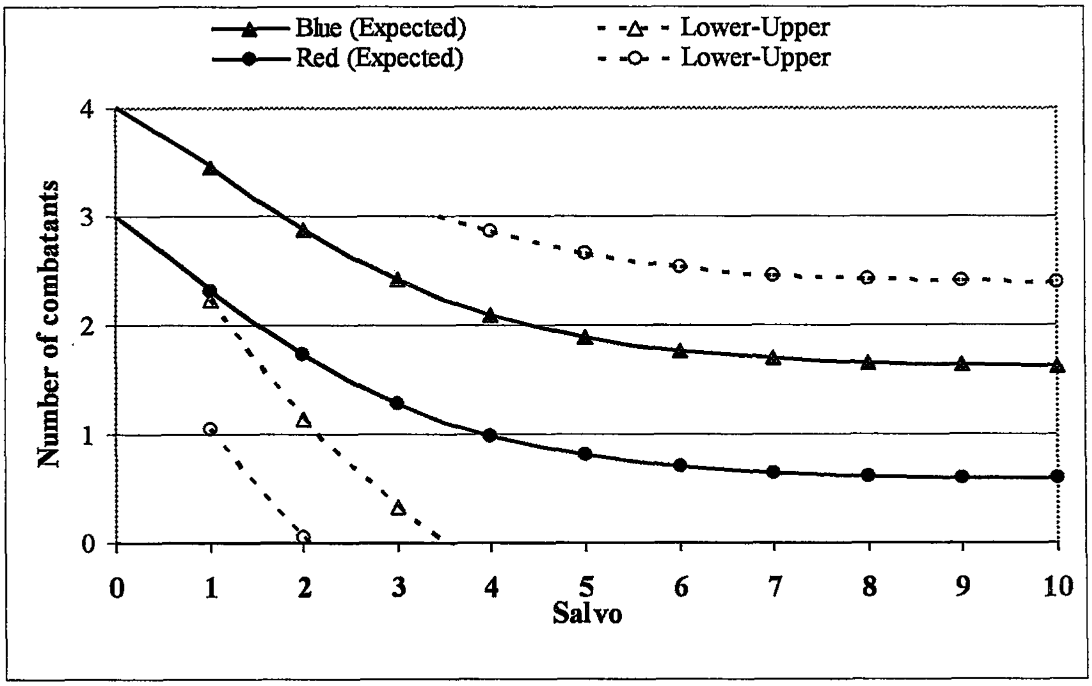


Figure 5.7 Plot for the first ten salvos

5.1.3 DSM for Heterogeneous Forces

In this section, combat between more than two units each having a small number of combatants is modeled as a multi-dimensional death process. Let the state definition be $(t, i_1, \ldots, i_I, j_1, \ldots, j_J)$ where $t = 0, 1, \ldots$ is the discrete time counter denoting the salvo number, $i_1 = 0, 1, \ldots, m_1, \cdots, i_I = 0, 1, \ldots, m_I$ and $j_1 = 0, 1, \ldots, n_1, \cdots, j_J = 0, 1, \ldots, n_J$ be the number of combatants remaining in I blue and J red units at the end of salvo t. Let $P(t, i_1, \ldots, i_I, j_1, \ldots, j_J)$ be the probability of having $i_1, \ldots, i_I$ blue and $j_1, \ldots, j_J$ red combatants alive at the end of salvo t. The initial condition is $P(0, m_1, \ldots, m_I, n_1, \ldots, n_J) = 1$. An engagement between two blue units and one red unit is given in Figure 5.8.

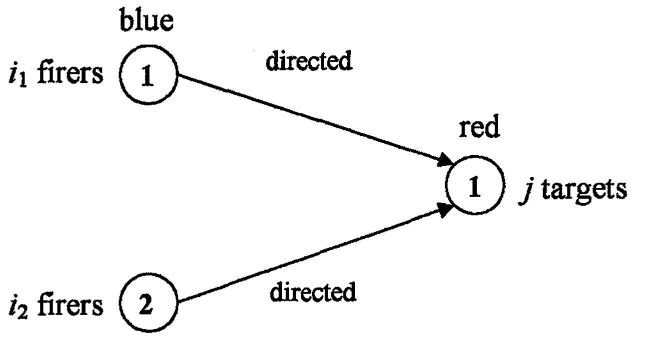


Figure 5.8 Heterogeneous combat situation

Directed Fire

If there are I blue units with i_1, i_2, ..., i_I, firers shooting at j targets in a certain red unit, the number of different arrangements is given by

$$\binom{i_1+j-1}{i_1}\times\binom{i_2+j-1}{i_2}\times\cdots\times\binom{i_I+j-1}{i_I}.$$

In particular, consider the combat situation (given in Figure 5.8) where there are i_1 and i_2 firers in two different units shooting independently at j=3 identical targets with SSKP values p_{k1} and p_{k2}. Let $\{A_1A_2|B_1B_2|C_1C_2\}$ be the pattern denoting the number of firers engaged with each of the j=3 targets such that $A_1+B_1+C_1= i_1$ and $A_2+B_2+C_2= i_2$. Then for i_1=4 and i_2=3, we have 28 patterns resulting from the cartesian products of main patterns of each unit where the main patterns of units are obtained as defined in homogeneous case (Section 5.1.2). The patterns for the above example are given in the first column of Table 5.7.

Table 5.7 Combined directed fire casualty probabilities for i_1=4, i_2=3, j=3, p_{k1}=0.2, p_{k2}= 0.3

$\{A_1A_2\|B_1B_2\|C_1C_2\}$	$n(\{A_1A_2\|B_1B_2\|C_1C_2\})$	$P_{4,3}(0)$	$P_{4,3}(1)$	$P_{4,3}(2)$	$P_{4,3}(3)$	Total
{43|00|00}	3	0.1405	0.8595	0.0000	0.0000	1.0000
{40|03|00}	6	0.1405	0.4716	0.3879	0.0000	1.0000
{42|01|00}	6	0.1405	0.6197	0.2398	0.0000	1.0000
{41|02|00}	6	0.1405	0.4957	0.3638	0.0000	1.0000
{40|02|01}	6	0.1405	0.4089	0.3602	0.0903	1.0000
{41|01|01}	3	0.1405	0.4699	0.3254	0.0642	1.0000
{33|10|00}	6	0.1405	0.6946	0.1649	0.0000	1.0000
{30|13|00}	6	0.1405	0.5054	0.3541	0.0000	1.0000
{30|10|03}	6	0.1405	0.4381	0.3572	0.0641	1.0000
{32|11|00}	6	0.1405	0.5299	0.3296	0.0000	1.0000
{32|10|01}	6	0.1405	0.5148	0.2997	0.0449	1.0000
{31|12|00}	6	0.1405	0.4694	0.3901	0.0000	1.0000
{31|10|02}	6	0.1405	0.4329	0.3612	0.0654	1.0000
{30|12|01}	6	0.1405	0.4120	0.3585	0.0890	1.0000
{30|11|02}	6	0.1405	0.3905	0.3595	0.1095	1.0000
{31|11|01}	6	0.1405	0.4221	0.3527	0.0847	1.0000
{23|20|00}	6	0.1405	0.5785	0.2810	0.0000	1.0000
{20|20|03}	3	0.1405	0.4272	0.3472	0.0851	1.0000
{22|21|00}	6	0.1405	0.4806	0.3789	0.0000	1.0000
{22|20|01}	6	0.1405	0.4467	0.3386	0.0741	1.0000
{21|20|02}	6	0.1405	0.3984	0.3598	0.1013	1.0000
{21|21|01}	3	0.1405	0.4064	0.3617	0.0914	1.0000
{23|10|10}	3	0.1405	0.5698	0.2585	0.0312	1.0000
{20|13|10}	6	0.1405	0.4857	0.3216	0.0522	1.0000
{22|11|10}	6	0.1405	0.4530	0.3461	0.0604	1.0000
{20|12|11}	6	0.1405	0.4073	0.3559	0.0963	1.0000
{21|12|10}	6	0.1405	0.4261	0.3662	0.0671	1.0000
{21|11|11}	3	0.1405	0.3939	0.3588	0.1069	1.0000
Total	150	0.1405	0.4818	0.3301	0.0476	1.0000

For example, pattern {21|12|10} indicates that two firers of first unit and one firer of second unit engage one red target, one firer of first unit and two firers of second unit engage another red target, whereas the last firer of first unit fires at the remaining red target. Note that this scheme may occur in six different ways. Number and probability of kills may, of course, change depending on the pattern. For example, maximum numbers of targets that can be killed are one for {43|00|00}, two for {31|12|00} and three for {21|12|10}. The total number of repetitions of $\{A_1A_2|B_1B_2|C_1C_2\}$ is defined as in Section 5.1.2. For example, if the three segments separated by | are all different, then the number of such repetitions amounts to 3!/(1!1!1!). The relative frequency of a pattern is then

$$P(\{A_1A_2 \mid B_1B_2 \mid C_1C_2\}) = \frac{n(\{A_1A_2 \mid B_1B_2 \mid C_1C_2\})}{\binom{i_1+3-1}{i_1}\binom{i_2+3-1}{i_2}} \tag{5.19}$$

Given a pattern $\{A_1A_2|B_1B_2|C_1C_2\}$, the probability of having l= 0, 1, 2, 3 casualties depends on whether or not the first target subject to A_1 shots from the first unit and A_2 shots from the second unit is killed, and so on. Then, simplifying p_{k1} and p_{k2} as p_1 and p_2,

$$P(l \text{ cas.}|\{A_1A_2|B_1B_2|C_1C_2\}) = \sum_{\substack{l_A,l_B,l_C:0 \text{ or } 1 \\ l_A+l_B+l_C=l}} \left[1-(1-p_1)^{A_1}(1-p_2)^{A_2}\right]^{l_A}\left[(1-p_1)^{A_1}(1-p_2)^{A_2}\right]^{1-l_A}$$

$$\left[1-(1-p_1)^{B_1}(1-p_2)^{B_2}\right]^{l_B}\left[(1-p_1)^{B_1}(1-p_2)^{B_2}\right]^{1-l_B}\left[1-(1-p_1)^{C_1}(1-p_2)^{C_2}\right]^{l_C}\left[(1-p_1)^{C_1}(1-p_2)^{C_2}\right]^{1-l_C} \tag{5.20}$$

Hence, the probability of l casualties under fire from two units is

$$P_{i_1,i_2}(l) = \sum_{\substack{A_1+B_1+C_1=i_1 \\ A_2+B_2+C_2=i_2}} \frac{n(\{A_1A_2 \mid B_1B_2 \mid C_1C_2\})}{\binom{i_1+3-1}{i_1}\binom{i_2+3-1}{i_2}} P(l \text{ } cas. \mid \{A_1A_2 \mid B_1B_2 \mid C_1C_2\}) \tag{5.21}$$

An example case where i_1=4, i_2=3 and j=3 with p_1=0.2 and p_2=0.3 is illustrated in Table 5.7 where, for example, the probability of two casualties for the pattern {21|11|11} is calculated as

P(2cas.|{21|11|11})

$=2[1-0.8^2(0.7)][1-0.8(0.7)][0.8(0.7)]+[0.8^2(0.7)][1-0.8(0.7)]\,[1-0.8(0.7)]$

$=2[0.552][0.44][0.56]+[0.448][0.44][0.44]$

$=2(0.1360128)+0.0867328=0.3587584.$

This pattern contributes to the overall probability of two casualties with $\frac{3}{(10)(15)}0.3587584 = 0.007175186.$

The analysis so far in this subsection is carried out from the viewpoint of firers. The same phenomenon could also be analyzed from the viewpoint of targets. When we consider the above example from the targets' perspective, there are two enemy units creating casualties with the probabilities given in Tables 5.1 and 5.2. The overall attrition process of targets is simply the convolution of the two attrition processes due to different firing units. This approach is similar to the one we use for the area fire presented in Section 5.1.2. If we apply the same analysis, which is summarized by Equation 5.11, we obtain the results given in Table 5.8. The overall casualty probabilities calculated in Tables 5.7 and 5.8 are exactly the same, meaning that we can treat directed fire as area fire. That is, we can use either view to analyze the same phenomenon. In Table 5.7, the total number of patterns depends on i_1, i_2 and j, and is of exponential order in general whereas Table 5.8 contains a smaller number of patterns, which depends mainly on j and the number of firing units. The order of total number of patterns in this case is $\Omega(j^2)$. Therefore, we propose taking the second viewpoint details of which will be discussed in area fire below.

Area Fire

Consider the case where there are three different area firing blue units with i_1, i_2, i_3 firers, and there are j red targets. Suppose values of $P_k(l)$=$P(l$ casualties with k firers) have already been calculated independently for each unit using Equation 5.11 where k=i_1,i_2,i_3. When three units are combined, the pattern (l_1,l_2,l_3|l_{12},l_{13},l_{23}|l_{123}) represents the number of casualties where l_1 is the number of kills only by the first unit, l_{12} denotes the number of kills only by the first and second units, and so on. Let

Table 5.8 Combined area fire casualty probabilities for i_1=4, i_2=3, j=3, p_{k1}=0.2, p_{k2}= 0.3

(l_1,l_2| l_{12})	l_A	l_B	$P_4(l_A)$	$P_3(l_B)$	$P_4(l_A)P_3(l_B)$	$n(l_A,l_B)$	$n(l_1,l_2\|l_{12})$	$P_{4,3}(0)$	$P_{4,3}(1)$	$P_{4,3}(2)$	$P_{4,3}(3)$
(0,0|3)	3	3	0.0029	0.0027	0.0000	1	1				0.0000
(1,0|2)	3	2	0.0029	0.1107	0.0003	3	3				0.0003
(2,0|1)	3	1	0.0029	0.5436	0.0016	3	3				0.0016
(3,0|0)	3	0	0.0029	0.3430	0.0010	1	1				0.0010
(0,1|2)	2	3	0.0931	0.0027	0.0003	3	3				
(0,0|2)	2	2	0.0931	0.1107	0.0103	9	3			0.0034	
(1,1|1)	2	2	0.0931	0.1107	0.0103	9	6				0.0069
(1,0|1)	2	1	0.0931	0.5436	0.0506	9	6			0.0337	
(2,1|0)	2	1	0.0931	0.5436	0.0506	9	3				0.0169
(2,0|0)	2	0	0.0931	0.3430	0.0319	3	3			0.0319	
(0,2|1)	1	3	0.4944	0.0027	0.0013	3	3				
(0,1|1)	1	2	0.4944	0.1107	0.0547	9	6			0.0365	
(1,2|0)	1	2	0.4944	0.1107	0.0547	9	3				
(0,0|1)	1	1	0.4944	0.5436	0.2688	9	3		0.0896		
(1,1|0)	1	1	0.4944	0.5436	0.2688	9	6			0.1792	
(1,0|0)	1	0	0.4944	0.3430	0.1696	3	3		0.1696		
(0,3|0)	0	3	0.4096	0.0027	0.0011	1	1				
(0,2|0)	0	2	0.4096	0.1107	0.0453	3	3			0.0453	
(0,1|0)	0	1	0.4096	0.5436	0.2227	3	3		0.2227		
(0,0|0)	0	0	0.4096	0.3430	0.1405	1	1	0.1405			
Total								**0.1405**	**0.4819**	**0.3300**	**0.0266**

$n(l_1,l_2,l_3|l_{12},l_{13},l_{23}|l_{123})$ denote the number of repetitions and $n(l_A,l_B,l_C)$ be the total number of patterns as defined in Section 5.1.2. With the same definition of S_3, the probability of l casualties under the simultaneous fire of three units is calculated as

$P_{i_1,i_2,i_3}(l) = P(l$ casualties with i_1, i_2, i_3 firers) =

$$\sum_{\substack{l_1,l_2,l_3,l_{12},l_{13},l_{23},l_{123} \in S_3 \\ l_1+l_2+l_3+l_{12}+l_{13}+l_{23}+l_{123}=l}} \frac{n(l_1,l_2,l_3\,|\,l_{12},l_{13},l_{23}\,|\,l_{123})}{n(l_A,l_B,l_C)} P_{i_1}(l_A)P_{i_2}(l_B)P_{i_3}(l_C) \qquad (5.22)$$

Note that here the area fire computations are carried out twice in a hierarchical manner. First, $P_{i_1}(l_A)$ is found by combining firers of the first unit, and this is repeated independently for the other two units. Then the three units are combined to find overall casualty probabilities $P_{i_1,i_2,i_3}(l)$. An example with two blue units is given in Table 5.8.

Mixed Fire

Suppose that, of the blue units combined against a red unit, some employ directed fire and others area fire. The approach proposed for area fire above, where we combine multiple area firing units, can also be used for combining these mixed units. We owe this to the equivalence of the combined casualty probabilities under the two viewpoints (viewpoints of firers and targets), as explained in directed fire case before in this section. For example, if the first unit employs directed fire, then $P_{i_1}(l_A) = P(l$ casualties with i_1 firers) must be calculated by Equation 5.6, otherwise by Equation 5.11. In other words, the fire type affects only the individual probabilities in the convolution $P_{i_1}(l_A)P_{i_2}(l_B)P_{i_3}(l_C)$ but given these, the combined casualty probabilities $P_{i_1,i_2,i_3}(l)$ are the same.

Salvo Treatment

The state transitions in heterogeneous DSM are calculated based on the binomial treatment of units as discussed in the previous subsections. Let us assume that there are I blue units $B_1, \ldots, B_I$ with $m_1, \ldots, m_I$ combatants each, and J red units $R_1, \ldots, R_J$ with $n_1, \ldots, n_J$ combatants each. The state space is then denoted by $(t, i_1, \ldots, i_I, j_1, \ldots, j_J)$ resulting in $(m_1+1)\cdots(m_I+1)(n_1+1)\cdots(n_J+1)$ many states in each salvo including the absorbing states where the number of combatants is zero for any unit. The initial condition is $P(0, m_1, \ldots, m_I, n_1, \ldots, n_J) = 1$.

It is possible for a blue unit to divide its force among multiple red units as explained in Chapter 4. Let $x_{B_1R_1}, \cdots, x_{B_1R_J}$ denote the allocation fractions from ALLM for the first blue unit where $x_{B_1R_1}i_1$ combatants of B_1 are allocated to R_1, $x_{B_1R_2}i_2$ combatants of B_1 is allocated to R_2, and so on.

An example heterogeneous combat situation is illustrated in Figure 5.9. There are two blue units, B_1 with 3 combatants and B_2 with 2 combatants, firing at a red unit R with 3 combatants. B_2 employs area fire, and all remaining fires are directed. R divides its force evenly between B_1 and B_2. The state space notation is (t,i_1,i_2,j) where we have (3+1)(2+1)(3+1) = 48 states in each salvo. The allocations are x_{B_1R}=1.0, x_{B_2R}=1.0, x_{R,B_1}=0.5, and x_{R,B_2}=0.5.

In salvo t, red unit R with j combatants is subject to attrition due to blue units allocated to R with fractions $x_{B_1R},\cdots,x_{B_IR}$. We have shown in the previous subsections how to find the probability of Δj casualties in R under the fire of $[x_{B_1,R}i_1]$ firers of B_1 and $[x_{B_2,R}i_2]$ firers of B_2. However, the number of allocated firers given in brackets should be natural integers. Suppose that B_1 allocates $x_{B_1,R}$=1/3 of its i_1=4 combatants to R. Then the number of B_1 firers engaging R, $x_{B_1,R}\,i_1$= (1/3)4=1.333, is not an integer. Hence, to compute the probability of having Δj casualties with 1.333 firers, first we compute the probabilities of having Δj casualties with 1 firer, $P_1^R(\Delta j)$, and having Δj casualties with 2 firers, $P_2^R(\Delta j)$, and then interpolate these results as follows.

$$P_4^R(\Delta j \text{ cas.} \mid \text{with } 1.333 \text{ firers}) = (2-1.333)\,P_1^R(\Delta j) + (1.333-1)\,P_2^R(\Delta j)$$

We can generalize this process as follows. Without loss of generality, let us assume that $[x_{B_1,R}i_1]$ is not an integer. Let $\check{i}_1 = \lfloor x_{B_1,R}i_1 \rfloor$ and $\hat{i}_1 = \lceil x_{B_1,R}i_1 \rceil$. Let $p_{\hat{i}_1} = x_{B_1,R}i_1 - \check{i}_1$ and $p_{\check{i}_1} = \hat{i}_1 - x_{B_1,R}i_1$ be the interpolation weights. Let $P(\Delta j|\check{i}_1)$ be the probability of having Δj casualties when $\check{i}_1$ firers of B_1 shoot at R. Then, the

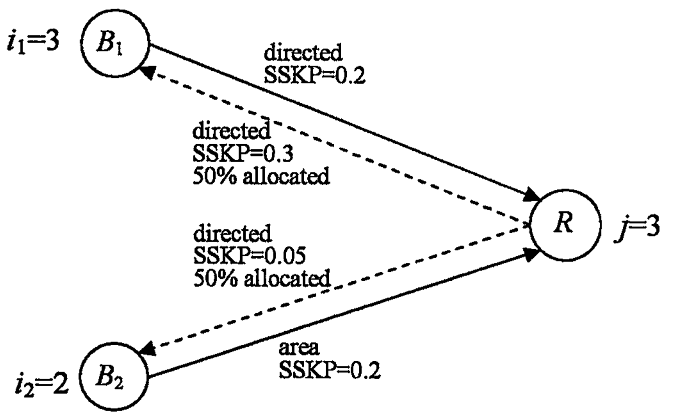


Figure 5.9 Example heterogeneous combat situation

probability of having Δj casualties in R is calculated as

$$P_{i_1}^R(\Delta j \mid [x_{B_1,R} i_1]) = p_{\check{i}_1} P_{\check{i}_1}^R(\Delta j) + p_{\hat{i}_1} P_{\hat{i}_1}^R(\Delta j) \tag{5.23}$$

Generalizing the above equation, we have

$$P_{i_1,\ldots,i_I}^R(\Delta j) = P_{i_1,\ldots,i_I}^R(\Delta j \mid [x_{B_1,R} i_1], \ldots, [x_{B_I,R} i_I])$$

$$= p_{\check{i}_1} \cdots p_{\check{i}_I} P_{\check{i}_1,\ldots,\check{i}_I}^R(\Delta j) + \cdots + p_{\hat{i}_1} \cdots p_{\hat{i}_I} P_{\hat{i}_1,\ldots,\hat{i}_I}^R(\Delta j) \tag{5.24}$$

In our example problem given in Figure 5.9, R divides its j=3 firers evenly between B_1 and B_2. Let us consider the attrition in B_1. We have $\check{j} = \lfloor 0.5(3) \rfloor = 1$, $\hat{j} = \lceil 0.5(3) \rceil = 2$ and $p_{\check{j}} = 0.5 = p_{\hat{j}}$. Since the probability of one casualty in B_1 is 0.465000 when j=2, and 0.300000 when j=1, $P_3^{B_1}(1)$=(0.5)0.465000+(0.5)0.300000 = 0.382500. Similarly, $P_3^{B_2}(1)$ = (0.5)0.096667 + (0.5)0.050000 = 0.073333.

The state transition $(t-1, i_1, \ldots, i_I, j_1, \ldots, j_J) \rightarrow (t, i_1-\Delta i_1, \ldots, i_I-\Delta i_I, j_1-\Delta j_1, \ldots, j_J-\Delta j_J)$ indicates that there are Δi_1 casualties in B_1, Δi_2 casualties in B_2, and so on. The corresponding state transition probability is

$$P((t-1, i_1, \ldots, i_I, j_1, \ldots, j_J) \rightarrow (t, i_1-\Delta i_1, \ldots, i_I-\Delta i_I, j_1-\Delta j_1, \ldots, j_J-\Delta j_J))=$$

$$P_{j_1,\ldots,j_J}^{B_1}(\Delta i_1) \cdots P_{j_1,\ldots,j_J}^{B_I}(\Delta i_I) P_{i_1,\ldots,i_I}^{R_1}(\Delta j_1) \cdots P_{i_1,\ldots,i_I}^{R_J}(\Delta j_J) \tag{5.25}$$

In our example, suppose an area firing combatant of B_2 has an effective radius of r=40, and the region containing j=3 red targets has radius R=100. The casualty probabilities for red are found as (Equation 5.7) $P_1^R(0)$=0.952764, $P_1^R(1)$=0.046476, $P_1^R(2)$=0.000756, $P_1^R(3)$=0.000004. If we combine two such firers, the effect of B_2 alone yields (Equation 5.11) $P_2^R(0)$=0.907759, $P_2^R(1)$=0.089282, $P_2^R(2)$=0.002927, $P_2^R(3)$=0.000032. The effect of B_1 alone with 3 firers is (Equation

5.6) $P_3^R(0)$=0.512000, $P_3^R(1)$=0.434400, $P_3^R(2)$=0.052800, $P_3^R(3)$=0.000800. If we combine the effects of B_1 and B_2, the red casualty probabilities are (Equation 5.22) $P_{3,2}^R(0)$=0.464773, $P_{3,2}^R(1)$=0.452971, $P_{3,2}^R(2)$=0.079326, $P_{3,2}^R(3)$=0.002930.

The state transition probabilities to the state $(t,2,1,1)$ in our example problem are listed in Table 5.9. Let us consider the state transition $(t-1,3,2,3) \rightarrow (t,2,1,1)$ given in the first row of the table, indicating that one B_1 combatant, one B_2 combatant and two R combatants are killed. Recall that, $P_3^{B_1}(1)$=0.382500, $P_3^{B_2}$=(1)=0.073333 and $P_{3,2}^R(2)$=0.079326. Hence,

$$P((t-1, 3, 2, 3) \rightarrow (t, 2, 1, 1)) = P_3^{B_1}(1)\, P_3^{B_2}(1)\, P_{3,2}^R(2)$$

$$= 0.382500\,(0.073333)\, 0.079326 = 0.002225.$$

Table 5.9 State transition probabilities to the state $(t,2,1,1)$

$(t-1, i_1, i_2, j)$	$P_j^{B_1}(i_1-2)$	$P_j^{B_2}(i_2-1)$	$P_{i_1,i_2}^R(j-1)$	Probability
$(t-1,3,2,3)$	0.382500	0.073333	0.079326	0.002225
$(t-1,3,2,2)$	0.300000	0.050000	0.469248	0.007039
$(t-1,3,2,1)$	0.150000	0.025000	0.495747	0.001859
$(t-1,3,1,3)$	0.382500	0.926250	0.066021	0.023390
$(t-1,3,1,2)$	0.300000	0.950000	0.460890	0.131354
$(t-1,3,1,1)$	0.150000	0.975000	0.503808	0.073682
$(t-1,2,2,3)$	0.595000	0.073333	0.042139	0.001839
$(t-1,2,2,2)$	0.700000	0.050000	0.375139	0.013130
$(t-1,2,2,1)$	0.850000	0.025000	0.619684	0.013168
$(t-1,2,1,3)$	0.595000	0.926250	0.030870	0.017013
$(t-1,2,1,2)$	0.700000	0.950000	0.361272	0.240246
$(t-1,2,1,1)$	0.850000	0.975000	0.629760	0.521914

The state probabilities for the first salvo can be seen in Figure 5.10. Note that these state probabilities are the same as the transition probabilities from the initial state due to the initial condition $P(0, m_1, \ldots, m_I, n_1, \ldots, n_J) = 1$. The state probabilities are in general determined as

Blue 1: $i_1=3$

Blue 2		Red $j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.256145	0.249641	0.043718	0.001615
Blue 2	$i_2=1$	0.020280	0.019765	0.003461	0.000128
	$i_2=0$	0.000115	0.000112	0.000020	0.000001

Blue 1: $i_1=2$

Blue 2		Red $j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.164665	0.160483	0.028105	0.001038
Blue 2	$i_2=1$	0.013037	0.012706	0.002225	0.000082
	$i_2=0$	0.000074	0.000072	0.000013	0.000000

Blue 1: $i_1=1$

Blue 2		Red $j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.009686	0.009440	0.001653	0.000061
Blue 2	$i_2=1$	0.000767	0.000747	0.000131	0.000005
	$i_2=0$	0.000004	0.000004	0.000001	0.000000

Blue 1: $i_1=0$

Blue 2		Red $j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.000000	0.000000	0.000000	0.000000
Blue 2	$i_2=1$	0.000000	0.000000	0.000000	0.000000
	$i_2=0$	0.000000	0.000000	0.000000	0.000000

Figure 5.10 State probabilities at the end of first salvo

$P((t, i_1, \ldots, j_J) =$

$$\sum_{\Delta i_1=0}^{m_1-i_1} \cdots \sum_{\Delta j_J=0}^{n_1-j_J} P(t-1, i_1+\Delta i_1, \ldots, j_J+\Delta j_J) P^{B_1}_{j_1,\ldots,j_J}(\Delta i_1) \cdots P^{R_J}_{i_1,\ldots,i_I}(\Delta j_J) \qquad (5.26)$$

Table 5.10 State probability of the state (2,2,1,1)

$(1,i_1,i_2,j)$	$P(1,i_1,i_2,j)$	Trans. P.	Contribution
(1,3,2,3)	0.256145	0.002225	0.000570
(1,3,2,2)	0.249641	0.007039	0.001757
(1,3,2,1)	0.043718	0.001859	0.000081
(1,3,1,3)	0.020280	0.023390	0.000474
(1,3,1,2)	0.019765	0.131354	0.002596
(1,3,1,1)	0.003461	0.073682	0.000255
(1,2,2,3)	0.164665	0.001839	0.000303
(1,2,2,2)	0.160483	0.013130	0.002107
(1,2,2,1)	0.028105	0.013168	0.000370
(1,2,1,3)	0.013037	0.017013	0.000222
(1,2,1,2)	0.012706	0.240246	0.003053
(1,2,1,1)	0.002225	0.521914	0.001161
Total			0.012950

To obtain the state probabilities for second salvo, $P(2,i_1,i_2,j)$, we first multiply the transition probabilities for each transition that yields $(2,i_1,i_2,j)$, say $P((1,i_1',i_2',j') \rightarrow (2, i_1,i_2,j))$, with corresponding state probabilities $P(1,i_1',i_2',j')$ of the first salvo (given in Table 5.9) to find individual contributions to state $P(2,i_1,i_2,j)$. Then summing all contributions to state $(2,i_1,i_2,j)$ we obtain the state probability, $P(2,i_1,i_2,j)$, for second salvo. Table 5.10 illustrates calculation of the state probability of (2,2,1,1) in our example problem. The transition probabilities to this state (found in Table 5.9) are multiplied with the state probabilities of the first salvo to determine individual contributions. For example, the contribution of (1,3,2,3) to (2,2,1,1) is

$$P(1,3,2,3)P((t-1,3,2,3) \rightarrow (t,2,1,1)) = 0.256145(0.002225) = 0.000570.$$

When all such contributions are added, the state probability $P(2,2,1,1)=0.012950$ is found as seen in Figure 5.11.

Blue 1: $i_1=3$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.065610	0.143631	0.107060	0.028722
Blue 2	$i_2=1$	0.010647	0.020739	0.013688	0.003085
	$i_2=0$	0.000499	0.000837	0.000467	0.000080

Blue 1: $i_1=2$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.094900	0.173405	0.102047	0.019739
Blue 2	$i_2=1$	0.015399	0.024923	0.012950	0.002104
	$i_2=0$	0.000721	0.001000	0.000438	0.000054

Blue 1: $i_1=1$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.040915	0.058311	0.025300	0.003190
Blue 2	$i_2=1$	0.006639	0.008368	0.003199	0.000338
	$i_2=0$	0.000311	0.000335	0.000108	0.000009

Blue 1: $i_1=0$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.003968	0.003819	0.000991	0.000076
Blue 2	$i_2=1$	0.000644	0.000545	0.000121	0.000008
	$i_2=0$	0.000030	0.000022	0.000004	0.000000

Figure 5.11 State probabilities at the end of second salvo

The expected number of blue and red combatants at the end of the first ten salvos and their respective variances are given in Table 5.11. As seen from these results, the rate of reduction in the expected number of combatants decreases as the salvo number increases, indicating the convergence as in the homogeneous DSM.

Table 5.11 Summary results for the first ten salvos

	Blue 1		Blue 2		Red	
Salvo	Expected	Variance	Expected	Variance	Expected	Variance
0	3.000000	0.000000	2.000000	0.000000	3.000000	0.000000
1	2.572500	0.289744	1.925833	0.069499	2.379586	0.411734
2	2.227588	0.531202	1.866770	0.125312	1.859099	0.716428
3	1.959507	0.742672	1.820757	0.168989	1.441089	0.891700
4	1.765284	0.904259	1.785343	0.202930	1.133023	0.935434
5	1.631606	1.021451	1.757784	0.229783	0.918030	0.911780
6	1.542525	1.105196	1.735742	0.251729	0.771864	0.867468
7	1.484273	1.164580	1.717495	0.270341	0.673054	0.823081
8	1.446561	1.206313	1.701862	0.286679	0.605628	0.784667
9	1.422250	1.235345	1.688070	0.301432	0.558648	0.752603
10	1.406589	1.255348	1.675617	0.315040	0.524932	0.725716

So far we have completed the basic computations of DSM including combat loss only. Before continuing with the extensions we briefly discuss the computational complexity of DSM. For a problem involving I blue units (B_1, …, B_I) and J red units (R_1, …, R_J) where initial number of combatants are m_1, …, m_I, n_1, …, n_J size of the state space is

$$v = \prod_{i=1}^{I}(m_i + 1)\prod_{j=1}^{J}(n_j + 1)$$

Once we compute state transition probabilities, $P(l \text{ kills})$ is done as a preprocessing which is exponential (2) due to number of patterns. Computation time for state transition probabilities (which is calculated once and used in every salvo), due to combat losses, is polynomial in $O(\nu^2)$. Considering the new state matrix (state probabilities at salvo t+1) and marginal probabilities the computation time for each is also polynomial in $O(\nu^2)$. Hence, computation time for one salvo is $O(\nu^4)$ yielding a total time of $O(\tau\nu^4)$ for τ salvos.

5.1.4 Extensions for DSM

We present two major and two minor extensions here. Major extensions are concerned with noncombat losses and the engagement process. They involve additional multiple single-dimensional discrete-time processes linked to DSM. Minor extensions are small changes in SSKPs to treat synergy effects, and shifts in states of military units that are subject to reinforcements.

Noncombat Loss

There are two sources of attrition in combat: combat loss due to the fire of the opposing force and noncombat loss due to reasons such as illness, accidents and desertions. While combat loss is determined by the interaction between two forces, noncombat loss depends only on own force sizes.

Noncombat losses can also be handled in DSM by means of the binomial process. We assume that noncombat loss probabilities (or rates), $q_{B_1},\ldots,q_{R_J}$ are specified for all military units and kept constant for all salvos. If a military unit is not subject to noncombat loss, its noncombat loss probability is fixed at zero. Single-dimensional noncombat loss transition probabilities for blue unit B_1 are calculated as

$$Q^{B_1}((t, i_1 + \Delta i_1) \rightarrow (t, i_1)) = \binom{i_1 + \Delta i_1}{\Delta i_1} q_{B_1}^{\Delta i_1} (1 - q_{B_1})^{i_1} \tag{5.27}$$

The marginal probability distribution of B_1, $P_{B_1}(t, i_1)$, i_1=1, …, m_1 can be determined from the joint state probabilities $P(t, i_1, \ldots, i_I, j_1, \ldots, j_J)$ resulting from combat loss. Hence, the single-dimensional state probabilities after noncombat loss in B_1 are

$$Q_{B_1}(t, i_1) = \sum_{\Delta i_1 = 0}^{m_1 - i_1} P_{B_1}(t, i_1 + \Delta i_1) \binom{i_1 + \Delta i_1}{\Delta i_1} q_{B_1}^{\Delta i_1} (1 - q_{B_1})^{i_1 - \Delta i_1} \tag{5.28}$$

Let $Q(t, i_1, \ldots, i_I, j_1, \ldots, j_J)$ be the joint state probabilities after noncombat loss at the end of salvo t. Then,

$$Q(t, i_1, \ldots, i_I, j_1, \ldots, j_J) = Q_{B_1}(t, i_1) \cdots Q_{B_I}(t, i_I) Q_{R_1}(t, j_1) \cdots Q_{R_J}(t, j_J) \tag{5.29}$$

Equivalently,

$$Q(t, i_1, \ldots, i_I, j_1, \ldots, j_J) = \sum_{\Delta i_1 = 0}^{m_1 - i_1} \cdots \sum_{\Delta j_J = 0}^{n_J - j_J} P(t, i_1 + \Delta i_1, \ldots, j_J + \Delta j_J)$$

$$\binom{i_1 + \Delta i_1}{\Delta i_1} q_{B_1}^{\Delta i_1} (1 - q_{B_1})^{i_1 - \Delta i_1} \cdots \binom{j_J + \Delta j_J}{\Delta j_J} q_{R_J}^{\Delta j_J} (1 - q_{R_J})^{j_J - \Delta j_J} \tag{5.30}$$

In order to incorporate noncombat losses in DSM, in the combat loss state probability calculation described in Equation 5.26, the state probabilities of the previous salvo, $P(t-1, i_1+\Delta i_1, \ldots, j_1+\Delta j_J)$, should be replaced with the probabilities after noncombat loss, $Q(t-1, i_1+\Delta i_1, \ldots, j_1+\Delta j_J)$, except for the first salvo.

The incorporation of noncombat loss process may lead to decomposition of the overall combat simulation into small-sized DSM simulations. Consider the situation given in Figure 5.12. Instead of dealing with this combat as a whole which involves

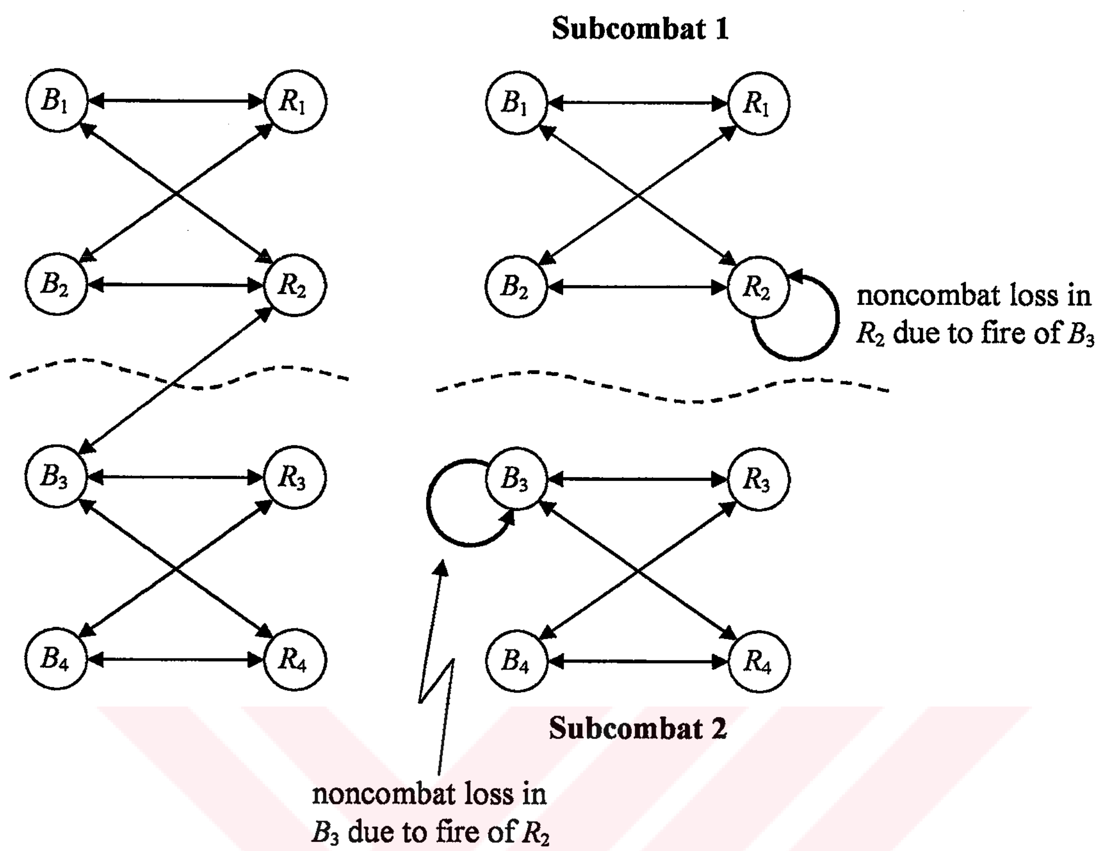


Figure 5.12 Decomposition of a combat using noncombat loss process

four units, we can decompose it into two subcombats as shown in the figure. Regarding the engagement between B_3 and R_2 we treat this interaction as if each one is subjet to a noncombat loss effect due to its opponent's fire. Note that noncombat rates of B_3 and R_2 change in every salvo.

Engagement

One of the assumptions of DSM so far is that all combatants of a military unit fire in every salvo. Target detection time, weapon preparation time, and rate of fire vary for different weapon systems, making perfect synchronization impossible. If we determine the salvo duration in terms of the most frequent firing weapon system, we can define an engagement probability for the other systems. For example, if salvo duration is determined in terms of the firing cycle of an infantry soldier with a rifle, and an antitank weapon system fires once in every four salvos, then we set its engagement probability as 0.25. Furthermore, command control problems and exhaustion in weapon systems may give rise to engagement probabilities even if technical capabilities permit to fire in each salvo.

Let $e_{B_1}, \ldots, e_{R_J}$ be engagement probabilities for military units. If all combatants of a military unit fire in every salvo, its engagement probability is one. Engagement probabilities induce independent single-dimensional binomial processes as in noncombat loss. However, the combatants that do not fire in a certain salvo stay in combat and are subject to attrition. Hence, the engagement process affects all the targets although the number of firers may decrease.

In plain DSM, we have assumed that R with j combatants is subject to attrition due to blue units allocated to R with the fractions $x_{B_1R}, \cdots, x_{B_IR}$. Let us consider the interaction of B_1 and R. In plain DSM, only the values of $\breve{i}_1 = \lfloor x_{B_1,R} i_1 \rfloor$ and $\hat{i}_1$ = $\lceil x_{B_1,R} i_1 \rceil$ are considered in calculating the combat loss of R. With the above engagement process, one has to consider $\hat{i}_1$, $\breve{i}_1$, $\breve{i}_1 - 1$, …, 1, 0 with respective probabilities calculated from the binomial engagement distribution.

The effect of the engagement process in our salvo calculations can be reflected in Equation (5.24), which is modified as

$$P^R_{i_1,\ldots,i_I}\left(\Delta j \mid [x_{B_1,R} i_1],\ldots,[x_{B_I,R} i_I]\right) =$$

$$\sum_{\nabla i_1=0}^{i_1}\cdots\sum_{\nabla i_I=0}^{i_I} P^R_{i_1,\ldots,i_I}\left(\Delta i_1 \mid [x_{B_1,R}(i_1-\nabla i_1)],\ldots,[x_{B_I,R}(i_I-\nabla i_I)]\right)$$

$$\binom{i_1}{\nabla i_1} e_{B_1}^{\nabla i_1}(1-e_{B_1})^{i_1-\nabla i_1}\cdots\binom{i_I}{\nabla i_I} e_{B_I}^{\nabla i_I}(1-e_{B_I})^{i_I-\nabla i_I} \qquad (5.31)$$

In our example problem given in Figure 5.9, let the engagement probabilities be e_{B_1}=1.0 indicating that all B_1 combatants engage, e_{B_2}=0.5 meaning each B_2 combatant fires once in every two salvos on the average, and e_R=0.8 since red combatants are under heavy fire and cannot engage all the time. Recall that the initial condition is $P(0,3,2,3)=1$. Thus, in the first salvo, exactly i_1=3 B_1 firers and i_2=2, 1 or 0 B_2 firers shoot at j=3 targets. The engagement probability distribution for B_2 firers is:

$P(i_2 = 2) = 0.5^2 = 0.25$, $P(i_2 = 1) = (2)0.5(0.5) = 0.50$, $P(i_2 = 0) = 0.5^2 = 0.25$.

Then,

$P^R(\Delta j_1|[i_1=3],[i_2=2, 1 \text{ or } 0]) =$

$0.25P^R(\Delta j_1|[i_1= 3],[i_2=2])+0.50P^R(\Delta j_1|[i_1= 3],[i_2=1])+0.25P^R(\Delta j_1|[i_1=3],[i_2=0])$.

Given the combat loss probabilities of R with i_2=2,1 or 0, loss probabilities after engagement are found as

$P^R(0)$=0.25(0.464773)+0.50(0.487815)+0.25(0.512000)=0.488101,

$P^R(1)$=0.25(0.452971)+0.50(0.444406)+0.25(0.434400)=0.444046,

$P^R(2)$=0.25(0.079326)+0.50(0.066021)+0.25(0.052800)=0.066042 and

$P^R(3)$=0.25(0.002930)+0.50(0.001758)+0.25(0.000800)=0.001812.

Similarly,

$P^{B_1}(0)$=0.663040, $P^{B_1}(1)$=0.325440, $P^{B_1}(2)$=0.011520, $P^{B_1}(3)$=0.000000, and

$P^{B_2}(0)$=0.940640, $P^{B_2}(1)$=0.059147, $P^{B_2}(2)$=0.000213.

The state probabilities calculated so far in the first salvo are given in Figure 5.13. For example, the state probability $P(1,2,1,1)=P^{B_1}(1)P^{B_2}(1)P^{R}(2)$=0.325440 (0.059147) 0.066042 = 0.001271.

The marginal probabilities at this point are

$P_{B_1}(3)$=0.663040, $P_{B_1}(2)$=0.325440, $P_{B_1}(1)$=0.011520, $P_{B_1}(0)$=0.000000;

$P_{B_2}(2)$=0.940640, $P_{B_2}(1)$=0.059147, $P_{B_2}(0)$=0.000213;

$P_R(3)$=0.488101, $P_R(2)$=0.444046, $P_R(1)$=0.066042, and $P_R(0)$=0.001812. Consequently, the expected values for B_1, B_2 and R are 2.651520, 1.940427 and 2.418436.

So far, we have incorporated the engagement process in our example and found combat loss figures after engagement for the first salvo. Before the battle moves on to the second salvo, noncombat loss can also be applied. Let the noncombat loss probabilities be q_{B_1}=0.05, q_{B_2}=0.001 and q_R=0.08. Then from Equation 5.27, noncombat loss probabilities for the transition (1,3,2,3) $\rightarrow$ (1,2,1,1) are found as

Blue 1: $i_1=3$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.304420	0.276943	0.041189	0.001130
Blue 2	$i_2=1$	0.019142	0.017414	0.002590	0.000071
	$i_2=0$	0.000069	0.000063	0.000009	0.000000

Blue 1: $i_1=2$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.149418	0.135932	0.020217	0.000555
Blue 2	$i_2=1$	0.009395	0.008547	0.001271	0.000035
	$i_2=0$	0.000034	0.000031	0.000005	0.000000

Blue 1: $i_1=1$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.005289	0.004812	0.000716	0.000020
Blue 2	$i_2=1$	0.000333	0.000303	0.000045	0.000001
	$i_2=0$	0.000001	0.000001	0.000000	0.000000

Blue 1: $i_1=0$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.000000	0.000000	0.000000	0.000000
Blue 2	$i_2=1$	0.000000	0.000000	0.000000	0.000000
	$i_2=0$	0.000000	0.000000	0.000000	0.000000

Figure 5.13 State probabilities after engagement and shooting in the first salvo

$Q^{B_1}((1,3) \rightarrow (1,2))=(3)0.05(0.95)^2=0.135375$,

$Q^{B_2}((1,2) \rightarrow (1,1))=0.001998$, and

$Q_R((1,3) \rightarrow (1,1))=0.017664$.

Thus, the contribution of the state (1,3,2,3) before noncombat loss to the state (1,2,1,1) after noncombat loss is 0.135375(0.001998)0.017664=0.000005. Recall from Figure 5.13 that $P(1,3,2,3)$=0.304420. Hence, using Equation 5.29, the contribution of (1,3,2,3) before noncombat loss to the state probability $Q(1,2,1,1)$ is 0.304420(0.000005)=0.0000015. If we add all possible contributions to (1,2,1,1) after noncombat loss as we did in Table 5.10, we have $Q(1,2,1,1)$=0.003150. The state probabilities found after noncombat loss are given in Figure 5.14. Using the information in Figure 5.14, the expected values calculated for B_1, B_2 and R are 2.518944, 1.938486 and 2.224961.

Blue 1: $i_1=3$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.202833	0.253484	0.071907	0.005436
Blue 2	$i_2=1$	0.013173	0.016462	0.004670	0.000353
	$i_2=0$	0.000059	0.000074	0.000021	0.000002

Blue 1: $i_1=2$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.136822	0.170990	0.048506	0.003667
Blue 2	$i_2=1$	0.008886	0.011105	0.003150	0.000238
	$i_2=0$	0.000040	0.000050	0.000014	0.000001

Blue 1: $i_1=1$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.016622	0.020772	0.005893	0.000445
Blue 2	$i_2=1$	0.001079	0.001349	0.000383	0.000029
	$i_2=0$	0.000005	0.000006	0.000002	0.000000

Blue 1: $i_1=0$ — Red

		$j=3$	$j=2$	$j=1$	$j=0$
	$i_2=2$	0.000525	0.000657	0.000186	0.000014
Blue 2	$i_2=1$	0.000034	0.000043	0.000012	0.000001
	$i_2=0$	0.000000	0.000000	0.000000	0.000000

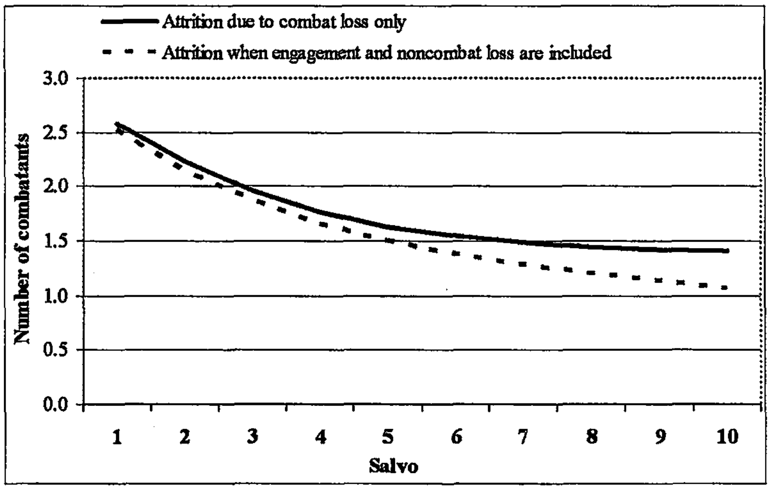
Figure 5.14 State probabilities after noncombat loss in the first salvo

Recalling that the expected value of B_1 is 2.651520 without noncombat loss, then the rate of reduction in its expected value is (2.651520–2.518944)/2.651520 = 0.05 due to noncombat loss, meaning that the rates of reduction in expected values are equal to the respective noncombat loss probabilities.

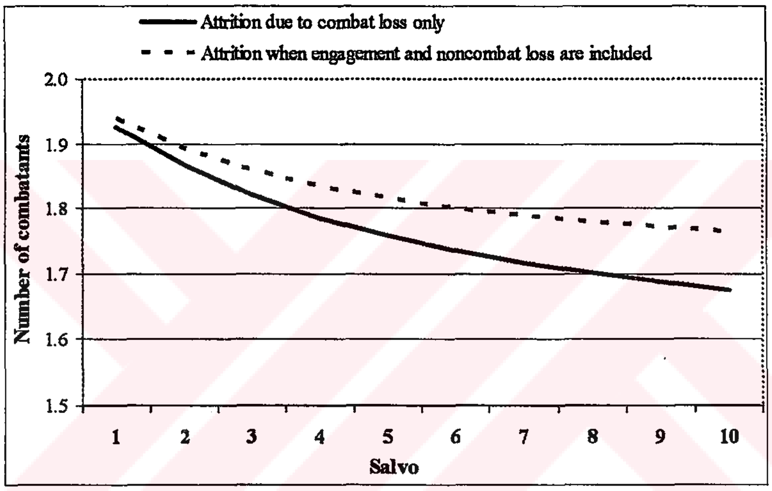
The expected number and variance of blue and red combatants over ten salvos, after the major extensions are applied to the example problem, are given in Table 5.12. The plots comparing the attrition of units due to combat losses only and the attrition after engagement and noncombat loss processes for the first ten salvos are given in Figure 5.15.

Table 5.12 Summary results with engagement and noncombat loss

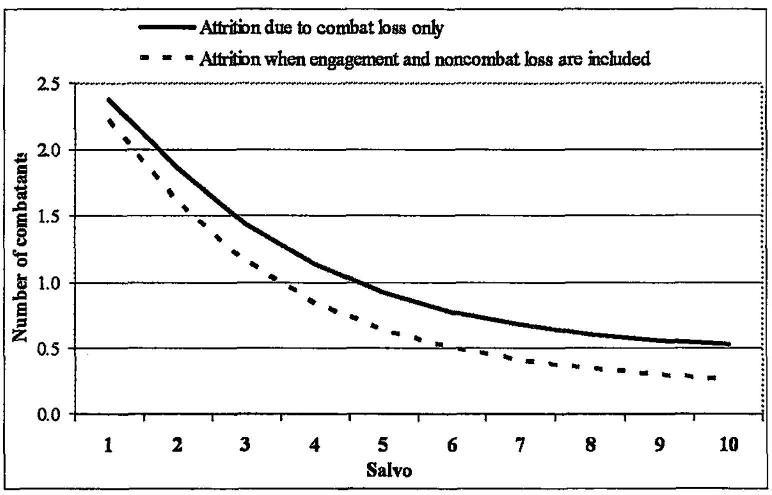
	Blue 1		Blue 2		Red	
Salvo	Expected	Variance	Expected	Variance	Expected	Variance
0	3.000000	0.000000	2.000000	0.000000	3.000000	0.000000
1	2.518944	0.351646	1.938486	0.058277	2.224961	0.504962
2	2.144604	0.602536	1.892272	0.102419	1.610222	0.783351
3	1.861228	0.773690	1.858394	0.135002	1.153151	0.845955
4	1.651866	0.874442	1.833731	0.158975	0.840067	0.773918
5	1.495940	0.924668	1.815386	0.177047	0.633149	0.664859
6	1.376101	0.941542	1.801219	0.191197	0.496469	0.562645
7	1.280010	0.937288	1.789815	0.202730	0.404095	0.478162
8	1.199587	0.920103	1.780283	0.212469	0.339291	0.410448
9	1.129747	0.895322	1.772068	0.220928	0.291765	0.355948
10	1.067339	0.866329	1.764821	0.228429	0.255308	0.311383



(a) Attrition of Blue 1



(b) Attrition of Blue 2



(c) Attrition of Red

Figure 5.15 Plots for the first ten salvos

The treatment of different engagement probabilities to regulate ammunition availability is possible as well. Consider an antitank weapon with four ammunitions each of which is fired in the first four salvos. Once we fix the engagement probability of this weapon system at the value of zero in salvos 5 and on, we can model the second phase during which this weapon system is subject to attrition but it cannot fire and give attrition.

Reinforcements

Although combat between two units can be modeled under various rates of continuous reinforcement as Jaiswal *et al.* (1997) did, we consider discrete reinforcements that simply extend the dimension of the state space and incur a shift in the state probabilities. Suppose that the red unit in our example gets two new combatants at the end of salvo t. Then

$$n \leftarrow 3+2=5,\ P(t, i_1, i_2, j+2) \leftarrow P(t, i_1, i_2, j), \text{ and } P(t, i_1, i_2, 1)=P(t, i_1, i_2, 0)=0 \qquad (5.32)$$

Division and Combination Effects

As explained in Chapter 4, we assume that firepower of a unit can be distributed over more than one opposing unit. Force division usually yields a reduction in the attrition potentials. Let λ_R denote the fractional loss in attrition power of R when it divides its force between two blue units as in our example. The SSKPs of R to blue units ($p_{k:R,B_1}$, $p_{k:R,B_2}$) are decreased by λ_R if the force division effect is to be observed. Let $p^{FD}_{k:R,B_1}$ and $p^{FD}_{k:R,B_2}$ be the SSKPs after force division. Then,

$$p^{FD}_{k:R,B_1} = (1-\lambda_R)\, p_{k:R,B_1} \text{ and } p^{FD}_{k:R,B_2} = (1-\lambda_R)\, p_{k:R,B_2} \tag{5.33}$$

As to the combination effect, when two or more units are allocated to the same target, this combination results in an increase in the attrition power due to synergy. Let ϕ_{B_1} and ϕ_{B_2} denote the fractional gain in attrition power of B_1 and B_2 combatants when they are combined against R. If we increase the SSKP of B_1 combatants by ϕ_{B_1} then it is possible that the resulting SSKP will be greater than one. Suppose the SSKP is 0.95 and this probability increases by ϕ_{B_1}=0.20 yielding (0.95)(1.20)=1.14 as the updated probability. To avoid this, instead of increasing the kill probability we reduce the probability of not kill by $(1-\phi_{B_1})$, which, for the above example, yields (0.05)(0.80)=0.04 resulting in a SSKP of 1–0.04=0.96. Hence the SSKPs of B_1 and B_2 after force combination are

$$p^{FC}_{k:B_1,R} = 1-(1-\phi_{B_1})(1-p_{k:B_1,R}) \text{ and } p^{FC}_{k:B_2,R} = 1-(1-\phi_{B_2})(1-p_{k:B_2,R}) \tag{5.34}$$

5.2 Lanchester Model (LM)

Lanchester model (LM) is the other module that ASM uses. LM simulates combats involving units each having more than 20 combatants by solving a set of differential equations depending on the fire type. These differential equations define the attrition rate (rate of change in force level) of unit, using the attrition rate coefficients (ARCs). LM employs three basic Lanchester equations according to fire type (directed, area, and Helmbold) which are defined in Chapter 2. Recall that closed form solutions of these equations are available for combats including only homogeneous forces where both sides use either are fire or directed fire. Since our work includes engagements involving heterogeneous forces and allows mixed use of directed and area fires we choose to solve these differential equations by numerical approximation.

The differential equation system for a combat situation is written according to the specific attrition conditions such as fire types of opposing units, reinforcement and noncombat loss rates, constant or variable attrition coefficients, division and combination effects. For instance, a Lanchester system of equations defining the attrition of red unit R_j subject to fires of blue units is given below.

$$\frac{dR_j}{dt} = -\sum_{i \in SB_j^D} x_{ij} b_{ij} B_i - \sum_{i \in SB_j^A} x_{ij} b_{ij} R_j B_i - \sum_{i \in SB_j^H} x_{ij} b_{ij} \left(\frac{R_j}{B_i}\right)^{1-\tau_{ij}} B_i + \delta_j - \gamma_j R_j \qquad (5.35)$$

where x_{ij} is the fraction of B_i allocated to R_j and b_{ij} is the corresponding attrition coefficient. Note that Equation (5.35) reflects the rate of change in a heterogeneous environment. The first summation in the right hand side is the attrition rate due to the fires of blue units using directed fire where the index set of these blue units is SB_j^D. Similarly the second summation is the attrition rate resulting from area firing blue units with index set SB_j^A. The third summation defines the attrition rate due to the fires of blue units, included in set SB_j^H, having characteristics of Helmbold fire. (Here τ_{ij} is the portion of B_i that can effectively be used against R_j when initial force ratio B_{i0}/R_{j0} is too large where $0 < \tau_{ij} < 1$.) Moreover, δ_j is the continuous renewal rate and γ_j is the noncombat loss coefficient of R_j.

Another variation might be incorporation of time (or weapon range) dependence into attrition coefficients by adjusting them as follows.

$$d_{ij}(t) = d_{ij0} - v_{ij}t \tag{5.36}$$

$$b_{ij}(d_{ij}(t)) = \begin{cases} b_{ij0}(1 - d_{ij} / q_{ij})^{rdp}, & \text{if } 0 \le d_{ij} \le q_{ij} \\ 0, & \text{otherwise} \end{cases}$$

where d_{ij0} and v_{ij} are the initial distance and the closing velocity between B_i and R_j. q_{ij} is the weapon range of B_i against R_j and rdp is the range dependency parameter.

As seen in Equation (5.35) traditional Lanchester equations for heterogeneous forces assume that individual attrition processes are additive. However, we partially relax this assumption with the help of division and combination parameters as seen in the directed fire equation below.

$$\frac{dR_j}{dt} = -\sum_{i \in SB_j^D} \left(1 + \phi(i, j, l_j)\right)\left(1 - \lambda(i, k_i)\right) x_{ij} b_{ij} B_i \tag{5.37}$$

Finally, use of numerical approximation to solve Lanchester equations is illustrated in Equation (5.37).

$$R_j(t + \Delta t) = R_j(t) - \sum_{i \in SB_j^D} \left(1 + \phi(i, j, l_j)\right)\left(1 - \lambda(i, k_i)\right) x_{ij} b_{ij}(t + \Delta t) B_i(t) \Delta t \tag{5.38}$$

where Δt is a very small time step. Further details about solving LM models by numerical approximation will be given in the next section and in Chapter 6 through computations on example problems.

5.3 Integration of LM and DSM

In this section, we demonstrate the combined use of DSM and LM in cases where a small unit and a large unit may engage each other. When one unit has less than 20 combatants while its opponent has 20 or more, we have the problem of integrating DSM and LM. Figure 5.16 shows an instance of such a situation.

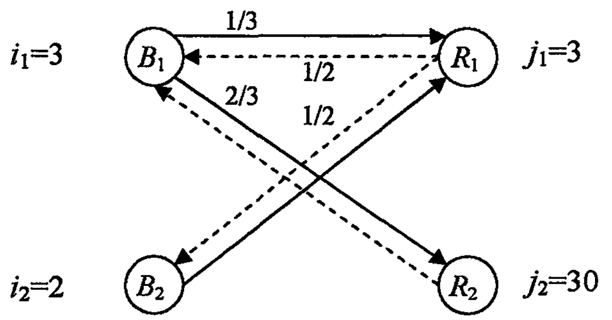


Figure 5.16 Combat between small and large units

Consider the engagement between B_1 and R_2 where two combatants of B_1 fire at R_2 but all three combatants (say tanks) of B_1 are subject to fire by all 30 combatants (say infantries) of R_2. Such an engagement is treated under following assumptions.

A1) R_2 divides its force equally among three combatants of B_1. Hence, initially 30/3=10 R_2 firers shoot at each B_1 firer.

A2) B_1 uses area fire against R_2, and R_2 uses directed fire against B_1.

A3) SSKP of B_1 >> SSKP of R_2.

The last two assumptions are reasonable since otherwise, with only two firers against 30, B_1 would hardly survive against R_2 for any length of time, and there would not exist a reasonable combat situation. As a result, force ratios such as 30/3 or 30/2 are meaningful in a real life combat, only with mixed fire types and substantially different SSKPs. Based on the assumptions stated above, attritions of B_1 and R_2 can be modeled as follows.

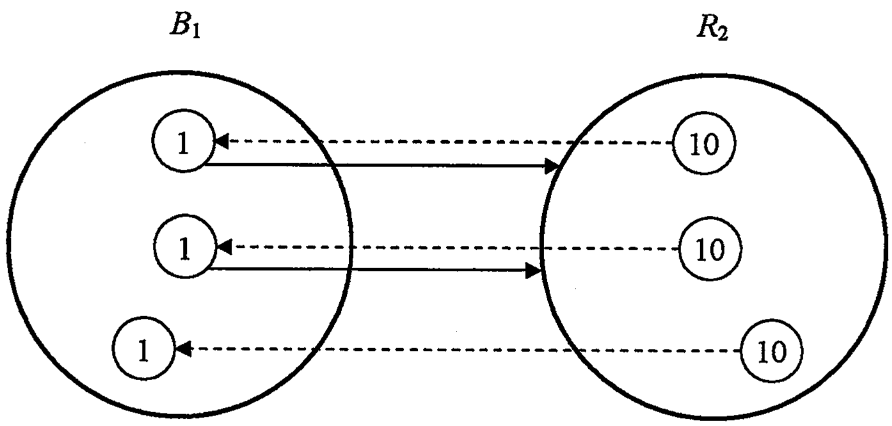


Figure 5.17 Engagement between B_1 and R_2

5.3.1 Attrition of a Small Unit

By assumption A1, R_2 allocates 10 combatants to each B_1 combatant as in Figure 5.17 and SSKP of R_2 combatants against B_1 is relatively small. Two combatants of B_1, on the other hand, area fire at R_2 as a whole.

Suppose the true SSKP of R_2 against B_1 is $p'_{k:R_2,B_1}$. Then, the SSKP value to be used in DSM for R_2 can be approximated by considering simultaneous fire of $x_{R_2,B_1} j_2 / i_1$ firers of R_2 at a single B_1 target as

$$p_{k:R_2,B_1} = 1-(1-p'_{k:R_2,B_1})^{x_{R_2,B_1} j_2 / i_1} \tag{5.39}$$

Let $p'_{k:R_2,B_1} = 0.001$ for the above example where $x_{R_2,B_1} = 1$, then the approximated SSKP of R_2 is found as $p_{k:R_2,B_1} = 1-(1-0.001)^{30/3} = 0.01$.

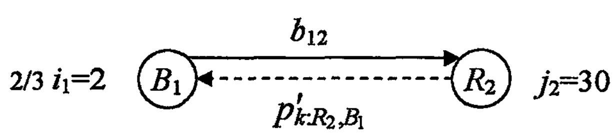
Since the attrition process of R_2 is modeled by LM, B_1 and R_2 need not interact in DSM. Hence, we can model the attrition process of B_1 as if B_1 is subject to (additional) noncombat loss with a probability of $q_{B_1} = p_{k:R_2,B_1}$. In the presence of an additional large red unit R_3 allocating (say 8) firers to a single B_1 target, this probability would be $q_{B_1} = 1-(1-p_{k:R_2,B_1})^{10}(1-p_{k:R_3,B_1})^{8}$.

Throughout the salvos, the number of firers allocated to a single B_1 target may not be integer according to LM results for R_2 or R_3. However, the above formula is still applicable with nonintegral group sizes.

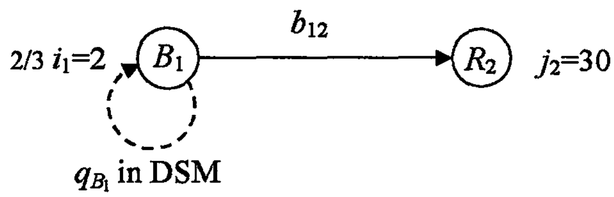
We can use this (additional) probability as in the noncombat loss process of DSM, and calculate the attrition of B_1. For the case where i_1=3 combatants of B_1 survive from fires of R_2 at the end of the first salvo is illustrated in Figure 5.18. Then using Equation 5.27 the probabilities of having l=0, 1, 2, or 3 B_1 combatants killed, $P_{30}(l)$, due to fires of 30 R_2 firers can be calculated. As an example, consider

$$Q^{B_1}((1,3,\bullet)\rightarrow(1,3,\bullet)) = P_{30}(0) = \binom{3}{0}(1-q_{B_1})^{3}(q_{B_1})^{0}$$

$$= (1)(1-0.01)^{3}(0.01)^{0} = 0.9703$$



(a) Before treatment (original)



(b) After treatment

Figure 5.18 Integration of LM and DSM

When similar calculations are performed for i_1=1, 2 and 3 B_1 casualties additional noncombat loss transition probabilities given Figure 5.19 are obtained.

In order to obtain the probability distribution of the number of B_1 survivors at the end of a salvo, we multiply the additional noncombat loss transition probabilities with the marginal probability distribution of B_1 obtained after engagement, combat loss (due to R_1) and noncombat loss processes; and sum the results for each number of survivors (Equation 5.28). Let the state probabilities for 3, 2, 1 and 0 B_1 combatants without additional noncombat loss be 0.85738, 0.13898, 0.00364, and 0.00001 respectively. Then the state probabilities for B_1 after additional noncombat loss are calculated. For instance,

$$Q_{B_1}(t, 2, \bullet) = P(t-1, 2, \bullet)\ Q^{B_1}(0 \text{ loss}) + P(t-1, 3, \bullet)\ Q^{B_1}(1 \text{ loss})$$

$$= (0.13898)(0.9801) + (0.85738)(0.029400) = 0.1614$$

5.3.2 Attrition of a Large Unit

As to the attrition of R_2, we use b_{12} which is the attrition coefficient of B_1 against R_2 where 2/3 of B_1 is allocated to R_2, and model the attrition of R_2 by LM. We can use two methods to compute the attrition of R_2.

	to $(1, i_1, \bullet)$				
from $(1, i_1+\Delta i_1, \bullet)$	i_1=3	i_1=2	i_1=1	i_1=0	Total
$i_1+\Delta i_1$=3	0.970300	0.029400	0.000297	0.000001	1.0000
$i_1+\Delta i_1$=2		0.980100	0.019800	0.000100	1.0000
$i_1+\Delta i_1$=1			0.990000	0.010000	1.0000
$i_1+\Delta i_1$=0				1.000000	1.0000

Figure 5.19 Noncombat loss transition probabilities for B_1

Method 1:

Using LM, attrition rate of R_2 is defined as $\frac{dj_2}{dt} = -b_{12}(x_{B_1,R_2}\bar{i}_1)j_2 = -b_{12}(\frac{2}{3}\bar{i}_1)j_2$

where $\bar{i}_1$ is the expected value of the number of B_1 combatants obtained from DSM. Then remaining force size of R_2 at the end of the current salvo can be found by numerical approximation as

$$j_2(t) = j_2(t-1) - b_{12}\ \tfrac{2}{3}\bar{i}_1(t-1)j_2(t-1)\Delta t.$$

Here $\bar{i}_1(t-1)$ is the expected number of B_1 combatants at the end of previous salvo. Details about this computation will be given in Appendix C.

Method 2:

Alternatively we can compute the attrition of R_2 as follows. Let $LM^A(x_{B_1,R_2}i_1, j_2, k\Delta t)$ be the attrition of R_2 during a salvo duration of length $k\Delta t$ when its j_2 combatants are subject to area fire of $x_{B_1,R_2}i_1$ firers of B_1. Here k is a constant selected depending on the time length, Δt, within which at most one kill may occur. Let d_s be the salvo duration. Then we perform $k = d_s/\Delta t$ LM runs for each salvo. Since B_1 has i_1=3 firers initially and 2/3 of B_1 is allocated to R_2, then for salvo t we make the following calculations to compute the number of R_2 kills.

$$\Delta_{j_2}^{i_1=3} = P(t-1, 3, \bullet, \bullet, j_2)\, P(x_{B_1,R_2}i_1=2)\, LM^A(2, j_2, k\Delta t)$$

$$= P(t-1, 3, \bullet, \bullet, j_2)\,(1)\, LM^A(2, j_2, k\Delta t)$$

$$\Delta_{j_2}^{i_1=2} = P(t-1, 2, \bullet, \bullet, j_2)$$

$$[P(x_{B_1,R_2} i_1{=}2)\, LM^A(2, j_2, k\Delta t) + P(x_{B_1,R_2} i_1{=}1)\, LM^A(1, j_2, k\Delta t)]$$

$$= P(t-1, 2, \bullet, \bullet, j_2)\, [(1/3)\, LM^A(2, j_2, k\Delta t) + (2/3)\, LM^A(1, j_2, k\Delta t)]$$

$$\Delta_{j_2}^{i_1=1} = P(t-1, 1, \bullet, \bullet, j_2)$$

$$[P(x_{B_1,R_2} i_1{=}1)\, LM^A(1, j_2, k\Delta t) + P(x_{B_1,R_2} i_1{=}0)\, LM^A(0, j_2, k\Delta t)]$$

$$= P(t-1, 1, \bullet, \bullet, j_2)\, [(2/3)\, LM^A(1, j_2, k\Delta t) + (1/3)\, LM^A(0, j_2, k\Delta t)]$$

where, for example, $\Delta_{j_2}^{i_1=3}$ is the number of R_2 kills when i_1=3. Therefore, the expected casualty of R_2 is computed as

$$\Delta j_2 = \Delta_{j_2}^{i_1=3} + \Delta_{j_2}^{i_1=2} + \Delta_{j_2}^{i_1=1}.$$

The comparison of these two approaches, based on the results for ten salvos, is given in Table 5.13. Since the differences between these two methods are very small, we prefer using first method which involves less computation.

Table 5.13 Comparison of two LM computation methods

Salvo	Expected Value of j_2		Difference	Percent difference
	Method 1	*Method 2*		
1	24.9876	24.9876	0.000000	0.000000
2	21.3995	21.4149	0.015376	0.071802
3	18.7587	18.7803	0.021600	0.115000
4	16.7623	16.7854	0.023048	0.137312
5	15.2092	15.2311	0.021973	0.144262
6	13.9656	13.9855	0.019861	0.142015
7	12.9431	12.9606	0.017524	0.135208
8	12.0824	12.0977	0.015317	0.126613
9	11.3436	11.3569	0.013363	0.117663
10	10.6990	10.7106	0.011676	0.109017

5.3.3 An Example

Numerical illustration is given in Appendix C for ten salvos of the instance given in Figure 5.16. We consider two cases regarding the incorporation of additional noncombat loss of B_1 due to fires of R_2 as summarized below. Let q_1 be the constant noncombat loss probability of B_1 (which is 0.05). Let q_2 be the additional noncombat loss probability of B_1 due to fires of R_2 where q_2 which varies throughout salvos.

Case-1. Let $q = 1-(1-q_1)(1-q_2)$ which is average noncombat loss probability.

- Compute transition probabilities using q for each salvo, each time updating noncombat loss probability q since q_1 changes throughout salvos.
- Run the noncombat loss process, after combat loss process, at each salvo.

Case-2. Compute transition probabilities of noncombat loss process using constant noncombat loss probability q_1 and store the values in advance (since they will not change throughout salvos). During salvo computations;

- First, compute transition probabilities due to additional noncombat loss for each salvo, updating variable noncombat loss probability q_2, and run additional noncombat loss process (LM combat) after fixed noncombat loss process.
- Then run the fixed noncombat loss process after additional noncombat loss (LM combat) process.

We use case-2 in illustrating the computations for the first two salvos in Appendix C where the overall computation processes and their sequence can be summarized as in Figure 5.20. Attrition of B_1 due to fires of R_2 is treated as an

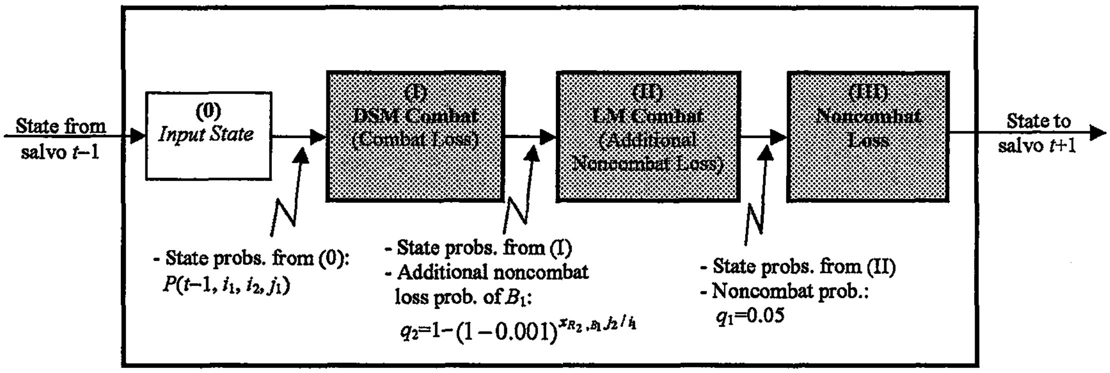


Figure 5.20 Computations for salvo t for case-2

additional noncombat loss process (LM combat) and computed right after combat loss process (DSM combat) for each salvo followed by noncombat loss process based on constant noncombat loss rate $q_1= 0.05$.

We make calculations using both cases and obtained the results given in Tables 5.14 and 5.15 for the first ten salvos. As seen in these tables, case-1 and case-2 give the same results. Regarding the additional noncombat loss effect due to LM combat, case-1 is easier to implement when compared to case-2. However, we prefer using the approach described in case-2 since it better reflects the sequential relations among processes.

Table 5.14 Summary results for first ten salvos (case 1)

Salvo	B_1		B_2		R_1		R_2
	Expected	**Variance**	**Expected**	**Variance**	**Expected**	**Variance**	**Expected**
0	3.000000	0.000000	2.000000	0.000000	3.000000	0.000000	30.000000
1	2.493868	0.369506	1.938060	0.059503	2.534949	0.371658	24.987575
2	2.067616	0.639484	1.885532	0.109781	2.145930	0.619215	21.399524
3	1.715139	0.814142	1.840991	0.152143	1.823198	0.767865	18.758723
4	1.433326	0.894305	1.803129	0.187920	1.558434	0.838446	16.762345
5	1.212678	0.907277	1.770770	0.218329	1.342300	0.854278	15.209160
6	1.041063	0.881289	1.742916	0.244398	1.165573	0.835389	13.965599
7	0.906968	0.836008	1.718748	0.266948	1.020105	0.796414	12.943069
8	0.800868	0.782970	1.697615	0.286624	0.899192	0.747113	12.082409
9	0.715430	0.728300	1.678999	0.303925	0.797551	0.693630	11.343574
10	0.645231	0.675015	1.662495	0.319240	0.711116	0.639660	10.698956

Table 5.15 Summary results for first ten salvos (case 2)

Salvo	B_1		B_2		R_1		R_2
	Expected	**Variance**	**Expected**	**Variance**	**Expected**	**Variance**	**Expected**
0	3.000000	0.000000	2.000000	0.000000	3.000000	0.000000	30.000000
1	2.493868	0.369506	1.938060	0.059503	2.534949	0.371658	24.987575
2	2.067616	0.639484	1.885532	0.109781	2.145930	0.619215	21.399524
3	1.715139	0.814142	1.840991	0.152143	1.823198	0.767865	18.758723
4	1.433326	0.894305	1.803129	0.187920	1.558434	0.838446	16.762345
5	1.212678	0.907277	1.770770	0.218329	1.342300	0.854278	15.209160
6	1.041063	0.881289	1.742916	0.244398	1.165573	0.835389	13.965599
7	0.906968	0.836008	1.718748	0.266948	1.020105	0.796414	12.943069
8	0.800868	0.782970	1.697615	0.286624	0.899192	0.747113	12.082409
9	0.715430	0.728300	1.678999	0.303925	0.797551	0.693630	11.343574
10	0.645231	0.675015	1.662495	0.319240	0.711116	0.639660	10.698956

CHAPTER 6

Illustration of Modeling on a Scenario

A tactical level land combat proceeds in stages enabling us to decompose the problem into mini-battles as explained in Chapter 3. This way we can model the combat in a semi-dynamic manner such that we keep the model parameters constant within a stage whereas we can change them as we move from one stage to the next.

The model base of our proposed decision support system (DSS), which models a mini-battle a combat stage, consists of three interacting models. These are: an integer programming model (ALLM) for allocating heterogeneous blue forces to heterogeneous red forces with minimum total weapon effectiveness, an attrition simulation model (ASM) for predicting force levels throughout the battle stage and determining whether or not battle targets have been reached, and a weapon effectiveness update model (WEI) for finding the "value" of blue units at the end of the current stage and modifying them for subsequent stages. The overall DSS framework is summarized in Figure 6.1.

In modeling a mini-battle, ALLM and ASM are solved iteratively until ASM indicates that the preset attrition goals for red units can be reached within the target combat duration. This iterative solution process will be described in Section 6.1. Once these targets are achieved, WEI finds how effective different blue units have

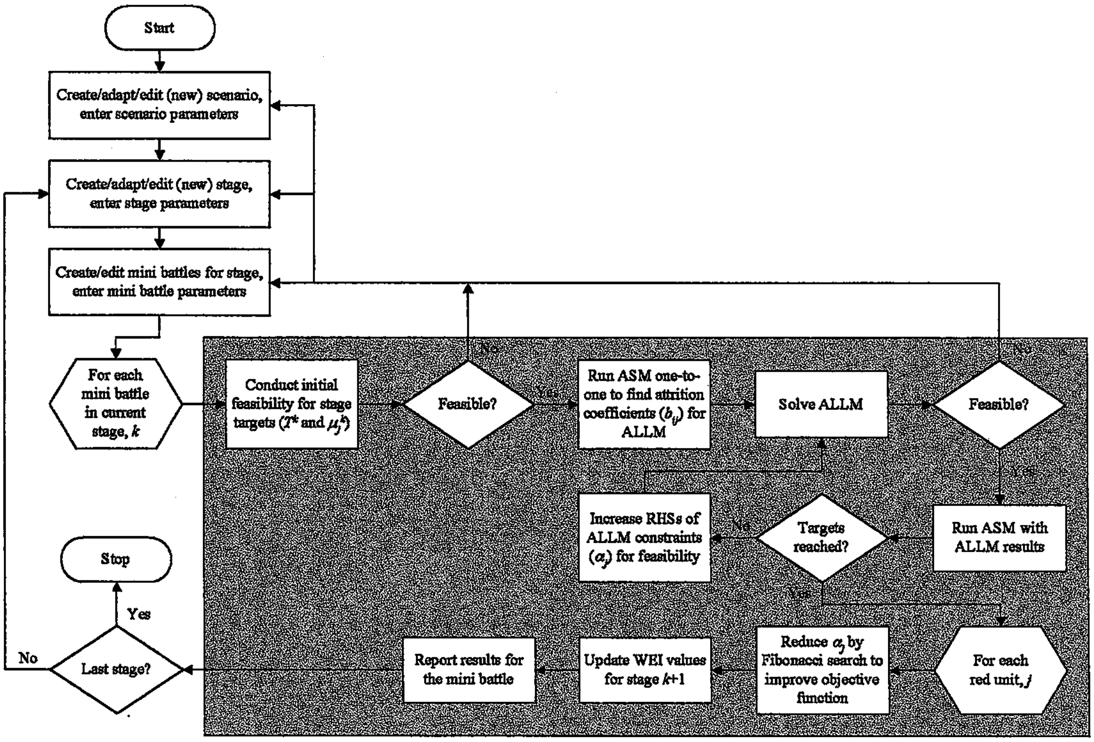


Figure 6.1 Overall DSS framework

been in reaching the targets, irrespective of their static effectiveness values, and updates them for the next stage. The WEI model is described in Section 6.2. In Section 6.3, we illustrate the overall modeling approach on a scenario.

6.1 ALLM-ASM Interaction

When ALLM is solved for the first time for allocating blue units to red units in a mini battle, the solution is fed as input to ASM. If simulation results indicate that target loss fractions (μ_j) for some red units are not achieved within target battle duration (T), right hand sides of respective attrition goal constraints (α_j) are increased and ALLM is solved once again. In deciding on the increase for an α_j, we make use of the attrition ratio, AR_j, computed using the ASM results as

$$AR_j = \frac{\mu_j R_{j0}}{R_{j0} - R_j(T)}$$

where $R_j(T)$ is the force level of red unit j at time T whose initial force level is R_{j0}.

$AR_j > 1$ when μ_j is not reached for red unit j before T. Then the new right hand side value for the respective attrition goal constraint is found as $AR_j \alpha_j$. ALLM-ASM interaction iterates in this fashion until target loss fractions are satisfied for all red units. This interaction is shown with double arrows in Figure 3.2.

Once a feasible ALLM solution validated by ASM is found, the interaction continues with the purpose of minimizing the objective function of ALLM. This time we try to reduce α_j values without violating feasibility. α_j values can be gradually reduced using line search techniques. Özdemirel and Kandiller (2001) proposed using Fibonacci search where they determined the search range based on the survival ratio SR_j as defined below.

$$SR_j = \frac{R_j(T)}{R_{j0} - \mu_j R_{j0}}$$

When attrition goal for a red unit j is satisfied then $SR_j < 1$ and Fibonacci search can be carried out in the range $(SR_j \alpha_j, \alpha_j)$ for each such red unit. Among the feasible solutions found in this manner, the one having the lowest objective function value is chosen as the final ALLM solution for the mini battle. We propose an alternative approach to reduce α_j values. This approach will be illustrated in Section 6.3.

Another point concerning the relationship between ALLM and ASM is the potential difference in interpretation of a military unit. In both ALLM and ASM, we assume that forces on both sides are heterogeneous, i.e. different types of blue and red units can take place in a mini battle. Furthermore, in ALLM, a blue unit can optionally be taken as a heterogeneous task force in itself, which may consist of different types of homogeneous subunits, such as infantry, anti-tank and rocket launcher subunits. ALLM allocates task forces to keep the unit integrity intact. In ASM, however, homogeneous subunits of a task force can be simulated individually to accommodate different fire types, attrition coefficients and simulation types (deterministic or stochastic). This also helps to predict the quantities of weapon systems (a tank or an infantry soldier) used to win a mini battle. The unit-subunit differentiation is not made for red units.

6.2 Weapon Effectiveness Index Update Model (WEI)

WEI provides information needed for better planning the use of blue units across the stages by incorporating the changes in battle environment into overall modeling. Static weapon effectiveness values of blue units can be updated through the stages by increasing the cost coefficient of a certain blue unit in all subsequent stages following the one in which that unit is of vital importance to win the battle. Therefore, the blue unit will not be used unless absolutely necessary, and hence its potential will be used as much as possible in the stage of vital importance. In this section, we present the weapon effectiveness update process developed by Özdemirel and Kandiller (2001).

WEI uses two pieces of update information supplied by the final solutions of ALLM and ASM. The procedure of deriving the update information and decoupling is described next.

WEI Update Based on ALLM

The dual value of a red attrition goal constraint indicates the necessary increase in total weapon effectiveness of blue units to overcome one unit of increase in the goal threshold. Since ALLM is an integer programming model, the following linear programming relaxation based on the final ALLM solution is used to determine the dual values for red units.

$$LP: \quad \min \quad \sum_{i=1}^{I}\sum_{j=1}^{J} WEI_i x''_{ij}$$

$$\text{s.t.} \quad \sum_{i \in SB_j} b''_{ij} x''_{ij} \geq \alpha_j \qquad j=1, \ldots, J$$

$$l_{ij} \leq x''_{ij} \leq u_{ij} \qquad i=1, \ldots, I,\ j=1, \ldots, J$$

where SB_j is the index set of blue units assigned to red unit j in the final ALLM solution, and l_{ij} and u_{ij} are the lower and upper bounds on x''_{ij} that keeps the first set of constraints binding so that they have dual values d_j. Note that here x''_{ij} is the continuous relaxation of x_{ij} used in the real formulation of ALLM, and b''_{ij} values are determined by using final values of the technical binary variables to reflect all division and combination effects. To solve this formulation, we can define bound constraints such as $0 \leq x''_{ij} \leq 1$. Fortunately, the formulation can be decomposed into J independent subproblems as

$$LP(j): \min \quad \sum_{i=1}^{I} WEI_i x''_{ij}$$

$$\text{s.t.} \quad \sum_{i \in SB_j} b''_{ij} x''_{ij} \geq \alpha_j$$

$$0 \leq x''_{ij} \leq 1 \qquad i=1, \ldots, I$$

such that their solutions dictate the solution of the main problem. Problem *LP(j)*, which corresponds to red unit *j*, is the continuous relaxation of the binary knapsack problem having a closed form solution. If we sort WEI_i / b''_{ij} ratios in ascending order, we have

$$\frac{WEI_{(1)}}{b''_{(1)j}} \le \frac{WEI_{(2)}}{b''_{(2)j}} \le \cdots \le \frac{WEI_{(k)}}{b''_{(k)j}} \le \cdots \le \frac{WEI_{(|SB_j|)}}{b''_{(|SB_j|)j}}$$

There exists an index *k* such that the conditions

$$b''_{(1)j} + b''_{(2)j} + \cdots + b''_{(k-1)j} < \alpha_j \text{ and } b''_{(1)j} + b''_{(2)j} + \cdots + b''_{(k-1)j} + b''_{(k)j} \ge \alpha_j$$

hold. The closed form solution of *LP(j)* is then

$$x''_{(1)j} = x''_{(2)j} = \cdots = x''_{(k-1)j} = 1,\ x''_{(k+1)j} = x''_{(k+2)j} = \cdots = x''_{(|S_j|)j} = 0 \text{, and}$$

$$x''_{(k)j} = \frac{\alpha_j - (b''_{(1)j} + b''_{(2)j} + \cdots + b''_{(k-1)j})}{b''_{(k)j}}$$

Hence, the dual values for the subproblems are determined as

$$d_j = \frac{WEI_{(k)}}{b''_{(k)j}} \qquad j=1,\ldots,J$$

The dual value for red unit *j* should be distributed among the associated blue units. The system of equations given below is solved for $X_i(j)$ to determine individual shares of blue units.

$$\sum_{i \in SB_j} b''_{ij} X_i(j) = d_j$$

$$X_i(j) / X_k(j) = WEI_i / WEI_k \qquad \forall\, i,k \in SB_j$$

The marginal weapon effectiveness value for blue unit *i* is then found as the sum of $X_i(j)$ values for all red units, i.e.

$$Z_i = \sum_{j=1}^{J} X_i(j) \qquad i=1,\ldots, I$$

Suppose the weapon effectiveness value of blue unit i used at the beginning of stage k is denoted by WEI_i^k. This value is updated for the next stage as

$$AWEI_i^{k+1} = (1+Z_i)WEI_i^k \qquad i=1,\ldots, I$$

WEI Update Based on ASM

Let us consider the following allocations that are determined by the final ALLM solution. Blue unit $i1$ is allocated to red unit $j1$ as a whole whereas another blue unit $i2$ is allocated to red units $i1$ and $i2$ with fractions 1/3 and 2/3, respectively. After ASM is run for this example, suppose we obtain final force levels of blue units $i1$ and $i2$ as $B_{i1}(T)$ and $B_{i2}(T)$, where T is the target battle duration. Let us analyze what will happen if we increase the initial force level of the first blue unit, $B_{i1,0}$, by a factor of 1/3 by adding one subunit. In this case, red unit $j1$ is going to be neutralized earlier and hence blue units $i1$ and $i2$ will be subject to less attrition. Since blue unit $i2$ saves some power, red unit $j2$ is also going to be neutralized earlier. Let $B'_{i1}(T)$ and $B'_{i2}(T)$ denote the final force levels after the initial force level of the first blue unit is increased. This yields the gains $g_{i1}(i1)=B'_{i1}(T)-B_{i1}(T)$ and $g_{i2}(i1)=B'_{i2}(T)-B_{i2}(T)$ in the two blue units. These gains can be weighted by unit weapon effectiveness values to determine the total gain when the force level of the first blue unit is increased, i.e.

$$TG_{i1} = \frac{WEI_{i1}}{B_{i1,0}} g_{i1}(i1) + \frac{WEI_{i2}}{B_{i2,0}} g_{i2}(i1)$$

The net total gain is then found by subtracting the value of the additional force,

$$NTG_{i1} = TG_{i1} - \frac{WEI_{i1}}{B_{i1,0}} \times \frac{1}{3} B_{i1,0}$$

Using this procedure, the new weapon effectiveness value of blue unit i at the beginning of stage k+1 is determined as

$$LWEI_i^{k+1} = WEI_i^k + NTG_i \qquad i=1,\ldots, I$$

Decoupling

The $AWEI_i^{k+1}$ and $LWEI_i^{k+1}$ values above update the actual WEI_i^k as if blue unit i had no casualties in stage k, and without considering the external changes. Let $SWEI_i^{k+1}$ denote the *standard* weapon effectiveness that takes into account stage specific uncontrollable factors such as terrain and weather conditions. $SWEI_i^{k+1}$ also considers the casualties in blue unit i as of the end of stage k as well as renewal and replenishments at the beginning of stage k+1. These values can be found by using one of the available systems such as NATO's TASCFORM (Regan and Wogt, 1988) or OLI (Dupuy, 1979).

The first step in decoupling is to calculate the weighted average of the values supplied by ALLM and ASM, where ω is the weight reflecting aspiration level of the decision maker in evaluating relative importance of the update information.

$$ALWEI_i^{k+1} = \omega\, AWEI_i^{k+1} + (1-\omega) LWEI_i^{k+1} \qquad 0 \le \omega \le 1$$

The second step is to find an intermediate estimate, which is obtained through exponential smoothing where θ is the smoothing constant chosen by the decision maker.

The final value to be used for the next stage is then determined as

$$WEI_i^{k+1} = \frac{SWEI_i^{k+1}}{SWEI_i^k} \times PWEI_i^{k+1}$$

Here, by using the ratio of standard weapon effectiveness values, we make an adjustment for dynamic uncontrollable factors and force level changes as we move from stage k to stage k+1.

6.2 Illustration of Modeling Approach on a Scenario

In this section we illustrate the overall modeling approach for a mini-battle on a scenario involving units with various sizes and division possibilities. Consider the allocation graph given in Figure 6.2 where five blue units and five red units engage each other. Initial sizes of units ($B_{10},\ldots,B_{50}$; $R_{10},\ldots,R_{50}$), weapon effectiveness indices ($WEI_1,\ldots,WEI_5$) of blue units, target loss fractions ($\mu_1,\ldots,\mu_5$) and attrition goals ($\alpha_1,\ldots,\alpha_5$) for red units are shown on the graph. Suppose that target duration is T=0.020 (hours) for the stage. For example, μ_3 =0.40 means that 40% of 20 R_3 combatants (i.e. 8 combatants) should be destroyed during current combat stage of duration 0.020 hours. Here B_1 uses area fire against R_2, and B_2 uses area fire against R_1, whereas all other fire types are directed. We assume that red units fire at the blue units that are allocated to them. If two or more blue units are combined against a red unit, red will react to them by dividing its force among the blue units in proportion with the blues' allocation fractions.

Coefficients of attrition goal constraints in ALLM (b'_{ij}), attrition rate coefficients of red units against blue units (r_{ji}), that are necessary for LM

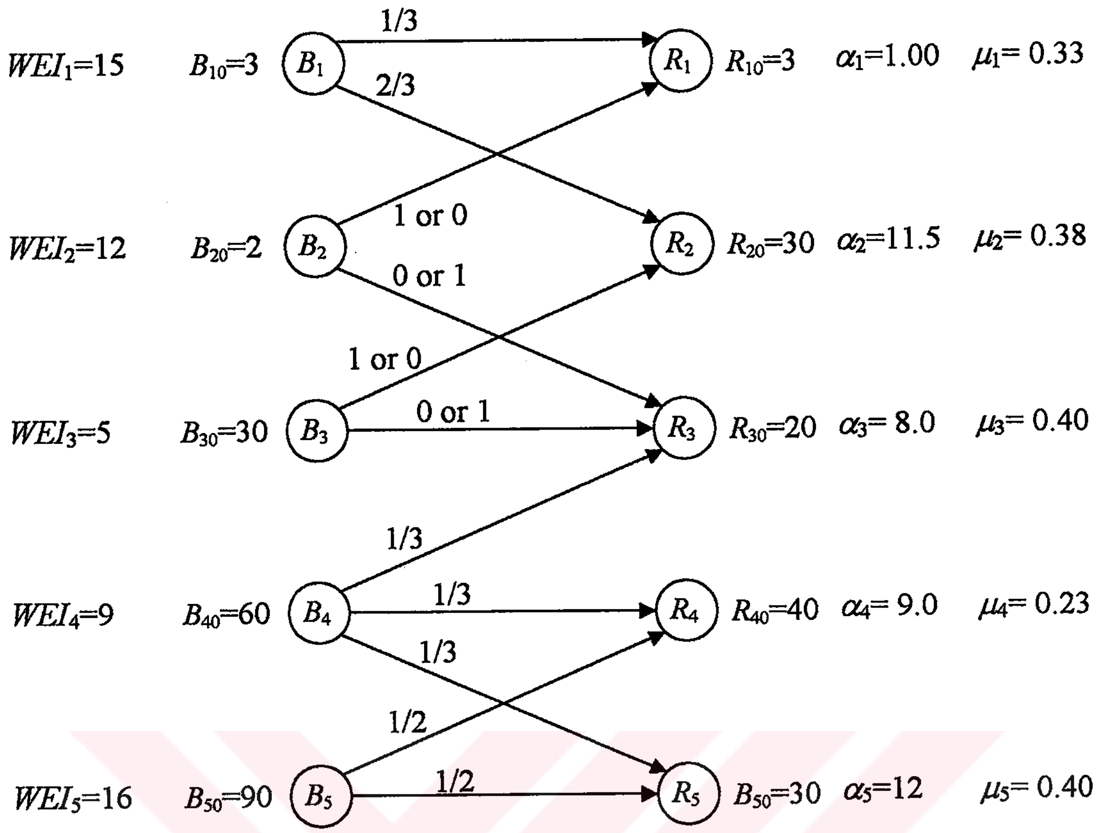


Figure 6.2 Allocation Graph

computations, and single shot kill probabilities (p_{ij}, q_{ji}) required for DSM computations are given in Figure 6.3. If any blue (red) unit is divided between multiple opposing units, corresponding b_{ij} (r_{ji}) values are reduced by λ_1=0.03 if the division is two-way, and by λ_2=0.05 if the division is three-way. In case of force combination, corresponding b_{ij} (r_{ji}), values are increased by ϕ_1=0.08 when two units are combined, and by ϕ_2=0.09 when three or more units are combined.

$b'_{ij} =$

(i) \ (j)	1	2	3	4	5
1	1.5	18.0	-	-	-
2	0.6	-	4.0	-	-
3	-	9.0	9.0	-	-
4	-	-	14.0	12.0	13.0
5	-	-	-	25.0	15.0

(a) ALLM input

$r_{ji} =$

(j) \ (i)	1	2	3	4	5
1	1.10	1.00	-	-	-
2	24.0	-	12.0	-	-
3	-	6.00	14.0	10.0	-
4	-	-	-	11.5	16.0
5	-	-	-	44.0	55.0

(b) LM input

$p_{ij} =$

(i) \ (j)	1	2	3	4	5
1	0.20	0.30	-	-	-
2	0.10	-	0.25	-	-
3	-	0.40	0.35	-	-
4	-	-	0.50	0.25	0.60
5	-	-	-	0.65	0.70

$q_{ji} =$

(j) \ (i)	1	2	3	4	5
1	0.300	0.050	-	-	-
2	0.001	-	0.005	-	-
3	-	0.002	0.40	0.30	-
4	-	-	-	0.25	0.35
5	-	-	-	0.45	0.30

(c) DSM input

Figure 6.3 Input data for the example problem

First we should allocate blue units to red units in an optimal way by solving ALLM using the above data. First solution of ALLM yields a total WEI value of 46 and we obtain the allocations given in Figure 6.4. Now we should check whether target loss fraction (μ_j) is achieved for each red unit. If we examine the Figure 6.4, we see that the mini battle is decomposed into three parts. The first part involves the engagements among B_1, B_2, R_1 and R_2 and attrition computations for these units require integrated use of DSM and LM as explained in Section 5.3. Due to large force sizes, attritions in the other two parts can be calculated using only LM. The attritions for these two parts are calculated by numerically solving the differential equations for directed fire. Here we set time step $\Delta t = 0.01*\min(\Delta t_3, \Delta t_4, \Delta t_5)$ which is sufficiently small for the accuracy of computations, where Δt_j is the time length

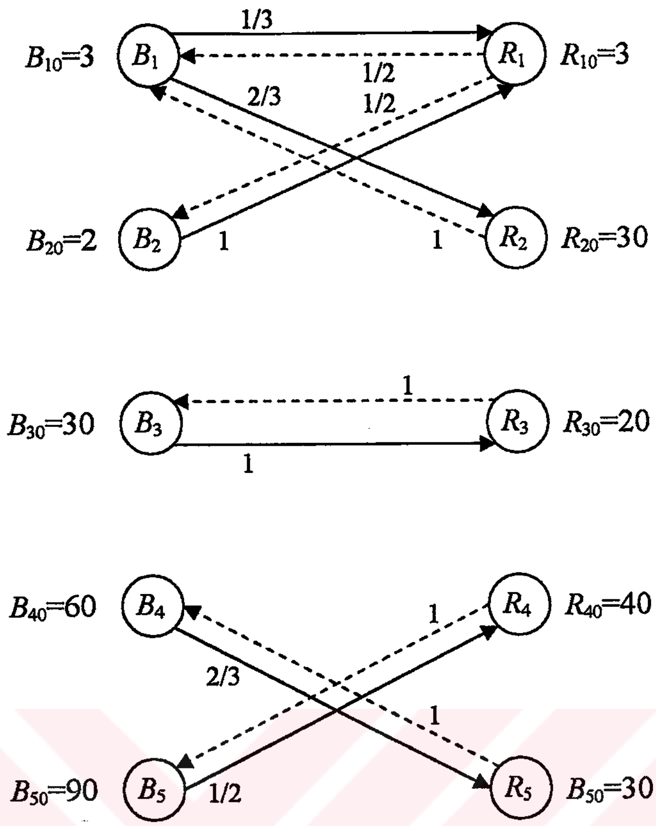


Figure 6.4 ALLM solution (first iteration)

within which at most one R_j kill may occur. For example, attrition equation for R_3 is

$$j_3(t) = j_3(t-\Delta t_3) - b_{33} * i_3(t) * \Delta t_3$$

Since the attrition of R_3 during Δt_3 should not be greater than one,

$$b_{33} * i_3(t) * \Delta t_3 \leq 1 \text{ yielding } \Delta t_3 \leq 1 / (b_{33} * i_3(t))$$

Hence, we set $\Delta t_3 = 1 / (b_{33} * i_3(t))$ where $i_3(t)$ is taken as the initial number of B_3 combatants. After setting Δt_4 and Δt_5 similarly and obtaining Δt as defined above, we perform the LM computations. The results for the mini battle for target duration (T=0.020) are given in Table 6.1.

Table 6.1 The results of first iteration

$\Sigma WEI = 46$	R_1	R_2	R_3	R_4	R_5
B_1	1/3	2/3	-	-	-
B_2	1	-	-	-	-
B_3	-	-	1	-	-
B_4	-	-	-	-	2/3
B_5	-	-	-	1/2	-
initial force level	3	30	20	40	30
target loss fraction (μ_j)	0.33	0.38	0.40	0.23	0.40
attrition goal (α_j)	1.00	11.5	8.0	9.0	12.0
force level at T=0.020	0.71	10.70	15.06	20.13	22.70
loss fraction at T=0.020	0.76	0.64	0.25	0.49	0.24
attrition ratio (AR_j)	0.98	0.84	1.62	0.46	1.64
action			$\alpha_3\uparrow$		$\alpha_5\uparrow$
updated (α_j) values	1.00	11.5	13.0	9.0	19.7

Since loss fractions for R_3 and R_5, which are 0.25 and 0.24, are below target levels of 0.40 for each, we need to increase the right hand sides (α_j values) of attrition goal constraints for R_3 and R_5, and then solve ALLM again. As defined in Section 6.1, in increasing an α_j value we use the attrition ratio, AR_j, computed according to ASM results as

$$AR_j = \mu_j R_{j0} \,/\, (R_{j0} - R_j(T))$$

where R_{j0} is the initial force size of red unit j. Recall that, if AR_j>1 then μ_j is not achieved for red unit j before T. In this case, new right hand side value for respective attrition goal constraint (updated α_j value) is computed as $AR_j\alpha_j$.

Therefore, the values of α_3 and α_5 can now be updated for the second iteration as

$$AR_3\alpha_3 = (1.62)(8) = 13.0$$

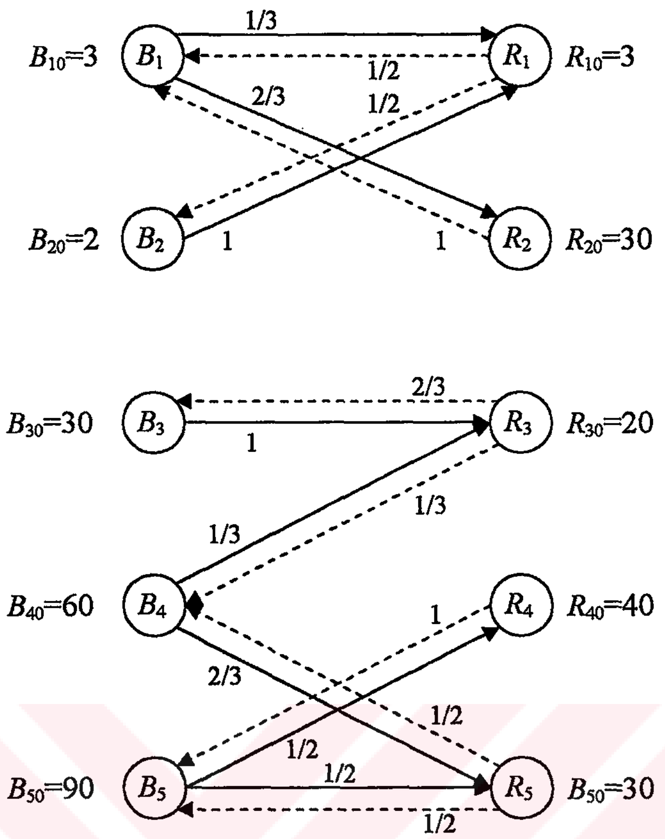


Figure 6.5 ALLM solution (second iteration)

$AR_5\alpha_5 = (1.64)(12) = 19.7.$

When we solve ALLM again replacing previous values of α_3 and α_5 (8.0 and 12.0) with updated values (13.0 and 19.7) the new optimal ALLM solution yields an increased total WEI value of 57, where new allocations resulting from second ALLM solution are given in Figure 6.5. The difference of second ALLM solution from the first one is that B_4 allocates 1/3 of its combatants (which were not used in the first iteration) to R_3 in combination with all B_3 combatants. Also, B_5 allocates 1/2 of its force to R_5 in combination with 2/3 of B_4 combatants. That is, new solution has increased the allocations to R_3 and R_5 as expected.

Table 6.2 The results of second iteration

$\Sigma WEI = 57$	R_1	R_2	R_3	R_4	R_5
B_1	1/3	2/3	-	-	-
B_2	1	-	-	-	-
B_3	-	-	1	-	-
B_4	-	-	1/3	-	2/3
B_5	-	-	-	1/2	1/2
initial force level	3	30	20	40	30
target loss fraction (μ_j)	0.33	0.38	0.40	0.23	0.40
attrition goal (α_j)	1.00	11.5	13.0	9.0	19.7
force level at T=0.020	0.71	10.7	9.14	21.1	7.9
loss fraction at T=0.020	0.76	0.64	0.54	0.47	0.74
attrition ratio (AR_j)	0.98	0.84	0.74	0.49	0.54
survival ratio (SR_j)	0.35	0.57	0.76	0.68	0.44
action					$\alpha_5 \downarrow$
updated (α_j) values	1.00	11.5	13.0	9.0	12.7

This time, the mini-battle is decomposed in two parts, and we need to run LM for the second part. The results for all units at target duration T=0.020 are given in Table 6.2.

Since target loss fractions for all red units are achieved ALLM solution is validated by LM and DSM results. Now we can investigate whether it is possible to minimize the total WEI value of ALLM by making some reductions in α_j values. Survival ratios of red units at T that can be used in Fibonacci search (explained in Section 6.1) are computed as follows.

$$SR_3 = R_3(T) / (R_{30} - \mu_3 R_{30}) = 9.14 / (20 - 0.50*20) = 0.76$$

$$SR_4 = R_4(T) / (R_{40} - \mu_4 R_{40}) = 21.1 / (40 - 0.23*40) = 0.68$$

$$SR_5 = R_5(T) / (R_{50} - \mu_5 R_{50}) = 7.9 / (30 - 0.40*30) = 0.44$$

Hence ranges for Fibonacci search are (9.88, 13.0), (6.12, 9.0) and (8.67, 19.7) for R_3, R_4 and R_5 respectively. Since SR_5 is the minimum we start the search with α_5

since this may bring a relatively larger gain in the objective function value. ALLM has to be solved for each new α_j value and ASM has to be run to check feasibility of the solution.

Alternatively, we can reduce α_j values by checking whether or not some portions of blue units can be saved. The process is as follows. For each blue unit currently allocated to red unit j, compute α_j reductions (starting from minimum reduction) such that each time some portion of a blue unit can be saved without violating feasibility. Then, we solve ALLM for each reduction level to obtain allocations and then run ASM. If target loss fractions for all red units are achieved

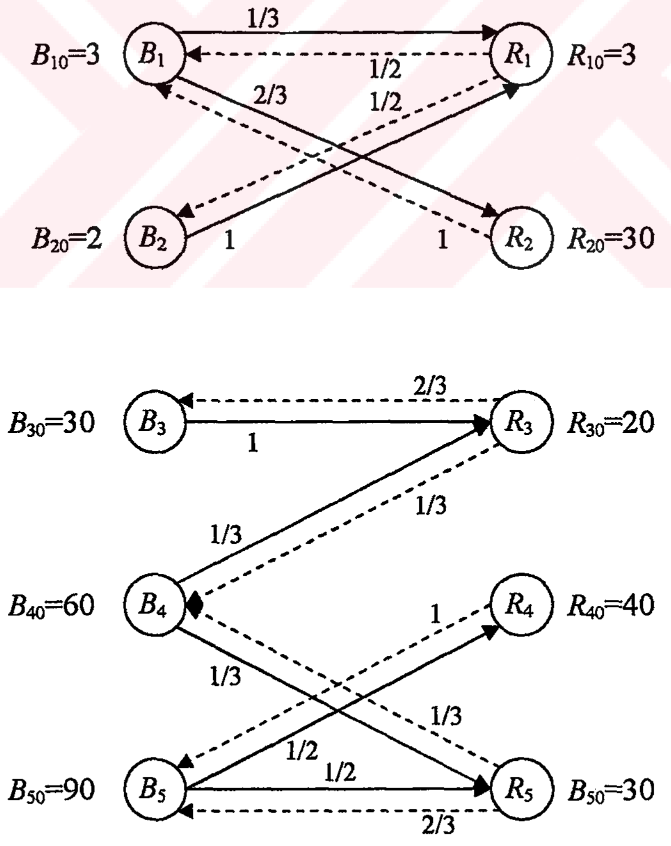


Figure 6.6 ALLM solution (third iteration)

then continue reducing the α_j value, otherwise skip the current red unit and select the next red unit having the minimum survival ratio among remaining ones.

In our example, starting from R_5 we first reduce α_5 value by

$$\text{Min}\ [(1/3)18.0,\ (2/3)18.0,\ (1/2)24.0] = \text{Min}\ [6.0,\ 12.0,\ 12.0] = 6.0$$

yielding $\alpha'_5 = 19.7 - 6.0 = 12.7$. When solving ALLM with $\alpha'_5 = 12.7$ we see that ALLM problem is still feasible and obtain the optimal solution given in Figure 6.6 where 1/3 of B_4 is saved and total WEI value is reduced to 54.

As 2/3 of B_4 has been allocated to R_5 at the end of the first iteration, and 1/3 of B_4 is now saved from R_5, we compute the attritions of units using ASM as in Table 6.3.

Table 6.3 The Results of Third Iteration

$\Sigma WEI = 54$	R_1	R_2	R_3	R_4	R_5
B_1	1/3	2/3	-	-	-
B_2	1	-	-	-	-
B_3	-	-	1	-	-
B_4	-	-	1/3	-	1/3
B_5	-	-	-	1/2	1/2
initial force level	3	30	20	40	30
target loss fraction (μ_j)	0.33	0.38	0.40	0.23	0.40
attrition goal (α_j)	1.00	11.5	13.0	9.0	12.7
force level at T=0.020	0.71	10.70	9.0	21.8	13.14
loss fraction at T=0.020	0.76	0.64	0.55	0.46	0.56
attrition ratio (AR_j)	0.98	0.84	0.73	0.51	0.72
survival ratio (SR_j)	0.35	0.57	0.75	0.71	0.73
action					
updated (α_j) values					

Since we have a feasible ALLM solution validated by ASM results we replace α_5 by $\alpha'_5 = 12.7$. Now we can reduce α_5 by

Min [(1/3)18.0, (1/2)24.0] = Min [6.0, 12.0] = 6.0,

which gives α'_5 = 12.7 –6.0 = 6.7. However we observe that ALLM solution is not validated for α'_5 = 6.7 by ASM since all target loss fractions are not achieved (R_5 is found to have 22.17 survivors yielding AR_5 = 1.53 >1). Hence we skip R_5 (leaving α_5 as 12.7) and select R_3 which has the next minimum survival ratio (0.71), reducing α_3 by

Min [(1)9.0, (1/3)14.0] = Min [9.0, 4.67] = 4.67.

Hence using α'_3 = 13.0 – 4.67 = 8.33 we solve ALLM again and observe that ASM computations do not validate ALLM solution (R_3 has 15.06 survivors yielding AR_3 = 1.62 >1). Then we skip R_3 (leaving α_3 as 13.0) and select another red unit having the minimum survival ratio. However, we observe that no more improvement is possible since, ALLM solution is either infeasible or not validated by ASM results.

To sum up, the allocations given in Figure 6.6 are the optimal allocations obtained from ALLM where specified attrition goals (α_1 , …, α_5) are achieved for the current mini-battle and the total values (weapon effectiveness values) of units used in this stage is found as 54. Under these allocations the attrition of red units at the end of the stage (with duration T=0.020) will be such that each one loses some fraction of its force level, where these loss fractions are greater than or equal to defined target levels (μ_1 , …, μ_5).

Now we illustrate how to update WEI_5 which is the weapon effectiveness index of B_5 where B_5 is allocated to R_4 and R_5. First, we update WEI_5 based on ALLM. According to the final solution of ALLM, B_5 is allocated to R_4 and R_5 with fractions (1/2,1/2) whereas 1/3 of B_4 is also allocated to R_5. Considering R_5, we compute $b''_{(i)5}$

values taking into account force combination and division as b''_{45} = 13(1–0.03)(1+0.08) = 13.619 and b''_{55} = 15(1–0.03)(1+0.08) = 15.714.

Recall that, in Section 6.2, we define x''_{ij} is the continuous relaxation of x_{ij} used in the real ALLM where we define bound constraints $0 \le x''_{ij} \le 1$. Since, WEI_4/b''_{45} = 9/13.619 = 0.661 and WEI_5/b''_{55} = 16/15.714 = 1.018 then k=1 indicating B_4. Further, since b''_{45} =13.619 > α_5 = 12, we have

$$x''_{55} = 0$$

whereas

$$x''_{45} = \alpha_5 / b''_{45} = 12 / 13.619 = 0.881.$$

Now the dual value of the constraint corresponding to attrition goal of R_5 is calculated as

$$d_5 = WEI_4/b''_{45} = 9 / 13.619 = 0.661.$$

Next, we solve the following system of equations for $X_i(j)$ to determine individual shares of blue units.

$$b''_{45}X_4(5) + b''_{55}X_5(5) = d_5 \rightarrow 13.619\,X_4(5) + 15.714\,X_5(5) = 0.661$$

$$X_5(5)/X_4(5) = WEI_5/WEI_4 = 16/9 \rightarrow X_5(5) = 1.778\,X_4(5)$$

and obtain $X_4(5)$ = 0.0159, and $X_5(5)$ = 0.0283.

When similar computations are made for R_5 we obtain $X_5(4)$ = 0.0233. Then the marginal weapon effectiveness value for B_5 is found as the sum of $X_5(j)$ values for j=4,5 as follows.

$$Z_5 = X_5(4) + X_5(5) = 0.0233 + 0.0283 = 0.0516$$

Hence weapon effectiveness index of B_5 is updated for second stages as

$$AWEI_5^2 = (1+Z_5)WEI_5^1 = (1+0.0516)16 = 16.83.$$

Now, we update WEI_5 based on ASM again considering the allocations by the final ALLM solution. After ASM is run we obtain final force levels of B_3, B_4 and B_5 as 27.39, 52.49 and 63.25, respectively. Next, we us analyze what will happen if we increase the initial force level of the B_5, $B_{i5,0}$, by a factor of 1/2 by adding one subunit. When we increase $B_{i5,0}$, from 90 to 135, ASM give the following force levels at the end of T=0.020.

$$B'_{i3}(T) = 27.39,\ B'_{i4}(T) = 53.60, \text{ and } B'_{i5}(T) = 112.93.$$

This yields the gains,

$$g_{i3}(i5)=B'_{i3}(T)-B_{i3}(T) = 27.39 - 27.39 = 0,$$

$$g_{i4}(i5)=B'_{i4}(T)-B_{i4}(T) = 53.60 - 52.49 = 1.11,$$

$$g_{i5}(i5)=B'_{i5}(T)-B_{i5}(T) = 112.93 - 63.25 = 49.68.$$

When we weigh these gains by unit weapon effectiveness values we obtain the total gain when the force level of B_5 is increased, i.e.

$$TG_{i5} = \frac{WEI_{i3}}{B_{i3,0}} g_{i3}(i5) + \frac{WEI_{i4}}{B_{i4,0}} g_{i4}(i5) + \frac{WEI_{i5}}{B_{i5,0}} g_{i5}(i5)$$

$$= 0 + (9/60)\ 1.11 + (16/90)\ 49.68$$

$$= 8.998.$$

Then, by subtracting the value of the additional force the net total gain is found as

$$NTG_{i5} = TG_{i5} - \frac{WEI_{i5}}{B_{i5,0}} \times \frac{1}{2} B_{i5,0} = 8.998 - (16/90)(1/2)(90)$$

$$= 0.998.$$

Hence, the new weapon effectiveness value of B_5 at the beginning of stage 2 is determined as

$$LWEI_5^2 = WEI_5^1 + NTG_5$$

$$= 16 + 0.998 = 17.998.$$

Now we illustrate the decoupling process. At first we calculate the weighted average of the values supplied by ALLM and ASM as

$$ALWEI_5^2 = \omega\, AWEI_5^2 + (1-\omega)LWEI_5^2$$

$$= \omega\,(16.83) + (1-\omega)17.998 \qquad\qquad 0 \le \omega \le 1$$

where ω is the weight reflecting aspiration level. Let $\omega = 0.20$, then

$$ALWEI_5^2 = 0.20\,(16.83) + 0.80(17.998) = 17.76.$$

Then we find an intermediate estimate obtained through exponential smoothing

$$PWEI_5^2 = \theta\, WEI_5^2 + (1-\theta)ALWEI_5^2 \qquad\qquad 0 \le \theta \le 1$$

$$= \theta\,(16) + (1-\theta)\,17.76$$

where θ is the smoothing constant chosen by the decision maker. Assuming that θ is selected as 0.75, we

$$PWEI_5^2 = 0.75\,(16) + 0.25(17.76) = 16.44.$$

The final value to be used for the next stage is then determined as

$$WEI_5^2 = \frac{SWEI_5^2}{SWEI_5^1} \times 16.44$$

where $SWEI_5^1$ and $SWEI_5^2$ denote the standard weapon effectiveness that takes into account stage specific uncontrollable factors as explained in Section 6.2. Here, by using the ratio of standard weapon effectiveness values, we can adjust for dynamic uncontrollable factors and force level changes for stage 2.

CHAPTER 7

Conclusion and Future Work

In this thesis, we present a modeling framework for analyzing tactical level land combat to generate information for weapon and ammunition planning. Considering the dynamic nature of combat, we decompose it into stages and mini-battles. We propose an integrated system which combines optimization and simulation within a dynamic setting, for modeling mini-battles in a combat stage. This system is composed of three models interacting with each other. These models are: the allocation model (ALLM) which is an optimization model used for allocating blue units to red units, the attrition simulation model (ASM) for predicting the attrition of units, and the weapon effectiveness index update model (WEI) for better planning the use of blue units. Each of these models deals with a different aspect which is an integral part of any combat. Starting from the first stage, we move forward stage by stage where the engagements within a stage are decomposed into mini-battles. Setting an input/output relationship between stages such that the output of an earlier stage is passed as input to the next stages, we obtain the solution for overall combat. Our approach assumes that the problem parameters remain constant within a stage while they may change as we move from one stage to the next, hence it analyses combat in a semi-dynamic environment.

Decomposition comes naturally, since real life combat proceeds in stages and, in each stage, independent local engagements (mini-battles) take place in different geographical locations. Decomposition has two advantages:

- We have the flexibility of changing input parameters including user goals from stage to stage.
- We reduce the complexity of problem substantially by modeling one mini-battle at a time rather than the entire combat.

The major disadvantage is that the overall solution can be suboptimal, as in almost any decomposition approach. However, the questions that what stages the combat should have and which local engagements should take place in each stage must be answered at a higher level of planning. This usually involves strategic and operational level planning by military experts and may be subject of another study.

As to the modeling of a mini-battle, our approach contributes the literature in following areas.

- The integrated system consisting of three basic models and combining them within a decision support framework facilitates a semi-dynamic modeling of combat.
- Optimal allocation of heterogeneous blue units becomes possible such that the minimum amount of total weapon effectiveness is used and the users' combat goals are satisfied. Synergistic effects due to force division and combination are also taken into consideration in the formulation. A solution methodology is explored as well.
- The allocations are validated by simulation which is carried out at a lower level of resolution.

- A discrete-time stochastic model (DSM) is developed based on SSKPs which facilitates analysis of combat problems involving small heterogeneous units.
- Treatment of different fire types (directed, area, and mixed fire) is possible in DSM by using binomial processes, which also take overkills into consideration.
- Use of salvos of fixed length as time advancement mechanism and allowing multiple kills in a salvo add more reality to DSM. Hence, we relax the classical assumption which permits at most one kill in one firing cycle.
- Extensions such as effects due to force combination and division, engagement, noncombat loss and reinforcement increase the realism. DSM integrates stochastic engagement, shooting and noncombat loss processes in calculating casualties in each salvo, and treats reinforcements as well.
- Each mini-battle may involve units of different sizes (small units, large units or a mixture of both). We can use the proper simulation tool needed for each mini-battle (pure DSM, pure LM or integrated use of LM and DSM).

However, there are some drawbacks in our approach requiring further investigation. We foresee the following short term research areas to further improve the overall model.

ALLM

Efficient solution methods should be developed for solving especially large problem instances. Since linear programming relaxation of ALLM can be obtained easily by decomposition and column generation, a method of getting an integer solution applied on the LP solution might be improved by employing the group problem approach sketched in Chapter 4. Also, for a fair comparison of column generation with genetic algorithm (Erdem and Özdemirel, 2002), the same improvement steps used in genetic algorithm should be implemented for column generation method.

DSM

Dimensional limits such as the total number of units and combatants should be determined by applying DSM for real combat scenarios and investigating the computational burden involved. If, for example, we have five units each having 19 combatants in a mini-battle of a combat stage then the number of states in each salvo is $(19+1)^5 = 32{,}000{,}000$. This requires the development of an efficient algorithm in order to extensively experiment with the size limits of DSM. Alternatively, one may consider aggregating individual combatants to reduce the problem size, or decomposition with the help of noncombat losses.

It may be possible to formulate DSM as a single-dimensional Markov chain and calculate the results for any salvo directly by applying matrix geometric analysis using the special structure of the transition matrix.

Finally, risk analyses on munition expenditures can be performed by means of DSM when different weapon systems are synchronized by means of the engagement process.

ARCs and SSKPs

ARCs, which are needed for large scale combat problems and difficult to find in real life, can be estimated from DSM results. This can be done by fitting the LM differential equations to the force level curves obtained from DSM throughout the salvos. An approximate LM can then be run using the estimated ARCs.

Considering the engagements where forces are not stationary but approach each other, range dependent (or time varying) ARCs could be utilized in LM for large forces. Similarly, since the distance between units may change throughout the salvos, salvo dependent SSKPs could be used in DSM.

Furthermore, adverse effects such as the influence of reduction in firepower due to wearing down of units (especially for large combat durations), and suppression effect of opposing fires could also be handled by reducing the values of ARCs and SSKPs.

Finally, the effect of the changes in force sizes (or force ratios) can also be reflected in these parameters. For example, a decrease in the number of combatants of a blue unit may result in a reduction in the fire support provided for remaining combatants and therefore may cause a relative increase in the suppression effect of opposing fires. This effect may be incorporated by decreasing or increasing SSKPs (or modifying synergy effects) depending on the size and direction of the change. Also, the alterations in force sizes (or force ratios) may change the time required for preparing weapons, target detection, aiming and shooting. Hence we should update the engagement probabilities for each weapon type.

Environmental Conditions

In the proposed system, the influence of weather and terrain conditions is assumed to be reflected in input parameters (weapon effectiveness indices, ARCs, SSKPs). One may try to develop a more explicit way of incorporating these effects into the modeling framework.

We can also identify the following long term future research issues.

- Extensive scenario analysis involving military experts should be made to test and validate the overall modeling framework. To evaluate the modeling framework, a group of military experts can replace ALLM, target allocations suggested by them can be simulated, and results can be compared with those obtained from our system. In addition, by comparing the results obtained from this system with real combat data available or with outcomes of field exercises, some modifications can be made in modeling (if necessary) to compensate its deficiencies and to adapt it to various combat environments such as conventional combats, guerilla warfare and combats involving highly sophisticated weapons.
- The eventual use of our approach would be in weapon and ammunition planning by providing information on the amount of munition required to win the combat. This information can be used in developing a new weapon and ammunition planning system that may yield a reduction in defense expenditures.
- In force allocation model, we consider only blue units and assume that red allocates its force to blue in proportion with the blues' allocation fractions. Hence, the overall system views combat from the blue's perspective. Alternatively, we could view combat as a two person game and let the red side allocate its forces using a similar allocation scheme. Such a game theoretic approach may lead to the development of a wargaming system. This system can be used in scenario development and hence in analysis of various threat situations, as well as in training military personnel.

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APPENDIX A

Reviewed Work

Here, we present the summary of the literature that we have reviewed during our research in the order that they are referred in the Literature Review (Chapter 2). The first column of the table shows title, author, source of the publication and the publication year. In the second column we stress the main issues addressed in each paper. The third column contains the information that we can possibly use in our approach, and the fourth column indicates deficiencies or possible extensions that can be considered for further study.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
1. An Introduction to Variable Resolution Modeling, (Davis P.K.), [Warfare Modeling], 1995.	• Variable-resolution modeling and development of integrated families of models with varied resolutions are introduced. • The following questions are addressed ➢ What is variable-resolution modeling? ➢ What is the need for variable-resolution modeling? ➢ What forms can it take and how does it relate to "families of models"? ➢ How should one go about it?	• Variable-resolution modeling allows the user to change the resolution either by turning the resolution knobs within a single model, or tuning from one model to another in a family. • Cross-resolution model connections links existing models with different resolutions either; ➢ In software (models can operate together) or ➢ External (outputs of one are transferred manually to become inputs of another). • The term Resolution can be used for very different concepts. A model can have high-resolution since; ➢ It deals with more fine-grained entities ➢ It ascribes a richer set of attributes to the entities ➢ It describes the relationships among those attributes (logical dependencies) in more details ➢ It incorporates detailed physical and command-control processes governing changes in entity attributes ➢ Its dimension in terms of spatial grid or time step is very high. • The needs for low-resolution modeling are ➢ Initial cuts (innovation, exploration, ...). ➢ Comprehension (seeing the forest rather than the trees). ➢ System analysis and policy analysis. ➢ Adaptability. ➢ Low cost and rapid analysis. ➢ Making use of low-resolution knowledge and data. • The needs for high-resolution modeling are ➢ Understanding phenomena. ➢ Representing knowledge.	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
		➢ Simulating reality. ➢ Calibrating or informing lower-resolution models. ➢ Making use of high-resolution knowledge and data. • Types of variable-resolution modeling ➢ Selected viewing (providing aggregate displays from a more detailed underlying simulation). ➢ Alternative submodels (most common; the alternative submodels and databases are inconsistent to varying degrees. ➢ Integrated hierarchical variable-resolution modeling (IHVR) (quite uncommon; describes critical processes as being composed hierarchically of subordinate processes.	
2. Towards a Formal Theory of Aggregation, (Fowler B.W.), [Military Operations Research], V4, N1, 1999.	• Two methods for aggregating the heterogeneous quadratic Lanchester system to a homogeneous quadratic Lanchester system are introduced. Mathematical forms of both systems are preserved. • An algorithm for aggregating the results of platform level simulation to homogeneous or heterogeneous quadratic Lanchester system is presented. • A comparison with the Potential Anti-Potential is made.	• Lanchester equations for heterogeneous forces are expressed as matrix (vector) differential equations where the attrition rate coefficients are defined in ARC tensor. • Two types of aggregation are defined: i) Proper aggregation: aggregation on the basis of identicality or equivalence of aggregated entities (eg. *Considering* all the tanks in a tank company as identical). ii) Formal aggregation: aggregation on the basis of functional description (combining the tanks and armored personnel carriers [high resolution] into armored fighting vehicles [low resolution]). Formal Aggregation is examined under two types: i) Formal Intensive Aggregation and ii) Formal Extensive Aggregation • Formal Intensive Aggregation: The aggregation that does not depend on the force strength elements,	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
		but only attrition rate coefficients. ➢ Based on numerical experimentation rather than mathematical theory ➢ Two restrictions are placed on ARC tensor: i) It must not include any fratricide contributions, ii) It must be dense in a combat sense; each force strength element must attrit at least one other force strength element and be attritted by at least one other force strength element. ➢ The mathematical restriction that the ARC tensor not be reducible in dimension is also imposed. ➢ Eigenvector solution of a particular form can be obtained yielding the aggregation of the heterogeneous Quadratic Lanchester problem to a unique homogeneous Quadratic Lanchester problem. • Formal Extensive Aggregation: The aggregation depends on the force strength elements whose ARC tensor is reducible. ➢ System to be considered has already been reduced either by intensive aggregation or by formulation to one which contains homogeneous subsystems. ➢ Using the insights of intensive aggregation a solution is postulated. • Aggregation from simulation: The composite output of most simulations includes a Killer-Victim Scoreboard (KVS), initial force strength and combat duration. KVS is a square matrix whose component Ki/j represents the number of ith element killed during the combat by jth element. KVS is assumed to have the same geometry as ARC tensor.	

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
3. Force Deployment in a Conventional Theatre-level Military Engagement, (Kevin Y.K. and Lam M.N.), [Journal of the Operational Research Society], 1995	• Six key parameters are introduced and discussed as the governing parameters (Firepower and effectiveness, Battle front movement, Military readiness, Disintegration of forces, Sustainability and augmentation, and Attrition) that affect the force deployment in a theatre-level military engagement. • All parameters are taken as deterministic. • A non-linear programming model is developed to describe the combat dynamics • Sensitivity analysis on the optimal solution is discussed	• The objective function of the model is the minimization of the total advancement of the Red force (delaying the Red as long as possible) • The combat is divided into phases of equal length (days) • Replacement of forces by units from the reserve is handled. • Inventory balance equations are used to describe the losses for each stage • In the dynamic programming formulation replacements of troops from reserve forces are also allowed in each stage.	• The stochastic effects can be handled by replacing the parameters with probabilistic equivalents • Length of phases (periods) may be different • The objective function can be stated as the maximization of the time delay by the Blue force
4. Lagrangean Relaxation Approach to the Targeting Problem, (Kwon O., Kang D., Lee K. and Park S.), [Naval Research Logistics V46, 640-653], 1999	❑ Subject: A weapon-target allocation problem is modeled with the objective of minimizing the total firing cost. • Modeling and solution approach: ➢ First, weapon-target allocation problem (TP1) is modeled as a nonlinear integer programming problem. ➢ Then, nonlinear constraints are transformed into linear ones taking logarithms on both sides (TP). ➢ Finally, after relaxing some constraints (LD), Lagrangean relaxation and branch-and-bound approach are used • Force Composition: Heterogeneous (i weapon types firing j targets). • Engagement Type: Directed • Implementation ➢ Parameters: - fi, number of rounds available for each	• Linearization of weapon-target allocation problems • Use of Lagrangean relaxation and branch-and-bound approaches in solving allocation problems	• Use of different objectives such as minimization of the threat or maximization of total attrition of opposing force

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	weapon type - pij, probability of destroying target j by one round of weapon system i - dj, minimum desired probability of destroying target j - cij, cost of firing one round from weapon system i to target j - uij, upper bound on the number of rounds that can be fired from weapon system i to target j ➢ Data generation: - Four groups of data are generated according to weapon-target size: 5x7, 5x15, 10x20, and 10x30. - Graph representing weapon-target pairs is a complete bipartite graph (a weapon can fire at every target). • Results (Computations): ➢ 15 test problems for each weapon-target size is generated ➢ Three different branching rules are used ➢ The results of test problems are provided involving - Average, maximum, and minimum values of the number nodes generated, - CPU times, Percentage of the relative ratio between optimal objective function values of TP and LD.		
5. Semi Dynamic Modeling of Tactical Level Land Combat, (Özdemirel N.E.,	• An approach decomposing a battle into stages and mini battles and using three models for each mini battle has been presented.	• The use of three models interactively allows the combination of mathematical programming and simulation is a semi-dynamic setting. These models are; - a mathematical programming model for	• Development and use of stochastic Lanchester models for small forces can be an area for further study yielding significant results for

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
Kandiller L., Vardar C. and Bayındır P.), Technical Report No: 01-11, Ankara, Turkey: Middle East Technical University, (2001).	• Tactical level battles are considered	optimizing force allocations (ALLM) - a Lanchester simulation model for predicting whether or not the stage targets are reached under the allocations (LM) - a weapon effectiveness update model from one stage to the next (WEI) • A decision support system with these three interacting models are proposed to help the user with allocation decisions and with the prediction of force size and weapon requirements to win the battle.	performing risk analysis and weapon and ammunition planning.
6. Theory of Measures of Effectiveness for General Purpose Military Forces: Part II. Lagrange Dynamic Programming, (Pugh G. E.), [Operations Research], 1973	• A combat is modelled as a time sequential two-person zero-sum game • Pure strategies and their associated payoffs are assumed to be known • Time dependent shadow values (Lagrange multipliers) are used for various types of combat resources • Expected values of resources are defined in two alternative way: (i) fixed form (applied to each pure strategy), (ii) free form (applied to mixed strategies) • The model is solved iteratively	• Requirements for strategies are defined and feasible strategies in each period are determined depending on the resources left from the previous period (constraints) • Resource availabilities in each period are controlled by recursive transformation equations (constraints) • Overall value of the game is obtained by summing each period's payoff (obj. function) • A set of discounting factors that accomplish the conventional discounting of future value of utilities are used in the calculation of payoffs.	• The combat scenario should be defined and pure strategies should be determined as a list • Lanchester systems can be incorporated into the dynamic programming approach and used to calculate the resource levels
7. Theory of Measures of Effectiveness for General-Purpose Military Forces: Part I. A Zero- Sum Payoff Appropriate for Evaluating Combat Strategies, (Pugh G.E. and	• The bargaining game situation is considered where each side has perfect information of his opponent's strategies and their associated payoff. • The purpose is to develop a theory of measure of effectiveness for general-purpose forces. • An approach that forms the basis for the selection of a zero-sum payoff function to compare alternative combat strategies is	• The Nash non-zero-sum-"bargaining game" analogy is used to evaluate the *dominated* and *non-dominated* (bargaining line) strategies and *enforceable* payoff level of players. • Non-dominated strategies are the cooperative strategies as in the usual non-zero-sum games. • Optimum strategies include a solution in the form of a zero-sum game whose payoff is a weighted difference of the actual payoffs to the two players in a non-zero-sum game.	• The reasons for preferring the zero-sum game are; (i) it is very familiar and (ii) it is mathematically tractable. Hence the computational advantages of keeping the problem in a simple form is preferred to the inclusion of other (secondary) objectives; humanitarian goals, matching

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
Mayberry J.P.), [Operations Research], 1973.	developed. • Non-zero-sum-payoff is transformed into a zero-sum-payoff. • Overall national war-time goals or objectives are derived from and related to peace-time goals.	• The solutions for some specific cases (where one side is relatively/very strong than the other side) are obtained by simultaneously solving the *bargaining game* and the *threat game*. • The solution of the bargaining game occurs where a hyperbola relative to the threat point is tangent to the bargaining surface. • Use of zero-sum game gives computational advantage since it is relatively tractable.	the pace of combat to the political response pace of nations. Realism and tractability act oppositely. • This approach (assuming perfect information is available for the overall combat) misses the time-sequential nature of both negotiation and conflict as both sides gain information and change their understanding of the game payoffs.
8. An Allocation Problem of Support Fire in Combat as a Differential Game, (Kawara Y.), [Operations Research], 1973.	• The problem of allocation of support fire between two different targets is modelled and analyzed as a two person zero sum game. • Each side has one major unit (infantry) and one support unit (artillery). • Two cases are considered; i) support units of both sides survive combat and ii) One side's or both sides' support units are annihilated during combat. • The relation between the value of the game and its duration is also examined.	• The ratio of two sides' primary units at the end of combat, x_1/x_2 (defied as the measure of superiority), is taken as the payoff to the first player (and its negative to the second player). • Optimal allocation of fire is obtained as a 0-1 rule (concentration of fire on enemy primary unit or support unit). • The attacker can maximize the payoff by choosing the proper combat duration. • If the specified combat duration exceeds a critical value, the battle is divided into two phases: in phase I, all supporting fire is concentrated on enemy support unit and in phase II (whose length is equal to the critical value), it is concentrated on enemy primary unit. • For the battles whose duration is less than the critical time, all fire is concentrated on enemy primary units.	• More realistic situations (where each side has more than two types of weapon/unit) should be considered. • Combats with longer duration can be divided into more than two stages and each stage can be handled as a combat where all support fire is concentrated on one of the enemy units. • Not only support fire but also fire distribution of the primary units can be planned.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
9. Determining the Class of Payoffs That Yield Force-Level-Independent optimal Fire-Support Strategies, (Taylor J.G.), [Operations Research], 1977.	• The dependence of the structure of optimal combat strategies in a time-sequential combat game on the quantification of the military objectives is examined. • The class of terminal payoffs that yield force-level-independent (state-variable-independent) optimal strategies to the fire support game of Kawara (1973) is determined.	• State-variable-independent strategies are the simplest ones to determine and implement (e.g. continuous state information is not necessary) • A mathematical technique is developed to determine the class of payoffs for which the optimal fire-support strategies have the property of being state-variable independent • Functional dependence of switching times in the 0-1 optimal fire support strategies is shown to be determined by the mathematical form of the criterion function (payoff).	• The quantification of combat objectives (of both sides) and their effect on the structure of optimal strategies should be studied • A method that includes all units (not only infantry) in the payoff function and tries to derive optimal strategies can be developed. (Note: including only infantry yields state- variable-independent strategies as shown in this paper).
10. Mini-Risk: Strategies for a Simplified Board Game, (Maliphant S.A. and Smith D.K.), [Journal of Operational Society], V41, N1, 1990.	• A popular board game named "risk" which is a game of global strategy is used as a means to develop a dynamic programming approach leading to useful ideas of strategy and objectives. • The main characteristics of the game are as follows. ➢ The game includes an attacker and a defender having various number of units in different territories. ➢ The game proceeds by a round of battles across territorial boundaries. ➢ The game ends when predefined termination conditions are realized.	• The process described here can help developing dynamic programming formulations in real combats where successive decisions are given. • Several objectives can be defined in a real combat for attacking or defending sides. The objective listed in this paper are; ➢ to maximize the probability that the attacker defeats the enemy, ➢ to maximize the expected number of pieces in the attacker's army at the end of the turn, ➢ to maximize the expected difference between two armies at the end of the turn, ➢ to minimize the expected number of pieces in the defender's army at the end of the turn. Dynamic Programming Formulation • The reasons that the problem posed in this game can be modelled as a dynamic programming problem are	• Some potential extensions to the models developed here can be used in dynamic programming formulations of real combat problems. • Different objective functions can be used and corresponding formulations can be developed. • More detailed statistical analysis of the optimal policy can be done.

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
		➢ Decisions are made sequentially one at a time. ➢ It is possible to list the possible decisions at any time ➢ The state of the "system" can adequately be described on the basis of the number of units held by each army across the frontier ➢ Once a decision has been made it is possible to write down the possible states which could be reached at the time of the next decision. ➢ A stochastic transformation operator changes the state of the system. ➢ There is an objective function which will link decisions in different states, and with appropriate terminal conditions, it can be used to identify which decision is best from any state. • The dynamic programming approach developed here uses the first objective: The attacker wishes to play so as to maximize the probability that he wins. • P(A,D) is defined as the probability of winning under an optimal strategy given that two armies have strengths A and D. • At the same time the defender will be trying to minimize the probability that the attacker wins. • So the probability P(A,D) is defined as a max-min function. • As an example (A,D) is taken (4,2) as the state: ➢ the transitions from different states and the corresponding winning probabilities are given as a table that can be treated as a payoff table of a two-person game. ➢ The corresponding expected losses of the attacker are also calculated and presented in a table.	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
11. On The Isbell and Marlow Fire Programming Problem, (Taylor J.G.), [Nav. Res. Log. Quart.], 1972.	• Isbell-Marlow fire programming problem (1956) is one of the first studies using heterogeneous forces (homogeneous Y force against heterogeneous X force having two type of units, X_1 and X_2) and developing an optimal fire distribution policy (target selection). They did not obtain a complete solution and could not determine when terminal states of combat were reached. • Here a complete solution of the Isbell and Marlow fire programming problem is derived.	• Allocation of fraction of fire is always 0 or 1 for square law attrition process. • Y always allocates his fire entirely to the target type yielding the greatest marginal return. • Y's target priorities only switch with time in the case when Y loses.	• Other possible situations like i) both sides are heterogeneous, ii) attrition processes other than square law and iii) time dependent attrition rate coefficients can be considered.
12. A Theory of Ideal Linear Weights for Heterogeneous Combat Forces, (Howes D.R. and Thrall R.M.), [Nav. Res. Log. Quart.], 1973.	• A procedure to compute the overall weight of a force, using the effectiveness values of each weapon against each weapon of the opposing force, is developed.	• Overall force effectiveness indices, which are the sum of the weighted averages of individual weapon effectiveness values are derived from inter-weapon effectiveness matrices assuming this matrices are given. • The classical Perron-Frobenius theory of eigenvalues and eigenvectors of nonnegative matrices is applied to the formulation developed to obtain the solution. • This approach can compensate the deficiency of Lanchester Equations which assumes homogeneity of forces and can be used in the analysis of combats between heterogeneous forces.	• This paper assumes that the weapon effectiveness indices are known. Hence the development of inter-weapon matrices can be the subject of another study.
13. Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent	• Optimal fire distribution rules are developed for a homogeneous Y fighting against heterogeneous X force having n types of weapon (fraction of the fire directed to each type of X force weapon is	• The rules that are developed to determine the distribution of fire can be used in strategy definitions. • Each period of a combat (in a dynamic programming formulation) can be taken as a	• Both sides should be considered as heterogeneous forces and the optimal fire allocation rules may be developed accordingly.

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
Attrition-Rate Coefficients, (Taylor J.G.), [Nav. Res. Log. Quart.], 1974a.	to be determined) • Prescribed duration battle is considered where none of the sides are annihilated at the end • A combination of Lanchester equations for directed fire (square law) and deterministic optimal control theory is used • Attrition rate coefficients are assumed to be time (range)-dependent • The objective is to maximize the worth of surviving Y forces while minimizing that of X forces	prescribed duration battle. • Assigning a value for each type of weapon and maximizing the total value of the survivors can be considered as an objective (or one of the objectives) for each stage of the battle. • 0-1 allocation rule can be used where the concentration of fire is necessary.	• The Other types of Lanchester Equations (area fire, mixed models, ...) or their mixture can be used for different places and different periods of a combat. • The cases where a termination criteria is given (such as force annihilation or realization of a force level) may be considered as well. • Instead of 0-1 fire allocation a continuous (fractional) allocation can be developed defining weights for the targets.
14. Lanchester-Type Models of Warfare and Optimal Control, (Taylor J.G.), [Nav. Res. Log. Quart.], 1974b.	• The optimization of the dynamics of combat (optimal fire distribution) is studied on five idealized combat models. • The models represent a period of a combat where the attrition occurs according to Lanchester-type equations. • A choice of tactics (fire distribution rules) is available to one side and subject to change within time. • The factors discussed, that affect the optimal fire distribution policies are ➢ combatant objectives, ➢ termination conditions, ➢ type of attrition process, and ➢ variable attrition rate coefficients. • The main questions addressed are; i) How should the fire be distributed?	• i)"Switching times" of fire distribution rules and ii) the target types upon which all fire is concentrated are determined for each idealized combat model. • When the targets undergo a *square-law* (directed fire) attrition process; i) All fire is always concentrated on one target type (0-1 allocation rule). ii) The allocation is not (directly) dependent upon the force levels. • When the targets undergo a *linear-law* (area fire) attrition process; i) Fire may be divided between target types (policies other than 0-1 rule are possible). ii) The allocation is (directly) dependent upon the force levels. • The connection between optimal control theory and differential games should be reviewed to obtain	• Only one side (homogeneous force) has the choice tactics. When determining one side's optimal fire allocation policy, the opponent's possible alternatives and their effect on the attrition process can be studied. • Both forces can be taken as heterogeneous and then optimal fire distribution policies can be derived.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	ii) Do target priorities change within time? iii) Does the number of target types affect the optimal distribution of fire? iv) Do termination circumstances affect the optimal allocation policies? v) How does the nature of the attrition process affect the optimal distribution of fire? vi) How does the uncertainty and confusion of combat affect the optimal distribution rule?	insight in determining the optimal tactics. • Target priorities change in relation to the evolution of marginal return of target destruction (value of dual variables) within time. This evolution depends on i) the goals of the combatants (utility assigned survivors at the end of the battle) and ii) termination conditions. • When intelligence and command control systems are very efficient, the optimal tactic is to concentrate fire on a specific target type. • When capability for redirection of fire from destroyed targets is poor, the optimal tactic may be to allocate fire in a proportional fashion over target types (where the ratios of target density in each target area are constant). • Battle commander must use his judgement to foresee to what end the battle is steered to so that he may select his strategy accordingly.	
15. On The Treatment of Force-Level Constraints In Time-Sequential Combat Problems, (Taylor J.G.), [Nav. Res. Log. Quart.], 1975.	• The treatment of force-level constraints in time sequential combat optimization problems is illustrated using Isbell-Marlow problem (homogeneous Y force against heterogeneous X force having two types of weapon) with constant attrition rate coefficients. • The necessary conditions of optimality for optimal fire-distribution policy are obtained. • Some military principles for target selection and the valuation of combat resources are deduced from the solution.	• The characteristics of the optimal fire distribution policy; i) Optimal policy is always concentration of fire on one type of target ($\phi^* = 0$ or 1). ii) Optimal policy depends on -whether Y wins or loses -$a_1b_1/(a_2b_2)$ -$a_1p/(a_2q)$ * a_ib_i is a measure of strategic value of firing at X_i (rate of destruction of X_i's kill capability against Y). * a_1p is a measure of short-run return to Y from firing at X_1 at the end of battle (rate of destruction of X_i value at the end of the battle). • Optimal fire-distribution policy (expressed as a	• Dependence of the optimal fire distribution policies on force levels in combat models other than Isbell-Marlow can be studied. • Instead of utility maximization of surviving combatants at the end other types of objectives should be considered. • Time dependent attrition rate coefficients should be considered as well.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
		closed loop control) depends on the force levels not on time. • Important military principles deduced from the solution of Isbell-Marlow problem; i) Always concentrate fire on one type of target ii) Fire is always concentrated on the available enemy target type with the largest $a_i b_i$ value. iii) We have a motivation for valuing enemy target types in direct proportion to their kill capability (effectiveness) since the optimal policy is very appealing and simple in this case. A commander must ascertain to what ends the battle can be steered to and so that he may devise his strategy accordingly.	
16. An Examination of the Effects of the Criterion Functional on Optimal Fire-Support Policies, (Taylor J.G. and Brown G.G.), [Nav. Res. Log. Quart.], 1978.	• The dependence of the structure of optimal time-sequential fire-support policies on the quantification of military objectives is examined by considering four problems each considering a different quantification of objectives (i.e. criterion functional). • Optimal time-sequential allocation of supporting fires during the "approach to contact" phase of infantry against enemy defensive positions. • The combat dynamics are modelled by deterministic Lanchester-type equations. • Both forces are heterogeneous (each consists of two type of units). X infantry (X_1 and X_2) approaches to the positions held by Y infantry (Y_1 and Y_2) where there is no "crossfire" (e.g. X_1 is not attrited by Y_2). • Optimal fire support policy is developed via	• Three essential parts of time-sequential combat optimization problems; i) The decision criteria (for both combatants) ii) The model of conflict termination (and / or unit breakpoints) iii) The model of combat dynamics • For the same combat dynamics the distribution of supporting fires between two enemy forces in any optimal control policy depends only on whether the terminal payoff reflects the objective of attaining an "overall" military advantage or a "local" one. • Switching times for changes in the ranking of target priorities are different when the decision criterion is the difference and the ratio of the military worths of total infantry survivors and also the difference and the ratio of the military worths of the combatants' total infantry losses. • Hence, the optimal fire support policy is significantly influenced by the quantification of	• Firepower of primary units (infantry) can be considered as well as the support fire. • Not only "approach to contact" phase but also the phases after both forces get close contact can be studied.

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	optimal control Theory. • The problems are all nonconvex.	military objectives. • Each of the four problems has one of the following criterion functional (objective functions) to be optimized. i) Weighted average of the force ratios of opposing number of infantry in both infantry combat zones. ii) The difference between the military worths of the surviving X and Y forces at the end of the "approach to contact". iii) The ratio of total military worths of the surviving X and Y forces. iv) The ratio of military worths of losses.	
17. Combat Modeling with Partial Differential Equations, (Protopopescu V., Santoro R.T. and Dockery J.), [European Journal of Operational Research], 1989.	• The purpose is to add the minimum in complexity to the original description to more accurately reflect the realities of combat. • One-dimensional spatial effects are included in the classical Lanchester differential equations of combat. • The coupled set of Lanchester Equations is replaced by a system of partial differential equations. • Shortcoming of Lanchester Equations are expressed as; i) they do not account for the movement of forces on the battlefield., ii) Command and Control (C^2) is usually not treated. • Deterministic models for homogeneous forces are used. • Some specific examples for 3-1, 6-1 and 1-1 force ratio are given.	• The motivation for such a study ➢ Without spatial dependence the maneuver is impossible. ➢ Without nonlinear effects the larger force always wins. • The formulation includes the following (none of which are presented in the classical Lanchester systems): i) *Diffusion*: expresses the natural tendency of any force to lose its original configuration as it moves, fights, etc., or simply just as time goes by, due to fatigue, loss of concentration, loss of motivation, etc. ii) *Advection*: Describes the large scale, ordered 'flow' of troops on the battlefield as opposed to 'chaotic' small-scale movement represented by diffusion. iii) *State dependent attrition* from forces which are closing on one another.	• Natural barriers and elevations can be handled by either / both boundary conditions and modifications in the advection and diffusion terms. • Once two dimensional results are available, heterogeneous forces and stochasticity can be included.

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18. Two Effects of Firepower: attrition and suppression, (Hughes W.P., Jr), [Military Operations Research], V1, N3, 1995.	• A quantitative approach to suppression of enemy fire is developed. • The principles from combat science that are relevant to the proportion of losses of forces' are reviewed. • This paper examines the measure of combat power's mental effect, which is the suppression of enemy actions.	• Two of the principles of Combat Science are; ➢ Military force, or combat power is a real phenomenon, the results of which are observed by its effects on the enemy in battle. ➢ The observable effects of combat power are i) (not merely) physical (producing casualties) but also ii) mental (persuading the enemy of our superiority, and iii) spiritual (diminishing enemy morale and will to continue fighting) • Four reasons that causes objections that the classical square law does not fit the historical battle casualty data for ground combat: ➢ The law cannot be applied when the required conditions are not present in the battle. ➢ Insufficient attention is given to the defender advantage. ➢ Battle is usually divided into episodes, such that different elements of a force play predominant roles at different phase of the battle. ➢ There is a shortage of data with which to measure the effect of suppression of enemy actions from firepower. • Since suppression produces no casualties and is a transient phenomenon that disappears when the battle is over, the effect of its presence is overlooked in almost all combat models, including high resolution simulations. • The square law is examined where the unit effectiveness (attrition rate coefficients, α and β) are variable. The reasons for this variability: ➢ The change in range between the fighting units	• A battle can be divided into phases in such a way that not only the changes in unit effectiveness but also the effect of all factors other than unit effectiveness should be handled in a formulation. • The suppression effect can be incorporated into the formulation together with the other combat degradation factors. • Other Lanchester systems can also be analyzed using the approach developed here.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
		as the battle continues, ➢ The suppressive effect of each side's fire on the other's fire. • The variables, a(t) and b(t), denoting the rate of fire in shots by each unit, and s indicating the rate of reduction of fire imposed on the other side per shot fired are used to develop a new formulation that handles the suppression effects. • A numerical example and detailed interpretation of the suppression calculations are provided. • Summary of the major points: ➢ The coefficients of unit effectiveness should be regarded as variables in time. ➢ Fire volume and unit effectiveness are often far more variable than the change due to casualties, in which case they have a greater immediate effect on the outcome than attrition. ➢ When unit firepower is diminished more rapidly than the surviving number of units (due to the effectiveness of enemy fire) then the unit firepower advantage has the square law effect, not numerical advantage.	
19. Efficient Concentration of Forces, or How to Fight Outnumbered and Win, (Bitters D.L.), [Warfare Modeling], 1995.	• This paper assumes that minimum loss of friendly forces is the objective to be considered while accomplishing the goal of mission. • The principle of efficient force concentration as a means of minimizing losses while defeating the enemy, particularly one that is numerically superior is explored by looking at several attrition mechanisms based on Lanchester models.	• The concept of elasticity borrowed from economics is used to develop a more meaningful measure of effectiveness called force elasticity, $\epsilon_F=(dR/R)/(dB/B)$ which is a dimensionless local measure comparing the percent change in the Red and Blue forces at a specific time t. • The conditions for parity, red win and Blue win are shown in terms of force elasticity in a fight to finish battle where one or both sides will annihilate	• Forces are assumed to be homogenous. A more general case may be obtained using heterogeneous forces. • Measures that serve to reduce the enemy's fighting effectiveness, such as deception, distruption, delay and demoralization may be incorporated into the

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
	• A new measure of effectiveness called *force elasticity* is introduced as the proper benchmark for comparing the relative effectiveness of forces.	eventually in three types of engagements: directed fire, area fire and mixed fire (ambush). • The well known military principle of divide and conquer is illustrated by applying a tactic where Blue is assumed to partition Red into four equal parts and engage each subsequentially in aimed-fire battles with remaining strength from previous battle. The annihilation condition is examined using force elasticity at each battle. • Optimization results are obtained developing a quadratic programming problem and using Lagrange multipliers yielding the optimum partitioning of Red forces that minimizes Blue casualties. The principle of force concentration is shown to yield no advantage in the area fire case. For this case the optimal tactic is to fight a battle under the most favourable circumstances possible. • Also the following three special cases are handled and optimal partitioning rules are obtained for each using percent effectiveness degradation factors. ➢ *Tireless soldier:* relative effectiveness of individual combatants remains unchanged across battles. ➢ *Geometric fatigue factor-red advantage:* Attrition ratios reduces since the relative effectiveness of Blue combatants decrease due to various human factors such as fear and fatigue. ➢ *Geometric effectiveness enhancement factor-red advantage:* Both sides lose effectiveness from battle to a battle but Red's effectiveness decrease faster than Blue's.	formulation. • This study does not take into consideration the replenishment. As a further study, the modifications that should be made if a replenishment policy were built into the model should be examined.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
20. An Analytical Model for Close Combat Dynamics (Gass N.) [Journal of the Operational Research Society V48, 132-141], 1997.	• A dynamic combat model is developed using the continuity equations of fluid dynamics. • Three main features (movement of battle front, replenishment of losses, and withdrawal of combat units are handled in the model. • The units are assumed to be in close contact with the enemy.	• This approach is used to model the combat at a micro level that may be a part of the overall campaign. • Although the combat units may have different speeds, the combat zone is assumed to move at an average velocity • Losses occur near the frontline and they are negligible while away from the frontline.	• Continuity equations of combat dynamics can be used to analyze the complex situations (surprise attacks, pursuits, head-to-head attacks) • These types of models can be used to obtain insights into the relationship between force ratio, effectiveness, withdrawals, replenishments, speed of the units, and the movement of the frontline. • Combat units of different types and functions can be used to model a more general combat scenario.
21. The Two-on-One Stochastic Duel (Gafarian A.V. and Ancker C.J.), [Naval Research Logistics Quarterly V31, 309-324], 1984.	❑ Subject: The general two-on-one stochastic duel (as an extension of one-on-one stochastic duel) is investigated. • Model: Stochastic Process (Stochastic Lanchester with ned, and a process with Erlang-2 distribution) • Force Composition: Homogeneous. • Force Sizes: Two combatants (A side)-on-one combatant (B side) • Engagement Type: Directed for both sides • Termination: Complete annihilation • Scenario: ➢ Two opponents conduct a continuous engagement until one side is completely destructed ➢ No time and ammunition limitations	• Use of exponential and Erlang-2 distributions in stochastic combat processes	• Consideration of time and ammunition limitations • Use of area fire • Use of other termination criteria

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	• Assumptions/Features: 3 principal features; ➢ Each firer is considered separately as a marksman firing at a passive target ➢ They concentrated only on firings which are kills rather than on every firing event ➢ Three marksmen's firing sequences are superposed and backward recurrence time technique is used to write the state probability equations. • Dependence / Independence ➢ Two opposing firing processes are independent ➢ Two combatants of A side fire independently of each other • Process (Stochastic) ➢ Time advance: Continuous ➢ States: (i, j, t); i and j combatants are alive at time t. ➢ Parameters: - Kill probabilities (SSKP); pA (same for both combatants of A) and pB ➢ Distribution: two cases are studied regarding interfiring times - exponentially distributed for both sides (Stochastic Lanchester version of two-on-one square law) - Erlang-2 distribution for A and exponential distribution for B • Results (Computations): ➢ State probabilities are derived for the first time for two-on-one stochastic duel ➢ From these probabilities also derived - Probability of win		

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	- Mean and variance of the number of survivors		
22. The Many-on-One Stochastic Duel, (Kress M.), [Naval Research Logistics V34, 713-720], 1987.	❑ Subject: The general many-on-one stochastic duel conditioned on the order in which targets are attacked is investigated. • Model: Stochastic Lanchester • Force Composition: Homogeneous. • Force Sizes: One blue unit-on-many red units (example one-on-three) • Engagement Type: Directed for both sides • Termination: Complete annihilation • Dependence / Independence ➢ Two opposing firing processes are independent • Process (Stochastic) ➢ Time advance: Continuous ➢ States: $W_{l,k}(t)$; l red units and k blue units are destroyed by time t. ➢ Parameters: - Round dependent kill probabilities ➢ Distribution: - Interkilling times are exponentially distributed for both sides - Special case; gamma distributed for blue • Results (Computations): ➢ State probabilities are derived (considering five cases). ➢ Relative firepower effectiveness of two sides are examined - Measured in terms of kill rates or reciprocal of the mean killing time	• Use of gamma distribution • Evaluation of firepower effectiveness and its value to achieve success	• Use of area fire • Use of heterogeneous forces • Many-on-many type of engagements

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
23. Some Two-on-Two Homogeneous Stochastic Combats, (Gafarian A.V. and Manion K.R.), [Naval Research Logistics V36, 721-764], 1989.	❑ Subject: Two versions of two-on-two homogeneous stochastic combat are considered with the motivation of developing more realistic firefight models. • Model: Stochastic Process • Force Composition: Homogeneous. • Force Sizes: Two combatants (blue)-on-two combatants (red) • Engagement Type: Directed for both sides • Termination: Combat terminates at breakpoints • Scenario: ➢ Two opponents conduct a continuous engagement until one side reaches its breakpoints ➢ No time and ammunition limitations ➢ Every combatant picks an opposing combatant at random ➢ Interkilling time random variable does not change from kill to kill and identical for all combatants of the same side. ➢ Two modes of resuming firing on a survivor are considered; - "reselect on": When a target of a firer is killed whether by him or the other firers on his side, he resumes afresh the interkilling process on the survivor. - "reselect off": (i) If the target of a firer is killed by him, he starts afresh the interkilling process on the survivor. (ii) If, on the other hand, his target is killed by the other firers on his side, his remaining time to a firing (or a	• Use of iid interkilling time • Independece of the firing process of combatants of the same side as well as the independece of opposing firing processes • Use of different modes for a combatant's resuming its fire	• Consideration of time and ammunition limitations • Use of area fire • Use of other termination criteria

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	killing) is carried over to the survivor. ➢ Five different models are addresses considered regarding the breakpoints and modes of resuming fire and two of these models are solved in this study. • Dependence / Independence ➢ Two opposing firing processes are independent ➢ The combatants of the same side fire independently of each other • Process (Stochastic) ➢ Time advance: Continuous ➢ States: (i, j, t); i and j combatants are alive at time t. ➢ Parameters: - Kill rates; rA and rB ➢ Distribution: interfiring times assumed to follow a Gamma-2 distribution • Approach ➢ Aiming configurations are delineated (total of four distinct configuration) ➢ States are decomposed with regard to initial aiming configurations • Results (Computations): ➢ State probabilities are derived ➢ From these probabilities also derived - Probability of win - Mean and variance of the number of survivors - Mean and variance of the battle duration ➢ Relative difference with equivalent exponential and deterministic Lanchester		

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	models are compared		
24. A Fast Approximation of Homogeneous Stochastic Combat, (Yang J. and Gafarian A.V.), [Naval Research Logistics V42, 503-533], 1995.	☐ Subject: An algorithm based on solving a set of exact Kolmogorov equations and approximating the kill rate of one combatant conditioned on the state of the system is developed for solving homogeneous stochastic combat models. • Force Composition: Homogeneous. • Force Sizes: Many-on-many • Model: ➢ Counting process - Killing process for one side - Killing process for an combatant ➢ Kolmogorov equations • Engagement Type: Directed vs. directed • Termination: Complete annihilation • Dependence / Independence ➢ Two opposing firing processes are independent ➢ Each combatant fires independently • Process (Deterministic) ➢ Time advance: Discrete ➢ States: (t, m, n) ➢ Parameters: - Kill rates(Conditional-state dependent). • Results (Computations) ➢ Exact Kolmogorov equations for states ➢ Computation of probabilities for - Interior transient states - Boundary transient states ➢ State dependent kill rates • Other	• Use of different level of coordination to prevent multi kills (overkills) • Treatment of munitions (use of munitions dependent SSKPs)	• Use of heterogeneous forces • Use of other termination criterion • Use of other fire types (such as area fire and mixed fire)

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
	➢ Solving battles greater than 4-on-4 requires huge amount of computation		
25. Modelling the Mobile Land Battle: The Lanchester Frame of Reference and Some Key Issues at the Tactical Level, (Speight L.R.), [Military Operations Research], V1, N3, 1995.	• The influence of Lanchester formulations (as the actual building blocks of large scale simulations and as a means of interpolating between or extrapolating from the results) is stressed. • The paper deals with the direct fire armour anti-armour battle between homogeneous forces the tactical level with the aim of establishing basic principles.	• Some modeling issues are discussed under the following six subheadings ➢ Opportunities for engagement ➢ Target acquisition given the opportunity to engage ➢ Kill probabilities given target acquisition ➢ Battle termination criteria ➢ Combat degradation ➢ The likely utility of Lanchester frame of reference • An engagement between two forces can be split into a series of parallel and/or sequential "mini-engagements" using the combination of tactics, terrain and manoeuvre. • The structuring of mobile engagements into a series of mini-battles is of important for the following reasons; ➢ Mini-battles will set a natural bound on the intensity of the putative engagement as a whole. ➢ The outcome of an ensemble of mini-battles would differ from that of a single Lanchester battle between the two complete sets of combatants. ➢ The conditions of each mini-battle will have a fundamental effect on the whole process of target acquisition. • A previously devised discrete time Markov model	• Homogeneous forces are used. • The issues are not pursued to a conclusion.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
		as an equivalent deterministic and continuous Lanchester formulation is used to compare with the Lanchester equations at the mini-battle level. ➢ The expected number predictions of the Lanchester and Markov formulations diverge as time progresses. ➢ The Markov model predicts lower attrition in each case. ➢ The divergence is worse for the "square law" than for the "linear law" and proportionately worse for "very small" as opposed to "small" numbers. • A very simple time-stepped Monte Carlo simulation is built to check whether the Markov formulation can reflect the interactions within a mini-battle.	
26. Stochastic analysis of Combat Models Under Different Termination Decision Rules, (Jaiswal N.K., Sangeeta Y. and Gaur S.C.), [European journal of Operations Research], 1995.	• Combat between two opposing forces under different termination decision rules has been modeled as a continuous-time discrete –state space Markov process. • Major characteristics of combat are evaluated and some numerical results are presented for stochastic Lanchester equations of directed fire, area fire and warfare with smart weapons.	• They constructed infinitesmal generator matrix of the Markov process and then computed the transition probabilities. • The following characteristics are computed using Markov process; - distribution of combat duration - mean and variance of combat duration - the probability of win of either side - the expected number of survivors at combat termination - joint probability distribution of two forces - ratio of expected losses • The concept of smart weapons (more sophisticated weapons with very high kill potential) and the modification of Lanchester equations for these	• The numerical solution approach that they used to solve combat problems and obtain combat characteristics can be applied to other stochastic combat models. • The interkilling time may not need to follow exponential distribution.

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
		types of weapons are introduced.	
27. Attrition Formulas for Deterministic Large Scale Combat, (Anderson L.B.), [Naval Research Logistics V42, 345-373], 1995.	❑ Subject: Attrition formulas are described for i) area fire and directed fire, ii) various levels of coordination of fire, iii) explicit consideration of the use of various types of munitions, iv) allowing maximum density of targets for area fire and v) meaningful allocations of directed fire. • Force Composition: Heterogeneous in the sense that various types of munitions are used. • Force Sizes: Many-on-many • Engagement Type: Directed and Area fires treated separately. • Termination: Complete annihilation • Dependence / Independence ➢ Two opposing firing processes are independent • Process (Deterministic) ➢ Time advance: Discrete ➢ Parameters: - pimj: SSKP of an combatant of type i if it fires target of type i with munitions of type m. • Results (Computations) ➢ Directed fire and area fire attrition equations for following cases - Uncoordinated fire - Partially coordinated fire - Coordinated across all shooters type • Regarding Area fire ➢ Targets are uniformly distributed in an	• Use of different level of coordination to prevent multi kills (overkills) • Treatment of munitions (use of munitions dependent SSKPs)	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	area ➢ Fires in each salvo may overlap ➢ A target is killed in a salvo with probability pimj if it is in the fatal area (otherwise it survives that salvo)		
28. General Solutions to Many-on-Many Heterogeneous Stochastic Combat, (Parkhideh S. and Gafarian A.V.), [Naval Research Logistics V43, 937-953], 1996.	❑ Subject: Development of general solutions to many-on-many heterogeneous stochastic combat. • Model: Continuous-time stochastic process (where combat is modeled as a sequence of aiming and killing events. • Force Composition: Heterogeneous. • Force Sizes: One-on-one (unit A vs. unit B) each having a number of combatants • Engagement Type: Directed for both sides • Termination: Combat terminates when any side reaches a predefined breakpoint. • Dependence / Independence ➢ Two opposing aiming and killing processes are independent • Process (Stochastic) ➢ Time advance: Continuous ➢ States: (t, A(t), B(t)) ➢ Parameters: - Time to kill (random variable) - Probabilities of aiming and killing events ➢ Distribution: - Time between consecutive kills are randomly distributed • Results (Computations): ➢ State probabilities are derived by - enumerating all possible routes that the	• Modeling of a combat as a sequence of aiming and killing events • Use of combinatorial analysis in combat modeling	• Use of different fire types (directed and area) • Use of other termination decision rules • Analysis of many-on-many combats (in terms of number of units) each having different attrition effects

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	combat may go through (defining sequences of aiming and killing events), - computing probabilities of aiming and killing events.		
29. Stochastic Modelling of combat with reinforcement, (Jaiswal N.K., Sangeeta Y. and Gaur S.C.), [European journal of Operations Research], 1997.	• Homogeneous combat with reinforcements has been modeled as a continuous-time discrete –state space Markov process. • The effect of reinforcement on combat characteristics are also studied.	• Two new concepts have been defined; - Reinforcement Effectiveness Index - Reinforcement Parity Curve • Reinforcements are assumed to made at prespecified force levels.	• Markov process is applied for directed fire only under absolute termination decision rule. Hence, - Other combat types (area and mixed) can also be studied - The effect of other termination decision rules can also be analyzed.
30. The Effects of Splitting Exponential Stochastic Lanchester Battles, (McNaught K.R.), [Journal of the Operational Research Society, V50, 244-254], 1999.	❑ Subject: The effects of applying Exponential Stochastic Lanchester (ESL- stochastic version of deterministic square law [directed fire]) to battles which have been split into smaller engagements (mini-battles) are explored. • Model Type: Markov Chain • Force Composition: Homogeneous • Force Sizes: many - on- many • Scenario: ➢ A two stage battle (l,1) is considered: 1 mini battles in 1st stage and one mini battle in 2nd stage. Outcome of the 1st stage is the income for 2nd stage ➢ 1st stage battle is split evenly (m1=m0/l	• Decomposition of a combat into mini-battles • Use of Monte Carlo simulation • Might's comment: 'The architecture of a model should separate those processes - The process that determine which engagements occur - The process that determine the outcome once they do occur'	• Use of other termination criteria (such as percent casualty and potential stopping criteria observed in real battles) • Increasing the number of stages • Use of non-exponential inter-firing times

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	and $n1=n0/l$ for each mini-battle) • Engagement Type: Blue (Directed) - Red (Directed) • Termination: Complete annihilation of one side (for each mini-battle) • Dependence / Independence ➢ Two opposing firing processes are independent • Process (Stochastic) ➢ Time advance: Discrete (at each kill) ➢ Distribution: - Inter-kill times: Exponential distribution (with λ) ➢ Kill parameters: individual kill rates (β and ρ), and total kill rate ($\lambda = m\beta + n\rho$) ➢ States: (m, n): # of blue and red at any time ➢ Max # of kills at each step(transition): 1 kill (either from blue or from red) ➢ k: cumulative number of kills ➢ Pk(m, n): bivariate distrib of the number of survivors after k kills. ($kmax= m0 + n0 - 1$) • Results (Computations) ➢ Probability of win (for blue and red) at each mini-battle ➢ Expected number of survivors (at each mini-battle) • Other Observations ➢ Increase in the number of mini-battles in 1st stage ➢ Change in the force ratio ➢ Random (uneven) split of battles using		

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	Monte Carlo simulation		
31. Markov Chain Models of One-on-One Combat (McNaught K.R.), [15th European Simulation Multiconference Proceedings, 280-289], (June 2001).	☐ Subject: Two variants of one-on-one duels are solved as continuous time Markov Chain models. • Model Type: Markov Chain • Force Composition: Homogeneous (tank – defender) • Force Sizes: One-on-one (combatant) • Scenario: S1: Blue defends in a fixed position – red attacks S2: Blue defends by changing its position after its third shot – red attacks • Engagement Type: Blue (Directed) - Red tank (Directed) • Dependence / Independence ➢ Two opposing firing processes are dependent on each other ➢ Each side must detect its opponent and then fire - Defender detects first easily by observing movement of attacker - Attacker detects defender only when defender fires • Process (Stochastic) ➢ Time advance: Continuous ➢ Distribution: - Inter-firing times (for both sides): 2-phase Erlang distribution in S1 - Inter-firing times (for both sides): Exponential distribution in S2 - Time for defender to detect attacker: Exponential	• Use of "first detect, and then fire" type of process • Dependence of opposing firing processes (violates the assumption that two firing processes are independent made in some analytical models) • Use of Erlang distribution • State definitions	• Combat including units having more than one combatants • Combat between heterogeneous forces

Title, Author, Year	*Major Points*	*Inspiration (what we can use)*	*Deficiencies (what else to be done)*
	- Time for change of position: Exponential (in S2) ➢ Kill parameter: SSKP for both ➢ States: - 9 states for S1 and 11 states for S2 ([both searching], [blue in first stage, red searching], …, [red wins])		
32. Suppression and the Theory of Stochastic Duels, (Armstrong M.J.), [Military Operations Research V6, N4, 31-41], 2001.	❑ Subject: A stochastic duel between two opposing units, in which both kills and suppression effects of firepower are possible, is formulated with the purpose of increasing the reality. • Model Type: Markov Chain • Force Composition: Homogeneous • Force Sizes: One-on-one (small unit or combatant) • Scenario: ➢ Each unit can be in one of three conditions; i) a: active (able to shoot) ii) s: suppressed (can not shoot until it recovers) iii) d: killed (destroyed) ➢ Each side has unlimited ammunition • Engagement Type: Blue (Directed) - Red (Directed) • Termination: Complete annihilation of one side (for each mini-battle) • Dependence / Independence ➢ Two opposing firing processes are dependent on each other (when one side is suppressed it cannot shoot until it recovers)	• Increasing the reality of the model by incorporating the moral and physical conditions of combatants • Computation of expected ammunition consumption	• Many-on-many type of engagements • Use of heterogeneous forces

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	• Process (Stochastic) ➢ Time advance: Continuous ➢ Distribution: - Inter-firing times: Exponential distribution ➢ Kill parameters: - individual rates of fire (b and r), - SSKP, probability of kill per shot (k, c). - probability of suppression per shot (s, z) - probability of miss per shot (m, n) - rate of recovery from suppression (R, B) ➢ States:(blue, red); (a,a), (a,s), (s,a), (a,d), (d,a) where (a,d), (d,a) are absorbing. • Results (Computations) ➢ Probability of win (for red) ➢ Expected duration of the duel ➢ Expected proportion of time that red is suppressed ➢ Expected number of rounds fired by red • Other Observations ➢ Benefit of surprise		
33. A Bayesian Stochastic Formulation of Lanchester Combat Theory, (Salim A.S.R. and Hamid W.M.), [Military Operations Research V6, N3, 69-	❑ Subject: A Bayesian stochastic model is formulated where beta distribution is chosen as a prior distribution for survivor probability. • Model Type: Bayesian • Force Composition: Homogeneous • Force Sizes: Many-on-many • Scenario: Blue tanks vs. red tanks (using simulated data)	• The concept of stochastic variable attrition rate coefficient (treatment of attrition rate coefficient in a time interval as a random variable). • Treatment of survivor probabilities as a random variable. • Use of Bayes method when data are scarce or data acquisition is prohibitive. • Update of information to make decisions based on prior information.	• Use of heterogeneous forces

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
76], 2001.	• Engagement Type: Blue (Directed) - Red (Directed) • Termination: Complete annihilation • Dependence / Independence ➢ Two opposing firing processes are independent • Process (Stochastic) ➢ Time advance: Continuous ➢ Distribution: - Number of survivors at time t: Binomial distribution ➢ Parameters: - survival probability - Initial number of units - Some physical/operational parameters indicating fighting power of a force • Results (Computations) ➢ The distribution of the number of survivors, P(Nt) ➢ Expected value of the attrition rate coefficient		
34. Modelling the Mobile Land Battle: Lanchester's Equations, Mini-Battle Formation and the Acquisition of Targets, (Speight L.R.), [Military Operations Research], V3, N5, 1997.	• This paper concentrates on the first two issues highlighted previously (in the first of the series of papers on modelling the mobile land battle): opportunity for engagement and target acquisition process. • Mini-battle formation and target acquisition, the evidence from trials and from battle records was first reviewed for each of the two selected combat factors. • The modelling issues were built into the "laboratory" environment of a very simple	• A Monte Carlo battle simulation was used as the "laboratory" tool where the emphasis is on mini-battle formation and target acquisition. • Target acquisition: ➢ The observer was taken to detect a target if, during a stationary dwell time or "glimpse", the eye could fixate within a "hard shelled" lobe centered on the target. ➢ "Single glimpse" probability of detection was taken as the ratio of the width of this lobe to that of the nominal search arc.	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	Monte Carlo simulation of a mobile armoured engagement.	• The aims of laboratory experiments: i) to judge whether Lanchester formulation could provide a description for the mobile engagement where the outcomes vary as a function of the opposing force strength, ii) to gauge how the parameters of such a function might vary as some major engagement conditions were altered. • Different sets of experimental conditions are defined and applied. • The strength of Lanchester approach lies in abstraction rather than detailed computation, hence it is sensible to assess its potential as a summarizing device at the tactical level. • A new state equation is obtained including the parameter set of a pseudo-Lanchester formulation. • This approach is used to fit different pseudo-Lanchester formulations to the experimental results by adopting a least-squares solution for each set of experimental solution. • Some conclusions are summarized as follows according to tactical issues and aggregation issues: i) The potential of Lanchester formulation as summary devices at the tactical/ operational level ii) The preferred form of pseudo-Lanchester Law iii) Estimating absolute levels of attrition and the course of action in time and space iv) Defeat criteria	
35. A Proposed Foundation for a Theory of Combat,	❑ Subject: Discussion of the status of combat theory. • He made a research on combat theory and	• Decomposition and analysis of a combat as a network of mini-battles	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
(Ancker C.J., Jr.), [Naval Research Logistics V42, 311-343], 1995.	revealed some deficiencies. • He proposed two axioms of combat to better understand combat and develop models: ➢ First axiom: "All combat is a hierarchical network of firefights." ➢ Second axiom: "A firefight is a terminating stochastic target attrition process on a discrete state space with a continuous time parameter." • Combat theorem (resulting from the two axioms) is stated as: "A combat network is a dynamic terminating stochastic target attrition process on a discrete state space with a continuous time parameter."		
36. Termination Decision Rules in Combat Attrition Models, (Jaiswal N.K. and Nagabhushana B.S.), [Warfare Modeling], 1995.	• Combat termination decision rules are defined depending on the casualties and the ratio of force strengths. • The effect of these termination rules on parity numbers (condition for Blue or Red winning) and the number of survivors at combat termination is analyzed using a generalized force strength equation , which satisfies a large class of attrition laws. • As a result of the analysis, it is pointed out that minimum number of survivors and the minimum force strength required to win the battle can possibly be determined given the following information. ➢ Type of attrition law ➢ Initial strengths of the opposing forces ➢ Attrition rate	• It is indicated that the situation where force sizes are grossly unequal is not taken into consideration in Lanchester Equations, and the approaches considering grossly unequal force sizes are reviewed. • The studies regarding the determination of combat termination rules are summarized. • Four termination rules are described: i) *Absolute decision rule (A):* The combat terminates when the force strength reaches a given threshold value. ii) *Proportional decision rule (P):* The combat terminates when the force ratio reaches a specified threshold value. iii) *AOP rule:* The combat terminates when the force strength curve crosses either the absolute or the proportional threshold lines.	• Attrition rates used in this paper are independent of spatial coordinates. The spatial effects can be incorporated into the model to obtain the generalized termination decision rules.

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
	➢ Decision rules ➢ Thresholds	iv) *AAP rule:* The combat terminates when the force strength curve crosses both the absolute and the proportional threshold lines. • The effect of termination decision rules on number of survivors and on parity number is developed for different decision rules for Blue and Red forces. • Combat duration (time taken by the force strength curve to reach decision line) for directed fire and area fire models of Lanchester is obtained for Blue and Red using *A* and *P* rules.	
37. Modelling the Mobile Land Battle: Combat Degradation and Criteria for Defeat, (Speight L.R. and Rowland D.), [Military Operations Research], V4, N3, 1999.	• This is the third paper in the series of modelling the mobile land battle • This paper provides an initial exploration and discussion of one alternative approach to the phenomena combat degradation and criteria for defeat. • Some evidence from trials and battle, regarding the combat degradation, on armoured warfare and infantry battle is presented. • A modelling approach, similar to the previous papers of this series, for combat degradation is introduced within Lanchester frame of reference.	• Combat Degradation: It is commonly accepted that military performance in actual battle tends to be significantly worse than that normally achieved in peacetime operations due to the following reasons. i) Task differences (different target acquisition conditions, motion characteristics, exposure times, etc.) ii) Deterioration of military skills (under stress and intense stimulation of war) iii) Active contribution to battle (refusal or inability to exercise the skills under direct warlike conditions) • Combat degradation effects are incorporated into a Monte Carlo simulation using the previously developed mini-battle formation and target acquisition. • Two postulates: i) There is a notional dimension of willingness or ability to contribute to the battle. Weapon crews are grouped into three classes: "heroes", "partially active" and "inactive" each having	

Title, Author, Year	Major Points	Inspiration (what we can use)	Deficiencies (what else to be done)
		different probabilities of responding appropriately when a target is acquired. ii) The parent populations of weapon crews of different armies are differently distributed along the notional 'ability' to contribute in battle dimension. • The battle simulation is run under 13 different experimental conditions for some selected population categories and the casualties as a function of initial force ratios are presented. • Two pseudo-Lanchester formulations are fitted to the simulation results using nonlinear least squares approximation. • Defeat Criteria: ➢ Lanchester formulations have nothing directly to say about the victory or defeat ➢ Summary of the data from 602 battles showed that the mean losses per day were about 4% of the mean initial strength for the attacker and *about 6% for the defender.* ➢ One overwhelming characteristics of success at the operational level: integrity of the opponent's defensive line had been fatally weakened (this can be accomplished by successful outranking manoeuvre or breakthrough). ➢ At the tactical level manoeuvre is again the key to the success. ➢ In the modelling approach to the determination of defeat at the tactical level, breakthrough success or failure in each run was assessed in terms of the "overwhelming strength advantage" in simulation trials.	

APPENDIX B

Column Generation Details

Here, we show that why we can not keep surplus variables in the basis, and hence can not obtain integer solutions, in our approach based on reformulation of ALLM as a column generation and decomposition problem. Recall that the alternative formulation of ALLM (to be solved as a column generation and decomposition problem) is as follows.

$$
\text{(P1):}\quad
\begin{aligned}
& Min\ z = c^T \sum_{l=1}^{L} \beta_l y^l \\
s.t.\quad & A_H \left(\sum_{l=1}^{L} \beta_l y^l \right) \geq \alpha \\
& \sum_{l=1}^{L} \beta_l = 1 \\
& \beta_l \geq 0 \qquad l = 1, \ldots, L
\end{aligned}
$$

where β_1, β_2, …, β_L are the weights (decision variables) satisfying $\beta_1+\beta_2+\ldots+\beta_L = 1$ and $\beta_l \geq 0$ for l=1,…,L. y^l is the vector of decision variables representing an allocation pattern which is, for example, defined as

$$y^l = [\,x_{11}^1 \quad x_{11}^2 \quad x_{12}^1 \quad x_{12}^2 \quad x_{21}^1 \quad x_{21}^2 \quad x_{21}^3 \quad x_{22}^1 \quad x_{22}^2 \quad x_{22}^3\,]^T$$

where

$$x_{ij}^{k_i} = \begin{cases} 1 & \text{if } k_i^{th} \text{ portion of blue unit } i, B_i, \text{ is allocated to red unit } j, R_j \\ 0 & \text{otherwise} \end{cases} \qquad k_i = 1, \ldots, K_i$$

The details of the formulation are given in Section 4.2.

Let y^1 be the initial pattern and B be the corresponding initial basis where initial basic variables are $BV=\{\beta_1, s_1,\ldots, s_J\}$. Recall that the initial basis matrix is in the following form.

$$B=\left[\begin{array}{c|c} A_H y^1 & -I \\ \hline 1 & 0\cdots0 \end{array}\right]_{(J+1)x(J+1)}$$

After some algebra, we see that the inverse of the initial basis matrix has the following structure,

$$B^{-1}=\left[\begin{array}{c|c} 0\cdots0 & 1 \\ \hline -I & A_H y^1 \end{array}\right]_{(J+1)x(J+1)}$$

Now, updated right hand side values (values of the initial basic variables) are computed as $\hat{d}=B^{-1}d$, where $d=[\alpha_1,\ldots,\alpha_1, 1]^T$ is the vector of right hand side values of (P1), as follows.

$$\hat{d}=B^{-1}d=\left[\begin{array}{c|c} 0\cdots0 & 1 \\ \hline -I & A_H y^1 \end{array}\right]\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_J \\ \hline 1\end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ \hline -\alpha_1+A_H(1,\bullet)y^1 \\ \vdots \\ -\alpha_J+A_H(J,\bullet)y^1 \end{bmatrix} \qquad (1)$$

where $A_H(j,\bullet)$ is the row of the coefficient matrix of hard constraints, A_H, corresponding to j^{th} attrition goal.

Suppose, in S2 of the algorithm, we obtained y^2 as the new (candidate) pattern, with weight β_2, to enter the basis and assume that its reduced cost is negative ($\hat{c}_l>0$).

Then the entering column is

$$D^2 = \begin{bmatrix} A_H y^2 \\ \hdashline 1 \end{bmatrix},$$

and the updated column of the basis is

$$\hat{B}_2 = B^{-1}D^2 = B^{-1}\begin{bmatrix} A_H y^2 \\ \hdashline 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \hdashline -A_H(1,\bullet)y^2 + A_H(1,\bullet)y^1 \\ \vdots \\ -A_H(J,\bullet)y^2 + A_H(J,\bullet)y^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \hdashline A_H(1,\bullet)(y^1 - y^2) \\ \vdots \\ A_H(J,\bullet)(y^1 - y^2) \end{bmatrix} \qquad (2)$$

Suppose that k^{th} basic variable, $BV(k)$, is the leaving variable. Now we face the problem of enforcing k=1, that is having decision variable β_1 as the leaving variable while keeping surplus variables $s_1,\ldots,s_J$ in the basis. To determine the leaving variable we apply the minimum ratio test, using Equations 1 and 2, and obtain the following ratios.

$$\frac{\hat{d}}{\hat{B}_2} = \begin{bmatrix} 1/1 \\ \hdashline \{-\alpha_1 + A_H(1,\bullet)y^1\}/\{A_H(1,\bullet)(y^1 - y^2)\} \\ \vdots \\ \{-\alpha_J + A_H(J,\bullet)y^1\}/\{A_H(J,\bullet)(y^1 - y^2)\} \end{bmatrix} = \begin{bmatrix} r_{\beta_1} \\ \hdashline r_{s_1} \\ \vdots \\ r_{s_J} \end{bmatrix} \qquad (3)$$

where r_{β_1} is the ratio corresponding to β_1, and r_{s_j} (j=1,…,J) is the ratio corresponding to surplus variable s_j. Here, in order to select β_1 as the leaving variable the ratio r_{β_1}=1 should be the minimum which requires that r_{s_j}>1 for j=1,…,J. That is,

$$\frac{\{-\alpha_j + A_H(j,\bullet)y^1\}}{\{A_H(j,\bullet)(y^1 - y^2)\}} > 1$$

$$\rightarrow \quad -\alpha_j + A_H(j,\bullet)y^1 > A_H(j,\bullet)y^1 - A_H(j,\bullet)y^2$$

$$\rightarrow \quad A_H(j,\bullet)y^2 > \alpha_j \qquad j=1,\ldots,J \tag{4}$$

In order to keep surplus variables in the basis, the condition defined by Equation 4 should be satisfied for all ratios given by Equation 3. It means that the entering pattern y^2 should be feasible (that is, y^2 must satisfy all attrition goals). Hence, to obtain an integer solution at the end, this condition needs to be satisfied at all subsequent iterations meaning that every entering pattern should be feasible. However, our column generation process does not guarantee this due to the reasons addressed below.

At every iteration we solve I independent linear programming problems where problem $i=1,\ldots,I$ corresponds to the allocation of a blue unit.

$$(LP_i): \quad \begin{aligned} \gamma_i &= \min \ (c^T - \hat{w}^T A_H)_i \, y_i^l = \sum_{k_i=1}^{K_i} (c^T - \hat{w}^T A_H)_{i,k_i} \, x_{i,j}^{k_i} \\ s.t. \quad & y_i^l \in P_i \end{aligned} \qquad i=1,\ldots,I \tag{5}$$

where constraints of (LP_i) assures only that any portion of B_i can be allocated to at most one red unit. That is, they do not consider the attrition goals (α_j values) of red units. Next, we combine the vectors obtained from each subproblem and obtain the overall allocation pattern as $y^l = y_1^l + \cdots + y_I^l$.

Since each LP (LP_i, $i=1,\ldots,I$) handles the allocation of a blue unit without taking the attrition goals into consideration we may face one of the following situations, regarding the overall solution (pattern y^l), at each iteration.

(i) y^l does not satisfy any of the attrition goals or satisfies only some of them. In either case, the solution is infeasible.

(ii) y^l satisfies all attrition goals meaning that the solution is feasible.

In almost all of the problems we tried, we face with the situation described in case (i). When objective function coefficients of surplus variables are zero (which is the normal case) we had a null vector, as the entering pattern, at first iteration where none of the attrition goals are satisfied. Then, we assigned different values as objective function coefficients of surplus variables and observed that for some values we obtained patterns satisfying only some of the attrition goals.

To sum up, our column generation and decomposition approach does not guarantee case (ii) and gives fractional solutions in almost all cases. Hence, we developed a heuristic to obtain integer solutions based on these fractional values as explained in Section 4.3.

APPENDIX C

LM and DSM Integration Example

We solve the example problem given in Figure 5.16 for first ten salvos. For LM computation salvo duration is set as d_s=5Δt where Δt =0.2 is the time length within which at most one kill may occur. That is, the number of R_2 combatants who are alive at the end of a salvo is computed running LM five times. Assuming that b_{12}=0.08, we make the following computations for first salvo where initially two B_1 combatants fire at R_2.

$$j_2(t_1) = j_2(0) - b_{12}\,[\tfrac{2}{3}\,\bar{i}_1(0)\,]\,j_2(0)\Delta t = 30-0.08(2)(30)(0.2) = 29.04$$

$$j_2(t_2) = j_2(t_1) - b_{12}\,[\tfrac{2}{3}\,\bar{i}_1(t_1)\,]j_2(t_1)\Delta t = 29.04-0.08(2)(29.04)(0.2) = 28.1107$$

$$j_2(t_3) = j_2(t_2) - b_{12}[\tfrac{2}{3}\,\bar{i}_1(t_2)\,]j_2(t_2)\Delta t = 28.1107-0.08(2)(28.1107)(0.2)= 27.2112$$

$$j_2(t_4) = j_2(t_3) - b_{12}[\tfrac{2}{3}\,\bar{i}_1(t_3)\,]j_2(t_3)\Delta t = 27.2112-0.08(2)(27.2112)(0.2) = 26.3404$$

$$j_2(t_5) = j_2(t_4) - b_{12}[\tfrac{2}{3}\,\bar{i}_1(t_4)\,]j_2(t_4)\Delta t = 26.3404-0.08(2)(26.3404)(0.2) = 25.4975$$

where t_k= $k\Delta t$, k=1,…, 5. Hence 25.5 R_2 combatants survive at the end of first salvo. We need the following additional assumptions.

A4) Number of B_1 combatants (which is three at the beginning) does not change within the duration of 5Δt.

A5) Also, R_2 is assumed to keep its force size (available at the beginning of salvo) throughout the salvo.

Recall that we can handle the attrition effect of R_2 as if B_1 is subject to an additional noncombat loss rate as explained in Section 5.3.1. Due to this approach the existence of a R_2 (which is a large unit) does not increase the state space of DSM. Letting the state definition be (t, i_1, i_2, j_1) where $i_1 = 0, 1, 2, 3$, $i_2 = 0, 1, 2$ and $j_1 = 0, 1, 2, 3$ be the number of combatants remaining in B_1, B_2 and R_1 respectively at the end of salvo t we have 48 states as in Section 5.1.

Initial Computations: Combat Loss Transition Probabilities for DSM

Since SSKPs are kept constant within a combat stage transition probabilities due to combat losses do not change throughout salvos. Hence we first compute and store these probabilities to ease the computations since they will be used in each salvo computation later. The data that we use throughout all computations are given in Table 1.

Table 1 Problem data

Unit	Initial Num.	Eng. Prob.	NC loss Prob.	Radius		Alloc. fraction to				SSKP against			ARC to	Attrition Process
				Fatal (r)	Pos. (R)	B_1	B_2	R_1	R_2	B_1	B_2	R_1	R_2	
B_1	3	1	0.05	-	-	-	-	1/3	2/3	-	-	0.2	0.08	DSM(D)+LM
B_2	2	0.5	0.001	40	-	-	-	1	-	-	-	0.1	-	DSM(D)
R_1	3	0.8	0.08	-	100	1/2	1/2	-	-	0.3	0.05	-	-	DSM(D+A)
R_2	30	1	0.02	-	-	1	-	-	-	0.001	-	-	-	LM(A)

Initially, we compute the probabilities of casualties (combat losses) for B_1, B_2 and R_1 using DSM under specified allocation fractions. As given in Table 1, B_1 allocates 1/3 of its combatants to R_1 whereas all B_2 combatants are allocated to R_1. R_1 allocates its combatants to B_1 and B_2 evenly. B_1 and B_2 use directed and area fire respectively against R_1 which employs directed fire against both B_1 and B_2. Example calculations of combat loss probabilities, for R_1 are given in Table 2 for states $(t,2,i_2,j_1)$.

Table 2 Combat loss probabilities of R_1 using allocation fractions

i_1	i_2	j_1	i_{11}= 0 (with probability 0.333)				i_{11}= 1 (with probability 0.667)				*Average Probabilities*			
			P_0(0 kill)	P_0(1 kill)	P_0(2 kill)	P_0(3 kill)	P_1(0 kill)	P_1(1 kill)	P_1(2 kill)	P_1(3 kill)	P(0 kill)	P(1 kill)	P(2 kill)	P(3 kill)
2	2	3	0.90776	0.08928	0.00293	0.00003	0.72621	0.25893	0.01464	0.00023	0.78673	0.20238	0.01073	0.00016
2	2	2	0.93752	0.06164	0.00084	0.00000	0.75002	0.24298	0.00701	0.00000	0.81252	0.18253	0.00495	0.00000
2	2	1	0.96826	0.03174	0.00000	0.00000	0.77461	0.22540	0.00000	0.00000	0.83916	0.16085	0.00000	0.00000
2	2	0	1.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
2	1	3	0.95276	0.04648	0.00076	0.00000	0.76221	0.23083	0.00690	0.00005	0.82573	0.16938	0.00485	0.00004
2	1	2	0.96826	0.03149	0.00026	0.00000	0.77461	0.22199	0.00341	0.00000	0.83916	0.15849	0.00236	0.00000
2	1	1	0.98400	0.01600	0.00000	0.00000	0.78720	0.21280	0.00000	0.00000	0.85280	0.14720	0.00000	0.00000
2	1	0	1.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
2	0	3	1.00000	0.00000	0.00000	0.00000	0.80000	0.20000	0.00000	0.00000	0.86667	0.13333	0.00000	0.00000
2	0	2	1.00000	0.00000	0.00000	0.00000	0.80000	0.20000	0.00000	0.00000	0.86667	0.13333	0.00000	0.00000
2	0	1	1.00000	0.00000	0.00000	0.00000	0.80000	0.20000	0.00000	0.00000	0.86667	0.13333	0.00000	0.00000
2	0	0	1.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000

$P_0(l$ kill) is the probability of l kills from R_1 when, in addition to area firing B_2 combatants, none of the two B_1 firers are allocated to R_1 whereas $P_1(l$ kill) denotes the same probability when one B_1 combatant fires at R_1 together with B_2 combatants. These basic probabilities are computed using the processes described for directed fire and area fire in Section 5.1. In Table 2, for example, the probability of no R_1 kill when i_2=2 and j_1=3 is computed as

$$P(0 \text{ kill}) = 1/3P_0(0 \text{ kill}) + 2/3\, P_1(0 \text{ kill}) = 1/3(0.90776) + 2/3\,(0.72621)$$

$$= 0.78673.$$

Similar computations are performed to obtained kill probabilities of all three units.

Then we update these kill probabilities by incorporating the engagement probabilities where engagement probabilities are 1, 0.5 and 0.8 for B_1, B_2 and R_1 respectively. These updated kill probabilities for R_1 are given in Table 3 for states $(t,2,i_2,j_1)$.

Table 3 Kill probabilities of R_1 after engagement

i_1	i_2	j_1	**P(0 kill)**	**P(1 kill)**	**P(2 kill)**	**P(3 kill)**
2	2	3	0.82621	0.16862	0.00511	0.00006
2	2	2	0.83937	0.15821	0.00242	0.00000
2	2	1	0.85286	0.14715	0.00000	0.00000
2	2	0	1.00000	0.00000	0.00000	0.00000
2	1	3	0.84620	0.15136	0.00243	0.00002
2	1	2	0.85291	0.14591	0.00118	0.00000
2	1	1	0.85973	0.14027	0.00000	0.00000
2	1	0	1.00000	0.00000	0.00000	0.00000
2	0	3	0.86667	0.13333	0.00000	0.00000
2	0	2	0.86667	0.13333	0.00000	0.00000
2	0	1	0.86667	0.13333	0.00000	0.00000
2	0	0	1.00000	0.00000	0.00000	0.00000

After completing the above computations we obtain the overall transition matrix (say *to go* matrix) consisting of transition probabilities due to combat losses. Partial entries of this matrix are given in Table 4.

Finally, we compute the transition probabilities due to noncombat losses only and yielding the transition matrix given in Table 5. Noncombat loss probabilities of B_1, B_2 and R_1 are taken as 0.05, 0.001 and 0.08 respectively.

Table 4 State transition probabilities due to combat losses (*to go* matrix)

TO \ FROM	(t,3,2,3)	(t,3,2,2)	(t,3,2,1)	(t,3,2,0)	(t,3,1,3)	(t,3,1,2)	(t,3,1,1)	(t,3,1,0)	(t,3,0,3)	(t,3,0,2)	(t,3,0,1)	(t,3,0,0)
(t,3,2,3)	0.47566											
(t,3,2,2)	0.14354	0.56530										
(t,3,2,1)	0.00443	0.16178	0.67893									
(t,3,2,0)	0.00005	0.00252	0.18347	1.00000								
(t,3,1,3)	0.02969	0.00000	0.00000	0.00000	0.48716							
(t,3,1,2)	0.00896	0.02355	0.00000	0.00000	0.13435	0.57442						
(t,3,1,1)	0.00028	0.00674	0.01386	0.00000	0.00215	0.15394	0.68440					
(t,3,1,0)	0.00000	0.00011	0.00374	0.00000	0.00002	0.00124	0.17800	1.00000				
(t,3,0,3)	0.00032	0.00000	0.00000	0.00000	0.03074	0.00000	0.00000	0.00000	0.53043			
(t,3,0,2)	0.00010	0.00000	0.00000	0.00000	0.00848	0.02393	0.00000	0.00000	0.13261	0.60800		
(t,3,0,1)	0.00000	0.00000	0.00000	0.00000	0.00014	0.00641	0.01397	0.00000	0.00000	0.15200	0.70400	
(t,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00005	0.00363	0.00000	0.00000	0.00000	0.17600	1.00000

Table 5 State transition probabilities due to noncombat losses

TO \ FROM	(t,3,2,3)	(t,3,2,2)	(t,3,2,1)	(t,3,2,0)	(t,3,1,3)	(t,3,1,2)	(t,3,1,1)	(t,3,1,0)	(t,3,0,3)	(t,3,0,2)	(t,3,0,1)	(t,3,0,0)
(t,3,2,3)	0.66629											
(t,3,2,2)	0.17382	0.72423										
(t,3,2,1)	0.01511	0.12595	0.78721									
(t,3,2,0)	0.00044	0.00548	0.06845	0.85566								
(t,3,1,3)	0.00133	0.00000	0.00000	0.00000	0.66696							
(t,3,1,2)	0.00035	0.00145	0.00000	0.00000	0.17399	0.72496						
(t,3,1,1)	0.00003	0.00025	0.00158	0.00000	0.01513	0.12608	0.78800					
(t,3,1,0)	0.00000	0.00001	0.00014	0.00171	0.00044	0.00548	0.06852	0.85652				
(t,3,0,3)	0.00000	0.00000	0.00000	0.00000	0.00067	0.00000	0.00000	0.00000	0.66763			
(t,3,0,2)	0.00000	0.00000	0.00000	0.00000	0.00017	0.00073	0.00000	0.00000	0.17416	0.72568		
(t,3,0,1)	0.00000	0.00000	0.00000	0.00000	0.00002	0.00013	0.00079	0.00000	0.01514	0.12621	0.78879	
(t,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00007	0.00086	0.00044	0.00549	0.06859	0.85738

Tables 4 and 5 include constant data that are needed to be computed and stored once. Then they will be used throughout salvo calculations since SSKPs and noncombat loss probabilities are assumed to remain constant during a combat stage.

Attrition of B_1 due to fires of R_2 is treated as an additional noncombat loss process (LM combat) and computed after combat loss process of DSM combat. Transition probabilities due to LM combat is given in Table 6 for first salvo where additional noncombat loss probability of B_1 is $q_{B_1} = 1-(1-0.001)^{30/3} = 0.009955$.

Table 6 State transition probabilities due to LM combat (additional noncombat loss of B_1) for the first salvo

TO \ FROM	(1,3,2,3)	(1,3,2,2)	(1,3,2,1)	(1,3,2,0)	(1,3,1,3)	(1,3,1,2)	(1,3,1,1)	(1,3,1,0)	(1,3,0,3)	(1,3,0,2)	(1,3,0,1)	(1,3,0,0)
(1,3,2,3)	0.97043											
(1,3,2,2)	0.00000	0.97043										
(1,3,2,1)	0.00000	0.00000	0.97043									
(1,3,2,0)	0.00000	0.00000	0.00000	0.97043								
(1,3,1,3)	0.00000	0.00000	0.00000	0.00000	0.97043							
(1,3,1,2)	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043						
(1,3,1,1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043					
(1,3,1,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043				
(1,3,0,3)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043			
(1,3,0,2)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043		
(1,3,0,1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043	
(1,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97043

Now sequential computations for the first salvo are done and results are provided in Table 7. Since this is the first salvo initial state probabilities are 1 for state (3, 2, 3) and zero for other states as seen in the first row of Table 7 (a). Transition probabilities from state (i_1, i_2, j_1) due to combat losses are computed by multiplying the initial probability of this state with the probabilities corresponding to transitions from state (i_1, i_2, j_1) to other states which are available in Table 4 (to go matrix). For example;

$$P((3, 2, 3) \rightarrow (3, 2, 3)) = (1.000)(0.475656) = 0.475656$$
$$P((3, 2, 3) \rightarrow (3, 2, 2)) = (1.000)(0.143539) = 0.143539.$$

Table 7 State probabilities at the end of first salvo

(a) State transitions after combat losses

initial state probabilities →		1.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	state prob.	FROM											
TO	$P(1,i_1,i_2,j_1)$	(1,3,2,3)	(1,3,2,2)	(1,3,2,1)	(1,3,2,0)	(1,3,1,3)	(1,3,1,2)	(1,3,1,1)	(1,3,1,0)	(1,3,0,3)	(1,3,0,2)	(1,3,0,1)	(1,3,0,0)
(1,3,2,3)	0.47566	0.47566	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,2)	0.14354	0.14354	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,1)	0.00443	0.00443	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,0)	0.00005	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,3)	0.02969	0.02969	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,2)	0.00896	0.00896	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,1)	0.00028	0.00028	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,3)	0.00032	0.00032	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,2)	0.00010	0.00010	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

(b) State transitions after (additional) noncombat losses (LM combat)

state probabilities after combat loss →		0.47566	0.14354	0.00443	0.00005	0.02969	0.00896	0.00028	0.00000	0.00032	0.00010	0.00000	0.00000
	state prob.	FROM											
TO	$P(1,i_1,i_2,j_1)$	(1,3,2,3)	(1,3,2,2)	(1,3,2,1)	(1,3,2,0)	(1,3,1,3)	(1,3,1,2)	(1,3,1,1)	(1,3,1,0)	(1,3,0,3)	(1,3,0,2)	(1,3,0,1)	(1,3,0,0)
(1,3,2,3)	0.46159	0.46159	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,2)	0.13930	0.00000	0.13930	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,1)	0.00430	0.00000	0.00000	0.00430	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,0)	0.00005	0.00000	0.00000	0.00000	0.00005	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,3)	0.02882	0.00000	0.00000	0.00000	0.00000	0.02882	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,2)	0.00870	0.00000	0.00000	0.00000	0.00000	0.00000	0.00870	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,1)	0.00027	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00027	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,3)	0.00031	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00031	0.00000	0.00000	0.00000
(1,3,0,2)	0.00009	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00009	0.00000	0.00000
(1,3,0,1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

(c) State transitions after noncombat losses

state probabilities after LM combat →		0.46159	0.13930	0.00430	0.00005	0.02882	0.00870	0.00027	0.00000	0.00031	0.00009	0.00000	0.00000
	state prob.	FROM											
TO	$P(1,i_1,i_2,j_1)$	(1,3,2,3)	(1,3,2,2)	(1,3,2,1)	(1,3,2,0)	(1,3,1,3)	(1,3,1,2)	(1,3,1,1)	(1,3,1,0)	(1,3,0,3)	(1,3,0,2)	(1,3,0,1)	(1,3,0,0)
(1,3,2,3)	0.30755	0.30755	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,2)	0.18111	0.08023	0.10088	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,1)	0.02791	0.00698	0.01754	0.00339	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,2,0)	0.00130	0.00020	0.00076	0.00029	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,3)	0.01983	0.00062	0.00000	0.00000	0.00000	0.01922	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,2)	0.01168	0.00016	0.00020	0.00000	0.00000	0.00501	0.00630	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,1)	0.00180	0.00001	0.00004	0.00001	0.00000	0.00044	0.00110	0.00021	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,1,0)	0.00008	0.00000	0.00000	0.00000	0.00000	0.00001	0.00005	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000
(1,3,0,3)	0.00023	0.00000	0.00000	0.00000	0.00000	0.00002	0.00000	0.00000	0.00000	0.00021	0.00000	0.00000	0.00000
(1,3,0,2)	0.00013	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00000	0.00000	0.00005	0.00007	0.00000	0.00000
(1,3,0,1)	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000
(1,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Then the probability of being in a state after combat losses, which are the probabilities in $P(1, i_1, i_2, j_1)$ column, is obtained by summing the transition probabilities in the row corresponding to that state. Since this is the first salvo only transitions from state (1, 3, 2, 3) to other states are possible. These computations complete the process (I) of Figure 5.20.

As to process (II) of Figure 5.20, calculation of transition and state probabilities due to LM combat (which incurs an additional noncombat loss to B_1 only), is similar to the above process and given in Table 7 (b). Here state probabilities obtained after combat losses are used as the initial state probabilities for LM combat. To compute transition probabilities from state $(1, i_1, i_2, j_1)$ due to LM combat we multiply the initial probability of this state with the transition probabilities from state $(1, i_1, i_2, j_1)$ which are available in Table 6. Transition probabilities due to LM combat for first salvo, as illustrated below.

$P((1, 3, 2, 3) \rightarrow (1, 3, 2, 3)) = (0.475656)(0.970431) = 0.461591$

$P((1, 3, 2, 3) \rightarrow (1, 3, 2, 2)) = (0.475656)(0.000000) = 0.000000$

Hence, probability of being in a certain state after LM combat, which are the probabilities in $P(1, i_1, i_2, j_1)$ column of Table 7 (b), is obtained by summing the transition probabilities in that row.

Then we carry out similar computations for process (III) of Figure 5.20. We use state probabilities obtained at the end of previous process (LM combat) as our initial state probabilities for noncombat loss process and multiply each of them with corresponding transition probabilities given in Table 5.18. Two examples of these calculations are,

$P((1, 3, 2, 3) \rightarrow (1, 3, 2, 3)) = (0.461591)(0.666293) = 0.307555$

$P((1, 3, 2, 3) \rightarrow (1,3, 2, 2)) = (0.461591)(0.173816) = 0.080232$

Finally we compute the state probabilities after noncombat loss by summing the transition probabilities in each row. These are the probabilities given in $P(1, i_1, i_2, j_1)$ column of Table 7 (c) where, at the same time, these are the overall state probabilities at the end of first salvo including all attrition mechanism (combat loss, LM combat and noncombat loss). For example, probabilities of being in states (1, 3, 2, 3), (1, 3, 2, 2) and (1, 3, 1, 1) at the end of first salvo are 0.307555, 0.181114 and 0.001800 respectively.

Then we calculate the expected values and variances of force levels at the end of first salvo as shown in Table 8 where (•) indicates summation of the state probabilities, i.e. probabilities in $P(1, i_1, i_2, j_1)$ column of Table 7 (c), over corresponding index. Here, for example, $P(1,3,\bullet,\bullet) = 0.551665$ is the sum of the entire probabilities since these are all the states where i_1=3. Therefore, expected numbers of B_1, B_2 and R_1 combatants at the end of first salvo are 2.493868, 1.93806 and 2.534949 respectively.

Table 8 Expected values and variances at the end of first salvo

***B*₁**				***B*₂**				***R*₁**			
i_1	$P(i_1,\bullet,\bullet)$	$i_1 \cdot P(i_1,\bullet,\bullet)$	$i_1^2 \cdot P(i_1,\bullet,\bullet)$	i_2	$P(\bullet, i_2, \bullet)$	$i_2 \cdot P(\bullet, i_2,\bullet)$	$i_2^2 \cdot P(\bullet, i_2,\bullet)$	j_1	$P(\bullet,\bullet,j_1)$	$j_1 \cdot P(\bullet,\bullet,j_1)$	$j_1^2 \cdot P(\bullet,\bullet,j_1)$
3	0.551665	1.654994	4.964982	2	0.938760	1.877519	3.755039	3	0.593872	1.781616	5.344849
2	0.392513	0.785026	1.570052	1	0.060541	0.060541	0.060541	2	0.349721	0.699442	1.398883
1	0.053847	0.053847	0.053847	0	0.000700	0.000000	0.000000	1	0.053891	0.053891	0.053891
0	0.001975	0.000000	0.000000					0	0.002516	0.000000	0.000000
Total	1.000000	2.493868	6.588882	Total	1.000000	1.938060	3.815579	Total	1.000000	2.534949	6.797623
$E_1[i_1]$	**2.493868**			$E_1[i_2]$	**1.938060**			$E_1[j_1]$	**2.534949**		
$V[i_1]$	**0.369506**			$V[i_2]$	**0.059503**			$V[j_1]$	**0.371658**		

As to the expected value of R_2 combatants, we make LM computations for area fire as explained before. Since initially R_2 is subject to area fires of 2 B_1 combatants, where the attrition coefficient of B_1 against R_2 is b_{12} = 0.08, we obtain the results given in Table 8 regarding the attrition of R_2. We assume that noncombat loss probability of R_2 is 0.02.

Table 9 LM computations for R_2 during first salvo

Beginning of Salvo 1		Expected number of R_2 combatants (j_2)					
						End of Salvo 1	
(2/3) i_1	j_2	1 $\Delta t = 0.2$	2 $\Delta t = 0.2$	3 $\Delta t = 0.2$	4 $\Delta t = 0.2$	**5** $\Delta t = 0.2$	**After noncombat loss**
2	30	29.0400	28.1107	27.2112	26.3404	**25.4975**	**24.9876**

As seen in Table 9, LM computations yield an expected value of 25.4975. Then we incorporate noncombat losses by simply reducing this value by noncombat loss rate. Hence expected number of R_2 combatants at the end of first salvo is (25.4975)(1−0.02) = 24.9876.

To make computations for second salvo we first recalculate the transition probabilities for LM combat updating additional noncombat loss probability of B_1 due to fires of R_2. Since the expected values of B_1 and R_2 are 2.493868 and 24.9876 respectively, then additional noncombat loss probability of B_1 is $q_{B_1} = 1-(1-0.001)^{(24.9876/2.493868)} = 0.00997$. Then transition probabilities due to LM combat can be calculated as given in Table 10.

Table 10 State transition probabilities due to LM combat (additional noncombat loss of B_1) for second salvo

	FROM											
TO	(1,3,2,3)	(1,3,2,2)	(1,3,2,1)	(1,3,2,0)	(1,3,1,3)	(1,3,1,2)	(1,3,1,1)	(1,3,1,0)	(1,3,0,3)	(1,3,0,2)	(1,3,0,1)	(1,3,0,0)
(1,3,2,3)	0.97037											
(1,3,2,2)	0.00000	0.97037										
(1,3,2,1)	0.00000	0.00000	0.97037									
(1,3,2,0)	0.00000	0.00000	0.00000	0.97037								
(1,3,1,3)	0.00000	0.00000	0.00000	0.00000	0.97037							
(1,3,1,2)	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037						
(1,3,1,1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037					
(1,3,1,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037				
(1,3,0,3)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037			
(1,3,0,2)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037		
(1,3,0,1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037	
(1,3,0,0)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.97037

Now we can make computations for second salvo in the same way described for the first salvo. Since this is second salvo, initial state probabilities for this salvo are the final state probabilities obtained in first salvo, i.e., the probabilities in the first column ($P(1, i_1, i_2, j_1)$ column) of Table 7 (c). Then we perform exactly the same sequence of operations explained in the first salvo and obtain the results given in Table 11. As seen in the first column ($P(2, i_1, i_2, j_1)$ column) of Table 11 (c), for example, overall probabilities of being in states (2, 3, 2, 3) and (2, 3, 2, 2) at the end of second salvo are 0.094584 and 0.127652 respectively which are lower than the values obtained at the end of first salvo, whereas the overall probability of state (2, 3, 1, 1) is 0.006042 which is higher than the overall probability of same state computed at the end of first salvo first salvo ((1, 3, 1, 1) = 0.001800.

Table 11 State probabilities at the end of second salvo

(a) State transitions after combat losses

initial state probabilities →		0.30755	0.18111	0.02791	0.00130	0.01983	0.01168	0.00180	0.00008	0.00023	0.00013	0.00002	0.00000
	state prob.	FROM											
TO	$P(1,i_1,i_2,j_1)$	(2,3,2,3)	(2,3,2,2)	(2,3,2,1)	(2,3,2,0)	(2,3,1,3)	(2,3,1,2)	(2,3,1,1)	(2,3,1,0)	(2,3,0,3)	(2,3,0,2)	(2,3,0,1)	(2,3,0,0)
(2,3,2,3)	0.14629	0.14629	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,2)	0.14653	0.04415	0.10238	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,1)	0.04961	0.00136	0.02930	0.01895	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,0)	0.00690	0.00002	0.00046	0.00512	0.00130	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,3)	0.01879	0.00913	0.00000	0.00000	0.00000	0.00966	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,2)	0.01640	0.00276	0.00427	0.00000	0.00000	0.00266	0.00671	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,1)	0.00477	0.00009	0.00122	0.00039	0.00000	0.00004	0.00180	0.00123	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,0)	0.00054	0.00000	0.00002	0.00010	0.00000	0.00000	0.00001	0.00032	0.00008	0.00000	0.00000	0.00000	0.00000
(2,3,0,3)	0.00083	0.00010	0.00000	0.00000	0.00000	0.00061	0.00000	0.00000	0.00000	0.00012	0.00000	0.00000	0.00000
(2,3,0,2)	0.00059	0.00003	0.00000	0.00000	0.00000	0.00017	0.00028	0.00000	0.00000	0.00003	0.00008	0.00000	0.00000
(2,3,0,1)	0.00014	0.00000	0.00000	0.00000	0.00000	0.00000	0.00007	0.00003	0.00000	0.00000	0.00002	0.00001	0.00000
(2,3,0,0)	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000

(b) State transitions after (additional) noncombat losses (LM combat)

state probabilities after combat loss →		0.14629	0.14653	0.04961	0.00690	0.01879	0.01640	0.00477	0.00054	0.00083	0.00059	0.00014	0.00001
	state prob.	FROM											
TO	$P(1,i_1,i_2,j_1)$	(2,3,2,3)	(2,3,2,2)	(2,3,2,1)	(2,3,2,0)	(2,3,1,3)	(2,3,1,2)	(2,3,1,1)	(2,3,1,0)	(2,3,0,3)	(2,3,0,2)	(2,3,0,1)	(2,3,0,0)
(2,3,2,3)	0.14196	0.14196	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,2)	0.14219	0.00000	0.14219	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,1)	0.04814	0.00000	0.00000	0.04814	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,0)	0.00669	0.00000	0.00000	0.00000	0.00669	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,3)	0.01824	0.00000	0.00000	0.00000	0.00000	0.01824	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,2)	0.01591	0.00000	0.00000	0.00000	0.00000	0.00000	0.01591	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,1)	0.00462	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00462	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,0)	0.00053	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00053	0.00000	0.00000	0.00000	0.00000
(2,3,0,3)	0.00081	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00081	0.00000	0.00000	0.00000
(2,3,0,2)	0.00057	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00057	0.00000	0.00000
(2,3,0,1)	0.00013	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00013	0.00000
(2,3,0,0)	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001

(c) State transitions after noncombat losses

state probabilities after LM combat →		0.14196	0.14219	0.04814	0.00669	0.01824	0.01591	0.00462	0.00053	0.00081	0.00057	0.00013	0.00001
	state prob.	FROM											
TO	$P(1,i_1,i_2,j_1)$	(2,3,2,3)	(2,3,2,2)	(2,3,2,1)	(2,3,2,0)	(2,3,1,3)	(2,3,1,2)	(2,3,1,1)	(2,3,1,0)	(2,3,0,3)	(2,3,0,2)	(2,3,0,1)	(2,3,0,0)
(2,3,2,3)	0.09458	0.09458	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,2)	0.12765	0.02467	0.10298	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,1)	0.05795	0.00215	0.01791	0.03790	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,2,0)	0.00986	0.00006	0.00078	0.00330	0.00573	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,3)	0.01235	0.00019	0.00000	0.00000	0.00000	0.01216	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,2)	0.01496	0.00005	0.00021	0.00000	0.00000	0.00317	0.01153	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,1)	0.00604	0.00000	0.00004	0.00008	0.00000	0.00028	0.00201	0.00364	0.00000	0.00000	0.00000	0.00000	0.00000
(2,3,1,0)	0.00088	0.00000	0.00000	0.00001	0.00001	0.00001	0.00009	0.00032	0.00045	0.00000	0.00000	0.00000	0.00000
(2,3,0,3)	0.00055	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00000	0.00054	0.00000	0.00000	0.00000
(2,3,0,2)	0.00057	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00014	0.00042	0.00000	0.00000
(2,3,0,1)	0.00020	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00007	0.00011	0.00000
(2,3,0,0)	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001

Then we again calculate the expected values and variances of force levels at the end of second salvo as shown in Table 12 where now the expected numbers of B_1, B_2 and R_1 combatants are 2.067616, 1.885532 and 2.145930 respectively.

Table 12 Expected value and variances at the end of second salvo

B_1				B_2				R_1			
i_1	$P(i_1,\bullet,\bullet)$	$i_1 \cdot P(i_1,\bullet,\bullet)$	$i_1^2 \cdot P(i_1,\bullet,\bullet)$	i_2	$P(\bullet, i_2, \bullet)$	$i_2 \cdot P(\bullet, i_2,\bullet)$	$i_2^2 \cdot P(\bullet, i_2,\bullet)$	j_1	$P(\bullet,\bullet,j_1)$	$j_1 \cdot P(\bullet,\bullet,j_1)$	$j_1^2 \cdot P(\bullet,\bullet,j_1)$
3	0.325635	0.976904	2.930713	2	0.889740	1.779480	3.558961	3	0.368110	1.104331	3.312994
2	0.446548	0.893097	1.786193	1	0.106052	0.106052	0.106052	2	0.434819	0.869638	1.739275
1	0.197615	0.197615	0.197615	0	0.004208	0.000000	0.000000	1	0.171961	0.171961	0.171961
0	0.030202	0.000000	0.000000					0	0.025110	0.000000	0.000000
Total	1.000000	2.067616	4.914521	Total	1.000000	1.885532	3.665012	Total	1.000000	2.145930	5.224230
$E_2[i_1]$	**2.067616**			**$E_2[i_2]$**	**1.885532**			**$E_2[j_1]$**	**2.145930**		
$V[i_1]$	**0.639484**			**$V[i_2]$**	**0.109781**			**$V[j_1]$**	**0.619215**		

Finally the calculations for the attrition of R_2 are provided in Table 13 where the expected number of R_2 combatants surviving at the end of first salvo is 24.9876 (see Table 9). Since the expected number of B_1 combatants at the end of first salvo is $E_1[i_1] = 2.493868$ (from Table 8) then the expected number of B_1 combatants firing at R_2 is (2/3) $E_1[i_1]$ = (2/3) 2.493868 = 1.6626 at second salvo. Hence, the expected number of R_2 combatants is found as 21.3995 at the end of second salvo as seen in Table 13.

Table 13 LM computations for R_2 during second salvo

Beginning of Salvo 2		Expected number of R_2 combatants (j_2)					
						End of Salvo 2	
(2/3) i_1	j_2	1 $\Delta t = 0.2$	2 $\Delta t = 0.2$	3 $\Delta t = 0.2$	4 $\Delta t = 0.2$	**5** **$\Delta t = 0.2$**	**After noncombat loss**
1.6626	24.9876	24.3229	23.6759	23.0460	22.4330	**21.8362**	**21.3995**

Computations for future salvos can be performed using the above procedure, which is illustrated for first and second salvos, where the results for the first ten salvos are provided in Section 5.3.3.

CURRICULUM VITAE

Haluk Aygüneş was born in Kayseri on January 30, 1965. Following his mid-scooling in Kayseri and İstanbul, he graduated from the Turkish Military Academy in 1987. He was assigned in several military units from 1988 to 1992. He received his M.S. Degree in Mechanical Engineering from the Naval Postgraduate School, California USA, in February 1994. Since then, he has acted as an instructor in the Turkish Military Academy in Ankara. He has a book in the field of Operations Research with the title of "Yöneylem Araştırması" published in 2001. His main areas of interest are combat modeling and facility location.