BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

ÖMÜR UĞUR

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Prof. Dr. Canan ÖZGEN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Ersan AKYILDIZ
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Hasan TAŞELİ
Co-Supervisor

Prof. Dr. Marat AKHMET
Supervisor

Examining Committee Members

Prof. Dr. Marat AKHMET

Prof. Dr. Hüseyin BEREKETOĞLU

Assoc. Prof. Dr. Tanıl ERGENÇ

Prof. Dr. Hasan TAŞELİ

Prof. Dr. Ağacık ZAFER

Abstract

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Uğur, Ömür
Ph.D., Department of Mathematics
Supervisor: Prof. Dr. Marat AKHMET
Co-Supervisor: Prof. Dr. Hasan TAŞELİ

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The theory of impulsive differential equations has become an important area of research in recent years. Linear equations, meanwhile, are fundamental in most branches of applied mathematics, science, and technology. The theory of higher order linear impulsive equations, however, has not been studied as much as the corresponding theory of ordinary differential equations.

In this work, higher order linear impulsive equations at fixed moments of impulses together with certain boundary conditions are investigated by making use of a Green's formula, constructed for piecewise differentiable functions. Existence and uniqueness of solutions of such boundary value problems are also addressed.

Properties of Green's functions for higher order impulsive boundary value problems are introduced, showing a striking difference when compared to classical boundary value problems of ordinary differential equations. Necessarily, instead of an ordinary Green's function there corresponds a sequence of Green's functions due to impulses.

Finally, as a by-product of boundary value problems, eigenvalue problems for higher order linear impulsive differential equations are studied. The conditions for the existence of eigenvalues of linear impulsive operators are presented. Basic properties of eigensolutions of self-adjoint operators are also investigated. In particular, a necessary and sufficient condition for the self-adjointness of Sturm-Liouville operators is given. The corresponding integral equations for boundary value and eigenvalue problems are also demonstrated in the present work.

Keywords: Theory of Differential Equations, Impulsive Differential Equations, Boundary Value Problems, Eigenvalue Problems, Green's Functions, Linear Operators, Adjoint Operators, Sturm-Liouville Systems.

Öz

YÜKSEK MERTEBEDEN LİNEER İMPALSİF DİFERANSİYEL DENKLEMLER İÇİN SINIR DEĞER PROBLEMLERİ

Uğur, Ömür
Doktora, Matematik Bölümü
Tez Yöneticisi: Prof. Dr. Marat AKHMET
Ortak Tez Yöneticisi: Prof. Dr. Hasan TAŞELİ

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İmpalsif diferansiyel denklemler teorisi son yıllarda önemli bir araştırma alanı olarak ortaya çıkmaktadır. Öte yandan, lineer denklemler, teknoloji, bilim ve uygulamalı matematiğin birçok dalında temel konulardan biridir. Fakat, yüksek mertebeden impalsif denklemler teorisi sıradan diferansiyel denklemler teorisinde olduğu kadar araştırılmamıştır.

Bu çalışmada, belirli sınır koşullarıyla birlikte impals anları sabit zamanlı yüksek mertebeden lineer impalsif denklemler, parça parça türevlenebilir fonksiyonlar için yapılandırılan Green formülü yardımıyla incelenmiştir. Bu türdeki sınır değer problemlerinin çözümünün varlığı ve tekliği de belirtilmiştir.

Klasik sınır değer problemleriyle karşılaştırıldığında şaşırtıcı farklılıklar gösteren yüksek mertebeden impalsif sınır değer problemleri için Green fonksiyonlarının özellikleri verilmiştir. İmpals etkileri sonucu, alışılagelen Green fonksiyonu yerine, Green fonksiyonlarından oluşan bir sisteme ihtiyaç duyulmaktadır.

Son olarak, bir tür sınır değer problemi olarak ele alabileceğimiz yüksek mertebeden lineer impalsif diferansiyel denklemler için özdeğer problemleri çalışılmıştır. Yüksek mertebeden lineer impalsif operatörlerin özdeğerlerinin varlığı için gerekli koşullar gösterilmiştir. Bu arada, kendine-eş operatörlerin özdeğer ve özfonksiyonlarının temel özellikleri de araştırılmıştır. Özel olarak, Sturm-Liouville operatörlerinin kendine-eşliği için gerek ve yeter koşulu belirtilmiştir. Yine bu çalışmada, sınır değer ve özdeğer problemlerine karşılık gelen integral denklemler de sunulmuştur.

Anahtar Kelimeler: Diferansiyel Denklemler Teorisi, İmpalsif Diferansiyel Denklemler, Sınır Değer Problemleri, Özdeğer Problemleri, Green Fonksiyonları, Lineer Operatörler, Eş Operatörler, Sturm-Liouville Sistemleri.

To my little one, Öykü

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Chapter 1

Introduction

Many evolution process are characterized by the fact that they are subject to short-time perturbation whose duration is negligible in comparison with the duration of the process. This results with a sudden change of the state of the process. For example, when a hammer hits a string which is already oscillating, it experiences a rapid change of velocity; a pendulum of a clock, meanwhile, undergoes a sudden change of momentum when it crosses its equilibrium position; and so on.

For the description of the continuous change of such processes, ordinary differential equations are used, while the short-time perturbations of those processes are described by sudden changes of their states at certain times. It becomes, therefore, necessary to study dynamical systems with discontinuous trajectories, or with *impulse effect*, for the sake of brevity as they called, *impulsive differential equations*, or sometimes, *differential equations with impulse actions*.

The theory of impulsive differential equations has become an important area of research in recent years because of the needs of modern technology, engineering and physics. Moreover, impulsive differential equations is richer in applications compared to the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations [10, 13, 15, 42, 45]. A basic peculiarity of the impulsive differential equations occurs when the impulses are not at fixed moments of time, but satisfying a certain space-time relation. In the presence of unfixed moments of impulse actions, the possibility of appearance of the so-called *beating of solutions* is one peculiarity of the theory of impulsive differential equations, for instance, see [7].

However, the basic principles of solutions of higher order impulsive differential equations have not yet been sufficiently elaborated when compared to that of first order equations, even in the linear case. Especially, boundary value problems for higher order impulsive differential equations have been studied by some authors in some special cases, see [1, 2, 16, 20, 22]. Among them there are also some studies on the eigenvalue problems [32, 50, 51] and construction of Green's functions for some higher order linear impulsive differential equations [20, 21].

The main aim of the present work, however, is to study the theory of higher order linear impulsive differential equations together with the boundary value and eigenvalue problems, and to emphasize some of the distinguishing properties of such problems from the ones of ordinary differential equations.

In this chapter, meanwhile, we shall state some basic theory of first order impulsive differential equations with fixed moments of impulse actions. The theory and auxiliary assertions within this chapter, will be used in the subsequent chapters to develop the theory of boundary value and eigenvalue problems for higher order linear impulsive differential equations.

1.1 Linear Homogeneous Equations

The theory of linear equations are fundamental in most branches of applied mathematics, as well as in science and technology. Very often, complicated nonlinear problems are studied through linearization in order to understand the basic properties of the dynamical systems.

In this section, we will establish some basic properties of linear system of impulsive differential equations of homogeneous type. These results, and the results in the following section, are mostly based on the studies in [18, 43].

Let J be an interval of $\mathbb{R}$, and $\{\theta_i\}$ be the given strictly increasing sequence of impulse points in J, such that it has no finite accumulation point. Then, it follows that $\{\theta_i\}$ is a finite sequence of isolated points such that $\theta_i < \theta_{i+1}$ in the case when J is a bounded interval of $\mathbb{R}$. However, the sequence $\{\theta_i\}$ may be an infinite sequence of such points having no finite accumulation point when J is an infinite interval. Throughout the present work, therefore, we will always assume the following assumption.

Assumption 1. Any compact interval $J \subset \mathbb{R}$ contains only a finite number of impulse points θ_i.

Later, in the development of the theory of linear impulsive differential equations we will impose other conditions, mainly to simplify the argument therein.

Let $\mathbb{F}$ denote either the complex field $\mathbb{C}$ or the real numbers $\mathbb{R}$. We will denote by $\mathcal{PLC}(J, \{\theta_i\}\,;\mathbb{F}^n)$ the set of all functions $\varphi : J \to \mathbb{F}^n$ that are continuous for $t \neq \theta_i$, left continuous and having discontinuities of the first kind at the points $t = \theta_i$ of J. A *linear impulsive differential equation* on J is an equation of the form

$$\begin{cases} x' = A(t)x + f(t), & t \neq \theta_i, \\ \Delta x|_{t=\theta_i} = B_i x + a_i, \end{cases} \tag{1.1}$$

where $A(t)$ is an $n \times n$ matrix with $a_{ij} \in \mathcal{PLC}(J, \{\theta_i\}\,;\mathbb{F})$, $f(t) \in \mathcal{PLC}(J, \{\theta_i\}\,;\mathbb{F}^n)$, and B_i is an $n \times n$ and a_i is an $n \times 1$ constant matrices for every i. Also, $\Delta x|_{t=\theta_i} = x(\theta_i^+) - x(\theta_i^-)$ denotes the jumps of x at the points of impulses θ_i.

When $f(t) \neq 0$ for some $t \in J$, or $a_i \neq 0 = [0, \ldots, 0]^T$ for some i such that $\theta_i \in J$, (1.1) is called a *nonhomogeneous* impulsive differential equation. On the other hand, the equation

$$\begin{cases} x' = A(t)x, & t \neq \theta_i, \\ \Delta x|_{t=\theta_i} = B_i x, \end{cases} \tag{1.2}$$

is called the corresponding *homogeneous* impulsive differential equation.

The zero function, $x(t) = 0$ for all $t \in J$, is obviously a solution of (1.2); this function is called the *trivial* solution of the homogeneous linear impulsive differential equation. It is worth nothing that any solution $x(t, t_0, x_0)$, passing through the point $(t_0, x_0) \in J \times \mathbb{F}^n$, of (1.1) or (1.2) is uniquely continuable to the right of t_0, if t_0 is not the right end point of J. In particular, if $J = \mathbb{R}$, the whole real line, then for any $t_0 \in \mathbb{R}$, a solution of a linear impulsive equation is defined for all $t \geq t_0$. However, for any given interval J we will define $J_{t_0}^+$ to be $J_{t_0}^+ = J \cap [t_0, \infty)$. Similarly, we will denote by $J_{t_0}^-$ the interval $J \cap (-\infty, t_0]$. Clearly, $J = J_{t_0}^+ \cup J_{t_0}^-$.

Unlike the classical ordinary differential equations, a solution of linear impulsive differential equation is not necessarily continuable to the left of an impulse point, say θ_j, for some j. Even if, it is continuable to the left of θ_j this continuation need not be unique. This is because of the impulse condition in (1.1): If $\det(E + B_j) = 0$ at that point of impulse action then either the solution is not continuable to the left of θ_j, or if continuable then this continuation is not unique. In the latter case the solution splits into an infinite set of solutions to pass left of impulse point θ_j. See, for instance [43].

One of the main results of the theory of linear impulsive differential equation is the existence and uniqueness of solutions, defined in the interval J. In other words, at every point $(t_0, x_0) \in J \times \mathbb{F}^n$, there passes one and only one solution. This is stated in the following theorem.

Theorem 1.1 (Existence and Uniqueness). *Let $A(t) \in \mathcal{PLC}(J, \{\theta_i\}; \mathbb{F}^{n\times n})$ and $B_i \in \mathbb{F}^{n\times n}$. Then, for any $(t_0, x_0) \in J \times \mathbb{F}^n$, there exists a unique solution $x(t) = x(t, t_0, x_0)$ of (1.2) such that*

$$x(t_0) = x(t_0, t_0, x_0) = x_0,$$

and this solution is uniquely defined for all $t \in J^+_{t_0}$.

If, moreover, $\det(E + B_i) \neq 0$ for every i, then the solution $x(t) = x(t, t_0, x_0)$ is uniquely defined for all $t \in J$.

Proof. Proof is trivial, however, we refer to [43] for a complete discussion. □

Because of the existence and uniqueness theorem, namely Theorem 1.1, and the discussions preceding it, we will always assume the following condition holds in order to have unique solution that is defined for all $t \in J$. For a given linear impulsive differential equation of the form (1.1), the following assumption is satisfied, unless otherwise stated explicitly.

Assumption 2. For every i such that $\theta_i \in J$, the matrices $E + B_i$ are all nonsingular.

Therefore, if $x = x(t)$ is any solution of (1.2) such that $x(t_0) = 0$ for some $t_0 \in J$ or $x(\theta_i^+) = 0$ for some i, then the uniqueness requires that $x(t) = 0$ for all $t \in J$. Namely, $x(t)$ is the trivial solution. This observation, however, leads to the following fact about the structure of the set Ω of all solutions of a linear homogeneous impulsive equation on an interval J.

Theorem 1.2. *The set Ω of all solutions of (1.2), which are defined on the interval J, is an n-dimensional vector space over $\mathbb{F}$.*

Recall that, in any finite dimensional vector space of dimension n, any set of n linearly independent vectors forms a basis. If $\{\phi_1, \ldots, \phi_n\}$ is any set of n linearly independent solutions of (1.2) on J, then this set is said to form a *basis*, or a *fundamental set of solutions* for the linear homogeneous impulsive equation (1.2). The linearly independent solutions $\phi_1, \ldots, \phi_n$ of (1.2) are called *fundamental solutions*. Also, a matrix $\Phi(t)$ will be called a *solution matrix* for (1.2) if its n columns are solutions of the equation.

On the other hand, if $\Phi(t)$ is a matrix solution whose columns are n linearly independent solutions of (1.2), for all $t \in J$, then $\Phi(t)$ is called a *fundamental matrix* of the homogenous equation (1.2), and the inverse, $\Phi^{-1}(t)$, exists for all $t \in J$. Moreover, $\Phi(\theta_i^+)$ is invertible for all i. Hence, any solution $x(t)$ of (1.2) can be written as

$$x(t) = \Phi(t)\, c,$$

where c is an arbitrary column vector in $\mathbb{F}^n$.

In fact, a fundamental matrix $\Phi(t)$ of the linear homogeneous impulsive differential equation (1.2) satisfying the condition that $\Phi(t_0) = E$ is called the *matriciant* for (1.2), and denoted by $\Phi(t, t_0)$. Moreover, the matriciant, $\Phi(t, t_0)$ for (1.2) is unique and satisfies the equality

$$\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0)$$

for all t_0 and t in J, and for any fundamental matrix $\Phi(t)$. In other words, the matriciant for a linear homogeneous impulsive equation is independent of the choice of a basis for the space Ω of solutions. The solution of (1.2) passing through the point $(t_0, x_0) \in J \times \mathbb{F}^n$ is the function $x = x(t)$, given by

$$x(t) = \Phi(t, t_0)\, x_0.$$

Therefore, if one chooses the fundamental solutions $\phi_1, \ldots, \phi_n$ of (1.2) in such a way that they satisfy the initial conditions $\phi_j(t_0) = e_j$, where e_j is the column vector whose components are zero except the jth, which is 1, then the matrix $[\phi_1, \ldots, \phi_n]$ becomes the matriciant for (1.2).

Using the theory of ordinary differential equations it is possible to determine, explicitly, the fundamental matrices as well as the matriciant for an impulsive differential equation. Let $X(t, s)$ be the solution of the *matrix initial value problem*, formally defined by

$$X' = A(t)X, \quad X(s, s) = E, \tag{1.3}$$

then a fundamental matrix $\Phi(t)$ of (1.2) can be expressed in terms of the matrix $X(t, s)$ as follows:

$$\Phi(t) = X(t, \theta_{j+k})(E + B_{j+k}) \prod_{\nu=k}^{1} X(\theta_{j+\nu}, \theta_{j+\nu-1})(E + B_{j+\nu-1}) X(\theta_j, t_0) X(t_0) \tag{1.4}$$

for $\theta_{j-1} < t_0 \leq \theta_j < \theta_{j+k} < t \leq \theta_{j+k+1}$, where $X(t)$ is any fundamental matrix of the ordinary differential equation

$$x' = A(t)\, x, \quad \theta_{j-1} < t \leq \theta_j.$$

Hence, the matriciant $\Phi(t, t_0)$ for (1.2) can be expressed similarly, namely

$$\Phi(t, t_0) = X(t, \theta_{j+k})(E + B_{j+k}) \prod_{\nu=k}^{1} X(\theta_{j+\nu}, \theta_{j+\nu-1})(E + B_{j+\nu-1})X(\theta_j, t_0)$$

provided that $\theta_{j-1} < t_0 \le \theta_j < \theta_{j+k} < t \le \theta_{j+k+1}$ holds. In particular,

- $\Phi(t, t_0) = X(t, t_0)$, for $\theta_{j-1} < t_0 \le t \le \theta_j$,
- $\Phi(t, t_0) = X(t, \theta_j)(E + B_j)X(\theta_j, t_0)$, for $\theta_{j-1} < t_0 \le \theta_j < t \le \theta_{j+1}$.

Equation (1.4) describing a fundamental matrix $\Phi(t)$ of (1.2), and the fact that $\Phi(t_0) = X(t_0)$ follows the Abel's formula,

$$\det \Phi(t) = \det \Phi(t_0) \exp \left(\int_{t_0}^{t} \mathrm{tr} A(s)\, ds \right) \prod_{\nu=1}^{k+1} \det(E + B_{j+\nu-1})$$

for $\theta_{j-1} < t_0 \le \theta_j < \theta_{j+k} < t \le \theta_{j+k+1}$, and hence, $\Phi(t)$ is nonsingular if and only if $\Phi(t_0)$ is nonsingular as required, because $(E + B_i)$ are assumed to be nonsingular by the *Assumption* 2 for every i. Hence, the following theorem for fundamental matrix, $\Phi(t)$, of (1.2) is valid.

Theorem 1.3. *Suppose $\Phi(t)$ is any fundamental matrix of the (1.2). Then $\Psi(t) = \Phi(t)\,C$ is a solution matrix of the corresponding matrix equation to (1.2) for every constant $n \times n$ matrix C.*

Conversely, for any solution matrix $\Psi(t)$ of (1.2), there exists a unique constant $n \times n$ matrix C such that $\Psi(t) = \Phi(t)\,C$ holds. Moreover, the solution matrix $\Psi(t)$ is a fundamental matrix of (1.2) if and only if C is nonsingular.

For some other basic properties of fundamental matrices and the matriciants of linear homogeneous systems of impulsive differential equations, we refer to [18, 43].

We conclude this section by defining the adjoint equation to the linear homogeneous equation (1.2), and stating a theorem about the properties of the solutions. The linear homogeneous impulsive differential equation

$$\begin{cases} y' = -A^*(t)y, \quad t \neq \theta_i, \\ \Delta y|_{t=\theta_i} = -(E + B_i^*)^{-1} B_i^* y \end{cases} \tag{1.5}$$

is called the *adjoint equation* to (1.2), where "*" represents the complex conjugate transpose of a matrix in the case when $\mathbb{F} = \mathbb{C}$; however, if $\mathbb{F} = \mathbb{R}$, then it represents only the transpose of a matrix.

Theorem 1.4. *Any fundamental matrices $\Phi(t)$ and $\Psi(t)$ of the equations (1.2) and (1.5), respectively, satisfy the identity*

$$\Psi^*(t)\Phi(t) = C \tag{1.6}$$

for all $t \in J$, where C is a constant $n \times n$ matrix.

If (1.6) holds, where $\Phi(t)$ is a fundamental matrix of (1.2) and C is nonsingular, then $\Psi(t)$ is a fundamental matrix of the adjoint equation (1.5).

Proof. See [18, 43]. □

1.2 Linear Nonhomogeneous Equations

In this section we will consider the following linear nonhomogeneous impulsive differential equation

$$\begin{cases} x' = A(t)x + f(t), \quad t \neq \theta_i, \\ \Delta x|_{t=\theta_i} = B_i x + a_i, \end{cases} \tag{1.7}$$

where $f \in \mathcal{PLC}(J, \{\theta_i\}; \mathbb{F}^n)$ and $a_i \in \mathbb{F}^n$ for every i. Here, we also remark that $A \in \mathcal{PLC}(J, \{\theta_i\}; \mathbb{F}^{n\times n})$ and $B_i \in \mathbb{F}^{n\times n}$ are such that $\det(E + B_i) \neq 0$ for every i.

The relationship between nonhomogeneous equation (1.1) and the corresponding homogeneous one

$$\begin{cases} x' = A(t)x, \quad t \neq \theta_i, \\ \Delta x|_{t=\theta_i} = B_i x \end{cases} \tag{1.8}$$

is the following: if $\varphi(t)$ is a solution of (1.8) and $\psi(t)$ is a solution of (1.7), then the function $\varphi(t) + \psi(t)$ is again a solution of (1.7). Conversely, if $\varphi_1(t)$ and $\varphi_2(t)$ are two solutions of (1.7), then the difference $\varphi_1(t) - \varphi_2(t)$ is a solution of (1.8). Hence the following theorem on the space Ω' of solutions of linear nonhomogeneous equation (1.7) can easily be proved.

Theorem 1.5. *Let Ω be the set of solutions of (1.8) on J, then the set Ω' of all solutions of (1.7) on J is the affine space*

$$\Omega' = \{x : x(t) = \psi(t) + u(t), \quad u \in \Omega\}$$

where $\psi(t)$ is a particular solution of (1.7).

As in the classical theory of ordinary differential equations it is possible to define a particular solution of the nonhomogeneous equation (1.7) by the help of the linearly independent solutions of the corresponding homogeneous equation. The following theorem, however, is known as the *variation of parameters* or *variation of constants* formula for linear impulsive differential equations. Moreover, it implicitly gives such particular solutions of (1.7) in terms of the fundamental solutions of the corresponding homogeneous equation. For the proof of this theorem, however, we refer [43].

Theorem 1.6 (Variation of Parameters). *Let $\Phi(t)$ be a fundamental matrix of (1.8). Then every solutions $x = x(t)$ of the associated nonhomogeneous equation (1.7) is given by the formula*

$$x(t) = \begin{cases} \Phi(t)\Big(c + \displaystyle\int_{t_0}^{t} \Phi^{-1}(s) f(s)\, ds + \sum_{t_0 \le \theta_i < t} \Phi^{-1}(\theta_i^+) a_i\Big), & t_0 \le t \\ \Phi(t)\Big(c + \displaystyle\int_{t_0}^{t} \Phi^{-1}(s) f(s)\, ds - \sum_{t \le \theta_i < t_0} \Phi^{-1}(\theta_i^+) a_i\Big), & t \le t_0 \end{cases} \tag{1.9}$$

In particular, if $\Phi(t, t_0)$ is the matriciant for the homogeneous equation (1.8), then the solution $x(t) = x(t, t_0, x_0)$ such that $x(t_0) = x_0$ is given by the formula

$$x(t) = \begin{cases} \Phi(t, t_0) x_0 + \displaystyle\int_{t_0}^{t} \Phi^{-1}(t, s) f(s)\, ds + \sum_{t_0 \le \theta_i < t} \Phi^{-1}(t, \theta_i^+) a_i, & t_0 \le t \\ \Phi(t, t_0) x_0 + \displaystyle\int_{t_0}^{t} \Phi^{-1}(t, s) f(s)\, ds - \sum_{t \le \theta_i < t_0} \Phi^{-1}(t, \theta_i^+) a_i, & t \le t_0 \end{cases} \tag{1.10}$$

Here, we remark that the summations are taken to be zero when $t = t_0$ in the variation of parameters formulas above. One may see [18] for variation parameters formula for nonlinear impulsive equations.

Chapter 2

Higher Order Linear Equations

In most areas of applied sciences there arise the higher order differential equations which govern the behavior of the state of an observable. In some cases, however, these observable quantities undergo sudden change of their state depending on the nature of the problems that they characterize. For example, when a hammer hits a string that is oscillating, it experiences a rapid change, only of its velocity. However, consider a fish population in a pool, and every week some fish, male or female, are taken out to be sold. This action will affect not only the number of population of fish in the pool, but it will also affect the rate of change of the population, depending on the number of male or female fish remained within the pool for reproduction.

Therefore, it is of great importance to investigate higher order impulsive differential equations not only to characterize such problems in applied sciences and technology, but also it is interesting in mathematical point of view. In fact, there are researches on the higher order impulsive differential equations such as [20, 32, 48, 50, 51] among which the boundary value and eigenvalue problems are also studied to some extent. However, the theory of higher order linear impulsive differential equations has not yet been fully studied as much as the corresponding theory of linear ordinary differential equations. In contrast, the theory of first order impulsive differential equations has been studied to a great extent in recent years, and has proved to be much richer than the theory of first order ordinary differential equations.

In this chapter of the present work we will see that with some simple modifications, the theory of higher order impulsive differential equations become very similar to that of first order system equations with impulse actions, and it can be extended in parallel to the theory of higher order ordinary differential equations. However, we should always keep in mind that the trajectories of impulsive differential equations are in general discontinuous.

However, it is important how to define a higher order impulsive differential equation and construct its corresponding system of first order equations, since the solution, if any, is not necessarily continuous at the points of impulse actions. In the following introductory section, therefore, we will introduce some basics of such construction.

2.1 Introduction

The problem of defining a higher order impulsive differential equation is mainly, due to the fact that a function having a discontinuity of the first kind at an impulse point will not possess a derivative at that point. Moreover, there is no way to define it unless the jump is zero. Hence, to talk about the jumps of the derivatives of such functions at impulse points is ambiguous. However, as some researches [18, 39] suggest one may consider the jumps of the limits of the derivatives at the impulse points. Similar arguments, hence, can be made for higher order derivatives. Moreover, one might consider functions which are not even defined at the impulse points, but have limits from the left and the right at the points of impulse actions. In this case, unfortunately, one needs to redefine the concept of a solution to an impulsive differential equation on an interval.

Logically, it is important, however, to keep the property that a solution to an impulsive differential equation is left continuous at the points of discontinuities. This, being defined at the points of discontinuities, also preserves the uniqueness concept of a solution. Otherwise, one can define the values of such solutions (if we can call them as solutions!) in an arbitrary way at the points of impulses.

Fortunately, if a function is left continuous at a point one may consider the left derivative at that point. Similarly, the right derivative can be considered when the function is right continuous. To be consistent with the natural development of impulsive differential equations we will, mainly, deal with functions that are left continuous at the points of their discontinuities.

Let $C([a,b]) = C([a,b];\mathbb{F})$ denotes the set of all functions $f : [a,b] \to \mathbb{F}$ that are continuous on the closed interval $[a,b]$, and let $C^1([a,b]) = C^1([a,b];\mathbb{F})$ be the set of all functions $f \in C([a,b])$ that are continuously differentiable in the open interval

(a,b) and the left and the right derivatives, respectively,

$$f'_-(b) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h},$$

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h},$$

exist. We define, similarly the higher order left and right derivatives of such functions, respectively, as follows:

$$f_-^{(n)}(b) = \lim_{h\to 0^-} \frac{f^{(n-1)}(b+h)-f_-^{(n-1)}(b)}{h},$$

$$f_+^{(n)}(a) = \lim_{h\to 0^+} \frac{f^{(n-1)}(a+h)-f_+^{(n-1)}(a)}{h},$$

recursively for every $n \geq 1$.

Therefore, let $C^n([a,b]) = C^n([a,b];\mathbb{F})$ denotes the set of all functions $f \in C([a,b])$ that are continuously differentiable in (a,b) and $f_-^{(n)}(b)$ and $f_+^{(n)}(a)$ exist. Recall that the existence of the left (respectively, right derivative) of a function f at a point implies that the function itself is left (respectively, right continuous) at that point.

Conversely, if a function $f \in C([a,b])$ is continuously differentiable in (a,b) and is such that

$$f'(b^-) = \lim_{h\to 0^-} f'(b+h)$$

exists, then by the Mean Value Theorem, there exists a $\xi \in (b+h,b),\ h<0$, such that

$$\begin{aligned} f'_-(b) &= \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \\ &= \lim_{\xi\to b^-} f'(\xi) \\ &= f'(b^-) \end{aligned}$$

holds. Namely, $f'_-(b) = f'(b^-)$. Similarly the existence of $f'(a^+)$ implies the existence of the right derivative $f'_+(a)$ and $f'_+(a) = f'(a^+)$.

These observations leads us to the following discussion. Let J be any interval of $\mathbb{R}$, and $\{\theta_i\} \subset J$ be any sequence of points θ_i such that $\theta_i < \theta_{i+1}$, and have no finite accumulation point. As in Chapter 1, let $\mathcal{PLC} = \mathcal{PLC}(J, \{\theta_i\};\mathbb{F})$ denote the set of all functions $f : J \to \mathbb{F}$ that are left continuous for all $t \in J$, with discontinuities of the first kind at $t = \theta_i$. In other words,

$$\mathcal{PLC} = \left\{ f : J \to \mathbb{F} \mid f \in C(J \setminus \{\theta_i\}) \text{ is left continuous on } J \text{ and } \Delta f|_{t=\theta_i} < \infty \right\},$$

where $\Delta f|_{t=\theta_i} = f(\theta_i^+) - f(\theta_i^-)$ is the difference of the right and left limits at the point $t = \theta_i$, with $f(\theta_i^-) = f(\theta_i)$.

Similarly, for a function $f : J \to \mathbb{F}$ that is jth times continuously differentiable for $t \neq \theta_i$ we define the jumps $\Delta f^{(j)}|_{t=\theta_i} = f^{(j)}(\theta_i^+) - f^{(j)}(\theta_i^-)$ at the points of impulses $t = \theta_i$ for every i. Then, $\mathcal{PLC}^n = \mathcal{PLC}^n(J, \{\theta_i\}; \mathbb{F})$ will denote the following set of functions:

$$\mathcal{PLC}^n = \left\{ f \in \mathcal{PLC} : f \in C^n(J \setminus \{\theta_i\}) \text{ such that } \Delta f^{(n)}|_{t=\theta_i} < \infty \right\}.$$

It should be remarked that we will not distinguish between $f_-^{(j)}(\theta_i)$ and $f^{(j)}(\theta_i^-)$ for any $j = 0, 1, \ldots, n$, provided that f is left continuous at the point θ_i. The existence of $f_-^{(n)}(\theta_i)$ for a function $f \in C^n(J \setminus \{\theta_i\})$ for $n \geq 1$ implies that $f_-^{(j)}(\theta_i) = f^{(j)}(\theta_i^-)$ for all $j \leq n$, and hence f is continuous from the left at $t = \theta_i$ for every i. The converse is also true by the arguments above, if the function is left continuous at the points $t = \theta_i$. Within this work, we will prefer limits rather than left derivatives, to be consistent with the conventions.

So, an impulsive differential equation on J of order $n \geq 1$, with fixed impulses at $t = \theta_i$, is of the following form

$$\begin{cases} x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}), \quad t \neq \theta_i, \\ \Delta x^{(j-1)}|_{t=\theta_i} = I_{i,j}(x(\theta_i^-), x'(\theta_i^-), \ldots, x^{(n-1)}(\theta_i^-)), \quad j = 1, \ldots, n \end{cases} \tag{2.1}$$

where f is a function of $(n+1)$ variables, defined for all $t \in J \setminus \{\theta_i\}$, and $I_{i,j}$ is a function of n variables for each i and j. Moreover, the jumps

$$\Delta x^{(j-1)}|_{t=\theta_i} = x^{(j-1)}(\theta_i^+) - x^{(j-1)}(\theta_i^-), \quad j = 1, \ldots, n$$

are defined by the differences of the right and the left limits of $x^{(j-1)}(t)$ at the point $t = \theta_i$. By a solution of an impulsive differential equation (2.1), of order n, we mean the following definition.

Definition 2.1. The function $\varphi : J \to \mathbb{F}$ is said to be a *solution* of an nth order impulsive differential equation (2.1) on J if it satisfies the following conditions:

1. The function $\varphi \in \mathcal{PLC}^{n-1}$ and $\varphi^{(n)}(t)$ exists at every point $t \neq \theta_i$ on J.
2. The function φ satisfies the following equalities
   (a) $\varphi^{(n)}(t) = f(t, \varphi(t), \ldots, \varphi^{(n-1)}(t))$, for all $t \in J \setminus \{\theta_i\}$,

(b) $\varphi^{(j-1)}(\theta_i^+) = \varphi^{(j-1)}(\theta_i^-) + I_{i,j}(\varphi(\theta_i^-), \ldots, \varphi^{(n-1)}(\theta_i^-))$, for all $1 \le j \le n$, and for all i.

Now, as is done in ordinary differential equations, [26, 27], let $\vec{x} = [x_1, \ldots, x_n]^T$ with components formally defined by $x_j(t) = x^{(j-1)}(t)$ for all $j = 1, \ldots, n$, in such a way that

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(n-1)}(t) \end{pmatrix}, \quad \text{for } t \neq \theta_i, \tag{2.2}$$

and without loss of generality

$$\vec{x}(\theta_i) = \begin{pmatrix} x_1(\theta_i) \\ x_2(\theta_i) \\ \vdots \\ x_n(\theta_i) \end{pmatrix} = \begin{pmatrix} x(\theta_i) \\ x'(\theta_i^-) \\ \vdots \\ x^{(n-1)}(\theta_i^-) \end{pmatrix}, \quad \text{for } t = \theta_i \tag{2.3}$$

for every i. Then, one can transform an nth order impulsive differential equation (2.1) to a first order impulsive differential equation of the form

$$\begin{cases} \vec{x}' = \vec{f}(t, \vec{x}), \quad t \neq \theta_i, \\ \Delta \vec{x}|_{t=\theta_i} = \vec{I}_i(\vec{x}(\theta_i)) \end{cases} \tag{2.4}$$

where the functions $\vec{f}(t, \vec{x})$ and $\vec{I}_i(\vec{x})$ are defined by

$$\vec{f}(t, \vec{x}) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1, \ldots, x_n) \end{pmatrix} \quad \text{and} \quad \vec{I}_i(\vec{x}) = \begin{pmatrix} I_{i,1}(x_1, \ldots, x_n) \\ I_{i,2}(x_1, \ldots, x_n) \\ \vdots \\ I_{i,n}(x_1, \ldots, x_n) \end{pmatrix}$$

for all $t \neq \theta_i$ and for all i, respectively.

It should be noted, however, if $\vec{x}$ is a solution of (2.4), then the first component of it, $x_1(t)$, is a solution of the nth order impulsive differential equation (2.1). Conversely, if $x(t)$ is a solution of (2.1), then $\vec{x}(t)$, defined by (2.2) and (2.3), is a solution of the corresponding first order equation (2.4).

Therefore, it is possible to investigate the basic properties of higher order impulsive differential equations, as well as their solutions, by the help of the properties of the corresponding first order system of impulsive equations. However, in the next section will consider higher order linear impulsive differential equations, and establish only some of their basic features.

2.2 Linear Impulsive Equations

The theory of linear equations are fundamentals in most branches of applied mathematics, as well as engineering and other natural sciences. Very often, one deals with linearization of nonlinear complicated problems in order to understand basic structure, and properties of the systems which are governed by some nonlinear equations.

In the present section of the current work, we will study the fundamental properties of higher order linear equations with impulse actions at fixed moments of time.

In order to simplify the discussion throughout this section, and within the rest of the work, we will fix the interval $J = [\alpha, \beta]$, and the sequences $\{\theta_i\}_{i=1}^{p}$ of impulse points $\theta_i \in (\alpha, \beta)$ in such a way that the following assumption holds.

Assumption 3. Let $J = [\alpha, \beta]$ and $\{\theta_i\}$ satisfy the following condition

$$\theta_0 = \alpha < \theta_1 < \theta_2 < \cdots < \theta_p < \beta = \theta_{p+1}. \tag{2.5}$$

An nth order linear impulsive differential equation on J is an equation of the form

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t), & t \neq \theta_i, \\ \Delta x^{(j-1)}|_{t=\theta_i} - \displaystyle\sum_{k=1}^{n} b_{ijk} x^{(k-1)}(\theta_i^-) = a_{ij}, & \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, p \end{array} \end{cases} \tag{2.6}$$

where the functions $p_0, p_1, \ldots, p_n$ and f are assumed to be at least left continuous on J with discontinuities of the first kind at $t = \theta_i$; and the scalars b_{ijk} and a_{ij} are in $\mathbb{F}$ for every $i = 1, \ldots, p$; $j = 1, \ldots, n$; $k = 1, \ldots, n$. The jumps $\Delta x^{(j-1)}|_{t=\theta_i}$ at θ_i, for each $j = 1, \ldots, n$ are, defined by

$$\Delta x^{(j-1)}|_{t=\theta_i} = x^{(j-1)}(\theta_i^+) - x^{(j-1)}(\theta_i^-), \quad j = 1, \ldots, n$$

for every $i = 1, \ldots, p$.

If $f(t) \neq 0$ for some $t \in J$, or $a_{ij} \neq 0$ for some $1 \leq i \leq p$ and $1 \leq j \leq n$, then the linear equation (2.6) is called a *nonhomogeneous linear impulsive differential equation.* While, the equation of the form

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0, & t \neq \theta_i, \\ \Delta x^{(j-1)}|_{t=\theta_i} - \displaystyle\sum_{k=1}^{n} b_{ijk} x^{(k-1)}(\theta_i^-) = 0, & \begin{array}{l} j = 1, \ldots, n \\ i = 1, \ldots, p \end{array} \end{cases} \tag{2.7}$$

is called the corresponding *homogeneous impulsive differential equation.*

It should be noted, however, that in (2.6) we have n number of impulse conditions, apart from the nth order linear differential equation. So, it is helpful to rewrite the impulse conditions in (2.6) using matrices and column vectors. As is done in the first section of this chapter, let $\widehat{x}$ denotes the column vector with components $x, x', \ldots, x^{(n-1)}$. Namely,

$$\widehat{x}(t) = \begin{pmatrix} x(t) \\ x'(t) \\ \vdots \\ x^{(n-1)}(t) \end{pmatrix} \quad \text{for } t \neq \theta_i, \tag{2.8}$$

and

$$\widehat{x}(\theta_i^{\pm}) = \lim_{h \to 0^{\pm}} \widehat{x}(\theta_i + h), \quad \text{for } i = 1, \ldots, p. \tag{2.9}$$

Then, the impulse conditions in (2.6) can be written simply,

$$\Delta \widehat{x}|_{t=\theta_i} - B_i \widehat{x}(\theta_i^-) = a_i, \quad i = 1, \ldots, p,$$

where B_i are $n \times n$ constant matrices, and a_i are $n \times 1$ column vectors defined, respectively, by

$$B_i = \begin{pmatrix} b_{i11} & b_{i12} & \cdots & b_{i1n} \\ b_{i21} & b_{i22} & \cdots & b_{i2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{in1} & b_{i12} & \cdots & b_{inn} \end{pmatrix}, \qquad a_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix}$$

for $i = 1, \ldots, p$. Hence an nth order linear impulsive differential equation on J can be written equivalently in the form

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t), \quad t \neq \theta_i, \\ \Delta \widehat{x}|_{t=\theta_i} - B_i \widehat{x}(\theta_i^-) = a_i, \quad i = 1, \ldots, p, \end{cases} \tag{2.10}$$

where $\widehat{x}(t) = [x(t), x'(t), \ldots, x^{(n-1)}(t)]^T$, for $t \neq \theta_i$, B_i are $n \times n$ constant matrices and a_i are constant $n \times 1$ column vectors. Throughout the work, by $\widehat{x}$, we mean, for a function $x \in C^{n-1}(J \setminus \{\theta_i\}; \mathbb{F})$, the vector valued function defined by (2.8), and $\widehat{x}(\theta_i^{\pm})$ will denote the limits in (2.9). Of course, in the case when the function $x \in \mathcal{PLC}^{n-1}$ and has left derivatives at $t = \theta_i$, we have the following equality,

$$\widehat{x}(\theta_i) = \begin{pmatrix} x(\theta_i) \\ x'_-(\theta_i) \\ \vdots \\ x_-^{(n-1)}(\theta_i) \end{pmatrix} = \widehat{x}(\theta_i^-).$$

Moreover, it is also possible and helpful to rewrite the ordinary differential equation

$$p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t), \quad t \neq \theta_i$$

in its corresponding first order form, provided that $p_0(t) \neq 0$ for all $t \in J$, as follows

$$\widehat{x}' = A(t)\widehat{x} + \frac{f(t)}{p_0(t)}\, e_n, \quad t \neq \theta_i,$$

where $e_n = [0, \ldots, 0, 1]^T$. The $n \times n$ matrix $A(t)$, defined by,

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_n(t)}{p_0(t)} & -\frac{p_{n-1}(t)}{p_0(t)} & -\frac{p_{n-2}(t)}{p_0(t)} & \cdots & -\frac{p_1(t)}{p_0(t)} \end{pmatrix} \tag{2.11}$$

is the so-called [27] *companion matrix* for the nth order linear differential equation. We should note that this matrix $A(t)$ is, in general, piecewise left continuous with discontinuities of the first kind at $t = \theta_i$.

Therefore, the first order linear impulsive differential equation corresponding to an nth order linear impulsive equation (2.10) or (2.6), can be written as follows

$$\begin{cases} \widehat{x}' = A(t)\widehat{x} + \frac{f(t)}{p_0(t)}\, e_n, \quad t \neq \theta_i, \\ \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-) = a_i, \quad i = 1, \ldots, p. \end{cases} \tag{2.12}$$

The corresponding homogeneous equation is given by

$$\begin{cases} \widehat{x}' = A(t)\widehat{x}, \quad t \neq \theta_i, \\ \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-) = 0, \quad i = 1, \ldots, p. \end{cases} \tag{2.13}$$

It is easy to investigate the existence and uniqueness of solutions of a linear nth order impulsive differential equations that is written in the form (2.12). Since a function $x = x(t)$ is a solution of (2.10) if and only if $y = \widehat{x}(t)$ is a solution of the associated first order impulsive equation (2.12).

Theorem 2.1. *Let $\frac{1}{p_0}, p_1, \ldots, p_n$ and f be functions in $\mathcal{PLC}$. For any $t_0 \in J$, and $\xi = [\xi_1, \ldots, \xi_n]^T \in \mathbb{F}^n$, there exists a unique solution $x(t) = x(t, t_0, \xi)$ of (2.10) on J, satisfying the initial condition*

$$x^{(j-1)}(t_0) = \xi_j, \quad j = 1, \ldots, n, \tag{2.14}$$

provided that $\det(E + B_i) \neq 0$ *for all* $i = 1, \ldots, p$

Proof. The proof of this theorem is similar to the proof of the existence and uniqueness theorem for systems of first order equations. Using the equivalent form (2.12), the proof follows. $\square$

Example 2.1. Consider the following initial value problem

$$\begin{cases} -x'' = 0, \quad t \neq 1, \\ \Delta\widehat{x}|_{t=1} - B\widehat{x}(1^-) = 0, \\ x(0) = x'(0) = 1, \end{cases} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The unique solution $x = x(t)$ of this problem can easily be computed as

$$x(t) = \begin{cases} 1 + t, & t \leq 1 \\ 2 + 4(t-1), & t > 1 \end{cases}$$

$\diamondsuit$

In this work, beside the basic assumptions on the interval J and the sequence of impulse points θ_i, explicitly stated in (2.5), we will always assume that for a given nth order linear impulsive differential equation of the form (2.12) the hypothesis of the existence and uniqueness theorem 2.1 holds, unless otherwise stated explicitly. Namely, all the functions $\frac{1}{p_0}, p_1, \ldots, p_n$ and f are of class $\mathcal{PLC}$. Note that since $\frac{1}{p_0} \in \mathcal{PLC}$ the following two conditions hold for $p_0(t)$ automatically:

$$\begin{aligned} &p_0(t) \neq 0, \quad t \in J, \\ &p_0(\theta_i^+) \neq 0, \quad i = 1, \ldots, p. \end{aligned}$$

2.2.1 Homogeneous Equations

In this subsection, we will establish some simple properties of higher order linear homogeneous impulsive differential equations given by (2.7), which could be written more simply as

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0, \quad t \neq \theta_i, \\ \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-) = 0, \quad i = 1, \ldots, p, \end{cases} \tag{2.15}$$

where $\frac{1}{p_0}, p_1, \ldots, p_n$ are assumed to functions in $\mathcal{PLC}$, $J = [\alpha, \beta]$, and the sequence $\{\theta_i\}$ are such that

$$\theta_0 = \alpha < \theta_1 < \cdots < \theta_p < \beta = \theta_{p+1}.$$

Furthermore, the matrices B_i are $n \times n$ with $\det(E + B_i) \neq 0$ for every $i = 1, \ldots, p$ to ensure the existence, uniqueness, and continuability of solutions throughout J. The first order equivalent for (2.15) is given by (2.13) in the previous section.

Clearly, $x(t) = 0$ for all $t \in J$ is a solution of (2.15). The solution $x(t) \equiv 0$ is called the *trivial* solution of the homogeneous equation. Moreover, if $x(t)$ is a solution of (2.15) with the initial data $\widehat{x}(t_0) = 0$ for some $t_0 \in J$ or $\widehat{x}(\theta_i^+) = 0$ for some i, then it follows from the uniqueness of solutions that $x(t) = 0$ for all $t \in J$, since $\det(E + B_i) \neq 0$ for all $i = 1, \ldots, p$. This leads to the following theorem on the space of solutions of homogeneous impulsive equations.

Theorem 2.2. *The set Ω_n of solutions of an nth order linear homogeneous impulsive differential equation (2.15) on J is an n-dimensional vector space over $\mathbb{F}$.*

In view of the theorem above, we may identify n linearly independent solutions $\phi_1, \ldots, \phi_n$ of the homogeneous impulsive differential equation as *fundamental solutions*. The set $\{\phi_1, \ldots, \phi_n\}$ of fundamental solutions, therefore, will be called a *fundamental set* of solutions, or simply a *basis* for Ω_n.

Let $\{\phi_1, \ldots, \phi_n\}$ be any set of n solutions of linear homogeneous equation (2.15). Then, we define a vector valued function $\Phi = \Phi(t)$ as follows

$$\Phi(t) = [\phi_1(t), \ldots, \phi_n(t)], \tag{2.16}$$

whose components are fundamental solutions, and a associated matrix valued function $\widehat{\Phi} = \widehat{\Phi}(t)$ defined by

$$\widehat{\Phi}(t) = [\widehat{\phi}_1(t), \ldots, \widehat{\phi}_n(t)]. \tag{2.17}$$

Hence, the columns of $\widehat{\Phi}(t)$ becomes solutions of the corresponding first order equation (2.13). The determinant, $\det \widehat{\Phi}(t)$, is called the *Wronskian* of solutions of higher order linear homogeneous equation (2.15).

Moreover, the matrix $\widehat{\Phi}(t)$ defined by (2.17) is a fundamental matrix of (2.13) if and only if its first row $\Phi(t)$ consists of n linearly independent solutions of (2.15).

For brevity, if $\{\phi_1, \ldots, \phi_n\}$ is a fundamental set of solutions of linear homogeneous equation (2.15), then we will call the row vector, $\Phi(t)$, defined by (2.16), as a *fundamental row vector* or a *row vector of fundamental solutions*, and the $n \times n$ matrix, $\widehat{\Phi}(t)$, defined by (2.17), as a *fundamental matrix* for (2.15), see [27].

Hence, by the above theorem, any solution $x = x(t)$ of a linear homogeneous impulsive differential equation (2.15) is of the form

$$x(t) = \Phi(t)\, c = c_1\phi_1(t) + \cdots + c_n\phi_n(t),$$

where $c = [c_1, \ldots, c_n]^T$ is any column vector, and the functions $\phi_1, \ldots, \phi_n$ are any fundamental solutions of (2.15).

Moreover, since $\widehat{\Phi}(t)$ is a matrix solution of the corresponding first order equation (2.13), the Wronskian, $\det \widehat{\Phi}(t)$, satisfies the following equality

$$\det \widehat{\Phi}(t) = \det \widehat{\Phi}(t_0) \exp\left(-\int_{t_0}^{t} \frac{p_1(s)}{p_0(s)}\, ds\right) \prod_{\nu=1}^{k+1} \det(E + B_{j-\nu-1})$$

for $\theta_{j-1} < t_0 \leq \theta_j < \theta_{j+k} < t \leq \theta_{j+k+1}$, which follows from the discussion in Section 1.1 and the fact that the companion matrix $A(t)$ satisfies

$$\mathrm{tr} A(t) = \frac{p_1(t)}{p_0(t)}$$

for all $t \in J$. Therefore, we proved the following theorem.

Theorem 2.3. *The vector valued function $\Phi(t) = [\phi_1(t), \ldots, \phi_n(t)]$ is a row vector of fundamental solutions of linear homogeneous impulsive differential equation (2.15) if and only if* $\det \widehat{\Phi}(t_0) \neq 0$ *for some $t_0 \in J$.*

Example 2.2. Consider the following impulsive differential equation with a single impulse point $\theta \in \mathbb{R}$,

$$\begin{cases} -x'' = 0, \quad t \neq \theta, \\ \Delta \widehat{x}|_{t=\theta} - B\widehat{x}(\theta^-) = 0, \end{cases} \qquad B = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad 1 + b \neq 0.$$

The fundamental solutions $x = \phi_1(t)$ and $x = \phi_2(t)$ can be calculated as,

$$\phi_1(t) = \begin{cases} 1, & t \leq \theta \\ 1 + a(t-\theta), & t > \theta \end{cases} \qquad \phi_2(t) = \begin{cases} t, & t \leq \theta \\ \theta + (a\theta + 1 + b)(t - \theta), & t > \theta \end{cases}$$

In general, we can construct the row vector, $\Phi(t)$, of fundamental solutions and the fundamental matrix, $\widehat{\Phi}(t)$, for this problem as follows

$$\Phi(t) = \begin{cases} \begin{pmatrix} 1, & t \end{pmatrix}, & t \leq \theta \\ \begin{pmatrix} 1 + a(t-\theta), & \theta + (a\theta + 1 + b)(t-\theta) \end{pmatrix}, & t > \theta \end{cases}$$

and

$$\widehat{\Phi}(t) = \begin{cases} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, & t \leq \theta \\ \begin{pmatrix} 1 + a(t-\theta) & \theta + (a\theta + 1 + b)(t-\theta) \\ a & (a\theta + 1 + b) \end{pmatrix}, & t > \theta \end{cases}$$

Notice that $\det \widehat{\Phi}(t) \neq 0$ for all $t \in \mathbb{R}$. So, the general solution of the problem can be written as a linear combination of these fundamental solutions $\phi_1(t)$ and $\phi_2(t)$, namely any solution $x = x(t)$ is of the form,

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t).$$

In particular, when $\theta = a = b = 1$, we have the unique solution in Example 2.1, that is,

$$x(t) = \begin{cases} 1+t, & t \leq 1 \\ 2+4(t-1), & t > 1 \end{cases}$$

which satisfies the initial conditions $x(0) = x'(0) = 1$. $\diamondsuit$

Finally, in this subsection, we remark that it is possible to construct an adjoint of the nth order homogeneous impulsive equation (2.15), using its corresponding first order system (2.13), as in the Section 1.1 of Chapter 1. However, in the next chapter we will establish such adjoint problems for nth order linear boundary value problems by constructing a Green's formula [27, 41] for functions which have discontinuities of the first kind at the impulse points $t = \theta_i$ of the interval $J = [\alpha, \beta]$. So, we postpone the discussion of adjoint equations to Chapter 3.

2.2.2 Nonhomogeneous Equations

In this subsection, we will consider an higher order linear nonhomogeneous impulsive differential equations of the form

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t), \quad t \neq \theta_i, \\ \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-) = a_i, \quad i = 1, \ldots, p, \end{cases} \tag{2.18}$$

together with the relationship between its corresponding homogeneous equation (2.15), which has been studied in the previous subsection to some extent.

Because of linearity of (2.18) it follows immediately that if $\varphi(t)$ is a solution of the corresponding homogeneous equation (2.15) and $\psi(t)$ is a solution of (2.18), then the sum $\varphi(t) + \psi(t)$ of the functions $\varphi(t)$ and $\psi(t)$ is again a solution of (2.18). Conversely, if $\varphi_1(t)$ and $\varphi_2(t)$ are two solutions of the nonhomogeneous equation (2.18), then the difference $\varphi_1(t) - \varphi_2(t)$ is a solution of the corresponding homogeneous equation (2.15).

By the above observation, therefore, the following theorem about of the solutions of a nonhomogeneous impulsive differential equations can easily be proved.

Theorem 2.4. *The set Ω_n' of solutions of an nth order linear nonhomogeneous impulsive equation (2.18) on J is the affine space*

$$\Omega_n' = \{x : x(t) = \varphi(t) + u(t), \quad u \in \Omega\},$$

where φ is a particular solution of (2.18).

As in the theory of differential equations, it is possible and helpful in many applications, to define a particular solution of an nth order nonhomogeneous equation (2.18) by means of the fundamental solutions of the corresponding homogeneous equation (2.15). To see this, we prove the following theorem; known as the *variation of parameters* formula.

Theorem 2.5 (Variation of Parameters). *Let $\Phi(t) = [\phi_1(t), \ldots, \phi_n(t)]$ be row vector of fundamental solutions of (2.15), then there is a solution $\psi(t)$ of the nonhomogeneous impulsive differential equation (2.18) of the form*

$$\psi(t) = \Phi(t) \left(\int_{t_0}^{t} \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n \, ds + \sum_{t_0 \le \theta_i < t} \widehat{\Phi}^{-1}(\theta_i^+) a_i \right)$$

for t_0 and t in J, such that $t_0 \le t$.

Similarly,

$$\psi(t) = \Phi(t) \left(\int_{t_0}^{t} \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n \, ds - \sum_{t \le \theta_i < t_0} \widehat{\Phi}^{-1}(\theta_i^+) a_i \right)$$

for $t < t_0$.

Proof. Let us consider the case when $t_0 \le t$, first. Since $\phi_1, \ldots, \phi_n$ are fundamental solutions of (2.15), it follows that $\widehat{\Phi}(t)\widehat{\Phi}^{-1}(t) = E$ so that

$$\Phi^{(j-1)}(t)\widehat{\Phi}^{-1}(t) = e_j^T = [0, \ldots, 0, 1, 0, \ldots, 0], \quad j = 1, \ldots n$$

for $t \neq \theta_i$, therefore, we have

$$\begin{aligned}
\psi'(t) &= \Phi'(t) \left(\int_{t_0}^{t} \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n \, ds + \sum_{t_0 \le \theta_i < t} \widehat{\Phi}^{-1}(\theta_i^+) a_i \right) \\
&\quad + \Phi(t)\widehat{\Phi}^{-1}(t) \frac{f(s)}{p_0(s)}\, e_n \\
&= \Phi'(t) \left(\int_{t_0}^{t} \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n \, ds + \sum_{t_0 \le \theta_i < t} \widehat{\Phi}^{-1}(\theta_i^+) a_i \right)
\end{aligned}$$

and hence, similarly we get

$$\psi^{(j-1)}(t) = \Phi^{(j-1)}(t) \left(\int_{t_0}^{t} \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n \, ds + \sum_{t_0 \le \theta_i < t} \widehat{\Phi}^{-1}(\theta_i^+) a_i \right)$$

for all $j = 1, \ldots, n$. However,

$$\begin{aligned}
\psi^{(n)}(t) &= \Phi^{(n)}(t)\left(\int_{t_0}^{t}\widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\,e_n\,ds + \sum_{t_0\le\theta_i<t}\widehat{\Phi}^{-1}(\theta_i^+)a_i\right)\\
&\quad + \Phi^{(n-1)}(t)\widehat{\Phi}^{-1}(t)\frac{f(s)}{p_0(s)}\,e_n\\
&= \Phi^{(n)}(t)\left(\int_{t_0}^{t}\widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\,e_n\,ds + \sum_{t_0\le\theta_i<t}\widehat{\Phi}^{-1}(\theta_i^+)a_i\right)\\
&\quad + \frac{f(s)}{p_0(s)}.
\end{aligned}$$

Substituting the functions $\psi(t)$ into (2.18) we see that $\psi(t)$ satisfies the differential equation.

On the other hand, by the above calculations we have

$$\widehat{\psi}(t) = \widehat{\Phi}(t)\left(\int_{t_0}^{t}\widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\,e_n\,ds + \sum_{t_0\le\theta_i<t}\widehat{\Phi}^{-1}(\theta_i^+)a_i\right)$$

for $t \neq \theta_i$, and hence it follows, remembering that θ_i are isolated points in J,

$$\begin{aligned}
\widehat{\psi}(\theta_k^-) &= \lim_{h\to 0^-}\widehat{\psi}(\theta_k + h)\\
&= \widehat{\Phi}(\theta_k^-)\left(\int_{t_0}^{\theta_k}\widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\,e_n\,ds + \sum_{t_0\le\theta_i<\theta_k}\widehat{\Phi}^{-1}(\theta_i^+)a_i\right),
\end{aligned}$$

but

$$\begin{aligned}
\widehat{\psi}(\theta_k^+) &= \lim_{h\to 0^+}\widehat{\psi}(\theta_k + h)\\
&= \widehat{\Phi}(\theta_k^+)\Bigg(\int_{t_0}^{\theta_k}\widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\,e_n\,ds\\
&\qquad + \sum_{t_0\le\theta_i<\theta_k}\widehat{\Phi}^{-1}(\theta_i^+)a_i + \widehat{\Phi}^{-1}(\theta_k^+)a_k\Bigg)
\end{aligned}$$

for all $k = 1, \ldots, p$. Therefore,

$$\Delta\widehat{\psi}|_{t=\theta_k} - B_k\widehat{\psi}(\theta_k^-) = a_k$$

holds. That is, $\psi(t)$ also satisfies the impulse condition in (2.15).

The case $t < t_0$ is similar. □

The following corollary of the variation of parameters formula proves helpful in investigation of the general solutions of an nth order linear impulsive differential equations.

Corollary 2.1. *Any solution $x = x(t)$ of (2.18) is of the form*

$$x(t) = \Phi(t)\left(c + \int_{t_0}^{t} \widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{t_0 \le \theta_i < t} \widehat{\Phi}^{-1}(\theta_i^+)a_i\right)$$

for $t_0 \le t$, where $\Phi(t)$ is any row vector of fundamental solutions of the corresponding homogeneous equation (2.15).

In particular,

$$x(t) = \Phi(t)\left(\widehat{\Phi}^{-1}(t_0)\,\xi + \int_{t_0}^{t} \widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{t_0 \le \theta_i < t} \widehat{\Phi}^{-1}(\theta_i^+)a_i\right)$$

for $t_0 \le t$ satisfies the initial conditions

$$x^{(j-1)}(t_0) = \xi_j, \quad j = 1, \ldots, n,$$

where the column vector ξ is $\xi = [\xi_1, \ldots, \xi_n]^T$.

A similar result holds for $t_0 > t$.

Example 2.3. A particular solution, $x = \psi(t)$, of the following problem,

$$\begin{cases} -x'' = 2, \quad t \neq \theta, \quad \theta \in \mathbb{R}, \\ \Delta\widehat{x}|_{t=1} - B\widehat{x}(1^-) = -2\begin{pmatrix} 0 \\ a+b \end{pmatrix}, \end{cases} \qquad B = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad 1 + b \neq 0$$

can be written as follows,

$$\psi(t) = \begin{cases} -t^2, & t \le \theta \\ 2(a+b)\theta + 2\theta^2 + a\theta^3 - (2(a+b) + 2\theta + a\theta^2)t - t^2, & t > \theta \end{cases}$$

Hence, the general solution, $x = x(t)$, of the impulsive differential equation can be given as

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \psi(t),$$

where c_1, c_2 are arbitrary constants and $\phi_1(t)$ and $\phi_2(t)$ are the fundamental solutions and given in Example 2.2. In particular, therefore, if $a = b = \theta = 1$ the general solution of the nonhomogeneous problem can be given as

$$x(t) = \begin{cases} c_1 + c_2 t - t^2, & t \le 1 \\ c_1 t + c_2(3t - 2) + 7 - 7t - t^2, & t > 1 \end{cases}$$

Hence, the solution satisfying the initial conditions

$$x(0) = x'(0) = 1$$

becomes

$$x(t) = \begin{cases} 1 + t - t^2, & t \le 1 \\ 2 + 4(t-1) + 7 - 7t - t^2, & t > 1 \end{cases}$$

$\diamondsuit$

Finally, in this section we will establish an integral representation for the solutions of nonhomogeneous impulsive differential equations, using the variation of parameters formula.

Let $s \in J$ be fixed, and let

$$x_s(t) = \frac{\Phi(t)\widehat{\Phi}^{-1}(s)}{p_0(s)}\, e_n, \quad t \in J,$$

where $\Phi(t)$ is a row vector of fundamental solutions of (2.15). Then, clearly, $x_s(t)$ satisfies the corresponding homogeneous equation (2.15) and

$$\widehat{x}_s(s) = \frac{1}{p_0(s)}\, e_n$$

holds. Denoting $x_s(t)$ by $g(t,s)$, namely

$$g(t,s) = x_s(t) \tag{2.19}$$

for all t and s in the interval J, it follows that the function $g(t,s)$ is independent of the choice of a fundamental solutions of (2.15). For, if $\Phi_1(t)$ is another row vector of fundamental solutions, then there would exists a nonsingular constant matrix C such that

$$\Phi_1(t) = \Phi(t)C, \quad \text{and} \quad \widehat{\Phi}_1(t) = \widehat{\Phi}(t)C$$

hold for all $t \in J$. Thus,

$$\Phi_1(t)\widehat{\Phi}_1^{-1}(s) = \Phi(t)C(\widehat{\Phi}(s)C)^{-1} = \Phi(t)\Phi^{-1}(s)$$

for all s and t in the interval J.

Similarly, we define, for each $i = 1, \ldots, p$, the functions

$$\begin{aligned} &x_{\theta_i^+}(t) = \Phi(t)\widehat{\Phi}^{-1}(\theta_i^+), \quad \text{and} \\ &h(t,\theta_i^+) = x_{\theta_i^+}(t), \quad i = 1, \ldots, p. \end{aligned} \tag{2.20}$$

The functions $h(t,\theta_i^+)$ for $i = 1, \ldots, p$ become solutions of (2.15), and they are also independent of the fundamental solutions of (2.15). Also notice that, $\widehat{x}_{\theta_i^+}(\theta_i^+) = E$, and $h(t,\theta_i^+)$ for $i = 1, \ldots, p$ are $1 \times n$ matrices, say, of the following form,

$$h(t,\theta_i^+) = [h_1(t,\theta_i^+), \ldots, h_n(t,\theta_i^+)], \quad i = 1, \ldots, p.$$

Now, for any row vector, $\Phi(t)$, of fundamental solutions, the solution $x = x(t)$ of nonhomogeneous impulsive differential equation (2.18) can be written as

$$x(t) = \Phi(t)\left(\widehat{\Phi}^{-1}(t_0)\xi + \int_{t_0}^{t} g(t,s)f(s)\,ds + \sum_{t_0\le\theta_i<t} h(t,\theta_i^+)a_i\right), \qquad (2.21)$$

where $\xi = [\xi_1, \ldots, \xi_n]^T$, and this solution satisfies the initial condition

$$x^{(j-1)}(t_0) = \xi_j, \quad j = 1, \ldots, n.$$

The representation (2.21), of the solution of nonhomogeneous impulsive differential equation (2.18) will be significant in the study of Green's functions for boundary value problems for higher order impulsive equations in the next chapter.

Chapter 3

Boundary Value Problems

In Chapter 2, we have been concerned with the existence and uniqueness, and properties of solutions of initial value problems for impulsive differential equations. However, the study of physical problems, and applications of impulsive differential equations also require consideration of *boundary value problems* for impulsive differential equations. Throughout this chapter, we will define boundary value problems for higher order linear impulsive differential equations, and study the basic characteristic nature of them, as well as their solutions, which are forced to satisfy mostly homogeneous boundary conditions at the end points of an interval.

We will develop this chapter in parallel with the development of the corresponding theory of boundary value problem for classical ordinary differential equations so that the differences and similarities between them become much clearer.

Although the theory of boundary value problems for ordinary differential equations is widely known, the corresponding theory for impulsive differential equations has not been studied as much, especially for higher order impulsive differential equations. The use of Green's functions for first order periodic boundary value problems is extensively studied by several authors (see [17, 20, 43] and references therein). Also, there is some research on the construction of Green's functions for specific problems [44, 46], among which are the problems with nonlinear boundary conditions, or nonlinear first order impulsive differential equations. As an output of boundary value problems the inverse eigenvalue problems are also studied by several authors [32, 50, 51]. However, the general theory for linear boundary value problems for higher order impulsive differential equations has not been considered.

Our aim in the present chapter, which constitutes for the main part of the present work, is to investigate linear boundary value problems for higher order impulsive differential equations. We note that a solution of an impulsive differential equation has jump discontinuities at the impulse points. In the case of higher order impulsive differential equations not only solutions but their derivatives may possess jump discontinuities at the impulse points.

3.1 Green's Formula

Let $J = [\alpha, \beta]$ be an interval of $\mathbb{R}$, and let $\{\theta_i\}_{i=1}^p$, $\theta_i \in J$, be the sequence of impulse points such that

$$\theta_0 = \alpha < \theta_1 < \cdots < \theta_p < \beta = \theta_{p+1}.$$

Let ℓ be the differential operator of order n ($n \geq 1$) defined by

$$\ell(x) = p_0(t)x^{(n)} + \cdots + p_n(t)x, \quad t \neq \theta_i, \tag{3.1}$$

where $p_k : J \to \mathbb{F}$ are functions of class $\mathcal{PLC}^{n-k}$ for $k = 0, 1, \ldots, n$ and $p_0(t) \neq 0$ for all $t \in J$, and $p_0(\theta_i^+) \neq 0$ for $i = 1, \ldots, p$. In short, $\frac{1}{p_0} \in \mathcal{PLC}$.

At the impulse points θ_i, we define the *linear impulse actions*, δ_i, formally by

$$\delta_i(x) = \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-), \quad i = 1, \ldots, p, \tag{3.2}$$

where $\widehat{x} = [x, x', \ldots, x^{(n-1)}]^T$ is as defined in the previous chapter, and $\Delta\widehat{x}|_{t=\theta_i} = \widehat{x}(\theta_i^+) - \widehat{x}(\theta_i^-)$. Furthermore, we will always assume that B_i are constant matrices satisfying the conditions that

$$\det(E + B_i) \neq 0$$

for all $i = 1, \ldots, p$.

Moreover, at the end points α and β of the interval J we define linear boundary forms U_ν as follows

$$U_\nu(x) = \sum_{j=1}^{n} M_{\nu j}x^{(j-1)}(\alpha) + N_{\nu j}x^{(j-1)}(\beta), \quad \nu = 1, \ldots, m, \tag{3.3}$$

where $M_{\nu j}$ and $N_{\nu j}$ are constants in $\mathbb{F}$, and m is a positive integer. These boundary forms may also be written as

$$U(x) = M\widehat{x}(\alpha) + N\widehat{x}(\beta), \tag{3.4}$$

where M and N are $m \times n$ matrices with entries $M_{\nu j}$ and $N_{\nu j}$, respectively.

It is worth noting here that when the impulse actions, δ_i, defined by (3.2), can also be written in the form

$$\delta_i(x) = \widehat{x}(\theta_i^+) - (E + B_i)\widehat{x}(\theta_i^-), \quad i = 1, \dots, p,$$

which resembles, for each fixed i, the boundary form defined by (3.4). In the subsequent sections the differences and similarities will be more clearer.

In the theory of boundary value problems for ordinary differential equations, the fundamental results follow from two important formulas: the *Green's formula* and the *boundary form formula* [27, 40, 41]. Unfortunately, these formulas require continuously differentiable functions on an interval. In this section, we shall introduce the Green's formula for functions belonging to $\mathcal{PLC}^n$; in other words, continuously differentiable functions up to of order n (inclusive) for $t \neq \theta_i$, and whose derivatives have discontinuities of the first kind at the impulse points $t = \theta_i$.

Let u and v be functions of class $\mathcal{PLC}^n$, then k times integration by parts yields

$$\begin{aligned} \int_\alpha^\beta \overline{v} p_{n-k} u^{(k)}\, ds &= \sum_{i=0}^{p} \int_{\theta_i}^{\theta_{i+1}} \overline{v} p_{n-k} u^{(k)}\, ds \\ &= \sum_{i=0}^{p} \Big\{ \Big[\overline{v} p_{n-k} u^{(k-1)} - (\overline{v} p_{n-k})' u^{(k-2)} + \cdots + \\ &\quad + (-1)^{k-1} (\overline{v} p_{n-k})^{(k-1)} u \Big]_{s=\theta_i^+}^{s=\theta_{i+1}^-} + \\ &\quad + (-1)^k \int_{\theta_i}^{\theta_{i+1}} (\overline{v} p_{n-k})^{(k)} u\, ds \Big\} \end{aligned} \tag{3.5}$$

for every $k = 0, \dots, n$. Here, $\overline{v}$ denotes the complex conjugate of the function v in the case when $\mathbb{F} = \mathbb{C}$. Summing these integrals over k form 0 to n, it follows that

$$\begin{aligned} \int_\alpha^\beta \overline{v}\, \ell(u)\, ds &= \sum_{k=0}^{n} \sum_{i=0}^{p} \Big[\overline{v} p_{n-k} u^{(k-1)} - (\overline{v} p_{n-k})' u^{(k-2)} + \cdots + \\ &\quad + (-1)^{k-1} (\overline{v} p_{n-k})^{(k-1)} u \Big]_{s=\theta_i^+}^{s=\theta_{i+1}^-} + \int_\alpha^\beta \overline{\ell^\dagger(v)}\, u\, ds, \end{aligned} \tag{3.6}$$

where the formal adjoint differential operator $\ell^\dagger$ denotes

$$\ell^\dagger(v) = (-1)^n (\overline{p}_0 v)^{(n)} + (-1)^{n-1} (\overline{p}_1 v)^{(n-1)} + \cdots + \overline{p}_n v, \quad t \neq \theta_i$$

similar to the adjoint operator in ordinary differential equations [27, 41]. If we set

$$\mathcal{S}(u, v) = \sum_{k=0}^{n} \Big[\overline{v} p_{n-k} u^{(k-1)} - \cdots + (-1)^{k-1} (\overline{v} p_{n-k})^{(k-1)} u \Big] \tag{3.7}$$

then (3.6) becomes

$$\int_\alpha^\beta \overline{v}\,\ell(u)\,ds - \int_\alpha^\beta \overline{\ell^\dagger(v)}\,u\,ds = \mathcal{S}(u,v)\Big|_{t=\alpha}^{t=\beta} - \sum_{i=1}^{p} \Delta\mathcal{S}(u,v)|_{t=\theta_i}. \tag{3.8}$$

It is readily seen that the right hand side of (3.8) depends on the values of u and v not only at the boundary points α, β, but also at the impulse points $\theta_i,\ i = 1, \ldots, p$ in the interval $J = [\alpha, \beta]$. Similar results were obtained in [1, 13] for first order equations.

The equality in (3.8) will be called the *Green's formula* for functions u and v in the space $\mathcal{PLC}$ of functions. The form $\mathcal{S}(u,v)$, however, is a bilinear form in the variables $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$, and is said to be the *bilinear form associated with the linear operator* ℓ. See, for instance [27]. Moreover, this $\mathcal{S}$ can be written as follows

$$\begin{aligned} \mathcal{S}(u,v) &= \sum_{k=1}^{n}\sum_{j=0}^{k-1}(-1)^j(\overline{v}p_{n-k})^{(j)}u^{(k-j-1)} \\ &= \sum_{m=1}^{n}\sum_{j+k=m-1}(-1)^j(\overline{v}p_{n-m})^{(j)}u^{(k)} \end{aligned} \tag{3.9}$$

and hence

$$\mathcal{S}(u,v) = \sum_{j,k=1}^{n} \overline{v}^{(j-1)}\,S_{jk}\,u^{(k-1)} = \widehat{v}^*\,S\,\widehat{u}, \tag{3.10}$$

where S is an $n \times n$ triangular matrix with entries S_{jk}, and it will be named as the *matrix of the bilinear form* $\mathcal{S}$. The form of the matrix S of the bilinear form $\mathcal{S}$ can be given as follows

$$S = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1(n-1)} & p_0 \\ S_{21} & S_{22} & \cdots & -p_0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{(n-1)1} & (-1)^{n-2}p_0 & \cdots & 0 & 0 \\ (-1)^{n-1}p_0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.11}$$

Since $p_0(t) \neq 0$ for all $t \in J$, and $p_0(\theta_i^+) \neq 0$ for every $i = 1, \ldots, p$, it follows that the matrix S of the bilinear form $\mathcal{S}$ is nonsingular for all $t \in J$. To be specific, S satisfies the following conditions,

$$\begin{aligned} \det S(t) &= (p_0(t))^n \neq 0, \quad \text{for all } t \in J, \\ \det S(\theta_i^+) &= (p_0(\theta_i^+))^n \neq 0, \quad \text{for all } i = 1, \ldots, p. \end{aligned} \tag{3.12}$$

The entries S_{jk} of the matrix S can be given in the following proposition:

Proposition 3.1. *The entries S_{jk}, $j+k \leq n+1$, of the matrix S in (3.11) are defined by*

$$S_{jk} = \sum_{s=j-1}^{n-k} (-1)^s \binom{s}{j-1} p_{n-s-k}^{(s-j+1)}, \quad j+k \leq n+1 \tag{3.13}$$

for $j = 1, \dots, n$ and $k = 1, \dots, n-j+1$.

In particular,

$$S_{jk} = 0, \quad \text{if } j+k > n+1, \text{ and}$$

$$S_{jk} = (-1)^{j-1} p_0, \quad \text{if } j+k = n+1.$$

Proof. By using (3.9), the result follows. □

Now we return (3.8) in which the summation contains the jumps, $\Delta\mathcal{S}(u,v)|_{t=\theta_i}$, of $\mathcal{S}(u,v)$ at the points θ_i of impulses. It is interesting to write these jumps, $\Delta\mathcal{S}(u,v)|_{t=\theta_i}$, in terms of the impulse actions, δ_i, defined by (3.2) and its *adjoint actions*, say $\delta_i^\dagger$, assuming they have the same form as δ_i. In other words, we write

$$\delta_i^\dagger(y) = \Delta\widehat{y}|_{t=\theta_i} - C_i \widehat{y}(\theta_i^-), \quad i = 1, \dots, p, \tag{3.14}$$

where $y \in \mathcal{PLC}^{n-1}$ and C_i are $n \times n$ constant matrices to be determined. To find C_i, we observe that

$$\Delta\mathcal{S}(u,v)|_{t=\theta_i} = \mathcal{S}(u,v)(\theta_i^+) - \mathcal{S}(u,v)(\theta_i^-).$$

In view of (3.10),

$$\begin{aligned} \Delta\mathcal{S}(u,v)|_{t=\theta_i} &= \widehat{v}^*(\theta_i^+)S(\theta_i^+)\widehat{u}(\theta_i^+) - \widehat{v}^*(\theta_i^-)S(\theta_i^-)\widehat{u}(\theta_i^-) \\ &= \widehat{v}^*(\theta_i^+)S(\theta_i^+)\left[\Delta\widehat{u}|_{t=\theta_i} - B_i\widehat{u}(\theta_i^-)\right] \\ &\quad + \left\{\widehat{v}^*(\theta_i^+) - \widehat{v}^*(\theta_i^-)S(\theta_i^-)\left[S(\theta_i^+)(E+B_i)\right]^{-1}\right\} \\ &\quad \times S(\theta_i^+)(E+B_i)\widehat{u}(\theta_i^-), \end{aligned}$$

from which it follows that the jumps of the bilinear form $\mathcal{S}(u,v)$ at the points of impulses θ_i can be written as follows

$$\Delta\mathcal{S}(u,v)|_{t=\theta_i} = \widehat{v}^* S(\theta_i^+)\delta_i(u) - (\delta_i^\dagger(v))^* \left[-S(\theta_i^+)(E+B_i)\widehat{u}(\theta_i^-)\right], \tag{3.15}$$

where

$$\delta_i^\dagger(y) = \Delta\widehat{y}|_{t=\theta_i} - \left\{\left[(E+B_i^*)S^*(\theta_i^+)\right]^{-1} S^*(\theta_i^-) - E\right\}\widehat{y}(\theta_i^-) \tag{3.16}$$

for $y \in \mathcal{PLC}^{n-1}$ and for all $i = 1, \dots, p$.

When (3.16) is compared with (3.14), the $n \times n$ matrices C_i are uniquely determined by the matrices B_i and S as follows

$$C_i = \left[(E+B_i^*)S^*(\theta_i^+)\right]^{-1} S^*(\theta_i^-) - E.$$

Moreover,

$$E + C_i = \left[(E + B_i^*)S^*(\theta_i^+)\right]^{-1} S^*(\theta_i^-), \quad i = 1, \ldots, p$$

are nonsingular if and only if $E + B_i$ are nonsingular for $i = 1, \ldots, p$, because $\det S^*(\theta_i^+) \neq 0$ and $\det S^*(\theta_i^-) \neq 0$ for every $i = 1, \ldots, p$.

Therefore, an equivalent form of Green's formula (3.8) can be presented in the following way

$$\begin{aligned} &\int_\alpha^\beta \overline{v}\ell(u)\, ds + \sum_{i=1}^p \widehat{v}^*(\theta_i^+)S(\theta_i^+)\delta_i(u) \\ &\quad - \int_\alpha^\beta \overline{\ell^\dagger(v)} u\, ds - \sum_{i=1}^p \left(\delta_i^\dagger(v)\right)^* \left[-S(\theta_i^+)(E + B_i)\widehat{u}(\theta_i^-)\right] \\ &\quad = \widehat{v}^*(t)\, S(t)\, \widehat{u}(t)\Big|_{t=\alpha}^{t=\beta} \end{aligned} \tag{3.17}$$

so that the left hand side of (3.17) depends on the operators ℓ and δ_i, and their adjoint operators $\ell^\dagger$ and $\delta_i^\dagger$; the right hand side of it, however, depends on the boundary points α and β, respectively.

It should be noted that, (3.15) or the Green's formula (3.17) uniquely determines the adjoint impulse actions $\delta_i^\dagger$, but the calculation of $\Delta\mathcal{S}(u, v)|_{t=\theta_i}$ can be carried out in different ways in order to find $\delta_i^\dagger$ in a form other than (3.14). However, in the next section we will show that those adjoint impulse actions are equivalent to $\delta_i^\dagger$, given in (3.16), in some way. To be specific, if $\tilde{\delta}_i^\dagger$ are other adjoint impulse actions, then

$$\tilde{\delta}_i^\dagger = F_i \delta_i^\dagger, \quad i = 1, \ldots, p$$

holds for some nonsingular $n \times n$ constant matrices F_i.

3.2 Boundary Forms and Impulse Actions

This section is about the properties of linear boundary forms

$$U_\nu(x) = \sum_{j=1}^n M_{\nu j} x^{(j-1)}(\alpha) + N_{\nu j} x^{(j-1)}(\beta), \quad \nu = 1, \ldots, m, \tag{3.18}$$

where $M_{\nu j}$ and $N_{\nu j}$ are constant in $\mathbb{F}$; and m is a positive integer, and α, β are the left and right boundary points of the interval $J = [\alpha, \beta]$, respectively. The properties of such boundary forms are extensively investigated in [27, 41], and within this work we will restate those properties in order to make the work be self-contained.

A set $\{U_1, \ldots, U_m\}$ of boundary forms (3.18) is said to be *linearly independent* if and only if

$$\sum_{\nu=1}^{m} c_i U_\nu(x) = 0$$

for all $x \in \mathcal{PLC}^{n-1}$ on J implies that $c_1 = c_2 = \cdots = c_m = 0$.

It is helpful to write the boundary forms U_ν defined by (3.18) in a vector form as follows

$$U(x) = M\widehat{x}(\alpha) + N\widehat{x}(\beta), \tag{3.19}$$

where $U = [U_1, \ldots, U_m]^T$ is a column vector with components $U_1, \ldots, U_m$, and M, N are $m \times n$ matrices with entries $M_{\nu j}, N_{\nu j}$, respectively. The *vector boundary form U* is said to have rank m if $U_\nu,\ \nu = 1, \ldots, m$ are linearly independent boundary forms. In other words, the vector boundary form U is of rank m if and only if

$$\operatorname{rank}(M : N) = m,$$

where the matrix $(M : N)$ is defined by

$$(M : N) = \begin{pmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{pmatrix}.$$

Unless stated otherwise, we assume that a set of boundary forms is a linearly independent set.

For any given vector boundary form U of rank m with components $U_1, \ldots, U_m$ it is always possible (in many ways) to construct a vector boundary form U_c of rank $2n-m$ with components $U_{m+1}, \ldots, U_{2n}$ such that the combined boundary forms $U_1, \ldots, U_{2n}$ constitutes a set of $2n$ linearly independent forms. If U is any vector form of rank m, and U_c is any form of rank $2n - m$ such that the vector boundary form with components $U_1, \ldots, U_{2n}$ has rank $2n$, then U and U_c are said to be *complementary boundary forms* [27].

The following theorem relates the bilinear form $\mathcal{S}(u, v)$, defined by (3.10), appearing in the Green's formula, with the complementary boundary forms U and U_c of ranks m and $2n - m$ respectively. Also, the theorem below states, explicitly, the existence of adjoint vector boundary form $U^\dagger$ of rank $2n - m$ for a given vector boundary form U of rank m. The theorem and its proof can be found in [27]. For determination of such adjoint boundary forms for higher order ordinary differential equations, see also [41].

Theorem 3.1 (Boundary Form Formula). *Given any boundary form U of rank m, and any complementary form U_c, there exist unique boundary forms $U_c^\dagger$ and $U^\dagger$ of rank m and $2n-m$, respectively, such that,*

$$\mathcal{S}(u,v)\Big|_{t=\alpha}^{t=\beta} = \left(U_c^\dagger(v)\right)^* U(u) + \left(U^\dagger(v)\right)^* U_c(u) \tag{3.20}$$

holds for any functions u, v of class $\mathcal{PLC}^{n-1}$.

If $\tilde{U}_c$ is any other complementary form to U, and $\tilde{U}_c^\dagger, \tilde{U}^\dagger$ are the corresponding forms of rank m and $2n-m$, respectively, then

$$\widetilde{U}^\dagger(y) = C^* U^\dagger(y) \tag{3.21}$$

holds for some nonsingular matrix C.

Since the impulse actions for each fixed i,

$$\delta_i(x) = \widehat{x}(\theta_i^+) - (E + B_i)\widehat{x}(\theta_i^-)$$

is similar to the boundary form $U(x) = M\widehat{x}(\alpha) + N\widehat{x}(\beta)$, and the jumps $\mathcal{S}(x,y)|_{t=\theta_i}$ can be considered as a bilinear form in the variables

$$\xi_i = \begin{pmatrix} x(\theta_i^-) \\ x(\theta_i^+) \end{pmatrix}, \qquad \eta_i = \begin{pmatrix} y(\theta_i^-) \\ y(\theta_i^+) \end{pmatrix}$$

with the nonsingular matrix $\tilde{\mathbf{S}}_i$ defined by

$$\tilde{\mathbf{S}}_i = \begin{pmatrix} -S(\theta_i^-) & 0 \\ 0 & S(\theta_i^+) \end{pmatrix},$$

where 0's in the above matrix are $n \times n$ zero matrices. Since δ_i is of rank n for each fixed i, it follows that

$$\delta_i(x) = [-(E + B_i) : E]\,\xi_i, \quad \text{with } \det(E + B_i) \neq 0$$

holds, and hence for a given complementary impulse action, say δ_i^c, of rank n, there exist unique $\delta_i^{c\dagger}$ and $\delta_i^\dagger$, each have rank n, for each fixed i, by Theorem 3.1. So, the following theorem is true.

Theorem 3.2 (Impulse Form Formula). *Let $i \in \{1, \ldots, p\}$ be fixed. Given any impulse action δ_i of the form*

$$\delta_i(x) = \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-)$$

with $E+B_i$ being nonsingular, and let δ_i^c be its complementary impulse action, defined by

$$\delta_i^c(x) = -S(\theta_i^+)(E+B_i)\widehat{x}(\theta_i^-)$$

where S is the matrix of the bilinear form $\mathcal{S}(x,y)$. Then there exist unique impulse actions $\delta_i^{c\dagger}$ and $\delta_i^\dagger$, with

$$\delta_i^{c\dagger}(y) = S^*(\theta_i^+)\widehat{y}(\theta_i^-),$$

$$\delta_i^\dagger(y) = \Delta\widehat{y}|_{t=\theta_i} - \left\{\left[(E+B_i^*)S^*(\theta_i^+)\right]^{-1} S^*(\theta_i^-) - E\right\}\widehat{y}(\theta_i^-)$$

such that

$$\Delta\mathcal{S}(x,y)|_{t=\theta_i} = \left(\delta_i^{c\dagger}(y)\right)^* \delta_i(x) - \left(\delta_i^\dagger(y)\right)^* \delta_i^c(x)$$

holds for all x, y in $\mathcal{PLC}^{n-1}$.

Recall that the adjoint impulse actions $\delta_i^\dagger$ for $i = 1, \ldots, p$ were obtained already by computation of $\Delta\mathcal{S}(x,y)|_{t=\theta_i}$ in Section 3.1, see the equations (3.15) and (3.15). From (3.15), however, one can extract also the complementary impulse actions δ_i^c, as well as the $\delta_i^{c\dagger}$, which are explicitly given in Theorem 3.2.

In the following section, we will define homogeneous boundary value problems for higher order linear impulsive differential equations and establish some basic features of such problems.

3.3 Homogeneous Boundary Value Problems

For any vector boundary form U of rank m, we consider the homogeneous boundary condition

$$U(x) = 0 = M\widehat{x}(\alpha) + N\widehat{x}(\beta)$$

for any function x in $\mathcal{PLC}^{n-1}$. Similarly, the homogeneous impulse conditions

$$\delta_i(x) = 0 = \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-), \quad i = 1, \ldots, p.$$

The problem of finding a function $x \in \mathcal{PLC}^n$ which satisfies

$$(\mathrm{BVP})_m \qquad \begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \ldots, p, \\ U(x) = 0 \end{cases}$$

is called a *homogeneous boundary value problem* of rank m. The associated boundary value problem,

$$(\text{A-BVP})_{2n-m} \qquad \begin{cases} \ell^{\dagger}(x) = 0, & t \neq \theta_i, \\ \delta_i^{\dagger}(x) = 0, & i = 1, \ldots, p, \\ U^{\dagger}(x) = 0 \end{cases}$$

is called the *adjoint boundary value problem* of rank $2n-m$, associated with $(\text{BVP})_m$. Here,

$$\ell^{\dagger}(y) = (-1)^n (\overline{p}_0 y)^{(n)} + (-1)^{n-1} (\overline{p}_1 y)^{(n-1)} + \cdots + \overline{p}_n y, \quad t \neq \theta_i$$

and

$$\delta_i^{\dagger}(y) = \Delta \widehat{y}|_{t=\theta_i} - \left\{ \left[(E + B_i^*) S^*(\theta_i^+) \right]^{-1} S^*(\theta_i^-) - E \right\} \widehat{y}(\theta_i^-) \tag{3.22}$$

for $i = 1, \ldots, p$, and the adjoint vector boundary for $U^{\dagger}$ of rank $2n - m$ is defined by the boundary form formula, Theorem 3.20. In fact, If $U^{\dagger}$ is assumed to be of the following form

$$U^{\dagger}(y) = P^* \widehat{y}(\alpha) + Q^* \widehat{y}(\beta),$$

where P and Q are matrices having n rows and $2n-m$ columns such that the matrix $(P^* : Q^*)$ is of rank $2n - m$. It is possible to characterize the adjoint boundary conditions $U^{\dagger}(y) = 0$ directly in terms of these matrices P and Q, as well as the matrices M and N of the boundary condition $U(x) = 0$.

The following theorem, see [27], relates the boundary conditions $U(x) = 0$ and $U^{\dagger}(y) = 0$.

Theorem 3.3. *The boundary condition $U(x) = 0$ is adjoint to $U^{\dagger}(y) = 0$ if and only if*

$$M S^{-1}(\alpha) P = N S^{-1}(\beta) Q \tag{3.23}$$

where S is the matrix associated with the bilinear form $\mathcal{S}$.

In view of the theorem above, it should be noted that from the impulse actions δ_i and $\delta_i^{\dagger}$, when compared to the boundary forms U and $U^{\dagger}$, the relation similar to (3.23) naturally holds, because

$$\begin{aligned} S^{-1}(\theta_i^+) &= (E + B_i) S^{-1}(\theta_i^-) S(\theta_i^+) \left[S(\theta_i^+)(E + B_i^*) \right]^{-1} \\ &= S^{-1}(\theta_i^+) \end{aligned}$$

holds for $i = 1, \ldots, p$.

On the other hand, the theorem above yields the following corollary on the self-adjoint boundary conditions $U(x) = 0$.

Corollary 3.1. *If $m = n$, the boundary condition $U(x) = 0$ is adjoint to itself if and only if the equality*

$$MS^{-1}(\alpha)M^* = NS^{-1}(\beta)N^* \tag{3.24}$$

holds.

In terms of the impulse actions, however, equality (3.24) can be replaced by

$$E + B_i = \left[(E + B_i^*)S^*(\theta_i^+)\right]^{-1} S^*(\theta_i^-)$$

for $i = 1, \ldots, p$, which simply means that the impulse actions δ_i and $\delta_i^\dagger$ are exactly of the same forms. In the following corollary we state a necessary and sufficient condition for an impulse action δ_i to be self-adjoint; that is, adjoint to itself.

Corollary 3.2. *Let $i \in \{1, \ldots, p\}$ be fixed. An impulse action δ_i, defined by*

$$\delta_i(x) = \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-), \quad i = 1, \ldots, p$$

is self-adjoint if and only if

$$S^{-1}(\theta_i^+) = (E + B_i)S^{-1}(\theta_i^-)(E + B_i^*), \tag{3.25}$$

where S is the matrix associated with the bilinear form $\mathcal{S}$.

Proof. The adjoint impulse action $\delta_i^\dagger$ is defined by (3.22), hence substituting the condition (3.25) into (3.22) we obtain

$$\begin{aligned}
&\left[(E + B_i^*)S^*(\theta_i^+)\right]^{-1} S^*(\theta_i^-) - E \\
&\qquad = (E + B_i)S^{-1}(\theta_i^-)(E + B_i^*)(E + B_i^*)^{-1}S^{-1}(\theta_i^-) - E \\
&\qquad = B_i.
\end{aligned}$$

Thus, $\delta_i^\dagger = \delta_i$ and the proof is complete. $\square$

In view of the above discussion, we call the boundary value problem

$$\begin{cases} \ell(x) = 0, \quad t \neq \theta_i, \\ \delta_i(x) = 0, \quad i = 1, \ldots, p, \\ U(x) = 0 \end{cases}$$

a *self-adjoint* boundary value problem if and only if

(a) $\ell = \ell^\dagger$,

(b) $S^{-1}(\theta_i^+) = (E + B_i)S^{-1}(\theta_i^-)(E + B_i^*)$, for all $i = 1, \ldots, p$,

(c) $MS^{-1}(\alpha)M^* = NS^{-1}(\beta)N^*$,

where S is the matrix associated with the bilinear form $\mathcal{S}$. It should be remarked that for a self-adjoint boundary value problem, the rank m of the boundary form U equals to the order n of the differential operator. Also, the adjoint impulse actions $\delta_i^\dagger$ are identical with the impulse actions δ_i for $i = 1, \ldots, p$.

Let us define two linear subspaces D and D_0 of $\mathcal{PLC}$ as follows:

$$D = \{x \in \mathcal{PLC} : U(x) = 0\}, \text{ and}$$

$$D_0 = \{x \in \mathcal{PLC} : \delta_i(x) = 0, \quad i = 1, \ldots, p\}.$$

Clearly,

$$D \cap D_0 = \{x \in \mathcal{PLC} : U(x) = 0 \text{ and } \delta_i(x) = 0, \quad i = 1, \ldots, p\}$$

is again a linear subspace of $\mathcal{PLC}$. Let $\mathcal{L} : D_0 \to \mathcal{PLC}$ be a linear operator defined on D_0 by the differential operator ℓ,

$$\mathcal{L}x = \ell(x),$$

then a homogeneous impulsive differential equations can simply be written as

$$\mathcal{L}x = 0.$$

Similarly, we define $\mathcal{L}^\dagger : D_0^\dagger \to \mathcal{PLC}$ by

$$\mathcal{L}^\dagger y = \ell^\dagger(y)$$

for $y \in D_0^\dagger$ where

$$D_0^\dagger = \left\{y \in \mathcal{PLC} : \delta_i^\dagger(y) = 0, \quad i = 1, \ldots, p\right\}.$$

The space $D^\dagger$ is similarly defined.

Let the functions $\phi_1, \ldots, \phi_n$ be fundamental solutions of $\mathcal{L}x = 0$, and let $\widehat{\Phi}$ be a fundamental matrix for $\mathcal{L}x = 0$. That is,

$$\widehat{\Phi} = [\phi_1, \ldots, \phi_n] = \begin{pmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix}.$$

Similarly, if $\psi_1, \ldots, \psi_n$ are the fundamental solutions of the corresponding adjoint impulsive differential equation $\mathcal{L}^\dagger y = 0$, then a fundamental matrix $\widehat{\Psi}$ for $\mathcal{L}^\dagger y = 0$ is given by

$$\widehat{\Psi} = [\psi_1, \ldots, \psi_n] = \begin{pmatrix} \psi_1 & \cdots & \psi_n \\ \psi_1' & \cdots & \psi_n' \\ \vdots & \ddots & \vdots \\ \psi_1^{(n-1)} & \cdots & \psi_n^{(n-1)} \end{pmatrix}.$$

Thus, we may extend, (see [27]), the definitions of vector boundary forms U and $U^\dagger$ to matrices by defining

$$U\widehat{\Phi} = M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta),$$

and

$$U^\dagger\widehat{\Psi} = M\widehat{\Psi}(\alpha) + N\widehat{\Psi}(\beta),$$

respectively. Similar extension is possible for the impulse actions δ_i and $\delta_i^\dagger$.

The following theorem ascertain a necessary and sufficient condition for the existence of nontrivial solutions of homogeneous boundary value problems for higher order linear impulsive differential equations.

Theorem 3.4. *The problem (BVP)$_m$ has exactly k, $0 \leq k \leq n$, linearly independent solutions if and only if $U\widehat{\Phi}$ has rank $n-k$, where $\widehat{\Phi}$ is any fundamental matrix for $\mathcal{L}x = 0$.*

Proof. The function φ satisfies $\mathcal{L}x = 0$ if and only if the corresponding vector $\widehat{\varphi}$ with components $\varphi, \varphi', \ldots, \varphi^{(n-1)}$ is of the form $\widehat{\varphi} = \widehat{\Phi}c$, where c is a constant vector, and $\widehat{\Phi}$ is any fundamental matrix for $\mathcal{L}x = 0$. Thus, $U(\varphi) = 0$ if and only if

$$U(\widehat{\Phi}c) = (U\widehat{\Phi})c = 0.$$

The number of linearly independent vectors c satisfying $(U\widehat{\Phi})c = 0$ is, however, $n - \operatorname{rank}(U\widehat{\Phi})$,

On the other hand, if Φ_1 is any other fundamental matrix for $\mathcal{L}x = 0$, then we know that $\widehat{\Phi}_1 = \widehat{\Phi}C$, for some nonsingular matrix C. Therefore,

$$\operatorname{rank}(U\widehat{\Phi}_1) = \operatorname{rank}(U\widehat{\Phi}C) = \operatorname{rank}(U\widehat{\Phi})$$

completes the proof. $\square$

If the boundary value problem $(\mathrm{BVP})_m$ has exactly k linearly independent solutions, say $\varphi_1, \ldots, \varphi_k$, then any linear combination

$$\varphi = \sum_{i=1}^{k} c_i \varphi_i,$$

where c_i are constants in $\mathbb{F}$, is again a solution. Moreover, any solution of the problem $(\mathrm{BVP})_m$ is of the form φ for some constants c_i. That is, the solutions of linear boundary value problem $(\mathrm{BVP})_m$ form a vector space over $\mathbb{F}$ with dimension k.

Example 3.1. Consider the following boundary value problem for an impulsive differential equation,

$$\begin{cases} -x'' = 0, \quad t \neq \theta, \\ \Delta\widehat{x}|_{t=\theta} - B\widehat{x}(\theta^-) = 0, \\ x(0) = x(\pi) = 0, \end{cases} \qquad B = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad 1 + b \neq 0$$

on the interval $[0, \pi]$. We may state the following:

(a) If $\theta + (a\theta + 1 + b)(\pi - \theta) \neq 0$, then the problem has only the trivial solution, $x(t) = 0$ for all $t \in [0, \pi]$, and

(b) If $\theta + (a\theta + 1 + b)(\pi - \theta) = 0$, then the problem has a nontrivial solution,

$$x(t) = c \begin{cases} t, & t \leq \theta \\ \theta + (a\theta + 1 + b)(t - \theta), & t > \theta \end{cases} \qquad c \in \mathbb{F}.$$

In particular, if $\theta = 1$, and the matrix B is as follows

$$B = \begin{pmatrix} 0 & 0 \\ -1 & -\dfrac{1}{\pi - 1} \end{pmatrix},$$

then we have $1 - \dfrac{1}{\pi - 1} \neq 0$, and hence, the one-parameter family of solutions of the given problem becomes

$$x(t) = c \begin{cases} t, & t \leq 1 \\ 1 - \dfrac{t-1}{\pi - 1}, & t > 1 \end{cases} \qquad c \in \mathbb{F}.$$

$\diamondsuit$

There exists a certain duality between the number of nontrivial solutions of $(\text{BVP})_m$ and its corresponding adjoint problem $(\text{A-BVP})_{2n-m}$. We close this section by presenting this duality in the following theorem.

Theorem 3.5. *If $(BVP)_m$ has k linearly independent solutions, then the adjoint problem $(A\text{-}BVP)_{2n-m}$ has $k+m-n$ linearly independent solutions.*

In particular, if $m = n$, they have the same number of linearly independent solutions.

Proof. See [27]. □

3.4 Nonhomogeneous Boundary Value Problems

A *nonhomogeneous boundary value problem* associated with the problem $(\text{BVP})_m$ is the problem

$$(\text{N-BVP})_m \qquad \begin{cases} \ell(x) = f(t), & t \neq \theta_i, \\ \delta_i(x) = a_i, & i = 1, \ldots, p, \\ U(x) = \gamma, \end{cases}$$

where $f \in \mathcal{PLC}$, and a_i, γ are column vectors if $\mathbb{F}^n$ such that $f(t) \neq 0$ for some $t \in J = [\alpha, \beta]$, or $a_i \neq 0$ for some $i = 1, \ldots, p$, or $\gamma \neq 0$. Here the vector boundary form U is assumed to be of rank m.

Clearly, if φ and ψ are two solutions of the nonhomogeneous boundary value problem $(\text{N-BVP})_m$, the difference $\varphi - \psi$ is a solution of the corresponding homogeneous problem $(\text{BVP})_m$, namely

$$(\text{BVP})_m \qquad \begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \ldots, p, \\ U(x) = 0. \end{cases}$$

Hence, if the homogeneous boundary value problem $(\text{BVP})_m$ has k linearly independent solutions, say $\psi_1, \ldots, \psi_k$ then

$$\varphi = \psi + \sum_{j=1}^{k} c_j \psi_j$$

holds for some constants $c_1, \ldots, c_k$.

It is well-known from the theory of boundary value problems for ordinary differential equations that a nonhomogeneous boundary value problem does not always possess a solution, see for instance [27, 41, 47], which is of course the case in boundary value problems for impulsive differential equations. The following theorem provides a necessary and sufficient condition for the existence of a solution of $(\text{N-BVP})_m$. A similar theorem for the existence of a periodic solution of periodic first order impulsive equations can be found in [43].

Theorem 3.6. *The nonhomogeneous boundary value problem* $(\text{N-BVP})_m$ *has a solution if and only if the condition,*

$$\int_\alpha^\beta \overline{\psi}(s) f(s)\, ds + \sum_{i=1}^{p} \widehat{\psi}^*(\theta_i^+) S(\theta_i^+) a_i = (U_c^\dagger(\psi))^*\, \gamma \tag{3.26}$$

holds for every solution of the adjoint homogeneous problem $(\text{A-BVP})_{2n-m}$*.*

Proof. Let φ be a solution of problem $(\text{N-BVP})_m$ and ψ be any solution of the adjoint homogeneous problem $(\text{A-BVP})_{2n-m}$, then the Green's formula (3.17), together with the boundary form formula (3.20) gives (3.26) immediately, proving the necessity.

Conversely, suppose (3.26) holds for every ψ of the problem $(\text{A-BVP})_{2n-m}$. We know that every solution φ of the nonhomogeneous impulsive differential equation

$$\begin{cases} \ell(x) = f(t), & t \neq \theta_i, \\ \delta_i(x) = a_i, & i = 1, \dots, p \end{cases} \tag{3.27}$$

is of the following form

$$\varphi = \sum_{i=1}^{n} c_i \phi_i + \varphi_p,$$

where $\phi_1, \dots, \phi_n$ are fundamental solutions of the corresponding homogeneous equation to (3.27), c_i are constants, and φ_p is a particular solution of (3.27). Thus, the problem $(\text{N-BVP})_m$ has a solution only if there exists constants $c_1, \dots, c_n$ such that

$$\sum_{i=1}^{n} c_i\, U(\phi_i) + U(\varphi_p) = \gamma,$$

or equivalently,

$$(U\widehat{\Phi})\, c = \gamma - U(\varphi_p), \tag{3.28}$$

where $\widehat{\Phi} = [\widehat{\phi}_1, \dots, \widehat{\phi}_n]$ is the fundamental matrix corresponding to $\phi_1, \dots, \phi_n$, and $c = [c_1, \dots, c_n]^T$ is a column vector with components $c_1, \dots, c_n$. Since the system (3.28) has a solution c if and only if $\gamma - U(\varphi_p)$ is orthogonal to every solution u of the corresponding adjoint homogeneous system, we have

$$u^*(\gamma - U(\varphi_p)) = 0 \tag{3.29}$$

for every u satisfying

$$(U\widehat{\Phi})^* u = 0. \tag{3.30}$$

On the other hand, if (A-BVP)_{2n-m} has exactly $\tilde{k}$ linearly independent solutions $\psi_1, \ldots, \psi_{\tilde{k}}$. It can easily be shown, (see for instance [27]) that the vectors $U_c^\dagger(\psi_1), \ldots, U_c^\dagger(\psi_{\tilde{k}})$ are linearly independent column vectors in $\mathbb{F}^m$, which are solutions of (3.30). However, (3.30) has $m - \text{rank}(U\widehat{\Phi}) = m - (n-k)$ linearly independent solutions, where k is the number of linearly independent solutions of (BVP)_m. Thus, Theorem 3.5 in the previous section implies that $\tilde{k} = m - n + k$.

Hence, (3.29) holds for every u satisfying (3.30) if and only if

$$(U_c^\dagger(\psi_j))^* \, (\gamma - U(\varphi_p)) = 0$$

holds for every $j = 1, \ldots, \tilde{k}$. Applying Green's formula to the functions φ_p and ψ_j, we have

$$\int_\alpha^\beta \overline{\psi}_j(s) f(s)\, ds + \sum_{i=1}^{p} \widehat{\psi}_j^*(\theta_i^+) S(\theta_i^+) a_i = (U_c^\dagger(\psi_j))^* \, U(\varphi_p)$$

for each $j = 1, \ldots, \tilde{k}$. But, condition (3.26) is assumed to be true for every solution ψ of (A-BVP)_{2n-m}. Thus, we obtain

$$(U_c^\dagger(\psi_j))^* \, U(\varphi_p) = (U_c^\dagger(\psi))^* \, \gamma,$$

or

$$(U_c^\dagger(\psi_j))^* \, (\gamma - U(\varphi_p)) = 0$$

for every $j = 1, \ldots, \tilde{k}$. Therefore, there exists a constant vector $c = [c_1, \ldots, c_n]^T$ such that the (3.28) holds. This completes the proof. ☐

The case $m = n$ is of great interest in many applications of differential equations in science, since well-posed problems require uniqueness of solutions as well as the existence.

Corollary 3.3. *If $m = n$ and the only solution of (BVP)_n is the trivial one, then the nonhomogeneous boundary value problem (N-BVP)_n has a unique solution.*

Proof. Let $m = n$, and (BVP)_n has only the trivial solution. Then, $\text{rank}(U\widehat{\Phi}) = n$, and hence (3.28) can be solved uniquely for c. This completes the proof. ☐

Example 3.2. Consider the nonhomogeneous boundary value problem,

$$\begin{cases} -x'' = 0, \quad t \neq \theta, \\ \Delta\widehat{x}|_{t=\theta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad 0 < \theta < \pi. \\ x(0) = x(\pi) = 0, \end{cases}$$

This problem has a unique solution, since the corresponding homogeneous equation has only the trivial solution. For,

$$\theta + (a\theta + 1 + b)(\pi - \theta) = \theta + (\pi - \theta) = \pi \neq 0$$

by the previous Example 3.1, with $a = b = 0$. The unique solution, $x = x(t)$, can be written as follows,

$$x(t) = \begin{cases} \left(1 - \frac{\theta}{\pi}\right) t, & 0 \leq t \leq \theta \\ \theta \left(1 - \frac{t}{\pi}\right), & \theta < t \leq \pi \end{cases}$$

$\diamondsuit$

Example 3.3. Consider the nonhomogeneous boundary value problem,

$$\begin{cases} -x'' = 2, \quad t \neq 1, \\ \Delta\widehat{x}|_{t=1} - B\widehat{x}(1^-) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ x(0) = x(\pi) = 0, \end{cases}$$

with homogeneous boundary conditions at $t = 0$ and $t = \pi$. This problem has also a unique solution, $x = x(t)$, and defined by

$$x(t) = \begin{cases} \dfrac{\pi^2 + 7\pi - 7}{3\pi - 2} t - t^2, & t \leq 1 \\ \dfrac{(7 - 2\pi)\pi}{3\pi - 2} + \dfrac{3\pi^2 - 7}{3\pi - 2} t - t^2, & t > 1 \end{cases}$$

which can be obtained by the use of Example 2.3. $\diamondsuit$

We will show, however, by the use of Green's functions for impulsive differential equations, the unique solution of the problem above, in Example 3.3, can be obtained by taking an integral over the region $[0, \pi]$. This will be studied in the following section.

3.5 Green's Functions

Suppose that the rank m of the vector boundary form U is equal to the order n of the differential operator ℓ, that is $m = n$, and that the homogeneous boundary value problem $(\mathrm{BVP})_n$ has only the trivial solution. Then, it is possible to express the unique solution of the nonhomogeneous boundary value problem

$$\begin{cases} \ell(x) = f(t), \quad t \neq \theta_i, \\ \delta_i(x) = a_i, \quad i = 1, \ldots, p, \\ U(x) = 0, \end{cases} \tag{3.31}$$

with $\gamma = 0$, explicitly, in terms of the so-called *Green's Functions*.

Clearly, because of the impulse conditions in (3.31) Green's function for an impulsive boundary value problem differs from the one for an ordinary boundary value problem. The former becomes a function which is defined piecewise, at the least. Moreover, within this section it will be shown that the name Green's functions for an impulsive boundary value problem will mean more than just a single function when the order of the differential operator is more than unity.

Green's function for a first order impulsive differential systems was studied, mainly, in [43] for periodic equations. However, the theory constructed therein needs some improvement.

If $\Phi(t)$ is any row vector of fundamental solutions of

$$\begin{cases} \ell(x) = 0, \quad t \neq \theta_i, \\ \delta_i(x) = 0, \quad i = 1, \ldots, p, \end{cases} \tag{3.32}$$

then the variation of parameters formula obtained in Chapter 2 yields that the solution $x = x(t)$ of the nonhomogeneous boundary value problem (3.31) is of the form,

$$x(t) = \Phi(t)\, c + \Phi(t) \left(\int_\alpha^t \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{\alpha < \theta_j < t} \widehat{\Phi}^{-1}(\theta_j^+)\, a_j \right), \tag{3.33}$$

where c is a constant column vector that is to be determined from the boundary conditions $U(x) = 0$ in (3.31). As it has already been computed in Section 2.2.2 we have

$$\widehat{x}(t) = \widehat{\Phi}(t)\, c + \widehat{\Phi}(t) \left(\int_\alpha^t \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{\alpha < \theta_j < t} \widehat{\Phi}^{-1}(\theta_j^+)\, a_j \right),$$

and thus, by using the homogeneous boundary conditions we deduce that

$$[M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]\, c = -N\widehat{\Phi}(\beta) \left(\int_\alpha^\beta \widehat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{\alpha < \theta_j < \beta} \widehat{\Phi}^{-1}(\theta_j^+)\, a_j \right).$$

Letting

$$K = -[M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]^{-1} N\widehat{\Phi}(\beta)$$

we obtain

$$c = K\left(\int_\alpha^\beta \widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{\alpha<\theta_j<\beta} \widehat{\Phi}^{-1}(\theta_j^+)\, a_j\right),$$

and hence,

$$\begin{aligned} x(t) &= \Phi(t)(E+K)\left(\int_\alpha^t \widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{\alpha<\theta_j<t} \widehat{\Phi}^{-1}(\theta_j^+)\, a_j\right) \\ &\quad +\Phi(t)K\left(\int_t^\beta \widehat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}\, e_n\, ds + \sum_{t\le\theta_j<\beta} \widehat{\Phi}^{-1}(\theta_j^+)\, a_j\right). \end{aligned}$$

Therefore, the solution $x(t)$ of the nonhomogeneous boundary value problem (3.33) can be written simply as

$$x(t) = \int_\alpha^\beta G(t,s) f(s)\, ds + \sum_{j=1}^p H(t,\theta_j^+)\, a_j, \tag{3.34}$$

where the functions $G(t,s)$ and $H(t,\theta_j^+)$ are uniquely defined by the following equations.

$$G(t,s) = \begin{cases} \Phi(t)(E+K)\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, & s<t \\ \Phi(t)K\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, & s\ge t \end{cases} \tag{3.35}$$

and

$$H(t,\theta_j^+) = \begin{cases} \Phi(t)(E+K)\widehat{\Phi}^{-1}(\theta_j^+), & \theta_j<t \\ \Phi(t)K\widehat{\Phi}^{-1}(\theta_j^+), & \theta_j\ge t \end{cases} \tag{3.36}$$

for all $t \in [\alpha,\beta]$ and θ_j for $j = 1,\ldots,p$. Simply the couple $\{G,H\}$ of functions G and H is called *Green's couple*, or each of the functions $G(t,s)$ and $H(t,\theta_j^+)$ for any $j = 1,\ldots,p$ is going to be called *Green's function* for the homogeneous problem

$$\begin{cases} \ell(x) = 0, & t \ne \theta_i, \\ \delta_i(x) = 0, & i = 1,\ldots,p, \\ U(x) = 0. \end{cases}$$

It can also be shown that the functions $G(t,s)$ and $H(t,\theta_j^+)$, $j = 1,\ldots,p$, are independent of the choice of the fundamental solutions of the linear homogeneous impulsive differential equation (3.32). For, if $\Phi_1(t)$ is any other row vector of fundamental solutions, then there exists a nonsingular matrix C such that $\Phi_1(t) = \Phi(t)C$ holds for all $t \in [\alpha, \beta]$. Hence, if

$$\tilde{K} = -[M\widehat{\Phi}_1(\alpha) + N\widehat{\Phi}_1(\beta)]^{-1}N\widehat{\Phi}_1(\beta)$$

then

$$\begin{aligned}\widehat{\Phi}_1(t)\tilde{K}\widehat{\Phi}_1^{-1}(s) &= \Phi(t)C\left\{-C^{-1}[M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]^{-1}N\widehat{\Phi}(\beta)C\right\}C^{-1}\widehat{\Phi}^{-1}(s) \\ &= \widehat{\Phi}(t)K\widehat{\Phi}^{-1}(s)\end{aligned}$$

for all t and s in $[\alpha, \beta]$. Similarly,

$$\widehat{\Phi}_1(t)\tilde{K}\widehat{\Phi}_1^{-1}(\theta_j^+) = \widehat{\Phi}(t)K\widehat{\Phi}^{-1}(\theta_j^+)$$

holds for $j = 1, \ldots, p$.

It is readily seen from (3.35) that the function $G(t, s)$ is a scalar function of the variable (t, s) in the square $J^2 = [\alpha, \beta] \times [\alpha, \beta]$; and on the other hand, from (3.36) it follows that the function $H(t, \theta_j^+)$ is a vector ($1 \times n$ matrix) valued function defined on $[\alpha, \beta]$ for each fixed $j \in \{1, \ldots, p\}$. Indeed, the term θ_j^+ in $H(t, \theta_j^+)$ must be understood as an index (of summation). In other words, for each $j \in \{1, \ldots, p\}$ there corresponds a function $H(t, \theta_j^+)$, hence a finite sequence of functions $H(t, \theta_1^+), \ldots, H(t, \theta_p^+)$, each of which is a row vector.

Although the properties of the Green's function $G(t, s)$ is similar to that of corresponding Green's function for ordinary differential equations, see for instance [27, 41, 47], we should remark that the functions $G(t, s)$ and $H(t, \theta_j^+)$ for $j = 1, \ldots, p$ are all piecewise continuous, and have discontinuities of the first kind at the points of impulses.

3.5.1 Properties of Green's Function G

Let us consider the regions

$$R_{11} = [\alpha, \theta_1] \times [\alpha, \theta_1]$$

$$R_{i1} = (\theta_{i-1}, \theta_i] \times [\alpha, \theta_1], \quad i = 2, \ldots, p+1$$

$$R_{1j} = [\alpha, \theta_1] \times (\theta_{j-1}, \theta_j], \quad j = 2, \ldots, p+1$$

$$R_{ij} = (\theta_{i-1}, \theta_i] \times (\theta_{j-1}, \theta_j], \quad i, j = 2, \ldots, p+1$$

with $\theta_{p+1} = \beta$. Among these subregions R_{ij} of $J^2 = [\alpha, \beta] \times [\alpha, \beta]$ we have regions R_{ii} for $i = 1, \ldots, p+1$ in the form of squares which are divided into two triangles T_{ii}^u and T_{ii}^l by the line $t = s$ defined by

$$T_{ii}^u = \{(t, s) \in R_{ii} : s > t\}, \qquad T_{ii}^l = \{(t, s) \in R_{ii} : s < t\}. \tag{3.37}$$

Also, the line $t = s$ divide the whole square J^2 into two triangles that are similarly defined as

$$T^u = \left\{(t, s) \in J^2 : s > t\right\}, \qquad T^l = \left\{(t, s) \in J^2 : s < t\right\}. \tag{3.38}$$

Now, consider the case $n \geq 2$, for convenience. The case $n = 1$ can be treated similarly. In each of these rectangles R_{ij} the function $G(t, s)$ is continuous and $n-2$ times differentiable with respect to t, and have jump discontinuity in its $(n-1)$st derivative with respect to t at $t = s$. That is, in each of the triangles T_{ii}^u and T_{ii}^l of the squares R_{ii}. In the rectangles $R_{ij},\ i \neq j$, however, the Green's function $G(t, s)$ is differentiable up to of order $n-1$ with respect to t. Moreover, the Green's function, $G(t, s)$, for every fixed $s \in J$ satisfies the homogeneous boundary conditions $U(x) = 0$.

The following proposition gives some of the properties of the Green's function, $G(t, s)$, defined on the rectangle J^2.

Proposition 3.2. *Let $G(t, s)$ be the Green's function defined by (3.35). Then, the following properties hold.*

G1) $\dfrac{\partial^\nu}{\partial t^\nu} G(t, s)$, *($\nu = 0, 1, \ldots, n-2$) are continuous and bounded for (t, s) on the rectangles R_{ij}, $i, j = 1, \ldots, p+1$.*

G2) $\dfrac{\partial^\nu}{\partial t^\nu} G(t, s)$, *($\nu = n-1, n$) are continuous and bounded on the rectangles R_{ij} with $i \neq j$ and the triangles T_{ii}^u and T_{ii}^l.*

G3)

$$\frac{\partial^{n-1}}{\partial t^{n-1}} G(s^+, s) - \frac{\partial^{n-1}}{\partial t^{n-1}} G(s^-, s) = \frac{1}{p_0(s)}, \quad s \neq \theta_j, \tag{3.39}$$

$$\widehat{G}(\theta_j^+, \theta_j) - (E + B_j)\widehat{G}(\theta_j^-, \theta_j) = (E + B_j)\frac{1}{p_0(\theta_j)}\, e_n. \tag{3.40}$$

G4) As a function of t, $G(t, s)$ is left continuous and satisfies the following equations

$$\begin{cases} \ell(x) = 0, & t \in J_s \setminus \{\theta_i\}, \\ \delta_i(x) = 0, & i \in \{i : \theta_i \in J_s\}, \\ U(x) = 0, \end{cases}$$

where J_s is any of the intervals $[\alpha, s)$ and $(s, \beta]$.

Proof. From (3.35), it follows that $G(t,s)$ is continuous in each of the rectangles R_{ij} for $i,j = 1,\dots,p+1$. So, let $s \in [\alpha,\beta]$ be fixed. Then from the definition of $G(t,s)$, considered as a function of t, we have

$$\widehat{G}(t,s) = \begin{cases} \widehat{\Phi}(t)(E+K)\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, & s<t \\ \widehat{\Phi}(t)K\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, & s \geq t \end{cases}$$

So, using the continuity of the fundamental matrix $\widehat{\Phi}(t)$ at $s \neq \theta_j$

$$\widehat{G}(s^+,s) - \widehat{G}(s^-,s) = \widehat{\Phi}(s^-)\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n.$$

Hence,

$$\widehat{G}(s^+,s) - \widehat{G}(s^-,s) = \frac{1}{p_0(s)}\, e_n, \quad s \neq \theta_j$$

proves the jump condition (3.39). Moreover, if $s = \theta_j$ for fixed $j \in \{1,\dots,p\}$ we consider the function

$$G(t,\theta_j) = \begin{cases} \Phi(t)(E+K)\widehat{\Phi}^{-1}(\theta_j)\frac{1}{p_0(\theta_j)}\, e_n, & \theta_j<t \\ \Phi(t)K\widehat{\Phi}^{-1}(\theta_j)\frac{1}{p_0(\theta_j)}\, e_n, & \theta_j \geq t \end{cases} \tag{3.41}$$

and direct calculation of the jump $\Delta\widehat{G}(t,\theta_j)|_{t=\theta_j}$ at $t=\theta_j$ gives

$$\begin{aligned} \Delta\widehat{G}(t,\theta_j)|_{t=\theta_j} &= \widehat{G}(\theta_j^+,\theta_j) - \widehat{G}(\theta_j^-,\theta_j) \\ &= \Phi(\theta_j^+)(E+K)\widehat{\Phi}^{-1}(\theta_j)\tfrac{1}{p_0(\theta_j)}\, e_n \\ &\qquad -\Phi(\theta_j^-)K\widehat{\Phi}^{-1}(\theta_j)\tfrac{1}{p_0(\theta_j)}\, e_n \\ &= (E+B_j)\tfrac{1}{p_0(\theta_j)}\, e_n + B_j\Phi(\theta_j^-)K\widehat{\Phi}^{-1}(\theta_j)\tfrac{1}{p_0(\theta_j)}\, e_n \\ &= (E+B_j)\tfrac{1}{p_0(\theta_j)}\, e_n + B_j\widehat{G}(\theta_j^-,\theta_j) \end{aligned}$$

which proves (3.40). The rest of the proof can easily be treated using the definition of the matrix K. □

In other words, the properties *G1)–G4)* determine $G(t,s)$ uniquely, defined for all $(t,s) \in J^2 = [\alpha,\beta]\times[\alpha,\beta]$ and so can be taken as a definition of the Green's function $G(t,s)$. That is, we have the following theorem.

Theorem 3.7. *If the boundary value problem $(BVP)_n$ has only the trivial solution then the properties G1)–G4) uniquely determine the Green's function $G(t,s)$.*

Proof. The condition *G4)* directly implies that

$$G(t,s) = \begin{cases} \Phi(t)c(s), & s<t \\ \Phi(t)d(s), & s>t \end{cases}$$

for every $s \in [\alpha, \beta]$, where $\Phi(t)$ is any row vector of fundamental solutions of

$$\begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \ldots, p, \end{cases}$$

and $c(s)$ and $d(s)$ are column vectors with n components.

It follows from *G3)* that if $s \neq \theta_j$ for any $j = 1, \ldots, p$ then,

$$\begin{aligned} \frac{1}{p_0(s)}\, e_n &= \widehat{G}(s^+, s) - \widehat{G}(s^-, s) \\ &= \widehat{\Phi}(s^+)c(s) - \widehat{\Phi}(s^-)d(s) \end{aligned}$$

implies

$$c(s) - d(s) = \widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, \quad s \neq \theta_j$$

On the other hand, however, if $s = \theta_j$ for some $j \in \{1, \ldots, p\}$, then again the property *G3)* yields

$$\begin{aligned} (E + B_j)\tfrac{1}{p_0(\theta_j)} &= \widehat{G}(\theta_j^+, \theta_j) - (E + B_j)\widehat{G}(\theta_j^-, \theta_j) \\ &= \widehat{\Phi}(\theta_j^+)\, c(\theta_j) - (E + B_j)\widehat{\Phi}(\theta_j^-)\, d(\theta_j) \\ &= (E + B_j)\widehat{\Phi}(\theta_j^-)\, c(\theta_j) - (E + B_j)\widehat{\Phi}(\theta_j^-)\, d(\theta_j) \end{aligned}$$

because $\Phi(t)$ is a row vector of fundamental solutions. Hence,

$$c(\theta_j) - d(\theta_j) = \widehat{\Phi}^{-1}(\theta_j)\frac{1}{p_0(\theta_j)}\, e_n$$

holds for all $j = 1, \ldots, p$. Therefore,

$$c(s) - d(s) = \widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n \tag{3.42}$$

holds for ever $s \in [\alpha, \beta]$, and further (3.42) uniquely defines the difference $c(s) - d(s)$.

Now, the boundary conditions $U(x) = 0$, that $G(t, s)$ must satisfy as a function of t, yields the relation,

$$0 = M\widehat{G}(\alpha, s) + N\widehat{G}(\beta, s) = M\widehat{\Phi}(\alpha)d(s) + N\widehat{\Phi}(\beta)c(s).$$

By using (3.42), we get

$$[M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]\, d(s) = -N\widehat{\Phi}(\beta)\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n.$$

Since the homogeneous boundary value problem is assumed to have only the trivial solution, we must have the rank of the matrix

$$U\widehat{\Phi} = M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)$$

equal to n, exactly, and hence $(U\widehat{\Phi})^{-1}$ exists. Letting

$$K = -(U\widehat{\Phi})^{-1}N\widehat{\Phi}(\beta)$$

we obtain uniquely that

$$d(s) = K\widehat{\Phi}(s)\frac{1}{p_0(s)}\, e_n,$$

and therefore,

$$c(s) = (E+K)\widehat{\Phi}(s)\frac{1}{p_0(s)}\, e_n$$

is uniquely determined for every $s \in [\alpha, \beta]$. Thus,

$$G(t,s) = \begin{cases} \Phi(t)(E+K)\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, & s < t \\ \Phi(t)K\widehat{\Phi}^{-1}(s)\frac{1}{p_0(s)}\, e_n, & s \geq t \end{cases}$$

exists and uniquely determined. This completes the proof. $\square$

The proof of the theorem, however, gives a practical approach for finding the Green's function $G(t,s)$ in the form

$$G(t,s) = \begin{cases} \displaystyle\sum_{i=1}^{n} c_i(s)\phi_i(t), & s < t \\ \displaystyle\sum_{i=1}^{n} d_i(s)\phi_i(t), & s > t \end{cases}$$

where $\phi_1, \ldots, \phi_n$ are fundamental (piecewise) solutions of

$$\begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \ldots, p. \end{cases}$$

Also, the proof, suggests that it is enough to find the constants $c_i(s)$ and $d_i(s)$ for all $s \neq \theta_j$ for any $j = 1, \ldots, p$. No need to consider the case, $s = \theta_j$. This does not mean that the properties *G1)–G4)* without the condition (3.40) can determine the Green's function $G(t,s)$ uniquely. For, at $s = \theta_j$ the jump condition (3.39) in the property *G3)* is no longer valid. Instead, we have (3.40).

Example 3.4. Consider the following boundary value problem,

$$\begin{cases} -x'' = 0, & t \neq 1, \\ \Delta\widehat{x}|_{t=1} - B\widehat{x}(1^-) = 0, \\ x(0) = x(\pi) = 0, \end{cases} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

which has been studied in Example 3.1. The Green's function, $G(t,s)$, for the problem can be given as follows,

$$G(t,s)=\begin{cases} s+\frac{\pi s}{2-3\pi}\,t, & 0\le s<t\le 1\\ t+\frac{\pi t}{2-3\pi}\,s, & 0\le t\le s\le 1\\ -\frac{1}{2}\,\frac{\pi(2-3s)}{2-3\pi}\,t+\frac{1}{2}\,st, & 0\le t\le 1<s\le\pi\\ \\ -\frac{\pi(2-3t)}{2-3\pi}\,s+ts, & 0\le s\le 1<t\le\pi\\ -\frac{1}{2}\left((2-3s)t+\frac{\pi(2-3s)}{2-3\pi}(-2+3t)\right), & 1<s<t\le\pi\\ -\frac{1}{2}\left((2-3t)s+\frac{\pi(2-3t)}{2-3\pi}(-2+3s)\right), & 1<t\le s\le\pi \end{cases}$$

$\diamondsuit$

Finally, we close this subsection by the following corollary of Theorem 3.7, that relates the Green's function $G(t,s)$ with the solution of the homogeneous boundary value problem of the form

$$\begin{cases} \ell(x)=f(t), \quad t\neq\theta_i,\\ \delta_i(x)=0, \quad i=1,\ldots,p,\\ U(x)=0. \end{cases} \tag{3.43}$$

Corollary 3.4. *Under the assumptions of Theorem 3.7, there exists a unique solution $x=x(t)$ of (3.43) defined by*

$$x(t)=\int_\alpha^\beta G(t,s)f(s)\,ds$$

Proof. We use the properties of the Green's function $G(t,s)$. Let $R^0_{ij}=(\theta_{i-1},\theta_i)\times(\theta_{j-1},\theta_j)$ be the interior of the rectangles R_{ij}. The following four steps will lead us to the proof.

1. If $(t,s)\in R^0_{ij}$ with $i\neq j$, then
$$\frac{\partial^\nu}{\partial t^\nu}\int_{\theta_{j-1}}^{\theta_j} G(t,s)f(s)\,ds=\int_{\theta_{j-1}}^{\theta_j}\frac{\partial^\nu}{\partial t^\nu}G(t,s)f(s)\,ds, \quad \nu=0,1,\ldots,n.$$

2. If $(t,s)\in R^0_{ii}$, then
$$\frac{\partial^\nu}{\partial t^\nu}\int_{\theta_{i-1}}^{\theta_i} G(t,s)f(s)\,ds=\int_{\theta_{i-1}}^{\theta_i}\frac{\partial^\nu}{\partial t^\nu}G(t,s)f(s)\,ds, \quad \nu=0,1,\ldots,n-1,$$
and
$$\frac{\partial^n}{\partial t^n}\int_{\theta_{i-1}}^{\theta_i} G(t,s)f(s)\,ds=\int_{\theta_{i-1}}^{\theta_i}\frac{\partial^n}{\partial t^n}G(t,s)f(s)\,ds+\frac{f(t)}{p_0(t)}.$$

3. Notice that for all $t \in [\alpha, \beta] = [\theta_0, \theta_{p+1}]$, we have the following definition

$$\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds = \begin{cases} \displaystyle\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds, & \theta_0 \le t \le \theta_1 \\ \vdots & \vdots \\ \displaystyle\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds, & \theta_{k-1} < t \le \theta_k \\ \vdots & \vdots \\ \displaystyle\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds, & \theta_p < t \le \theta_{p+1} \end{cases}$$

for every $k = 1, \ldots, p+1$. Therefore, we have

$$\frac{\partial^\nu}{\partial t^\nu} \int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds = \int_{\theta_{k-1}}^{\theta_k} \frac{\partial^\nu}{\partial t^\nu} G(t,s)f(s)\,ds, \quad \nu = 0, 1, \ldots, n-1,$$

but for $\nu = n$, we have the following equality,

$$\frac{\partial^n}{\partial t^n} \int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds = \int_{\theta_{k-1}}^{\theta_k} \frac{\partial^n}{\partial t^n} G(t,s)f(s)\,ds + \kappa_k(t)\,\frac{f(t)}{p_0(t)},$$

where

$$\kappa_k(t) = \begin{cases} 1, & t \in (\theta_{k-1}, \theta_k) \\ 0, & \text{otherwise} \end{cases}$$

Finally,

4. The solution, $x = x(t)$, is defined by

$$x(t) = \int_\alpha^\beta G(t,s)f(s)\,ds = \sum_{k=1}^{p+1} \int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds.$$

So

$$x^{(\nu)}(t) = \sum_{k=1}^{p+1} \int_{\theta_{k-1}}^{\theta_k} \frac{\partial^\nu}{\partial t^\nu} G(t,s)f(s)\,ds, \quad \nu = 0, 1, \ldots, n-1, \tag{3.44}$$

and

$$x^{(n)}(t) = \sum_{k=1}^{p+1} \left[\int_{\theta_{k-1}}^{\theta_k} \frac{\partial^n}{\partial t^n} G(t,s)f(s)\,ds + \kappa_k(t)\,\frac{f(t)}{p_0(t)} \right]. \tag{3.45}$$

Hence, the use of (3.44) and (3.45) will eventually complete the proof. □

3.5.2 Properties of Green's Function H

As it is seen obviously from Corollary 3.4, the Green's function $G(t,s)$ is not sufficient in order to represent the solutions of impulsive boundary value problems with nonhomogeneous impulse actions. Namely, to represent the unique solution of the boundary value problem

$$\begin{cases} \ell(x) = f(t), & t \neq \theta_i, \\ \delta_i(x) = a_i, & i = 1, \ldots, p, \\ U(x) = 0, \end{cases} \tag{3.46}$$

provided that the corresponding homogeneous problem $(\mathrm{BVP})_n$ has only the trivial solution, one needs the help of the Green's functions $H(t, \theta_j^+)$ for each $j = 1, \ldots, p$.

Recall that for each $j \in \{1, \ldots, p\}$, the (row) vector valued function $H(t, \theta_j^+)$ is defined by (3.36), that is,

$$H(t, \theta_j^+) = \begin{cases} \Phi(t)(E + K)\widehat{\Phi}^{-1}(\theta_j^+), & \theta_j < t \\ \Phi(t)K\widehat{\Phi}^{-1}(\theta_j^+), & \theta_j \geq t \end{cases} \tag{3.47}$$

for every $t \in [\alpha, \beta]$. If we write the Green's function $H(t, \theta_j^+)$ for each j in the following form,

$$H(t, \theta_j^+) = [H_1(t, \theta_j^+), \ldots, H_n(t, \theta_j^+)]$$

with components $H_1(t, \theta_j^+), \ldots, H_n(t, \theta_j^+)$, then it is readily seen from the definition of $H(t, \theta_j^+)$ that each component $H_k(t, \theta_j^+)$, for $(1 \leq k \leq n)$ is a linear combinations of the fundamental solutions of the homogeneous impulsive equation,

$$\begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \ldots, p, \end{cases}$$

and hence satisfies

$$\ell(H_k(t, \theta_j^+)) = 0, \quad t \neq \theta_i, \quad \text{and} \quad t \neq \theta_j$$

$$\delta_i(H_k(t, \theta_j^+)) = 0, \quad i = 1, \ldots, j-1, j+1, \ldots, p$$

for $k = 1, \ldots, n$ Moreover, the functions $H_k(t, \theta_j^+)$, $k = 1, \ldots, n$ satisfy also the boundary conditions, since

$$\begin{aligned} U\widehat{H} &= M\widehat{H}(\alpha, \theta_j^+) + N\widehat{H}(\beta, \theta_j^+) \\ &= M\widehat{\Phi}(\alpha)K\widehat{\Phi}^{-1}(\theta_j^+) + N\widehat{\Phi}(\beta)(E + K)\widehat{\Phi}^{-1}(\theta_j^+) \\ &= \widehat{\Phi}(\beta)\widehat{\Phi}^{-1}(\theta_j^+) + [M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]K\widehat{\Phi}^{-1}(\theta_j^+) \end{aligned}$$

and hence, substituting $K = -[M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]^{-1}N\widehat{\Phi}(\beta)$ we deduce that

$$U\widehat{H} = 0.$$

An important property of the Green's function $H(t, \theta_j^+)$ is its behavior at the point $t = \theta_j$. The following proposition characterizes the Green's functions $H(t, \theta_j^+)$ for each $j \in \{1, \ldots, p\}$.

Proposition 3.3. *Let $j \in \{1, \ldots, p\}$ be arbitrarily fixed, and let $H(t, \theta_j^+)$ be the Green's function defined by (3.47) with components $H_1(t, \theta_j^+), \ldots, H_n(t, \theta_j^+)$. Then, each $H_k(t, \theta_j^+)$ is of class $\mathcal{PLC}^n$ and satisfies the following equation*

$$\begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \ldots, j-1, j+1, \ldots, p, \\ \delta_j(x) = e_k, \\ U(x) = 0. \end{cases}$$

Proof. Directly, from (3.47) it follows that

$$\begin{aligned} &\widehat{H}(\theta_j^+, \theta_j^+) - (E + B_j)\widehat{H}(\theta_j^-, \theta_j^+) \\ &\quad = \widehat{\Phi}(\theta_j^+)(E + K)\widehat{\Phi}^{-1}(\theta_j^+) - (E + B_j)\widehat{\Phi}(\theta_j^-)K\widehat{\Phi}^{-1}(\theta_j^+) \\ &\quad = (E + B_j)\widehat{\Phi}(\theta_j^-)(E + K)\widehat{\Phi}^{-1}(\theta_j^+) - (E + B_j)\widehat{\Phi}(\theta_j^-)K\widehat{\Phi}^{-1}(\theta_j^+) \\ &\quad = (E + B_j)\widehat{\Phi}(\theta_j^-)\widehat{\Phi}^{-1}(\theta_j^+) \\ &\quad = \widehat{\Phi}(\theta_j^+)\widehat{\Phi}^{-1}(\theta_j^+) \\ &\quad = E \end{aligned}$$

since $\widehat{\Phi}$ is a fundamental matrix. Hence, this proves the jump condition at $t = \theta_j$. The argument preceding the proposition completes the proof. □

We remark that the jump at $t = \theta_j$ of the Green's function $H(t, \theta_j^+)$ is similar to the property of the function $G(t, \theta_j)$ at the point $t = \theta_j$ in (3.40). Also, the jump at $t = \theta_j$ can be written as

$$\widehat{H}_k(\theta_j^+, \theta_j^+) - (E + B_j)\widehat{H}_k(\theta_j^-, \theta_j^+) = e_k, \quad k = 1, \ldots, n.$$

Moreover, Proposition 3.3 uniquely characterize the Green's functions $H(t, \theta_j^+)$ for each $j \in \{1, \ldots, p\}$.

Theorem 3.8. *If the boundary value problem $(BVP)_n$ has only the trivial solution, then Proposition 3.3 uniquely determines the Green's functions $H(t, \theta_j^+)$ for each $j \in \{1, \ldots, p\}$.*

Proof. Let $j \in \{1, \ldots, p\}$ be arbitrary, and let $\Phi(t)$ be a fundamental row vector. Then, it follows that

$$H(t, \theta_j^+) = \begin{cases} \Phi(t)C(j), & \theta_j < t \\ \Phi(t)D(j), & \theta_j \geq t \end{cases}$$

where $C(j)$ and $D(j)$ are $n \times n$ matrices for each j. The jump property of $H(t, \theta_j^+)$, at $t = \theta_j$, or equivalently in terms of matrices

$$\widehat{H}(\theta_j^+, \theta_j^+) - (E + B_j)\widehat{H}(\theta_j^-, \theta_j^+) = E$$

leads to

$$\widehat{\Phi}(\theta_j^+)C(j) - (E + B_j)\widehat{\Phi}(\theta_j^-)D(j) = E,$$

and hence,

$$C(j) - D(j) = \widehat{\Phi}^{-1}(\theta_j^+), \tag{3.48}$$

that is, the difference $C(j) - D(j)$ is uniquely determined. On the other hand, the boundary conditions $U(x) = 0$ implies that

$$M\widehat{\Phi}(\alpha)D(j) + N\widehat{\Phi}(\beta)C(j) = 0,$$

and hence, using (3.48) we obtain

$$[M\widehat{\Phi}(\alpha) + N\widehat{\Phi}(\beta)]D(j) = -N\widehat{\Phi}(\beta)\widehat{\Phi}^{-1}(\theta_j^+).$$

Since the homogeneous boundary value problem has only the trivial solution it follows that

$$D(j) = K\widehat{\Phi}^{-1}(\theta_j^+), \qquad C(j) = (E + K)\widehat{\Phi}^{-1}(\theta_j^+).$$

Therefore, the Green's functions $H(t, \theta_j^+)$ are uniquely defined as in (3.47). This completes the proof. □

The above theorem suggests a method for constructing Green's functions $H(t, \theta_j^+)$ for each $j = 1, \dots, p$: Let $\phi_1, \dots, \phi_n$ be fundamental solutions of

$$\begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \dots, p. \end{cases}$$

Then, each Green's function $H(t, \theta_j^+)$ can be represented by the equation,

$$H(t, \theta_j^+) = \begin{cases} \Big[\sum_{i=1}^{n} c_{1i}(j)\phi_i(t), \dots, \sum_{i=1}^{n} c_{ni}(j)\phi_i(t)\Big], & \theta_j < t \\ \Big[\sum_{i=1}^{n} d_{1i}(j)\phi_i(t), \dots, \sum_{i=1}^{n} d_{ni}(j)\phi_i(t)\Big], & \theta_j \geq t \end{cases}$$

However, this formula is not practical at all, in general. Instead, we prefer using (3.36). Also, we note that for every fixed $j \in \{1, \dots, p\}$, the row vectors $H(t, \theta_j^+)$ are piecewise continuous functions.

The following corollary gives the unique solution of the nonhomogeneous boundary value problem,

$$\begin{cases} \ell(x) = 0, & t \neq \theta_i, \\ \delta_i(x) = a_i, & i = 1, \dots, p, \\ U(x) = 0, \end{cases} \tag{3.49}$$

in terms of the Green's functions $H(t, \theta_j^+)$ for every $j = 1, \dots, p$.

Corollary 3.5. *Under the assumptions of Theorem 3.8, there exists a unique solution $x = x(t)$ of (3.49). Namely,*

$$x(t) = \sum_{j=1}^{p} H(t, \theta_j^+)\, a_j$$

for all $t \in [\alpha, \beta]$.

Proof. Obviously, $x(t)$ satisfies the differential equation $\ell(x) = 0$ and the boundary conditions $U(x) = 0$. In the case of impulse actions δ_i for $i = 1, \ldots, p$, however, by the properties of $H(t, \theta_j^+) = [H_1(t, \theta_j^+), \ldots, H_n(t, \theta_j^+)]$ we have

$$\delta_i(H_k(t, \theta_j^+)) = \begin{cases} 0, & i \neq j \\ e_k, & i = j \end{cases} \qquad k = 1, \ldots, n,$$

and hence, the impulse actions

$$\begin{aligned} \delta_i(x) &= \delta_i\left(\sum_{j=1}^{p} H(t, \theta_j^+)\, a_j\right) \\ &= \sum_{j=1}^{p} \Big(\Delta\widehat{H}(t, \theta_j^+)|_{t=\theta_i} - B_i \widehat{H}(\theta_i^-, \theta_j^+)\Big)\, a_j \\ &= E\, a_i = a_i \end{aligned}$$

for all $i = 1, \ldots, p$. This completes the proof. $\square$

Example 3.5. Consider the following boundary value problem,

$$\begin{cases} -x'' = 0, \quad t \neq 1, \\ \Delta\widehat{x}|_{t=1} - B\widehat{x}(1^-) = 0, \\ x(0) = x(\pi) = 0, \end{cases} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

which has been studied in Example 3.1 and Example 3.4. The Green's function, $G(t, s)$, in fact, was given in Example 3.4. Now, the Green's function, $H(t, 1^+)$, which is a row vector, can be given in the following form,

$$H(t, 1^+) = \begin{cases} \frac{1}{2}\left[\ \frac{2}{2-3\pi}\, t, \quad \frac{-2+2\pi}{2-3\pi}\, t\ \right], & 0 \leq t \leq 1 \\ \\ \frac{1}{2}\left[\ 3t + \frac{3\pi}{2-3\pi}(-2+3t), \quad -t - \frac{\pi}{2-3\pi}(-2+3t)\ \right], & 1 < t \leq \pi \end{cases}$$

$\diamondsuit$

Consider, finally, the problem of finding solutions of the nonhomogeneous boundary value problem (3.46), namely,

$$\begin{cases} \ell(x) = f(t), \quad t \neq \theta_i, \\ \delta_i(x) = a_i, \quad i = 1, \dots, p, \\ U(x) = 0, \end{cases} \tag{3.50}$$

provided that the corresponding homogeneous boundary value problem $(\mathrm{BVP})_n$ has only the trivial solution. Combining the properties of the Green's functions $G(t,s)$ and $H(t,\theta_j^+)$ for $j = 1, \dots, p$ the unique solution of the problem (3.50) can be given in terms of these functions. We state this result without proof in the following theorem.

Theorem 3.9. *If the homogeneous boundary value problem has only the trivial solution, then the solution $x = x(t)$ of (3.50) exists and unique. Moreover, this solution is expressed by*

$$x(t) = \int_\alpha^\beta G(t,s) f(s)\, ds + \sum_{j=1}^{p} H(t,\theta_j^+) a_j, \tag{3.51}$$

where $G(t,s)$ and $H(t,\theta_j^+)$ for $j = 1, \dots, p$ are the Green's functions.

Example 3.6. The following nonhomogeneous boundary value problem,

$$\begin{cases} -x'' = 2, \quad t \neq 1, \\ \Delta\widehat{x}|_{t=1} - B\widehat{x}(1^-) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\ x(0) = x(\pi) = 0, \end{cases}$$

has a unique solution, $x = x(t)$, that can be obtained by the help of the Green's functions $G(t,s)$ and $H(t,1^+)$ obtained in Example 3.4 and Example 3.4, respectively. Specifically, we have

$$\begin{aligned} x(t) &= 2\int_0^\pi G(t,s)\, ds + H(t,1^+)\begin{pmatrix} 0 \\ -4 \end{pmatrix} \\ &= \begin{cases} 2\displaystyle\int_0^\pi G(t,s)\, ds + H(t,1^+)\begin{pmatrix} 0 \\ -4 \end{pmatrix}, & 0 \le t \le 1 \\ 2\displaystyle\int_0^\pi G(t,s)\, ds + H(t,1^+)\begin{pmatrix} 0 \\ -4 \end{pmatrix}, & 1 < t \le \pi \end{cases} \\ &= \begin{cases} \dfrac{\pi^2 + 7\pi - 7}{3\pi - 2} t - t^2, & 0 \le t \le 1 \\ \dfrac{(7 - 2\pi)\pi}{3\pi - 2} + \dfrac{3\pi^2 - 7}{3\pi - 2} t - t^2, & 1 < t \le \pi \end{cases} \end{aligned}$$

$\diamondsuit$

Chapter 4

Eigenvalue Problems

In some cases, especially when solving a partial differential equation by the method of separation of variables one needs to deal with an ordinary differential equation containing a parameter. That parameter stands, roughly speaking, for the eigenvalue of that problem. However, it is of great importance to study, separately, the eigenvalue problems which mostly appear in many applications of differential equations in science and technology.

This chapter, in fact, is a consequence of the previous chapters. Generally speaking, eigenvalue problems for impulsive differential equations (or for classical ordinary differential equations) are problems that are characterized by a boundary value problem containing a parameter, mostly a complex parameter λ. Since a boundary value problem cannot have nontrivial solutions most of the time, nor even solutions at all, the parameter λ plays an important role for the existence of such nontrivial solutions of so-called *eigenvalue problems*.

However, in most of the theory of eigenvalue problems one encounters with the theory of operators, see [41, 47]. Eigenvalues of the problems, therefore, correspond to the eigenvalues of the operators, and the eigenfunctions corresponding to those eigenvalues are nothing but nontrivial solutions of the boundary value problems, in the domain of those operators.

In the first section of the present chapter, therefore, we define the eigenvalues and eigenfunctions for impulsive boundary value problems as well as their representative operators together with their domain of definitions.

Of course, the determination of these eigenvalues, as in the classical theory [27, 41, 47], depends mainly on the holomorphic solutions of impulsive differential equations with respect to parameters. See for instance [3, 10], for the investigation of the analytic dependence on parameters. So, in the first section, we study holomorphic solutions and their contribution to the determination of eigenvalues.

4.1 Eigenvalues and Eigenfunctions

In principal, one may consider the problem of finding the values of the parameter λ for which the homogeneous boundary value problem for impulsive differential equation,

$$\begin{cases} \ell(x) = \lambda x, \quad t \neq \theta_i, \\ \delta_i(x) = 0, \quad i = 1, \ldots, p, \\ U(x) = 0, \end{cases} \tag{4.1}$$

has nontrivial solutions. Each of these nontrivial solutions is called an *eigenfunction* corresponding to the *eigenvalue* λ. Here, we remark that the differential operator ℓ, and the impulse actions δ_i are as previously defined, and the vector boundary for U is of rank m unless otherwise stated, explicitly.

Generally speaking, a number λ is called an *eigenvalue* of an operator $\mathcal{L}_0$ if there exists, in the domain of definition, $\mathcal{D}_0$, of the operator $\mathcal{L}_0$, a function $x \not\equiv 0$ such that

$$\mathcal{L}_0 x = \lambda x$$

holds. The function x is called the *eigenfunction*, of the operator $\mathcal{L}_0$, corresponding to that eigenvalue λ.

In our case of the eigenvalue problem (4.1) the operator $\mathcal{L}_0 : \mathcal{D}_0 \to \mathcal{PLC}$ defined by the differential operator ℓ, on the linear subspace

$$\mathcal{D}_0 = \{x \in \mathcal{PLC}^n : U(x) = 0,\ \delta_i(x) = 0,\ i = 1, \ldots, p\} \tag{4.2}$$

of the space $\mathcal{PLC}$. In other words, if a function $x \in \mathcal{PLC}$ satisfies the boundary conditions $U(x) = 0$ and the impulse conditions $\delta_i(x) = 0$ for every $i = 1, \ldots, p$, then $\mathcal{L}_0 x = \ell(x)$. In general, $\ell(x) \notin \mathcal{PLC}$, however, it can be continued from the left at $t = \theta_i$. Therefore, it is possible to rewrite the eigenvalue problem (4.1) in a simpler form as follows,

$$\mathcal{L}_0 x = \lambda x, \tag{4.3}$$

and hence, the *eigenvalues* of an operator $\mathcal{L}_0$ are those values of the parameter λ for which the homogeneous boundary value problem (4.1) has nontrivial solution; each of these nontrivial solutions is an *eigenfunction* of the operator $\mathcal{L}_0$ corresponding to the value of λ. If λ is an eigenvalue and x is the corresponding eigenfunction, then the pair $\{\lambda, x\}$ is called an *eigensolution* of $\mathcal{L}_0$.

Linearity of the differential operator ℓ, and the fact that the $\mathcal{D}_0$ is a linear subspace implies that the operator $\mathcal{L}_0$ is a linear operator, which may be called a *linear impulsive differential operator*. Linearity of the operator $\mathcal{L}_0$ implies that a linear combination of eigenfunctions which correspond to one and the same eigenvalue is itself an eigenfunction corresponding to the same eigenvalue.

Moreover, since the homogeneous boundary value problem (4.1) can have, for a given value of λ, not more than n linearly independent solutions, it follows that the set of all eigenfunctions which belong to one and the same eigenvalue of the operator $\mathcal{L}_0$ forms a finite dimensional vector space with dimension not more than the order, n, of the impulsive differential equation. The dimension of this space is simply the number of linearly independent solutions of the homogeneous boundary value problem (4.1), for the given value of λ; and this number is called the *multiplicity* of the eigenvalue λ.

The set $\sigma(\mathcal{L}_0)$ of all eigenvalues of the impulsive differential operator $\mathcal{L}_0$ is called the *spectrum* of $\mathcal{L}_0$, and if $\lambda_0 \in \sigma(\mathcal{L}_0)$ the *eigenspace*, $\mathcal{E}(\mathcal{L}_0, \lambda_0)$ of $\mathcal{L}_0$ for the eigenvalue λ_0 is the set of all solutions of (4.3), including the zero function. In other words, $\mathcal{E}(\mathcal{L}_0, \lambda_0)$ is the null space of the operator $\mathcal{L}_0 - \lambda_0 1$, where 1 is the identity operator from $\mathcal{PLC}^n$ to $\mathcal{PLC}$.

In order to characterize the eigenvalues of the operator $\mathcal{L}_0$, we need to investigate holomorphic properties of solutions with respect to the parameter λ. However, in [3] it is presented that the holomorphic solutions with respect to the parameters of an impulsive differential equations can be obtain by substitution of convergent power series into another convergent power series with nonzero radii of convergence [38, 47]. In fact, this corresponds to the fact that the composition of holomorphic functions is again a holomorphic one. More briefly, if $f : \Omega_f \to \mathbb{F}$ and $g : \Omega_g \to \mathbb{F}$ are two holomorphic functions in their respective domains, and such that $g(\Omega_g) \subset \Omega_f$, then $(f \circ g) : \Omega_g \to \mathbb{F}$ is holomorphic in its domain Ω_g. Let us denote by $\mathcal{H}(\Omega)$, the set of holomorphic functions in a domain Ω.

In our discussion of eigenvalue problems (4.1) for an nth order impulsive differential equation, we may simplify the argument of holomorphic solutions. For, if we transform the linear impulsive differential equation,

$$\begin{cases} \ell(x) = \lambda x, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \dots, p, \end{cases} \tag{4.4}$$

into a first order system of equation of the form,

$$\begin{cases} w' = A(t, \lambda)\, w, & t \neq \theta_i \\ \Delta w|_{t=\theta_i} = B_i\, w(\theta_i^-), & i = 1, \dots, p, \end{cases} \tag{4.5}$$

where $A(t, \lambda)$ satisfies the existence and uniqueness of solution $w = w(t, \xi, \lambda)$, defined for all $t \in J = [\alpha, \beta]$ and such that

$$w(t_0, \xi, \lambda) = \xi, \quad t_0 \in J, \quad \xi \in \mathbb{C}^n \tag{4.6}$$

provided that $E + B_i$ are all nonsingular for $i = 1, \dots, p$. More precisely, $A(t, \lambda)$ is the companion matrix for the differential equation $\ell(x) = \lambda x$ and is defined by

$$A(t, \lambda) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{\lambda - p_n(t)}{p_0(t)} & -\frac{p_{n-1}(t)}{p_0(t)} & -\frac{p_{n-2}(t)}{p_0(t)} & \cdots & -\frac{p_1(t)}{p_0(t)} \end{pmatrix},$$

and the vector $w = [w_1, \dots, w_n]^T$ is such that $w_j(t) = x^{(n-1)}(t)$ for $t \neq \theta_i$, and $w_j(\theta_i) = x^{(n-1)}(\theta_i^-) = x_-^{(n-1)}(\theta_i)$ for each $j = 1, \dots, n$.

The following lemma states the solution $w = w(t, \xi, \lambda)$ of (4.5) satisfying the initial condition (4.6) is holomorphic in the variables $(\xi, \lambda) \in \mathbb{C}^n \times \mathbb{C}$ for every fixed $t \in J$.

Lemma 4.1. *The solution $w = w(t, \xi, \lambda)$ of (4.5) satisfying the initial condition (4.6) is continuous for $(t, \xi, \lambda) \in J_i \times \mathbb{C}^n \times \mathbb{C}$, and for each fixed $t \in J$ it is holomorphic in the variable $(\xi, \lambda) \in \mathbb{C}^n \times \mathbb{C}$, where $J = [\alpha, \beta]$ and the J_i are such that*

$$J_0 = [\alpha, \theta_1], \qquad J_i = (\theta_i, \theta_{i+1}], \quad i = 1, \dots, p,$$

provided that $\alpha = \theta_0 < \theta_1 < \cdots < \theta_p < \theta_{p+1} = \beta$.

Proof. For simplicity, we will assume $t_0 = \alpha$.

Let $\xi_0 = \xi$ and consider the following system of initial value problems for ordinary differential equations,

$$\begin{cases} w' = A(t, \lambda)\, w, & t \in J_i, \\ w(\theta_i) = \xi_i, & i = 0, 1, \dots, p, \end{cases} \tag{4.7}$$

where $\xi_i \in \mathbb{F}^n$. Let $\varphi_i = \varphi_i(t, \xi_i, \lambda)$ be the solution of the (4.7) for each $i = 0, 1, \ldots, p$. These solutions $\varphi_i(t, \xi_i, \lambda)$ are all continuous for $(t, \xi_i, \lambda) \in J_i \times \mathbb{C}^n \times \mathbb{C}$ and are holomorphic in the variable (ξ_i, λ) for fixed $t \in J_i$, by the general theory of ordinary differential equations [27].

Now, let $\varphi = \varphi(t, \xi_0, \lambda)$, $\varphi(t_0, \xi_0, \lambda) = \xi_0$, be the solution of (4.5). That is,

$$\begin{cases} w' = A(t, \lambda)\, w, \quad t \neq \theta_i \\ w(\theta_i^+) = (E + B_i)\, w(\theta_i^-), \quad i = 1, \ldots, p. \end{cases} \tag{4.8}$$

Clearly the restriction, $\varphi|_{J_0}$, of φ onto J_0 is $\varphi_0(t, \xi_0, \lambda)$, and hence holomorphic in (ξ_0, λ) fixed $t \in J_0$. In particular, it is holomorphic in (ξ_0, λ) for $t = \alpha$, and $t = \theta_1$. Moreover, at $t = \theta_1$ the function $\zeta_1(\xi_0, \lambda)$ defined by

$$\zeta_1(\xi_0, \lambda) = (E + B_1)\varphi_0(\theta_1^-, \xi_0, \lambda)$$

is holomorphic function for (ξ_0, λ).

Now, if ξ_1 is chosen to be ζ_1 it follows from the composition of holomorphic functions that

$$\varphi_1(t, \xi_0, \lambda) = \varphi_1(t, \zeta_1(\xi_0, \lambda), \lambda)$$

is holomorphic in (ξ_0, λ), for fixed $t \in J_1$; moreover,

$$\varphi|_{J_1} = \varphi_1$$

and hence, $\varphi = \varphi(t, \xi_0, \lambda)$ is holomorphic, for fixed $t \in [\alpha, \theta_2]$, in the variable (ξ_0, λ). In particular at $t = \theta_2$. Continuing this argument until

$$\varphi|_{J_p} = \varphi_p,$$

we conclude that the solution $\varphi = \varphi(t, \xi_0, \lambda)$ of (4.5) satisfying $\varphi(t, \xi_0, \lambda) = \xi_0 = \xi$ is holomorphic in (ξ, λ), for fixed $t \in [\alpha, \beta]$. The proof is completed. $\square$

An immediate corollary of Lemma 4.1, above is the following:

Corollary 4.1. *The solution $\varphi = \varphi(t, \xi, \lambda)$ of (4.5) satisfying $\varphi(t, \xi, \lambda) = \xi$ is an entire function of the parameter λ.*

Proof. Proof directly follows from the fact that the companion matrix $A(t, \lambda)$, for fixed $t \in [\alpha, \beta]$, is an entire function of $\lambda \in \mathbb{C}$. $\square$

The following theorem, however, states that the holomorphic solutions of linear impulsive differential equations in the form (4.4) on an interval $J = [\alpha, \beta]$ are, in fact, entire functions of the parameter λ.

Theorem 4.1. *Any solution $x = x(t, \lambda)$ of (4.4) is entire in the variable λ, for fixed $t \in [\alpha, \beta]$. Moreover, $x^{(j)}(t, \lambda)$ for fixed $t \neq \theta_i$ and $x^{(j)}(\theta_i^{\pm}, \lambda)$, are entire in the variable λ for every $j = 1, \dots, n-1$, and $i = 1, \dots, p$.*

Proof. Proof directly follows from Lemma (4.1) and Corollary (4.1), by writing (4.4) in the form (4.5). □

We turn our attention to conditions for the determination of the eigenvalues of linear impulsive differential equations. in other words, the eigenvalues of the operator $\mathcal{L}_0$. In order to achieve this, let

$$x_1(t, \lambda), \dots, x_n(t, \lambda) \tag{4.9}$$

denote the fundamental solutions of the linear homogeneous impulsive differential equation,

$$\begin{cases} \ell(x) = \lambda x, & t \neq \theta_i, \\ \delta_i(x) = 0, & i = 1, \dots, p, \end{cases} \tag{4.10}$$

satisfying, in particular, the following initial conditions

$$x_j^{(\nu-1)}(\alpha, \lambda) = \begin{cases} 0, & j \neq \nu \\ 1, & j = \nu \end{cases} \qquad j, \nu = 1, \dots, n$$

at the left end point $t = \alpha$ of the interval $J = [\alpha, \beta]$. The fundamental solutions, defined in (4.9), and their derivatives upto of order $n-1$, inclusive, are all entire functions of λ by Theorem 4.1. In order to have nontrivial solutions of the boundary value problem (4.1) we use the results of Chapter 3. Namely, since the solutions $x_j(t, \lambda)$ are linearly independent, the homogeneous boundary value problem (4.1) has λ as an eigenvalue if and only if there exist constants c_j, $j = 1, \dots, n$, not all zero, such that

$$x(t, \lambda) = \sum_{j=1}^{n} c_j x_j(t, \lambda)$$

satisfies the boundary conditions $U(x) = 0$. However, this is the case if and only if the system of equations

$$\sum_{j=1}^{n} c_j U_\nu(x_j) = 0, \quad \nu = 1, \dots, m \tag{4.11}$$

has nontrivial solutions, where U_ν are the components of the vector boundary form U, that is $U = [U_1, \ldots, U_m]^T$. When the system of equations is written in terms of a matrix equation,

$$\begin{pmatrix} U_1(x_1) & U_1(x_2) & \cdots & U_1(x_n) \\ U_2(x_1) & U_2(x_2) & \cdots & U_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ U_m(x_1) & U_m(x_2) & \cdots & U_m(x_n) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.12}$$

it turns out that (4.12) has nontrivial solutions for $c = [c_1, \ldots, c_n]^T$ if and only if the rank of the coefficient matrix,

$$\Gamma(\lambda) = \begin{pmatrix} U_1(x_1) & \cdots & U_1(x_n) \\ \vdots & \ddots & \vdots \\ U_m(x_1) & \cdots & U_m(x_n) \end{pmatrix} \tag{4.13}$$

is less than n. On the other hand however, this matrix $\Gamma(\lambda)$ is an entire function of λ. Therefore, we have the following immediate consequence.

Lemma 4.2. *If $m < n$, then any value of λ is an eigenvalue of the operator $\mathcal{L}_0$.*

Proof. If $m < n$ then we have $\operatorname{rank}\Gamma(\lambda) \leq m < n$, which proves the lemma. □

Moreover, there can only be two possibilities for the values of λ. This is proved in the following theorem.

Theorem 4.2. *For any impulsive differential operator $\mathcal{L}_0$ only the following two possibilities can occur.*

1. *Every number λ is an eigenvalue of $\mathcal{L}_0$, or*

2. *The operator $\mathcal{L}_0$ has at most enumerable eigenvalues (in particular, none at all), and the these eigenvalues can have no finite accumulation point.*

Proof. The case $m < n$ is already proved in Lemma 4.2. So, let $m \geq n$. Then, the rank of the matrix $\Gamma(\lambda)$ will be less than n if and only if all its minors of order n vanish, See [41]. On the other hand, each of these minors are entire functions of λ, and hence we have either (a) all the nth order minors of the matrix $\Gamma(\lambda)$ vanish identically, or (b) at least one nth order minor of $\Gamma(\lambda)$ does not vanish identically. In the case (a) we have, by Lemma 4.2, that any value of λ is an eigenvalue. However, in the case (b), if there is an nth order minor which is not identically zero, then only the zeros of that minor can be eigenvalues of the operator $\mathcal{L}_0$. But, a zero of this minor can be eigenvalue only if it makes all the other minors of $\Gamma(\lambda)$ not identically zero, and the minors of order n vanish. If the latter cannot happen, then the operator $\mathcal{L}_0$ has no eigenvalues. On the contrary, if those zeros are eigenvalues, then these zeros (zeros of a non-vanishing entire function) are isolated. So that they cannot have an accumulation point. Hence, the operator $\mathcal{L}_0$ has at most enumerable eigenvalues, and these eigenvalues can have no finite accumulation point. This completes the proof. □

The case, when $m = n$ is of particular interest in many applications of eigenvalue problems. In the rest of the work we shall consider this case, unless nothing to the contrary is stated, precisely.

Hence, instead of the rank of the square matrix $\Gamma(\lambda)$, we deal with the determinant of this matrix,

$$\gamma(\lambda) = \det \Gamma(\lambda), \tag{4.14}$$

which is called *characteristic determinant* of the operator $\mathcal{L}_0$, or of the boundary value problem (4.1), simply written as $\mathcal{L}_0 x = \lambda x$. By the preceding discussion, the characteristic determinant $\gamma(\lambda)$ is an entire function of λ, and the eigenvalues of the operator $\mathcal{L}_0$ are the zeros of the function $\gamma(\lambda)$, if there is any such zeros. Therefore, we have the following theorem.

Theorem 4.3. *The eigenvalues of the operator $\mathcal{L}_0$ are the zeros of the characteristic determinant $\gamma(\lambda)$. If $\gamma(\lambda)$ vanish identically, then any number λ is an eigenvalue. However, if $\gamma(\lambda)$ is not identically zero, then the operator has at most enumerable eigenvalues, and these eigenvalues can have no finite accumulation point.*

If, in particular, the characteristic determinant $\gamma(\lambda)$ has no zeros at all, then the operator $\mathcal{L}_0$ has no eigenvalues.

It is also possible to have λ as a multiple zero of the characteristic determinant $\gamma(\lambda)$. Recalling that the multiplicity of an eigenvalue is the number of linearly independent eigenfunctions corresponding to that eigenvalue, we state the following theorem.

Theorem 4.4. *If λ_0 is a zero of the characteristic determinant $\gamma(\lambda)$ with multiplicity k, then the multiplicity of the eigenvalue λ_0 cannot be greater than k.*

Proof. Let r be the rank of the matrix $\Gamma(\lambda_0)$, corresponding to $\gamma(\lambda_0)$. Then, the multiplicity of the eigenvalue λ_0 is equal to $n - r$. On the other hand, differentiating the determinant $\gamma(\lambda)$ shows that all derivatives of order less than $n-r$ vanish for $\lambda = \lambda_0$. But, we know that λ_0 is a zero of multiplicity k. Hence, $n-r \leq k$ completing the proof. $\square$

In the case of a zero of the characteristic determinant $\gamma(\lambda)$, of multiplicity 1, that is a simple zero of $\gamma(\lambda)$, we may state the following corollary.

Corollary 4.2. *An eigenvalue of the operator $\mathcal{L}_0$ is simple if it is a simple zero of the characteristic determinant $\gamma(\lambda)$.*

Proof. Let λ_0 be a simple zero of $\gamma(\lambda)$. Then, $n-r \leq 1$, follows from the above theorem. On the other hand, $n-r \geq 1$, because $\gamma(\lambda_0) = 0$. Thus, the number of linearly independent eigenfunctions corresponding to the eigenvalue λ_0 is $n-r = 1$, and completes the proof. $\square$

4.2 Adjoint Eigenvalue Problems

Adjoint operator, say $\mathcal{L}_0^\dagger$, of the impulsive differential operator $\mathcal{L}_0$ is operator $\mathcal{L}_0^\dagger : \mathcal{D}_0^\dagger \to \mathcal{PLC}$ defined by the linear differential operator $\ell^\dagger$, in the domain

$$\mathcal{D}_0^\dagger = \left\{ y \in \mathcal{PLC}^n : U^\dagger(y) = 0,\ \delta_i^\dagger(y) = 0,\ i = 1, \ldots, p \right\},$$

which is a linear subspace of $\mathcal{PLC}$. Namely, if $y \in \mathcal{D}_0^\dagger$ then $\mathcal{L}_0^\dagger y = \ell^\dagger(y)$. Here, the $U^\dagger$ and $\delta_i^\dagger$ for each $i = 1, \ldots, p$ are the corresponding adjoint vector boundary form to U, and adjoint impulse actions to δ_i, respectively. We remark, again that the vector boundary form U is of rank n, and hence the adjoint $U^\dagger$ is of rank n. The adjoint eigenvalue problem, therefore, can be written in the form

$$\begin{cases} \ell^\dagger(y) = \mu y, & t \neq \theta_i, \\ \delta_i^\dagger(y) = 0, & i = 1, \ldots, p, \\ U^\dagger(y) = 0, \end{cases} \tag{4.15}$$

or simply

$$\mathcal{L}_0^\dagger y = \mu y, \tag{4.16}$$

where μ is, in general, a complex parameter.

If λ is an eigenvalue of the operator $\mathcal{L}_0$, then it is easy to show that $\mu = \overline{\lambda}$ is an eigenvalue of the adjoint operator $\mathcal{L}_0^\dagger$. For, if $\ell_1(x) = \ell(x) - \lambda x$, then by the Green's formula obtained in Section 3.1 it follows that $\ell_1^\dagger = \ell^\dagger(y) - \overline{\lambda} y$, since the bilinear form $\mathcal{S}(x,y)$ does not explicitly depends on the coefficient $p_n(t)$ of x in $\ell(x)$, and the impulse actions δ_i does not depend on the parameter λ. Moreover, the following theorem on the multiplicity of $\overline{\lambda}$ of the adjoint eigenvalue problem is valid.

Theorem 4.5. *If λ is an eigenvalue of multiplicity k, of the operator $\mathcal{L}_0$, then $\overline{\lambda}$ is an eigenvalue of the adjoint operator $\mathcal{L}_0^\dagger$, and has the same multiplicity k.*

Proof. Let λ be an eigenvalue with multiplicity k, of the operator $\mathcal{L}_0$. Then, the homogeneous boundary value problem,

$$\begin{cases} \ell(x) - \lambda x = 0, \quad t \neq \theta_i, \\ \delta_i(x) = 0, \quad i = 1, \dots, p, \\ U(x) = 0, \end{cases}$$

has exactly k linearly independent solutions. But, this means the corresponding adjoint problem,

$$\begin{cases} \ell^\dagger(y) - \overline{\lambda} y = 0, \quad t \neq \theta_i, \\ \delta_i^\dagger(y) = 0, \quad i = 1, \dots, p, \\ U^\dagger(y) = 0, \end{cases}$$

has the same number k linearly independent solutions, since the vector boundary form U is assumed to be of rank n. This means that $\overline{\lambda}$ is an eigenvalue with multiplicity k of the adjoint operator $\mathcal{L}_0^\dagger$. This completes the proof. □

Now, suppose that x is an eigenfunction of the operator $\mathcal{L}_0$ corresponding to an eigenvalue λ and y is an eigenfunction corresponding to an eigenvalue μ of the adjoint operator $\mathcal{L}_0^\dagger$. That is, $\mathcal{L}_0 x = \lambda x$ and $\mathcal{L}_0^\dagger y = \mu y$. By Green's formula we obtain,

$$0 = \int_\alpha^\beta \overline{y} \mathcal{L}_0 x \, dt - \int_\alpha^\beta \overline{(\mathcal{L}_0^\dagger y)} x \, dt = (\lambda - \overline{\mu}) \int_\alpha^\beta \overline{y} x \, dt \tag{4.17}$$

for $x \in \mathcal{D}_0$ and $y \in \mathcal{D}_0^\dagger$. If we further, denote the standard inner product, $\langle f \,,\, g \rangle$, of functions f and g in $\mathcal{PLC}$, by

$$\langle f \,,\, g \rangle = \int_\alpha^\beta \overline{g} f \, dt,$$

then (4.17) can be written simply as

$$0 = \langle \mathcal{L}_0 x \,,\, y \rangle - \left\langle x \,,\, \mathcal{L}_0^\dagger y \right\rangle = (\lambda - \overline{\mu}) \langle x \,,\, y \rangle \,. \tag{4.18}$$

The following theorem is obvious.

Theorem 4.6. *Eigenfunctions x and y of the operators $\mathcal{L}_0$ and $\mathcal{L}_0^\dagger$ corresponding to the eigenvalues λ and μ, respectively, are orthogonal if $\lambda \neq \overline{\mu}$ in the sense that $\langle x \,,\, y \rangle = 0$.*

It is easy to see that (4.18), and hence Theorem 4.6 has particular corollaries in the case when the operator $\mathcal{L}_0$ is self-adjoint. The operator $\mathcal{L}_0$ is self-adjoint, however, if and only if $\ell^\dagger = \ell$, and $U(x) = 0$ and $\delta_i(x) = 0$ for every $i = 1, \ldots, p$ are self-adjoint boundary form and impulse actions, respectively. For, when the latter holds, it follows by the boundary form and impulse action formulas, see Section 3.2, that the domains of formally adjoint operators $\mathcal{L}_0$ and $\mathcal{L}_0^\dagger$ coincide. Since $\ell^\dagger = \ell$, it follows that $\mathcal{L}_0^\dagger = \mathcal{L}_0$. Hence we may state the following theorem, similar to the one for self-adjoint differential operators without impulse effect.

Theorem 4.7. *Eigenvalues of a self-adjoint operator $\mathcal{L}_0$ are real.*

Proof. For a self-adjoint operator $\mathcal{L}_0$, we have immediately from (4.18) that

$$0 = \langle \mathcal{L}_0 x \, , \, x \rangle - \langle x \, , \, \mathcal{L}_0 \rangle = (\lambda - \overline{\lambda}) \, \langle x \, , \, x \rangle \, ,$$

where x is the eigenfunction corresponding to the eigenvalue λ of the self-adjoint operator $\mathcal{L}_0$. Since $\langle x \, , \, x \rangle \neq 0$ for eigenfunctions, it follows $\overline{\lambda} = \lambda$, in other words, λ is real. □

Moreover, if x and y are eigenfunctions belonging to different eigenvalues λ and μ, respectively, of a self-adjoint impulsive differential operator $\mathcal{L}_0$, (4.18) implies that

$$0 = (\lambda - \mu) \, \langle x \, , \, y \rangle \, .$$

This proves the following corollary.

Corollary 4.3. *Eigenfunctions of a self-adjoint operator $\mathcal{L}_0$ corresponding to different eigenvalues are orthogonal.*

Since it is always possible to choose mutually orthogonal eigenfunctions corresponding to one and the same eigenvalue (of an impulsive operator $\mathcal{L}_0$) by the Gram-Schmidt orthogonalization process, it follows from the above corollary that (in the case when $\mathcal{L}_0$ is self-adjoint) it is possible to obtain a set of mutually orthogonal eigenfunctions of a self-adjoint operator $\mathcal{L}_0$.

4.3 Nonhomogeneous Problems Containing a Parameter

In the previous sections of this chapter we have studied the eigenvalue problems

$$\mathcal{L}_0 x = \lambda x, \tag{4.19}$$

where the domain of the operator is the linear subspace $\mathcal{D}_0$ of $\mathcal{PLC}^n$, defined by

$$\mathcal{D}_0 = \{x \in \mathcal{PLC}^n : U(x) = 0,\ \delta_i(x) = 0,\ i = 1, \ldots, p\}\,.$$

However, these types of eigenvalue problems correspond to the study of homogeneous boundary value problems of the form

$$\begin{cases} \ell(x) = \lambda x, \quad t \neq \theta_i, \\ \delta_i(x) = 0, \quad i = 1, \ldots, p, \\ U(x) = 0, \end{cases} \tag{4.20}$$

where λ is a certain parameter, and U is a boundary form of rank n.

On the other hand, we studied nonhomogeneous boundary value problems in Section 3.4 of Chapter 3, and their solutions represented by the Green's functions $G(t, s)$ and $H(t, \theta_j^+)$ for every $j = 1, \ldots, p$. These nonhomogeneous problems, therefore, necessarily force us to change the operator $\mathcal{L}_0$ to some other impulsive differential operator, say $\mathcal{L}_a$, defined in a domain, say $\mathcal{D}_a$. We make this in the following way.

Let $a = \{a_i\} = \{a_i\}_{i=1}^p$ be a finite sequence of vectors $a_i \in \mathbb{F}^n$, then we define $\mathcal{L}_a : \mathcal{D}_a \to \mathcal{PLC}$ to be an impulsive differential operator defined by ℓ, on the domain

$$\mathcal{D}_a = \{x \in \mathcal{PLC}^n : U(x) = 0,\ \delta_i(x) = a_i,\ i = 1, \ldots, p\}$$

where U is of rank n. Clearly $\mathcal{D}_a$ is not a linear space.

We consider the following problem, containing a parameter λ:

$$\mathcal{L}_a x = \lambda x + f(t), \tag{4.21}$$

which corresponds to a nonhomogeneous boundary value problem of the form

$$\begin{cases} \ell(x) = \lambda x + f(t), \quad t \neq \theta_i, \\ \delta_i(x) = a_i, \quad i = 1, \ldots, p, \\ U(x) = 0. \end{cases} \tag{4.22}$$

The corresponding homogeneous problem for the problem in (4.21) is given by (4.19).

As in the classical theory of ordinary differential equations, the eigenvalues of the operator $\mathcal{L}_0$ strongly effects the existence of solutions of (4.21). We recall that the values of λ for which the homogeneous problem (4.20) has nontrivial solutions are obviously the eigenvalues of the operator $\mathcal{L}_0$, and each nontrivial solution is a related eigenfunction. Now, suppose λ is not an eigenvalue of the impulsive differential operator $\mathcal{L}_0$. This means the problem $\mathcal{L}_0 x = \lambda x$ has only the trivial solution, and hence, by the results of the homogeneous boundary value problems, it follows that there exist unique Green's functions $G(t,s,\lambda)$ and $H(t,\theta_j^+,\lambda)$ for each $j = 1,\dots,p$, depending on λ. The unique solution $x = x(t,\lambda)$ of (4.21) for every $f \in \mathcal{PLC}$ and $a = \{a_i\}$ with $a_i \in \mathbb{F}^n$ can be given as follows,

$$x(t,\lambda) = \int_\alpha^\beta G(t,s,\lambda) f(s)\, ds + \sum_{j=1}^p H(t,\theta_j^+,\lambda)\, a_i$$

in terms of the Green's functions. Note that these functions, $G(t,s,\lambda)$ and $H(t,\theta_j^+,\lambda)$ for each $j = 1,\dots,p$, satisfy the properties of Green's functions for every λ, provided that λ is not an eigenvalue of $\mathcal{L}_0$. So, we have the following theorem.

Theorem 4.8. *If λ is not an eigenvalue of $\mathcal{L}_0$, then for any $f \in \mathcal{PLC}$ and $a = \{a_i\}$ with $a_i \in \mathbb{F}^n$, the problem (4.21) has a unique solution $x = x(t,\lambda)$ defined by*

$$x(t,\lambda) = \int_\alpha^\beta G(t,s,\lambda) f(s)\, ds + \sum_{j=1}^p H(t,\theta_j^+,\lambda)\, a_i,$$

where $G(t,s,\lambda)$ and $H(t,\theta_j^+,\lambda)$ for each $j = 1,\dots,p$ are the Green's functions for (4.19).

If λ of the operator $\mathcal{L}_0$, then we may rewrite the theorem 3.6 for the nonhomogeneous boundary value problems obtained in Section (3.4).

Theorem 4.9. *If λ is an eigenvalue of $\mathcal{L}_0$, then the problem (4.21) has a solution if and only if the equality*

$$\int_\alpha^\beta \overline{\psi}(s) f(s)\, ds + \sum_{i=1}^p \widehat{\psi}^*(\theta_i^+) S(\theta_i^+) a_i = 0$$

holds for every solution ψ of the adjoint homogeneous problem $\mathcal{L}_0^\dagger y = \overline{\lambda} y$.

Now, suppose that $\lambda = 0$ is not an eigenvalue of the operator $\mathcal{L}_0$. In other words, suppose that $\mathcal{L}_0 x = 0$ has only the trivial solution. This implies, however, the existence of unique Green's functions

$$G(t,s) = G(t,s,0)$$

and

$$H(t,\theta_j^+) = H(t,\theta_j^+,0)$$

so that for any $f \in \mathcal{PLC}$ and $a = \{a_i\}$ with $a_i \in \mathbb{F}^n$ the unique solution $x = x(t)$ of the nonhomogeneous problem (4.21) can be written, formally, in terms of the parameter λ as follows

$$x(t) = \lambda \int_\alpha^\beta G(t,s)x(s)\,ds + \int_\alpha^\beta G(t,s)f(s)\,ds + \sum_{j=1}^{p} H(t,\theta_j^+)\,a_i,$$

and this yields the well-known Fredholm type integral equation,

$$x(t) = \lambda \int_\alpha^\beta G(t,s)x(s)\,ds + g(t), \tag{4.23}$$

where $g(t)$ is a function of class $\mathcal{PLC}$ and defined by

$$g(t) = \int_\alpha^\beta G(t,s)f(s)\,ds + \sum_{j=1}^{p} H(t,\theta_j^+)\,a_i.$$

Also notice that, the function $g(t)$, nonhomogeneous part of the integral equation (4.23), is identically zero function if and only if $f(t) = 0$ for all $t \in [\alpha, \beta]$ and $a_i = 0$ for every $i = 1, \ldots, p$. Since $g(t)$ satisfies $\mathcal{L}_a x = f(t)$ whose corresponding homogeneous problem is assumed to have only the trivial solution.

Our aim, here, is not to investigate the corresponding integral equation (4.23), where $G(t,s)$ stands for the kernel of some integral operator. However, we emphasize that this kernel $G(t,s)$ is of class $\mathcal{PLC}$ for fixed s or t in the interval $J = [\alpha, \beta]$. Also, the function $g(t)$ is of class $\mathcal{PLC}$. Moreover, $G(t,s)$ and $g(t)$ are square integrable functions over the domains J^2 and J, respectively. Hence, the general theorems concerning the these types of integral equations, namely Fredholm second type, can be applied to (4.23). We refer [37] for the study of integral equations with L_2-kernels. Also, [44, 46] includes such types of integral equations for functions with bounded variations.

4.4 Sturm-Liouville Operators

In this section we shall investigate a special kind of impulsive operator $\mathcal{L}_0$, which is defined by a particular second order differential operator ℓ of the form

$$\ell(x) = -\frac{d}{dt}\left(p_0(t)\frac{dx}{dt}\right) + q(t)\,x, \quad t \neq \theta_i \tag{4.24}$$

on the linear subspace

$$\mathcal{D}_0 = \{x \in \mathcal{PLC}^n : U(x) = 0,\ \delta_i(x) = 0, \quad i = 1, \ldots, p\} \tag{4.25}$$

of $\mathcal{PLC}$, where $p_0 \in \mathcal{PLC}^1$, $\frac{1}{p_0} \in \mathcal{PLC}$, and $q \in \mathcal{PLC}$ are real valued functions of $t \in [\alpha, \beta]$. Also, the boundary form U and impulse actions δ_i for $i = 1, \ldots, p$ are defined in the general settings as follows,

$$U(x) = M\widehat{x}(\alpha) + N\widehat{x}(\beta) \tag{4.26}$$

and

$$\delta_i(x) = \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-), \quad i = 1, \ldots, p, \tag{4.27}$$

where $M = (M_{ij})$, and $N = (N_{ij})$ are 2×2 matrices such that the $\text{rank}(M : N) = 2$, and $B_i = (b_{j\nu}(i))$ are all 2×2 matrices such that

$$C_i = E + B_i$$

are nonsingular for every $i = 1, \ldots, p$.

Our aim, in this section, is to give necessary and sufficient conditions for $\mathcal{L}_0$ to be self-adjoint. Then, we want to define *Sturm-Liouville* impulsive differential operator in the case when $\mathbb{F} = \mathbb{R}$. We define

$$\begin{aligned} U_1(x) &= M_{11}x(\alpha) + M_{12}x'(\alpha) \\ U_2(x) &= N_{21}x(\beta) + N_{22}x'(\beta) \end{aligned} \tag{4.28}$$

with $M_{21} = M_{22} = N_{11} = N_{12} = 0$, provided that

$$\begin{aligned} |\, M_{11}\,| + |\, M_{12}\,| &\neq 0 \\ |\, N_{21}\,| + |\, N_{22}\,| &\neq 0. \end{aligned}$$

In order to achieve our goal we need to concentrate mainly on the impulse actions δ_i, since the rest will follows from the classical theory of Sturm-Liouville boundary value problems. Of course, the impulse conditions $\delta_i(x) = 0$ play an important role in the self-adjointness of the impulsive differential operator $\mathcal{L}_0$.

In the case when $p_0(t)$ and $q(t)$ are real valued functions on $[\alpha, \beta]$, it follows from the Green's formula that $\ell^\dagger = \ell$. Moreover, we know from the Section 3.3 that

$$\begin{cases} \ell(x) = 0, \quad t \neq \theta_i, \\ \delta_i(x) = 0, \quad i = 1, \ldots, p, \\ U(x) = 0 \end{cases}$$

is self-adjoint if and only if the conditions

(a) $S^{-1}(\theta_i^+) = (E + B_i)S^{-1}(\theta_i^-)(E + B_i^*), \quad i = 1, \ldots, p,$

(b) $MS^{-1}(\alpha)M^* = NS^{-1}(\beta)N^*$

hold. Here, S is the matrix of the bilinear form associated with the operator ℓ. An immediate calculation shows that S is

$$S(t) = \begin{pmatrix} 0 & -p_0(t) \\ p_0(t) & 0 \end{pmatrix}$$

with $\det S(t) = (p_0(t))^2 \neq 0$ for all $t \in [\alpha, \beta]$, and $\det S(\theta_i^+) = (p_0(\theta_i^+))^2 \neq 0$ for all $i = 1, \ldots, p$. Hence, a necessary condition for the operator $\mathcal{L}_0$ to be self-adjoint can be obtained from (a) by taking the determinant of both sides, namely,

$$|\det(E + B_i)|^2 = \frac{(p_0(\theta_i^-))^2}{(p_0(\theta_i^+))^2}. \tag{4.29}$$

Of course the necessary condition (4.29) becomes

$$|\det(E + B_i)|^2 = 1$$

if $p_0(t)$ is continuous on $[\alpha, \beta]$.

On the other hand, if we calculate directly from condition (a), denoting $C_i = E + B_i$ with entries $c_{j\nu}(i)$, we see that

$$\frac{p_0(\theta_i^-)}{p_0(\theta_i^+)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} c_{11}(i)\overline{c_{12}(i)} - c_{12}(i)\overline{c_{11}(i)} & c_{11}(i)\overline{c_{22}(i)} - c_{12}(i)\overline{c_{21}(i)} \\ c_{21}(i)\overline{c_{12}(i)} - c_{22}(i)\overline{c_{11}(i)} & c_{21}(i)\overline{c_{22}(i)} - c_{22}(i)\overline{c_{21}(i)} \end{pmatrix}$$

holds as a necessary and sufficient condition for δ_i to be self-adjoint. In other words, the impulse actions δ_i are self-adjoint if and only if the following three conditions,

$$c_{11}(i)\overline{c_{12}(i)} - c_{12}(i)\overline{c_{11}(i)} = 0 \tag{4.30}$$

$$c_{21}(i)\overline{c_{22}(i)} - c_{22}(i)\overline{c_{21}(i)} = 0 \tag{4.31}$$

$$c_{11}(i)\overline{c_{22}(i)} - c_{12}(i)\overline{c_{21}(i)} = \frac{p_0(\theta_i^-)}{p_0(\theta_i^+)} \tag{4.32}$$

holds for all $i = 1, \ldots, p$. If B_i for every $i = 1, \ldots, p$ are real matrices, then the necessary and sufficient condition reduces to

$$\det(E + B_i) = \frac{p_0(\theta_i^-)}{p_0(\theta_i^+)}, \quad i = 1, \ldots, p. \tag{4.33}$$

Similar calculations can be carried out for the boundary conditions U to obtain necessary and sufficient conditions for U to be self-adjoint. We state the following theorem for self-adjointness of the operator $\mathcal{L}_0$.

Theorem 4.10. *The impulsive operator $\mathcal{L}_0$ is self-adjoint if and only if the following six conditions hold.*

1. $[1+b_{11}(i)]\overline{b_{12}(i)} - b_{12}(i)[1+\overline{b_{11}(i)}] = 0$,
2. $b_{21}(i)[1+\overline{b_{22}(i)}] - [1+b_{22}(i)]\overline{b_{21}(i)} = 0$,
3. $[1+b_{11}(i)][1+\overline{b_{22}(i)}] - b_{12}(i)\overline{b_{21}(i)} = \dfrac{p_0(\theta_i^-)}{p_0(\theta_i^+)}$, *for all* $i = 1, \ldots, p$,
4. $\dfrac{\overline{M_{11}}M_{12} - M_{11}\overline{M_{12}}}{p_0(\alpha)} = \dfrac{\overline{N_{11}}N_{12} - N_{11}\overline{N_{12}}}{p_0(\beta)}$,
5. $\dfrac{\overline{M_{21}}M_{22} - M_{21}\overline{M_{22}}}{p_0(\alpha)} = \dfrac{\overline{N_{21}}N_{22} - N_{21}\overline{N_{22}}}{p_0(\beta)}$,
6. $\dfrac{\overline{M_{11}}M_{22} - M_{21}\overline{M_{12}}}{p_0(\alpha)} = \dfrac{\overline{N_{11}}N_{22} - N_{21}\overline{N_{12}}}{p_0(\beta)}$.

If all those matrices M, N and B_i for every $i = 1, \ldots, p$ were real then we would have only the conditions in (3) and (6). This is given in the following corollary.

Corollary 4.4. *If M, N and B_i for $i = 1, \ldots, p$ are real then $\mathcal{L}_0$ is self-adjoint if and only if*

(a) $\det(E + B_i) = \dfrac{p_0(\theta_i^-)}{p_0(\theta_i^+)}, \quad i = 1, \ldots, p,$

(b) $\dfrac{M_{11}M_{22} - M_{21}M_{12}}{p_0(\alpha)} = \dfrac{N_{11}N_{22} - N_{21}N_{12}}{p_0(\beta)}.$

Now, the following homogeneous boundary value problem for impulsive differential equation is called homogeneous *Sturm-Liouville boundary value problem.* Namely,

$$\begin{cases} -\dfrac{d}{dt}\left(p_0(t)\dfrac{dx}{dt}\right) + q(t)\,x = 0, \quad t \neq \theta_i, \\ \Delta\widehat{x}|_{t=\theta_i} - B_i\widehat{x}(\theta_i^-) = 0, \quad i = 1, \ldots, p, \\ a_1x(\alpha) + a_2p_0(\alpha)x'(\alpha) = 0, \\ b_1x(\beta) + b_2p_0(\beta)x'(\beta) = 0, \end{cases} \tag{4.34}$$

provided that

i. $p_0 \in \mathcal{PLC}^1$, $\frac{1}{p_0} \in \mathcal{PLC}$, and $q \in \mathcal{PLC}$ are real valued functions of $t \in [\alpha, \beta]$,

ii. a_i, b_i are all real for $i = 1, 2$, such that $a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$,

iii. The matrices B_i are real 2×2 with $\det(E + B_i) = \dfrac{p_0(\theta_i^-)}{p_0(\theta_i^+)}$, for all $i = 1, \ldots, p$.

The corresponding nonhomogeneous Sturm-Liouville problem can be defined in an obvious manner.

Under the above assumptions, the impulsive differential operator $\mathcal{L}_0$, defined by ℓ, on the domain prescribed by the homogeneous impulse and boundary conditions is self-adjoint, and this operator is called *Sturm-Liouville operator*. Hence, all the results obtained in Section 4.2 for self-adjoint operators are valid. In particular, all eigenvalues of a Sturm-Liouville operator are real, and eigenfunctions corresponding to different eigenvalues are orthogonal.

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Vita

Ömür UĞUR was born in İstanbul, on 8 March, 1971. After completing his elementary and secondary education he was accepted to Uluğ Bey (Private) High School with scholarship from İstanbul Eğitim ve Kültür (İSTEK) Vakfı.

He started his undergraduate studies at the Department of Mathematics, M.E.T.U. (Middle East Technical University) in 1991 and took his B.S. degree in 1995. At the same time he completed courses in Physics Minor Programme. He became research assistant at the Department of Mathematics, M.E.T.U., in the same year, and started his graduate studies. He received his M.Sc. degree in Mathematics in 1998, with a thesis entitled "Hypervirial Analysis of Quantum Mechanical Systems", and published his first international article "An eigenfunction expansion for the Schrödinger equation with arbitrary non-central potentials" in Journal of Mathematical Chemistry, Vol. 32, No. 4, 2002.