THE IDENTIFICATION OF A BIVARIATE MARKOV CHAIN MARKET MODEL

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Approval of the Graduate School of Social Sciences

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ABSTRACT

THE IDENTIFICATION OF A BIVARIATE MARKOV CHAIN MARKET MODEL

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This work is an extension of the classical Cox-Ross-Rubinstein discrete time market model in which only one risky asset is considered. We introduce another risky asset into the model. Moreover, the random structure of the asset price sequence is generated by bivariate finite state Markov chain. Then, the interest rate varies over time as it is the function of generating sequences. We discuss how the model can be adapted to the real data. Finally, we illustrate sample implementations to give a better idea about the use of the model.

ÖZ

İKİ DEĞİŞKENLİ MARKOV PİYASA MODELİ

Yıldırak, Ş. Kasırga

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Bu tez tek bir riskli finansal varlıktan oluşan Cox-Ross-Rubinstein kesikli zaman piyasa modelini iki riskli finansal varlıgin bulunduğu bir modele dönüştürerek, söz konusu bu iki enstrüman arasındaki ilişkiyi analiz eder. Varlık fiyatlarının rassal yapısı iki değişkenli iki durumlu markov zincir varsayımından oluşturulmaktadır. Ayrıca modelin tanımlı ve analitik çözümünün olduğu gösterilmiştir. Son olarak, modelin tatbiki ve piyasa verilerine uygunluğu tartışılmıştır.

To Övida Mou,

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CHAPTER 1

INTRODUCTION

> You can say, well, this orbit is really not a complete orbit. Actually, at every moment the electron has only an inaccurate position and an inaccurate velocity, and between these two inaccuracies there is this uncertainty relation. And only by this idea it was possible to say what such an orbit was. Werner Heisenberg

This dissertation develops a financial market model. The model aims at explaining the co-movements of returns of two risky assets and contributes to the solution of the constant interest rate problem by allowing the interest rate to vary over time.

The model developed in this study is a modification of Cox-Ross-Rubinstein (1979) (CRR) single asset binomial option pricing model. To modify the CRR model, first, we introduce a bivariate model in which the random structure follows an independent sequences of innovations. Then we extend it immediately to the case where the random structure follows a Markov chain which allows the generating sequences to be conditionally independent. In the independent case, we take the interest rate as constant. On the other hand, in the case where we introduce the Markov property, the interest rate gains a more realistic structure and becomes dependent on time.

Our model is developed by assuming that the financial markets are complete with no transaction costs, no taxes, no restrictions on short sales, and assets are infinitely divisible. Also no arbitrage opportunities are allowed in modeling the financial markets.

In Chapter 2 of the thesis, we construct the model in a world of the independent innovations. We assume there are two generating sequences related to two different risky assets. There is a risk free asset whose price is the discount factor. Two generating sequences are also independent and iid. Since the randomness of the model depends only on these sequences, the risky asset prices follow a markov chain. We find that the unconditional expectations of the both generating sequences are equal to the interest rate which is the price of the risk-free asset. Thus the interest rate is constant under the assumption of independency. We take advantage of the fact that the discounted prices are martingales. We use this feature to solve the inverse problem for the parameters and show that the model has a closed-form solution. Moreover, we obtain unbiased estimators for cross-effect coefficients. After having simulated the model we check the efficiency of the estimators in the simulated sample. In the estimation procedure, we introduce an interesting solution to the inverse problem. We take advantage of the occurrences of an random event and of the series being non-stationary. We have two different estimators for each cross-effect parameters,ε and δ. It is very convenient, because we estimate the parameters using second method so-called *occurrences* when the price series is stationary.

We introduce the Markov property into the model in Chapter 3. We loose the assumption that the generating sequences are independent from each other by having the generating sequences followed a first order markov chain. In this case, Interest rate becomes the function of joint realization of the generating sequences. Then we obtain four different values that the interest rate that depends on the realization of the random sequences at the previous period[1]. We assume that the sequences are conditionally independent to ease the calculations. Moreover when they are not conditionally independent, there exists more than one probability that gives the same conditional joint probability that satisfies the martingale measure. Then the market is said to be incomplete. We show how one can obtain a numerical solution for the problem of portfolio optimization under complete and incomplete market, at appendix C. Again in the simulated sample we search for the properties of the estimators we obtained under the markov chain hypothesis.

In Chapter 4 we first illustrate the way we simulate the model under both assumptions. Then we obtain a data from the Central Bank of Turkey to analyze the results and the methods to fit the model to the real world. We discuss the results and suggest possible extensions to make the model more efficient. We also show that the model is more efficient than the single asset

[1]Notice that, as we have more than two states interest rate takes increasing numbers of values.

CRR model.

Finally in Chapter 5 of this thesis, we analyze the use of the model on option pricing implementations. We try out different options with two underlying options. We find that the model explains the pay-off path of the spread options. This is expected due to the fact that the model is built to explain the co-movements of two different assets.

The assumption of complete markets with no arbitrage opportunities may be seen as less realistic when compared with the real world phenomena. But these assumptions do not restrict our model since we can extend to an incomplete market. Although we do not focus on this case in this thesis we show how to extend the model in Appendix C.

Indeed, two biggest difficulties in modelling a financial market have been to model the correlation structure and let the interest rate vary. This study aims to overcome the difficulties caused by such problems by using different approaches. Moreover, these approaches are made as simple as possible without loosing accuracy.

CHAPTER 2

THE MODEL

In this chapter, we will build our model and identify its parameters.

2.1 Introduction

The model we study in this chapter is a bivariate extension of Cox-Ross-Rubinstein's binomial single asset pricing model [9]. CRR model introduces one riskless asset that has a return of periodic compounded interest rate and a risky asset whose return is stochastic.

There has been some works on Multivariate, Multi-Dimensional and Multi-State extension of CRR model. Boyle [3] makes the first extension, where he adds another risky asset to the model. He suggests that the mean and the variance of the discrete-time model should be the same as the mean and the variance of the continuous model, as a solution to the system of two equations. Boyle, Evnine, and Gibbs [5] extends the Boyle model to k generating sequences. He [10] extends the model to a multi-asset case. Another extension that is made by Boyle and Vorst [7] considers of adding a transaction cost to the model as an additional variable.

Extensions made so far only consider the time discretion of continuous time models, while our contribution follows discrete time methodology all the time.

For the sake of simplicity we first consider a bivariate generating sequence consisting of two mutually independent iid sequences. We construct the model at the second section. In Section 3, we identify the model, and solve the inverse problem for the parameters of the model. We test the model by simulating the model in Matlab environment in Section 4. In the next chapter, we discuss the model in which the generating bivariate sequence constitutes a Markov chain and show that the identification procedure remains as simple as in the iid case

2.2 The Cox-Ross-Rubinstein Model

In this section, we will give a brief description of the CRR model. There is only one risky asset whose price S_n at time n, $0 \leq n \leq N$, and a riskless asset whose return is over one period of time, $S_n^0 = (1+r)^n$. The risky asset is modelled as follows: between two consecutive period the relative price change is either a or b:

$$S_{n+1} = \begin{cases} S_n(1+a) \\ S_n(1+b) \end{cases}$$

The initial stock price S_0 is given. The set of all possible states then $\Omega = \{1+a, 1+b\}^N$. Each N-tuple represents the successive values of the ratio S_{n+1}/S_n, $n = 0, 1, ..., N-1$. It is assumed that $\mathcal{F}_0 = \{\phi, \Omega\}$ and $(F) = \mathcal{P}(\Omega)$. The $\sigma - algebra$ $\mathcal{F}_n$ is equal to $\sigma(S_1, ..., S_2)$ generated by the random variables $S_1, ..., S_n$ Let's introduce the variables $T_n = S_n/S_{n+1}$, for $n = 1, ..., N$ If $(x_1, ..., x_n) \in \Omega$, then $\mathbb{P}\{(x_1, ..., x_n)\} = \mathbb{P}\{T_1 = x_1, ..., T_N = x_N\}$ As a result, knowing $\mathbb{P}$ equivalent to knowing the law of the N-tuple $(T_1, ..., T_N)$. Also notice that for $n \geq 1$, $\mathcal{F}_n = \sigma(T_1, ..., T_n)$

Proposition 2.2.1. *The discounted prices* $(\tilde{S}_n)$ *is a martingale under* $\mathbb{P}$, *iff* $\mathbb{E}(T_{n+1}|\mathcal{F}_n) = 1 + r$, $\forall n \in \{0, 1, ..., N-1\}$

Proof.

$$\frac{\tilde{S}_{n+1}}{\tilde{S}_n} = \frac{(1+r)^{-(n+1)} S_{n+1}}{(1+r)^{-n} S_n} = \frac{T_{n+1}}{1+r}$$

By the martingale property

$$\mathbb{E}[\frac{\tilde{S}_{n+1}}{\tilde{S}_n}|\mathcal{F}_n] = 1$$

So

$$\mathbb{E}[T_{n+1}|\mathcal{F}_n] = 1 + r$$

□

By the theorem $B.0.2$ we can write the following

$$\mathbb{E}[T_{n+1}|\mathcal{F}_n] = 1 + r$$

Then

$$(1 + a)p + (1 + b)(1 - p) = 1 + r$$

Solving for p

$$p = \frac{b - r}{b - a}$$

Notice that if $r \notin (a, b) \Rightarrow \mathbb{E}[T_{n+1}|\mathcal{F}_n] \neq 1 + r$.

2.3 The Model

We suppose that the time parameter n takes its values in the set of nonnegative integers. We consider a riskless asset whose value at time n is given by

$$S_n^0 = \prod_{k=1}^{n} (1 + r_k) \tag{2.3.1}$$

where r_k is the interest rate at time $k-1$. S_n^0 is the value at time n of one unit of currency at time 0. The values at time n of the two risky assets that we consider in this model are denoted by S_n^1 and S_n^2.The transpose of a matrix M is denoted by M'. The price vector S_n is the column vector $[S_n^1\ S_n^2]'$. The random structure of the sequence (S_n) is supposed to be generated by two independent and identically distributed sequences (U_n^1) and (U_n^2), where (U_n^1)(resp. (U_n^2)) takes its values a and b, $a < b$ (resp. c and d, $c < d$) with probabilities p and $(1 - p)$ (resp. q and $(1 - q)$). To avoid trivial cases we suppose that $0 < p < 1$, $0 < q < 1$. We also denote $U_n = [U_n^1\ U_n^2]'$. The sample space Ω consists of the cartesian product $\mathcal{U}_N^1 \times \mathcal{U}_N^2$, where $\mathcal{U}_N^1$ (resp.

$\mathcal{U}_N^2$) is the set of words of length $N < \infty$ formed by two-letter alphabet $\{a, b\}$, (resp. $\{c, d\}$. Therefore Ω is a finite space. We define a probability measure $\mathbb{P}$ on Ω as the product of two Bernoulli distributions. For instance, if $\omega = (x, y) \in \mathcal{U}_N^1 \times \mathcal{U}_N^2$

$$\mathbb{P}(x, y) = p^k (1-p)^{N-k} q^j (1-q)^{N-j} \tag{2.3.2}$$

where k (resp.j) is the number of times that a (resp.c) figures in x (resp.y). According to our hypothesis $\forall \omega = (x, y) \in \Omega, \mathbb{P}(\omega) > 0$. It is obvious that this probability generates a probability measure on the algebra of all subsets of Ω. We always denote it by $\mathbb{P}$. According to this construction, random variables U_n^1 and U_n^2 are defined as follows: $U_n^1(\omega) = U_n^1(x, y)\,(resp.\, U_n^2(\omega) = U_n^2(x, y))$ is only the function of the nth letter in word $x\,(resp.\, y)$. It is obvious that (U_n^1) and $(U_n^2)$are independent iid sequences. The $\sigma - algebra$ (algebra) $\mathcal{F}_n$ generated by $\{U_k, k \leq n\}$, $n \leq N$, is the set of all subsets of $\mathcal{U}_n^1 \times \mathcal{U}_n^2$, where $\mathcal{U}_n^1\,(resp.\mathcal{U}_n^2)$ is the subset of($\mathcal{U}_N^1$) (resp. $(\mathcal{U}_N^2)$) whose prefixes are the given word of length n. To make things simpler we identify $\mathcal{U}_n^1 \times \mathcal{U}_n^2$ with the cartesian product of the sets of words of length n with alphabets $\{a, b\}$ and $\{c, d\}$, respectively.

The prices of the risky assets will form a Markov chain described by

$$\begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} = \begin{pmatrix} 1 + U_n^1 & \varepsilon(U_n^2 + \alpha) \\ \delta(U_n^1 + \beta) & 1 + U_n^2 \end{pmatrix} \begin{pmatrix} S_{n-1}^1 \\ S_{n-1}^2 \end{pmatrix} \tag{2.3.3}$$

with $\varepsilon \neq 0$ or $\delta \neq 0$.

Now, we want $\mathbb{P}$ to be the risk neutral probability under which the discounted asset prices become martingales with respect to filtration $(\mathcal{F}_n)$. Discounted asset prices are:

$$\tilde{S}_n^1 = S_n^1 \prod_{k=1}^{n} (1 + r_k)^{-1} \tag{2.3.4}$$

$$\tilde{S}_n^2 = S_n^2 \prod_{k=1}^{n} (1 + r_k)^{-1} \tag{2.3.5}$$

Therefore equation 2.3.3 becomes:

$$\begin{pmatrix} \prod_{k=1}^{n}(1+r_k)\tilde{S}_n^1 \\ \prod_{k=1}^{n}(1+r_k)\tilde{S}_n^2 \end{pmatrix} = \begin{pmatrix} 1+U_n^1 & \varepsilon(U_n^2+\alpha) \\ \delta(U_n^1+\beta) & 1+U_n^2 \end{pmatrix} \begin{pmatrix} \prod_{k=1}^{n-1}(1+r_k)\tilde{S}_{n-1}^1 \\ \prod_{k=1}^{n-1}(1+r_k)\tilde{S}_{n-1}^2 \end{pmatrix} \tag{2.3.6}$$

From this we get:

$$\begin{pmatrix} \tilde{S}_n^1 \\ \tilde{S}_n^2 \end{pmatrix} = (1+r_n)^{-1} \begin{pmatrix} 1+U_n^1 & \varepsilon(U_n^2+\alpha) \\ \delta(U_n^1+\beta) & 1+U_n^2 \end{pmatrix} \begin{pmatrix} \tilde{S}_{n-1}^1 \\ \tilde{S}_{n-1}^2 \end{pmatrix} \tag{2.3.7}$$

We want the discounted prices $\tilde{S}_n$ to constitute a martingale under $\mathbb{P}$. We also assume that (r_n) is predictable (i.e. $\mathcal{F}_{n-1}$ measurable). Taking the conditional expectations of both sides of the equation (2.3.7), we get

$$\mathbb{E}(\tilde{S}_n|\mathcal{F}_{n-1}) = \frac{1}{1+r_n} \begin{pmatrix} 1+\mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & 1+\mathbb{E}(U_n^2) \end{pmatrix} \tilde{S}_{n-1} \tag{2.3.8}$$

The martingale property allows us to write the following expression

$$\tilde{S}_{n-1} = \frac{1}{1+r_n} \begin{pmatrix} 1+\mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & 1+\mathbb{E}(U_n^2) \end{pmatrix} \tilde{S}_{n-1} \qquad (2.3.9)$$

This also implies that

$$\tilde{S}_{n-1} = \frac{1}{r_n} \begin{pmatrix} \mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & \mathbb{E}(U_n^2) \end{pmatrix} \tilde{S}_{n-1} \qquad (2.3.10)$$

To simplify notation, let us put $\tilde{S}_{n-1} = [x\ y]'$. Therefore this last equation takes the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (2.3.11)$$

i.e., $x = Ax + By$, $y = Cx + Dy$. This implies that $(x, y)\perp(A-1, B)$ and $(x, y)\perp(C, D-1)$. Since $x = \tilde{S}_{n-1}^1$ and $y = \tilde{S}_{n-1}^2$ are random sequences, with $0 < p < 1$ and $0 < q < 1$, there is no chance that $\tilde{S}_{n-1}$ takes successive collinear values. Thus, the above orthogonality relations are satisfied only if $x = 0$ and $y = 0$. Therefore, $\mathbb{E}(U_n^1) = \mathbb{E}(U_n^2) = r_n$ and $\alpha = \beta = -r_n$. We have proved that under the independency assumption, the interest rate r_n is constant. We then can write the following

$$\mathbb{E}(U_n^1) = \mathbb{E}(U_n^2) = r_n = r. \qquad (2.3.12)$$

Equation 2.3.3 becomes

$$\begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} = \begin{pmatrix} 1+U_n^1 & \varepsilon(U_n^2 - r) \\ \delta(U_n^1 - r) & 1+U_n^2 \end{pmatrix} \begin{pmatrix} S_{n-1}^1 \\ S_{n-1}^2 \end{pmatrix} \tag{2.3.13}$$

Remark 2.3.1. The above model (D.0.7) is an extension of the model known as the Cox, Ross and Rubinstein (CRR) [9] model where only one risky asset is considered.

In further models we shall consider markovian extension of this model. For the moment we are going to work on the identification of the parameters for the model with two risky assets.

Remark 2.3.2. Normally ε and δ in the model should be small numbers. They express the amplitude of the perturbation caused by S^2 on S^1 and by S^1 on S^2, respectively. That is to say, although generating sequences are assumed to be independent, assets are kind of correlated via ε and δ. We suppose $\varepsilon\delta \neq 1$

2.4 Identification of The Model

2.4.1 p and q

Here we suppose that the sequence (S_n) satisfies our model, i.e. the fitness of the model to a given market is perfect. It is realistic to suppose that

$$0 < r < 1$$

We have

$$\mathbb{E}(U_n^1) = pa + (1-p)b = r \tag{2.4.1}$$

and

$$\mathbb{E}(U_n^2) = qc + (1-q)d = r \tag{2.4.2}$$

From this we get

$$p = \frac{b-r}{b-a} \tag{2.4.3}$$

$$q = \frac{d-r}{d-c} \tag{2.4.4}$$

As $0 < p < 1$, we must have $a < r < b$ and similarly, $c < r < d$, i.e.

$$max(a,c) < r < min(b,d) \tag{2.4.5}$$

Now, we are able to simulate artificial series for asset prices by giving initial values for r, a, b, c, d, ε and δ. The question is whether one can estimate the parameters just observing the price series. To answer this question, we begin with the estimation of ε and δ

2.4.2 Estimation of ε and δ

Let us express Equation D.0.7 in terms of discounted prices,

$$\begin{pmatrix} (1+r)^n \tilde{S}_n^1 \\ (1+r)^n \tilde{S}_n^2 \end{pmatrix} = \begin{pmatrix} 1+U_n^1 & \varepsilon(U_n^2 - r) \\ \delta(U_n^1 - r) & 1+U_n^2 \end{pmatrix} \begin{pmatrix} (1+r)^{n-1} \tilde{S}_{n-1}^1 \\ (1+r)^{n-1} \tilde{S}_{n-1}^2 \end{pmatrix} \tag{2.4.6}$$

Then

$$\begin{pmatrix} (1+r)\tilde{S}_n^1 \\ (1+r)\tilde{S}_n^2 \end{pmatrix} - \begin{pmatrix} \tilde{S}_{n-1}^1 \\ \tilde{S}_{n-1}^2 \end{pmatrix} = \begin{pmatrix} U_n^1 & \varepsilon(U_n^2 - r) \\ \delta(U_n^1 - r) & U_n^2 \end{pmatrix} \begin{pmatrix} \tilde{S}_{n-1}^1 \\ \tilde{S}_{n-1}^2 \end{pmatrix} \tag{2.4.7}$$

Then

$$\begin{pmatrix} \Delta\tilde{S}_n^1 + r\tilde{S}_n^1 \\ \Delta\tilde{S}_n^2 + r\tilde{S}_n^2 \end{pmatrix} = \begin{pmatrix} U_n^1 & \varepsilon(U_n^2 - r) \\ \delta(U_n^1 - r) & U_n^2 \end{pmatrix} \begin{pmatrix} \tilde{S}_{n-1}^1 \\ \tilde{S}_{n-1}^2 \end{pmatrix} \tag{2.4.8}$$

where $\Delta\tilde{S}_n^i = \tilde{S}_n^i - \tilde{S}_{n-1}^i$

Taking U_n^1 and U_n^2 out,

$$\begin{pmatrix} \Delta\tilde{S}_n^1 + r\tilde{S}_n^1 \\ \Delta\tilde{S}_n^2 + r\tilde{S}_n^2 \end{pmatrix} = \begin{pmatrix} \tilde{S}_{n-1}^2 & \varepsilon\tilde{S}_{n-1}^2 \\ \delta\tilde{S}_{n-1}^1 & \tilde{S}_{n-1}^1 \end{pmatrix} \begin{pmatrix} U_n^1 \\ U_n^2 \end{pmatrix} - \begin{pmatrix} r\varepsilon\tilde{S}_{n-1}^2 \\ r\delta\tilde{S}_{n-1}^1 \end{pmatrix} \tag{2.4.9}$$

$$\begin{pmatrix} \tilde{S}_{n-1}^2 & \varepsilon\tilde{S}_{n-1}^2 \\ \delta\tilde{S}_{n-1}^1 & \tilde{S}_{n-1}^1 \end{pmatrix} \begin{pmatrix} U_n^1 \\ U_n^2 \end{pmatrix} = \begin{pmatrix} \Delta\tilde{S}_n^1 + r\tilde{S}_n^1 + r\varepsilon\tilde{S}_{n-1}^2 \\ \Delta\tilde{S}_n^2 + r\tilde{S}_n^2 + r\delta\tilde{S}_{n-1}^1 \end{pmatrix} \tag{2.4.10}$$

Solving for U_n

$$\begin{pmatrix} U_n^1 \\ U_n^2 \end{pmatrix} = \frac{1}{(1-\varepsilon\delta)\tilde{S}_{n-1}^1\tilde{S}_{n-1}^2} \begin{pmatrix} \tilde{S}_{n-1}^2 & -\varepsilon\tilde{S}_{n-1}^2 \\ -\delta\tilde{S}_{n-1}^1 & \tilde{S}_{n-1}^1 \end{pmatrix} \begin{pmatrix} \Delta\tilde{S}_n^1 + r\tilde{S}_n^1 + r\varepsilon\tilde{S}_{n-1}^2 \\ \Delta\tilde{S}_n^2 + r\tilde{S}_n^2 + r\delta\tilde{S}_{n-1}^1 \end{pmatrix} \tag{2.4.11}$$

Remark 2.4.1. The market is such that $\tilde{S}_n^i > 0$

This gives

$$U_n^1 = \frac{1}{(1-\varepsilon\delta)\tilde{S}_{n-1}^1}\left(\ \Delta\tilde{S}_n^1 + r\tilde{S}_n^1 + r\varepsilon\tilde{S}_{n-1}^2 - \varepsilon(\Delta\tilde{S}_n^2 + r\tilde{S}_n^2 + r\delta\tilde{S}_{n-1}^1)\ \right) \tag{2.4.12}$$

$$U_n^2 = \frac{1}{(1-\varepsilon\delta)\tilde{S}_{n-1}^2}\left(\ \Delta\tilde{S}_n^2 + r\tilde{S}_n^2 + r\delta\tilde{S}_{n-1}^1 - \delta(\Delta\tilde{S}_n^1 + r\tilde{S}_n^1 + r\varepsilon\tilde{S}_{n-1}^2)\ \right) \tag{2.4.13}$$

Simplifying and centering the equations 2.4.12 and 2.4.13 around r give the following representations,

$$U_n^1 - r = \frac{1+r}{(1-\varepsilon\delta)}\left(\ \frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^1} - \varepsilon\frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^1}\ \right) \tag{2.4.14}$$

$$U_n^2 - r = \frac{1+r}{(1-\varepsilon\delta)}\left(\ \frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^2} - \delta\frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^2}\ \right) \tag{2.4.15}$$

Since (U_n^i) is an iid sequence such that $E(U_n^i) = r$, by taking the time averages both sides of equations (2.4.14) and (2.4.15), we get the following two equations,

$$0 = avg\left(\ \frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^1}\ \right) - \hat{\varepsilon}avg\left(\ \frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^1}\ \right) \tag{2.4.16}$$

$$0 = avg\left(\ \frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^2}\ \right) - \hat{\delta}avg\left(\ \frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^2}\ \right) \tag{2.4.17}$$

where avg refers to the time average and $\hat{\varepsilon}$,$\hat{\delta}$ indicate the estimations for ε and δ. Solving above equations for ε and δ respectively, we finally reach the expressions for the moment estimators for ε and δ.

$$\hat{\varepsilon} = \frac{avg(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^1_{n-1}})}{avg(\frac{\Delta \tilde{S}^2_n}{\tilde{S}^1_{n-1}})} \tag{2.4.18}$$

$$\hat{\delta} = \frac{avg(\frac{\Delta \tilde{S}^2_n}{\tilde{S}^2_{n-1}})}{avg(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^2_{n-1}})} \tag{2.4.19}$$

Remark 2.4.2. Formulas (2.4.18) and (2.4.19) are only valid if the time averages are different from 0. Various simulations show that this is the case at least for these simulations. In these simulated cases we do not observe the stationarity of $(\tilde{S}_n)$. If it were the case, the time averages would be equal to 0. We indeed would have

$$avg\left(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^1_{n-1}} \right) = \mathbb{E}\left(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^1_{n-1}} \right) \tag{2.4.20}$$

$$\mathbb{E}\left(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^1_{n-1}} \right) = \mathbb{E}\left(\mathbb{E}(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^1_{n-1}} | \mathcal{F}_{n-1}) \right) \tag{2.4.21}$$

$$\mathbb{E}\left(\mathbb{E}(\frac{\Delta \tilde{S}^1_n}{\tilde{S}^1_{n-1}} | \mathcal{F}_{n-1}) \right) = \mathbb{E}\left(\frac{1}{\tilde{S}^1_{n-1}} \mathbb{E}(\Delta \tilde{S}^1_n | \mathcal{F}_{n-1}) \right) = 0 \tag{2.4.22}$$

This expression is 0 due to the martingale property of $\tilde{S}^1_n$. If by any chance, the values taken by the parameters are such that (S_n) is stationary, then we have another method for the identification of ε and δ.

2.4.3 Occurrences of ε and δ

We know that the values of (U_n^1) have positive probabilities. Therefore, U_n^1 takes the value a infinitely many times. In this case, we can write equation (2.4.14) for two different time indices m and n for which $U_n^1 = a$. Then, we can write the following two equations:

$$a - r = \frac{1+r}{(1-\varepsilon\delta)} \left(\frac{\Delta \tilde{S}_n^1}{\tilde{S}_{n-1}^1} - \varepsilon \frac{\Delta \tilde{S}_n^2}{\tilde{S}_{n-1}^1} \right) \tag{2.4.23}$$

$$a - r = \frac{1+r}{(1-\varepsilon\delta)} \left(\frac{\Delta \tilde{S}_m^1}{\tilde{S}_{m-1}^1} - \varepsilon \frac{\Delta \tilde{S}_m^2}{\tilde{S}_{m-1}^1} \right) \tag{2.4.24}$$

Equalizing the two equations and solving for ε, we get

$$\varepsilon_{m,n} = \frac{\frac{\tilde{S}_m^1}{\tilde{S}_{m-1}^1} - \frac{\tilde{S}_n^1}{\tilde{S}_{n-1}^1}}{\frac{\Delta \tilde{S}_m^2}{\tilde{S}_{m-1}^1} - \frac{\Delta \tilde{S}_n^2}{\tilde{S}_{n-1}^1}} \tag{2.4.25}$$

$\varepsilon_{m,n}$ is a random variable coinciding with ε if $U_n^1 = U_m^1 = a$ for $m \neq n$. Suppose now that we plot the ratio (2.4.25) for various (m, n), m≠n. We should observe that the true value of ε appears often. On the other hand, when the pairs (m, n) do not correspond to the true value of ε, then the ratio should present a nonconstant path. Similarly for δ, we have

$$\delta_{k,j} = \frac{\frac{\tilde{S}_k^1}{\tilde{S}_{k-1}^1} - \frac{\tilde{S}_j^1}{\tilde{S}_{j-1}^1}}{\frac{\Delta \tilde{S}_k^2}{\tilde{S}_{k-1}^1} - \frac{\Delta \tilde{S}_j^2}{\tilde{S}_{j-1}^1}} \tag{2.4.26}$$

for some (k, j), $k \neq j$, and $\delta_{k,j}$ can be observed by the values taken by the right hand side of the equation (2.4.26).

2.4.4 The estimators of the Values of the Generating Sequences

Recall the equations (2.4.14) and (2.4.15). Having estimated ε and δ, we can construct the series (U_n) by means of the asset prices. All the parameters at the right hand sides are known to us. All we have to do is to construct the time series U_n^i, $i = 1, 2$ and compute the relative frequencies of the two states for each series in order to estimate a, b, c, d. We also impose a method to estimate the amplitudes. We know that second central moment for innovations are

$$\sigma_1^2 = p(a - r)^2 + (1 - p)(b - r)^2 \tag{2.4.27}$$

$$\sigma_2^2 = q(c - r)^2 + (1 - q)(d - r)^2 \tag{2.4.28}$$

Inserting right hand sides of the equations (2.4.3) and 2.4.4 for p and q respectively

$$\sigma_1^2 = \frac{b - r}{b - a}(a - r)^2 + \frac{r - a}{b - a}(b - r)^2 \tag{2.4.29}$$

$$\sigma_2^2 = \frac{d - r}{d - c}(c - r)^2 + \frac{r - a}{d - c}(d - r)^2 \tag{2.4.30}$$

They reduce to

$$\sigma_1^2 = (b - r)(r - a) \tag{2.4.31}$$

$$\sigma_2^2 = (d - r)(r - c) \tag{2.4.32}$$

Solving (2.4.3) and (2.4.31) for a and b, and (2.4.4) and (2.4.32) for c and d, we get the expressions for the estimators

$$\hat{a} = r - \sigma_1(\frac{1-p}{p})^{1/2}$$

$$\hat{b} = r + \sigma_1(\frac{p}{1-p})^{1/2}$$

$$\hat{c} = r - \sigma_2(\frac{1-q}{q})^{1/2}$$

$$\hat{d} = r + \sigma_2(\frac{q}{1-q})^{1/2} \tag{2.4.33}$$

2.5 Simulation of The Model

We simulated the model giving initial values for r, a, b, c, d, ε, δ and prices at time $n = 0$. Interest rate r could anytime be obtained from the market. As a result of simulation, we ended up with having price series for two assets with the length of n. Then, we solve the inverse problem by conducting formulas first in equations (2.4.18) and (2.4.19), and then in (2.4.14) and (2.4.15). Then, we checked if (2.4.33) holds. After sufficient iterations, we obtained the initial values given at the very beginning of the simulation, indicating that our model is statistically identifiable.

We also simulated the ratios for ε and δ in equations (2.4.25) and (2.4.26). Results for ε can be seen from the table 3.1,

As seen from the table, values often pass through the true value of ε. Moreover, other numbers do not follow any trend and wander around, as

Table 2.1: The Occurrences: $\hat{\varepsilon}_{m,n}$

$m \downarrow \backslash \overrightarrow{n}$												
	na											
	8.66	na										
	1.24	ε	na									
	ε	0.81	0.61	na								
	ε	1.07	0.70	ε	na							
	0.96	ε	ε	0.55	0.65	na						
	ε	0.46	0.41	ε	ε	0.40	na					
	ε	-0.20	-0.42	ε	ε	-0.60	ε	na				
	-0.1	ε	ε	-104.2	-1.66	ε	0.65	0	na			
	-0.4	ε	ε	-0.65	ε	ε	1.10	0.05	ε	na		
	0.45	ε	ε	0.56	0.40	ε	0.55	0.84	ε	ε	na	
	0.54	ε	ε	0.72	0.52	ε	0.29	0.41	ε	ε	ε	na

na = not a number (i.e. $\frac{0}{0}$)

expected.

Since the model assumes that the the generating sequences are independent iid sequences, high frequency data is more appropriate. Indeed, once the information starts flowing, it establishes certain correlation for the generating sequences .

2.6 Properties of the Estimators

In order to find out whether the moments estimators are unbiased, we make use of the sample mean and the sample standard deviation.

$$\widehat{E[\Theta_{i,t}]} = \bar{\hat{\Theta}}_{i,t} = \frac{\sum \hat{\Theta}_{i,t}}{T}$$

$$E[\hat{\Theta}_{i,t} - \widehat{E[\Theta_{i,t}]}] = S_{\hat{\Theta}_{i,t}} = \frac{\sum(\hat{\Theta}_{i,t} - \bar{\hat{\Theta}}_{i,t})}{T}$$

where $\hat{\Theta}_i = (\hat{\varepsilon}, \hat{\delta}, \hat{a}, \hat{b}, \hat{c}, \hat{d})$, and t is the t^{th} simulation with t=1,...,T. For each trajectory t, we find a value for the estimators. Then we can calculate the sample mean and the sample standard deviation by the above formulae. Moreover, we calculate the variation of the coefficient

$$VC = \frac{S_{\hat{\Theta}_{i,t}}}{\bar{\hat{\Theta}}_{i,t}}$$

to see how volatile the estimators are. In order to see the dispersion from the initial values, we treat the initial values as the corresponding means while we calculate the standard deviations and the VCs and we denote it by VC*. Having run 1000 simulations, we found the following results;

$$|\Theta_{\hat{\varepsilon},0} - \bar{\hat{\Theta}}_{\hat{\varepsilon}}| = .0007, VC*_{\hat{\varepsilon}} = .0034$$

$$|\Theta_{\hat{\delta},0} - \bar{\hat{\Theta}}_{\hat{\delta}}| = .0043, VC*_{\hat{\delta}} = .0480$$

$$|\Theta_{\hat{a},0} - \bar{\hat{\Theta}}_{\hat{a}}| = .3051, VC*_{\hat{a}} = .6418$$

$$|\Theta_{\hat{b},0} - \bar{\hat{\Theta}}_{\hat{b}}| = .3241, VC*_{\hat{b}} = .6203$$

$$|\Theta_{\hat{c},0} - \bar{\hat{\Theta}}_{\hat{c}}| = .3414, VC*_{\hat{c}} = .8703$$

$$|\Theta_{\hat{d},0} - \bar{\hat{\Theta}}_{\hat{d}}| = .2703, VC*_{\hat{d}} = .6150$$

These results show that $\hat{\varepsilon}$ and $\hat{\delta}$ are unbiased estimators. On the other hand, $\hat{a}$,$\hat{b}$,$\hat{c}$ and $\hat{d}$ are unlikely unbiased. We calibrate the model satisfying the minimum squared error between the theoretical and realized asset prices with respect to the magnitudes.

CHAPTER 3

MARKOV CHAIN MODEL

3.1 Introduction

In this chapter, we study the model under Markov chain hypothesis. Suppose that $[U_n^1 \; U_n^2]'$ constitute a Markov chain, then

$$\mathbb{P}(U_n|U_1, U_2, ..., U_{n-1}) = \mathbb{P}(U_n|U_{n-1}) \, a.s. \tag{3.1.1}$$

by the Markov property. Following the same reasoning as in the independent case above, we obtain

$$\mathbb{E}[U_n^1|U_{n-1}] = \mathbb{E}[U_n^2|U_{n-1}] = r(U_{n-1}) \, a.s. \tag{3.1.2}$$

This shows that $r(U_{n-1})$ is only a function of U_{n-1}.

3.2 Probability Law Under Markov Chain Hypothesis

Adapting the equations (2.4.3) and (2.4.4) to the new conditions, we get the following two expressions for probabilities of the generating sequences.

$$\mathbb{E}[U_n^1|U_{n-1}] = p_{(a),(U_{n-1})}a + (1 - p_{(a),(U_{n-1})})b = r(U_{n-1}) \tag{3.2.1}$$

$$\mathbb{E}[U_n^2|U_{n-1}] = p_{(c),(U_{n-1})}c + (1 - p_{(c),(U_{n-1})})d = r(U_{n-1}) \tag{3.2.2}$$

where $p_{(a),(U_{n-1})} = \mathbb{P}(U_n^1 = a|U_{n-1})$ and $p_{(c),(U_{n-1})} = \mathbb{P}(U_n^2 = c|U_{n-1})$ and $p_{(a),(U_{n-1})} = p_{(a,c),(U_{n-1})} + p_{(a,d),(U_{n-1})}$, so on.
Solving the above equations for p and q respectively

$$p_{(a),(U_{n-1})} = \frac{b - r(U_{n-1})}{b - a} \tag{3.2.3}$$

$$q_{(c),(U_{n-1})} = \frac{d - r(U_{n-1})}{d - c} \tag{3.2.4}$$

where $p_{(a),(U_{n-1})}$ and $q_{(c),(U_{n-1})}$ are only the functions of U_{n-1}. According to the values of U_{n-1}, $r(U_{n-1})$ may take 4 different values.

Remark 3.2.1. By equations (3.2.3) and (3.2.4), we obtained the conditional marginal probabilities of U_n^1 and U_n^2. This does not enable us to obtain the joint conditional probability of U_n^1 and U_n^2. In order to carry out the simulation and our identification procedures we will assume that U_n^1 and U_n^2 are conditionally independent given U_{n-1}. We insist on the fact that the simulation and the identification procedures would be the same in the case where the transitions are not conditionally independent. Equations (3.2.1) and (3.2.2) are not sufficient enough to determine the joint conditional probability of U_n given U_{n-1}. Therefore, there are more than one transition probability for which the martingale condition (3.1.2) is satisfied. Therefore, (if no other condition on U_n is imposed in our model) the corresponding market is incomplete. This situation should be taken into consideration for the problem of portfolio optimization (see appendix C.

3.3 Estimation of ε and δ Under Markov Chain Hypothesis

We make use of the equations 2.4.14 and 2.4.15 to insert the Markov property into our model

$$U_n^1 - r(U_{n-1}) = \frac{1 + r(U_{n-1})}{(1 - \varepsilon\delta)} \left(\frac{\Delta \tilde{S}_n^1}{\tilde{S}_{n-1}^1} - \varepsilon \frac{\Delta \tilde{S}_n^2}{\tilde{S}_{n-1}^1} \right) \tag{3.3.1}$$

$$U_n^2 - r(U_{n-1}) = \frac{1 + r(U_{n-1})}{(1 - \varepsilon\delta)} \left(\frac{\Delta \tilde{S}_n^2}{\tilde{S}_{n-1}^2} - \delta \frac{\Delta \tilde{S}_n^1}{\tilde{S}_{n-1}^2} \right) \tag{3.3.2}$$

It is very rational to suppose that the Markov chain in our model is irreducible. Then, $\{U_n\}$ is ergodic. Thus, one can show that

$$\begin{aligned} \mathbb{E}\left(\frac{U_n^i - r(U_{n-1})}{1 + r(U_{n-1})} \right) &= \mathbb{E}\left(\mathbb{E}\left(\frac{U_n^i - r(U_{n-1})}{1 + r(U_{n-1})} | \mathcal{F}_{n-1}\right) \right) \\ &= \mathbb{E}\left(\frac{1}{1 + r(U_{n-1})} \mathbb{E}[(U_n^1 - r(U_{n-1})) | \mathcal{F}_{n-1}] \right) \\ &= 0 \end{aligned}$$

Then taking the time averages of both sides in equations (3.3.1) and (3.3.2) and solving for ε and δ respectively, we reach the same estimators $\hat{\varepsilon}$ and $\hat{\delta}$ shown at equations (2.4.18) and (2.4.19) . Replacing these estimators in equations (3.3.1) and (3.3.2), we can observe the histogram of U_n. Histogram of (U_n) will enable us to observe the values of a, b, c, and d. Then, one can estimate the following transition matrix where $\pi_s = \mathbb{P}(U_n^1 = u_n^1 | U_{n-1} =$

Table 3.1: Probability Transition Matrix

	a,c	a,d	b,c	b,d
a,c	π_1	π_2	π_3	π_4
a,d	π_5	π_6	π_7	π_8
b,c	π_9	π_{10}	π_{11}	π_{12}
b,d	π_{13}	π_{14}	π_{15}	π_{16}

$u_{n-1}).\mathbb{P}(U_n^2 = u_n^2 | U_{n-1} = u_{n-1})$, s=1,....16.

3.3.1 a, b, c, d Under Markov Chain Hypothesis

In applications, one may run into a case where histogram is not clear enough to distinguish the amplitudes. Then, in this situation we again need estimators for the amplitudes of U_n^1 and U_n^2. Estimators we found under the assumption of independency (2.4.33), are not good as we have more than one interest rate. However, one can write the conditional second central moments,

$$\sigma_{n,1}^2 = \frac{b - r(U_{n-1})}{b-a}(a - r(U_{n-1}))^2 + \frac{r(U_{n-1}) - a}{b-a}(b - r(U_{n-1}))^2 \qquad (3.3.3)$$

$$\sigma^1_{n,2} = \frac{d - r(U_{n-1})}{d - c}(c - r(U_{n-1}))^2 + \frac{r(U_{n-1}) - a}{d - c}(d - r(U_{n-1}))^2 \quad (3.3.4)$$

They reduce to

$$\sigma^2_{n,1} = (b - r(U_{n-1}))(r(U_{n-1}) - a) \quad (3.3.5)$$

$$\sigma^2_{n,2} = (d - r(U_{n-1}))(r(U_{n-1}) - c) \quad (3.3.6)$$

We write the equations (3.2.3),(3.2.4), (3.3.5),(3.3.6) in the mean form to overcome the varying interest rate problem for the estimators,

$$\bar{p}_n = \frac{b - \bar{r}(U_{n-1})}{b - a} \quad (3.3.7)$$

$$\bar{q}_n = \frac{d - \bar{r}(U_{n-1})}{d - c} \quad (3.3.8)$$

$$\overline{\sigma^2}_{n,1} = b\bar{r}(U_{n-1}) - ba - \overline{r(U_{n-1})^2} + a\bar{r}(U_{n-1}) \quad (3.3.9)$$

$$\overline{\sigma^2}_{n,2} = d\bar{r}(U_{n-1}) - dc - \overline{r(U_{n-1})^2} + c\bar{r}(U_{n-1}) \quad (3.3.10)$$

Solving (3.3.7) and (3.3.9) for a and b, and (3.3.8) and (3.3.10) for c and d, we can get the expressions for the estimators

$$\hat{a} = \bar{r}(U_{n-1})\bar{P}_{(a),(U_{n-1})} \mp \frac{[(\overline{\sigma^2}_{n,1} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)P_{(a),(\bar{U}_{n-1})}\bar{P}_b]^{1/2}}{P_{(a),(\bar{U}_{n-1})}}$$

$$\hat{b} = \bar{r}(U_{n-1})\bar{P}_{(a),(U_{n-1})} - \bar{r}(U_{n-1}) \mp \frac{[(\overline{\sigma^2}_{n,1} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)P_{(a),(\bar{U}_{n-1})}\bar{P}_b]^{1/2}}{\bar{P}_b}$$

$$\hat{c} = \bar{r}(U_{n-1})\bar{P}_c \mp \frac{[(\overline{\sigma^2}_{n,2} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)\bar{P}_c\bar{P}_d]^{1/2}}{\bar{P}_c}$$

$$\hat{d} = \bar{r}(U_{n-1})\bar{P}_c - \bar{r}(U_{n-1}) \mp \frac{[(\overline{\sigma^2}_{n,2} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)\bar{P}_c\bar{P}_d]^{1/2}}{\bar{P}_d} \quad (3.3.11)$$

where $\bar{P}_a$,$\bar{P}_b$,$\bar{P}_c$,$\bar{P}_d$ are the mean of corresponding probabilities of amplitudes, $\overline{\sigma^2}_{n,1}$, $\overline{\sigma^2}_{n,2}$ are the mean of conditional variances of U_n^1 and U_n^2 respectively, and finally $\bar{r}(U_{n-1})$ is the mean of $r(U_{n-1})$. All the estimators have two roots. We select the one that satisfies the following conditions

$$a < \bar{r}(U_{n-1}) < b$$

$$c < \bar{r}(U_{n-1}) < d$$

Notice that we do not have to insert the mean of r, while finding the histogram of U_n, but the corresponding r series as we have them available.

3.4 Estimation of Transition Probabilities π

Having obtained the series of U_n, we can calculate the frequency for each possible transition cases by simply counting the number of occurrences from u_n series. For instance in order to find the frequency for $P(a, d|a, c)$ first, we count the occurrences of (a, c) for denominator. Then, we count the number of (a, d) couples every time they occur after (a, c) couple for nominator. Then this ratio will give us the empirical frequency.

3.5 Simulation of the Model Under Markov Chain Hypothesis

We simulate the model under the conditionally independent transition hypothesis case (See Remark 3.2.1). We first create some given transition probability matrix. We assign the previous given values to the model parameters except for the interest rate r. This is because r is no longer constant and it takes on (preassigned) values in terms of U_{n-1}. Then, we follow the same testing procedure in which we first generate artificial series of prices based on mentioned algorithm, and check for the values of the parameters. Again, following sufficient iterations we find what we expected to see confirming the model's identifiability under Markov chain hypothesis. The following results are the results about the properties of the estimators mentioned in 2.6

$$|\Theta_{\hat{\varepsilon},0} - \bar{\hat{\Theta}}_{\hat{\varepsilon}}| = .0026, VC*_{\hat{\varepsilon}} = .0260$$

$$|\Theta_{\hat{\delta},0} - \bar{\hat{\Theta}}_{\hat{\delta}}| = .0021, VC*_{\hat{\delta}} = .0071$$

$$|\Theta_{\hat{a},0} - \bar{\hat{\Theta}}_{\hat{a}}| = .1540, VC*_{\hat{a}} = .5116$$

$$|\Theta_{\hat{b},0} - \bar{\hat{\Theta}}_{\hat{b}}| = .1182, VC*_{\hat{b}} = .2956$$

$$|\Theta_{\hat{c},0} - \bar{\hat{\Theta}}_{\hat{c}}| = .1463, VC*_{\hat{c}} = .3751$$

$$|\Theta_{\hat{d},0} - \bar{\hat{\Theta}}_{\hat{d}}| = .0753, VC*_{\hat{d}} = .2150$$

These results show that $\hat{\varepsilon}$ and $\hat{\delta}$ are unbiased estimators. On the other hand, $\hat{a}$,$\hat{b}$,$\hat{c}$ and $\hat{d}$ are unlikely unbiased. In Chapter 4, we will show a way to get better estimates for $\hat{a}$,$\hat{b}$,$\hat{c}$ and $\hat{d}$.

CHAPTER 4

APPLICATIONS I

> I believe that the existence of the classical "path" can be pregnantly formulated as follows: The "path" comes into existence only when we observe it. Werner Heisenberg

In this chapter, we will show the simulations for the independent case and the conditionally independent case, optimal portfolio selection under both complete and incomplete market and discuss the adaption of the models to the real data.

4.1 Simulations

4.1.1 Independent Case

We recall that the independent case is the one in which U_n^1 and U_n^2 are mutually independent i.i.d sequences. We give the following initial values to parameters $r = .05$, $a = -0.6$, $b = 0.45$, $c = -0.21$, $d = 0.31$, $\varepsilon = 0.21$, $\delta = 0.9$, $S_{n-1}^1 = 1000$ and $S_{n-1}^2 = 1000$. After 10000 iterations, we obtained the results to check whether our model under independence assumption holds. We generate the price series of two assets. Having these artificially generated series as the observations related to two assets, we calculate the parameters by the estimators, first $\hat{\varepsilon}$ and $\hat{\delta}$ by (2.4.18) and (2.4.19) and then $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ by (2.4.33). Table 6.1 exhibits the results of the first fifteen iterations for innovations and estimated parameters using the estimation formulae resulting from 10000 iterations.

Table 4.1: Simulation Results

U_n^1	U_n^2	Estimates
.431	.300	$\hat{\varepsilon}$=.2058
.431	.300	$\hat{\delta}$=.0833
.429	-.194	$\hat{a}$=-.62
-.569	-.206	$\hat{b}$=.486
.431	.300	$\hat{c}$=-.234
.429	-.195	$\hat{d}$=.33
-.568	.288	
.429	-.195	
.429	-.194	
-.568	-.206	
-.569	-.202	
.432	.299	
-.566	.294	
.434	.298	
.433	.298	

As seen from Table 6.1, results are very close to the initial values, indicating that the model holds. This proves the efficiency of the estimation procedure.

4.1.2 Conditionally Independent Case

This is the case where $\mathbb{P}(U_n^1 = u_n^1, U_n^2 = u_n^2 | U_{n-1}) = \mathbb{P}(U_n^1 = u_n^1 | U_{n-1} = u_{n-1}).\mathbb{P}(U_n^2 = u_n^2 | U_{n-1} = u_{n-1})$. In this section, we will analyze the simulation and its results for the estimators in the case where innovations constitute a Markov chain. We know that interest rate r_n is then the function of U_{n-1}. Since there are 4 different possible values that U_{n-1} can take, we determine 4 different initial $r_n(U_{n-1})$ values to begin the simulation. These values are $r(ac) = .01$, $r(ad) = .12$, $r(bc) = .076$, $r(bd) = .14$, where $r(ac)$ refers to the value of interest rate r_n given that $U_{n-1} = (a, c)$ and so on. The other parameters take on these initial values: $\varepsilon = .1$, $\delta = .29$, $a = -.301$, $b = .4$, $c = -.39$, $d = .35$, $S_{n-1}^1 = 10000$ and $S_{n-1}^2 = 10000$. Having these values we can tabulate our transition probability matrix π. For example, from the equations (3.2.3) and (3.2.4) we can calculate $(Pa|ad)$=$P(U_n^1 = a | U_{n-1}^1 = a, U_{n-1}^2 = d) = .39942$ and $(Pc|ad)$=$P(U_n^2 = c | U_{n-1}^1 = a, U_{n-1}^2 = d) = .31081$, given that $(U_{n-1}^1 = a, U_{n-1}^2 = d)$. Since we assume that U_n^1 and U_n^2 are conditionally independent, $P(ac|ad) = (Pa|ad)(Pc|ad) = .1241)$. The columns of

Table 4.2: Probability Transition Matrix

$\tilde{\pi}$	ac	ad	bc	bd
ac	.2556	.3007	.2038	.2398
ad	.1241	.2753	.1867	.4139
bc	.1781	.2926	.2003	.3290
bd	.1053	.2656	.1785	.4506

the Table 6.2 indicate the possible state at time $n-1$, while the rows indicate the possible state at time n. Then, for instance, the value .2556 is basically the transition probability of migrating from state $U_{n-1} = a, c$ at time $n-1$ to state $U_n = a, c$ at time n, $P(ac|ac)$.

Now we are ready to run our model explained in chapter 4 to generate the price series. Having created the artificial price series, we then apply our estimators.

After having 10000 iterations we reached the results shown on the table 6.3. Table 6.3 shows the results again for any consecutive 15 iterations of 10000 iterations for innovation series and estimators of 10000 iterations.

Table 4.3: Simulation Results: The Conditionally Independent Case

U_n^1	U_n^2	Estimates
-.295	-.374	$\hat{\varepsilon}$=.1032
-.295	.293	$\hat{\delta}$=.2914
-.296	.294	$\hat{a}$ = -.3386
.429	.296	$\hat{b}$ =.4266
.436	.303	$\hat{c}$=-.3659
.436	.303	$\hat{d}$=.3471
-.291	-.369	
-.296	-.372	
.431	-.372	
.432	.292	
.431	-.370	
.437	-.368	
.437	.302	
.437	.307	
-.294	.306	

Results again are in the favor of the model. An increase in the number of the states will certainly increase the accuracy of the model. There is always a trade off between simplicity and accuracy.

One must also check with the transition probability matrix after the simulation to analyze the convergence of the model. Table 6.4 is the probability transition matrix of innovations generated by the simulation.

Table 4.4: Probability Transition Matrix: A Conditionally Independent Simulation

π	ac	ad	bc	bd
ac	.2621	.2172	.2359	.2849
ad	.1561	.2885	.1986	.3569
bc	.1677	.2642	.1929	.3751
bd	.0922	.2927	.1565	.4586

We can calculate the stationary probabilities of the chain by

$$\pi_j = \sum_{i \in \xi} \pi_i p_{ij}$$

Then it can be shown that transition probabilities given at pre-simulation, and found at post-simulation converge to the stationary probability distribution, as they are supposed to. The results shown in Table 6.5 imply that pre-simulation and post-simulation conditional probabilities converge very similar stationary probability laws.

Table 4.5: Stationary Probabilities: Before and After

π	π_{ac}	π_{ad}	π_{bc}	π_{bd}
Pre	.1373	.2785	.1886	.3866
Post	.1492	.2750	.1867	.3891

4.1.3 Reality Application

We collected daily data from the Central Bank of Turkey on the Euro and the US Dollar from $02 - January - 2002$ to $12 - August - 2003$. We have selected the overnight interest rate as a daily discount factor rate. Having discounted the series, we proceed to obtain our estimators for ε, δ, a transition probability matrix and amplitudes of the innovations.

Table 6.6 shows the results for the first 114 days to analyze the series first in sample. Again the first two columns have some fifteen consecutive occurrences of U_n^1 and U_n^2. Then we discuss the out of sample results.

As seen from the table 6.6, innovations U_n^1 and U_n^2 are exposed to more states than only two states.

We know that $a < r_n < b$ and $c < r_n < d$. We also know that conditional expectations of U_n^1 and U_n^2 are r_n. So, for instance, every time we have a U_n^1 value that is smaller than the interest rate r_n, we count that value as an occurrence of a. That is to say,

$$u_n^1 < r_n \Rightarrow u_n^1 = a$$

Table 4.6: Empirical results

U_n^1	U_n^2	Estimators
.0776	-.1249	$\hat{\varepsilon}$=.6910
.1784	-.2822	$\hat{\delta}$=1.4860
-.0163	.0444	$\hat{a}$ = -.1871
.1644	-.2496	$\hat{b}$ =.1753
-.2187	.3680	$\hat{c}$=-.2750
.3077	.4963	$\hat{d}$=.3000
.4660	-.7410	
-.1023	.1557	
-.0314	.0529	
.1030	-.1441	
-.1923	.3173	
-.2003	-3346	
-.2803	.4752	
-.2587	.4106	
.1043	-.1562	

$$u_n^1 > r_n \Rightarrow u_n^1 = b$$

$$u_n^2 < r_n \Rightarrow u_n^2 = c$$

$$u_n^2 > r_n \Rightarrow u_n^2 = d$$

What we really do is to fit the realized U_n values to a binomial distribution. The transition probability matrix estimation is presented at table 6.7.

For the out of sample performance, we use the estimates we obtained for a, b, c, d, ε, δ and the transition matrix above. We simulate the model 20000 times for the next 7 days and take the average of the 20000 outcomes. That is to say, we created 20000 trajectories as scenarios to run and we find an

Table 4.7: Empirical Probability Transition Matrix

π	ac	ad	bc	bd
ac	.091	.11	.72	.079
ad	.0003	.4845	.5052	.01
bc	.009	.4615	.5288	.0006
bd	.133	.67	.096	.101

average value for each day of the next 7 days. Out of sample results for 7-day prediction are presented in the Table 6.8. Table 6.8 has the compar-

Table 4.8: Empirical Forecast Results

Dollar	Dollar*	Euro	Euro*	Parity	Parity*
1.6530	1.6530	1.6108	1.6108	0.9745	0.9745
1.6476	1.6525	1.6073	1.6102	0.9755	0.9744
1.6523	1.6565	1.6161	1.6162	0.9781	0.9756
1.6612	1.6666	1.6181	1.6310	0.9741	0.9786
1.6682	1.6857	1.6303	1.6590	0.9773	0.9841
1.6682	1.7185	1.6303	1.7069	0.9773	0.9933
1.6680	1.7711	1.6398	1.7838	0.9831	1.0072
1.6604	1.8523	1.6383	1.9027	0.9867	1.0272

($*$) indicates the simulated values and monetary units are divided by 1000000 Turkish Lira (TL).

ison between realized values and the simulated values for the next 7 days. These comparisons have been made in nominal prices and Euro/Dollar parity. As seen the simulated price levels tend to shoot out. There could be many reasons for that. First of all, the empirical distribution is obviously not binomial. So the estimators for amplitudes of innovations cause the model to shoot out, as they are bigger than the best fitting values[1]. We fit innovation series to various distributions to see whether they coincide with any theoretical distribution. Imposing the Kolmogorov-Smirnov test for the period the data is collected, we found that normalized series of U_n^1 and U_n^2 come from standard normal distributions. The descriptive statistics at table 6.9

Table 4.9: Decriptive Statistics

Descriptive Statistics	U_n^1	U_n^2
Skewness	-.1802	.1393
Kurtosis	3.6877	3.4393

also support that the distribution of the innovations follow a Normal distribution. We also run the same process for some different intervals of data. Although for some intervals kurtosis values is little higher than 4, we found that the distribution of the innovations could be approximated by gaussian distribution. The skewness of innovation's empirical distribution is always close to zero implying that the distributions are nearly symmetric.

Some other difficulty we face with is to model the interest rate when forecasting the model. It is very hard to find a conditional interest rate that satisfies both (3.2.3) and (3.2.4). That is again because the innovations in real data are subject to more than 2 states. In this case, the empirical transition probability law is not the as same the one that would give the convenient interest rate in terms of the model. Moreover, we assumed that

[1] It would have been the other way around, if the estimates had been smaller than the best fitting values.

U_n follows a Markov chain with one lag. That is to say, we claim that interest rate depends only on the jumps at the previous day. This assumption is also hard to hold for most of the intervals.

Another assumption that is hard to hold is the independency and conditional independency of U_n^1 and U_n^2. High frequency data such as hourly or 5 minutes data is more appropriate. Indeed, it is logical to think that once the information starts flowing, investors reflect their reaction against the assets' move. Even if the jumps are not correlated, reasons causing the jumps could be.

We also check to see whether calibrating the model for better estimates of the parameters would work it out. For this purpose, we fit the model to the data satisfying the minimization of the sum of the squared error of price series.

$$\min_{a,b,c,d} \sum [\tilde{S}_n - \hat{\tilde{S}}_n]^2 \tag{4.1.1}$$

where $\hat{\tilde{S}}_n$ is an estimated risk neutral price series. We fit our model for the past 114 days to get better estimates for the parameters a, b, c, d. For instance, for $a = -.0416$, $b = .0365$, $c = -.055$ and $d = .06$, we obtained the results in table 6.10. Table 6.10 shows that adjusting the estimators give better results than in the previous case[2]. Especially in terms of matching the end period value in parities we get much better results. This is expected due to the fact that the co-integration of two same order of linearly related

[2]We simulated the model many times for different intervals as well.

Table 4.10: Adjusted Model Results

Dollar	Dollar*	Euro	Euro*	Parity	Parity*
1.6530	1.6530	1.6108	1.6108	0.9745	0.9745
1.6476	1.6552	1.6073	1.6108	0.9755	0.9749
1.6523	1.6552	1.6161	1.6116	0.9781	0.9753
1.6612	1.6653	1.6181	1.6137	0.9741	0.9760
1.6682	1.6655	1.6303	1.6171	0.9773	0.9770
1.6682	1.6657	1.6303	1.6217	0.9773	0.9782
1.6680	1.6662	1.6398	1.6278	0.9831	0.9799
1.6604	1.6666	1.6383	1.6355	0.9867	0.9817

non-stationary series produces a stationary vector [8]. However, the forecast can still not capture the ups and downs (volatility). Simulated prices follow a smoothly increasing process, if the end value is higher than the initial. When the end value is lower, price series tend to show a smooth decrease with no ups and downs. Therefore, it is good at finding the probability of hitting a certain value or exceeding it. We will deal with this point where we study the two-color rainbow option pricing examples in chapter 5. Also notice that under this kind of stochastic environment, the process for the satisfaction of minimized mean squared error function is a process that takes a very long time to give reliable results.

Some other issue to mention related to hitting exercises is to adjust the steps of binomial tree simulations. We assumed that every step is one-day. When we search for the answer to the question of what the prices will be at the 180^{th} day from today, we basically iterate the model 180 times. However, we do not know for sure that the effect of the experience of each day are equally weighted. Boyle and Lau [6] showed that it is possible to adjust the number of the time steps to make the stopping point fall exactly on or very close to the nodes.

4.1.4 Comparison with CRR

In this subsection we will compare the results we obtained implementing our model with results that would be obtained from classical CRR model implementation. The most widely used method that applies the CRR model to the real world problems is the following.

The price of an asset S is either to increase proportionally by up with probability p or to decrease by $down$ with probability (1-p). Then, $p = Prob(S_n.up)$ and $(1-p) = Prob(S_n.down)$.

$$u = up = e^{\sigma(\Delta n)^{1/2}}$$

$$d = down = e^{-\sigma(\Delta n)^{1/2}}$$

where σ is the standard deviation on the return of the price series.

$$p = \frac{e^{r\Delta n} - d}{u - d}$$

For the period, we implemented our model in subsection 4.1.3, the parameters of CRR model take on the following values.

$u_{dollar} = 1.0092$, $d_{dollar} = .9908$, $u_{euro} = 1.011$, $d_{euro} = .9889$, $p_{dollar} =$

.5634 and $p_{euro} = .5514$

Now, we are ready to simulate the CRR model for the next 7 days in the same fashion as in subsection 4.1.3. Dollar(CRR) and Euro(CRR) are

Table 4.11: Comparison with CRR-Levels

Dollar	Dollar*	Dollar(CRR)	Euro	Euro*	Euro(CRR)
1.6530	1.6530	1.6530	1.6108	1.6108	1.6108
1.6476	1.6552	1.6547	1.6073	1.6108	1.6129
1.6523	1.6552	1.6567	1.6161	1.6116	1.6145
1.6612	1.6653	1.6587	1.6181	1.6137	1.6162
1.6682	1.6655	1.6604	1.6303	1.6171	1.6179
1.6682	1.6657	1.6625	1.6303	1.6217	1.6199
1.6680	1.6662	1.6640	1.6398	1.6278	1.6219
1.6604	1.6666	1.6660	1.6383	1.6355	1.6245

the Dollar prediction of CRR model and the Euro prediction of CRR model respectively. As seen from the table 6.11, although the Dollar forecast does well, the Euro forecast is no good because the CRR model does not count the cross effects whereas our model does. Obviously, the prices of Euro and the Dollar are very much related to each other. Even in the case where the estimators are not adjusted via calibration, comparison between two models have results in favor of the bivariate model. When we analyze the parity we see this fact in a better way.

Table 4.12: Comparison with CRR-Parity

Parity	Parity*	Parity**	Parity(CRR)
0.9745	0.9745	.9745	.9745
0.9755	0.9744	.9749	.9747
0.9781	0.9756	.9753	.9745
0.9741	0.9786	.9760	.9744
0.9773	0.9841	.9770	.9744
0.9773	0.9933	.9782	.9744
0.9831	1.0072	.9799	.9746
0.9867	1.0272	.9817	.9751

($*$) indicates the model estimation and ($**$) indicates the adjusted model estimation.

CHAPTER 5

APPLICATIONS II - OPTION PRICING

In this chapter, we study the pricing of various options with two underlying assets.

5.1 Introduction

In this chapter, we will deal with some examples of two-color rainbow options that are defined as options on two risky underlying assets that can not by some trick of reasoning be valued as if they were options on one underlying asset [19] [20]. One can find great numbers of treatments to these problems in Black-Scholes [2] environment. For spread options see [22], [24]. For the maximum of two risky assets see [22]. For Dual-Strike options see again [19] [20].

5.2 Option Pricing

First we show how the call option of one underlying asset is priced. Let us notate the strike price of the asset at some terminal date K. Then the pay-off function of the related option is ,

$$C_N = \max[0, S_N - K]^1 \tag{5.2.1}$$

where C is the call option pay off. By the no-arbitrage condition we know that the following equation must be satisfied by risk-neutral probabilities $\tilde{\mathbb{P}}$,

$$C_n = \frac{1}{1+r} E^{\tilde{\mathbb{P}}}[C_{n+1}] \tag{5.2.2}$$

Either repeating (5.2.2) backward or by an appropriate numerical solution, one can find a solution for initial value C_0. For a monte carlo algorithm, the problem would be stated as the below equation [21].

$$C_0 = \frac{1}{(1+r)^n} E^{\tilde{\mathbb{P}}}[C_N] \tag{5.2.3}$$

[1] For simplicity, we prefer using the call option design of the pay-off function for every option we deal with. For the put option version, it would be $K - S_N$ istead.

Applying the one period option pricing function (5.2.2) to our conditionally independent model, we get,

$$C_n = \frac{1}{1+r_n}[Pac.C_{n+1}(a,c)+Pad.C_{n+1}(a,d)+Pbc.C_{n+1}(b,c)+Pbd.C_{n+1}(b,d)] \tag{5.2.4}$$

where Pac, Pad, Pbc, Pbd are the conditional joint probabilities used as in $(C.2.2)$. For more than single period, we simulate the problem by implementing the adaptation of the equation (5.2.3) to the varying interest rate.

$$C_0 = \prod_{n=1}^{N} \frac{1}{(1+r_n)} E^{\tilde{\mathbb{P}}}[C_N] \tag{5.2.5}$$

Let us notate C_N^1 as the pay-off expression of a call option with the first underlying asset and C_N^2 as the pay-off expression of the option with the second underlying asset. It is also possible to calculate the probability of C_N^1 exceeding C_N^2, by simulating the model.

$$Prob(C_N^2 \leq C_N^1)$$

5.3 Rainbow Options Implementation

5.3.1 Dual-Strike Options

Dual strike options are options with two underlying assets and two regarding strike prices K_1 and K_2. Following is the regarding pay-off function dual-strike option.

$$C_N = \max[0, (\tilde{S}_N^1 - K_1), (\tilde{S}_N^2 - K_2)] \tag{5.3.1}$$

Example: Euro-Dollar Application and Results

We implement a pay-off pricing simulation with the calibrated parameters model in subsection 4.1.3. We implement the equation(5.2.5) for 10000 times and store the values. We selected the strike prices K_1 and K_2, 1660400 TL and 1638300 TL respectively. These values are the realized values taken on the 7^{th} day by the US Dollar and the Euro, respectively. Then, the realized pay-offs are 0 TL for both currencies. That is to say that we set strike prices K_1 and K_2 at the values that two currencies take at the expiration date. We seek whether our model produces 0 as the pay-off value for this option. Then we run the simulations. We store 0 if the 7^{th} day value of the simulated trajectory is smaller than strike prices and store the biggest difference when the currencies are greater than the corresponding the strike prices[2]. Then, we sum all these stored values and divide them by the number of the trajectories (10000). Our model produces 11101 TL as a pay off price, when realized the pay off 0 TL. For another experiment, we set the strike prices 50000 TL less than the realized values of the 7^{th} day, $K1 = 1660400$ and $K_2 = 1588300$. The model produces 67043 TL for this option. According to these realized values, our model is said to overprice the option.

[2] When one of the differences is 0, we store the positive amount.

5.3.2 Maximum of Two Risky Assets

These kinds of exotic options are simply a special case of Dual-Strike options where strike prices K_1 and K_2 are the same.

$$K_1 = K_2 = K$$

Pay-off function of these options are formulated as,

$$C_N = \max[0, \max(\tilde{S}_N^1, \tilde{S}_N^2) - K] \tag{5.3.2}$$

It could also be written as

$$C_N = \max[\tilde{S}_N^1, \tilde{S}_N^2, K] - K$$

Example: Euro-Dollar Application and Results

We go through the same procedure as in 5.3.1. We picked the point where the realized values of the assets almost coincide and we set the value as a strike pricing to test again whether the model produce O price. We again found that the model overprices this option as well.

5.3.3 Spread Options

Spread options are written on the difference between two prices (or rates, or indices). The mathematical expression of the pay-off function for this option is

$$C_N = \max[0, (\tilde{S}_N^1 - \tilde{S}_N^2) - K] \tag{5.3.3}$$

Example: Euro-Dollar Application and Results

For the spread option pricing performance, we set the strike price as the last day difference between two assets $K = \tilde{S}_7^1 - \tilde{S}_7^2$. In the last day, this value is $C_{realized} = 22082.5562$ TL. After 100000 iterations our model prices this spread option $C_{model} = 21170.5467$. As expected, our model works very well in terms of relating two asset prices' movements and gives very good estimates. We also analyzed the same example in terms of log prices in order to see option on parity performance. We simulated the model again on zero pricing scenario, $S_7 = log(S_7^2/S_7^1)$ and $K = log(S_7^2/S_7^1)$. The model produces the value .0000012 which is very close to "0". It turns out that the model gets to give much better results under logarithmic differences approach.

CHAPTER 6

CONCLUSIONS

The considered model is fully identifiable. Moreover, the multi-state extension of the model is valid. Another extension that could possibly improve the model's fitting to real data is to increase its memory by replacing (U_n) by a k^{th} order Markov chain, with $k > 1$. Clearly, in both cases the identification procedures would take more time, thus implying longer periods of observations. An extension with a more than 2-dimensional generating process (U_n) can also be considered. But, in this case, the coupling coefficients like ε and δ would not all be identifiable. We then would be faced with calibration problems similar to those seen in the classical Black and Scholes model.

In terms of the applications, we intended to show the instrument types that the model should be used to price. Model's explanation of the relation between two risky variables is fairly good. Notice that we do not impose any traditional correlation coefficient in the model and we let the model produce the joint move of the assets together. As of the spread option implementation, the model works better comparing with the other instruments. The model also could be used in other jointly hitting problems such as financial risk assessment.

The generating sequence with a multinomial distribution could improve the results as the distribution converges to a normal distribution. Moreover, in this case the model generates infinitely many interest rates which is similar to the real market situation. For the case where the generating sequences are normally distributed, we could have the following presentation,

$$\begin{pmatrix} U^1_{n+1} \\ U^2_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} U^1_n \\ U^2_n \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \xi^1_n \\ \xi^2_n \end{pmatrix}$$

REFERENCES

REFERENCES

[1] Bellman, R., *Dynamic Programming* Princeton Univ. Press, Princeton, New Jersey,(1957)

[2] Black, F., and Scholes, M., 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, p. 659-683.

[3] Boyle, P., 1988 (March), "A Lattice Framework for Option Pricing with Two State Variables," *Journal of Financial and Quantitative Analysis*, Vol. 23, No. 1, p. 1-12.

[4] Boyle, P., Y. K. Tse, 1990, "An Algorithm for Computing Values of Options on the maximum or Minumum of Several Assets," *Journal of Financial and Quantitative Analysis*, Vol. 25, p. 215-27.

[5] Boyle, P., J. Evnine, and S. Gibbs, 1989, "Numerical Evaluations of Multivariate Contingent Claims," *Journal of Derivatives*, 1, p. 6-14.

[6] Boyle, P., J. Evnine, and S. H. Lau, 1994, "Bumping up against the Barrier with the binomial method," *Review of Financial Studies*, Vol. 2, p. 241-250.

[7] Boyle, P., J. Evnine, and S. H. Lau, 1994, "Bumping up against the Barrier with the binomial method," *Review of Financial Studies*, Vol. 2, p. 241-250.

[8] Charamza,W, W., Deadman, D.,F.: *New Directions in Econometric Practice* Edwar-Elgar Publishing NH USA (1997)

[9] Cox, J., S. Ross, and M. Rubinstein, 1979 (October), "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, Vol. 7, p. 44-50.

[10] He, H., 1990, "Convergence from Discrete to Continuos Time Contingent Claims Prices," *Review of Financial Studies*, Vol. 3, No. 4, p. 523-546.

[11] Garman, M., 1992, "Charm School," RISK 57(11): 53 and 56

[12] Haug, E., G.:*The Complete Guide to Optiom Pricing Formulas* McGraw-Hill (1998)

[13] Hull, J., C. *Options, Futures, and Other Derivatives* Fourth Edition Prentice-Hall Internationl (2000)

[14] Körezlioğlu, H. and Hayfavi,A.: *Elements of Probability Theory* METU Basim Isligi, (2001)

[15] Lamberton, D. and Lapeyre,B.: *Stochastic Calculus Applied to Finance* Chapman and Hall / CRC, (1996)

[16] Neftci, S.: *Introduction to the Mathematics of Financial Derivarives* Second Edition Academic Press (2000)

[17] Pliska, S.: *Introduction to Mathematical Finance* BlackwellPublishers (1997)

[18] Ravindran, K., 1993, "Low-Fat Spreads," RISK 6(10): 67-70

[19] Rubinstein, M., 1991, "Somewhere Over the Rainbow," RISK 4(11)

[20] Rubinstein, M., 1994b, "Return to Oz," RISK 7(11): 67-70

[21] Shreve, S., *Stochastic Calculus and Finance* Forthcoming Springer (2004)

[22] Stultz, R. M., 1982, "Options on the minumum or the maximum of two risky assets," *Journal of Financial Economics* 10: 161-185

[23] Williams, D., *Probability with Martingales* University Press, Cambridge (1991)

[24] Zhang, Z.: *Exotic Options*. World Scientific Publishing, (1998)

APPENDIX A

FINITE STATE MARKOV CHAINS

In this work, we only consider Markov chains with finite state spaces.

Let ξ be a finite set. A Markov Chain with values in ξ is a discrete-time process $\mathcal{X} = \{x_n, m \in \mathbb{N}\}$ taking its values in ξ such that $\forall n \in \mathbb{N}$ and $\forall j, i$, $\forall n \geq 1$,

$$\mathbb{P}\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0\} = \mathbb{P}\{X_{n+1} = j | X_n = i\}$$

This property is called the *Markov property.* A Markov chain is said to be homogenous if $\mathbb{P}\{X_{n+1} = j | X_n = i\}$ is independent of n. From now on, $\mathcal{X}$ will be a homogenous Markov chain and we shall put

$$p_{ij} = \mathbb{P}\{X_{n+1} = j | X_n = i\}$$

$$P = [P_{ij}]_{i \in \xi, j \in \xi}$$

The entries of P^n will be denoted by $p_{ij}(m)$. This is nothing but the probability that $\mathcal{X}$ passes from i to j at the m^{th} transition. A Markov chain $\mathcal{X}$ is said to be irreducible if for any pair (i, j) of points in ξ, $\exists m > 0, n > 0$ such that $P_{ij}(m) > 0$ and $P_{ij}(n) > 0$. An irreducible Markov chain has always a stationary probability $\pi = \{\pi_i, i \in \xi\}$,in the sense that,

$$\pi_j = \sum_{i \in \xi} \pi_i p_{ij}$$

If the probability of X_0 is chosen to be π then $\mathcal{X}$ becomes a stationary process, i.e. $\forall n, n_1, n_2, ..., n_k \in \mathbb{N}$, the probability distribution of $\{X_{n_{1+h}}, X_{n_{2+h}}, ..., X_{n_{k+h}}\}$ does not depend on h. Moreover $\mathcal{X}$ is also ergodic in the sense that for any continuous function $f(x_1, ..., x_k)$, we have

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N} \sum_{h=1}^{N} f(X_{n_{1+h}}, X_{n_{2+h}}, ..., X_{n_{k+h}}) &= \mathbb{E}_\pi[f(X_{n_1}, ..., X_{n_k})] \\ &= \mathbb{E}_i[f(X_{n_1}, ..., X_{n_k})] a.s. \end{aligned}$$

for all $i \in \xi$, where $\mathbb{E}_\pi$ denotes the expectation when the probability of X is π and $\mathbb{E}_i$ is the expectation when $X_0 = i$.

Definition A.0.1. $\{\mathcal{F}_n : n \geq 0\}$ is a filtration, if it is an increasing family of sub σ-algebras of $\mathcal{F}$: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subset \mathcal{F}_2 ..... \subset \mathcal{F}$. We define

$$\mathcal{F}_\infty := (\bigcup_n \mathcal{F}_n) \subseteq \mathcal{F}$$

APPENDIX B

THE FINANCIAL MARKETS

A discrete time market model is built in a finite probability space (Ω,$\mathcal{A}$, $\mathbb{P}$)[1] equipped with a filtration, i.e. an increasing sequence $\mathcal{F}_0$, $\mathcal{F}_1$,.....,$\mathcal{F}_N$ of sub-σ-algebras of $\mathcal{A}$. $\mathcal{F}_n$ represents the information available at time n and we call it the σ-algebra of events up to time n. The horizon N is supposed to be finite. In other words, a filtration is the history of the market. We denote it by $\mathcal{F} = \{\mathcal{F}_n, n = 0, 1, ..., N\}$. There is no loss of generality in taking $\mathcal{A} = \mathcal{F}_N$. We take $\mathcal{F}_0 = \{\phi, \Omega\}$. we will suppose that $\forall \omega \in \Omega$, $\mathbb{P}(\{\omega\}) > 0$.

The market consists of $d + 1$ financial assets whose prices at time n are non-negative random variables S_n^0, S_n^1,...,S_n^d measurable w.r.t $\mathcal{F}_n$. The vector $S_n = (S_n^0, S_n^1, ..., S_n^d)$ is the vector of prices at time n. The asset indexed by '0' is the riskless asset and we have $S_0^0 = 1$. If the return of the riskless asset at tim $k-1$ is r_k then $S_n^0 = \prod_{k=1}^{n}(1 + r_k)$. This is the value at time n of one unit money put onto a bank account at time 0. The coefficient $B_n = \frac{1}{S_n^0}$ is

[1] Ω is a finite set.

the discount factor (from time n to 0). The assets indexed by $i = 1, 2, ..., d$ are called risky assets.

A trading strategy is defined as a stochastic process $\Phi = \{(\phi_n^0, \phi_n^1, ...\phi_n^d)\}_0 \leq n \leq N$ in $\mathbb{R}^{d+1}$ where ϕ_n^i denotes the number of shares of the asset i held in the portfolio at time n. Φ is predictable in the sense that $\forall i = 1, ..., d$, ϕ_0^i is $\mathcal{F}_0$ measurable and $\forall n \geq 1$ ϕ_n^i is $\mathcal{F}_{n-1}$ measurable.

The value of the portfolio at time n is the inner product

$$V_n(\Phi) = \Phi_n S_n = \sum_{i=0}^{d} \Phi_n^i S_n^i$$

Its discounted value is

$$\tilde{V}_n(\Phi) = B_n(\Phi_n S_n) = \Phi_n \tilde{S}_n$$

with $B_n = \frac{1}{S_n^0}$ and $\tilde{S}_n = (1, B_n S_n^1, ..., B_n S_n^d)$

Definition B.0.2. A strategy is called self-financing if the following equations is satisfied for all $n = 0, 1, ..., N$

$$\Phi_n S_n = \Phi_{n+1} S_n$$

The interpretation of this is the following:

At time n, once the new prices are quoted, the investor readjusts its positions from Φ_n to Φ_{n+1} without bringing or consuming any wealth. the equality $\Phi_n S_n = \Phi_{n+1} S_n$ is equivalent to

$$\Phi_{n+1}(S_{n+1} - S_n) = \Phi_{n+1} S_{n+1} - \Phi_n S_n$$

or to

$$V_{n+1}(\Phi) - V_n(\Phi) = \Phi_{n+1}(S_{n+1} - S_n)$$

Using this last equality we can prove the following [15],

Proposition B.0.1. *The following are equivalent,*

1. *The strategy Φ is self financing*

2. *For any $n \in \{1, ..., .N\}$*

$$V_n(\Phi) = V_0(\Phi) + \sum_{j=1}^{N} \phi_j \Delta S_j$$

   where $\Delta S_j = S_j - S_{j-1}$

3. *For any $n\{1, ..., N\}$*

$$\tilde{V}_n(\Phi) = V_0(\Phi) + \sum_{j=1}^{N} \phi_j \Delta \tilde{S}_j$$

   where $\Delta \tilde{S}_j = \tilde{S}_j - \tilde{S}_{j-1} = B_j \tilde{S}_j - B_{j-1} \tilde{S}_{j-1}$

Definition B.0.3. A strategy Φ is admissible if it is self-financing and if $V_n(\Phi) \geq 0$ for any $m = 0, 1, ..., N$

Definition B.0.4. An arbitrage strategy is an admissible strategy with zero initial value and non-zero final value.

Arbitrage and option pricing problem are related to martingale problems. We summarize the results.

Definition B.0.5. [23] A sequence of random variables $\{M_n : 0 \leq n \leq N\}$ is said to be adapted to the filtration $\{\mathcal{F}_n\}$, if for each n, M_n is $\mathcal{F}_n$ measurable.

Definition B.0.6. A sequence $\{M_n : 0 \leq n \leq N)$ is called a martingale (relative to $(\{\mathcal{F}_n\}, \mathbb{P})$) if

1. M_n is adapted

2. $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$, a.s for all $n \leq N - 1$

a supermartingale, if $\mathbb{E}[M_n|\mathcal{F}_{n-1}] \leq M_{n-1}$, a.s for all $n \leq N-1$, a submartingale, if $\mathbb{E}[M_n|\mathcal{F}_{n-1}] \geq M_{n-1}$, a.s for all $n \leq N - 1$

These definitions [15] can be extended to d-dimensional process, for instance $\{M_n : 0 \leq n \leq N\}$ is a martingale if its components are martingales.

Definition B.0.7. An adapted sequence of $(H_n)_{0 \leq n \leq N}$ of random variables is predictable $\forall n \geq 1$, H_n is $\mathcal{F}_{n-1}$ measurable.

Definition B.0.8. The market is said to be viable if there is no arbitrage opportunity.

Now we can announce the following important results,

Theorem B.0.2. *The market is viable iff there exists a probability* $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *such that the discounted prices of assets are martingales under* $\mathbb{P}^*$.

$\mathbb{P}^*$ and $\mathbb{P}$ are said to be equivalent iff $\forall A$ such that $\mathbb{P}A = 0$ then $\mathbb{P}^*(A) = 0$ and $\forall A$ such that $\mathbb{P}^*A = 0$ then $\mathbb{P}(A) = 0$.

In our market model where we suppose that $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$, any probability measure $\mathbb{P}^*$ such that $\mathbb{P}^*(\{\omega\}) > 0$ for all $\omega \in \Omega$ is equivalent to $\mathbb{P}^*$.

The probability measure $\mathbb{P}^*$ of the above theorem is called an *equivalent martingale measure* or *risk neutral probability.*

An important notion in financial markets is the notion of completeness. A contingent claim is the wealth, an investor wishes to attend at time N. A contingent claim is a random variable depending on the history of the market, i.e. a $\mathcal{F}_N$-measurable random variable.

Definition B.0.9. The contingent claim h is attainable if there exists an admissible strategy worth h at time N.

Definition B.0.10. The market is complete if every contingent claim is attainable.

Finally, we announce the following important result.

Theorem B.0.3. *A viable market is complete if and only if there exists a unique probability measure* $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *under which discounted prices are martingale.*

APPENDIX C

OPTIMAL PORTFOLIO UNDER INCOMPLETE MARKET

We show that under the viable market assumption, optimization of a portfolio with two risky assets has a numerical solution. We also introduce the incompleteness of the market into dynamic programming problem in which investor hedges himself/herself against the worst case for every period.

C.1 Value of the Portfolio

The value of our portfolio at time n is

$$V_n(\phi) = \phi_n^0 S_n^0 + \phi_n^1 S_n^1 + \phi_n^2 S_n^2 \qquad \text{(C.1.1)}$$

where ϕ_n^i is the amount of the corresponding asset i held by the investor. we denote by ϕ_n the three dimensional vector $(\phi_n^0,\ \phi_n^1,\ \phi_n^2)$. We can write the following by the definition of self-financing strategy and proposition $B.0.1$,

$$V_{n+1}(\phi) = V_n(\phi) + \phi_{n+1}^0 \Delta S_n^0 + \phi_{n+1}^1 \Delta S_n^1 + \phi_{n+1}^2 \Delta S_n^2 \qquad \text{(C.1.2)}$$

Discounting the expression above to get the discounted price presentation, we have

$$\tilde{V}_{n+1}(\phi) = V_n(\phi) + \phi^0_{n+1}\Delta\tilde{S}^0_n + \phi^1_{n+1}\Delta\tilde{S}^1_n + \phi^2_{n+1}\Delta\tilde{S}^2_n$$

We know that $\Delta\tilde{S}^0_n$ is 0.

$$\tilde{V}_{n+1}(\phi) = V_n(\phi) + \phi^1_{n+1}\Delta\tilde{S}^1_n + \phi^2_{n+1}\Delta\tilde{S}^2_n \tag{C.1.3}$$

This also implies

$$\tilde{V}_n(\phi) = V_0 + \sum_{j=1}^{n}(\phi^1_j\Delta\tilde{S}^1_j + \phi^2_j\Delta\tilde{S}^2_j) \tag{C.1.4}$$

Under martingale probability measure $\tilde{\mathbb{P}}$, we have

$$\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{V}_n] = \mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{V}_0]$$

We assume the rational economic agent maximizes his/her expected utility.

Definition C.1.1. A utility function G is a strictly concave and continuously differentiable real function on $(0, \infty)$ satisfying

$$G'(0) := \lim_{x \downarrow 0} G'(x) = +\infty$$

$$G'(0) := \lim_{x \to \infty} G'(x) = 0$$

The investor has to choose optimal set of strategies ϕ for each period to maximize the expected utility. Since the random structure of our model follows a markov chain, investor can set up the expectations at the beginning of every period depending on the filtration whose generation depends on the order of the chain. So it reduces to a single period optimization problem.

C.2 Stochastic Optimization Under Complete Market

We do not impose any constraint on the sign of strategies ϕ_n. Negative sign indicates borrowing (shortselling). Borrowing is allowed as long as total wealth remains positive.

C.2.1 Dynamic Programming

Suppose an investor wants to optimize the expected utility G of wealth at time n.

$$\psi(n, V) := \sup_{\phi_n \in \Phi} \mathbb{E}[G(\tilde{V}_N)]$$

We multiply the portfolio value under martingale probability by the discount factor. The corresponding Bellman equation which is the solution to the problem is

$$\upsilon(u_{N-1}, N, V) = \mathbb{E}[G(\tilde{V}_N)] \tag{C.2.1}$$

for all $V > 0$

$$\begin{aligned} \upsilon(u_{n-1}, n, V) : &= \sup_{\phi_n \in R} \mathbb{E}[P_{(ac),(U_{n-1})}.G(\tilde{V}_{n+1}(a,c)) \\ &+ P_{(bc),(U_{n-1})}.G(\tilde{V}_{n+1}(b,c)) \\ &+ P_{(ad),(U_{n-1})}.G(\tilde{V}_{n+1}(a,d)) \\ &+ P_{(bd),(U_{n-1})}.G(\tilde{V}_{n+1}(b,d))] \end{aligned} \tag{C.2.2}$$

$n = 0, 1, 2, ..., N-1$ for all $V > 0$. Where $P_{(U_n),(U_{n-1})} = P(U_n^1 = u_n^1, U_n^2 = u_n^2 | U_{n-1}^1 = u_{n-1}^1, U_{n-1}^2 = u_{n-1}^2)$ Solving equations (2.4.14) and (2.4.15) for $\Delta\tilde{S}_n^1$ and $\Delta\tilde{S}_n^2$ we get the following expressions

$$\Delta\tilde{S}_n^1 = \frac{(U_n^1 - r(U_{n-1}))\tilde{S}_{n-1}^1 + \varepsilon(U_n^2 - r(U_{n-1}))\tilde{S}_{n-1}^2}{1 + r(U_{n-1})} \tag{C.2.3}$$

$$\Delta\tilde{S}_n^2 = \frac{\delta(U_n^1 - r(U_{n-1}))\tilde{S}_{n-1}^1 + (U_n^2 - r(U_{n-1}))\tilde{S}_{n-1}^2}{1 + r(U_{n-1})} \tag{C.2.4}$$

Let our utility function have the following form

$$G(V) = \tfrac{V^\Theta}{\Theta}\ \text{, where } 0 < \Theta < 1$$

Now, we can write the Bellman equation in terms of the observable parameters except for the control actions Φ. Then, all we have to do is to solve the problem for ϕs.

C.3 Stochastic Optimization Under an Incomplete Market

At remark (3.2.1), we have explained that the martingale measure is not unique[1]. That is to say the market in our model is incomplete. In an incomplete market, the investor seeks for hedging himself against the worst case.

The problem is then a constrained optimization problem. The constraints are the conditional marginal probabilities of U_n^1 and U_n^2 for a given joint probability in the set of probabilities satisfying the martingale condition. Incomplete market problem can be considered as a model specification problem with joint probabilities being the parameters. Then, there are more than one objective functions to maximize subject to weights. The investor, in this case, would choose the model with the minimum expected utility subject to joint probabilities. Then the problem is

$$\inf_{\tilde{\mathbb{P}}(U_n^1,U_n^2)} \sup_{\phi_n \in \Phi} \mathbb{E}[G(\tilde{V}_N)]$$

,

such that

$$Pac + Pad = \tilde{\mathbb{P}}a$$

$$Pbc + Pbd = \tilde{\mathbb{P}}b$$

$$Pac + Pbc = \tilde{\mathbb{P}}c$$

[1]Notice that it holds as long as U_n^1 and U_n^2 are not (conditionally) independent.

$$Pad + Pbd = \tilde{\mathbb{P}}d$$

with given initial wealth v_0

As seen, the investor maximizes the expected utility by choosing the strategy corresponding to a given joint probability and then selects the joint probability that minimizes the expected utility. The difference from the complete market case is that we have more than one optimal expected portfolio to maximize. This selection would naturally vary with the type of the investor. Assuming that the investor is risk adverse type, we claim that he/she would choose the strategies corresponding to the worst case.

APPENDIX D

TURKISH SUMMARY

Bu tezin amacı iki adet finansal varlıktan oluşan bir piyasanın işleyişine ilişkin bir model geliştirmek ve bu modelin parametreler için tahmin edicilerini formüle ederek modelin kendi içinde tanımlı olduğunu göstermektedir. Modelin ana amacı, bir stokastik süreçi takip eden bu iki varlık arasındaki korelasyon yapısının incelenmesi ve faiz haddi ile olan ilişkinin analizinin yapılmasıdır.

Hakikaten de bu alanda yapılan neşriyat korelasyon ve değişen faiz problematiği üzerinde yoğunlaşmıştır. Genelde korelasyon problemi probabilistik varsayımlarla evvelden belirlenmiş bir korelasyon yapısının modellere adaptasyonu ile giderilmeğe çalışılmış ve faiz haddide sabit alınmıştır. Bu tezde anlatılan bu model ise korelasyon yapısının modelin kendi içinde belirlenmesini sağlar. Korelasyon modelde direk olarak gözlenemesede, kendi işleyişi içerisinde kendi ürettiği korelasyon yapısını sisteme tanıtır. Bu model markov zinciri modellemesi yardımı ile faizi bir dönem önce oluşmuş olan rassal patlamaların bir fonksiyonu olarak algılar. Böylece bir dönem önceki informasyona dayalı olarak model içinde faiz haddi değişkenlik kazanır.

Modeli oluşturan fiyatların faiz haddi ile iskonto edilmiş halinin martingale olduğunu göstererek martingale fiyatlaması yapmaktayız. Martingale varsayımı piyasayı arbitrajdan arındırarak adil fiyatlama yapan bir mekanizmaya sahip olmamızı sağlar.

İki değişkenimizin n zamanındaki fiyatını sıra ile S_n^1 ve S_n^2 ile, bu fiyatlara ilişkin rassal dizileride sıra ile U_n^1 ve U_n^2 olarak ifade ettik. U_n^1, p olasılığı ile a ve $(1-p)$ olasılığı ile b değerini alırken, U_n^2 q olasılığı ile c ve $(1-q)$ olasılığı ile d değerini almaktadır. İlk bölümde bu dizilerin biribirlerinden bağımsız ve kendi içindede bağımsız ve denk olarak dağıldığını varsaydık. Bu halde aşağıdaki model bir markov zinciri oluşturur.

$$\begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} = \begin{pmatrix} 1+U_n^1 & \varepsilon(U_n^2+\alpha) \\ \delta(U_n^1+\beta) & 1+U_n^2 \end{pmatrix} \begin{pmatrix} S_{n-1}^1 \\ S_{n-1}^2 \end{pmatrix} \qquad \text{(D.0.1)}$$

$\varepsilon \neq 0$, $\delta \neq 0$.

İskonto fiyatların belli bir olasılık kanunu altında martingale olmalarından yararlanarak modelimizi sadeleştirebiliriz. Çünkü martingalin koşullu beklentisi kendisini verir. Ayrıca faiz haddinin bir gün önceden belirlenebildiğini varsayarsak,

$$\mathbb{E}(\tilde{S}_n|\mathcal{F}_{n-1}) = \frac{1}{1+r_n} \begin{pmatrix} 1+\mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & 1+\mathbb{E}(U_n^2) \end{pmatrix} \tilde{S}_{n-1} \qquad \text{(D.0.2)}$$

Martingale özelliği sebebi ile

$$\tilde{S}_{n-1} = \frac{1}{1+r_n} \begin{pmatrix} 1+\mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & 1+\mathbb{E}(U_n^2) \end{pmatrix} \tilde{S}_{n-1} \tag{D.0.3}$$

Basitleştirirsek,

$$\tilde{S}_{n-1} = \frac{1}{r_n} \begin{pmatrix} \mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & \mathbb{E}(U_n^2) \end{pmatrix} \tilde{S}_{n-1} \tag{D.0.4}$$

Ve

$$r_n I = \begin{pmatrix} \mathbb{E}(U_n^1) & \varepsilon(\mathbb{E}(U_n^2)+\alpha) \\ \delta(\mathbb{E}(U_n^1)+\beta) & \mathbb{E}(U_n^2) \end{pmatrix} \tag{D.0.5}$$

Artik aşağıdaki kanunu yazabiliriz.

$$\mathbb{E}(U_n^1) = \mathbb{E}(U_n^2) = r_n = r. \tag{D.0.6}$$

Böylelikle modelimiz D.0.1 aşağıdaki hali alır.

$$\begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} = \begin{pmatrix} 1+U_n^1 & \varepsilon(U_n^2-r) \\ \delta(U_n^1-r) & 1+U_n^2 \end{pmatrix} \begin{pmatrix} S_{n-1}^1 \\ S_{n-1}^2 \end{pmatrix} \tag{D.0.7}$$

Bundan sonra amacımız elimizde varolan bu modelin parametrelerini gözlemlenebilir değişkenler cinsinden yazıp tanımlanabilir olup olmadıklarını göstermektir. Bu sayede modelin analitik çözümünün olup olmadığınıda irdelemiş olacağız.

Binomial dağılım aşağıdaki ifadeyi yazabilmemize olanak verir.

$$\mathbb{E}(U_n^1) = pa + (1-p)b = r \tag{D.0.8}$$

and

$$\mathbb{E}(U_n^2) = qc + (1-q)d = r \tag{D.0.9}$$

Bu ifadeden aşağıdaki ifade elde edilir.

$$p = \frac{b-r}{b-a} \tag{D.0.10}$$

$$q = \frac{d-r}{d-c} \tag{D.0.11}$$

$0 < p < 1$ olduğundan $a < r < b$ ve $c < r < d$, i.e.

Bazı düzenlemeler sonucunda U_n için aşağıdaki ifadeleri yazıp r etrafında merkezledik

$$U_n^1 - r = \frac{1+r}{(1-\varepsilon\delta)}\left(\frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^1} - \varepsilon\frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^1} \right) \tag{D.0.12}$$

$$U_n^2 - r = \frac{1+r}{(1-\varepsilon\delta)}\left(\frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^2} - \delta\frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^2} \right) \tag{D.0.13}$$

U_n serisi durağan olduğundan ve bu serinin koşulsuz beklenen değeri r olduğundan bu farkın zamana göre averajı 0 dır. Bu sebep ile her iki tarafın averajını alıp ε ve δ için çözersek aşağıdaki tahmin edicileri buluruz.

$$\hat{\varepsilon} = \frac{avg(\frac{\Delta\tilde{S}_n^1}{\tilde{S}_{n-1}^1})}{avg(\frac{\Delta\tilde{S}_n^2}{\tilde{S}_{n-1}^1})} \tag{D.0.14}$$

$$\hat{\delta} = \frac{avg(\frac{\Delta \tilde{S}_n^2}{\tilde{S}_{n-1}^2})}{avg(\frac{\Delta \tilde{S}_n^1}{\tilde{S}_{n-1}^2})} \tag{D.0.15}$$

ε ve δ bilindiğne göre artık a,b,c ve d değerleri için problemi çözebiliriz. Birinci ve ikinci merkezi momentler yardımı ile bu değerler içinde birer tahmin edici buluruz.

$$\begin{aligned}
\hat{a} &= r - \sigma_1(\frac{1-p}{p})^{1/2} \\
\hat{b} &= r + \sigma_1(\frac{p}{1-p})^{1/2} \\
\hat{c} &= r - \sigma_2(\frac{1-q}{q})^{1/2} \\
\hat{d} &= r + \sigma_2(\frac{q}{1-q})^{1/2}
\end{aligned} \tag{D.0.16}$$

Şimdi Markov özelliğini modele giydirebiliriz.

$$\mathbb{E}[U_n^1|U_{n-1}] = \mathbb{E}[U_n^2|U_{n-1}] = r(U_{n-1}) \tag{D.0.17}$$

Bu durumda olasılık kanunu aşağıdaki gibidir.

$$p_{(a),(U_{n-1})} = \frac{b - r(U_{n-1})}{b-a} \tag{D.0.18}$$

$$q_{(c),(U_{n-1})} = \frac{d - r(U_{n-1})}{d-c} \tag{D.0.19}$$

U_{n-1} 4 farklı değer alacağı için bu olasılıklarda bir önceki döneme bağımlı olarak 4 farklı değer alabilirler. Bu koşullu marjinal olasılıkları verir ama birlikte dağılım hakkında bir şey söylemez. Biz de koşullu olarak bağımsız olduklarını kabul ettik. Koşullu bağımsız olmadıkları takdirde aynı martingale şartlarını sağlayan bir çok birlikte dağılım yazabiliriz. Bu market 'Incomplete' demektir. Bağımsız durumda yaptıklarımızı buradada yapıp ε ve δ için aynı tahmin edicilere ulaşırız. a,b,c ve d ise

$$\hat{a} = \bar{r}(U_{n-1})\bar{P}_{(a),(U_{n-1})} \mp \frac{[(\overline{\sigma^2}_{n,1} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)\bar{P}_{(a),(U_{n-1})}\bar{P}_b]^{1/2}}{\bar{P}_{(a),(U_{n-1})}}$$

$$\hat{b} = \bar{r}(U_{n-1})\bar{P}_{(a),(U_{n-1})} - \bar{r}(U_{n-1}) \mp \frac{[(\overline{\sigma^2}_{n,1} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)\bar{P}_{(a),(U_{n-1})}\bar{P}_b]^{1/2}}{\bar{P}_b}$$

$$\hat{c} = \bar{r}(U_{n-1})\bar{P}_c \mp \frac{[(\overline{\sigma^2}_{n,2} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)\bar{P}_c\bar{P}_d]^{1/2}}{\bar{P}_c}$$

$$\hat{d} = \bar{r}(U_{n-1})\bar{P}_c - \bar{r}(U_{n-1}) \mp \frac{[(\overline{\sigma^2}_{n,2} + \overline{r(U_{n-1})^2} - \bar{r}(U_{n-1})^2)\bar{P}_c\bar{P}_d]^{1/2}}{\bar{P}_d} \quad \text{(D.0.20)}$$

Daha sonra simulasyonlar yardımı ile modelimizin analitik olarak çözümlü olduğunu gösterdik. Daha sonraki bölümlerde ise modelin kullanabileceği yerleri gösterdik ve verilere tatbik ettik. Sonuçlar bir hayli entereasan çıktı. Model tek bir fiyatın değişimini iyi açıklayamasa bile ikisinin birlikte gelişimlerini oldukça iyi açıklamaktadır. Özellikle Spread Opsiyon türevi üzerindeki uygulamalar bunun en büyük kanıtı olarak sunulabilir. Opsiyon fiyatlamaları için Monte Carlo metodunu kullandık. Kontrat fiyatı olarak gerçekleşmiş bir ileri gün fiyatını seçerek modelimizin opsiyon için 0 lira değer biçip biçmeyeceğini görmek istedik. Sonuçta modelin arasında ilişki bulunan iki varlığın hareketini oldukça iyi gözlüyor olduğna tatmin olduk. Burada tahmin edicilerimiz hakında söylememiz gereken bir önemli husus daha var. ε ve δ moment tahmin edicileri oldukça iyi iken a b c ve d için aynısını söylemek mümkün değil. Bu sebep ile modelin kalibrasyonu modelin tahmin etme gücünü yükseltmektedir. Sonuç olarak modelimiz tamamen tanımlı ve amacına uygun olarak uygulanabilmektedir. Basit ama efektif bir model olması en büyük özelliğidir. Bir başka önemli özelliği ise kolylıkla geliştirilebilir olmasıdır. Hakikatende tatbikat sırasında rassallığın normal dağılımdan geldiğini görülmektedir. Gauss-Markov gösterimi ile aşağıdaki şekilde modelimizi geliştirebiliriz.

$$\begin{pmatrix} U_{n+1}^1 \\ U_{n+1}^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} U_n^1 \\ U_n^2 \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \xi_n^1 \\ \xi_n^2 \end{pmatrix}$$

VITA

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