SECOND ORDER ROTATIONAL EFFECT ON NONRADIAL OSCILLATIONS IN δ-SCUTI STARS

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ZIYAD MATALGAH

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Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Sinan Bilikmen
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Rikkat Civelek
Co-Supervisor

Prof. Dr. Halil Kırbıyık
Supervisor

Examining Committee Members

Prof. Dr. Halil Kırbıyık

Assoc. Prof. Dr. Rikkat Civelek

Prof. Dr. Nilgün Kızıloğlu

Asst. Prof. Dr. Sinan Kaan Yerli

Asst. Prof. Dr. Zekeriya Müyesseroğlu

ABSTRACT

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Matalgah, Ziyad

MSc., Department of Physics

Supervisor: Prof. Dr. Halil Kırbıyık

Co-Supervisor: Assoc. Prof. Dr. Rikkat Civelek

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In this work the effect of rotation on oscillation frequencies have been investigated . Rotation has been treated as a perturbation and detailed calculations were done on the influence of second order rotation . We used an evolutionary model of$\delta - Scuti$ star V1162 Ori with a mass of $1.8M_{\odot}$. The eigenfrequencies were calculated in two cases , the slow rotation case with $vsini = 46kms^{-1}$ and the fast rotation case with $vsini = 61.9kms^{-1}$. Calculation were carried out by a modified oscillation program and results were compared to observations .

Keywords: Rotation , V1162 , Oscillations , Variable Stars , Stellar Evolution

ÖZ

δ-SCUTI YILDİZLARİNİN ÇAPSAL OLMAYAN SALİNİMLARİNA
DÖNMENİN IKINCI DERECEDAN ETKILERI

Matalgah, Ziyad
Master, Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Halil Kırbıyık
Ortak Tez Yöneticisi: Assoc. Prof. Dr. Rikkat Civelek

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Bu çalışmada salınım yapan yıldızlarda frekanslarına etkileri ikinci derece bozulmada katilarak detaylı incelerdi .1.8 Güneş kütleli evrim modeli , $\delta - Scuti$ yıldızı olan V1162 Ori yıldızı için model olarak kullanıldı . Salınım frekanslarındakı dönmenın sebep oldğu değişim iki farklı dönme hızı alınarak ayrı ayrı hesplandı ($46kms^{-1}$ ve $61.9kms^{-1}$) . Hesaplamalarda geliştirilmiş salınım program kullanıldı ve sonuçlar gözlemsel-verilerle karşılaştırıldı.

Anahtar Kelimeler: Dönme , V1162 , Salınımlar , Değişken Yıldızlar , Evrim Modeli

TO MY PARENTS

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CHAPTER I

INTRODUCTION

It is well known that our universe is like a mechanical machine , which is changable through time , in other words , we are living in an absolutely dynamical universe . We can say that everything in our universe is variable with time , and since stars are the major ingredients of our universe , they should be changeable accordingly , which can be seen theorytically or through observations . In fact , stars are variable through their properties , just like human being's life , they born , develop , pass through different stages and die .

There is no dought that the oscillatory motion of stars is something trivial , a pulsating star is a star changing it's physical properties with time , i.e , pressure , temperature , luminosity ... etc . These oscillations are due to a restoring forces of two different nature , one is due to pressure force and the other is due to gravitational force , the two kinds are resulting from the gravitational potential and the hydrogen burning (nuclear fusion of hydrogen) .

Most kinds of stars are pulsating stars , either single or binary ones . The former theory of stellar pulsation was first developed to give an explanation for only radial pulsations as a special case , it explained the behavior of the classical variables such as Cepheids and RR Lyrae stars , but now it is improved to include both radial and nonradial pulsations alike . Stellar variations and pulsations was studied early by Ritter (1880) , later stydies was described by Rosseland (1949) , Osaki (1974) , Smith (1980) and Unno (1981) . The theory of stellar pulsation is very important in recognizing the stellar structure , it plays an important role in modern cosmology , modern astrophysics , solar and stellar seismology , helioseismology and astroseismology . Stellar modelling is also an interesting tool used for studying these oscillations and probing the stellar interiors .

A star is like a musical instrument . It can oscillate at frequency giving certain different tunes . In fact a star can oscillate radially by maintining its spherical shape , or nonradially by desturbing its spherical shape , the latter case is the most general case . Both kinds of scillations are observed in most of the oscillating stars including the binary ones . Nonradial oscillations was first studied by Lord Kelvin (1863) , the adiabaticity of these oscillations was discussed by Pekeris (1938) . Other manipulations were performed by Cowling (1941) , Ledoux (1951) , Cox (1976) , Cox (1980) , Unno (1989) and Walraven (1958) .

$\delta - scuti$ stars of spectral class A-F are pulsating stars situated on or just above the main sequence . They are either main sequence (MS) or early post main sequence objects . Their pulsation periods range from one to a few houres and this is clear throgh changes in their light curves and their line profile variations which are confirmed by observations . Also δ-scuti stars show a wide range of pulsation amplitude from milimagnitude level up to almost one magnitude . The different patterns of frequencies in their oscillations are clues for existing nonradial oscillations that appear at low degree p-modes ($l \leq 3$) where l is the spherical harmonic degree . In some δ-scuti stars rotation has been observed , fast rotating kinds of these stars have rotational velocity ($vsini \geq 50km/s$) . Radial and nonradial oscillations of these stars were studied by Stellingwert (1979) , Chevalier (1971) , Lee (1985) , Sofi et al (1998) Pamyatnykh (2000) and Dziembowski (1977) .

In fact there are some factors affecting the stellar pulsation like rotation , magnetic field , tidal effect and relativistic effects . Rotation destroys the spherical symmetry of the star and remove the azimuthal degeneracy of the oscillating frequencies of a nonrotating star producing rotational frequency splitting . In the treatment of the problem rotation is assumed to be uniform and the rotational frequency (Ω) was considered as perturbation . Nonrotating and nonmagnetic sperical symmetry distribution of stellar mass element is also assummed . Radiative viscosity and turbulence were neglected . The hydrostatic equilibrium and the adiabaticity are assummed . Many studies have been done for rotating fluid stars , either by investigating the Coriolis force effect up to first order in (Ω) (e.g . Hansen et al.1979 ; Hansen et al.1978 ; Carroll and Hansen.1982; Strohmayer.1991 ; Strohmayer.1996) or by ivistigating both Coriolis force and centrifugal force up to second order in (Ω) (e.g . Simon.1982 ; Smeyers and Dennis.1971 ; Saio.1981 ; Dziembowski and Goode .1992) .

The principal goal of our work is to study the effect of systemmatic fast rotation on stellar pulsation for the noneradial case . Oscillations are calculated up to second order in Ω by applying a modified oscillatory program - written by Al-Murad and Kırbıyık (1995)- to the V1162 Ori δ-scuti star . Computations were carried out for low spherical harmonic degrees ($l \leq 3$) , we choosed a suitable model for our star at two different rotational velocities i.e fast and slow rotational velocities . Our work contained five chapters in addition to two appendices . In chapter two we discussed some properties of nonradial oscillations , the fundamental equations of fluid mechanics and hydrodynamics concerning the stellar structure and their linearized form using the material derivative concept , we made use of the relation between Lagrangian and Eulerian point of views . The first order rotational effect at the eigenfrequencies and eigenfunctions was explained in chapter three by assuming slow rotation . In chapter four the second order rotational effect and eigenfrequencies were discussed assuming fast rotation and all derivations were explained . In chapter five we offered our results and give conclusions with some comments on results . Finally , appendix one included the MATHEMATICA results for some of the angular integrals while appendix two included the modified version of the oscillatory program .

CHAPTER II

NONRADIAL OSCILLATIONS

Nonradial oscillations are the general case of stellar pulsation , any star can be considered as a fluid gaseouse sphere , we can apply the basic equations of fluid mechanics and make use of the basic laws of hydrodynamics , by assuming some constraints and utilizing the concepts of linear theory in perturbation . The basic principles of fluid mechanics and stellar structure were discussed in details by Landau (1966) and Cox (1968) . In this chapter we will discuss some properties of nonradial oscillations and give the final form of the linearized basic equations of the stellar structure and oscillations under the adiabatic condition .

II.1 Fundamental properties of nonradial oscillations

When a star starts to oscillate in such away , without maintaining its radial and spherical symmetry , it is oscillating nonradially , i.e. , some regions of its surface are expanding while others are contracting . These oscillations occure at certain modes and tunes . Their normal modes can be described by three integers (n , l , m) , the quantum number l relates to the zonal lines of the stellar surface which propogate at different phases , while the quantum number m indicates the number of nodal lines at the longitude and it takes the values [-l , ... , l] , the degeneracy of the modes depends on the value of m . The nodal surfaces in the radial direction are assigned by the quantum number n . Both pressure and boyancy forces of the star work as a restoring force , and hence , two kinds of modes appears , the pressure (p) modes and the gravity (g) mode . Furthermore , a fundamental third mode (f) also appears which is an intermediate mode between the two previouse ones . The vibrational properties of the medium is characterized by two kinds of frequencies , the Lamb (L) freqeuency and the Brünt Väisälä (N) frequency.These frequencies are given by

$$L^2 = \frac{l(l+1)c^2}{r^2} \tag{II.1}$$

$$N^2 = g(\frac{d\ln(p)}{\Gamma_1 dr} - \frac{d\ln\rho}{dr}) \tag{II.2}$$

where c , g , and Γ_1 are the speed of sound , the gravitational acceleration and the adiabatic exponent respectively . They are given by

$$c^2 = \frac{\Gamma_1 p}{\rho} \tag{II.3}$$

$$g = \frac{GM_r}{r^2} \tag{II.4}$$

$$\Gamma_1 = (\frac{d\ln p}{d\ln\rho})_{ad} \tag{II.5}$$

the eigenfrequency (σ) or the frequency of oscillations can be represented in term of adimentionless frequency (ω) as

$$\omega^2 = \sigma^2 R^3/GM \tag{II.6}$$

where R is the surface radius , M is the total mass and G is the gravitational constant . The plot of (ω^2) versus r/R is called the propagation diagram . The dominating number of radial nodes K is given by

$$k = (p - nodes) - (g - nodes) \tag{II.7}$$

These nodes appear at the propagation diagram when plotting the Lamp and Brünt Väisälä frequenceis . Figures II.1 , II.2 and II.3 show the propagation diagram of V1162 Ori with 1.80 M_0 . In case of $\sigma^2 > N^2, L^2$ and $\sigma^2 \geq 0$

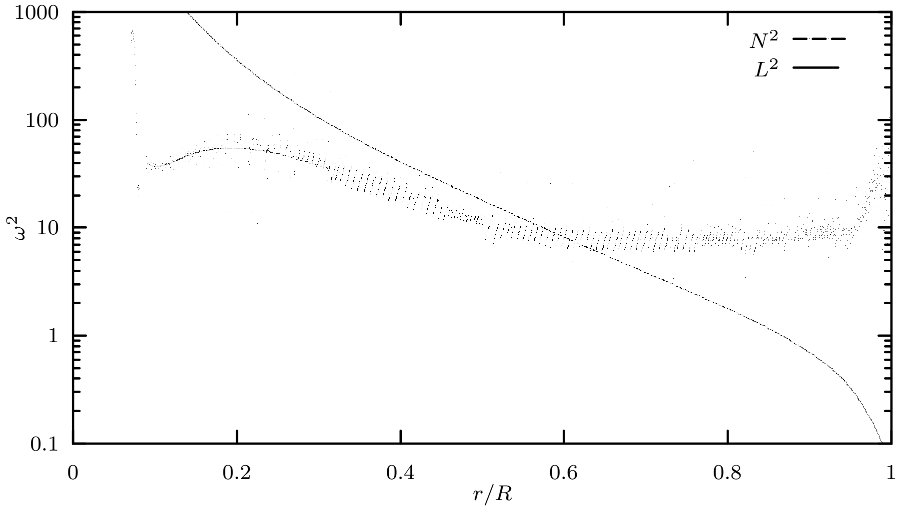

Figure II.1: Propagation diagram for l=3 , $v = 46kms^{-1}$

, the oscillations are of p-modes while in case of $\sigma^2 < N^2, L^2$ the oscillations are of g-modes . If σ^2 is lying between L^2 and N^2 , we obtain a mixed case f-mode . The eigenfunctions corresponding to these modes are proportional to the spherical harmonics $Y_l^m(\theta, \phi)$, here a spherical polar coordinate system is used to describe the eigenfunctions . The perturbation of a physical variable(quantity) is proportional to $Y_l^m(\theta, \phi) \exp(i\sigma t)$, this perturbation can be written as

$$f'(r, \theta, \phi) = f'(r) Y_l^m(\theta, \phi) \exp(i\sigma t). \tag{II.8}$$

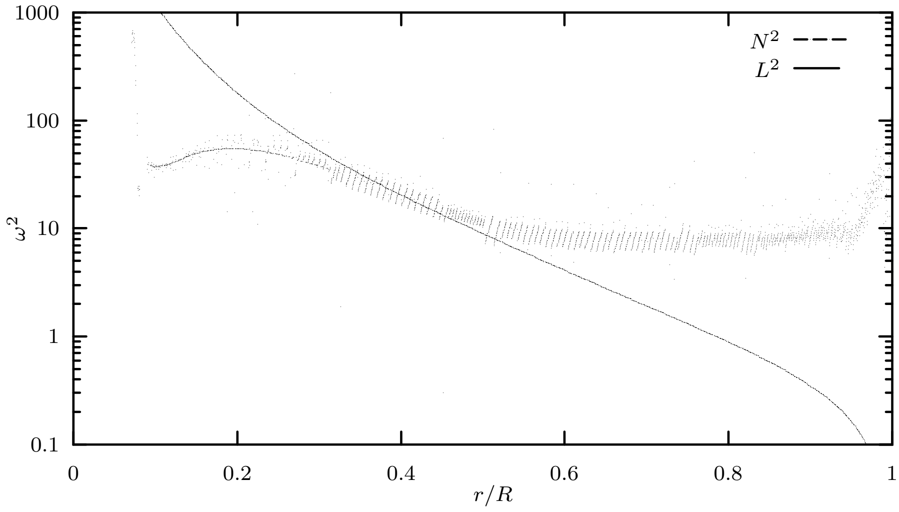

Figure II.2: Propagation diagram for l=2 , $v = 46kms^{-1}$

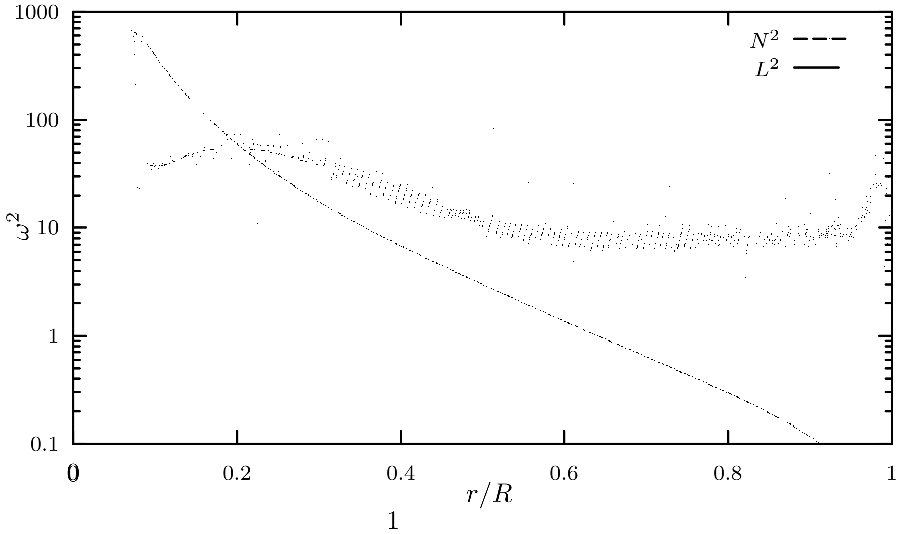

Figure II.3: Propagation diagram for l=1 , $v = 46kms^{-1}$

The position of the mass element is represented by a three dimentional vector , namely , the displacement vector as

$$\vec{\xi} = [\xi_r, \xi_h \frac{\partial}{\partial\theta}, \frac{\xi_h}{\sin\theta}\frac{\partial}{\partial\phi}] Y_l^m(\theta,\phi)\exp(i\sigma t) \tag{II.9}$$

where ξ_h is given by

$$\xi_h = \frac{1}{\sigma^2 r}(\frac{p'}{\rho} + \varphi') \tag{II.10}$$

II.2 Lagrangian and Eulerian perturbations

There are two basic coordinate systems which may be employed or used to describe the motion of a fluid mass and to derive the main conservation laws , i.e , the conservation of mass , energy and momentum . The first one is called the eulerian in which the spatial coordinates (x , y , z) are considered to be independent variables . The second coordinate system is called the lagrangian in which the coordinates (x , y , z) are no longer independent , they are a function of time , since in this case we are observing the motion of a fluid from acertain frame of reference . The velocity vector is a function of time and position with rspect to the first point of view while it is a function of time only with reaspect to the second . The substantial (material) derivative is used to signify the rate of change with respect to time , as seen by an observer moving with the particle , it is denoted by (d/dt) . The rate of change with respect to time at a fixed position in space is denoted by $(\partial/\partial t)$. If we assume that 'f' is our quantity , its variations can be represented as

$$f'(\vec{r},t) = f(\vec{r},t) - f_0(\vec{r},t) \tag{II.11}$$

$$\delta f = f(\vec{r},t) - f_0(\vec{r_0},t) \tag{II.12}$$

the above two equations relates to the eulerian and lagrangian perspectives respectively . The relation between the two perturbations is written as

$$\delta f = f' + (\delta\vec{r}).\vec{\nabla} f_0 \tag{II.13}$$

the above equation can be rewritten as ,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}. \tag{II.14}$$

The Taylor sereis expansion may be employed to get the above equation , the displacement vector is taken to the first order , in addition the linear terms in $\delta\vec{r}$ were considered . For more details see (Cox , 1980) .

II.3 Basic equations of hydrodynamics

A fluid dynamical system can be described by basic equations called the conservation laws , i.e , the conservation of mass (the continuity equation) , momentum (the equation of motion) and energy . In addition we have Poisson equation which is needed as a supplementary equation to comlete the stellar structure equations . By considering a nonrotating , nonmagnetic star without convection , these equations are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0 \tag{II.15}$$

$$\rho(\frac{\partial}{\partial t} + \vec{v}.\vec{\nabla})\vec{v} = -\vec{\nabla}p - \rho\vec{\nabla}\varphi \tag{II.16}$$

$$\frac{\partial E}{\partial t} + \vec{v}.\vec{\nabla}E + \frac{p}{\rho}\vec{\nabla}.\vec{v} = T\frac{ds}{dt} \tag{II.17}$$

$$\nabla^2\varphi = 4\pi G\rho \tag{II.18}$$

where φ, s, p, ρ, E and T are the gravitational potential , specific intropy , pressure , density , internal energy and temperature respectively . In fact these equations constitute nonlinear , nonadiabatic system of partial defferential equations

. The assumption of adiabaticity is used as an approximation , it is charactarized by the following equations

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta \rho}{\rho} \tag{II.19}$$

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta \rho}{\rho} \tag{II.20}$$

$$\frac{ds}{dt} = 0 \tag{II.21}$$

here Γ_1 and Γ_2 are the adiabatic exponents given by

$$\Gamma_3 = (\frac{\partial \ln T}{\partial \ln \rho})_{ad} + 1 \tag{II.22}$$

II.4 Linearized set of equations

Since equations II.15 - II.18 are nonlinear , we seek for a linearization technique (method) to tackle the problem . In such a theory , two solutions are assumed , a particular or unperturbed solution which is the principal one and the perturbed (small) solution . According to these solutions , any dependent variable can be expressed as the sum of perturbed and unperturbed solutions . Using the eularian point of view as a small perturbation , our physical dependent variables (φ, ρ, v, p) can be expressed as

$$\varphi = \varphi_0 + \varphi' \tag{II.23}$$

$$\rho = \rho_0 + \rho' \tag{II.24}$$

$$v = v_0 + v' \tag{II.25}$$

$$p = p_0 + p' \tag{II.26}$$

where the "0" subscript denotes the unperturbed quantities . Substituting these new variables in equations II.15 - II.18 and making use of the relation between eularian and lagrangian perturbations , we get , after some manipulations , the linearized form up to first order variation as

$$\frac{\partial \rho'}{\partial t} + \vec{\nabla}.(\rho_0 \vec{v'}) = 0 \quad \text{(II.27)}$$

$$\frac{\partial v'}{\partial t} = \frac{\rho'}{\rho^2}\vec{\nabla}p - \frac{1}{\rho}\vec{\nabla}p' - \nabla\vec{\varphi'} \quad \text{(II.28)}$$

$$\frac{p'}{p} = \Gamma_1 \frac{\rho'}{\rho} + \vec{\xi_r}(\frac{\Gamma_1}{\rho}\vec{\nabla}\rho - \frac{1}{p}\vec{\nabla}p) \quad \text{(II.29)}$$

$$\nabla^2 \varphi' = 4\pi G\rho'. \quad \text{(II.30)}$$

The above set of equations can be reduced to

$$\frac{1}{r^2}\frac{d}{dr}(r^2\xi_r) - \frac{g}{c^2}\xi_r + (1 - \frac{l^2}{\sigma^2})\frac{p'}{\rho c^2} = \frac{l(l+1)}{\sigma^2 r^2}\varphi' \quad \text{(II.31)}$$

$$\frac{1}{\rho}\frac{dp'}{dr} + \frac{g}{\rho c^2}p' + (N^2 - \sigma^2)\xi_r = -\frac{d\varphi'}{dr} \quad \text{(II.32)}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\varphi'}{dr}) - \frac{l(l+1)}{r^2}\varphi' = 4\pi G\rho(\frac{p'}{\rho c^2} + \frac{N^2}{g}\xi_r). \quad \text{(II.33)}$$

If we introduce the following dimensionless new variables (Dziembowski , 1971) as

$$y_1 = \frac{\xi_r}{r} \quad \text{(II.34)}$$

$$y_2 = \frac{1}{gr}(\frac{p'}{\rho} + \varphi') \quad \text{(II.35)}$$

$$y_3 = \frac{1}{gr}\varphi' \quad \text{(II.36)}$$

$$y_4 = \frac{1}{g}\frac{d\varphi'}{dr} \quad \text{(II.37)}$$

Substituting equations (II.34 - II.37)into equations (II.31- II.33) we get the following four first-order differentiall equations

$$x\frac{dy_1}{dx} = (V_g - 3)y_1 + [\frac{l(l+1)}{c_1\omega^2} - V_g]y_2 + V_g y_3 \quad \text{(II.38)}$$

$$x\frac{dy_2}{dx} = (c_1\omega^2 - A^*)y_1 + (A^* - U + 1)y_2 - A^* y_3 \quad \text{(II.39)}$$

$$x\frac{dy_3}{dx} = (1-U)y_3 + y_4 \tag{II.40}$$

$$x\frac{dy_4}{dx} = UA^*y_1 + UV_g y_2 + [l(l+1) - UV_g]y_3 - Uy_4 \tag{II.41}$$

where we have

$$V_g = \frac{V}{\Gamma_1} = \frac{-1}{\Gamma_1}\frac{d\ln p}{dr} = \frac{gr}{c^2} \tag{II.42}$$

$$c_1 \equiv (r/R)^3/(M_r/M) \tag{II.43}$$

$$A^* = -rA = rg^{-1}N^2. \tag{II.44}$$

Here A is the Schwarzschild discriminant given by

$$A = \frac{d\ln\rho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln p}{dr} \tag{II.45}$$

. Equations II.38 - II.41 are subjected to the following boundary conditions

$$\frac{c_1\omega^2}{l}y_1 - y_2 = 0 \tag{II.46}$$

$$ly_3 - y_4 = 0 \tag{II.47}$$

$$(l+1)y_3 + y_4 = 0 \tag{II.48}$$

$$y_1 - y_2 + y_3 = 0. \tag{II.49}$$

Equations II.46 and II.47 are related to the centre of the star while the last two ones are related to the surface .

CHAPTER III

FIRST ORDER ROTATIONAL EFFECT

Since the eigenfrequencies and the corresponding eigenvectors for a nonrotating oscillating star are independent of the quantum number (m) , the degeneracy of the oscillating frequency will be removed (splitted) under the first order rotational effect , where the rotational frequency (Ω) is supposed to be uniform and small . Many studies have been made for first order effect , for example (Lee and Strohmayer , 1996) , (Gough and Thompson , 1990) and (Simon , 1969) In this chapter we will discuss the change of both the the eigenfunctions and eigenfrequencies under the Coriolis force effect making use of the Cowling approximation condition (Cowling , 1941) .

III.1 The change of the basic equations

The momentum equation will be affected by an extra term due to the Coriolis force effect , with respect to a rotational frame of reference we have ,

$$(\frac{d\vec{v}}{dt})_r = (\frac{d\vec{v}}{dt})_0 + 2(\vec{\Omega} \times \vec{v}) \tag{III.1}$$

the linearized momentum equation II.28 now can be written as

$$\frac{\partial v'}{\partial t} + 2(\vec{\Omega} \times \vec{v'}) = -\frac{1}{\rho}\vec{\nabla p'} + \frac{\rho'}{\rho^2}\vec{\nabla p} - \vec{\nabla \varphi'} \tag{III.2}$$

this equation can be rewritten again as

$$-\sigma'^2\vec{\xi} + 2i\sigma'(\vec{\Omega} \times \vec{\xi}) = -\frac{1}{\rho}\vec{\nabla p'} + \frac{\rho'}{\rho^2}\vec{\nabla p} - \vec{\nabla \varphi'} \tag{III.3}$$

where we have made use of the following relations

$$\delta\vec{r} = \delta r \exp i\sigma t \tag{III.4}$$

$$\vec{v'} = \frac{d}{dt}(\delta r) = i\sigma\delta r \exp i\sigma t \tag{III.5}$$

$$\frac{d}{dt}(\vec{v'}) = -\sigma^2\delta r \exp i\sigma t \tag{III.6}$$

III.2 The perturbed momentum equation

The eigenvalue frequency (σ') and the eigenfunction (f) can be expanded in powers of the rotational frequency (Ω) (Simon , 1969) as

$$\sigma' = \sigma_0 + \sigma_1 \tag{III.7}$$

$$f = f_0 + f_1 \tag{III.8}$$

accordingly the displacement vector ($\vec{\xi}$) reads

$$\vec{\xi} = \vec{\xi_0} + \vec{\xi_1} \tag{III.9}$$

where σ_1 , f_1 , $\vec{\xi_1}$ are in order of Ω while σ_0 , f_0 , $\vec{\xi_0}$ have the zeroth order . Substituting the values of σ' , $\vec{\xi}$ into equation III.3 we get

$$-(\sigma_0+\sigma_1)^2(\vec{\xi_0}+\vec{\xi_1})+[2i(\sigma_0+\sigma_1)(\vec{\Omega}\times(\vec{\xi_0}+\vec{\xi_1})] = -\frac{1}{\rho}\vec{\nabla}(p_0+p_1)+\frac{(\rho_0+\rho_1)}{\rho^2}\vec{\nabla p}-\vec{\nabla}(\varphi_0+\varphi_1) \tag{III.10}$$

up to first order the first left-hand side of the above equation reads

$$-(\sigma_0+\sigma_1)^2(\vec{\xi_0}+\vec{\xi_1}) = -(\sigma_0^2+\sigma_1^2+2\sigma_0\sigma_1)(\vec{\xi_0}+\vec{\xi_1}) \quad \text{(III.11)}$$

while the second term gives

$$2i(\sigma_0+\sigma_1)[\vec{\Omega}\times(\vec{\xi_0}+\vec{\xi_1})] = 2i\sigma_0(\vec{\Omega}\times\vec{\xi_0}) \quad \text{(III.12)}$$

the final form of the momentum equation becomes

$$-\sigma_0^2\vec{\xi_0}-\sigma_0^2\vec{\xi_1}-2\sigma_0\sigma_1\vec{\xi_0}+2i\sigma_0(\vec{\Omega}\times\vec{\xi_0}) = \frac{-1}{\rho}\vec{\nabla}p_0+\frac{\rho_0}{\rho^2}\vec{\nabla}p-\vec{\nabla}\phi_0+\frac{-1}{\rho}\vec{\nabla}p_1+\frac{\rho_1}{\rho^2}\vec{\nabla}p-\vec{\nabla}\varphi 1 \quad \text{(III.13)}$$

III.3 The change of the eigenfrequencies

The displacement vector ($\vec{\xi}$) and any of the eigenfunctions , say ($\vec{f_1}$) can be expanded in terms of their zeroth order (Zahn , 1966) as

$$\vec{\xi_1} = \sum_n a_n\vec{\xi_0} \quad \text{(III.14)}$$

$$\vec{f_1} = \sum_n a_n\vec{f_0} \quad \text{(III.15)}$$

where the coefficient (a_n) is given by

$$a_n = \int_0^M \vec{\xi_1}.\vec{\xi_0^*}dM_r \quad \text{(III.16)}$$

where the integral has been taken over the hole star . The zeroth and first order order of III.13 can be written as

$$-\sigma_0^2\vec{\xi_0} = \frac{-1}{\rho}\vec{\nabla}p_0+\frac{\rho_0}{\rho^2}\vec{\nabla}p-\vec{\nabla}\varphi_0 \quad \text{(III.17)}$$

$$-\sigma_0^2\vec{\xi_1}-2\sigma_0\sigma_1\vec{\xi_0}+2i\sigma_0(\vec{\Omega}\times\vec{\xi_0}) = \frac{-1}{\rho}\vec{\nabla}p_1+\frac{\rho_1}{\rho^2}\vec{\nabla}p-\vec{\nabla}\varphi_1 \quad \text{(III.18)}$$

if we expand the first order quantities φ_1, ρ_1, p_1 in terms of their zeroth order φ_0, ρ_0, p_0 we get

$$\varphi_1 = \sum_n a_n \varphi_0 \tag{III.19}$$

$$p_1 = \sum_n a_n p_0 \tag{III.20}$$

$$\rho_1 = \sum_n a_n \rho_0 \tag{III.21}$$

making use of the zeroth order equation (III.17) we get

$$-\sigma_0^2 \sum_n a_n \vec{\xi_0} - 2\sigma_0\sigma_1\vec{\xi_0} + 2i\sigma_0(\vec{\Omega} \times \vec{\xi_0}) = \sum_n a_n(\frac{-1}{\rho_0}\vec{\nabla}p_0 + \frac{\rho_0}{\rho^2}\vec{\nabla}p - \vec{\nabla}\varphi_0) \tag{III.22}$$

after some manipulations the above equation gives

$$\sigma_1\vec{\xi_0} = i(\vec{\Omega} \times \vec{\xi_0}) \tag{III.23}$$

multiplying both sides of equation III.23 by $\vec{\xi}^*$ and integrate over the hole star we get

$$\sigma_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_0}.\vec{\xi_0}^*)r^2 \sin\theta\rho dr d\theta d\phi = i\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[(\vec{\Omega} \times \vec{\xi_0}).\vec{\xi_0^*}]r^2 \sin\theta\rho dr d\theta d\phi \tag{III.24}$$

where $\vec{\Omega}$, $\vec{\xi_0}$, $\vec{\xi_0^*}$ are given by

$$\vec{\Omega} = [\Omega\cos\theta, -\Omega\sin\theta, 0] \tag{III.25}$$

$$\vec{\xi_0} = (\xi_{0r}, \xi_{0h}\frac{\partial}{\partial\theta}, \frac{\xi_{0h}}{\sin\theta}\frac{\partial}{\partial\phi})Y_l^m(\theta,\phi)(\exp i\sigma t) \tag{III.26}$$

$$\vec{\xi_0^*} = (\xi_{0r}, \xi_{0h}\frac{\partial}{\partial\theta}, \frac{\xi_{0h}}{\sin\theta}\frac{\partial}{\partial\phi})Y_l^{m*}(\theta,\phi)(\exp -i\sigma t) \tag{III.27}$$

hence we have

$$\sigma_1 = \frac{i\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[(\vec{\Omega} \times \vec{\xi_0}).\vec{\xi_0^*}]r^2 \sin\theta\rho dr d\theta d\phi}{\sigma\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_0}.\vec{\xi_0}^*)r^2\rho \sin\theta dr d\theta d\phi} \tag{III.28}$$

the scalar product $[(\vec{\Omega} \times \vec{\xi_0}).\vec{\xi_0^*}]$ is given by

$$[\vec{\Omega} \times \vec{\xi_0}].\vec{\xi_0^*} = -im\Omega[2\xi_{0r}\xi_{0h}|Y_l^m|^2 + \xi_{0h}^2\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}|Y_l^m|^2] \tag{III.29}$$

integrate both sides of equation III.29 over the hole star we get

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_0}].\vec{\xi_0^*}\rho\sin\theta r^2drd\theta d\varphi = -im\Omega\int_0^{R}[2\xi_{0r}\xi_{0h}+\xi_{0h}^2]\rho\sin\theta r^2dr \tag{III.30}$$

and finally we obtain

$$\sigma_1 = m\Omega C \tag{III.31}$$

where C is given by

$$C = \frac{\int_0^R[2\xi_{0r}\xi_{0h}+\xi_{0h}^2]\rho r^2dr}{\int_0^R[2\xi_{0r}^2+l(l+1)\xi_{0h}^2]\rho r^2dr} \tag{III.32}$$

now according to equation III.7 we have

$$\sigma' = \sigma_0 + m\Omega C \tag{III.33}$$

where σ' is the frequency of oscillations in the rotating system , and with respect to an inertial frame of reference, the frequency σ is given by

$$\sigma = \sigma' - m\Omega \tag{III.34}$$

and then

$$\sigma = \sigma_0 - m\Omega(1-C) \tag{III.35}$$

For rotational models of stars , the nonradial oscillations modes are described by couple infinity number of differential equations , so just the r - component of these oscillations are taken and under the effect of slow rotation equations II.37 - II.40 can be written as

$$r\frac{dY_1}{dr} = (\frac{V}{\Gamma_1}-3)+[\frac{l(l+1)}{C_1\omega_0^2}-\frac{V}{\Gamma_1}]Y_2+\frac{V}{\Gamma_1}Y_3+\frac{2m\Omega}{\sigma_0}\{Y_{0,1}+[\frac{1}{C_1\omega_0^2}-\frac{\sigma_1}{m\Omega}\frac{l(l+1)}{C_1\omega_0^2}]Y_{0,2}\} \tag{III.36}$$

$$r\frac{dY_2}{dr} = (C_1\omega_0^2+rA)Y_1+(1-U-rA)Y_2+rAY_3+\frac{2m\Omega}{\sigma_0}(\frac{\sigma_1}{m\Omega}C_1\omega_0^2Y_{0,1}-Y_{0,2}) \tag{III.37}$$

$$r\frac{dY_3}{dr} = (1-U)Y_3 + Y_4 \tag{III.38}$$

$$r\frac{dY_3}{dr} = -UrAY_1 + \frac{UV}{\Gamma_1}Y_2 + [l(l+1) - \frac{UV}{\Gamma_1}]Y_3 - UY_4 \tag{III.39}$$

where

$$V = -\frac{d\ln p}{d\ln r} = \frac{gr}{C^2} \tag{III.40}$$

$$U = -\frac{d\ln M_r}{d\ln r} = \frac{4\pi\rho r^3}{M_r} \tag{III.41}$$

$$C_1 = (\frac{r}{R})^3 \tag{III.42}$$

$$\omega_0^2 = \frac{\sigma_0^2 R^3}{GM} \tag{III.43}$$

$$A = \frac{d\ln p}{d\ln r} - \frac{1}{\Gamma_1}\frac{d\ln p}{dr} \tag{III.44}$$

and $Y_{0,i}(i = 1, ..., 4)$ are the variables in the nonrotating case . Our equations are subject to the following boundary conditions

$$C_1\omega_0^2 Y_1 - lY_2 + \frac{2m\Omega}{\sigma_0}C_1\omega_0^2[\frac{\sigma_1}{m\Omega} - \frac{1}{l}]Y_{0,1} = 0 \tag{III.45}$$

$$lY_3 - Y_4 = 0 \tag{III.46}$$

$$(1 - (\frac{4 + C_1\omega_0^2}{V})Y_1 + \{\frac{l(l+1)}{C_1\omega_0^2} - 1\}Y_2 + (1 - \frac{l+1}{V})Y_3$$
$$+\frac{2m\Omega}{\sigma_0}\{1 - \frac{\sigma_1}{m\Omega}C_1\omega_0^2)Y_{0,1} + [1 - \frac{\sigma_1}{m\Omega}l(l+1) + C_1\omega_0^2]\frac{Y_{0,2}}{C_1\omega_0^2} = 0 \tag{III.47}$$

$$UY_1 + (l+1)Y_3 + Y_4 = 0 \tag{III.48}$$

where equations III.45 , III.46 are related to the centre of the star while the last two ones are related to the surface . Under the cowling approximation (Cowling , 1941) equations III.36 - III.39 can be reduced to the two following equations

$$r\frac{dY_1}{dr} = (\frac{V}{\Gamma_1} - 3) + [\frac{l(l+1)}{C_1\omega_0^2} - \frac{V}{\Gamma_1}]Y_2 + \frac{V}{\Gamma_1}Y_3 + \frac{2m\Omega}{\sigma_0}\{Y_{0,1} + [\frac{1}{C_1\omega_0^2} - \frac{\sigma_1}{m\Omega}\frac{l(l+1)}{C_1\omega_0^2}]Y_{0,2}\} \tag{III.49}$$

$$r\frac{dY_2}{dr} = (C_1\omega_0^2 + rA)Y_1 + (1 - U - rA)Y_2 + rAY_3 + \frac{2m\Omega}{\sigma_0}(\frac{\sigma_1}{m\Omega}C_1\omega_0^2 Y_{0,1} - Y_{0,2}) \tag{III.50}$$

these last two equations were manipulated numerically by Kırbıyık and Al-Murad by using the Runge Kutta method (Kırbıyık and Al-Murad , 1995) .

CHAPTER IV

SECOND ORDER ROTATIONAL EFFECT

As mentioned in chapter three , the degeneracy of the oscillating frequencies for nonradial oscillation will be removed or splitted according to the quantum number m , the same thing can be said here concerning the second order because this degeneracy is due to the effect of the centrifugal force . The resolution of the degeneracy will give different frequencies for a given mode . The rotational frequency is uniform , but here it is supposed to be higher (faster) . the displacement vector($\vec{\xi}$) is no longer a function of the radial distance (r) as in the radial oscillation case , but a function of the angular variables (θ, ϕ) . Accurate treatment for the second order effect were done by (Chelebowski , 1978) , (Gough and Thompson , 1990) and (Sofi et al , 1998) In this chapter we will discuss the change in the momentum equation and the eigenfrequencies by giving a detailed derivation for the second order eigenfrequency (σ_2) .

IV.1 The change of the momentum equation

Like the first order case , the momentum equation will be affected by an additional term due to the centrifugal force . With respect to a rotating frame of reference it takes the form

$$(\frac{d\vec{v}}{dt})_r = (\frac{d\vec{v}}{dt})_0 + 2(\vec{\Omega} \times \vec{v}) + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{IV.1}$$

the third term of the right hand side of above equation represents the centrifugal force effect and by using the material derivative concept the equation reads

IV.2 The Perturbed Momentum Equation

If we expand the eigenfrequency (σ') and the displacement vector ($\vec{\xi}$) in powers of the rotational frequency (Ω) as

$$\sigma' = \sigma_0 + \sigma_1 + \sigma_2 \tag{IV.3}$$

$$\vec{\xi} = \vec{\xi_0} + \vec{\xi_1} + \vec{\xi_2} \tag{IV.4}$$

where σ_2 and $\vec{\xi_2}$ are of the order of (Ω^2) . Substituting the values of σ , $\vec{\xi}$ into equation **??** we get

$$-(\sigma_0 + \sigma_1 + \sigma_2)(\vec{\xi_0} + \vec{\xi_1} + \vec{\xi_2}) + 2i(\sigma_0 + \sigma_1 + \sigma_2)[\vec{\Omega} \times (\vec{\xi_0} + \vec{\xi_1} + \vec{\xi_2})] =$$
$$-\frac{1}{\rho}\vec{\nabla}(p_0 + p_1 + p_2) + \frac{(\rho_0 + \rho_1 + \rho_2)}{\rho^2}\vec{\nabla}p - \vec{\nabla}(\varphi_0 + \varphi_1 + \varphi_2) \tag{IV.5}$$

up to second order the first left hand - side term of the above equation gives

$$-\sigma'^2\vec{\xi} = -\sigma_0^2\vec{\xi_2} - \sigma_1^2\vec{\xi_0} - 2\sigma_0\sigma_1\vec{\xi_1} - 2\sigma_0\sigma_2\vec{\xi_0} \tag{IV.6}$$

while the second term gives

$$2i\sigma'(\vec{\Omega} \times \vec{\xi}) = 2i\sigma_0(\vec{\Omega} \times \vec{\xi_0}) + 2i\sigma_1(\vec{\Omega} \times \vec{\xi_0}) \tag{IV.7}$$

and the third term can be written as

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \vec{\Omega} \cdot (\vec{\Omega} \times \vec{r}) - \Omega^2 \vec{r} \tag{IV.8}$$

similar to first order case , the displacement vector $\vec{\xi_2}$ and the eigenfunction $\vec{f_2}$ can be expanded as

$$\vec{\xi_2} = \sum_n a_n \vec{\xi_0} \tag{IV.9}$$

$$\vec{f_2} = \sum_n a_n \vec{f_0} \tag{IV.10}$$

and hence we have

$$-\sigma_0^2\vec{\xi_2} - \sigma_1^2\vec{\xi_0} - 2\sigma_0\sigma_1\vec{\xi_1} - 2\sigma_0\sigma_2\vec{\xi_0} + 2i\sigma_0(\vec{\Omega} \times \vec{\xi_1})$$
$$+2i\sigma_1(\vec{\Omega} \times \vec{\xi_0}) + \vec{\Omega} \cdot (\vec{\Omega} \times \vec{r}) - \Omega^2\vec{r} = -\frac{1}{\rho}\vec{\nabla}(p_2) + \frac{(\rho_2)}{\rho^2}\vec{\nabla}p - \vec{\nabla}(\varphi_2) \tag{IV.11}$$

if we expand the quantities φ_2 , p_2 , ρ_2 , in terms of their zeroth values φ_0 , p_0 , ρ_0 as

$$\varphi_2 = \sum_n a_n \varphi_0 \tag{IV.12}$$

$$p_2 = \sum_n a_n p_0 \tag{IV.13}$$

$$\rho_2 = \sum_n a_n \rho_0 \tag{IV.14}$$

we obtain

$$-\sigma_1^2\vec{\xi_0} - 2\sigma_0\sigma_1\vec{\xi_1} - 2\sigma_0\sigma_2\vec{\xi_0} + 2i\sigma_0(\vec{\Omega} \times \vec{\xi_1})$$
$$+2i\sigma_1(\vec{\Omega} \times \vec{\xi_0}) + \vec{\Omega} \cdot (\vec{\Omega} \times \vec{r}) - \Omega^2\vec{r} = 0 \tag{IV.15}$$

where we have made use of equations III.17 - III.18 . Dotting equation IV.15 by the complex conjugate $\vec{\xi_0^*}$ and integrate over the hole star , we obtain

$$\sigma_1^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_0}\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi + 2\sigma_0\sigma_2\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_0}\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi$$
$$+2\sigma_0\sigma_1\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_1}\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi = +2i\sigma_0\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_1}].\vec{\xi_0^*}\rho r^2\sin\theta drd\theta d\phi$$
$$-2i\sigma_1\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_0}].\vec{\xi_0^*}\rho r^2\sin\theta drd\theta d\phi + \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\Omega}\cdot\vec{\xi_0}).(\Omega\cdot\vec{r})\rho r^2\sin\theta drd\theta d\phi$$
$$-\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\Omega^2(\vec{r}\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi \tag{IV.16}$$

By definition we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_1}\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi = <\vec{\xi_1} \mid \vec{\xi_0^*}> = 0 \tag{IV.17}$$

where the notation $<\vec{\xi_1} \mid \vec{\xi_0}>$ relates to inner product vector space . If we define the quantity J as

$$J = \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\xi_0}\cdot\vec{\xi_0^*})\rho r^2 drd\theta d\phi \tag{IV.18}$$

we can rewrite the first order rotational frequency σ_1 in terms of J as

$$\sigma_1 = +\frac{i}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_0}].\vec{\xi_0^*}\rho r^2\sin\theta drd\theta d\phi \tag{IV.19}$$

and equation IV.16 again reads

$$\sigma_1^2 J + 2\sigma_0\sigma_2 J = 2i\sigma_0\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_1}].\vec{\xi_0^*}\rho r^2\sin\theta drd\theta d\phi$$
$$-\Omega^2\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[r\cdot\vec{\xi_0^*}]\rho r^2\sin\theta drd\theta d\phi + 2\sigma_1^2 J + \int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\Omega}\cdot\vec{\xi_0^*})(\Omega\,\vec{\cdot}\,\xi_0^*)\rho r^2\sin\theta drd\theta d\phi \tag{IV.20}$$

dividing by J we obtain

$$\sigma_1^2 + 2\sigma_0\sigma_2 = +\frac{2i}{J}\sigma_0\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_1}].\vec{\xi_0^*}\rho r^2\sin\theta drd\theta d\phi$$
$$+\frac{1}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\Omega}\cdot r)(\vec{\Omega}\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi + 2\sigma_1^2 - \frac{\Omega^2}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(r\cdot\vec{\xi_0^*})\rho r^2\sin\theta drd\theta d\phi$$

(IV.21

so σ_2 can be expressed as

$$\sigma_2 = \frac{\sigma_1^2}{2\sigma_0} + \frac{i}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_1}].\vec{\xi_0^*}\rho r^2 \sin\theta dr d\theta d\phi + \frac{1}{2\sigma_0 J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\Omega}\cdot r)(\vec{\Omega}\cdot\vec{\xi_0^*})\rho r^2 \sin\theta dr d\theta d\phi - \frac{\Omega^2}{2\sigma_0 J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(r\cdot\vec{\xi_0^*})\rho r^2 \sin\theta dr d\theta d\phi \quad \text{(IV.22)}$$

now let

$$\sigma_2 = \sigma_{21} + \sigma_{22} + \sigma_{23} + \sigma_{24} \quad \text{(IV.23)}$$

where we have

$$\sigma_{21} = \frac{\sigma_1^2}{2\sigma_0} \quad \text{(IV.24)}$$

$$\sigma_{22} = +\frac{i}{J}\sigma_0\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[\vec{\Omega}\times\vec{\xi_1}].\vec{\xi_0^*}\rho r^2 \sin\theta dr d\theta d\phi \quad \text{(IV.25)}$$

$$\sigma_{23} = -\frac{\Omega^2}{2\sigma_0 J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(r\cdot\vec{\xi_0^*})\rho r^2 \sin\theta dr d\theta d\phi \quad \text{(IV.26)}$$

$$\sigma_{24} = \frac{1}{2\sigma_0 J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\vec{\Omega}\cdot r)(\vec{\Omega}\cdot\vec{\xi_0^*})\rho r^2 \sin\theta dr d\theta d\phi \quad \text{(IV.27)}$$

IV.3 Finding the displacement vector

The general linearized equation of motion in spherical polar coordinates (r , θ , ϕ) under the effect of rotation can be expressed as

$$-2\sigma_0\sigma_1\vec{\xi_0} - \sigma_0^2\vec{\xi_1} = -2i\sigma_0(\vec{\Omega}\times\vec{\xi_0}) - \vec{\nabla}\chi_1 + \frac{\Gamma_1 p}{\rho}(\vec{\nabla}\cdot\vec{\xi_1})\vec{A} \quad \text{(IV.28)}$$

where $\vec{A}$ is given by

$$\vec{A} = \frac{1}{\rho}\vec{\nabla}\rho + \frac{1}{\Gamma_1}p\vec{\nabla}p \quad \text{(IV.29)}$$

where we made use of the spherical symmetry property given by

$$-\frac{1}{\rho^2}(\vec{\nabla}p\cdot\vec{\xi})\vec{\nabla}\rho+\frac{1}{\rho^2}(\vec{\nabla}\rho\cdot\vec{\xi})\vec{\nabla}p=0. \quad \text{(IV.30)}$$

For a spherically symmetric star , the quantity $\vec{A}$ has only radial component , i.e , $\vec{A}=A\hat{e_r}$, and χ_1 is given by

$$\chi_1=\frac{p_1}{\rho}+\varphi_1 \quad \text{(IV.31)}$$

and hence $\vec{\xi_1}$ can be written as

$$\vec{\xi_1}=\frac{2i}{\sigma_0}(\vec{\Omega}\times\vec{\xi_0})+\frac{1}{\sigma_0^2}\vec{\nabla}\chi_1-\frac{\Gamma_1 p}{\rho\sigma_0}A\hat{e_r}-\frac{2\sigma_1}{\sigma_0}\vec{\xi_0}. \quad \text{(IV.32)}$$

Let us denote the four right - hand side terms by $\vec{T_1}$, $\vec{T_2}$, $\vec{T_3}$ and $\vec{T_4}$ respectively and find these terms one by one .

IV.3.1 the first term

$$\vec{T_1}=\frac{2i}{\sigma_0}(\vec{\Omega}\times\vec{\xi_0})\begin{pmatrix}\hat{e_r} & \hat{e_\theta} & \hat{e_\phi}\\ \Omega\cos\theta & -\Omega\sin\theta & 0\\ \xi_{0r}Y_l^m & \xi_{0h}\frac{\partial Y_l^m}{\partial\theta} & \xi_{0h}\frac{im}{\sin\theta}Y_l^m\end{pmatrix} \quad \text{(IV.33)}$$

by evaluating the cross product , $\vec{T_1}$ gives

$$\vec{T_1}=(\frac{2m\Omega\xi_{0h}Y_l^m}{\sigma_0})\hat{e_r}+(\frac{2i\Omega\cot\theta\xi_{0h}}{\sigma_0})\hat{e_\theta}$$
$$+[\frac{2m\Omega\cos\theta\xi_{0h}}{\sigma_0}\frac{\partial Y_l^m}{\partial\theta}+\frac{2i\Omega\sin\theta\xi_{0h}Y_l^m}{\sigma_0}]\hat{e_\phi} \quad \text{(IV.34)}$$

IV.3.2 the second term

$$\vec{T_2}=\frac{1}{\sigma_0^2}\vec{\nabla}\chi_1=\frac{1}{\sigma_0^2}[\frac{\partial\chi_1}{\partial r}\hat{e_r}+\frac{1}{r}\frac{\partial\chi_1}{\partial\theta}\hat{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial\chi_1}{\partial\phi}\hat{e_\phi}] \quad \text{(IV.35)}$$

if we define the term b_1 as

$$b_1 = \frac{\chi_1}{\sigma_0^2 r Y_l^m} \tag{IV.36}$$

we get

$$\vec{T_2} = \frac{1}{\sigma_0^2}\frac{\partial \chi_1}{\partial r}\hat{e_r} + b_1\frac{\partial Y_l^m}{\partial \theta}\hat{e_\theta} + \frac{imb_1 Y_l^m}{\sin\theta}\hat{e_\phi} \tag{IV.37}$$

IV.3.3 the third term

The term $\vec{T_3}$ is straight forward it is given by the following relation

$$\vec{T_3} = -\frac{\Gamma_1 p}{\rho\sigma_0}(\vec{\nabla}\cdot\vec{\xi_1})A\hat{e_r} \tag{IV.38}$$

IV.3.4 the forth term

The term $\vec{T_4}$ is given by

$$\vec{T_4} = \frac{-2\sigma_1}{\sigma_0}[\xi_{0r}Y_l^m\hat{e_r}, \xi_{0h}\frac{\partial Y_l^m}{\partial\theta}\hat{e_\theta}, \frac{im\xi_{0h}}{\sin\theta}Y_l^m\hat{e_\phi}] \tag{IV.39}$$

IV.3.5 The components of the displacement vector

There are three components for $\vec{\xi_1}$ which are given as follow

$$\vec{\xi_{1r}} = [\frac{2m\Omega\xi_{0h}}{\sigma_0} + \frac{1}{\sigma_0^2}\frac{\partial R}{\partial r} - \frac{2\sigma_1\xi_{0r}}{\sigma_0} - \frac{\Gamma_1 p}{\rho\sigma_0}(\vec{\nabla}\cdot\vec{\xi_1})B]Y_l^m \tag{IV.40}$$

where we have $B = A/Y_l^m$

$$\vec{\xi_{1\theta}} = (b_1 - \frac{2\sigma_1\xi_{0h}}{\sigma_0})\frac{\partial Y_l^m}{\partial\theta} + \frac{2m\Omega\xi_{0h}}{\sigma_0}\cot\theta Y_l^m \tag{IV.41}$$

$$\vec{\xi_{1\phi}} = (b_1 - \frac{2\sigma_1\xi_{0h}}{\sigma_0})\frac{imY_l^m}{\sin\theta} + \frac{2i\Omega}{\sigma_0}[\xi_{0r}\sin\theta Y_l^m + \xi_{0h}\cos\theta\frac{\partial Y_l^m}{\partial\theta}] \tag{IV.42}$$

IV.4 The change of the eigenfrequencies

In this part we will find the value of σ_2 by evaluating the terms σ_{21} , σ_{22} , σ_{23} and σ_{24} one by one .

IV.4.1 the term σ_{22}

According to equation IV.25 the quantity $[\vec{\Omega} \times \vec{\xi_1}].\vec{\xi_0^*}$ is given by

$$[\vec{\Omega} \times \vec{\xi_1}].\vec{\xi_0^*} = \begin{vmatrix} \xi_{0r} Y_l^{m^*} & \frac{\xi_{0h} \partial Y_l^{m^*}}{\partial \theta} & \frac{\xi_{0h} \partial Y_l^{m^*}}{\sin\theta \partial \phi} \\ \Omega \cos\theta & -\Omega \sin\theta & 0 \\ \xi_{1r} & \xi_{1\theta} & \xi_{1\phi} \end{vmatrix} \tag{IV.43}$$

now let us denote the quantity $[\vec{\Omega} \times \vec{\xi_1}].\vec{\xi_0^*}$ by the term S , then we will have

$$S = -\xi_{0h}\xi_{1\phi}\Omega\cos\theta\frac{\partial Y_l^{m^*}}{\partial\theta} + \frac{im}{\sin\theta}\xi_{0h}\xi_{1\theta}\Omega\cos\theta Y_l^{m^*}$$
$$-\xi_{0r}\xi_{1\phi}\Omega\sin\theta Y_l^{m^*} + im\Omega\xi_{0h}\xi_{1r}Y_l^{m^*}Y_l^m \tag{IV.44}$$

again if we denote for the write hand - side terms of the above equation by S_1 , S_2 , S_3 and S_4 we will get

$$\sigma_{22} = +\frac{i}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^R (S_1 + S_2 + S_3 + S_4)\rho r^2 \sin\theta dr d\theta d\phi \tag{IV.45}$$

where we have

$$S_1 = im\Omega\xi_{0h}\xi_{1r}Y_l^{m^*} \tag{IV.46}$$

$$S_2 = \frac{im}{\sin\theta}\xi_{0h}\xi_{1\theta}\Omega\cos\theta Y_l^{m^*} \tag{IV.47}$$

$$S_3 = -\xi_{0r}\xi_{1\phi}\Omega\sin\theta Y_l^{m^*} \tag{IV.48}$$

$$S_4 = -\xi_{0h}\xi_{1\phi}\Omega\cos\theta\frac{\partial Y_l^{m^*}}{\partial\theta} \tag{IV.49}$$

sustituting the values of ξ_{1r} , $\xi_{1\theta}$ and $\xi_{1\phi}$ from equations IV.40 - IV.42 into equations IV.46 - IV.49 we obtain

$$S_1 = -im\Omega\xi_{0h}Y_l^{m^*}Y_l^m[\frac{2m\Omega\xi_{0h}}{\sigma_0} + \frac{1}{\sigma_0^2}\frac{\partial R}{\partial r} - \frac{2\sigma_1\xi_{0r}}{\sigma_0} - \frac{\Gamma_1 p}{\rho\sigma_0}(\vec{\nabla}\cdot\vec{\xi_1})B] \tag{IV.50}$$

$$S_2 = \frac{-im}{\sin\theta}\xi_{0h}\Omega\cos\theta Y_l^{m^*}[(b_1 - \frac{2\sigma_1\xi_{0h}}{\sigma_0})\frac{\partial Y_l^m}{\partial\theta} + \frac{2m\Omega\xi_{0h}}{\sigma_0}\cot\theta Y_l^m] \qquad \text{(IV.51)}$$

$$S_3 = -\xi_{0r}\Omega\sin\theta Y_l^{m^*}[(b_1 - \frac{2\sigma_1\xi_{0h}}{\sigma_0})\frac{imY_l^m}{\sin\theta} + \frac{2i\Omega}{\sigma_0}(\xi_{0r}\sin\theta Y_l^m + \xi_{0h}\cos\theta\frac{\partial Y_l^m}{\partial\theta})] \qquad \text{(IV.52)}$$

$$S_4 = -\xi_{0h}\Omega\cos\theta\frac{\partial Y_l^{m*}}{\partial\theta}[(b_1 - \frac{2\sigma_1\xi_{0h}}{\sigma_0})\frac{imY_l^m}{\sin\theta} + \frac{2i\Omega}{\sigma_0}(\xi_{0r}\sin\theta Y_l^m + \xi_{0h}\cos\theta\frac{\partial Y_l^m}{\partial\theta})] \qquad \text{(IV.53)}$$

substituting the last values of S_1 , S_2 , S_3 and S_4 in equation IV.45 we obtain the following seven integrals denoted by $(I_{1t}, I_{2t}, I_{3t}, I_{4t}, I_{5t}, I_{6t}, I_{7t})$ respectively

$$\frac{-i}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(im\Omega\xi_{0h}\xi_{1r}|Y_l^m|^2)r^2\sin\theta drd\theta d\phi = \frac{2m^2\Omega^2 I_1}{J\sigma_0}\int_0^R(\xi_{0h}^2 - \xi_{0h}\xi_{0r})r^2\rho dr \qquad \text{(IV.54)}$$

$$\frac{-i}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(im\Omega\xi_{0h}C_1\cot\theta\frac{\partial}{\partial\theta}|Y_l^m|^2)r^2\sin\theta drd\theta d\phi = \frac{-2m^2\Omega^2 CI_2}{J\sigma_0}\int_0^R(\xi_{0h}^2)r^2\rho dr \qquad \text{(IV.55)}$$

$$\frac{-i}{J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(im\Omega\xi_{0r}C_1|Y_l^m|^2)r^2\sin\theta drd\theta d\phi = \frac{-2m^2\Omega^2 CI_3}{J\sigma_0}\int_0^R(\xi_{0h}\xi_{0r})r^2\rho dr \qquad \text{(IV.56)}$$

$$\frac{2\Omega^2}{J\sigma_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\xi_{0r}\xi_{0h}\sin\theta\cos\theta\frac{\partial}{\partial\theta}|Y_l^m|^2)r^2\sin\theta drd\theta d\phi = \frac{2\Omega^2 I_4}{J\sigma_0}\int_0^R(\xi_{0h}\xi_{0r})r^2\rho dr \qquad \text{(IV.57)}$$

$$\frac{2\Omega^2}{J\sigma_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\xi_{0r}^2\sin^2\theta|Y_l^m|^2)r^2\sin\theta dr d\theta d\phi =$$
$$\frac{2\Omega^2 I_5}{J\sigma_0}\int_0^{R}(\xi_{0r}^2)r^2\rho dr \qquad \text{(IV.58)}$$

$$\frac{2m^2\Omega^2}{J\sigma_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\xi_{0h}^2\cot^2\theta|Y_l^m|^2)r^2\sin\theta dr d\theta d\phi =$$
$$\frac{2m^2\Omega^2 I_6}{J\sigma_0}\int_0^{R}(\xi_{0h}^2)r^2\rho dr \qquad \text{(IV.59)}$$

$$\frac{2\Omega^2}{J\sigma_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\xi_{0h}^2\cos^2\frac{\partial}{\partial\theta}Y_l^m\frac{\partial}{\partial\theta}Y_l^{m^*})r^2\sin\theta dr d\theta d\phi =$$
$$\frac{2\Omega^2 I_7}{J\sigma_0}\int_0^{R}(\xi_{0h}^2)r^2\rho dr \qquad \text{(IV.60)}$$

where I_1 , I_2 , I_3 , I_4 , I_5 , I_6 and I_7 are the angular integrals given by

$$I_1 = \int_0^{2\pi}\int_0^{\pi}\sin\theta|Y_l^m|^2 d\theta d\phi = 1 \qquad \text{(IV.61)}$$

$$I_2 = \int_0^{2\pi}\int_0^{\pi}\cot\theta\frac{\partial}{\partial\theta}|Y_l^m|^2 d\theta d\phi = 1 \qquad \text{(IV.62)}$$

$$I_3 = \int_0^{2\pi}\int_0^{\pi}\sin\theta|Y_l^m|^2 d\theta d\phi = 1 \qquad \text{(IV.63)}$$

$$I_4 = \int_0^{2\pi}\int_0^{\pi}\sin^2\theta\cos\theta\frac{\partial}{\partial\theta}|Y_l^m|^2 d\theta d\phi =$$
$$\frac{-3}{2l+1}[\frac{l+1-m^2}{2l+3}+\frac{l^2-m^2}{2l-1}]+1 \qquad \text{(IV.64)}$$

$$I_5 = \int_0^{2\pi}\int_0^{\pi}\sin^3\theta|Y_l^m|^2 d\theta d\phi =$$
$$1-[\frac{l^2-m^2}{4l^2-1}+\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}] \qquad \text{(IV.65)}$$

$$I_6 = \int_0^{2\pi}\int_0^{\pi} \sin\theta\cot^2\theta |Y_l^m|^2 d\theta d\phi = m - m^2\frac{(l-m)!}{(l+m)!} \tag{IV.66}$$

$$\begin{aligned} I_7 = \int_0^{2\pi}\int_0^{\pi} \sin\theta\cos^2\theta \frac{\partial}{\partial\theta}Y_l^m \frac{\partial}{\partial\theta}Y_l^{m^*} d\theta d\phi = \\ \frac{(l-m)(l-m-1)(l+m+1)^2}{4(4l^2-1)} + (l+m)(l-m+1)^2(l+m-1) \\ +\frac{(l-m)^2(l+m+1)}{4(2l+1)(2l+3)} + (l+m)^2(l-m+1)(l-m+2) \end{aligned} \tag{IV.67}$$

the last integral was obtained by making use of the following three relations

$$(1-x^2)^{\frac{1}{2}}P_l^m(\cos\theta) = \frac{1}{2}P_l^{m+1}(\cos\theta) - \frac{1}{2}(l+m)(l-m+1)P_l^{m-1}(\cos\theta) \tag{IV.68}$$

$$\sin\theta\frac{dP_l^m}{d\theta} = \frac{(l+1)(l+m)}{(2l+1)}P_{l-1}^m + \frac{(l)(l-m+1)}{(2l+1)}P_{l+1}^m \tag{IV.69}$$

$$\cos\theta\frac{dP_l^m}{d\theta} = \frac{(l+m)}{(2l+1)}P_{l-1}^m + \frac{(l)(l-m+1)}{(2l+1)}P_{l+1}^m \tag{IV.70}$$

where the equation V.69 represent the recurrence relation of the associated legender polynomial . See table V.1

IV.4.2 the term σ_{21}

This term can be given directly by

$$\sigma_{21} = \frac{m^2\Omega^2C^2}{2\sigma_0} \tag{IV.71}$$

where we made use of first order term ($\sigma_1 = m\Omega C$)

IV.4.3 the other terms

tThe sum of these two terms is given by

$$\begin{aligned} \sigma_{24} + \sigma_{23} = \frac{-\Omega^2}{2\sigma_0 J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[r^3\xi_{0r}\theta\sin^3\theta Y_l^{m^*}]\rho dr d\theta d\phi \\ -\frac{\Omega^2}{2\sigma_0 J}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}[r^3\xi_{0h}\cos\theta\sin^2\theta\frac{\partial Y_l^{m^*}}{\partial\theta}]\rho dr d\theta d\phi \end{aligned} \tag{IV.72}$$

the above equation can be rewritten as

$$\sigma_{24}+\sigma_{23}=\frac{\Omega^2 I_8}{2\sigma_0 J}\int_0^R[r^3\xi_{0r}]dr-\frac{\Omega^2 I_9}{2\sigma_0 J}\int_0^R[r^3\xi_{0h}]dr \tag{IV.73}$$

where I_8 , I_9 are the angular integrals given by

$$I_8-I_9=\int_0^{2\pi}\int_0^{\pi}[\sin^3\theta Y_l^{m^*}-\cos\theta\sin^2\theta\frac{\partial Y_l^{m^*}}{\partial\theta}]d\theta d\phi=$$
$$\delta_{m0}\delta_{l2}\frac{4\sqrt{\pi}}{3}-\delta_{m0}\delta_{l2}(\frac{4\sqrt{5\pi}}{15}+2\sqrt{\frac{4\pi}{5}}) \tag{IV.74}$$

these two integrals were evaluated by parts with the help of the following relation

$$\int_0^{2\pi}\int_0^{\pi}Y_1^1 Y_l^m Y_{l+1}^{m+1}\sin\theta d\theta d\phi=[\frac{3(l+m+1)(l+m+2)}{8\pi(2l+1)(2l+2)}]^{\frac{1}{2}}. \tag{IV.75}$$

In fact the two integrals will vanish , i.e. , they will be equal to zero except for l=2 and m=0 .

CHAPTER V

RESULTS AND CONCLUSION

Rotation is an important topic , it does not only affect the structure of stars but also affect the oscillation frequency . It has been seen that the second order perturbation effect is an important agent in changing the stellar oscillations . The rotational angular velocity of the star can be treated as a perturbation which can be studied up to second order or even up to third order (Sofi et al , 1998) .

In fact the mathematical treatment of the second order effect is greatly difficult . Since we are interesting only in the second order effect of rotation , we restricted our work at the oscillation frequencies and we did not attempt to solve the eigenfunctions when the second order is included .This may be a work for a future study .

A linear set of oscillatory equations has been developed , so the problem can be manipulated linearly under the adiabatic condition . In order to achieve our main aim we applied the modyfied program to V1162 Ori $\delta - Scuti$ star with $1.80M_{\odot}$ .In our calculations we found the terms σ_0 , σ_1 , σ_2 , σ_{1t} and σ_{2t} , where σ_{1t} and σ_{2t} are given as ($\sigma_{1t} = \sigma_0 + \sigma_1$) and ($\sigma_{2t} = \sigma_{1t} + \sigma_2$) .

The angular integral concerning (σ_2) were evaluated numerically by MATHEMATICA . The values of these integrals are tabulated in table V.1 for certain values of l and m . We see that I_8 and I_9 vanish except for l=2 , m=0 and I_6 goes to infinity for m=0 .

The radial integral parts of these terms were also evaluated numerically by the modyfied program . The total terms for σ_2 including both radial and angular parts are quoted in table V.2 for $V = 61.9 kms^{-1}$. we see that the term I_{5t} is the dominating term which has the maximum value . We can say that this term can be taken into consideration alone while finding the second order frequency and the other terms are too small that they can be neglected .

Ratios with respect to first order were also calculated for$V = 61.9 kms^{-1}$. The ratios σ_2/σ_1 and σ_{2t}/σ_{1t} are tabulated in table V.3 . We see that the ratio σ_2/σ_1 is less than one while the ratio σ_{2t}/σ_{1t} is greater than one . We can conclude that the second order term σ_2 has a considerable rotational effect and it can not be neglected .

Finally the total second order oscillation frequencies have been obtained and compared with observations for rotating and non rotating models . As seen from table V.4 , we obtained different frequencies for different modes . The agreement with observations was plausible and the order of magnitude of the absolute difference $|f_c - f_o|$ ranges from 0.01 to 0.1 for the majority of the calculated values .

Table V.1: Angular integral terms (l=1,2,3)

m	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9
1	1	1	1	2/5	4/5	1/2	3/10	0	0
0	1	1	1	-4/5	2/5	0	2/5	0	0
-1	1	1	1	2/5	4/5	1/2	3/10	0	0
2	1	1	1	4/7	6/7	1/4	3/7	0	0
1	1	1	1	-2/7	4/7	3/2	11/14	0	0
0	1	1	1	-4/7	10/21	0	18/7	$-4/3\sqrt{0.2\pi}$	$-6\sqrt{0.2\pi}$
-1	1	1	1	-2/7	4/7	3/2	11/14	0	0
-2	1	1	1	4/7	6/7	1/4	3/7	0	0
3	1	1	1	2/3	8/9	1/6	1/2	0	0
2	1	1	1	-1	2/3	3/4	1	0	0
1	1	1	1	-11/105	8/15	5/2	27/10	0	0
0	1	1	1	-56/105	22/45	0	28/5	0	0
-1	1	1	1	-11/15	8/15	5/2	27/10	0	0
-2	1	1	1	-1	2/3	3/4	1	0	0
-3	1	1	1	2/3	8/9	1/6	1/2	0	0

Table V.2: Second order terms (l=1,2,3) , $v = 61.9kms^{-1}$

m	I_{1t} $\times 10^{-7}$	I_{2t} $\times 10^{-7}$	I_{3t} $\times 10^{-7}$	I_{4t} $\times 10^{-7}$	I_{5t} $\times 10^{-7}$	I_{6t} $\times 10^{-7}$	I_{7t} $\times 10^{-7}$	σ_{21} $\times 10^{-7}$
1	-0.0024	0.008	0.4840	-0.0684	20.990	0.2416	0.1449	0.0007
0	0	0	0	0.1368	10.4950	0	0.1932	0
-1	-0.0024	0.008	0.4840	-0.0684	20.990	0.2416	0.1449	0.0007
2	-0.0375	0.0148	5.0130	-0.2826	23.310	1.2500	0.5355	0.0078
1	-0.0093	0.0037	1.2530	0.1413	15.540	1.8740	0.9818	0.0019
0	0	0	0	0.0971	12.4940	0	1.2420	0
-1	-0.0093	0.0037	1.2530	0.1413	15.540	1.8740	0.9818	0.0019
-2	-0.0375	0.0148	5.0130	-0.2826	23.310	1.2500	0.5355	0.0078
3	-0.2186	-0.1143	4.5870	0.1821	14.860	0.7836	0.2612	0.4471
2	-0.0971	-0.0507	2.039	-0.2731	11.150	1.5670	0.5224	0.1987
1	-0.0242	-0.0127	0.5097	-0.0286	8.9160	1.3060	1.4100	0.0496
0	0	0	0	0.0912	12.8272	0	2.7048	0
-1	-0.0242	-0.0127	0.5097	-0.0286	8.9160	1.3060	1.4100	0.0496
-2	-0.0971	-0.0507	2.039	-0.2731	11.150	1.5670	0.5224	0.1987
-3	-0.2186	-0.1143	4.5870	0.1821	14.860	0.7836	0.2612	0.4471

Table V.3: Eigenfrequency ratios for $(l = 1, 2, 3), v = 61.9 kms^{-1}$

m	σ_0 Rad/s	$\sigma_1 \times 10^{-4}$ Rad/s	$\sigma_2 \times 10^{-4}$ Rad/s	$\sigma_{1t} \times 10^{-2}$ Rad/s	$\sigma_{2t} \times 10^{-2}$ Rad/s	σ_2/σ_1	σ_{2t}/σ_{1t}
1	0.0012	0.4039	0.0217	0.1171	0.1174	0.0517	1.002
0	0.0012	0	0.0010	0.1212	0.1212	∞	1.008
-1	0.0012	-0.4039	0.0217	0.1250	0.1254	-0.0517	1.002
2	0.0009	0.8059	0.0298	0.0869	0.0872	0.0369	1.003
1	0.0009	0.4030	0.0197	0.0910	0.0911	0.0491	1.002
0	0.0009	0	0.0013	0.0950	0.0950	∞	1.013
-1	0.0009	-0.4030	0.0298	0.0990	0.0992	-0.0491	1.002
-2	0.0009	-0.8059	0.0197	0.1031	0.1034	-0.0369	1.003
3	0.0014	0.1610	0.0207	0.1318	0.1319	0.0179	1.002
2	0.0014	0.7743	0.0150	0.1357	0.1358	0.0194	1.002
1	0.0014	0.3871	0.0121	0.1396	0.1397	0.0313	1.001
0	0.0014	0	0.0015	0.1434	0.1434	∞	1.010
-1	0.0014	-0.3871	0.0121	0.1473	0.1474	-0.0313	1.001
-2	0.0014	-0.7743	0.0150	0.1512	0.1513	-0.0194	1.001
-3	0.0014	-0.1610	0.0207	0.1550	0.1552	-0.0179	1.001

Table V.4: Oscillation frequencies for $(1.80 M_0)$ model

V_{rot} kms^{-1}	l	m	N_p	N_g	f_c d^{-1}	f_o d^{-1}	$\lvert\nabla f\rvert$ d^{-1}	Q_v d	$\omega \times 10^2$ Rad/s
0.0	0	0	0	0	12.889	12.941	0.052	0.211	0.061
	1	0	0	0	13.057	12.941	0.116	0.135	0.076
	2	0	0	1	13.227	12.708	0.519	0.139	0.075
46.0	0	0	0	0	12.526	12.708	0.182	0.126	0.079
	1	0	0	0	12.939	12.941	0.002	0.134	0.007
	1	1	1	0	16.122	15.990	0.132	0.209	0.062
	2	0	0	1	12.995	12.941	0.054	0.135	0.076
	2	-2	1	1	16.096	15.990	0.106	0.208	0.062
	3	1	2	1	19.235	19.170	0.065	0.297	0.052
	3	-1	2	0	21.928	21.719	0.209	0.378	0.045
61.9	0	0	0	0	12.653	12.708	0.055	0.125	0.079
	1	1	1	0	16.136	15.990	0.146	0.204	0.062
	1	0	0	0	13.053	12.941	0.112	0.133	0.076
	2	-1	1	1	15.936	15.990	0.054	0.199	0.062
	3	-1	3	0	25.360	25.416	0.056	0.505	0.039
	3	-1	2	0	21.928	21.719	0.209	0.378	0.045
	3	1	2	1	19.206	19.170	0.036	0.289	0.052

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APPENDIX ONE

Int5[0,0] = 2/3

Int5[1,-1] = 4/5

Int5[1,0] = 2/5

Int5[1,1] = 4/5

Int5[2,-2] = 6/7

Int5[2,-1] = 4/7

Int5[2,0] = 10/21

Int5[2,1] = 4/7

Int5[2,2] = 6/7

Int5[3,-3]= 8/9

Int5[3,-2] = 2/3

Int5[3,-1] = 8/15

Int5[3,0] = 22/45

Int5[3,1] = 8/15

Int5[3,2] = 2/3

Int5[3,3] = 8/9

Int6[0,0] = DIVERGES

Int6[1,-1] = 1/2

Int6[1,0] = DIVERGES

Int6[1,1] = 1/2

Int6[2,-2] = 1/4

Int6[2,-1] = 3/2

Int6[2,0] = DIVERGES

Int6[2,1] = 3/2

Int6[2,2] = 1/4

Int6[3,-3] = 1/6

Int6[3,-2] = 3/4

Int6[3,-1] = 5/2

Int6[3,0] = DIVERGES

Int6[3,1] = 5/2

Int6[3,2] = 3/4

Int7[0,0] = 0

Int7[1,-1] = 3/10

Int7[1,0] = 2/5

Int7[1,1] = 3/10

Int7[2,-2] = 3/7

Int7[2,-1] = 11/14

Int7[2,0] = 18/7

Int7[2,1] = 11/14

Int7[2,2] = 3/7

Int7[3,-3] = 1/2

Int7[3,-2] = 1

Int7[3,-1] = 27/10

Int7[3,0] = 28/5

Int7[3,1] = 27/10

Int7[3,2] = 1

Int7[3,3] = 1/2

APPENDIX TWO

```
IMPLICIT REAL*8(A-H,O-Z)
cfirst order correction
cGAMA1 her shellde degisiyor
        save
c en son modelara bakmak icin yaptigim prog
       DIMENSION X(10000),Y(10000),Z(10000),YY(10000),
     *  ZZ(10000),RR(10000),DTERM(2000,2000),FQV(2000,2000),
     *  RM(10000),PR(10000),DNS(10000),RMI(10000),PRI(10000),
     *  DLTH(10000),FIRSTI(2000,2000),SECONDI(2000,2000),
     *
DNSI(10000),SD(10000),BV(10000),FL(10000),FRATIO(2000,2000),
     *  COMEGA(2000,2000),THIRDI(2000,2000),FORTHI(2000,2000),
     *  DNSD(10000),DLTR(10000),DLPP(10000),FIFTHI(2000,2000),
     *  SIGMA(2000,2000),QV(2000,2000),SIXTHI(2000,2000),
     *  CNUM(10000),CDEN(10000),XPHASE(10000),SQV(2000,2000),
     *
YPHASE(10000),ERPR(10000),THIRDC(10000),SRATIO(2000,2000),
     *  A1(10000),A2(10000),B1(10000),B2(10000),FORTHC(10000),
     *  tl(10000),alrl(10000),el(10000),bwl(10000),wl(10000),
     *
psi(10000),beta(10000),il(10000),rrr(10000),TRATIO(2000,2000),
     *  rmm(10000),rdns(10000),FIFTHC(10000),FSIGMA(2000,2000),
     *  rpr(10000),gama1(10000),gamval(10000),rgamval(10000),
     *  SSIGMA(2000,2000),SIGMAT(2000,2000),SCOMEGA(2000,2000),
     *  FCOMEGA(2000,2000)
        CHARACTER*4 bdchar
C       CHARACTER*1 mode
        character*4 aster/'  * '/
c       data aster/'*   '/
        G=6.6732D-8
        PI=3.141593d0
        AMSUN=1.985d33
        RSUN=6.9598d10
        ALRSUN=3.844d33
        aSIGMA=5.6724d-05
        J=1
        np=0
        NA=0
        NG=0
        IORD=0
        L=2
        nfe=0
        jjj=1
c        GAMA1=5./3.
c        vsini=0.d0
c        vsini=195.0d+5
        amin=1.d0
        amax=120.d0
        tol0=0.1d-3
        tol1=0.1d-3
        del0=1.0d-2
c        del0=(amax-amin)/50000.d0
        H=3.d-4
        tol=tol0
        DELTA =DEL0
```

```
        OMEGA=AMIN
c
PRINT*,'=================================================================
c     *  ====================='
c        PRINT*,'               THE FOLLOWING MODES HAS BEEN
APPROACHED'
c
PRINT*,'=================================================================
c     *  ======================'
c        PRINT*,' '
        TR=1.5361d11
        TM=3.5730d33
        open(7,file='m4.dta',status='OLD')
        OPEN (11,FILE='l21v46m80',STATUS='UNKNOWN')
        OPEN (12,FILE='l22v46m80',STATUS='UNKNOWN')
        OPEN (13,FILE='l23v46m80',STATUS='UNKNOWN')
        OPEN (14,FILE='l24v46m80',STATUS='UNKNOWN')
        OPEN (15,FILE='l25v46m80',STATUS='UNKNOWN')
        OPEN (16,FILE='l26v46m80',STATUS='UNKNOWN')
        OPEN (17,FILE='l27v46m80',STATUS='UNKNOWN')
c       OPEN (18,FILE='l28v46m80',STATUS='UNKNOWN')
        DO 15 M=1,341
c10     READ (7,*) RM(M), RR(M), DNS(M), PR(M), gamval(i)
        read(7,119)
bdchar,il(m),rm(m),rr(m),dns(m),tl(m),pr(m),alrl(m),
     *         el(m),bwl(m),wl(m),psi(m),beta(m),gamval(m)
  119   format(A4,I3,8e12.5,e12.4,f9.5,1x,f6.4,e12.5)
c       PRINT*,'TEST',m,RM(M),RR(M),DNS(M),PR(M)
c       write(5,1020) bdchar,m,rm(m),rr(m),dns(m),pr(m),gamval(m)
c1020   format(A4,I4,5(e11.4))
c       if(m.eq.357) go to 15
        if(m.gt.100) go to 15
c       write(5,107)aster
c       print*,aster
c107    format(A4)
        if(bdchar. eq. aster) m11=m
c       x11=rr(m)/tr
c       x11=x11+0.0009d0
c       x11=x11+0.0005d0
c       x11=x11+0.025d0
c       endif
c       write(5,1035) x11
c1035   format(e11.4)
 15     CONTINUE
        x11=rr(m11)/tr
        m=m-1
        vsini=bwl(m)*TR
        DNSM= TM/((4.D0/3.D0)*PI*TR**3.D0)
        adnsm=amsun/((4.D0/3.D0)*PI*rsun**3.D0)
        rhom=dsqrt(dnsm/1.408d0)
        atr=tr/rsun
        atm=tm/amsun
        alr=alrl(m)/alrsun
        ate1=alrl(m)/(4.d0*pi*tr*tr*asigma)
        ate2=dsqrt(ate1)
        ate3=dsqrt(ate2)
        ate=dlog(ate3)
        ads=dnsm/adnsm
        aads=atm/(atr**3.d0)
```

```
c        write (44,439)vsini,atm,alr,atr,ate3,ads
c        write (9,439)vsini,atm,alr,atr,ate3,ads
 439    format(3(2x,d11.4),//,3(2x,d11.4))
 1000    FORMAT(E11.3,4(E11.3),2I4,2(E11.3))
c       j2=2
c108   continue
c*******************************************************
 187    continue
        I=1
c       OPEN (5,FILE='pulsout.dta',STATUS='UNKNOWN')
        X(1)=0.D0
        i1=0
        tp=pr(1)
        td=dns(1)
        do 3 j=1,m
        rrr(j)=rr(j)/tr
c       if(rrr(j).eq.rrr(j-1))print *,j
        rmm(j)=rm(j)/tm
        rdns(j)=dns(j)/td
        rpr(j)=pr(j)/tp
        rgamval(j)=gamval(j)
  3      continue
 20     X1=X(I)
c       if(rrr(i).lt.0.1d0) h=1.0d-6
        CALL INTRP (X1,RM1,RRR,RMM,m)
        CALL INTRP (X1,DNS1,RRR,rDNS,m)
        CALL INTRP (X1,PR1,RRR,rPR,m)
        call intrp (x1,gm1,rrr,rgamval,m)
        gama1(i)=gm1
        RMI(I)=RM1
        dns1=dns1*td
        pr1=pr1*tp
        DNSI(I)=DNS1
        PRI(I)=PR1
c       WRITE(5,1000) X(I),PRI(I),RMI(I),DNSI(I),i,m
        if(y1.gt.1.0d8) go to 33
c       WRITE(5,1000) X1,PR1,RM1,DNS1,gm1,i,m
  33    continue
        I=I+1
        x(i)=x(i-1)+H
c       if(x(i).lt.0.084d0)go to 33
        if(x(i).lt.x11)go to 33
        if(i1.eq.2)go to 34
        x(2)=x(i)
        x11=x(2)
        i=2
        i1=2
  34    continue
c  **************************************m=1 ? m=385?
        if(x(i).gt.1.0d0) go to 25
        GOTO 20
 25     RMI(I)=1.0D0
        DNSI(I)=0.0D0
        PRI(I)=0.0D0
        CALL DNSDIF (DNSI,DNSD,TR,H,I)
c       OPEN (5,FILE='output1.dat',STATUS='NEW')
c       WRITE(5,101)
c       WRITE(5,104)
c       WRITE(5,101)
```

```
        DO 35 N=2,i-1
        SD(N)=DNSD(N)/DNSI(N)+DNSI(N)*G*TM*RMI(N)/
     *  (GAMA1(N)*PRI(N)*TR*TR*X(N)*X(N))
        BV(N)=-SD(N)*G*TM*RMI(N)/(TR*TR*X(N)*X(N))
        BV(N)=BV(N)*TR**3/(G*TM)
c       if(bv(n).lt.0.0d0) bv(n)=bv(n-1)
        tr2=tr*tr

FL(N)=DFLOAT(L*(L+1))*GAMA1(N)*PRI(N)/(DNSI(N)*(TR2*X(N)*X(N)))
        FL(N)=FL(N)*TR**3/(G*TM)
        if(y1.gt.1.0d8) go to 36
C       WRITE(8,2000) X(N),BV(N),FL(N)
c       WRITE(88,2001) X(N),FL(N)
c       WRITE(89,2001) X(N),BV(N)
2001    format(F10.4,E11.4)
 2000   FORMAT(F10.4,2(E20.4))
 35     CONTINUE
 36     continue
c104    FORMAT(7X,'X',11X,'BRUNT V.FRQ.',9X,'LAMB FRQ.')
c       CLOSE(5)
c*******************************************************c
 40     CONTINUE
        Y(2)=1.D0
        IF (L.EQ.0) THEN
           Z(2)=1.d0-
3.D0*GAMA1(2)*PRI(2)*TR*X(2)/(G*TM*RMI(2)*DNSI(2))
c       print*, z(2),x(2),rmi(2),dnsi(2),gama1(2),pri(2)
           GOTO 5
        ENDIF
        Z(2)=OMEGA*X(2)**3/(RMI(2)*DFLOAT(L))
c**********************************************
c5      DO 50 N=3,300
   5    do 50 n=2,I-1
        X0=X(N)
        Y0=Y(N)
        Z0=Z(N)
        RMI0=RMI(N)
        DNSI0=DNSI(N)
        PRI0=PRI(N)
        DNSD0=DNSD(N)
        gama0=gama1(N)
        CONST1=0.D0
        CONST2=0.D0
c       print*, gama0,rmi0,dnsi0,pri0,dnsd0,y0,z0
        CALL
RUNGE(X0,Y0,Z0,Y1,Z1,G,TM,RMI0,DNSI0,PRI0,TR,GAMA0,L,OMEGA,
     *  PI,DNSD0,H,CONST1,CONST2)
         if(y1.gt.1.0d8) then
        x11=x11+h
        go to 187
        endif
c       print*, x0,y0,z0,y1,z1
        Y(N+1)=Y1
        Z(N+1)=Z1
 50     CONTINUE
c*****************************************************************
        BC=Y(I-1)-Z(I-1)
        bc1=Y(i-2)-z(i-2)
        if (nfe.eq.0) then
```

```
      bc3=bc
      bc2=omega
      jj=1
      nfe=11
      go to 888
      endif
c      SIGM=DSQRT(G*TM*OMEGA/(TR**3))
c      fr=1.d6*sigm/(2.d0*pi)
      if(bc.ge.0.d0.and.bc3.le.0.d0)then
      if(jjj.eq.3) go to 589
      if(jj.eq.1)then
      vlv=omega
      yly=bc
      jjj=3
      endif
  589 continue
      rbc=dabs(omega-bc2)
      rbc1=dabs(bc-tol)
      if (rbc.lt.1.d-9) goto 568
      if (rbc1.lt.1.d-6) go to 568
      jj=2
      delta=delta/10
      omega=bc2+delta
      go to 40
      endif
      if(bc.le.0.d0.and.bc3.ge.0.d0)then
      if(jjj.eq.3) go to 579
      if(jj.eq.1)then
      vlv=omega
      yly=bc
      jjj=3
      endif
  579 continue
      rbc=dabs(omega-bc2)
      rbc1=dabs(bc-tol)
      if (rbc.lt.1.d-9) goto 568
      if (rbc1.lt.1.d-6) go to 568
       jj=2
      delta=delta/10
      omega=bc2+delta
      go to 40
      endif
      bc3=bc
      bc2=omega
      omega=omega+delta
      IF (OMEGA.GT.AMAX) GOTO 999
      jj=1
      go to 40
 568  continue
      jj=1
      jjj=1
      SIGM=DSQRT(G*TM*OMEGA/(TR**3))
      fr=1.d6*sigm/(2.d0*pi)
      QVA  =2.D0*PI/SIGM/86400.D0
      rrqva=1.d0/qva
      RQVA=rhom*QVA
c      if(dabs(bc).lt.0.5) print *,omega,bc,fr
c     DO 55 N=4,300
c     print*,sigm,qva
```

```
        ng=0.d0
        np=0.d0
        do 55 n=2,I-1
        ERPR(N)=G*TM*RMI(N)*DNSI(N)*Z(N)/(TR*X(N))
        DLPP(N)=(ERPR(N)-
y(n)*G*TM*RMI(N)*DNSI(N)/(TR*X(N)))/PRI(N)
        DLTR(N)=y(n)*X(N)*TR
        DLTH(N)=(ERPR(N)/DNSI(N))/ (X(N)*TR*SIGM**2.D0)
        IF (Y(N)*Y(N+1).LT.0.D0) THEN
c       print *,y(n),y(n+1)
        IF (OMEGA.LT.BV(N).AND.OMEGA.LT.FL(N)) NG=NG+1
        IF (OMEGA.GT.BV(N).AND.OMEGA.GT.FL(N)) NP=NP+1
        ENDIF
        CNUM(N)=(2.D0*DLTR(N)*DLTH(N)+DLTH(N)*DLTH(N))*DNSI(N)*
     *  (X(N)*TR)**2.D0

CDEN(N)=(DLTR(N)*DLTR(N)+L*(L+1)*DLTH(N)*DLTH(N))*DNSI(N)*
     *  (X(N)*TR)**2.D0
        THIRDC(N)=(DLTR(N)*X(N)*TR)**2.D0*DNSI(N)
        FORTHC(N)=(DLTH(N)*X(N)*TR)**2.D0*DNSI(N)
        FIFTHC(N)=DLTR(N)*DLTH(N)*(X(N)*TR)**2.D0*DNSI(N)
 55     CONTINUE
c*****************************************************************
****
        IORD=NP-NG
        IF (IORD.LT.0.D0) MODE='G'
        IF (IORD.GT.0.D0) MODE='P'
        IF (IORD.EQ.0.D0) MODE='F'
c        print *,l,np,ng,iord
c        WRITE(5,105)l,np,ng,iord
 105    FORMAT(//,I2,5X,3(I3,5X))
        IF (L.EQ.0) THEN
c        OPEN (5,FILE='output1.dat',STATUS='NEW')
        if(omega.lt.1.1d0) go to 403
c        write(5,348)rrqva,omega,fr,rqva
c        write(9,748)l,rfr
 748    format(1x,i2,3x,f10.4)
 348    format (2x,f10.4,2x,f10.4,2x,f10.4,2x,f10.4)
         goto 403
c        WRITE(5,101)
c        WRITE(5,102)
c        WRITE(5,101)
         WRITE(90,352)K,SIGM,OMEGA,QVA,FR,RRQVA
352      FORMAT(I5,3e14.4,2f10.5)
c        WRITE(5,101)
c        WRITE(5,103)
c        WRITE(5,101)
  403    continue
         DO 71 N= 2,I-1
         XPHASE(N)=DLOG10(1.D0+DABS(DLTR(N)/TR))
         YPHASE(N)=DLOG10(1.D0+DABS(Z(N)*RMI(N)/X(N)))
         IF (DLTR(N).LT.0.D0) XPHASE(N)= -XPHASE(N)
         IF(ZZ(N).LT.0.D0) YPHASE(N)= -YPHASE(N)
  71     CONTINUE
         GOTO 70
c         go to 999
         ENDIF
c****************************************************
        CALL ROTAT(CNUM,CDEN,CINTGRL,DEN,I,H)
```

```
        CALL
SECONDFRQ(THIRDC,FORTHC,FIFTHC,THIRD,FORTH,FIFTH,I,H)
c       ROTFRQ=(61.9D+5)/(2.D0*PI*TR)
        ROTFRQ=(61.9D+5)/(TR)
c*********************************************************
c       OPEN (7,FILE='output3.dat',STATUS='NEW')
c        lal=l*(l+1)
        DO 65 K=-L,L,1
        SIGMA(J,K) =SIGM+DFLOAT(K)*ROTFRQ*(1.D0-CINTGRL)
        FSIGMA(J,K)=DFLOAT(K)*ROTFRQ*(1.D0-CINTGRL)
        DTERM(J,K)=(DFLOAT(K)*ROTFRQ*(1-CINTGRL))**2.d0/SIGM*2.d0
c       rfr=sigma(j,k)*1.d+6/(2.d0*pi)
        FIRSTI(J,K)=-(2.D0*(DFLOAT(K)*ROTFRQ)**2.D0)/(SIGM*DEN)*
     *  (FORTH+CINTGRL*FIFTH)
        SECONDI(J,K)=-(2.D0*(DFLOAT(K)*ROTFRQ)**2.D0*CINTGRL)/
     *  (SIGM*DEN)*(FIFTH+FORTH)
        THIRDI(J,K)=((2.D0*(ROTFRQ)**2.D0)/(SIGM*DEN))*FIFTH
        FORTHI(J,K)=((2.D0*(ROTFRQ)**2.D0)/(SIGM*DEN))*THIRD

FIFTHI(J,K)=(2.D0*(DFLOAT(K)*ROTFRQ)**2.D0)/(SIGM*DEN)*FORTH
        SIXTHI(J,K)=(2.D0*(ROTFRQ)**2.D0)/(SIGM*DEN)*FORTH
        IF(K.EQ.2) THEN
        THIRDI(J,K)=THIRDI(J,K)*(4.D0/7.D0)
        FORTHI(J,K)=FORTHI(J,K)*(6.D0/7.D0)
        FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/4.D0)
        SIXTHI(J,K)=SIXTHI(J,K)*(3.D0/7.D0)
        ELSE IF(K.EQ.1) THEN
        THIRDI(J,K)=THIRDI(J,K)*(-2.D0/7.D0)
        FORTHI(J,K)=FORTHI(J,K)*(4.D0/7.D0)
        FIFTHI(J,K)=FIFTHI(J,K)*(3.D0/2.D0)
        SIXTHI(J,K)=SIXTHI(J,K)*(11.D0/14.D0)
        ELSE IF(K.EQ.0) THEN
        THIRDI(J,K)=0.D0
        FORTHI(J,K)=0.D0
        FIFTHI(J,K)=0.D0
        SIXTHI(J,K)=0.D0
        ELSE IF(K.EQ.-1) THEN
        THIRDI(J,K)=THIRDI(J,K)*(-2.D0/7.D0)
        FORTHI(J,K)=FORTHI(J,K)*(4.D0/7.D0)
        FIFTHI(J,K)=FIFTHI(J,K)*(3.D0/2.D0)
        SIXTHI(J,K)=SIXTHI(J,K)*(11.D0/14.D0)
        ELSE IF(K.EQ.-2) THEN
        THIRDI(J,K)=THIRDI(J,K)*(4.D0/7.D0)
        FORTHI(J,K)=FORTHI(J,K)*(6.D0/7.D0)
        FIFTHI(J,K)=FIFTHI(J,K)*(1.D0/4.D0)
        SIXTHI(J,K)=SIXTHI(J,K)*(3.D0/7.D0)
        ENDIF

SSIGMA(J,K)=FIRSTI(J,K)+SECONDI(J,K)+THIRDI(J,K)+FORTHI(J,K)+
     *  FIFTHI(J,K)+DTERM(J,K)+SIXTHI(J,K)
        SIGMAT(J,K)=SIGMA(J,K)+SSIGMA(J,K)
        FRATIO(J,K)=FSIGMA(J,K)/SIGM
        SRATIO(J,K)=SSIGMA(J,K)/SIGM
        TRATIO(J,K)=SSIGMA(J,K)/FSIGMA(J,K)
        IFR=SIGM*1.D+6/(2.D0*PI)
        SFR=SIGMAT(J,K)*1.D+6/(2.D0*PI)
        FFR=SIGMA(J,K)*1.D+6/(2.D0*PI)
        FQV(J,K)=2.D0*PI/SIGMA(J,K)/86400.D0
        IQV=2.D0*PI/SIGM/86400.D0
```

```
         SQV(J,K) =2.D0*PI/SIGMAT(J,K)/86400.D0
         rqva=rhom*qv(j,k)
         FRFR=1.d0/FQV(J,K)
         IRFR=1.d0/IQV
         SRFR=1.d0/SQV(J,K)
         ICOMEGA=SIGM**2.D0*TR**3.D0/(G*TM)
         FCOMEGA(J,K)=SIGMA(J,K)**2.D0*TR**3.D0/(G*TM)
         SCOMEGA(J,K)=SIGMAT(J,K)**2.D0*TR**3.D0/(G*TM)
         com=comega(j,k)
c         write(9,343)k,com,rqva,rfr,rrqva
c 343     format (2x,f10.4,2x,e11.4,2(f10.4,2x))
c         write(5,343)k,rrqva,com,rfr,rqva
c         print*,l,mode,iord,np,ng
 743     format(1x,i2,3x,f10.4)
 343     format (i2,2x,f10.3,2x,f10.4,2x,f10.4,2x,f10.2)
c         print *,rqva,vrqva,rrqva,com
c        WRITE(5,101)
c        WRITE(5,102)
c        WRITE(5,101)
C
WRITE(9,100)k,COMEGA(J,K),SIGMA(J,K),QV(J,K),MODE,IORD,rfr,rqv
c    *   ,rrqva

WRITE(12,402)K,THIRDI(J,K),FORTHI(J,K),FIFTHI(J,K),SIXTHI(J,K)
         WRITE(11,401)K,DTERM(J,K),FIRSTI(J,K),SECONDI(J,K)
         WRITE(13,402)K,SIGM,FSIGMA(J,K),SSIGMA(J,K)
         WRITE(15,401)K,FRATIO(J,K),SRATIO(J,K),TRATIO(J,K)
         WRITE(14,400)K,SIGMA(J,K),SIGMAT(J,K)
         WRITE(16,409)K,SIGMA(J,K),FCOMEGA(J,K),FQV(J,K),FFR,FRFR
         WRITE(17,409)K,SIGMAT(J,K),SCOMEGA(J,K),SQV(J,K),SFR,SRFR
c        WRITE(18,409)K,SIGM,ICOMEGA,IQV,IFR,IRFR
         YY(2)=1.D0
         ZZ(2)=((X(2)**3/RMI(2))*OMEGA*(1.D0+2.D0*K*ROTFRQ/
     *   SIGM*(-CINTGRL-1.D0/L)*Y(2)))/L
         DO 75 N=2,I-1
         X0=X(N)
         Y0=YY(N)
         Z0=ZZ(N)
         RMI0=RMI(N)
         DNSI0=DNSI(N)
         PRI0=PRI(N)
         DNSD0=DNSD(N)
         gama0=gama1(N)
         CONST1=2.D0*K*ROTFRQ/SIGM*(Y(N)+(RMI(N)/(X(N)**3*
     *   OMEGA)+CINTGRL*L*(L+1)*RMI(N)/(X(N)**3*OMEGA))*Z(N))
         CONST2=2.D0*K*ROTFRQ/SIGM*(-CINTGRL*OMEGA*X(N)**3*
     *   Y(N)/RMI(N)-Z(N))
         CALL
RUNGE(X0,Y0,Z0,Y1,Z1,G,TM,RMI0,DNSI0,PRI0,TR,GAMA0,L,OMEGA
     *   ,PI,DNSD0,H,CONST1,CONST2)
         YY(N+1)=Y1
         ZZ(N+1)=Z1
         ERPR(N)=G*TM*RMI(N)*DNSI(N)*ZZ(N)/(TR*X(N))
         DLPP(N)= (ERPR(N)-
YY(N)*G*TM*RMI(N)*DNSI(N)/(TR*X(N)))/PRI(N)
         DLTR(N)= YY(N)*X(N)*TR
         DLTH(N)=(ERPR(N)/DNSI(N))/(X(N)*TR*SIGM**2.D0)
         XPHASE(N)=DLOG10(1.D0+DABS(DLTR(N)/TR))
         YPHASE(N)=DLOG10(1.D0+DABS(ZZ(N)*RMI(N)/X(N)))
```

```
        IF (DLTR(N).LT.0.D0) XPHASE(N)= -XPHASE(N)
        IF (ZZ(N).LT.0.D0) YPHASE(N)=-YPHASE(N)
C       WRITE(12,3050)X(N),Y(N),DLPP(N)
 75     CONTINUE
 65     CONTINUE
c       CLOSE(5)
c 100    FORMAT(I4,F10.3,E11.4,f10.4,A5)
c 100    FORMAT(I5,F13.3,E14.4,f13.4,A3,I3,F13.4,f13.5,/f13.5)
 101    FORMAT(60('='))
 102    FORMAT(4X,'M',7X,' OMEGA',7X,'SIGMA',6X,'P-
VALUE',6X,'MODE')
 103
FORMAT(4X,'X',8X,'DR/R',8X,'DP/P',6X,'XPHASE',6X,'YPHASE')
 400    FORMAT(I5,2e14.4)
 401    FORMAT(I5,3e14.4)
 409    FORMAT(I5,3e14.4,2x,f10.5,2x,f10.5)
 402    FORMAT(I5,4e14.4)
c403    FORMAT(I5,e18.8)
 405    FORMAT(I5,e18.8)
 3000   FORMAT(2(E11.4))
 3050   FORMAT(F7.3,2(E12.3))
 70     CONTINUE
c        go to 999
c******************************************************
        J=J+1
        TOL=TOL0
        jj=1
        jjj=1
        omega=vlv
        bc2=omega
        bc3=yly
        DELTA=DEL0
        NP=0
        NG=0
        IORD=0
        OMEGA=OMEGA+DELTA
GOTO 40
c******************************************************
  888   IF (DABS(BC).LT.TOL) THEN
c         PRINT*,'THE',J,'.APPROACH','TOL = ',TOL,OMEGA
c        print *,omega,fr
        TOL=TOL/10.D0
        DELTA=DELTA/10.D0
        OMEGA=OMEGA+DELTA
GOTO 40
        ENDIF
c*******************************************************
 80     OMEGA=OMEGA+DELTA
        rbcc=dabs(omega-bc2)
         rbcac1=dabs(bc-tol)
c        print *,rbcc,omega,rbcac1,jj
GOTO 40
CONTINUE   999
c
PRINT*,'=================================================
c     *  ======================   THE CHARECTERISTICS OF THESE
MODES'
c        PRINT*,'                   ARE IN /NONRAD OUT A1'
```

```
c
PRINT*,'==============================================================
c      *  ================='
c         PRINT*,'END'
        STOP
        END
c*******{{*****************************************************
****
c   INTERPOLATION.  REF: S.CONTE&C.de.BOORE,1981,numerical
analysis
c**************************************************************
**
        SUBROUTINE INTRP(XU,YU,X,F,NP)
        IMPLICIT REAL*8 (A-H,O-Z)
c       DIMENSION F(NP),X(NP),A(20),XK(20)
        DIMENSION F(5000),X(5000),A(20),XK(20)
c       do 113 j=1,385
c113        continue
c1002   format(I4, 2(e11.4))
c        TOL11=1.0D-2
        TOL11=1.0D-3
        IF (XU.GT.X(1).AND.XU.LE.X(NP)) THEN
            DO 11 NEXT=2,NP
              IF (XU.LE.X(NEXT)) GOTO 12
 11         CONTINUE
        ENDIF
        IF (XU.LE.X(1)) THEN
           YU=F(1)
           ELSE
             YU=F(NP)
        ENDIF
        IFLAG=3
        RETURN
 12     XK(1)=X(NEXT)
        NEXTL=NEXT-1
        NEXTR=NEXT+1
        A(1)=F(NEXT)
        YU=A(1)
        PSIK=1.D0
        KP1MAX=MIN(20,NP)
        DO 21 KP1=2,KP1MAX
        IF (NEXTL.EQ.0.D0) THEN
           NEXT=NEXTR
           NEXTR=NEXTR+1
        ELSEIF (NEXTR.GT.NP) THEN
            NEXT=NEXTL
        NEXTL=NEXTL-1
        ELSEIF (XU-X(NEXTL).GT.X(NEXTR)-XU) THEN
        NEXT=NEXTR
        NEXTR=NEXTR+1
        ELSE
          NEXT=NEXTL
          NEXTL=NEXTL-1
c       print*, xu,kp1
        ENDIF
        XK(KP1)=X(NEXT)
        A(KP1)=F(NEXT)
        DO 13 J=KP1-1,1,-1
c        if(xk(kp1).eq.xk(j)) go to 133
```

```
        A(J)=(A(J+1)-A(J))/(XK(KP1)-XK(J))
c        go to 13
c 133    a(j)=a(j-1)
c        print *,j,xu
 13     CONTINUE
        PSIK=PSIK*(XU-XK(KP1-1))
        ERROR=A(1)*PSIK
        YU=YU+ERROR
c       print*,yu,psik,a(1),a(2),a(3)
        IF (DABS(ERROR).LE.TOL11) THEN
        IFLAG=1
        RETURN
        ENDIF
 21     CONTINUE
        IFLAG=2
        RETURN
        END
c*********************************************************
c    NUMERICAL DIFFERENTIATION using 3 points formula
c*******************************************************
        SUBROUTINE DNSDIF(DNSI,DNSD,TR,H,I)
        IMPLICIT REAL*8 (A-H,O-Z)
        save
        DIMENSION DNSI(5000),DNSD(5000)
        DO 30 N=1,i-3
        DNSD(N)=(-3.D0*DNSI(N)+4.D0*DNSI(N+1)-
DNSI(N+2))/(2.D0*H*TR)
 30     CONTINUE
        DNSD(I-2)=(-DNSI(I-3)+DNSI(I-1))/(2.D0*H*TR)
        DNSD(I-1)=(DNSI(I-3)-4.D0*DNSI(I-2)+3.D0*DNSI(I-
1))/(2.D0*H*TR)
        RETURN
        END
c********************************************************
c             NUMERICAL INTEGRATION
c*******************************************************
        SUBROUTINE ROTAT(CNUM,CDEN,CINTGRL,DEN,I,H)
        IMPLICIT REAL*8 (A-H,O-Z)
        save
        DIMENSION CNUM(5000),CDEN(5000)
        SNUM1=0.D0
        SNUM2=0.D0
        SDEN1=0.D0
        SDEN2=0.D0
        SNUM0=CNUM(2)+CNUM(I+1)
        SDEN0=CDEN(2)+CDEN(I+1)
        DO 60 N=2,I-4,2
        SNUM1=SNUM1+CNUM(N+1)
        SNUM2=SNUM2+CNUM(N+2)
        SDEN1=SDEN1+CDEN(N+1)
        SDEN2=SDEN2+CDEN(N+2)
 60     CONTINUE
        CNUMI=H/3.D0*(SNUM0+4.D0*SNUM1+2.D0*SNUM2)
        CDENI=H/3.D0*(SDEN0+4.D0*SDEN1+2.D0*SDEN2)
        DEN=CDENI
        CINTGRL=CNUMI/CDENI
        RETURN
        END
c********************************************************
```

```
c               RUNGE-KUTTA METHOD
c*******************************************************
        SUBROUTINE
RUNGE(X,Y0,Z0,Y,Z,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA
     *  ,PI,DNSD,H,CONST1,CONST2)
        IMPLICIT REAL*8 (A-H,O-Z)

A=H*F1(X,Y0,Z0,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1/X

AA=H*F2(X,Y0,Z0,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+H*
     *  CONST2/X
        X=X+0.5D0*H
        Y1=Y0+0.5D0*A
        Z1=Z0+0.5D0*AA

B=H*F1(X,Y1,Z1,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1/X

BB=H*F2(X,Y1,Z1,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+H*
     *  CONST2/X
        Y1=Y0+0.5D0*B
        Z1=Z0+0.5D0*BB

C=H*F1(X,Y1,Z1,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1/X

CC=H*F2(X,Y1,Z1,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+(H*
     *  CONST2)/X
        X=X+0.5D0*H
        Y1=Y0+C
        Z1=Z0+CC

D=H*F1(X,Y1,Z1,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)+H*CONST1/X

DD=H*F2(X,Y1,Z1,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)+H*
       *  CONST2/X
        Y=Y0+(A+2.D0*B+2.D0*C+D)/6.D0
        Z=Z0+(AA+2.D0*BB+2.D0*CC+DD)/6.D0
        RETURN
        END
c***********************************************************
        FUNCTION F1(X,Y0,Z0,G,TM,RMI,DNSI,PRI,TR,GAMA1,L,OMEGA)
        IMPLICIT REAL*8 (A-H,O-Z)
        F1 = ((G*TM*RMI*DNSI/(PRI*TR*X*GAMA1)-
3.D0)*Y0+(L*(L+1)/(OMEGA*
     *  X**3/RMI)-G*TM*RMI*DNSI/(PRI*TR*X*GAMA1))*Z0)/X
        RETURN
        END
c************************************************************
        FUNCTION
F2(X,Y0,Z0,G,PI,TM,RMI,DNSI,PRI,TR,GAMA1,L,DNSD,OMEGA)
        IMPLICIT REAL*8 (A-H,O-Z)
        F2 = ((1.D0-4.D0*PI*DNSI*(TR*X)**3/(TM*RMI)-
TR*X*(DNSD/DNSI+
     *
DNSI*G*TM*RMI/(GAMA1*PRI*TR*TR*X*X)))*Z0+(OMEGA*X**3/RMI+X*TR*
     *  (DNSD/DNSI+DNSI*G*TM*RMI/(GAMA1*PRI*TR*TR*X*X)))*Y0)/X
        IF (L.EQ.0) THEN
        F2=F2+((4.D0*PI*DNSI*(TR*X)**3/(TM*RMI))*Y0)/X
        ENDIF
        RETURN
```

```
        END
******************************************************************
***************
c         FINDING THE SECOND ORDER EIGEN FREQUENCY
******************************************************************
**************
        SUBROUTINE
SECONDFRQ(THIRDC,FORTHC,FIFTHC,THIRD,FORTH,FIFTH,I,H)
        IMPLICIT REAL*8 (A-H,O-Z)
        SAVE
        DIMENsION THIRDC(5000),FIFTHC(5000),FORTHC(5000)
        STHIRDC1=0.D0
        SFORTHC1=0.D0
        SFIFTHC1=0.D0
        STHIRDC2=0.D0
        SFORTHC2=0.D0
        SFIFTHC2=0.D0
        STHIRDC0=THIRDC(2)+THIRDC(I+1)
        SFORTHC0=FORTHC(2)+FORTHC(I+1)
        SFIFTHC0=FIFTHC(2)+FIFTHC(I+1)
        DO 60 N=2,I-4,2
        STHIRDC1=STHIRDC1+THIRDC(N+1)
        SFORTHC1=SFORTHC1+FORTHC(N+1)
        SFIFTHC1=SFIFTHC1+FIFTHC(N+1)
        STHIRDC2=STHIRDC2+THIRDC(N+2)
        SFORTHC2=SFORTHC2+FORTHC(N+2)
        SFIFTHC2=SFIFTHC2+FIFTHC(N+2)
 60     CONTINUE
        THIRDCI=H/3.D0*(STHIRDC0+4.D0*STHIRDC1+2.D0*STHIRDC2)
        FORTHCI=H/3.D0*(SFORTHC0+4.D0*SFORTHC1+2.D0*SFORTHC2)
        FIFTHCI=H/3.D0*(SFIFTHC0+4.D0*SFIFTHC1+2.D0*SFIFTHC2)
        THIRD=THIRDCI
        FORTH=FORTHCI
        FIFTH=FIFTHCI
        RETURN
        END
```