PERIODIC SOLUTIONS AND STABILITY OF LINEAR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

JEHAD ALZABUT

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
THE DEPARTMENT OF MATHEMATICS

APRIL 2004

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Canan ÖZGEN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Şafak ALPAY
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Marat AKHMET
Co-Supervisor

Prof. Dr. Ağacık ZAFER
Supervisor

Examining Committee Members

Prof. Dr. Hüseyin BEREKETOĞLU

Prof. Dr. Hüseyin HÜSEYİNOV

Prof. Dr. Hasan TAŞELİ

Prof. Dr. Münevver TEZER

Prof. Dr. Ağacık ZAFER

ABSTRACT

PERIODIC SOLUTIONS AND STABILITY OF LINEAR IMPULSIVE DELAY DIFFERENTIAL EQUATIONS

ALZABUT, Jehad

Ph.D., Department of Mathematics

Supervisor: Prof. Dr. Ağacık ZAFER

Co-Supervisor: Prof. Dr. Marat AKHMET

April 2004, 96 pages.

In this thesis, we investigate impulsive differential systems with delays of the form

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t-\tau), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i, j \in \mathbb{Z} \end{cases}$$

and more generally of the form

$$\begin{cases} x'(t) = & \int_{-\tau}^{0} d_s \eta(t,s) x(t+s), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & A_{i0} x(\theta_i) + \sum_{-j \leq k < -1} A_{ik} x(\theta_{i+k}), \quad i, j, k \in \mathbb{Z} \end{cases}$$

The dissertation consists of five chapters. The first chapter serves as an introduction, contains preliminary considerations and assertions that will be encountered in the sequel. In chapter 2, we construct the adjoint systems and obtain the variation of parameters formulas of the solutions in terms of fundamental matrices. The asymptotic behavior of solutions of systems satisfying the *Perron* condition is investigated in chapter 3. In chapter 4, we give a result that characterizes the behavior of solutions in the case there is a bounded solution. Moreover, a necessary and sufficient condition for the existence of periodic solutions is obtained. In the last chapter, a series of consequences on the existence of periodic solutions of functionally equivalent impulsive systems with delays is established.

Keywords: Impulse, Delay, Adjoint System, Variation of Parameters, *Perron* Condition, Periodic Solution, Functional-Equivalence.

ÖZ

LİNEER "IMPULSIVE" GECİKMELİ DİFERENSİYEL SİSTEMLERİNİN PERİYODİK ÇÖZÜMLERİ VE KARARLILIĞI

ALZABUT, Jehad

Doktora, Matematik Bölümü

Tez Yöneticisi: Prof. Dr. Ağacık ZAFER

Ortak Tez Yöneticisi: Prof. Dr. Marat AKHMET

Nisan 2004, 96 sayfa.

Bu tezde

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t-\tau), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i, j \in \mathbb{Z} \end{cases}$$

ve

$$\begin{cases} x'(t) = & \displaystyle\int_{-\tau}^{0} d_s \eta(t,s) x(t+s), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & A_{i0} x(\theta_i) + \displaystyle\sum_{-j \leq k < -1} A_{ik} x(\theta_{i+k}), \quad i, j, k \in \mathbb{Z} \end{cases}$$

"impulsive" gecikmeli diferensiyel sistemleri incelenmiştir. Tez beş bölümden oluşmaktadır. Birinci bölüm, daha sonraki bölümlerde kullanılacak bazı temel kavramlar ve sonuçlar içermektedir. İkinci bölümde, eş sistemleri kurduktan sonra çözümler için bazı gösterimler elde edilmiştir. Daha sonra, *Perron* koşulunu sağlayan çözümlerin asimtotik davranışları üçüncü bölümde verilmiştir. Dördüncü bölümde ise, periyodik çözümün varlığı için gerek ve yeter koşullar elde edilmiştir. Son olarak, beşinci bölümde, fonksiyonel denk sistemlerin yardımıyla periyodik çözümlerin varlığı hakkında sonuçlar elde edilmiştir.

Anahtar Kelimeler: Gecikmeli, Eş Sistem, *Perron* Koşulu, Periyodik çözümü, Fonksiyonel Denklik.

Dedicated to

My parents FAYZEH and OTHMAN

My brothers and sisters

My little angel NEDİME

The ones I love more than myself...

ACKNOWLEDGMENTS

I would like to express my thank and indebtedness to my supervisors Prof. Dr. Ağacık ZAFER and Prof. Dr. Marat AKHMET, for their precious guidance, motivation and encouragement throughout this work.

Many thanks go to the jury members for their valuable comments and constructive suggestions and criticisms.

I am grateful to Prof. Dr. Yurdahan GÜLER, the chairman of the department of mathematics and computer science at Çankaya University for furnishing excellent facilities.

I feel fortunate to have had so many of my colleagues communicate with me throughout this research. I would particularly like to acknowledge the following whose efforts I greatly appreciate. Thabet ABDELJAWAD, Abdullah ÖZBEKLER, Mert ÇAĞLAR and Ömür UĞUR.

To my parents... I offer my sincere thank and gratitude for their faith in me and for their endless moral and financial support.

And... I deeply thank my wife Nedime for everything. I can only say "*I love you sweetheart*".

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

The impulsive differential systems describe evolution processes which at certain moments change their state rapidly. In the mathematical simulation of such processes it is convenient to assume that this change takes place momentarily and the process changes its state abruptly. In certain cases the motion of the processes described by the impulsive differential systems might depend on the data used previously, and this often turns out to be the cause of phenomena substantially affecting the motions. This past dependence causes the presence of delay in the considered equations. To describe such processes impulsive differential systems with delays are used. These systems have plenty of applications in the theory of automatic control, the theory of self oscillating systems, the study of problems connected with combustion in rocket motion, the problem of long range planning in economics, a series of biological problems and in many other areas of science and technology the number of which is steadily expanding, see for instance [14, 18, 21, 22, 29]. The abundance of applications is recently simulating a rapid development of the theory of impulsive differential systems with delays.

The first use of impulsive differential systems was the well known elegant example of model of a clock introduced by *N. M. Kruylov* and *N. N. Bogoliubov* in their classical monograph *Introduction to Nonlinear Mechanics* 1937, in which they showed that by studying systems of impulsive differential systems, it is possible to apply approximation methods used in nonlinear mechanics. However, the most systematic and in-depth studies started in the middle of the last century by the *Kiev school of Nonlinear Mechanics*. It is the mathematicians of this school who could broadly approach this problem, consider it in a general form, formulate and solve a number of problems which are important for applications. The theory of delay differential equations, meanwhile, has been under considerations. It was *Minorsky* who first pointed out clearly the importance of the considerations of the delay in the feedback mechanism. The fifties and the sixties of the last century witnessed the first monographs appearance of delay differential equations by *A. D. Myshkis, R. Bellman, J. M. Danskin, N. N. Krasovskii* and *K. L. Cooke.*

The great interest in applications during these and later years has certainly contributed significantly to the rapid development of the theory of impulsive differential systems and the theory of delay differential equations, and recently these theories have been elaborated to a considerable extent. In 1989, there appeared the first article of impulsive differential systems with delay [83]. In that article, sufficient conditions are obtained for the asymptotic stability of solution and for the existence of a nonoscillatory solution. During the last two decades, several articles dealing with stability, periodicity and oscillation of solutions of impulsive differential systems with delays have been published. Most of the systems considered in these articles have involved the delays only in the differential equation, some rather, have the same delay in the differential equation as well as at discontinuity points, see for instance [37, 39, 40, 41, 43, 45, 51, 53, 74, 83]. The theory of impulsive differential systems with delays, however, has been rarely considered with comparison to the theory of impulsive differential systems and the theory of delay differential equations.

In the present work we deal with impulsive differential systems that involve delays in both the differential equation and the impulse condition. The classical and the general linear systems are considered. We will construct adjoint systems and obtain the variation of parameters formulas of these systems. The properties of these systems with the emphasis on the qualitative behavior of the solutions are to be investigated.

1.2 Impulsive Delay Differential Equation

In the study of ordinary and delay differential equations, the solutions are continuous functions over the domain of definitions. Unfortunately, this is not the case for systems involving impulses. Usually, a solution $x(t)$ of a given impulsive differential system is a piecewise continuous function with points of discontinuities, say $\{\theta_i\}$, $i \in \mathbb{Z}$. These discontinuity points are often referred to as *moments of impulses*. According to the way in which the moments of impulses are determined, the impulsive differential systems are classified as follows [17, 18]:

(i) Equations with fixed moments of impulses;

(ii) Equations with variable moments of impulses.

Throughout the present work, however, we shall restrict ourselves to the investigations of systems involving fixed moments of impulses. Therefore, in order to simplify the statements of the assertions later, we introduce a set PLC of functions defined as follows. Let $\mathcal{J} \subset \mathbb{R}$ and a sequence $\{\theta_i\}$, $i \in \mathbb{Z}$, be fixed in $\mathcal{J}$ such that $\theta_{i+1} > \theta_i$ with $\lim_{i\to\infty} \theta_i = \infty$. We denote by $PLC(\mathcal{J},\mathbb{R}^n)$ the set of all piecewise left continuous functions $\psi : \mathcal{J} \to \mathbb{R}^n$ for $t \in \mathcal{J}$ having discontinuities of the first kind at $\theta_i \in \mathcal{J}$, $i \in \mathbb{Z}$. By $PLC^1(\mathcal{J}, \mathbb{R}^n)$ we mean as usual the set of functions $\psi : \mathcal{J} \to \mathbb{R}^n$ such that $\psi, \psi' \in PLC(\mathcal{J}, \mathbb{R}^n)$.

The mathematical model of a real process which experiences certain impulses at fixed moments $\{\theta_i\}$ could be given by an impulsive differential system

$$\begin{cases} x'(t) = & g(t, x(t)), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & J_i(\theta_i, x(\theta_i)), \quad i \in \mathbb{Z} \end{cases} \tag{1.1}$$

where $x' = dx/dt$ and $\Delta x(\theta_i) = x(\theta_i^+) - x(\theta_i^-)$. For an impulsive differential systems, we suppose that the solution $x = \phi(t)$ of system (1.1) is always continuous from the left, that is, $\phi(\theta_i^-) = \phi(\theta_i)$ where $\phi(\theta_i^-) = \lim_{t \to \theta_i^-} \phi(t)$.

The rate of change of physical system might depend not only on their present state but also on their past history. This dependence may take place in the differential part as well as in the jump. In this case, system (1.1) may be written as

$$\begin{cases} x'(t) = & f(t, x(t), x(\tau(t))), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & I_i(\theta_i, x(\theta_i), x(\theta_{v(i)})), \quad i \in \mathbb{Z} \end{cases} \tag{1.2}$$

where the following conditions (C1.1) are assumed :

(a) $\tau \in PLC(\mathcal{J}, \mathbb{R}^+)$, $v : \mathbb{Z} \to \mathbb{Z}$ such that $\tau(t) < t$ and $v(i) < i$ respectively, and $\lim_{t \to \infty} \tau(t) = \infty$;

(b) $f(\cdot, x, y) \in PLC(\mathcal{J}, \mathbb{R}^n)$ and $f(t, \cdot, \cdot) \in C(\Omega, \mathbb{R}^n)$ where $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$;

(c) $I_i \in C(\mathcal{J} \times \Omega, \mathbb{R}^n)$, $i \in \mathbb{Z}$.

System (1.2) contains delays both in the differential equation and in the impulse condition whereas most of the systems considered in the literature have involved a delay only in the differential equation. Some, rather, have the same delay in the differential equation as well as in the impulse condition, see [33, 37, 39, 40, 41, 45, 51, 53, 74, 83] for further information. We shall refer to system (1.2) as the *impulsive differential system with delays*.

For each initial point $t_0 \in \mathbb{R}$, we let

$$t_{min} = \inf\{\tau(t) : t \geq t_0\}$$

then, the function

$$x(t) = \phi(t), \quad t \in [t_{min}, t_0] \tag{1.3}$$

is referred to as *initial function* associated with the impulsive differential system with delays (1.2). If $t_0 \in \mathbb{R}$, $\phi(t)$ is continuous on $[t_{min}, t_0]$ and (C1.1) is satisfied, then a solution of the initial function problem (1.2), (1.3) is equivalent to solving the integral equation [33]

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(s, x(s), x(\tau(s)))ds + \sum_{t_0 \leq \theta_i < t} I_i(\theta_i, x(\theta_i), x(\theta_{v(i)})), \quad t \geq t_0 \\ x(t) &= \phi(t), \quad t \in [t_{min}, t_0]. \end{aligned}$$

1.3 Existence and Uniqueness of the Solutions

Consider system (1.2) associated with the initial function (1.3). Let $\mathcal{J} \subset \mathbb{R}$ and $\{\theta_i\}$, $i \in \mathbb{Z}$, be fixed in $\mathcal{J}$. Denote by $i(a, b)$ the number of θ_i in (a, b). Assume the conditions (C1.2):

(a) $\theta_{i+1} > \theta_i$, $i \in \mathbb{Z}$ with no finite accumulation points;

(b) $i(t_{min}, t_0) > r$ where $r = \sup_i (i - v(i))$.

Definition 1.3.1 *A function $x(t)$ is said to be a* solution *of (1.2) defined on an interval $[t_0, T)$ where $t_0 < T \leq +\infty$ if $x \in PLC^1([t_{min}, T), \mathbb{R}^n)$ and satisfies system (1.2).*

Definition 1.3.2 *A function $x(t)$ is said to be a* solution *of the initial function problem (1.2), (1.3) in the interval $[t_0, T)$ where $t_0 < T \leq +\infty$ if $x(t)$ is a solution of system (1.2) in the interval $[t_0, T)$ and satisfies (1.3).*

We also need the following conditions (C1.3):

(a) There exists a constant $L \geq 0$ such that

$$\|f(t, x', y') - f(t, x'', y'')\| \leq L\Big(|x' - x''| + |y' - y''|\Big)$$

for all $t \in \mathcal{J}$ and $x', x'', y', y'' \in \mathbb{R}^n$;

(b) There exist constants $L_i \geq 0$, $i \in \mathbb{Z}$ such that

$$\|I_i(\theta_i, x', y') - I_i(\theta_i, x'', y'')\| \leq L_i\Big(|x' - x''| + |y' - y''|\Big)$$

for all $\theta_i \in \mathcal{J}$ and $x', x'', y', y'' \in \mathbb{R}^n$.

The following theorem on global existence and uniqueness of the solutions of the initial function problem (1.2), (1.3) is valid. For existence and uniqueness theorems for delay differential equations and for impulsive differential systems, we refer to [12, 14, 20] and [17, 18, 21], respectively.

Theorem 1.3.1 *Let conditions (C1.1)-(C1.3) be fulfilled. Let $t_0 \in \mathbb{R}$ and $\phi \in PLC([t_{min}, t_0], \mathbb{R}^n)$ be given. Then, the initial function problem (1.2), (1.3) has a unique solution defined on $\mathcal{J}$.*

Example 1.3.1 *Consider the linear system*

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t - \tau), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}) \end{cases} \tag{1.4}$$

under the assumptions that $A(t)$, $B(t) \in PLC\ (\mathcal{J}\ ,\mathbb{R}^{n\times n})$ *and* C_i, $D_i \in \mathbb{R}^{n\times n}$, $i, j \in \mathbb{Z}$. *Then (1.4) has a solution defined on the interval* $\mathcal{J}$.

1.4 Basic Definitions and Theorems

We shall provide some basic definitions and some auxiliary results needed in the subsequent chapters. As most of them can be easily found in the plentiful literature, we give only the necessary references without including their proofs, except that of *Arzela-Ascoli* Lemma.

Definition 1.4.1 *Let* f *be a real valued function on an interval* $[a, b]$ *of* $\mathbb{R}$. *If* $\Delta = \{x_j\}_{0\leq j\leq n}$ *is a partition of* $[a, b]$, *we write*

$$V(f, \Delta) = \sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)|;$$

we also write $V(f, a, b) = \sup_{\Delta} V(f, \Delta)$. *We say that* f *is of* bounded variation *on* $[a, b]$ *if there exists* N *such that*

$$|V(f, a, b)| \leq N.$$

Theorem 1.4.1 *[16] (**Banach-Steinhaus Theorem**) Let* $\{T_n\}$ *be a sequence of bounded linear operators* $T_n : X \to Y$ *from a Banach space* X *into a normed space* Y *such that* $(\|T_n x\|)$ *is bounded for every* $x \in X$, *say,*

$$\|T_n x\| \leq C_x, \qquad n \in \mathbb{N}$$

where C_x *is a real number. Then, there is a real number* $C \geq 0$ *such that*

$$\|T_n\| \leq C, \qquad n \in \mathbb{N}.$$

One of the most important results that arises when dealing with theory of operators is the *Arzela-Ascoli* Lemma. This result was first stated in [10] and then proved in [11]. Indeed, the author considered the general case where different functions of a family F have different points of discontinuities. In what follows, however, we shall give the proof of a simple version of this theorem which is stated without proof in [33]. Consider the space $PLC(\mathcal{J}, \mathbb{R}^n)$ where $\mathcal{J} \subset \mathbb{R}$. Let the sequence $\{\theta_i\}, i \in \mathbb{Z}$ have a finite number of terms lying in the interval $[t_0, T]$ with no accumulation points.

Definition 1.4.2 *Let F denote the set of functions $f_n \in PLC([t_0, T], \mathbb{R}^n)$. Then, the set F is said to be* quasi equicontinuous *in $[t_0, T]$ if for a given $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in (\theta_i, \theta_{i+1}] \bigcap [t_0, T]$, $i \in \mathbb{Z}$, and $|t_1 - t_2| < \delta$ then $|f_n(t_1) - f_n(t_2)| < \epsilon$ for all $f_n \in \mathrm{F}$.*

Lemma 1.4.1 *(**Arzela-Ascoli Lemma**) Let $F \subset PLC([t_0, T], \mathbb{R}^n)$. If F is quasi equicontinuous in $[t_0, T]$ and uniformly bounded, that is, $\|f_n\| \leq c$ for all $f_n \in F$ and for some $c > 0$, then it is relatively compact.*

Proof. Without loss of generality, let $[t_0, T] = \cup_{i=0}^{2} I_i$ where $I_0 = [t_0, \theta_1]$, $I_1 = (\theta_1, \theta_2]$ and $I_2 = (\theta_2, T]$. That is, F has two points of discontinuities θ_1 and θ_2. For arbitrary number of points of discontinuities, the proof is similar. Consider the sequence $\{f_n\} \subset F \subset PLC([0, T], \mathbb{R}^n)$. Let $f_n^0 = f_n|_{I_0}$ then $\{f_n^0\} \subset F|_{I_0} \subset C(I_0)$ where $C(I_0)$ is the space of continuous functions on I_0. From the hypothesis of the lemma, F is quasi equicontinuous and uniformly bounded which implies that $F|_{I_0}$ is equicontinuous and uniformly bounded, therefore, by *Arzela-Ascoli* Lemma for continuous functions, $F|_{I_0}$ is relatively compact, so there exists a subsequence $\{f^0_{n_k}\} \subset \{f^0_n\}$ such that $f^0_{n_k} \to g_0(t) \in C(I_0)$. Thus, one can say that the subsequence $\{f_{n_k}\} \subset \{f_n\}$ uniformly converges to g_0 on I_0. On the interval $I_1 = (\theta_1, \theta_2]$, let

$$f^1_{n_k} = \begin{cases} f_{n_k}(t), & t \in (\theta_1, \theta_2] \\ f_{n_k}(\theta_1^+), & t = \theta_1 \end{cases}$$

then $\{f^1_{n_k}\} \subset F|_{I_1} \subset C(I_1)$, but $F|_{I_1}$ is equicontinuous and uniformly bounded. Hence, by *Arzela-Ascoli* Lemma, $F|_{I_1}$ is relatively compact, so there exists $\{f^1_{n_{kl}}\} \subset \{f^1_{n_k}\}$ such that $f^1_{n_{kl}} \to g_1(t) \in C(I_1)$. Let

$$f^2_{n_{kl}} = \begin{cases} f_{n_{kl}}(t), & t \in (\theta_2, T] \\ f_{n_{k_l}}(\theta_2^+), & t = \theta_2 \end{cases}$$

then, similarly, there exists $\{f^2_{n_{k_{lm}}}\} \subset \{f^2_{n_{kl}}\}$ such that $f^2_{n_{klm}} \to g_2(t) \in C(I_2)$. Therefore, we end up with the subsequence

$$\{f_{n_{k_{lm}}}\} \subset \{f_{n_{k_l}}\} \subset \{f_{n_k}\} \subset \{f_n\}$$

which converges to $g(t)$ on $[0, T]$ where

$$g(t) = \begin{cases} g_0(t), & t \in I_0 = [t_0, \theta_1] \\ g_1(t), & t \in I_1 = (\theta_1, \theta_2] \\ g_2(t), & t \in I_2 = (\theta_2, T] \end{cases}$$

This completes the proof. $\square$

CHAPTER 2

THE ADJOINT SYSTEMS

The adjoint systems as well as the variation of parameters formulas play an important role in studying differential equations. In particular, they provide an essential and significant insights for the investigations of stability and periodicity of solutions. Therefore, in the study of impulsive differential systems with delays, the need for these two notions also arises.

In this chapter, and on the basis of the results obtained in the literature for delay differential equations, we study impulsive differential systems with delays in the differential equation and in the impulse condition. We obtain the adjoint systems and the variation of parameters formulas for these systems. Moreover, some generalizations of the main results are established.

2.1 Introduction

The results of this chapter stems from the studies in [12, 14, 17, 18, 20]. Indeed, it was shown in [12] that the equation

$$y'(t) = -A^T(t)y(t) - B^T(t+\tau)y(t+\tau), \tag{2.1}$$

is the adjoint equation for

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) \tag{2.2}$$

with respect to the function

$$< y(t), x(t) >= y^T(t)x(t) + \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)ds, \tag{2.3}$$

that is, $< y(t), x(t) >= C$ for all solutions $y(t)$ and $x(t)$ of systems (2.1) and (2.2) respectively. Moreover, it is also known [17] that the system

$$\begin{cases} y'(t) = & -A^T(t)y(t), \quad t \neq \theta_i \\ \Delta y(\theta_i) = & -(I + C_i^T)^{-1} C_i^T y(\theta_i), \quad i \in \mathbb{Z} \end{cases} \tag{2.4}$$

is the adjoint system for

$$\begin{cases} x'(t) = & A(t)x(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i), \quad i \in \mathbb{Z} \end{cases} \tag{2.5}$$

in the sense that $y^T(t)x(t) = C$ for all solutions $y(t)$ and $x(t)$ of (2.4) and (2.5) respectively. Furthermore, the representations of the solutions of (2.2) and (2.5) are available, see [12, 17, 18, 20].

We consider the linear impulsive differential system with delays of the form

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t - \tau), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i \in \mathbb{Z} \end{cases} \tag{2.6}$$

where the conditions (C2.1) hold:

(a) $A(t)$, $B(t) \in PLC(\mathcal{J}, \mathbb{R}^{n\times n})$;

(b) C_i, $D_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{Z}$.

We should remark that in the case when $C_i = D_i \equiv 0$, system (2.6) reduces to (2.2) and if $B(t) = D_i \equiv 0$, it reduces to system (2.5).

2.2 The Adjoint System

The purpose of this section is to construct an adjoint system for (2.6). To do this, we introduce a function which is analogous to (2.3) as follows

$$< y(t), x(t) > \;=\; y^T(t)x(t) + \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)ds$$

$$+ \sum_{n(t)\leq k<n(t)+j} y^T(\theta_k^+)D_k x(\theta_{k-j}), \tag{2.7}$$

where

$$n(t) = \min\{k \in \mathbb{Z} : \theta_k \geq t\}.$$

Theorem 2.2.1 *Let $x(t)$ be a solution of (2.6). If $y(t)$ is a solution of*

$$\begin{cases} y'(t) = & -A^T(t)y(t) - B^T(t+\tau)y(t+\tau), \quad t \neq \theta_i \\ \Delta y(\theta_i) = & -(I+C_i^T)^{-1}C_i^T y(\theta_i) - (I+C_i^T)^{-1}D_{i+j}^T y(\theta_{i+j}^+), \quad i \in \mathbb{Z} \end{cases} \tag{2.8}$$

then $< y(t), x(t) >= C$ where $< y(t), x(t) >$ is given by (2.7).

Proof. Taking the derivative of function (2.7), for $t \in (\theta_i, \theta_{i+1})$, we get

$$\begin{aligned} \frac{d}{dt} < y(t), x(t) > &= y^{T\prime}(t)x(t) + y^T(t)x'(t) + y^T(t+\tau)B(t+\tau)x(t) \\ &- y^T(t)B(t)x(t-\tau), \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt} < y(t), x(t) > &= -y^T(t)A(t)x(t) - y^T(t+\tau)B(t+\tau)x(t) \\ &+ y^T(t)A(t)x(t) + y^T(t)B(t)x(t-\tau) \\ &+ y^T(t+\tau)B(t+\tau)x(t) - y^T(t)B(t)x(t-\tau) \\ &= 0, \end{aligned}$$

therefore,

$$< y(t), x(t) >= C_i, \quad t \in (\theta_i, \theta_{i+1}).$$

It remains to show that the constants C_i are all equal, that is, $C_i = C$ for all $i \in \mathbb{Z}$. Clearly

$$\Delta < y(t), x(t) > |_{t=\theta_i} = C_{i+1} - C_i.$$

In view of function (2.7), we get

$$\begin{aligned} C_{i+1} - C_i &= y^T(\theta_i^+)x(\theta_i^+) - y^T(\theta_i)x(\theta_i) + \sum_{n(\theta_i^+)\le k\le n(\theta_i^+)+j} y^T(\theta_k^+)D_k x(\theta_{k-j}) \\ &- \sum_{n(\theta_i)\le k\le n(\theta_i)+j} y^T(\theta_k^+)D_k x(\theta_{k-j}). \end{aligned}$$

Since $n(\theta_i^+) = i+1$ and $n(\theta_i) = i$, we have

$$\begin{aligned} C_{i+1} - C_i &= y^T(\theta_i^+)x(\theta_i^+) - y^T(\theta_i)x(\theta_i) - y^T(\theta_i^+)D_i x(\theta_{i-j}) \\ &+ y^T(\theta_{i+j}^+)D_{i+j}x(\theta_i). \end{aligned}$$

Using

$$x(\theta_i^+) = x(\theta_i) + C_i x(\theta_i) + D_i x(\theta_{i-j}),$$

and

$$D_{i+j}^T y(\theta_{i+j}^+) = y(\theta_i) - y(\theta_i^+) - C_i^T y(\theta_i^+),$$

we deduce that $C_{i+1} - C_i = 0$ for all $i \in \mathbb{Z}$ and thus $< y(t), x(t) >= C$. □

Definition 2.2.1 *System (2.8) is called the adjoint of (2.6).*

2.3 The Variation of Parameters Formula

The representations of the solutions of differential systems were derived by many authors using various techniques, see [12, 17, 18, 20]. In this section, however, we derive a variation of parameters formulas for impulsive differential systems with delays in terms of their fundamental matrices.

Consider the nonhomogeneous impulsive differential system with delays

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}) + J_i, \quad i \in \mathbb{Z} \end{cases} \tag{2.9}$$

where in addition to (C2.1), we assume the conditions (C2.2):

(a) $f(t) \in PLC(\mathcal{J},\mathbb{R}^n)$;

(b) $J_i \in \mathbb{R}^n$, $i \in \mathbb{Z}$.

Let $X(\alpha, t)$ be matrix solution of (2.6) defined for $\alpha > t$ where $t \in \mathbb{R}$ is fixed. This matrix satisfies the conditions $X(\alpha, t) \equiv 0$ for $\alpha < t$ and $X(t, t) = I$, where I is the identity matrix. Such a matrix is called a fundamental matrix of (2.6). Similarly, one can define the fundamental matrix of the adjoint system (2.8). Indeed, a matrix $Y(\alpha, t)$ which satisfies system (2.8) for $\alpha < t$, the conditions $Y(\alpha, t) \equiv 0$ for $\alpha > t$ and $Y(t, t) = I$, is a fundamental matrix of (2.8). It is clear that the matrix $X(\alpha, t)$ is defined for all $\alpha \geq t,\ \alpha \in \mathcal{J}$ while $Y(\alpha, t)$ is defined for all $\alpha \leq t,\ \alpha \in \mathcal{J}$.

Remark 2.3.1 *The fundamental matrices can be simply constructed by the method of steps, see [12, 27].*

The following two lemmas are required in the proofs of the main theorems

Lemma 2.3.1 *Let $Y(\alpha, t)$ be the fundamental matrix of (2.8) and $x(t)$ a solution of (2.9). Then, we have*

$$\sum_{n(\sigma)\leq k<n(t)} \Big[Y^T(\theta_k^+, t)x(\theta_k^+) \;-\; Y^T(\theta_k, t)x(\theta_k)\Big] =$$

$$\sum_{n(\sigma)-j\le k<n(\sigma)} Y^T(\theta_{k+j}^+,t)D_{k+j}x(\theta_k)$$
$$+\sum_{n(\sigma)\le k<n(t)} Y^T(\theta_k^+,t)J_k.$$

Proof. We observe that

$$\begin{aligned}\sum_{n(\sigma)\le k<n(t)}\Big[Y^T(\theta_k^+,t)x(\theta_k^+)-Y^T(\theta_k,t)x(\theta_k)\Big] &= \sum_{n(\sigma)\le k<n(t)} Y^T(\theta_k^+,t)D_kx(\theta_{k-j})\\ &- \sum_{n(\sigma)\le k<n(t)} Y^T(\theta_{k+j}^+,t)D_{k+j}x(\theta_k)\\ &+ \sum_{n(\sigma)\le k<n(t)} Y^T(\theta_k^+,t)J_k \qquad (2.10)\end{aligned}$$

where the impulse conditions of systems (2.8) and (2.9) are, respectively, used. In view of the right side of (2.10), substituting $m=k-j$ in the first summation and using $Y^T(\theta_{m+j}^+,t)\equiv 0$ for $n(t)-j\le m<n(t)$ in the second summation complete the proof. $\square$

Lemma 2.3.2 *Let $X(\alpha,t)$ be the fundamental matrix of (2.6) and $y(t)$ be a solution of (2.8). Then, we get*

$$\begin{aligned}\sum_{n(\sigma)\le k<n(t)}\Big[X^T(\theta_k,t)y(\theta_k) \;-\; X^T(\theta_k^+,t)y(\theta_k^+)\Big] &=\\ \sum_{n(\sigma)\le k<n(t)} X^T(\theta_k,t)D_{k+j}^Ty(\theta_{k+j}^+)&\\ -\sum_{n(\sigma)\le k<n(t)} X^T(\theta_{k-j},t)D_k^Ty(\theta_k^+).&\end{aligned}$$

The proof of this lemma can be accomplished simply using the impulse conditions of systems (2.6) and (2.8) respectively.

One of the main results in this section is contained in the following theorem

Theorem 2.3.1 *A solution $x(t)$ of (2.9) has the representation*

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha)d\alpha + \int_{\sigma}^{t} X(t,\alpha)f(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le k<n(\sigma)} X(t,\theta_{k+j}^{+})D_{k+j}x(\theta_k) + \sum_{n(\sigma)\le k<n(t)} X(t,\theta_k^{+})J_k, \end{aligned}$$

where $X(t,r)$ is the fundamental matrix of (2.6) and $x(\sigma)$ is fixed.

Proof. Multiplying the differential equation in (2.9) by the matrix $Y^T(\alpha,t)$ from left and integrating with respect to α from σ to t, we obtain

$$\begin{aligned} \int_{\sigma}^{t} Y^T(\alpha,t)x'(\alpha)d\alpha &= \int_{\sigma}^{t} Y^T(\alpha,t)A(\alpha)x(\alpha)d\alpha + \int_{\sigma}^{t} Y^T(\alpha,t)B(\alpha)x(\alpha-\tau)d\alpha \\ &+ \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha. \end{aligned}$$

Clearly, the left side of the above equation can be written as

$$\begin{aligned} \int_{\sigma}^{\theta_{n_1}^-} Y^T(\alpha,t)x'(\alpha)d\alpha &+ \int_{\theta_{n_1}^+}^{\theta_{n_1+1}^-} Y^T(\alpha,t)x'(\alpha)d\alpha + \dots \\ \dots &+ \int_{\theta_{n_2-1}^+}^{\theta_{n_2}^-} Y^T(\alpha,t)x'(\alpha)d\alpha + \int_{\theta_{n_2}^+}^{t} Y^T(\alpha,t)x'(\alpha)d\alpha. \end{aligned}$$

Integrating by parts and taking into account that $Y(\alpha,t)$ is a fundamental matrix of (2.8), we get

$$\begin{aligned} x(t) &= Y^T(\sigma,t)x(\sigma) - \int_{\sigma}^{t} Y^T(\alpha+\tau,t)B(\alpha+\tau)x(\alpha)d\alpha \\ &+ \int_{\sigma}^{t} Y^T(\alpha,t)B(\alpha)x(\alpha-\tau)d\alpha + \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha \\ &+ \sum_{n(\sigma)\le k<n(t)} \Big[Y^T(\theta_k^+,t)x(\theta_k^+) - Y^T(\theta_k,t)x(\theta_k)\Big]. \end{aligned}$$

Substituting $\gamma = \alpha - \tau$ in the second integral and using $Y^T(\alpha+\tau, t) \equiv 0$ for $t - \tau < \alpha \le t$, we have

$$\begin{aligned} x(t) &= Y^T(\sigma,t)x(\sigma) + \int_{\sigma-\tau}^{\sigma} Y^T(\alpha+\tau,t)B(\alpha+\tau)x(\alpha)d\alpha + \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha \\ &+ \sum_{n(\sigma)\le k<n(t)} \Big[Y^T(\theta_k^+,t)x(\theta_k^+) - Y^T(\theta_k,t)x(\theta_k)\Big]. \end{aligned}$$

Using Lemma 2.3.1 in the summation leads to

$$\begin{aligned} x(t) &= Y^T(\sigma,t)x(\sigma) + \int_{\sigma-\tau}^{\sigma} Y^T(\alpha+\tau,t)B(\alpha+\tau)x(\alpha)d\alpha + \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le k<n(\sigma)} Y^T(\theta_{k+j}^+,t)D_{k+j}x(\theta_k) + \sum_{n(\sigma)\le k<n(t)} Y^T(\theta_k^+,t)J_k. \end{aligned} \tag{2.11}$$

If the solution $x(t)$ is defined to be zero for $t < \sigma$ and if $f \equiv 0$, $J_i \equiv 0$, then

$$x(t) = Y^T(\sigma,t)x(\sigma).$$

Therefore, using $X(\sigma,\sigma) = I$, we see at once that $X(t,\sigma) = Y^T(\sigma,t)$.

Thus, (2.11) becomes

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha)d\alpha + \int_{\sigma}^{t} X(t,\alpha)f(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le k<n(\sigma)} X(t,\theta_{k+j}^+)D_{k+j}x(\theta_k) + \sum_{n(\sigma)\le k<n(t)} X(t,\theta_k^+)J_k. \quad \square \end{aligned} \tag{2.12}$$

The representation in this form will be referred to as the *variation of constants* or the *variation of parameters* formula.

Remark 2.3.2 *In the case when the impulses are absent, (2.12) is reduced to a well known formula*

$$x(t) = X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t,\alpha+\tau)B(\alpha+\tau)x(\alpha)d\alpha$$

given in [12, 20].

In a similar manner, we may use the matrix $X(\alpha, t)$ to obtain the variation of parameters formula for the solution of the adjoint system (2.8).

Theorem 2.3.2 *A solution $y(t)$ of (2.8) has the representation*

$$\begin{aligned} y(t) &= Y(t,\sigma)y(\sigma) + \int_{\sigma}^{\sigma+\tau} Y(t, s-\tau)B^T(s)y(s)ds \\ &- \sum_{n(\sigma)\le k<n(\sigma)+j} Y(t,\theta_{k-j})D_k^T y(\theta_k^+), \end{aligned}$$

where $Y(t,r)$ is the fundamental matrix of (2.8) and $y(\sigma)$ is fixed.

Proof. Multiplying the differential equation in (2.8) by $X^T(s,t)$ from left and integrating from t to σ, we have

$$\begin{aligned} \int_t^{\sigma} X^T(s,t)y'(s)ds &= -\int_t^{\sigma} X^T(s,t)A^T(s)y(s)ds \\ &- \int_t^{\sigma} X^T(s,t)B^T(s+\tau)y(s+\tau)ds. \end{aligned}$$

The left side of the above equation can be written as

$$\begin{aligned} \int_t^{\theta_{n_1}^-} X^T(s,t)y'(s)ds &+ \int_{\theta_{n_1}^+}^{\theta_{n_1+1}^-} X^T(s,t)y'(s)ds + \dots \\ \dots &+ \int_{\theta_{n_2-1}^+}^{\theta_{n_2}^-} X^T(s,t)y'(s)ds + \int_{\theta_{n_2}^+}^{\sigma} X^T(s,t)y'(s)ds. \end{aligned}$$

Further, we have

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) - \int_t^{\sigma} X_s^T(s,t)y(s)ds + \int_t^{\sigma} X^T(s,t)A^T(s)y(s)ds \\ &+ \int_t^{\sigma} X^T(s,t)B^T(s+\tau)y(s+\tau)ds \\ &+ \sum_{n(\sigma)\le k<n(t)} \Big[X^T(\theta_k,t)y(\theta_k) - X^T(\theta_k^+,t)y(\theta_k^+)\Big], \end{aligned}$$

where $X_s^T(s,t) = \partial/ds X^T(s,t)$. Taking into account that $X(s,t)$ is a fundamental matrix of (2.6), changing the variable of the third integral $r = s + \tau$, and using $X^T(s-\tau,t) \equiv 0$ for $s - \tau < t$, we deduce

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_\sigma^{\sigma+\tau} X^T(s-\tau,t)B^T(s)y(s)ds \\ &+ \sum_{n(\sigma)\le k<n(t)} \Big[X^T(\theta_k,t)y(\theta_k) - X^T(\theta_k^+,t)y(\theta_k^+)\Big]. \end{aligned}$$

Using Lemma 2.3.2 in the summation gives

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_\sigma^{\sigma+\tau} X^T(s-\tau,t)B^T(s)y(s)ds \\ &+ \sum_{n(\sigma)\le k<n(t)} X^T(\theta_k,t)D_{k+j}^T y(\theta_{k+j}^+) \\ &- \sum_{n(\sigma)\le k<n(t)} X^T(\theta_{k-j},t)D_k^T y(\theta_k^+). \end{aligned}$$

Substituting $r = k + j$ in the first summation and using $X^T(\theta_{k-j},t) \equiv 0$ for $k - j < n(t)$, we get

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_\sigma^{\sigma+\tau} X^T(s-\tau,t)B^T(s)y(s)ds \\ &- \sum_{n(\sigma)\le k<n(\sigma)+j} X^T(\theta_{k-j},t)D_k^T y(\theta_k^+). \end{aligned}$$

From the identity $X^T(\sigma,t) = Y(t,\sigma)$, it follows that

$$\begin{aligned} y(t) &= Y(t,\sigma)y(\sigma) + \int_\sigma^{\sigma+\tau} Y(t,s-\tau)B^T(s)y(s)ds \\ &- \sum_{n(\sigma)\le k<n(\sigma)+j} Y(t,\theta_{k-j})D_k^T y(\theta_k^+). \quad \Box \end{aligned} \tag{2.13}$$

2.4 Some Generalizations

In this section we carry out the results obtained in the previous sections for the system of the form

$$\begin{cases} x'(t) = & \int_{-\tau}^{0} d_s\eta(t,s)x(t+s), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & A_{i0}x(\theta_i) + \sum_{-j\leq k<-1} A_{ik}x(\theta_{i+k}), \quad i \in \mathbb{Z}, \end{cases} \tag{2.14}$$

where the conditions (C2.3) are assumed to hold:

(a) $\eta(t,s)$ is measurable in $(t,s) \in \mathbb{R}^2$, normalized so that $\eta(t,s) \equiv 0$ for $s \geq 0$ and $\eta(t,s) = \eta(t,-\tau)$ for $s \leq -\tau$;

(b) $\eta(\cdot,s) \in PLC(\mathcal{J},\mathbb{R}^{n\times n})$, uniformly with respect to $s \in [-\tau,0]$;

(c) $\eta(t,s)$ has bounded variation in $s \in [-\tau,0]$ for $t \in \mathcal{J}$ so there exists a positive constant N such that $Var_{[-r,0]}\eta(t,\cdot) \leq N$, t $\in \mathcal{J}$;

(d) There exists a positive constant W such that $\sum_{k=1}^{n} |A_{ik}| \leq W$ for $i \in \mathbb{Z}$.

The above hypothesis on η guarantees that for any $\phi \in C([-\tau,0],\mathbb{R}^n)$, system (2.14) has a unique solution on $[-\tau,\infty)$. The integral in (2.14) is often referred to as *Stieltjes* integral.

The following property of the *Stieltjes* integral proves very useful.

Theorem 2.4.1 *[12] If $x(s) \in PLC([a,b],\mathbb{R}^n)$, $\gamma(t)$ is of bounded variation in $[c,d]$, $\eta(\cdot,s) \in PLC(\mathcal{J},\mathbb{R}^{n\times n})$ uniformly with respect to $s \in [a,b]$ and is of bounded variation in $s \in [a,b]$ for $t \in [c,d]$, then*

$$\int_{c}^{d} \Big[\int_{a}^{b} d_s\eta(t,s)x(s)\Big] d\gamma(t) = \int_{a}^{b} d_s\Big[\int_{c}^{d} \eta(t,s)d\gamma(t)\Big]x(s).$$

Let us note that the same result remains valid in the case when the functions which occur are matrix valued. Indeed, we have

$$\int_a^b \int_c^d x^T(s) d_s \eta(\alpha, s-\alpha) y(\alpha) d\alpha = \int_c^d x^T(s) d_s \int_a^b \eta(\alpha, s-\alpha) y(\alpha) d\alpha.$$

Remark 2.4.1 *System (2.6) considered in section 2.2 is a particular case of (2.14).*

By virtue of the generalized linear system (2.14), the analogous form of function (2.7) becomes

$$\begin{aligned} < x(t), y(t) > &= x^T(t) y(t) + \int_{t-\tau}^{t} x^T(s) d_s \int_t^{s+\tau} \eta^T(\alpha, s-\alpha) y(\alpha) d\alpha \\ &+ \sum_{n(t)-j \le m < n(t)} x^T(\theta_m) \sum_{n(t) \le i < m+j} A^T_{i(m-i)} y(\theta_i^+). \end{aligned} \tag{2.15}$$

The following theorem is similar to Theorem 2.2.1.

Theorem 2.4.2 *Let $x(t)$ be a solution of (2.14). If $y(t)$ is a solution of*

$$\begin{aligned} y'(t) &= -d_t \int_{-\tau}^{0} \eta^T(t-s, s) y(t-s), \quad t \neq \theta_i \qquad (2.16) \\ \Delta y(\theta_i) &= -(I + A^T_{i0})^{-1} A^T_{i0} y(\theta_i) - (I + A^T_{i0})^{-1} \sum_{-j \le k < -1} A^T_{(i-k)k} y(\theta^+_{i-k}) \end{aligned}$$

then $< x(t), y(t) >= C$ where $< x(t), y(t) >$ is given by (2.15).

We will rely on the following two lemmas. The first lemma can be simply verified by interchanging the order of summations.

Lemma 2.4.1 *For any functions $x(t), y(t) : \mathbb{R} \to \mathbb{R}^n$ and a matrix $A_{im} \in \mathbb{R}^{n\times n}$, we have*

$$\begin{aligned}\sum_{n(\sigma)\le i<n(t)} \Big[\sum_{i-1\le m<i-j} x^T(\theta_m) \quad A^T_{i(m-i)}\Big] y(\theta_i^+) = &\\ &\sum_{n(\sigma)-j\le m<n(\sigma)} x^T(\theta_m) \sum_{n(\sigma)\le i<m+j} A^T_{i(m-i)} y(\theta_i^+)\\ + &\sum_{n(\sigma)\le m<n(t)-j} x^T(\theta_m) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+)\\ + &\sum_{n(t)-j\le m<n(t)} x^T(\theta_m) \sum_{m+1\le i<n(t)} A^T_{i(m-i)} y(\theta_i^+).\end{aligned}$$

Lemma 2.4.2 *Let $x(t)$ and $y(t)$ be solutions of systems (2.14) and (2.16), respectively. Then, we get*

$$\begin{aligned}\sum_{n(\sigma)\le i<n(t)} \Big[x^T(\theta_i^+)y(\theta_i^+) \quad - \quad x^T(\theta_i)y(\theta_i)\Big] = &\\ &\sum_{n(\sigma)\le i<n(t)} \Big[\sum_{-j\le k<-1} x^T(\theta_{i+k})A^T_{ik}\Big] y(\theta_i^+)\\ - &\sum_{n(\sigma)\le i<n(t)} x^T(\theta_i) \sum_{-j\le k<-1} A^T_{(i-k)k} y(\theta^+_{i-k}).\end{aligned}$$

Proof. It suffices to note that

$$x(\theta_i^+) = x(\theta_i) + A_{i0}x(\theta_i) + \sum_{-j\le k<-1} A_{ik}x(\theta_{i+k})$$

and

$$\sum_{-j\le k<-1} A^T_{(i-k)k} y(\theta^+_{i-k}) = y(\theta_i) - (I + A^T_{i0})y(\theta_i^+). \quad \square$$

Proof of Theorem 2.4.2. Multiplying the differential equation in (2.14) by

$y(t)$ and integrating from σ to t, we get

$$\int_\sigma^t x^{T\prime}(\alpha)y(\alpha))d\alpha = \int_\sigma^t \Big[\int_{-\tau}^0 x^T(\alpha+s)d_s\eta^T(\alpha,s)\Big]y(\alpha)d\alpha.$$

Clearly, the left side of the above equation can be written as

$$\begin{aligned}\int_\sigma^{\theta_{n_1}^-} x^{T\prime}(\alpha)y(\alpha)d\alpha \;&+\; \int_{\theta_{n_1}^+}^{\theta_{n_1+1}^-} x^{T\prime}(\alpha)y(\alpha)d\alpha + \dots \\ \dots \;&+\; \int_{\theta_{n_2-1}^+}^{\theta_{n_2}^-} x^{T\prime}(\alpha)y(\alpha)d\alpha + \int_{\theta_{n_2}^+}^{t} x^{T\prime}(\alpha)y(\alpha)d\alpha.\end{aligned}$$

It follows that

$$\begin{aligned}x^T(t)y(t) - x^T(\sigma)y(\sigma) \;&=\; \int_\sigma^t \Big[\int_{-\tau}^0 x^T(\alpha+s)d_s\eta^T(\alpha,s)\Big]y(\alpha)d\alpha \\ &+\; \int_\sigma^t x^T(\alpha)y'(\alpha)d\alpha \\ &+\; \sum_{n(\sigma)\le i<n(t)} \Big[x^T(\theta_i^+)y(\theta_i^+) - x^T(\theta_i)y(\theta_i)\Big].\end{aligned}$$

Using Lemma 2.4.2 in the summation and taking into account the system verified by y, we have

$$\begin{aligned}x^T(t)y(t) - x^T(\sigma)y(\sigma) \;&=\; \int_\sigma^t \Big[\int_{-\tau}^0 x^T(\alpha+s)d_s\eta^T(\alpha,s)\Big]y(\alpha)d\alpha \\ &-\; \int_\sigma^t x^T(\alpha)d_\alpha \int_{-\tau}^0 \eta^T(\alpha-s,s)y(\alpha-s)ds \\ &+\; \sum_{n(\sigma)\le i<n(t)} \Big[\sum_{-j\le k<-1} x^T(\theta_{i+k})A_{ik}^T\Big]y(\theta_i^+) \\ &-\; \sum_{n(\sigma)\le i<n(t)} x^T(\theta_i) \sum_{-j\le k<-1} A_{(i-k)k}^T y(\theta_{i-k}^+).\end{aligned}$$

Substituting $r = \alpha + s$ in the first integral, $r = \alpha - s$ in the second integral, $m = i + k$ in the first summation and $m = i - k$ in the second summation, we obtain

$$x^T(t)y(t) - x^T(\sigma)y(\sigma) \;=\; \int_\sigma^t \Big[\int_{\alpha-\tau}^\alpha x^T(s)d_s\eta^T(\alpha,s-\alpha)\Big]y(\alpha)d\alpha$$

$$
\begin{aligned}
- & \int_{\sigma}^{t} x^T(s)d_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha))d\alpha \\
+ & \sum_{n(\sigma)\le i<n(t)} \Big[\sum_{i-1\le m<i-j} x^T(\theta_m)A^T_{i(m-i)} \Big] y(\theta_i^+) \\
- & \sum_{n(\sigma)\le i<n(t)} x^T(\theta_i) \sum_{i+1\le m<i+j} A^T_{m(i-m)} y(\theta_m^+).
\end{aligned}
$$

Using Theorem 2.4.1 in the first integral and Lemma 2.4.1 in the first summation, we have

$$
\begin{aligned}
x^T(t)y(t) - x^T(\sigma)y(\sigma) = & \int_{\sigma-\tau}^{\sigma} x^T(s)d_s \int_{\sigma}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
+ & \int_{\sigma}^{t-\tau} x^T(s)d_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
+ & \int_{t-\tau}^{t} x^T(s)d_s \int_{s}^{t} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
- & \int_{\sigma}^{t} x^T(s)d_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
+ & \sum_{n(\sigma)-j\le m<n(\sigma)} x^T(\theta_m) \sum_{n(\sigma)\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
+ & \sum_{n(\sigma)\le m<n(t)-j} x^T(\theta_m) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
+ & \sum_{n(t)-j\le m<n(t)} x^T(\theta_m) \sum_{m+1\le i<n(t)} A^T_{i(m-i)} y(\theta_i^+) \\
- & \sum_{n(\sigma)\le m<n(t)} x^T(\theta_m) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+). \quad (2.17)
\end{aligned}
$$

The last integral and the last summation in (2.17) can be written, respectively, as

$$
\begin{aligned}
& \int_{\sigma}^{t-\tau} x^T(s)d_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
& \quad + \int_{t-\tau}^{t} x^T(s)d_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha,
\end{aligned}
$$

and

$$
\sum_{n(\sigma)\le m<n(t)-j} x^T(\theta_m) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+)
$$

$$+ \sum_{n(t)-j\leq m<n(t)} x^T(\theta_m) \sum_{m+1\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+).$$

Thus, (2.17) becomes

$$\begin{aligned}
x^T(t)y(t) - x^T(\sigma)y(\sigma) &= \int_{\sigma-\tau}^{\sigma} x^T(s)d_s \int_{\sigma}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
&+ \int_{t-\tau}^{t} x^T(s)d_s \int_{s}^{t} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
&- \int_{t-\tau}^{t} x^T(s)d_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
&+ \sum_{n(\sigma)-j\leq m<n(\sigma)} x^T(\theta_m) \sum_{n(\sigma)\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&+ \sum_{n(t)-j\leq m<n(t)} x^T(\theta_m) \sum_{m+1\leq i<n(t)} A^T_{i(m-i)} y(\theta_i^+) \\
&- \sum_{n(t)-j\leq m<n(t)} x^T(\theta_m) \sum_{m+1\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+).
\end{aligned}$$

Therefore

$$\begin{aligned}
x^T(t)y(t) - x^T(\sigma)y(\sigma) &= \int_{\sigma-\tau}^{\sigma} x^T(s)d_s \int_{\sigma}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
&- \int_{t-\tau}^{t} x^T(s)d_s \int_{t}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
&+ \sum_{n(\sigma)-j\leq m<n(\sigma)} x^T(\theta_m) \sum_{n(\sigma)\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&- \sum_{n(t)-j\leq m<n(t)} x^T(\theta_m) \sum_{n(t)\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+),
\end{aligned}$$

or

$$\begin{aligned}
< x(t), y(t) > &= x^T(t)y(t) + \int_{t-\tau}^{t} x^T(s)d_s \int_{t}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha \\
&+ \sum_{n(t)-j\leq m<n(t)} x^T(\theta_m) \sum_{n(t)\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+). \qquad (2.18)
\end{aligned}$$

Remark 2.4.2 *For any two solutions $x(t)$ and $y(t)$ of systems (2.14) and (2.16) we have*

$$\frac{d}{dt} < x(t), y(t) >= 0. \tag{2.19}$$

Definition 2.4.1 *In view of (2.19), system (2.16) is called the adjoint of (2.14).*

Remark 2.4.3 *In comparison with the results obtained in the literature for delay differential equations without impulses, relation (2.18) closely resembles the function considered in [12, 28]. In particular, the function (2.18) reduces to*

$$< x(t), y(t) >= x^T(t)y(t) + \int_{t-\tau}^{t} x^T(s)d_s \int_{t}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)d\alpha$$

whenever no impulse is imposed.

We shall now consider the nonhomogeneous system

$$\begin{cases} x'(t) = & \int_{-\tau}^{0} d_s\eta(t,s)x(t+s) + f(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & A_{i0}x(\theta_i) + \sum_{-j\leq k<-1} A_{ik}x(\theta_{i+k}) + J_i, \quad i \in \mathbb{Z}, \end{cases} \tag{2.20}$$

where conditions (C2.2) and (C2.3) are assumed to hold.

Before introducing the representation of the solution of system (2.20) we need to give the following lemma whose proof is similar to that of Lemma 2.4.2 and hence is omitted.

Lemma 2.4.3 *Let $Y(\alpha, t)$ be the fundamental matrix of (2.16) and $x(t)$ be a solution of system (2.20). Then, we have*

$$\sum_{n(\sigma)\leq i<n(t)} \Big[Y^T(\theta_i^+, t)x(\theta_i^+) \; - \; Y^T(\theta_i, t)x(\theta_i)\Big] =$$

$$
\begin{aligned}
&\sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t)\sum_{-j\le k<-1} A_{ik}x(\theta_{i+k})\\
-&\sum_{n(\sigma)\le i<n(t)}\Big[\sum_{-j\le k<-1} Y^T(\theta_{i-k}^+,t)A_{(i-k)k}\Big]x(\theta_i)\\
+&\sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t)J_i.
\end{aligned}
$$

Theorem 2.4.3 *A solution $x(t)$ of (2.20) has the representation*

$$
\begin{aligned}
x(t) &= X(t,\sigma)x(\sigma)+\int_{\sigma-\tau}^{\sigma} d_s\Big[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)\\
&+ \int_{\sigma}^{t} X(t,\alpha)f(\alpha)d\alpha\\
&+ \sum_{n(\sigma)-j\le m<n(\sigma)}\Big[\sum_{n(\sigma)\le i<m+j} X(t,\theta_i^+)A_{i(m-i)}\Big]x(\theta_m)\\
&+ \sum_{n(\sigma)\le i<n(t)} X(t,\theta_i^+)J_i,
\end{aligned}
$$

where $X(t,r)$ is the fundamental matrix of (2.14) and $x(\sigma)$ is fixed.

Proof. Multiplying the differential equation in (2.20) by $Y^T(\alpha,t)$ from left and integrating from σ to t, we get

$$
\int_{\sigma}^{t} Y^T(\alpha,t)x'(\alpha)d\alpha=\int_{\sigma}^{t} Y^T(\alpha,t)\Big[\int_{-\tau}^{0} d_s\eta(\alpha,s)x(\alpha+s)\Big]d\alpha+\int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha.
$$

Integrating by parts and taking into account that $Y(\alpha,t)$ is a fundamental matrix of (2.16), we have

$$
\begin{aligned}
x(t)-Y^T(\sigma,t)x(\sigma) &= \int_{\sigma}^{t} Y^T(\alpha,t)\Big[\int_{-\tau}^{0} d_s\eta(\alpha,s)x(\alpha+s)\Big]d\alpha\\
&- \int_{\sigma}^{t} d_\alpha\Big[\int_{-\tau}^{0} Y^T(\alpha-s,t)\eta(\alpha-s,s)ds\Big]x(\alpha)\\
&+ \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha\\
&+ \sum_{n(\sigma)\le i<n(t)}\Big[Y^T(\theta_i^+,t)x(\theta_i^+)-Y^T(\theta_i,t)x(\theta_i)\Big].
\end{aligned}
$$

Using Lemma 2.4.3 in the summation, we obtain

$$
\begin{aligned}
x(t) - Y^T(\sigma,t)x(\sigma) &= \int_\sigma^t Y^T(\alpha,t)\Big[\int_{-\tau}^0 d_s\eta(\alpha,s)x(\alpha+s)\Big]d\alpha \\
&- \int_\sigma^t d_\alpha\Big[\int_{-\tau}^0 Y^T(\alpha-s,t)\eta(\alpha-s,s)ds\Big]x(\alpha) \\
&+ \int_\sigma^t Y^T(\alpha,t)f(\alpha)d\alpha \\
&+ \sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t) \sum_{-j\le k<-1} A_{ik}x(\theta_{i+k}) \\
&- \sum_{n(\sigma)\le i<n(t)} \Big[\sum_{-j\le k<-1} Y^T(\theta_{i-k}^+,t)A_{(i-k)k}\Big]x(\theta_i) \\
&+ \sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t)J_i.
\end{aligned}
$$

Substituting $r = \alpha + s$ in the first integral, $r = \alpha - s$ in the second integral, $m = i+k$ in the first summation and $m = i-k$ in the second summation, we get

$$
\begin{aligned}
x(t) &= Y^T(\sigma,t)x(\sigma) + \int_\sigma^t Y^T(\alpha,t)\Big[\int_{\alpha-\tau}^\alpha d_s\eta(\alpha,s-\alpha)x(s)\Big]d\alpha \\
&- \int_\sigma^t d_\alpha\Big[\int_\alpha^{\alpha+\tau} Y^T(r,t)\eta(r,\alpha-r)dr\Big]x(\alpha) \\
&+ \int_\sigma^t Y^T(\alpha,t)f(\alpha)d\alpha \\
&+ \sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t)\Big[\sum_{i-1\le m<i-j} A_{i(m-i)}x(\theta_m)\Big] \\
&- \sum_{n(\sigma)\le i<n(t)} \Big[\sum_{i+1\le m<i+j} Y^T(\theta_m^+,t)A_{m(i-m)}\Big]x(\theta_i) \\
&+ \sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t)J_i.
\end{aligned}
$$

Using Theorem 2.4.1 in the first integral and changing the order of summations in the first sum leads to

$$
\begin{aligned}
x(t) &= Y^T(\sigma,t)x(\sigma) + \int_{\sigma-\tau}^\sigma d_s\Big[\int_\sigma^{s+\tau} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s) \\
&+ \int_\sigma^{t-\tau} d_s\Big[\int_s^{s+\tau} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{t-\tau}^{t} d_s\Big[\int_s^t Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)\\
&- \int_{\sigma}^{t} d_s\Big[\int_s^{s+\tau} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)\\
&+ \int_{\sigma}^{t} Y^T(\alpha,t)f(\alpha)d\alpha\\
&+ \sum_{n(\sigma)-j\le m<n(\sigma)}\Big[\sum_{n(\sigma)\le i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m)\\
&+ \sum_{n(\sigma)\le m<n(t)-j}\Big[\sum_{m+1\le i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m)\\
&+ \sum_{n(t)-j\le m<n(t)}\Big[\sum_{m+1\le i<n(t)} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m)\\
&- \sum_{n(\sigma)\le m<n(t)}\Big[\sum_{m+1\le i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m)\\
&+ \sum_{n(\sigma)\le i<n(t)} Y^T(\theta_i^+,t)J_i. \qquad (2.21)
\end{aligned}
$$

However,

$$
\begin{aligned}
\int_{\sigma}^{t} d_s\Big[\int_s^{s+\tau} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)
&= \int_{\sigma}^{t-\tau} d_s\Big[\int_s^{s+\tau} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)\\
&+ \int_{t-\tau}^{t} d_s\Big[\int_s^{s+\tau} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s), \qquad (2.22)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n(\sigma)\le m<n(t)}\Big[\sum_{m+1\le i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m)
&= \sum_{n(\sigma)\le m<n(t)-j}\Big[\sum_{m+1\le i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m)\\
&+ \sum_{n(t)-j\le m<n(t)}\Big[\sum_{m+1\le i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)}\Big]x(\theta_m). \qquad (2.23)
\end{aligned}
$$

The second integral in (2.22), nevertheless, can be written as

$$
\int_{t-\tau}^{t} d_s\Big[\int_s^{t} Y^T(\alpha,t)\eta(\alpha,s-\alpha)d\alpha\Big]x(s)
$$

$$+ \quad \int_{t-\tau}^{t} d_s \Big[\int_{t}^{s+\tau} Y^T(\alpha, t)\eta(\alpha, s-\alpha)d\alpha \Big] x(s), \tag{2.24}$$

while the second summation in (2.23) is

$$\sum_{n(t)-j \le m < n(t)} \Big[\sum_{m+1 \le i < n(t)} Y^T(\theta_i^+, t) A_{i(m-i)} \Big] x(\theta_m)$$
$$+ \sum_{n(t)-j \le m < n(t)} \Big[\sum_{n(t) \le i < m+j} Y^T(\theta_i^+, t) A_{i(m-i)} \Big] x(\theta_m). \tag{2.25}$$

The second integral in (2.24) and the second summation in (2.25) vanish since $Y^T(\alpha, t) \equiv 0$ for $\alpha > t$ and $Y^T(\theta_i^+, t) \equiv 0$ for $i > n(t)$ respectively. Thus, (2.21) becomes

$$\begin{aligned} x(t) &= Y^T(\sigma, t)x(\sigma) + \int_{\sigma-\tau}^{\sigma} d_s \Big[\int_{\sigma}^{s+\tau} Y^T(\alpha, t)\eta(\alpha, s-\alpha)d\alpha \Big] x(s) \\ &+ \int_{\sigma}^{t} Y^T(\alpha, t) f(\alpha) d\alpha \\ &+ \sum_{n(\sigma)-j \le m < n(\sigma)} \Big[\sum_{n(\sigma) \le i < m+j} Y^T(\theta_i^+, t) A_{i(m-i)} \Big] x(\theta_m) \\ &+ \sum_{n(\sigma) \le i < n(t)} Y^T(\theta_i^+, t) J_i. \end{aligned}$$

If the solution $x(t)$ is defined to be zero for $t < \sigma$ and if $f \equiv 0$, $J_i \equiv 0$, then

$$x(t) = Y^T(\sigma, t)x(\sigma).$$

Using $X(\sigma, \sigma) = I$, we see at once that $X(t, \sigma) = Y^T(\sigma, t)$. Thus

$$\begin{aligned} x(t) &= X(t, \sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} d_s \Big[\int_{\sigma}^{s+\tau} X(t, \alpha)\eta(\alpha, s-\alpha)d\alpha \Big] x(s) \\ &+ \int_{\sigma}^{t} X(t, \alpha) f(\alpha) d\alpha \\ &+ \sum_{n(\sigma)-j \le m < n(\sigma)} \Big[\sum_{n(\sigma) \le i < m+j} X(t, \theta_i^+) A_{i(m-i)} \Big] x(\theta_m) \\ &+ \sum_{n(\sigma) \le i < n(t)} X(t, \theta_i^+) J_i. \quad \square \end{aligned} \tag{2.26}$$

Corollary 2.4.1 *A solution $x(t)$ of the homogeneous system (2.14) can be written in the form*

$$
\begin{aligned}
x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} d_s\Big[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha, s-\alpha)d\alpha\Big]x(s) \\
&+ \sum_{n(\sigma)-j\leq m<n(\sigma)} \Big[\sum_{n(\sigma)\leq i<m+j} X(t,\theta_i^+)A_{i(m-i)}\Big]x(\theta_m), \qquad (2.27)
\end{aligned}
$$

where $X(t,r)$ is the fundamental matrix of (2.14) and $x(\sigma)$ is fixed.

Remark 2.4.4 *The presence of the summations in the formula above is due to the impulses. If there is no impulse, then we get*

$$
x(t) = X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} d_s\Big[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha, s-\alpha)d\alpha\Big]x(s)
$$

as in [12, 28].

To give an analogous result for the solution of the adjoint system (2.16) we need the following lemma.

Lemma 2.4.4 *Let $X(\alpha,t)$ be the fundamental matrix of (2.14) and $y(t)$ be a solution of system (2.16). Then, we have*

$$
\begin{aligned}
\sum_{n(t)\leq i<n(\sigma)} \Big[X^T(\theta_i,t)y(\theta_i) &- X^T(\theta_i^+,t)y(\theta_i^+)\Big] = \\
&\sum_{n(t)\leq i<n(\sigma)} X^T(\theta_i,t)\sum_{-j\leq k<-1} A^T_{(i-k)k}y(\theta^+_{i-k}) \\
&- \sum_{n(t)\leq i<n(\sigma)}\Big[\sum_{-j\leq k<-1} X^T(\theta_{i+k},t)A^T_{ik}y(\theta_i^+)\Big].
\end{aligned}
$$

The proof of this statement is similar to the proof of Lemma 2.4.2 and hence is omitted.

Theorem 2.4.4 *A solution $y(t)$ of (2.16) has the representation*

$$\begin{aligned} y(t) &= Y(t,\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} Y(t,\beta)d_\beta \int_{\sigma}^{\beta+\tau} \eta^T(\alpha,\beta-\alpha)y(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le m<n(\sigma)} Y(t,\theta_m) \sum_{n(\sigma)\le i<m+j} A^T_{i(m-i)}y(\theta_i^+), \end{aligned}$$

where $Y(t,r)$ is the fundamental matrix of (2.16) and $y(\sigma)$ is fixed.

Proof. Multiplying the differential equation in (2.16) by $X^T(\alpha,t)$ from left and integrating from t to σ, we get

$$\int_t^\sigma X^T(\alpha,t)y'(\alpha)d\alpha = -\int_t^\sigma X^T(\alpha,t)d_\alpha\Big[\int_{-\tau}^0 \eta^T(\alpha-s,s)y(\alpha-s)ds\Big],$$

which implies that

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_t^\sigma X^T(\alpha,t)d_\alpha\Big[\int_{-\tau}^0 \eta^T(\alpha-s,s)y(\alpha-s)ds\Big] \\ &- \int_t^\sigma \Big[\int_{-\tau}^0 X^T(\alpha+s,t)d_s\eta^T(\alpha,s)\Big]y(\alpha)d\alpha \\ &+ \sum_{n(t)\le i<n(\sigma)} \Big[X^T(\theta_i,t)y(\theta_i) - X^T(\theta_i^+,t)y(\theta_i^+)\Big]. \end{aligned}$$

Using Lemma 2.4.4 in the summation results in

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_t^\sigma X^T(\alpha,t)d_\alpha\Big[\int_{-\tau}^0 \eta^T(\alpha-s,s)y(\alpha-s)ds\Big] \\ &- \int_t^\sigma \Big[\int_{-\tau}^0 X^T(\alpha+s,t)d_s\eta^T(\alpha,s)\Big]y(\alpha)d\alpha \\ &+ \sum_{n(t)\le i<n(\sigma)} X^T(\theta_i,t) \sum_{-j\le k<-1} A^T_{(i-k)k}y(\theta_{i-k}^+) \\ &- \sum_{n(t)\le i<n(\sigma)} \Big[\sum_{-j\le k<-1} X^T(\theta_{i+k},t)A^T_{ik}y(\theta_i^+)\Big]. \end{aligned}$$

Similar to the proof of Theorem 2.4.3, we change the variables and the indices to obtain

$$y(t) = X^T(\sigma,t)y(\sigma) + \int_t^\sigma X^T(\beta,t)d_\beta\Big[\int_\beta^{\beta+\tau} \eta^T(\alpha,\beta-\alpha)y(\alpha)d\alpha\Big]$$

$$
\begin{aligned}
&- \int_t^{\sigma} \Big[\int_{\alpha-\tau}^{\alpha} X^T(\beta,t) d_\beta \eta^T(\alpha,\beta-\alpha)\Big] y(\alpha) d\alpha \\
&+ \sum_{n(t)\le i<n(\sigma)} X^T(\theta_i,t) \sum_{i+1\le m<i+j} A^T_{(m)(i-m)} y(\theta_m^+) \\
&- \sum_{n(t)\le i<n(\sigma)} \Big[\sum_{i-j\le m<i-1} X^T(\theta_m,t) A^T_{i(m-i)} y(\theta_i^+)\Big].
\end{aligned}
$$

Using Theorem 2.4.1 in the second integral and changing the order of summations in the second sum, we have

$$
\begin{aligned}
y(t) &= X^T(\sigma,t)y(\sigma) + \int_t^{\sigma} X^T(\beta,t) d_\beta \int_\beta^{\beta+\tau} \eta^T(\alpha,\beta-\alpha) y(\alpha) d\alpha \\
&- \int_{t-\tau}^{t} X^T(\beta,t) d_\beta \int_t^{\beta+\tau} \eta^T(\alpha,\beta-\alpha) y(\alpha) d\alpha \\
&- \int_t^{\sigma-\tau} X^T(\beta,t) d_\beta \int_\beta^{\beta+\tau} \eta^T(\alpha,\beta-\alpha) y(\alpha) d\alpha \\
&- \int_{\sigma-\tau}^{\sigma} X^T(\beta,t) d_\beta \int_\beta^{\sigma} \eta^T(\alpha,\beta-\alpha) y(\alpha) d\alpha \\
&+ \sum_{n(t)\le m<n(\sigma)} X^T(\theta_m,t) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&- \sum_{n(t)-j\le m<n(t)} X^T(\theta_m,t) \sum_{n(t)\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&- \sum_{n(t)\le m<n(\sigma)-j} X^T(\theta_m,t) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&- \sum_{n(\sigma)-j\le m<n(\sigma)} X^T(\theta_m,t) \sum_{m+1\le i<n(\sigma)} A^T_{i(m-i)} y(\theta_i^+). \qquad (2.28)
\end{aligned}
$$

The second integral and summation in (2.28), however, vanish since $X^T(\alpha,t) \equiv 0$ for $\alpha < t$ and $X^T(\theta_i^+,t) \equiv 0$ for $i < n(t)$. Moreover, it is possible to write, respectively, the first integral and the first summation in (2.28) as

$$
\begin{aligned}
\int_t^{\sigma-\tau} & X^T(\beta,t) \quad d_\beta \int_\beta^{\beta+\tau} \eta^T(\alpha,\beta-\alpha) y(\alpha) d\alpha \\
& + \qquad \int_{\sigma-\tau}^{\sigma} X^T(\beta,t) d_\beta \int_\beta^{\beta+\tau} \eta^T(\alpha,\beta-\alpha) y(\alpha) d\alpha,
\end{aligned}
$$

and

$$\sum_{n(t)\le m<n(\sigma)-j} X^T(\theta_m,t) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) + \sum_{n(\sigma)-j\le m<n(\sigma)} X^T(\theta_m,t) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+).$$

Thus, (2.28) becomes

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_{\sigma-\tau}^{\sigma} X^T(\beta,t)d_\beta \int_\beta^{\beta+\tau} \eta^T(\alpha,\beta-\alpha)y(\alpha)d\alpha \\ &- \int_{\sigma-\tau}^{\sigma} X^T(\beta,t)d_\beta \int_\beta^{\sigma} \eta^T(\alpha,\beta-\alpha)y(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le m<n(\sigma)} X^T(\theta_m,t) \sum_{m+1\le i<m+j} A^T_{i(m-i)} y(\theta_i^+) \\ &- \sum_{n(\sigma)-j\le m<n(\sigma)} X^T(\theta_m,t) \sum_{m+1\le i<n(\sigma)} A^T_{i(m-i)} y(\theta_i^+). \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= X^T(\sigma,t)y(\sigma) + \int_{\sigma-\tau}^{\sigma} X^T(\beta,t)d_\beta \int_\sigma^{\beta+\tau} \eta^T(\alpha,\beta-\alpha)y(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le m<n(\sigma)} X^T(\theta_m,t) \sum_{n(\sigma)\le i<m+j} A^T_{i(m-i)} y(\theta_i^+). \end{aligned}$$

Therefore

$$\begin{aligned} y(t) &= Y(t,\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} Y(t,\beta)d_\beta \int_\sigma^{\beta+\tau} \eta^T(\alpha,\beta-\alpha)y(\alpha)d\alpha \\ &+ \sum_{n(\sigma)-j\le m<n(\sigma)} Y(t,\theta_m) \sum_{n(\sigma)\le i<m+j} A^T_{i(m-i)} y(\theta_i^+). \quad \square \end{aligned} \tag{2.29}$$

CHAPTER 3

THE PERRON CONDITION

3.1 Introduction

An important condition for examining the asymptotic behavior of solutions of differential systems without impulses or delays is the one obtained by *O. Perron* in [12, 44]. Specifically, he proved that if for every continuous function $f(t)$, bounded on the semi axis $t \geq 0$, the solution of the initial value problem

$$\begin{cases} x'(t) = & A(t)x(t) + f(t), \\ x(0) = & 0, \end{cases}$$

is bounded on $t \geq 0$, then the trivial solution of the homogeneous equation

$$x'(t) = A(t)x(t)$$

is uniformly asymptotically stable. This condition is referred to as the condition of *Perron*. The same condition was defined analogously for delay differential equations having the initial function equal to zero on $[-\tau, 0]$, see [12, 52, 68, 70, 76].

In this chapter, by using the results obtained in chapter 2, we obtain a *Perron* condition for impulsive differential systems with delays of the form

$$\begin{cases} x'(t) = & \int_{-\tau}^{0} d_s\eta(t,s)x(t+s), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & A_{i0}x(\theta_i) + \sum\limits_{-j \leq k < -1} A_{ik}x(\theta_{i+k}), \quad i \in \mathbb{N}. \end{cases} \tag{3.1}$$

Conditions (C2.3) on page 20 are assumed to hold throughout this chapter.

As usual, $|x|$ will denote a norm of x. Also, if A is any real $m \times n$ matrix, then $|A|$ will denote the norm of a matrix A induced by the norm in $\mathbb{R}^n$, that is,

$$|A| = \sup_{|x|=1} |Ax| = \sup_{x \neq 0} \frac{|Ax|}{|x|}.$$

The norm of a linear operator is defined similarly. Very often we will use the fact

$$\left| \int_a^b d_s \eta(t,s) x(t+s) \right| \leq |V(\eta, a, b)| \cdot \|x\|, \tag{3.2}$$

where

$$\|x\| = \sup_{s \in [a,b]} |x(t+s)|.$$

Definition 3.1.1 *System (3.1) is said to verify the condition of* Perron *if for every vector function $f(t) \in PLC(\mathbb{R}^+, \mathbb{R}^n)$ bounded on $t \geq 0$ and every bounded sequence $\{J_i\}$, the solution of*

$$\begin{cases} x'(t) = & \displaystyle\int_{-\tau}^{0} d_s \eta(t,s) x(t+s) + f(t), \quad t \neq \theta_i,\ t \geq 0 \\ \Delta x(\theta_i) = & A_{i0} x(\theta_i) + \displaystyle\sum_{-j \leq k < -1} A_{ik} x(\theta_{i+k}) + J_i, \quad i \in \mathbb{N} \\ x(t) = & 0, \quad t \in [-\tau, 0] \end{cases} \tag{3.3}$$

is bounded on $t \geq 0$.

3.2 Auxiliary Assertions

In preparations for the main results, three essential lemmas are presented. The first result is analogous to *Fubini's* theorem

Lemma 3.2.1 *Let $X(t,r)$ be the fundamental matrix of system (3.1) and*

$$\zeta(n(\sigma),\theta_i) = \sum_{i-j\le m<n(\sigma)} A_{i(m-i)}x(\theta_m;t_0,\phi),$$

where x is a solution of (3.1). Then, we have

$$\int_{t_0}^{t}\Big[\sum_{n(\sigma)\le i<n(\sigma)+j} X(t,\theta_i^+)\quad \zeta(n(\sigma),\theta_i)\Big]d\sigma =$$

$$\sum_{n(t_0)\le i<n(t_0)+j}\Big[\int_{t_0}^{\theta_i} X(t,\theta_i^+)\zeta(n(\sigma),\theta_i)d\sigma\Big]$$

$$+\qquad \sum_{n(t_0)+j\le i<n(t)}\Big[\int_{\theta_{i-j}}^{\theta_i} X(t,\theta_i^+)\zeta(n(\sigma),\theta_i)d\sigma\Big].$$

Proof. Changing the order of integration and summation and using $X(t,\theta_i^+)\equiv 0$ for $n(t)<i$ complete the proof. □

The following lemma provides condition for boundedness of the fundamental matrix $Y(\alpha,t)$ of (2.16).

Lemma 3.2.2 *Let $Y(\alpha,t)$ be the fundamental matrix of solutions of the adjoint system (2.16). If there is a constant $C>0$ such that*

$$\int_0^t |Y(\alpha,t)|d\alpha + \sum_{0\le i<n(t)} |Y(\theta_i^+,t)| < C \qquad for \quad t\ge 0,$$

then

$$|Y(\alpha,t)| < M \qquad for \quad 0\le\alpha\le t.$$

Proof. Using formula (2.29) we have

$$Y(\alpha,t) \;=\; Y(t,\sigma)Y(\sigma,t) + \int_{\sigma-\tau}^{\sigma} Y(t,\beta)d_\beta \int_{\sigma}^{\beta+\tau} \eta^T(\gamma,\beta-\gamma)Y(\gamma,t)d\gamma$$

$$+ \sum_{n(\sigma)-j\le m<n(\sigma)} Y(t,\theta_m) \sum_{n(\sigma)\le i<m+j} A^T_{i(m-i)} Y(\theta_i^+,t).$$

Using Theorem 2.4.1 and interchanging the order of summations, we get

$$\begin{aligned} Y(\alpha,t) &= I + \int_{\sigma}^{\sigma+\tau} \Big[\int_{\gamma-\tau}^{\sigma} Y(t,\beta) d_\beta \eta^T(\gamma,\beta-\gamma)\Big] Y(\gamma,t)d\gamma \\ &+ \sum_{n(\sigma)\le i<n(\sigma)+j} \Big[\sum_{i-j\le m<n(\sigma)} Y(t,\theta_m)A^T_{i(m-i)}\Big] Y(\theta_i^+,t). \end{aligned}$$

Taking the norm of both sides leads to

$$\begin{aligned} |Y(\alpha,t)| &\le 1 + \left|\int_{\sigma}^{\sigma+\tau} \Big[\int_{\gamma-\tau}^{\sigma} Y(t,\beta) d_\beta \eta^T(\gamma,\beta-\gamma)\Big] Y(\gamma,t)d\gamma\right| \\ &+ \left|\sum_{n(\sigma)\le i<n(\sigma)+j} \Big[\sum_{i-j\le m<n(\sigma)} Y(t,\theta_m)A^T_{i(m-i)}\Big] Y(\theta_i^+,t)\right|. \end{aligned}$$

Conditions (C2.3) and relation (3.2) yield that

$$|Y(\alpha,t)| \le 1 + N\|\varphi\| \int_{\sigma}^{\sigma+\tau} |Y(\gamma,t)|d\gamma + W\|\varphi\| \sum_{n(\sigma)\le i<n(\sigma)+j} |Y(\theta_i^+,t)|,$$

where

$$\|\varphi\| = \sup_r |Y(t,r)|.$$

Using that $Y(\gamma,t)\equiv 0$ for $\gamma > t$ and $Y(\theta_i^+,t)\equiv 0$ for $i > n(t)$, it follows that

$$|Y(\alpha,t)| \le 1 + N\|\varphi\| \int_0^t |Y(\gamma,t)|d\gamma + W\|\varphi\| \sum_{0\le i<n(t)} |Y(\theta_i^+,t)|,$$

hence

$$|Y(\alpha,t)| \le 1 + \|\varphi\| \max\{N,W\}\Big[\int_0^t |Y(\gamma,t)|d\gamma + \sum_{0\le i<n(t)} |Y(\theta_i^+,t)|\Big].$$

By the condition of the lemma, the right side of the above inequality is bounded and, thus we have

$$|Y(\alpha,t)| < M \quad for \quad 0\le \alpha\le t,$$

where

$$M < 1 + \|\varphi\| \max\{N, W\} C. \quad \square$$

Remark 3.2.1 *A similar result can be obtained for the fundamental matrix $X(\alpha, t)$ of (3.1).*

The following result is similar to the one obtained by *Bellman* and *Cooke* in [23] for delay differential equations. It provides a useful information for systems satisfying the *Perron* condition.

Lemma 3.2.3 *If system (3.1) verifies the* Perron *condition, then there exists a constant C such that*

$$\int_0^t |X(t, \alpha)| d\alpha + \sum_{0 \le m < n(t)} |X(t, \theta_m^+)| < C \quad for \quad t \ge 0,$$

where $X(t, r)$ is the fundamental matrix of (3.1).

Proof. In view of formula (2.26), the solution satisfying $x(0) = 0$ is given by

$$x(t) = \int_0^t X(t, \alpha) f(\alpha) d\alpha + \sum_{0 \le m < n(t)} X(t, \theta_m^+) J_m. \tag{3.4}$$

The *Perron* condition implies that $x(t)$ is bounded. For fixed $t \ge 0$, consider the operator $U(f, J_m)$ on the *Banach* space of the bounded vector functions f on $t \ge 0$ and bounded sequence $\{J_m\}$ defined by

$$U(f, J_m) = \int_0^t X(t, \alpha) f(\alpha) d\alpha + \sum_{0 \le m < n(t)} X(t, \theta_m^+) J_m.$$

Let $\{t_k\}$ be the sequence of the rational positive numbers, and

$$U_k(f, J_m) = \int_0^{t_k} X(t_k, \alpha) f(\alpha) d\alpha + \sum_{0 \le m < n(t_k)} X(t_k, \theta_m^+) J_m.$$

From the boundedness of $x(t)$, it follows that

$$|U_k(f, J_m)| \le M_{f,J_m},$$

where M_{f,J_m} is a positive real number. Therefore, we may apply Theorem 1.4.1 and deduce that there exists an M such that

$$|U_k(f, J_m)| \le M\Big(|f| + |J_m|\Big)$$

for every f and J_m. Hence

$$\left| \int_0^{t_k} X(t_k, \alpha) f(\alpha) d\alpha + \sum_{0 \le m < n(t_k)} X(t_k, \theta_m^+) J_m \right| \le M\Big(|f| + |J_m|\Big).$$

For any real $t \ge 0$ there exists a sequence $\{t_{n_k}\}$ of rational numbers whose limit is t. It follows from

$$\left| \int_0^{t_{n_k}} X(t_{n_k}, \alpha) f(\alpha) d\alpha + \sum_{0 \le m < n(t_{n_k})} X(t_{n_k}, \theta_m^+) J_m \right| \le M\Big(|f| + |J_m|\Big)$$

that

$$\left| \int_0^{t} X(t, \alpha) f(\alpha) d\alpha + \sum_{0 \le m < n(t)} X(t, \theta_m^+) J_m \right| \le M\Big(|f| + |J_m|\Big) \qquad (3.5)$$

for every f and J_m.

Let x_{rk} be the elements of matrix $X(t, \alpha)$ and y_{rk} be the elements of $X(t, \theta_m^+)$. For fixed $t \ge 0$, we consider the vectors f^r and J_m^r whose components are equal to signx_{rk} and to signy_{rk} respectively. The vectors Xf^r and XJ_m^r will have, respectively, their r-th components

$$\sum_{k=1}^{n} |x_{rk}| \qquad and \qquad \sum_{k=1}^{n} |y_{rk}|.$$

From (3.5), we can write

$$\left| \int_0^t X(t,\alpha) f^r(\alpha) d\alpha + \sum_{0 \le m < n(t)} X(t,\theta_m) J_m^r \right| \le M_1,$$

where

$$M_1 = M\Big(|f^r| + |J_m^r| \Big).$$

Hence

$$\int_0^t \sum_{k=1}^{n} |x_{rk}(t,\alpha)| d\alpha + \sum_{0 \le m < n(t)} \sum_{k=1}^{n} |y_{rk}(t,\theta_m)| \le M_1.$$

Since this relation is true for every r, taking the summation over r of both sides, we deduce that there exists C such that

$$\int_0^t |X(t,\alpha)| d\alpha + \sum_{0 \le m < n(t)} |X(t,\theta_m^+)| \le C,$$

which completes the proof. □

Remark 3.2.2 *The result of Lemma 3.2.2 is an immediate consequence of Lemma 3.2.3. Indeed, Lemma 3.2.3 provides the condition*

$$\int_0^t |X(t,\alpha)| d\alpha + \sum_{0 \le m < n(t)} |X(t,\theta_m^+)| \le C,$$

which is sufficient by Lemma 3.2.2 to have $|X(t,\alpha)| < M$ *for* $t > \alpha$.

3.3 The Main Results

In this section, we provide stability theorems for impulsive differential systems with delays which satisfy the *Perron* condition.

Theorem 3.3.1 *If system (3.1) verifies the condition of* Perron*, then its trivial solution is uniformly stable.*

Proof. To emphasize the dependence of the solution x on the initial point t_0 and the initial function ϕ, we denote $x(t)$ by $x(t; t_0, \phi)$. The solution of system (3.1) equal with ϕ on $[t_0 - \tau, t_0]$ has the form

$$\begin{aligned} x(t; t_0, \phi) &= X(t, t_0)\phi(t_0) + \int_{t_0-\tau}^{t_0} d_s\Big[\int_{t_0}^{s+\tau} X(t, \alpha)\eta(\alpha, s - \alpha)d\alpha\Big]\phi(s) \\ &+ \sum_{n(t_0)-j\le m<n(t_0)} \Big[\sum_{n(t_0)\le i<m+j} X(t, \theta_i^+)A_{i(m-i)}\Big]\phi(\theta_m). \end{aligned}$$

Using Theorem 2.4.1 and changing the order of summations lead to

$$\begin{aligned} x(t; t_0, \phi) &= X(t, t_0)\phi(t_0) + \int_{t_0}^{t_0+\tau} X(t, \alpha) \int_{\alpha-\tau}^{t_0} d_s\eta(\alpha, s - \alpha)\phi(s)d\alpha \\ &+ \sum_{n(t_0)\le i<n(t_0)+j} X(t, \theta_i^+) \sum_{i-j\le m<n(t_0)} A_{i(m-i)}\phi(\theta_m). \end{aligned}$$

By Lemma 3.2.2, $|X(t, r)| < M$. Moreover, using conditions (C2.3) and relation (3.2), we have

$$\begin{aligned} |x(t; t_0, \phi)| &\le M\|\phi\| + \tau MN\|\phi\| + jMW\|\phi\| \\ &= M(1 + \tau N + jW)\|\phi\| \\ &= M_1\|\phi\|, \end{aligned}$$

where

$$M_1 = M(1 + \tau N + jW) \quad \text{and} \quad \|\phi\| = \sup_{\mathrm{r}} |\phi(\mathrm{r})|.$$

Thus, the zero solution of (3.1) is uniformly stable. □

Furthermore, and on the basis of the result of Theorem 3.3.1, systems satisfying the *Perron* condition possesses the following property

Theorem 3.3.2 *If system (3.1) verifies the condition of* Perron*, then its solution* $x(t; t_0, \phi)$ *equal with* ϕ *on* $[t_0 - \tau, t_0]$ *satisfies the inequality*

$$|x(t; t_0, \phi)| \le \frac{M_2}{t - t_0} \|\phi\|.$$

Proof. For $\sigma \ge t_0$, the solution $x(t; t_0, \phi)$ has the form

$$\begin{aligned} x(t; t_0, \phi) &= X(t, \sigma) x(\sigma; t_0, \phi) \\ &+ \int_{\sigma}^{\sigma+\tau} X(t, \alpha) \int_{\alpha-\tau}^{\sigma} d_s \eta(\alpha, s - \alpha) x(s; t_0, \phi) d\alpha \\ &+ \sum_{n(\sigma) \le i < n(\sigma)+j} X(t, \theta_i^+) \sum_{i-j \le m < n(\sigma)} A_{i(m-i)} x(\theta_m; t_0, \phi). \end{aligned}$$

Integrating both sides from t_0 to t with respect to σ we get

$$\begin{aligned} (t - t_0) x(t; t_0, \phi) &= \int_{t_0}^{t} X(t, \sigma) x(\sigma; t_0, \phi) d\sigma \\ &+ \int_{t_0}^{t} \Big[\int_{\sigma}^{\sigma+\tau} X(t, \alpha) \xi(\sigma, \alpha) d\alpha \Big] d\sigma \\ &+ \int_{t_0}^{t} \Big[\sum_{n(\sigma) \le i < n(\sigma)+j} X(t, \theta_i^+) \zeta(n(\sigma), \theta_i) \Big] d\sigma, \end{aligned}$$

where we set

$$\xi(\sigma, \alpha) = \int_{\alpha-\tau}^{\sigma} d_s \eta(\alpha, s - \alpha) x(s; t_0, \phi),$$

and

$$\zeta(n(\sigma), \theta_i) = \sum_{i-j \le m < n(\sigma)} A_{i(m-i)} x(\theta_m; t_0, \phi).$$

Changing the order of integrations and using Lemma 3.2.1 we get

$$\begin{aligned} (t - t_0) x(t; t_0, \phi) &= \int_{t_0}^{t} X(t, \sigma) x(\sigma; t_0, \phi) d\sigma \\ &+ \int_{t_0}^{t_0+\tau} \Big[\int_{t_0}^{\alpha} X(t, \alpha) \xi(\sigma, \alpha) d\sigma \Big] d\alpha \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_0+\tau}^{t} \Big[\int_{\alpha-\tau}^{\alpha} X(t,\alpha)\xi(\sigma,\alpha)d\sigma\Big]d\alpha \\
&+ \sum_{n(t_0)\le i<n(t_0)+j} \Big[\int_{t_0}^{\theta_i} X(t,\theta_i^+)\zeta(n(\sigma),\theta_i)d\sigma\Big] \\
&+ \sum_{n(t_0)+j\le i<n(t)} \Big[\int_{\theta_{i-j}}^{\theta_i} X(t,\theta_i^+)\zeta(n(\sigma),\theta_i)d\sigma\Big], \qquad (3.6)
\end{aligned}
$$

where we have used that $X(t,\alpha)\equiv 0$ for $t<\alpha$. The stability of the trivial solution yields

$$
|\xi(\sigma,\alpha)| \le N \sup_{s\in[\alpha-\tau,\sigma]} |x(s;t_0,\phi)| \le NM_1\|\phi\|,
$$

and

$$
|\zeta(n(\sigma),\theta_i)| \le W \sup_{m\in[i-j,n(\sigma)]} |x(\theta_m;t_0,\phi)| \le WM_1\|\phi\|.
$$

Thus, we obtain

$$
\left|\int_{\alpha-\tau}^{\alpha} \xi(\sigma,\alpha)d\sigma\right| \le \tau NM_1\|\phi\|,
$$

and

$$
\left|\int_{\theta_{i-j}}^{\theta_i} \zeta(n(\sigma),\theta_i)d\sigma\right| \le kWM_1\|\phi\|,
$$

where $k=\theta_i-\theta_{i-j}$.

From (3.6), we also get

$$
\begin{aligned}
(t-t_0)|x(t;t_0,\phi)| \le\ & \left|\int_{t_0}^{t} X(t,\sigma)x(\sigma;t_0,\phi)d\sigma\right| \\
&+ \left|\int_{t_0}^{t_0+\tau} \Big[\int_{t_0}^{\alpha} X(t,\alpha)\xi(\sigma,\alpha)d\sigma\Big]d\alpha\right| \\
&+ \left|\int_{t_0+\tau}^{t} \Big[\int_{\alpha-\tau}^{\alpha} X(t,\alpha)\xi(\sigma,\alpha)d\sigma\Big]d\alpha\right| \\
&+ \left|\sum_{n(t_0)\le i<n(t_0)+j} \Big[\int_{t_0}^{\theta_i} X(t,\theta_i^+)\zeta(n(\sigma),\theta_i)d\sigma\Big]\right| \\
&+ \left|\sum_{n(t_0)+j\le i<n(t)} \Big[\int_{\theta_{i-j}}^{\theta_i} X(t,\theta_i^+)\zeta(n(\sigma),\theta_i)d\sigma\Big]\right|.
\end{aligned}
$$

Then, it readily follows that

$$
\begin{aligned}
(t-t_0)|x(t;t_0,\phi)| &\le M_1\|\phi\|\int_{t_0}^{t}|X(t,\sigma)|d\sigma \\
&+ \int_{t_0}^{t_0+\tau}\Big[\int_{t_0}^{\alpha}|X(t,\alpha)||\xi(\sigma,\alpha)|d\sigma\Big]d\alpha \\
&+ \int_{t_0+\tau}^{t}\Big[\int_{\alpha-\tau}^{\alpha}|X(t,\alpha)||\xi(\sigma,\alpha)|d\sigma\Big]d\alpha \\
&+ \sum_{n(t_0)\le i<n(t_0)+j}\Big[\int_{t_0}^{\theta_i}|X(t,\theta_i^+)||\zeta(n(\sigma),\theta_i)|d\sigma\Big] \\
&+ \sum_{n(t_0)+j\le i<n(t)}\Big[\int_{\theta_{i-j}}^{\theta_i}|X(t,\theta_i^+)|\zeta(n(\sigma),\theta_i)|d\sigma\Big],
\end{aligned}
$$

which implies

$$
\begin{aligned}
(t-t_0)|x(t;t_0,\phi)| &\le \tau^2MNM_1\|\phi\|+\tau NM_1\|\phi\|\int_{t_0+\tau}^{t}|X(t,\alpha)|d\alpha \\
&+ jkMWM_1\|\phi\|+kWM_1\|\phi\|\sum_{n(t_0)+j\le i<n(t)}|X(t,\theta_i^+)| \\
&+ M_1\|\phi\|\int_{t_0}^{t}|X(t,\sigma)|d\sigma.
\end{aligned}
$$

Hence

$$
(t-t_0)|x(t;t_0,\phi)|\le MM_1(\tau^2N+jkW)\|\phi\|+M_1\|\phi\|\int_{t_0}^{t}|X(t,\sigma)|d\sigma
$$
$$
+\|\phi\|M_1\max\{\tau N,kW\}\Big(\int_0^t|X(t,\alpha)|d\alpha+\sum_{0\le i<n(t)}|X(t,\theta_i^+)|\Big).
$$

In view of Lemma 3.2.3, the right side of the above inequality is bounded. Dividing both sides by $t-t_0$, we obtain

$$
|x(t;t_0,\phi)|\le\frac{M_2}{t-t_0}\|\phi\|, \tag{3.7}
$$

where M_2 is chosen so that

$$
M_2 \;<\; MM_1(\tau^2N+jkW)+M_1D+M_1\max\{\tau N,kW\}C. \quad \square
$$

We are now in a position to give the main theorem of this chapter on the uniform asymptotic stability of the trivial solution of (3.1).

Theorem 3.3.3 *If system (3.1) verifies the condition of* Perron*, then its trivial solution is uniformly asymptotically stable.*

Proof. The uniform stability of the trivial solution follows from Theorem 3.3.1. It remains to prove that $\lim_{t\to\infty} x(t; t_0, \phi) = 0$ uniformly with respect to t_0 and ϕ and this follows simply by taking the limit as t tends to ∞ of both sides of (3.7). Thus the trivial solution is uniformly asymptotically stable. $\square$

CHAPTER 4

EXISTENCE OF PERIODIC SOLUTIONS

Periodic solutions of differential equations are of great interests both in applications and theory. Various methods have been developed in order to ensure the existence of periodic solutions, see for instance [12, 14, 15, 20, 29] for functional differential equations and [7, 8, 9, 10, 17, 18, 21, 24] for impulsive differential systems. However, little has been done concerning the periodicity of impulsive differential systems with delays [37, 43, 53].

It is known from the theory of ordinary differential equations that if the equation

$$x'(t) = A(t)x(t) \tag{4.1}$$

admits periodic solutions of period ω, then the adjoint equation

$$y'(t) = -A^T(t)y(t) \tag{4.2}$$

admits the same number of linearly independent periodic solutions as equation (4.1). Moreover, a necessary and sufficient condition in order that the equation

$$x'(t) = A(t)x(t) + f(t)$$

admits periodic solutions is that f be orthogonal to all periodic solutions $y(t)$ of period ω of equation (4.2), namely,

$$\int_0^\omega y^T(t)f(t)dt = 0. \tag{4.3}$$

For further details, see [12, 29, 49, 50, 56]. In [12] *Halanay* considered this problem for delay differential equations. In [13], however, *Samoilenko* and *Perestyuk* considered the same problem for impulsive differential systems. The techniques that they have used to prove this result were totally different. In this chapter we shall study this problem in parallel to the terminology used by *Halanay* adapted to systems involving impulses and delays. Moreover, a result that characterizes the behavior of solution in the case it admits a bounded solution is given.

4.1 Introduction

Consider the system

$$\begin{cases} x'(t) = \displaystyle\int_{-\tau}^{0} d_s\eta(t,s)x(t+s), \quad t \neq \theta_i \\ \Delta x(\theta_i) = A_{i0}x(\theta_i) + \displaystyle\sum_{-j\leq k<-1} A_{ik}x(\theta_{i+k}), \quad i \in \mathbb{Z} \end{cases} \tag{4.4}$$

where the conditions (C4.1) are assumed:

(a) $\eta(t,s)$ is periodic with respect to t of period $\omega > \tau$;

(b) There exists $p \in \mathbb{N}$ such that $A_{i0+p} = A_{i0}$, $A_{ik+p} = A_{ik}$ and $\theta_{i+p} = \theta_i + \omega$ for $i, k \in \mathbb{Z}$.

We define an operator U through the relation $U\phi(s) = x(\omega+s;\phi),\ s \in [-\tau, 0]$ where $x(t,\phi)$ is a solution of system (4.4) which coincides with ϕ on $[-\tau, 0]$. By the solution representation (2.27) obtained in chapter 2, we have

$$\begin{aligned} U\phi(s) &= X(\omega+s,0)\phi(0) + \int_{-\tau}^{0} d_\sigma\Big[\int_0^{\sigma+\tau} X(\omega+s,\alpha)\eta(\alpha,\sigma-\alpha)d\alpha\Big]\phi(\sigma) \\ &+ \sum_{-j\leq m<0}\Big[\sum_{0\leq i<m+j} X(\omega+s,\theta_i^+)A_{i(m-i)}\Big]\phi(\theta_m). \end{aligned}$$

The operator U is compact: it maps every bounded set into a relative compact set. Indeed, from $\|\phi\| \leq M$ we obtain $\|U\phi\| \leq M_1$ and from $\omega > \tau$ we obtain

$\omega + s > 0$ for $s \in [-\tau, 0]$, therefore

$$d/ds\ x(\omega + s; \phi) = \int_{-\tau}^{0} d_\sigma \eta(s, \sigma) x(\omega + s + \sigma; \phi), \quad s \in (\theta_k, \theta_{k+1})$$

hence $|d/ds\ x(\omega + s; \phi)| < M_2$. The set of functions $\{U\phi\}$, thus, forms a set of uniformly bounded and quasi equicontinuous functions, consequently, on the basis of Lemma 1.4.1, it is a relative compact set.

The following lemma provides a classical condition for periodicity of solutions of (4.4)

Lemma 4.1.1 *[29] Let conditions (C4.1) be satisfied. Then (4.4) has a solution of period* ω *if and only if there is a* $t_0 \in \mathbb{R}$ *and* $\phi : [t_0 - \tau, t_0] \to \mathbb{R}^n$ *with* $x(t + \omega; t_0, \phi) = \phi(t)$ *for* $t_0 - \tau \le t \le t_0$.

Proof. Suppose that $x(t; t_0, \phi)$ is a solution of period ω of (4.4). Then $x(t + \omega; t_0, \phi) = x(t; t_0, \phi) = \phi(t)$ for $t_0 - \tau \le t \le t_0$. Next, suppose that there is a t_0 and ϕ with

$$x(t + \omega; t_0, \phi) = \phi(t) \quad for \quad t_0 - \tau \le t \le t_0.$$

Then $x(t; t_0, \phi)$ and $x(t + \omega; t_0, \phi)$ are both solutions with the same initial condition. By the uniqueness of solutions, they are identical. $\square$

4.2 A Massera Type Theorem

Consider the nonhomogeneous system corresponding to (4.4)

$$\begin{cases} x'(t) = & \int_{-\tau}^{0} d_s\eta(t,s)x(t+s) + f(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & A_{i0}x(\theta_i) + \sum_{-j \leq k < -1} A_{ik}x(\theta_{i+k}) + J_i, \quad i \in \mathbb{Z}, \end{cases} \tag{4.5}$$

where in addition to (C4.1) the following conditions (C4.2) are assumed to be valid:

(a) $f(t)$ is continuous for $t \neq \theta_i$, $f \not\equiv 0$ for all $t \in (\theta_i, \theta_{i+1})$ and f is periodic of period $\omega > \tau$;

(b) There exists $p \in \mathbb{N}$ such that $J_{i+p} = J_i$ for $i \in \mathbb{Z}$.

Moreover, we assume that conditions (C2.3) on page 20 are valid throughout this section.

Let $x(t;\phi)$ be the solution of (4.5) defined for $t \geq 0$ by the function ϕ given in $[-\tau, 0]$. From conditions (C4.1)-(C4.2), it follows that $x(t+\omega;\phi)$ is likewise a solution of (4.5) defined for $t+\omega \geq \tau$. Lemma 4.1.1 implies that $x(t+\omega;\phi) = x(t;\phi)$ for all $t \geq -\tau$ and hence the solution is periodic. Thus, the periodicity condition of the solution is written as $x(\omega + s;\phi) = \phi(s)$ for $s \in [-\tau, 0]$. Let V be the operator defined by $V\phi = x(\omega + s;\phi)$; the function ϕ is an initial function for a periodic solution of the system if and only if $V\phi = \phi$, in other words, the periodic solutions of the system correspond to the fixed points of the operator V. Let $z(t;\phi)$ be the solution of the corresponding homogeneous system (4.4), defined for $t \geq 0$ by the initial function ϕ given in $[-\tau, 0]$. Then

$$x(t;\phi) = z(t;\phi) + \int_0^t X(t,\alpha)f(\alpha)d\alpha + \sum_{0 \leq i < n(t)} X(t,\theta_i)J_i.$$

If U is the operator defined by the relation $U\phi = z(\omega + s; \phi)$, we have

$$V\phi = U\phi + \int_0^{\omega+s} X(\omega + s, \alpha) f(\alpha) d\alpha + \sum_{0 \le i < n(\omega)+s} X(\omega + s, \theta_i) J_i,$$

where $n(\omega) = p$. The periodicity condition implies that

$$\phi = U\phi + \int_0^{\omega+s} X(\omega + s, \alpha) f(\alpha) d\alpha + \sum_{0 \le i < n(\omega)+s} X(\omega + s, \theta_i) J_i,$$

that is,

$$(I - U)\phi = \int_0^{\omega+s} X(\omega + s, \alpha) f(\alpha) d\alpha + \sum_{0 \le i < n(\omega)+s} X(\omega + s, \theta_i) J_i.$$

The operator U is compact, hence, $I - U$ has inverse if and only if the equation $(I - U)\phi = 0$ has no other solutions except $\phi = 0$. The equation $U\phi = \phi$ determines the initial functions of the periodic solutions of the homogeneous system. We have, thus, proved the following theorem

Theorem 4.2.1 *Let conditions (C4.1)-(C4.2) be satisfied. Then a necessary and sufficient condition in order that system (4.5) have a periodic solution of period ω, is that the corresponding homogeneous system (4.4) does not have periodic solutions of period ω different from the zero solution.*

Remark 4.2.1 *The above theorem was proved for delay differential equations without impulses and for impulsive differential systems without delays, see [12, 17].*

Existence of periodic solutions of period ω of system (4.5) is closely related to this having bounded solutions. This relation was proved by *Massera* for differential equations without delays and without impulses, see [17, 29, 50, 78]. Indeed, the system

$$x'(t) = A(t)x(t) + f(t) \tag{4.6}$$

was considered. It was shown that if system (4.6) does not have periodic solutions of period ω, then all the solutions are unbounded. This result is obtained for delay differential equations [12] and for impulsive differential systems [17]. In what follows, we shall show that this theorem can be applied for impulsive differential systems with delays.

Theorem 4.2.2 *If system (4.5) does not have periodic solutions, then all the solutions are unbounded.*

Proof. Let ψ be defined by

$$\psi(s) = \int_0^{\omega+s} X(\omega+s,\alpha)f(\alpha)d\alpha + \sum_{0\le i<n(\omega)+s} X(\omega+s,\theta_i)J_i,$$

where $n(\omega) = p$. The initial functions of the periodic solutions of system (4.5) verify the relation $(I-U)\phi = \psi$. If system (4.5) does not have periodic solutions, this equation does not have solutions, hence ψ does not belong to the space $(I-U)\Phi$ where Φ is the space of initial functions $\phi(t),\ t\in[-\tau,0]$. But the operator U is compact and thus $(I-U)\Phi$ is a closed subspace of Φ. Therefore, there exists a linear functional y such that $y[\phi]=0$ for $\phi\in(I-U)\Phi$ and $y[\psi]=1$. For every $\phi\in\Phi$ we have $y[(I-U)\phi]=0$, hence $y[\phi-U\phi]=y[\phi]-y[U\phi]=0$ and $y[\phi]=y[U\phi]$. From $x(\omega+s;\phi)=U\phi+\psi$, we obtain

$$y[x(\omega+s;\phi)] = y[U\phi]+y[\psi] = y[\phi]+1. \tag{4.7}$$

We have the relation

$$\begin{aligned} x(t+n\omega;\phi) &= z(t;x(n\omega+s;\phi)) + \int_0^t X(t,\alpha)f(\alpha)d\alpha \\ &+ \sum_{0\le i<n(t)} X(t,\theta_i)J_i. \end{aligned}$$

Indeed, in the second member we have a solution of system (4.5), and this solution coincides in $[-\tau, 0]$ with $x(t+nw;\phi)$. From this relation we deduce that

$$x[(n+1)\omega+s;\phi] = Ux(n\omega+s;\phi)+\psi.$$

From here, by the induction principle, we obtain

$$y[x(n\omega+s;\phi)] = y[\phi]+n.$$

Indeed, we have previously obtained this relation in (4.7) for $n=1$. Further,

$$\begin{aligned} y[x((n+1)\omega+s;\phi)] &= y[Ux(n\omega+s;\phi)+\psi] \\ &= y[Ux(n\omega+s;\phi)]+y[\psi] \\ &= y[x(n\omega+s;\phi)]+y[\psi] \\ &= y[\phi]+n+1 \end{aligned}$$

and the relation is proved by induction. From this we deduce that the sequence $y[x(n\omega+s;\phi)]$ is unbounded, which shows that the solution $x(t;\phi)$ can not be bounded. □

Corollary 4.2.1 *(**Massera**) If the system (4.5) admits at least one bounded solution, then it also admits a periodic solution.*

Remark 4.2.2 *The homogeneous system (4.4) corresponding to system (4.5) can only have a finite number of linearly independent periodic solutions. Indeed, the* *corresponding initial functions are solutions of the equation $U\phi = \phi$ and from the fact that the operator U is compact it follows that this system has only a finite number of independent solutions.*

4.3 A Necessary and Sufficient Condition

In this section, we shall establish a necessary and sufficient condition for the nonhomogeneous impulsive differential systems with delays to have periodic solutions. In particular, a refinement of condition (4.3) adapted to systems involving impulses and delays is obtained. Meanwhile, it will be shown that the number of linearly independent periodic solutions of impulsive differential system with delays is the same as those of the corresponding adjoint system. We shall present this study in details for the classical linear impulsive differential systems with delays, the arguments can be carried out for the general linear case in the same way.

Consider the nonhomogeneous system

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t-\tau) + f(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}) + J_i, \quad i \in \mathbb{Z}, \end{cases} \tag{4.8}$$

where in addition to (C4.2) the following conditions (C4.3) are fulfilled:

(a) $A(t), B(t) \in PLC(\mathcal{J}, \mathbb{R}^{n\times n})$ and $A(t), B(t)$ are periodic of period $\omega > \tau$;

(b) $C_i, D_i \in \mathbb{R}^{n\times n}$ and there exists $p \in \mathbb{N}$ such that $C_{i+p} = C_i$, $D_{i+p} = D_i$ and $\theta_{i+p} = \theta_i + \omega$, $i \in \mathbb{Z}$.

First, we shall give a result that provides a necessary condition for the existence of periodic solutions of period ω. The proof of this result follows simply using the function (2.7) constructed in chapter 2.

Theorem 4.3.1 *If system (4.8) has periodic solutions of period ω, then*

$$\int_0^{\omega} y^T(t)f(t)dt + \sum_{0\le i<n(\omega)} y^T(\theta_i^+)J_i = 0,$$

for all periodic solutions $y(t)$ of period ω of the adjoint system

$$\begin{cases} y'(t) = & -A^T(t)y(t) - B^T(t+\tau)y(t+\tau), \quad t \neq \theta_i \\ \Delta y(\theta_i) = & -(I+C_i^T)^{-1}C_i^T y(\theta_i) - (I+C_i^T)^{-1}D_{i+j}^T y(\theta_{i+j}^+), \quad i \in \mathbb{Z} \end{cases} \tag{4.9}$$

Proof. Let $x(t)$ be a solution of system (4.8) defined for $t \ge -\tau$ and $y(t)$ a solution of the adjoint system (4.9) defined for $t \le \omega + \tau$. For $0 \le t \le \omega$, we define the function $< y(t), x(t) >$ as in chapter 2, then

$$\frac{d}{dt} < y(t), x(t) >= y^T(t)f(t), \quad t \neq \theta_i. \tag{4.10}$$

If $x(t)$ and $y(t)$ are periodic solutions of period ω then we see at once that $< y(t), x(t) >$ is a periodic function of period ω, thus, we obtain, by integrating relation (4.10) from 0 to ω, the following

$$\int_0^{\omega} y^T(t)f(t)dt + \sum_{0\le i<n(\omega)} y^T(\theta_i^+)J_i = 0,$$

where we have used the fact that

$$\Delta < y(\theta_i), x(\theta_i) >= y^T(\theta_i^+)J_i. \quad \square$$

We shall prove now that the homogeneous system

$$\begin{cases} x'(t) = & A(t)x(t) + B(t)x(t-\tau), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i \in \mathbb{Z} \end{cases} \tag{4.11}$$

has the same number of independent periodic solutions as the adjoint system (4.9). Moreover, it will be shown that the condition of Theorem 4.3.1 is sufficient in order that system (4.8) have periodic solutions. To do this we will need a series of preliminaries.

Let $y(t;\psi)$ be the solution of system (4.9) defined for $t \leq \omega + \tau$ such that $y(t;\psi) = \psi$ for $t \in [\omega, \omega+\tau]$. It follows from the periodicity of the system that $y(t-\omega;\psi)$ will likewise be a solution defined for $t \leq 2\omega + \tau$. Using Lemma 4.1.1 we have $y(t-\omega;\psi) = y(t;\psi)$ and hence the solution will be periodic. Taking into account the solution formula (2.13), the periodicity condition becomes

$$\begin{aligned} \psi(t) &= X^T(\omega, t-\omega)\psi(\omega) + \int_\omega^{\omega+\tau} X^T(\xi-\tau, t-\omega)B^T(\xi)\psi(\xi)d\xi \\ &- \sum_{n(\omega)\leq k < n(\omega)+j} X^T(\theta_{k-j}, t-\omega)D_k^T\psi(\theta_k^+), \quad t \in [\omega, \omega+\tau] \end{aligned}$$

where $n(\omega) = p$. Setting $t = s + \omega + \tau$, we have

$$\begin{aligned} \psi(s+\omega+\tau) &= X^T(\omega, s+\tau)\psi(\omega) + \int_\omega^{\omega+\tau} X^T(\xi-\tau, s+\tau)B^T(\xi)\psi(\xi)d\xi \\ &- \sum_{n(\omega)\leq k < n(\omega)+j} X^T(\theta_{k-j}, s+\tau)D_k^T\psi(\theta_k^+). \end{aligned}$$

Writing $\tilde{\varphi}(s) = \psi(s+\omega+\tau)$, $s \in [-\tau, 0]$, we get

$$\begin{aligned} \tilde{\varphi}(s) &= X^T(\omega, s+\tau)\tilde{\varphi}(-\tau) + \int_\omega^{\omega+\tau} X^T(\xi-\tau, s+\tau)B^T(\xi)\tilde{\varphi}(\xi-\omega-\tau)d\xi \\ &- \sum_{n(\omega)\leq k < n(\omega)+j} X^T(\theta_{k-j}, s+\tau)D_k^T\tilde{\varphi}(\theta_k^+ - \omega - \tau). \end{aligned}$$

Changing the variable $\eta = \xi - \omega - \tau$ and the index $\tau_i^+ = \theta_k^+ - \omega - \tau$ for $k = i + p + j$, we obtain

$$\begin{aligned} \tilde{\varphi}(s) &= X^T(\omega, s+\tau)\tilde{\varphi}(-\tau) + \int_{-\tau}^{0} X^T(\eta+\omega, s+\tau)B^T(\eta+\tau)\tilde{\varphi}(\eta)d\eta \\ &- \sum_{-j \le i < 0} X^T(\theta_{i+p}, s+\tau)D_{i+j}^T\tilde{\varphi}(\tau_i^+). \end{aligned} \tag{4.12}$$

The problem, thus, is reduced to proving that equation (4.12) has the same number of linearly independent solutions as the equation $\varphi(s) - U\varphi(s) = 0$ and that the equation $\varphi(s) - U\varphi(s) = \phi(s)$ has a solution if and only if

$$\phi^T(0)\tilde{\varphi}(-\tau) + \int_{-\tau}^{0} \phi^T(\eta)B^T(\eta+\tau)\tilde{\varphi}(\eta)d\eta - \sum_{-j \le i < 0} \phi^T(\theta_i)D_{i+j}^T\tilde{\varphi}(\tau_i^+) = 0,$$

for all the solutions $\tilde{\varphi}$ of (4.12).

Lemma 4.3.1 *Let $X(t, \alpha)$ be the fundamental matrix of system (4.11) and $\psi(t)$ an initial function of the adjoint system (4.9) defined for $t \in [\omega, \omega + \tau]$. Then*

$$\begin{aligned} \sum_{-j \le r < 0} \psi^T & (\tau_r^+ + \omega + \tau)D_{r+j}\Big(\int_0^{\omega+\theta_r} X(\omega+\theta_r, \alpha)f(\alpha)d\alpha\Big) \\ &= \int_0^{\omega} \sum_{-j \le r < 0} \psi^T(\tau_r^+ + \omega + \tau)D_{r+j}X(\omega+\theta_r, \alpha)f(\alpha)d\alpha, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \int_{-\tau}^{0} \psi^T & (\eta+\omega+\tau)B(\eta+\tau)\Big(\sum_{0 \le i < n(\omega)+\eta} X(\omega+\eta, \theta_i^+)J_i\Big)d\eta \\ &= \sum_{0 \le i < n(\omega)} \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau)B(\eta+\tau)X(\omega+\eta, \theta_i^+)J_i d\eta. \end{aligned}$$

Proof. Changing the order of summation and integration in (4.13), we obtain that the left side is equal to

$$\int_0^{\omega - j} \sum_{-j \le r < 0} \psi^T(\tau_r^+ + \omega + \tau)D_{r+j}X(\omega+\theta_r, \alpha)f(\alpha)d\alpha$$

$$+\int_{\omega-j}^{\omega}\sum_{\alpha-\omega\le r<0}\psi^T(\tau_r^+ +\omega+\tau)D_{r+j}X(\omega+\theta_r,\alpha)f(\alpha)d\alpha.$$

However, since $X(\omega+\theta_r^+,\alpha)\equiv 0$ for $\omega+\theta_r<\alpha$, the second member of the above relation can be written as

$$\int_{\omega-j}^{\omega}\sum_{-j\le r<0}\psi^T(\tau_r^+ +\omega+\tau)D_{r+j}X(\omega+\theta_r,\alpha)f(\alpha)d\alpha.$$

Thus, (4.13) is obtained by summing up all the members. The proof of the second statement is similar and hence is omitted. □

It is convenient to make use of the following notation in the remainder of this section. That is, for the matrix functions ψ and φ defined in $[-\tau,0]$ for which multiplication is possible, we define the operation

$$\begin{aligned} <\psi(s),\varphi(s)> &= \psi^T(-\tau)\varphi(0)+\int_{-\tau}^{0}\psi^T(\xi)B(\xi+\tau)\varphi(\xi)d\xi \\ &- \sum_{-j\le k<0}\psi^T(\tau_k^+)D_{k+j}\varphi(\theta_k). \end{aligned} \tag{4.14}$$

Using the above notation we have the following lemma which will be a key tool in later analysis.

Lemma 4.3.2 *For the matrix functions $N,M,L\in\mathbb{R}^{n\times n}$, we have*

$$\langle <L(\sigma),M(\alpha,\sigma)>^T,N(\alpha)\rangle=\langle L(\sigma),<M^T(\alpha,\sigma),N(\alpha)>\rangle.$$

Proof. Let $K(\alpha) =< L(\sigma), M(\alpha,\sigma) >^T$. Then

$$
\begin{aligned}
< K(\alpha), N(\alpha) > &= K^T(-\tau)N(0) + \int_{-\tau}^{0} K^T(\xi)B(\xi+\tau)N(\xi)d\xi \\
&- \sum_{-j\le i<0} K^T(\tau_i^+)D_{i+j}N(\theta_i),
\end{aligned}
$$

and

$$
\begin{aligned}
K^T(\alpha) &= L^T(-\tau)M(\alpha,0) + \int_{-\tau}^{0} L^T(\xi)B(\xi+\tau)M(\alpha,\xi)d\xi \\
&- \sum_{-j\le i<0} L^T(\tau_i^+)D_{i+j}M(\alpha,\theta_i).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
< K(\alpha), N(\alpha) > &= \Big[L^T(-\tau)M(-\tau,0) + \int_{-\tau}^{0} L^T(\xi)B(\xi+\tau)M(-\tau,\xi)d\xi \\
&- \sum_{-j\le i<0} L^T(\tau_i^+)D_{i+j}M(-\tau,\theta_i)\Big]N(0) \\
&+ \int_{-\tau}^{0} \Big[L^T(-\tau)M(\xi,0) + \int_{-\tau}^{0} L^T(\eta)B(\eta+\tau)M(\xi,\eta)d\eta \\
&- \sum_{-j\le i<0} L^T(\tau_i^+)D_{i+j}M(\xi,\theta_i)\Big]B(\xi+\tau)N(\xi)d\xi \\
&- \sum_{-j\le i<0} \Big[L^T(-\tau)M(\tau_i^+,0) + \int_{-\tau}^{0} L^T(\xi)B(\xi+\tau)M(\tau_i^+,\xi)d\xi \\
&- \sum_{-j\le k<0} L^T(\tau_k^+)D_{k+j}M(\tau_i^+,\theta_k)\Big]D_{i+j}N(\theta_i),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
< K(\alpha), N(\alpha) > &= L^T(-\tau)M(-\tau,0)N(0) + \int_{-\tau}^{0} L^T(\xi)B(\xi+\tau)M(-\tau,\xi)N(0)d\xi \\
&- \sum_{-j\le i<0} L^T(\tau_i^+)D_{i+j}M(-\tau,\theta_i)N(0) \\
&+ L^T(-\tau)\int_{-\tau}^{0} M(\xi,0)B(\xi+\tau)N(\xi)d\xi \\
&+ \int_{-\tau}^{0} \Big(\int_{-\tau}^{0} L^T(\eta)B(\eta+\tau)M(\xi,\eta)d\eta\Big)B(\xi+\tau)N(\xi)d\xi
\end{aligned}
$$

$$
\begin{aligned}
& - \int_{-\tau}^{0} \Big(\sum_{-j\le i<0} L^T(\tau_i^+) D_{i+j} M(\xi,\theta_i) \Big) B(\xi+\tau) N(\xi) d\xi \\
& - L^T(-\tau) \sum_{-j\le i<0} M(\tau_i^+,0) D_{i+j} N(\theta_i) \\
& - \sum_{-j\le i<0} \Big(\int_{-\tau}^{0} L^T(\xi) B(\xi+\tau) M(\tau_i^+,\xi) d\xi \Big) D_{i+j} N(\theta_i) \\
& + \sum_{-j\le i<0} \Big(\sum_{-j\le k<0} L^T(\tau_k^+ D_{k+j} M(\tau_i^+,\theta_k) D_{i+j} N(\theta_i).
\end{aligned}
$$

Changing the order of integrations, the order of summations and the order of sum and integration, we obtain

$$
\begin{aligned}
< K(\alpha), N(\alpha) > \;=\; & L^T(-\tau) \Big[M(-\tau,0) N(0) + \int_{-\tau}^{0} M(\xi,0) B(\xi+\tau) N(\xi) d\xi \\
& - \sum_{-j\le i<0} M(\tau_i^+,0) D_{i+j} N(\theta_i) \Big] \\
& + \int_{-\tau}^{0} L^T(\eta) B(\eta+\tau) \Big[M(-\tau,\eta) N(0) \\
& + \int_{-\tau}^{0} M(\xi,\eta) B(\xi+\tau) N(\xi) d\xi \\
& - \sum_{-j\le i<0} M(\tau_i^+,\eta) D_{i+j} N(\theta_i) \Big] d\eta \\
& - \sum_{-j\le k<0} L^T(\theta_k^+) D_{k+j} \Big[M(-\tau,\theta_i) N(0) \\
& + \int_{-\tau}^{0} M(\xi,\theta_k) B(\xi+\tau) N(\xi) d\xi \\
& - \sum_{-j\le i<0} M(\tau_i^+,\theta_k) D_{i+j} N(\theta_i) \Big].
\end{aligned}
$$

Hence

$$
\langle < L(\sigma), M(\alpha,\sigma) >^T, N(\alpha) \rangle = \langle L(\sigma), < M^T(\alpha,\sigma), N(\alpha) > \rangle. \quad \square
$$

Thus, with this notation, the operator U can be written as

$$
U\varphi = < X^T(\omega+s, \eta+\tau), \varphi(\eta) > .
$$

If we define $\tilde{U}$ by

$$\tilde{U}\tilde{\varphi} =< \tilde{\varphi}(\eta), X(\omega+\eta, s+\tau) >^T,$$

then we observe the relation $< \tilde{U}\tilde{\varphi}, \varphi >=< \tilde{\varphi}, U\varphi >$ which is valid for every pair of functions $\tilde{\varphi}, \varphi$ continuous in $[-\tau, 0]$. Indeed, we have

$$< \tilde{U}\tilde{\varphi}, \varphi >=<< \tilde{\varphi}(\eta), X(\omega+\eta, s+\tau) >^T, \varphi(s) > .$$

Using Lemma 4.3.2, we obtain

$$< \tilde{U}\tilde{\varphi}, \varphi >=< \tilde{\varphi}(\eta), < X^T(\omega+\eta, s+\tau), \varphi(s) >>=< \tilde{\varphi}, U\varphi > .$$

Lemma 4.3.3 *Let* $K_l, \tilde{K}_l \in \mathbb{R}^{n\times n}$ *be defined by*

$$K_l(s,\eta) =< K_1^T(s,\alpha), K_{l-1}(\alpha,\eta) >,$$

and

$$\tilde{K}_l(\eta,s) =< K_{l-1}^T(\eta,\alpha), K_1(\alpha,s) >,$$

where $K_1 \in \mathbb{R}^{n\times n}$. *Then*

$$\tilde{K}_l(\eta,s) = K_l(\eta,s).$$

Proof. We have $K_2(\eta,s) =< K_1^T(\eta,\alpha), K_1(\alpha,s) >= \tilde{K}_2(\eta,s)$. Suppose that the equality holds for $j \leq l$. That is,

$$\begin{aligned} K_l(\eta,s) &= < K_1^T(\eta,\alpha), K_{l-1}(\alpha,s) > \\ &= \tilde{K}_l(\eta,s) \\ &= < K_{l-1}^T(\eta,\alpha), K_1(\alpha,s) > . \end{aligned}$$

Hence

$$\begin{aligned} K_{l+1}(\eta, s) &= < K_1^T(\eta, \alpha), K_l(\alpha, s) > \\ &= < K_1^T(\eta, \alpha), < K_{l-1}^T(\alpha, \beta), K_1(\beta, s) >> \end{aligned}$$

Using Lemma 4.3.2, we get

$$\begin{aligned} K_{l+1}(\eta, s) &= << K_1^T(\eta, \alpha), K_{l-1}(\alpha, \beta) >^T, K_1(\beta, s) > \\ &= < K_l^T(\eta, \beta), K_1(\beta, s) > \\ &= \tilde{K}_{l+1}(\eta, s). \quad \square \end{aligned}$$

Let $\tilde{V}$ be defined by $\tilde{V}\psi = y(\sigma - \omega; \psi)$, $\sigma \in [\omega, \omega + \tau]$ where ψ is continuous in $[\omega, \omega + \tau]$. If ρ is an eigenvalue of $\tilde{V}$ we will say that $\frac{1}{\rho}$ is a multiplier of the adjoint system. This definition is justified by the fact that if ρ is an eigenvalue of $\tilde{V}$ then system (4.9) admits a solution with the property $y(t + \omega; \psi) = \frac{1}{\rho} y(t; \psi)$ and conversely. Indeed, if $\tilde{V}\psi = \rho\psi$ then $y(\sigma - \omega; \psi) = \rho y(\sigma; \psi)$, since both members of the equality are solutions of system (4.9) and these solutions coincide in $[\omega, \omega + \tau]$; conversely, if this relation holds for $t \in [\omega, \omega + \tau]$, it yields $\tilde{V}\psi = \rho\psi$. The relation obtained can also be represented in the form $y(\sigma + \omega; \psi) = \frac{1}{\rho} y(\sigma; \psi)$. We have for $\sigma \in [\omega, \omega + \tau]$,

$$\begin{aligned} \tilde{V}\psi &= X^T(\omega, \sigma - \omega)\psi(\omega) + \int_{\omega}^{\omega+\tau} X^T(\xi - \tau, \sigma - \omega) B^T(\xi)\psi(\xi) d\xi \\ &- \sum_{n(\omega) \le k < n(\omega)+j} X^T(\theta_{k-j}, \sigma - \omega) D_k^T \psi(\theta_k^+), \end{aligned}$$

where $n(\omega) = p$. If ρ is an eigenvalue of $\tilde{V}$, then there exists a nonzero solution of the equation

$$\rho\psi(\sigma) = X^T(\omega, \sigma - \omega)\psi(\omega) + \int_{\omega}^{\omega+\tau} X^T(\xi - \tau, \sigma - \omega) B^T(\xi)\psi(\xi) d\xi$$

$$- \sum_{n(\omega)\le k<n(\omega)+j} X^T(\theta_{k-j}, \sigma-\omega) D_k^T \psi(\theta_k^+).$$

Setting $\tilde{\varphi}(s) = \psi(s+\omega+\tau), s \in [-\tau, 0]$, we may write

$$\begin{aligned}\rho\tilde{\varphi}(s) &= X^T(\omega, s+\tau)\tilde{\varphi}(-\tau) + \int_{\omega}^{\omega+\tau} X^T(\xi-\tau, s+\tau) B^T(\xi)\tilde{\varphi}(\xi-\omega-\tau)d\xi \\ &- \sum_{n(\omega)\le k<n(\omega)+j} X^T(\theta_{k-j}, s+\tau) D_k^T \tilde{\varphi}(\theta_k^+ - \omega - \tau),\end{aligned}$$

or

$$\begin{aligned}\rho\tilde{\varphi}(s) &= X^T(\omega, s+\tau)\tilde{\varphi}(-\tau) + \int_{-\tau}^{0} X^T(\eta+\omega, s+\tau) B^T(\eta+\tau)\tilde{\varphi}(\eta)d\eta \\ &- \sum_{-j\le i<0} X^T(\theta_{i+p}, s+\tau) D_{i+j}^T \tilde{\varphi}(\tau_i^+).\end{aligned}$$

The right side of the above equation is nothing but $\tilde{U}\tilde{\varphi}$ which implies, consequently, that the eigenvalues of the operators $\tilde{U}$ and $\tilde{V}$ coincide and in addition, if ψ is an eigenfunction for $\tilde{V}$, then $\tilde{\varphi} = \psi(s+\omega+\tau)$ is an eigenfunction for $\tilde{U}$. In the following lines, we shall prove that the eigenvalues of the operators U and $\tilde{U}$ coincide and that for each eigenvalue ρ both operators have the same number of linearly independent eigenfunctions. Thus, for $\rho = 1$ the assertion relative to periodic solutions will follow since the equation (4.12) is just $\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0$.

Theorem 4.3.2 *Systems (4.11) and (4.9) have the same number of linearly independent periodic solutions of period $\omega > \tau$. Moreover, if*

$$\int_0^{\omega} y^T(t) f(t)dt + \sum_{0\le i<n(\omega)} y^T(\theta_i^+) J_i = 0,$$

for all periodic solutions y of period ω of system (4.9), then system (4.8) has periodic solutions of period ω.

Proof. Consider the equation

$$\rho\varphi(s) - U\varphi(s) = F(s). \tag{4.15}$$

Since $X(\omega+s,\eta+\tau)$ is uniformly piecewise continuous for $s,\eta \in [-\tau,0]\times[-\tau,0]$, we may write

$$X(\omega+s,\eta+\tau)=\sum_{k=1}^{n} a_k(s)b_k(\eta)+K_1(s,\eta),$$

where $a_k(s)$ are column vectors and $b_k(\eta)$ are row vectors, linearly independent vectors, K_1 is a matrix and $|K_1|$ can be taken arbitrarily small. Clearly, we have

$$X^T(\omega+s,\eta+\tau)=\sum_{k=1}^{n} b_k^T(\eta)a_k^T(s)+K_1^T(s,\eta).$$

Then, by using the fact that $< b_k^T(\eta)a_k^T(s),\varphi(s) >= a_k(s) < b_k^T(\eta),\varphi(s) >$, (4.15) becomes

$$\rho\varphi(s)-\sum_{k=1}^{n} a_k(s) < b_k^T(\eta),\varphi(\eta) > - < K_1^T(s,\eta),\varphi(\eta) >= F(s).$$

Setting

$$\nu(s)=\frac{1}{\rho}\sum_{k=1}^{n} a_k(s) < b_k^T(\eta),\varphi(\eta) > +\frac{1}{\rho}F(s), \tag{4.16}$$

we obtain

$$\nu(s)=\varphi(s)-\frac{1}{\rho} < K_1^T(s,\eta),\varphi(\eta) > . \tag{4.17}$$

Now consider equations of the form

$$\nu(s)=\varphi(s)-\lambda < K_1^T(s,\eta),\varphi(\eta) > . \tag{4.18}$$

We seek a solution of the form $\varphi(s)=\sum_{i=0}^{\infty}\lambda^i\varphi_i(s)$. Substituting this into (4.18) and identifying the coefficients of the powers of λ, we obtain

$$\varphi_0(s)=\nu(s) \quad \text{and} \quad \mathrm{\varphi_i(s) =< K_1^T(s,\alpha),\varphi_{i-1}(\alpha) >,\quad i=1,2,\ldots.}$$

It follows that $|\varphi_i(s)| \leq M^i \sup_s |\nu(s)|$, where $M=\sup|K_1^T|$ and $i=1,2,\ldots$. Therefore, using the *Weierstrass* M-test, the series converges uniformly and absolutely if $|\lambda|M<1$. We have

$$\varphi_1(s) =< K_1^T(s,\alpha),\nu(\alpha) > .$$

By the induction principle, we obtain

$$\varphi_l(s) =< K_l^T(s,\alpha), \nu(\alpha) >,$$

where

$$K_l(s,\eta) =< K_1^T(s,\alpha), K_{l-1}(\alpha,\eta) > .$$

Indeed, we have

$$\varphi_{l+1}(s) =< K_1^T(s,\alpha), \varphi_l(\alpha) >=< K_1^T(s,\alpha), < K_l^T(\alpha,\eta), \nu(\eta) >> .$$

Using Lemma 4.3.2, we have

$$\varphi_{l+1}(s) =<< K_1^T(s,\alpha), K_l(\alpha,\eta) >^T, \nu(\eta) >=< K_{l+1}^T(s,\eta), \nu(\eta) > .$$

It follows that, if $|\lambda| < \frac{1}{M}$ then the solution of equation (4.18) can be written as

$$\varphi(s) = \nu(s) + \sum_{l=1}^{\infty} \lambda^l \varphi_l(s) = \nu(s) + \sum_{l=1}^{\infty} \lambda^l < K_l^T(s,\alpha), \nu(\alpha) > .$$

Thus

$$\varphi(s) = \nu(s) + < \Gamma^T(s,\alpha), \nu(\alpha) >,$$

where

$$\Gamma^T(s,\alpha) = \sum_{l=1}^{\infty} \lambda^l K_l^T(s,\alpha).$$

Therefore, if $\frac{1}{|\rho|} < \frac{1}{M}$ and $\sup |K_1^T| < |\rho|$, we deduce that

$$\varphi(s) = \nu(s) + < \Gamma^T(s,\alpha), \nu(\alpha) > \qquad (4.19)$$

is a solution of (4.17).

On the other hand, consider the equation

$$\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0,$$

which can be written as

$$\rho\tilde{\varphi}(s) =< \tilde{\varphi}(\alpha), \sum_{k=1}^{n} a_k(\alpha)b_k(s) + K_1(\alpha, s) >^T,$$

or

$$\rho\tilde{\varphi}(s) = \sum_{k=1}^{n} b_k^T(s) < \tilde{\varphi}(\alpha), a_k(\alpha) >^T + < \tilde{\varphi}(\alpha), K_1(\alpha, s) >^T .$$

Setting

$$\tilde{\nu}(s) = \frac{1}{\rho}\sum_{k=1}^{n} b_k^T(s) < \tilde{\varphi}(\alpha), a_k(\alpha) >^T, \tag{4.20}$$

we obtain

$$\tilde{\nu}(s) = \tilde{\varphi}(s) - \frac{1}{\rho} < \tilde{\varphi}(\alpha), K_1(\alpha, s) >^T . \tag{4.21}$$

We again seek a solution of the form $\tilde{\varphi}(s) = \sum_{i=0}^{\infty} \lambda^i \tilde{\varphi}_i(s)$. Substituting this into (4.21), we obtain

$$\tilde{\varphi}_0(s) = \tilde{\nu}(s) \quad \text{and} \quad \tilde{\varphi}_{\mathrm{i}}(\mathrm{s}) =< \tilde{\varphi}_{\mathrm{i}-1}(\alpha), \mathrm{K}_1(\alpha, \mathrm{s}) >^{\mathrm{T}}, \quad \mathrm{i} = 1, 2, \ldots .$$

Likewisely, the series converges uniformly and absolutely if $\frac{1}{|\rho|} < \frac{1}{M}$. By the induction principle, we get

$$\tilde{\varphi}_l(s) =< \tilde{\nu}(\alpha), \tilde{K}_l(\alpha, s) >^T,$$

where

$$\tilde{K}_l(\eta, s) =< K_{l-1}^T(\eta, \alpha), K_1(\alpha, s) > .$$

We finally obtain that the solution of (4.21) is in the form

$$\tilde{\varphi}(s) = \tilde{\nu}(s) + < \tilde{\nu}(\alpha), \tilde{\Gamma}(\alpha, s) >^T, \tag{4.22}$$

where

$$\tilde{\Gamma}(\alpha, s) = \sum_{l=1}^{\infty} \lambda^l \tilde{K}_l(\alpha, s).$$

Applying Lemma 4.3.3, we obtain

$$\tilde{\Gamma}(\eta, s) = \Gamma(\eta, s).$$

From equation (4.16), we have

$$\rho\nu(s) = \sum_{k=1}^{n} a_k(s) < b_k^T(\eta), \varphi(\eta) > +F(s). \tag{4.23}$$

But $\varphi(s) = \nu(s) + < \Gamma^T(s, \alpha), \nu(\alpha) >$. So

$$\rho\nu(s) = \sum_{k=1}^{n} a_k(s) < b_k^T(\eta), \nu(\eta) + < \Gamma^T(\eta, \alpha), \nu(\alpha) >> +F(s),$$

which can be written as

$$\rho\nu(s) = \sum_{k=1}^{n} a_k(s)\Big(< b_k^T(\eta), \nu(\eta) > + < b_k^T(\eta), < \Gamma^T(\eta, \alpha), \nu(\alpha) >> \Big) + F(s).$$

Using Lemma 4.3.2, we obtain

$$\rho\nu(s) = \sum_{k=1}^{n} a_k(s)\Big(< b_k^T(\eta), \nu(\eta) > + << b_k^T(\eta), \Gamma(\eta, \alpha) >^T, \nu(\alpha) > \Big) + F(s),$$

or

$$\rho\nu(s) = \sum_{k=1}^{n} a_k(s) < b_k^T(\alpha) + < b_k^T(\eta), \Gamma(\eta, \alpha) >^T, \nu(\alpha) > +F(s).$$

Hence

$$\rho\nu(s) = \sum_{k=1}^{n} a_k(s) < \bar{b}_k^T(\alpha), \nu(\alpha) > +F(s), \tag{4.24}$$

where

$$\bar{b}_k^T(\alpha) = b_k^T(\alpha) + < b_k^T(\eta), \Gamma(\eta, \alpha) >^T .$$

Setting $\lambda_k =< \bar{b}_k^T(\alpha), \nu(\alpha) >$, it follows from (4.24) that

$$\rho\nu(s) - F(s) = \sum_{k=1}^{n} \lambda_k a_k(s) \tag{4.25}$$

is the form of the solution of (4.24).

Analogously, from (4.20)

$$\rho\tilde{\nu}(s) = \sum_{k=1}^{n} b_k^T(s) < \tilde{\varphi}(\alpha), a_k(\alpha) >^T .$$

But, $\tilde{\varphi}(s) = \tilde{\nu}(s) + < \tilde{\nu}(\alpha), \tilde{\Gamma}(\alpha, s) >^T$. Therefore,

$$\rho\tilde{\nu}(s) = \sum_{k=1}^{n} b_k^T(s) < \tilde{\nu}(\alpha) + < \tilde{\nu}(\eta), \tilde{\Gamma}(\eta, \alpha) >^T, a_k(\alpha) >^T,$$

which we may write

$$\rho\tilde{\nu}(s) = \sum_{k=1}^{n} b_k^T(s) < \tilde{\nu}(\eta), \bar{a}_k(\eta) >^T, \tag{4.26}$$

where

$$\bar{a}_k(\eta) = a_k(\eta) + < \tilde{\Gamma}^T(\eta, \alpha), a_k(\alpha) > .$$

The solution of (4.26) has the form

$$\rho\tilde{\nu}(s) = \sum_{k=1}^{n} \mu_k b_k^T(s), \tag{4.27}$$

where

$$\mu_k =< \tilde{\nu}(\eta), \bar{a}_k(\eta) >^T .$$

In view of (4.24), (4.25) becomes

$$\sum_{k=1}^{n} \lambda_k a_k(s) = \sum_{k=1}^{n} a_k(s) < \bar{b}_k^T(\alpha), \frac{1}{\rho} F(\alpha) + \frac{1}{\rho} \sum_{j=1}^{n} \lambda_j a_j(\alpha) > . \tag{4.28}$$

Similarly, from (4.26), (4.27) can be written as

$$\sum_{k=1}^{n} \mu_k b_k^T(s) = \sum_{k=1}^{n} b_k^T(s) < \frac{1}{\rho} \sum_{j=1}^{n} \mu_j b_j^T(\eta), \bar{a}_k(\eta) >^T . \tag{4.29}$$

Taking into account that the vectors $\{a_k\}$ are linearly independent, we obtain from (4.28) the algebraic equation

$$\rho\lambda_k = \sum_{j=1}^{n} \gamma_{kj}\lambda_j + f_k, \tag{4.30}$$

where

$$\gamma_{kj} =< \bar{b}_k^T(\alpha), a_j(\alpha) > \quad \textit{and} \quad f_k =< \bar{b}_k^T(\alpha), F(\alpha) > .$$

Similarly, the vectors $\{b_k\}$ are linearly independent, thus we get from (4.29) the algebraic equation

$$\rho\mu_k = \sum_{j=1}^{n} \tilde{\gamma}_{jk}^T \mu_j, \tag{4.31}$$

where

$$\tilde{\gamma}_{jk}^T =< b_j^T(\eta), \bar{a}_k(\eta) > .$$

We know that equation (4.30) for λ_k has a solution if and only if $\sum_{k=1}^{n} \mu_k f_k = 0$ for all the solutions μ_k of the equation

$$\rho\mu_k = \sum_{j=1}^{n} \gamma_{jk}\mu_j. \tag{4.32}$$

But, we have $\tilde{\gamma}_{jk}^T = \gamma_{jk}$, since

$$\tilde{\gamma}_{jk}^T =< b_j^T(\eta), a_k(\eta)+ < \tilde{\Gamma}^T(\eta,\alpha), a_k(\alpha) >>$$

or

$$\tilde{\gamma}_{jk}^T =< b_j^T(\eta), a_k(\eta) > + < b_j^T(\eta), < \tilde{\Gamma}^T(\eta,\alpha), a_k(\alpha) >> .$$

Using Lemma 4.3.2, we have

$$\tilde{\gamma}_{jk}^T =< b_j^T(\eta), a_k(\eta) > + << b_j^T(\eta), \tilde{\Gamma}(\eta,\alpha) >^T, a_k(\alpha) >$$

or

$$\begin{aligned} \tilde{\gamma}_{jk}^T &= < b_j^T(\alpha) + < b_j^T(\eta), \tilde{\Gamma}(\eta,\alpha) >^T, a_k(\alpha) > \\ &= < \tilde{b}_j^T(\alpha), a_k(\alpha) >= \gamma_{jk}, \end{aligned}$$

where we have used that $\tilde{\Gamma}(\eta,\alpha) = \Gamma(\eta,\alpha)$. Therefore, systems (4.31) and (4.32) coincide.

The systems

$$\rho\lambda_k = \sum_{j=1}^{n} \gamma_{kj}\lambda_j \tag{4.33}$$

and

$$\rho\mu_k = \sum_{j=1}^{n} \gamma_{jk}\mu_j \tag{4.34}$$

have the same number of linearly independent solutions. To a solution of system (4.33) corresponds $\nu(s) = \frac{1}{\rho}\sum_{k=1}^{n} \lambda_k a_k(s)$, and to this corresponds the solution $\varphi(s) = \nu(s) + < \Gamma^T(s,\alpha), \nu(\alpha) >$ for the equation $\rho\varphi(s) - U\varphi(s) = 0$, linearly independent solutions corresponding to the linearly independent solutions of system (4.33). Likewise, a solution of the equation $\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0$ will correspond to a solution of system (4.31), which coincides with (4.34), linearly independent solutions corresponding to linearly independent solutions. It follows from here that the equations $\rho\varphi(s) - U\varphi(s) = 0$ and $\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0$ have the same number of independent solutions, which implies in particular the fact that U and $\tilde{U}$ have the same eigenvalues, hence if ρ is a multiplier of the system, $\frac{1}{\rho}$ is a multiplier of the adjoint system.

The condition

$$\sum_{k=1}^{n} \mu_k f_k = 0 \tag{4.35}$$

which is a necessary and sufficient in order that $\rho\varphi(s) - U\varphi(s) = F(s)$ has solutions can be explicitly written as

$$\sum_{k=1}^{n} \mu_k < b_k^T(\alpha) + < b_k^T(\eta), \Gamma(\eta,\alpha) >^T, F(\alpha) >= 0.$$

In view of (4.27), condition (4.35) becomes

$$\rho < \tilde{\nu}(\alpha), F(\alpha) > + \rho << \tilde{\nu}(\eta), \Gamma(\eta,\alpha) >^T, F(\alpha) >= 0.$$

However, $\tilde{\nu}(\alpha) + < \tilde{\nu}(\eta), \Gamma(\eta,\alpha) >^T = \tilde{\varphi}(\alpha)$ where we have used that $\Gamma(\eta,\alpha) = \tilde{\Gamma}(\eta,\alpha)$, therefore, we obtain the condition $< \tilde{\varphi}(\alpha), F(\alpha) >= 0$. We have, thus, shown that a necessary and sufficient condition in order that the equation $\rho\varphi(s) - U\varphi(s) = F(s)$ have solutions is $< \tilde{\varphi}(\alpha), F(\alpha) >= 0$ for all solutions $\tilde{\varphi}$ of $\rho\tilde{\varphi}(s) - \tilde{U}\tilde{\varphi}(s) = 0$.

In the equation $\rho\varphi(s) - U\varphi(s) = F(s)$ consider the case where $\rho = 1$, therefore, $F(s) = \varphi(s) - U\varphi(s)$, that is,

$$F(s) = \int_0^{\omega+s} X(\omega+s,\alpha) f(\alpha) d\alpha + \sum_{0 \le i < n(\omega)+s} X(\omega+s, \theta_i^+) J_i,$$

where $n(\omega) = p$. In view of formula (4.14), we deduce that

$$\begin{aligned} < \tilde{\varphi}(\alpha), F(\alpha) > &= \tilde{\varphi}^T(-\tau) F(0) + \int_{-\tau}^{0} \tilde{\varphi}^T(\xi) B(\xi+\tau) F(\xi) d\xi \\ &- \sum_{-j \le k < 0} \tilde{\varphi}^T(\tau_k^+) D_{k+j} F(\theta_k) = 0. \end{aligned}$$

Therefore

$$< \tilde{\varphi}(\alpha), F(\alpha) > = \tilde{\varphi}^T(-\tau) \Big[\int_0^{\omega} X(\omega,\alpha) f(\alpha) d\alpha + \sum_{0 \le i < n(\omega)} X(\omega, \theta_i^+) J_i \Big]$$

$$
\begin{aligned}
&+ \int_{-\tau}^{0} \tilde{\varphi}^T(\xi) B(\xi+\tau) \Big[\int_0^{\omega+\xi} X(\omega+\xi,\alpha) f(\alpha) d\alpha \\
&+ \sum_{0 \le i < n(\omega)+\xi} X(\omega+\xi,\theta_i^+) J_i \Big] d\xi \\
&- \sum_{-j \le k < 0} \tilde{\varphi}^T(\tau_k^+) D_{k+j} \Big[\int_0^{\omega+\theta_k} X(\omega+\theta_k,\alpha) f(\alpha) d\alpha \\
&+ \sum_{0 \le i < n(\omega)+\theta_k} X(\omega+\theta_k,\theta_i^+) J_i \Big] = 0.
\end{aligned}
$$

Setting $\tilde{\varphi}(s) = \psi(s+\omega+\tau)$

$$
\begin{aligned}
< \tilde{\varphi}(\alpha), F(\alpha) > &= \psi^T(\omega) \int_0^{\omega} X(\omega,\alpha) f(\alpha) d\alpha + \psi^T(\omega) \sum_{0 \le i < n(\omega)} X(\omega,\theta_i^+) J_i \\
&+ \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) \Big(\int_0^{\omega+\eta} X(\omega+\eta,\alpha) f(\alpha) d\alpha \Big) d\eta \\
&+ \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) \Big(\sum_{0 \le i < n(\omega)+\eta} X(\omega+\eta,\theta_i^+) J_i \Big) d\eta \\
&- \sum_{-j \le r < 0} \psi^T(\tau_r^+ +\omega+\tau) D_{r+j} \Big(\int_0^{\omega+\theta_r} X(\omega+\theta_r,\alpha) f(\alpha) d\alpha \Big) \\
&- \sum_{-j \le r < 0} \psi^T(\tau_r^+ +\omega+\tau) D_{r+j} \Big(\sum_{0 \le i < n(\omega)+\theta_r} X(\omega+\theta_r,\theta_i^+) J_i \Big) = 0.
\end{aligned}
$$

Interchanging the order of integrations in the second integral and the order of summations in the last sum, we have

$$
\begin{aligned}
< \tilde{\varphi}(\alpha), F(\alpha) > &= \psi^T(\omega) \int_0^{\omega} X(\omega,\alpha) f(\alpha) d\alpha \\
&+ \int_0^{\omega-\tau} \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta,\alpha) f(\alpha) d\eta d\alpha \\
&+ \int_{\omega-\tau}^{\omega} \int_{\alpha-\omega}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta,\alpha) f(\alpha) d\eta d\alpha \\
&- \sum_{-j \le r < 0} \psi^T(\tau_r^+ +\omega+\tau) D_{r+j} \Big(\int_0^{\omega+\theta_r} X(\omega+\theta_r,\alpha) f(\alpha) d\alpha) \Big) \\
&+ \psi^T(\omega) \sum_{0 \le i < n(\omega)} X(\omega,\theta_i^+) J_i
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{0\le i<n(\omega)-j} \Big(\sum_{-j\le r<0} \psi^T(\tau_r^+ + \omega + \tau) D_{r+j} X(\omega+\theta_r, \theta_i^+) J_i \Big) \\
&- \sum_{n(\omega)-j\le i<n(\omega)} \\
&\times \sum_{i-n(\omega)\le r<0} \psi^T(\tau_r^+ + \omega + \tau) D_{r+j} X(\omega+\theta_r, \theta_i^+) J_i \\
&+ \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) \Big(\sum_{0\le i<n(\omega)+\eta} X(\omega+\eta, \theta_i^+) J_i \Big) d\eta = 0.
\end{aligned}
$$

Looking at the third integral and the fourth summation we have for $\alpha > \omega + \eta$, $X(\omega+\eta, \alpha) \equiv 0$ and for $i > n(\omega) + r$, $X(\omega + \theta_r^+, \theta_i^+) \equiv 0$, so in the third integral and in the fourth sum we can replace $-\tau$ instead of $\alpha - \omega$ and $-j$ instead of $i - n(\omega)$ respectively. The condition, thus, can be written as

$$
\begin{aligned}
< \tilde{\varphi}(\alpha), F(\alpha) > \; &= \psi^T(\omega) \int_0^{\omega} X(\omega, \alpha) f(\alpha) d\alpha \\
&+ \int_0^{\omega} \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta, \alpha) f(\alpha) d\eta d\alpha \\
&- \sum_{-j\le r<0} \psi^T(\tau_r^+ + \omega + \tau) D_{r+j} \Big(\int_0^{\omega+\theta_r} X(\omega+\theta_r, \alpha) f(\alpha) d\alpha \Big) \\
&+ \psi^T(\omega) \sum_{0\le i<n(\omega)} X(\omega, \theta_i^+) J_i \\
&- \sum_{0\le i<n(\omega)} \sum_{-j\le r<0} \psi^T(\tau_r^+ + \omega + \tau) D_{r+j} X(\omega+\theta_r, \theta_i^+) J_i \\
&+ \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) \Big(\sum_{0\le i<n(\omega)+\eta} X(\omega+\eta, \theta_i^+) J_i \Big) d\eta = 0.
\end{aligned}
$$

Using Lemma 4.3.1 in the third and the last line, we have

$$
\begin{aligned}
< \tilde{\varphi}(\alpha), F(\alpha) > \; &= \psi^T(\omega) \int_0^{\omega} X(\omega, \alpha) f(\alpha) d\alpha \\
&+ \int_0^{\omega} \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta, \alpha) f(\alpha) d\eta d\alpha \\
&- \int_0^{\omega+\theta_r} \sum_{-j\le r<0} \psi^T(\tau_r^+ + \omega + \tau) D_{r+j} X(\omega+\theta_r, \alpha) f(\alpha) d\alpha \\
&+ \psi^T(\omega) \sum_{0\le i<n(\omega)} X(\omega, \theta_i^+) J_i
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{0\le i<n(\omega)} \sum_{-j\le r<0} \psi^T(\tau_r^+ + \omega+\tau) D_{r+j} X(\omega+\theta_r, \theta_i^+) J_i \\
&+ \sum_{0\le i<n(\omega)} \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta, \theta_i^+) J_i d\eta = 0.
\end{aligned}
$$

It readily follows that

$$
\begin{aligned}
< \tilde{\varphi}(\alpha), F(\alpha) > \;=\; & \int_0^{\omega} \Big[\psi^T(\omega) X(\omega,\alpha) \\
&+ \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta,\alpha) d\eta \\
&- \sum_{-j\le r<0} \psi^T(\tau_r^+ +\omega+\tau) D_{r+j} X(\omega+\theta_r,\alpha) \Big] f(\alpha) d\alpha \\
&+ \sum_{0\le i<n(\omega)} \Big[\psi^T(\omega) X(\omega,\theta_i^+) \\
&+ \int_{-\tau}^{0} \psi^T(\eta+\omega+\tau) B(\eta+\tau) X(\omega+\eta,\theta_i^+) d\eta \\
&- \sum_{-j\le r<0} \psi^T(\tau_r^+ +\omega+\tau) D_{r+j} X(\omega+\theta_r,\theta_i^+) \Big] J_i = 0.
\end{aligned}
$$

Changing the variable $\xi = \eta+\omega+\tau$ and the index $\theta_k^+ = \tau_r^+ +\omega+\tau$ for $k = r+p+j$ where $n(\omega) = p$, we have

$$
\begin{aligned}
< \tilde{\varphi}(\alpha), F(\alpha) > \;=\; & \int_0^{\omega} \Big[\psi^T(\omega) X(\omega,\alpha) + \int_{\omega}^{\omega+\tau} \psi^T(\xi) B(\xi) X(\xi-\tau,\alpha) d\xi \\
&- \sum_{n(\omega)\le k<n(\omega)+j} \psi^T(\theta_k^+) D_k X(\theta_{k-j},\alpha) \Big] f(\alpha) d\alpha \\
&+ \sum_{0\le i<n(\omega)} \Big[\psi^T(\omega) X(\omega,\theta_i^+) \\
&+ \int_{\omega}^{\omega+\tau} \psi^T(\xi) B(\xi) X(\xi-\tau,\theta_i^+) d\xi \\
&- \sum_{n(\omega)\le k<n(\omega)+j} \psi^T(\theta_k^+) D_k X(\theta_{k-j},\theta_i^+) \Big] J_i = 0.
\end{aligned}
$$

In view of formula (2.13), we obtain

$$\int_0^{\omega} y^T(\alpha)f(\alpha)d\alpha + \sum_{0\le i<n(\omega)} y^T(\theta_i^+)J_i = 0,$$

for all periodic solutions y of system (4.9). □

CHAPTER 5

FUNCTIONALLY EQUIVALENT SYSTEMS

5.1 Introduction

In general, the possibility of reducing impulsive differential systems with delays into impulsive differential systems without delays in such away that they have in common the same periodic solutions remains an ambiguous problem. However, in the case when the delays satisfy certain functional relations, this property becomes possible and could be achieved.

In [15], *El'sgol'ts* and *Norkin* considered the delay differential equation

$$x'(t) = f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)) \tag{5.1}$$

where the following conditions (C5.1) are satisfied:

(a) There exists $m_i \in \mathbb{N}$ such that $\tau_i = m_i\omega$ where $\omega > 0$ and $i = 1, \ldots, n$;

(b) $f \in C(\mathcal{J}\times\Omega, \mathbb{R}^n)$ is periodic of period ω with respect to the first argument where $\mathcal{J} \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n \times \ldots \times \mathbb{R}^n$.

Conditions (C5.1) imply that the periodic solutions of period ω of equation (5.1) are also solutions of the equation

$$x'(t) = f(t, x(t), x(t), \ldots, x(t)) \tag{5.2}$$

and the periodic solutions of period ω of equation (5.2) satisfy (5.1). In this case, equations (5.1) and (5.2) are said to be *functionally equivalent* relative to the functional relation

$$x(t+\omega) \equiv x(t). \tag{5.3}$$

Using this remarkable property, *El'sgol'ts* and *Norkin* obtained several results concerned with the existence of periodic solutions of these equations. In this chapter, we obtain similar results for impulsive differential systems with delays. Moreover, some general remarks on impulsive differential systems with deviating arguments are presented.

Consider the impulsive differential system with delays

$$\begin{cases} x'(t) = & f(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)), \quad t \neq \theta_k \\ \Delta x(\theta_k) = & I_k(\theta_k, x(\theta_k), x(\theta_{k-j_1}), \ldots, x(\theta_{k-j_n})), \quad k \in \mathbb{Z} \end{cases} \tag{5.4}$$

where the conditions (C5.2) hold:

(a) There exists $m_i \in \mathbb{N}$ such that $\tau_i = m_i\omega$ for some $\omega > 0$ and $i = 1, \ldots, s$;

(b) There exists $n_r \in \mathbb{N}$ such that $j_r = n_r p$ and $\theta_{k+p} = \theta_k + \omega$ for some $p > 0$ and $r = 1, \ldots, s$;

(c) $f(t, \cdot, \ldots, \cdot) \in C(\Omega, \mathbb{R}^n)$ and $f(\cdot, x_1, \ldots, x_n) \in PLC(\mathcal{J}, \mathbb{R}^n)$ is periodic of period ω;

(d) $I_i \in C(\mathcal{J} \times \Omega, \mathbb{R}^n)$ are periodic of period p, $i \in \mathbb{Z}$.

Before starting the formulation of the main results, we introduce the definition of functional equivalence for systems involving impulses.

Definition 5.1.1 *Conditions (C5.2) imply that the periodic solutions of period ω of system (5.4) are also solutions of*

$$\begin{cases} x'(t) = & f(t, x(t), x(t), \ldots, x(t)), \quad t \neq \theta_k \\ \Delta x(\theta_k) = & I_k(\theta_k, x(\theta_k), x(\theta_k), \ldots, x(\theta_k)), \quad k \in \mathbb{Z} \end{cases} \tag{5.5}$$

and the periodic solutions of period ω of system (5.5) satisfy (5.4). In this sense, systems (5.4) and (5.5) are said to be functionally equivalent *relative to the functional relation (5.3).*

Remark 5.1.1 *In those cases when system (5.4) is functionally equivalent to system (5.5), the periodic solutions of (5.4) may be obtained by the method of steps.*

5.2 Existence of Periodic Solutions

In this section, we shall use the definition of functional equivalence in the considered sense of systems (5.4) and (5.5) to establish the existence of periodic solutions of impulsive differential systems with delays. A series of nontrivial consequences are obtained. The first result is trivial and is simply verified by using the results of the theory of impulsive differential systems.

Theorem 5.2.1 *Let conditions (C5.2) be satisfied. Then system (5.4) does not have more than an n-parameter family of solutions of period ω.*

Proof. Suppose that ϕ is a periodic solution of period ω of (5.4). Then,

$$\begin{cases} \phi'(t) = & f(t, \phi(t), \phi(t-\tau_1), \ldots, \phi(t-\tau_n)), \quad t \neq \theta_k \\ \Delta\phi(\theta_k) = & I_k(\theta_k, \phi(\theta_k), \phi(\theta_{k-j_1}), \ldots, \phi(\theta_{k-j_n})), \quad k \in \mathbb{Z} \end{cases}$$

Conditions (C5.2) and the periodicity of $\phi(t)$ imply that

$$\begin{cases} \phi'(t) = & f(t, \phi(t), \phi(t), \ldots, \phi(t)), \quad t \neq \theta_k \\ \Delta\phi(\theta_k) = & I_k(\theta_k, \phi(\theta_k), \phi(\theta_k), \ldots, \phi(\theta_k)), \quad k \in \mathbb{Z} \end{cases} \tag{5.6}$$

System (5.6) is system of n first order impulsive differential equations whose general solution containing n arbitrary constants. Thus, since (5.6) has not more than n parameter family of solutions of period ω, (5.4) can not have more than n parameter family of solutions of period ω. $\square$

Let us now consider the linear system

$$\begin{cases} x'(t) + A_0(t)x(t) + \sum_{s=1}^{m} A_s(t)x(t-\tau_s) = & h(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) + B_i^0 x(\theta_i) + \sum_{j=1}^{n} B_i^j x(\theta_{i-k_j}) = & J_i, \quad i \in \mathbb{Z} \end{cases} \tag{5.7}$$

together with the following conditions (C5.3):

(a) There exist $\nu_s \in \mathbb{N}$ such that $\tau_s = \nu_s \omega$ for some $\omega > 0$ and for all $s = 1, 2, \ldots, m$;

(b) There exist $r_j \in \mathbb{N}$ such that $k_j = r_j p$ and $\theta_{i-r_j p} = \theta_i - r_j \omega$ for some $p > 0$ and for $j = 1, 2, \ldots, n$;

(c) $h \in PLC(\mathcal{J}, \mathbb{R}^n)$, $h \not\equiv 0$ for all $t \in (\theta_i, \theta_{i+1})$, and h is ω-periodic;

(d) There exists $p \in \mathbb{N}$ such that $J_{i+p} = J_i$, $i \in \mathbb{Z}$;

(e) $A_s \in PLC(\mathcal{J}, \mathrm{R}^{n\times n})$ for $s = 0, 1, \ldots, m$, $B_i^j \in \mathbb{R}^{n\times n}$ for $j = 0, 1, \ldots, n$ and $A_s(t+\omega) = A_s(t)$, $s = 0, 1, \ldots, m$ and $B_{i+p}^j = B_i$ $j = 0, 1, \ldots, n$ and for $i \in \mathbb{Z}$.

Theorem 5.2.2 *Let conditions (C5.3) be fulfilled. Then system (5.7) has a unique ω-periodic solution if*

$$\begin{cases} x'(t) + \sum_{s=0}^{m} A_s(t)x(t) = & 0, \quad t \neq \theta_i \\ \Delta x(\theta_i) + \sum_{j=0}^{n} B_i^j x(\theta_i) = & 0, \quad i \in \mathbb{Z} \end{cases} \tag{5.8}$$

has no periodic solutions of period ω, except the trivial one.

Proof. Consider the system

$$\begin{cases} x'(t) + \sum_{s=0}^{m} A_s(t)x(t) = & h(t), \quad t \neq \theta_i \\ \Delta x(\theta_i) + \sum_{j=0}^{n} B_i^j x(\theta_i) = & J_i, \quad i \in \mathbb{Z} \end{cases} \tag{5.9}$$

which is functionally equivalent to (5.7). However, in view of Theorem 4.2.1, system (5.9) has a unique periodic solution of period ω. Thus, system (5.7) has a unique periodic solution of period ω. □

In case $n = 1$, we may establish a more specific condition for (5.7) not to have a periodic solution. Let us rewrite (5.7) as follows:

$$\begin{cases} x'(t) + a_0(t)x(t) + \sum_{s=1}^{m} a_s(t)x(t-\tau_s) = & 0, \quad t \neq \theta_i \\ \Delta x(\theta_i) + b_i^0 x(\theta_i) + \sum_{j=1}^{n} b_i^j x(\theta_{i-k_j}) = & 0, \quad i \in \mathbb{Z} \end{cases} \tag{5.10}$$

where the conditions (C5.4) are satisfied:

(a) There exist $\nu_s \in \mathbb{N}$ such that $\tau_s = \nu_s \omega$ where $\omega > 0$;

(b) There exist $r_j \in \mathbb{N}$ such that $k_j = r_j p$ and $\theta_{i-r_j p} = \theta_i - r_j \omega$ where $p > 0$;

(c) $\sum_{j=0}^{n} b_i^j > -1$;

(d) $a_0(t), a_s(t) \in PLC(\mathcal{J}, \mathbb{R})$ are periodic of period ω;

(e) There exists $p \in \mathbb{N}$ such that $b_{i+p}^j = b_i^j$ for $j = 0, 1, \ldots, n$ and $i \in \mathbb{Z}$.

Define the function

$$H(r) = \begin{cases} \sum_{s=0}^{m} a_s(r) - \frac{1}{\theta_1}\ln(1+\sum_{j=0}^{m} b_i^j), & r \in [0,\theta_1], \\ \sum_{s=0}^{m} a_s(r) - \frac{1}{\theta_{i+1}-\theta_i}\ln(1+\sum_{j=0}^{m} b_i^j), & r \in (\theta_i,\theta_{i+1}], \\ \sum_{s=0}^{m} a_s(r) - \frac{1}{\omega-\theta_p}\ln(1+\sum_{j=0}^{m} b_i^j), & r \in (\theta_p, w]. \end{cases}$$

Theorem 5.2.3 *Let conditions (C5.4) be satisfied. Then, system (5.10) has no periodic solutions of period* ω *if*

$$measure\{r \in [0,\omega] :\ H(r) \neq 0\} > 0.$$

Proof. Let system (5.10) have a periodic solution $\psi(t)$ of period ω. Clearly

$$\begin{cases} \psi'(t) + \sum_{s=0}^{m} a_s(t)\psi(t) = 0, & t \neq \theta_i \\ \Delta\psi(\theta_i) + \sum_{j=0}^{n} b_{ji}\psi(\theta_i) = 0, & i \in \mathbb{Z}, \end{cases}$$

It follows that [17, 33]

$$\psi(t) = x_0 e^{-\int_0^t \sum_{s=0}^{m} a_s(r)dr} \prod_{0\le\theta_i<t} (1+\sum_{j=0}^{m} b_{ji}),$$

where $\psi(0) = x_0$. From the periodicity of ψ, we have

$$e^{-\int_0^\omega \sum_{s=0}^{m} a_s(r)dr} \prod_{0\le\theta_i<\omega} (1+\sum_{j=0}^{m} b_{ji}) = 1.$$

It readily follows that

$$\int_0^w \sum_{s=0}^{m} a_s(r)dr - \sum_{0\le\theta_i<w} [\ln(1+\sum_{j=0}^{m} b_{ji})] = 0,$$

which can be written as

$$\int_0^{\theta_1} H(r)dr + \sum_{i=1}^{p-1} \int_{\theta_i}^{\theta_{i+1}} H(r)dr + \int_{\theta_p}^{w} H(r)dr = 0. \tag{5.11}$$

Since, if measure$\{r \in [0,\omega] : \ H(r) \neq 0\} > 0$, then there exists an interval $I = (t_0 - \delta, t_0 + \delta)$ such that $H(r) < 0$ or $H(r) > 0$ for all $r \in I$ which contradicts the equality in (5.11). $\square$

Let us now consider a general relation

$$x(u(t)) = x(t) \tag{5.12}$$

where $u(t) \leq t$. We consider a scalar system of the form

$$\begin{cases} x'(t) = & a(t)x(t) + b(t)x(u(t)), \quad t \neq \theta_i \\ \Delta x(\theta_i) = & c_i x(\theta_i) + d_i x(\theta_{i-k}), \quad i \in \mathbb{Z} \end{cases} \tag{5.13}$$

where $a, b \in PLC(\mathcal{J}, \mathbb{R})$ and $u \in PLC(\mathcal{J}, \mathbb{R}^+)$. Let there exist a number $q > 0$ and a monotonic differentiable function $s = \psi(t)$ for $t \neq \theta_i$ such that $\psi(t) - \psi(u(t)) = q$ and $\psi(\theta_i) = \tau_i$ for $t = \theta_i$. Assume that $x(t) = z(\psi(t)) = z(s)$, $t \neq \theta_i$, then

$$x'(t) = z'(\psi(t))\psi'(t) = z'(s)\psi'(\psi^{-1}(s)), \quad s \neq \tau_i,$$

and

$$x(u(t)) = z(\psi(u(t))) = z(s - q), \quad s \neq \tau_i.$$

Therefore

$$z'(s) = a_1(s)z(s) + b_1(s)z(s - q), \quad s \neq \tau_i$$

where

$$a_1(s) = \frac{a(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))} \quad and \quad b_1(s) = \frac{b(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))}.$$

Moreover, since

$$\Delta x(\theta_i) = \Delta z(\psi(\theta_i)) = \Delta z(\tau_i) \quad and \quad x(\theta_{i-k}) = z(\psi(\theta_{i-k})) = z(\tau_{i-k}),$$

we have

$$\Delta z(\tau_i) = c_i z(\tau_i) + d_i z(\tau_{i-k}),$$

where

$$c_i = c_i(\psi^{-1}(\tau_i)) \quad and \quad d_i = d_i(\psi^{-1}(\tau_i)).$$

Hence (5.13) is reduced to the system

$$\begin{cases} z'(s) = & a_1(s)z(s) + b_1(s)z(s-q), \quad s \neq \tau_i \\ \Delta z(\tau_i) = & c_i z(\tau_i) + d_i z(\tau_{i-k}), \quad i \in \mathbb{Z} \end{cases} \tag{5.14}$$

The system

$$\begin{cases} z'(s) = & (a_1(s) + b_1(s))z(s), \quad s \neq \tau_i \\ \Delta z(\tau_i) = & (c_i + d_i)z(\tau_i), \quad i \in \mathbb{Z} \end{cases} \tag{5.15}$$

is equivalent to system (5.14) relative to relation (5.12). Thus, if the function $A(s) = a_1(s) + b_1(s)$ is periodic of period q and $B_i = c_i + d_i > -1$ is periodic of period k (the delays are the periods) such that $\tau_{i-k} = \tau_i - q$ and satisfy the condition

$$\int_0^\omega A(s)ds + \sum_{0 \leq \tau_i < \omega} [\ln(1 + B_i)] = 0,$$

then, system (5.15) has the solution of period $|q|$

$$z(s) = ce^{\int_0^s A(\xi)d\xi} \prod_{0 \leq \tau_i < s} (1 + B_i).$$

Consequently, system (5.13) under the mentioned conditions has the solution

$$x(t) = ce^{\int_0^{\psi(t)} A(s)ds} \prod_{0 \leq \theta_i < \psi(t)} (1 + B_i),$$

possessing property (5.12).

The presentations considered above can be extended to impulsive differential systems with deviating arguments of the form

$$\begin{cases} x^{(n)} = & f(t, x(t), x(t-p_i), x(t-q_j), \dots, x^{(n)}(t-p_i), x^{(n)}(t-q_j)), \quad t \neq \theta_k \\ \Delta x^{(n-1)} = & I_k(\theta_k, x(\theta_k), x(\theta_{k-u_r}), x(\theta_{r-v_q}), \dots, x^{(n-1)}(\theta_{k-u_r}), x^{(n-1)}(\theta_{k-v_q})) \end{cases} \tag{5.16}$$

provided that

(a) There exist m_i such that $p_i = m_i\omega$, $i = 1, 2, \dots, n$, for $\omega > 0$ and the constants $q_j > 0$, $j = 1, 2, \dots, s$;

(b) There exist z_r such that $u_r = z_r p$, $r = 1, 2, \dots, n$, for $p > 0$, the constants $v_q > 0$, $q = 1, 2, \dots, s$ and $\theta_{k+p} = \theta_k + \omega$;

(c) $f(\cdot, x_1, \dots, x_n) \in PLC(\mathcal{J}, \mathbb{R}^n)$ and $f(t, \cdot, \dots, \cdot) \in C(\Omega, \mathbb{R}^n)$ is periodic of period ω, where $\mathcal{J} \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n \times \dots \times \mathbb{R}^n$;

(d) $I_k \in C(\mathcal{J} \times \Omega, \mathbb{R}^n)$ are periodic of period p, $k \in \mathbb{Z}$.

In this case, we may consider a functionally equivalent system as

$$\begin{cases} x^{(n)} = & f(x(t), x(t), x(t-q_j), ..., x^{(n)}(t), x^{(n)}(t-q_j)), \quad t \neq \theta_k \\ \Delta x^{(n-1)} = & I_k(x(\theta_k), x(\theta_k), x(\theta_{k-v_q}), \dots, x^{(n-1)}(\theta_k), x^{(n-1)}(\theta_{k-v_q})) \end{cases} \tag{5.17}$$

where functional equivalence is understood in a similar manner.

We note that if system (5.16) is of neutral type, then its periodic solutions may be n-times piecewise differentiable functions. Then, as for system (5.17), there is a possibility of the case when its order turns out to be less than n, consequently, some of its solutions may remain piecewise differentiable a smaller number of times and may even be only piecewise continuous. There is also the possibility of the case, for instance if system (5.17) has order n and $s = 0$, for which all solutions of system (5.17) are n-times piecewise differentiable. Consequently, generally speaking, it is impossible to assert that systems (5.16) and (5.17) have in common all periodic solutions of period ω.

If system (5.16) is system with delays, that is, the right side of this system does not explicitly depend on $x^{(n)}(t-p_i)$ and $x^{(n)}(t-q_j)$, then all solutions of period ω of systems (5.16) and (5.17) are piecewise differentiable no more than n-times and , consequently, systems (5.16) and (5.17) are equivalent to any solution of period ω.

A scalar system of neutral type of the form (5.16) for $s = 0$, has not more than an n parameter family of solutions of period ω. If $q_j = v_q = 0$ in system (5.17), then any n-times piecewise differentiable periodic function of period ω will be a solution of system (5.16).

Remark 5.2.1 *In those cases when system (5.17) equivalent, in the considered sense, to system (5.16), the periodic solutions of (5.16) may be obtained by the method of steps.*

Example 5.2.1 *Consider the system of neutral type*

$$\begin{cases} x'(t) + \sum_{i=1}^{m} a_i(t)x'(t-p_i) + \sum_{j=0}^{n} b_j(t)x(t-q_j) = & 0, \quad t \neq \theta_k \\ \Delta x(\theta_k) + \sum_{r=1}^{m} c_k^r \Delta x(\theta_{k-u_r}) + \sum_{s=0}^{n} d_k^s x(\theta_{k-v_s}) = & 0, \quad k \in \mathbb{Z} \end{cases} \tag{5.18}$$

where the following conditions are satisfied (C5.6):

(a) $a_i, b_j \in PLC(\mathcal{J}, \mathbb{R})$ *are periodic of period* ω*;*

(b) There exists $p \in \mathbb{N}$ *such that* $c_{k+p}^r = c_k^r$ *for* $r = 1, 2, \ldots, m$ *and* $d_{k+p}^s = d_k^s$ *for* $s = 0, 1, \ldots, n$ *and* $k \in \mathbb{Z}$*;*

(c) p_i, q_j, u_r and v_s are nonnegatives, $p_i = m_i\omega$, $q_j = m_j\omega$ for some $\omega > 0$ and $u_r = q_r p$, $v_s = r_s p$ for some $p > 0$ such that $\theta_{k+p} = \theta_k + \omega$.

Let

$$1 + \sum_{i=1}^{m} a_i(t) \neq 0 \quad \textit{for} \quad t \neq \theta_i, \quad \textit{and} \quad 1 + \sum_{r=1}^{m} c_{rk} \neq 0. \tag{5.19}$$

Denote

$$A(t) = \frac{\sum_{j=0}^{n} b_j(t)}{1 + \sum_{i=1}^{m} a_i(t)} \quad \textit{and} \quad B_k = \frac{\sum_{s=0}^{n} d_{sk}}{1 + \sum_{r=1}^{m} c_{rk}} > -1.$$

In order that system (5.18) have a nontrivial periodic solution $x(t)$ of period ω, it is necessary and sufficient that the functions $A(t)$ and B_k are periodic of period ω and p, respectively, and

$$\int_0^{\omega} A(t)dt - \sum_{0 \leq \theta_k < w} [\ln(1 + B_k)] = 0. \tag{5.20}$$

To show this, consider the system

$$\begin{cases} x'(t) + A(t)x(t) = 0, & t \neq \theta_k \\ \Delta x(\theta_k) + B_k x(\theta_k) = 0, & k \in \mathbb{Z} \end{cases} \tag{5.21}$$

which is equivalent to system (5.18), relative to relation (5.3). Then, the solution of (5.21) has the form

$$x(t) = x_0 e^{-\int_0^t A(s)ds} \prod_{0 \leq \theta_k < t} (1 + B_k), \quad t \geq 0 \tag{5.22}$$

where x_0 is an arbitrary constant. The necessity of the condition is obvious. Sufficiency is proved by substituting the function (5.22) into system (5.18).

REFERENCES

[1] M. AKHMETOV and A. ZAFER, *The Contobillity of Boundary Value Problems for Quasilinear Impulsive Systems*, Nonlinear Anal. 34, 1055-1065, 1998.

[2] M. AKHMETOV and A. ZAFER, *Successive Approximations Method for Quasilinear Impulsive Differential Equations with Control*, Appl. Math. Letters. 13, 99-105, 2000.

[3] M. U. AKHMETOV and R. SEJILOVA, *The Control of the Boundary Value Problem for Linear Impulsive Integro-Differential Systems*, J. Math. Anal. Appl. 236, 312-326, 1999.

[4] M. U. AKHMETOV and A. ZAFER, *Stability of the Zero Solution of Impulsive Differential Equations by the Lyapunov Second Method*, J. Math. Anal. Appl. 13, 99-105, 2000.

[5] M. U. AKHMETOV, A. ZAFER and R. D. SEJILOVA, *The Control of Boundary Value Problem for Quasilinear Impulsive Integro-Differential Equations*, Nonlinear Anal. 48, 271-286, 2002.

[6] M. U. AKHMET, *On the General Problrm of Stability for Impulsive Differential Equations*, J. Math. Anal. Appl. 288(1), 182-196, 2003.

[7] M. U. AKHMETOV, *On Periodic Solutions of Cerain Systems of Differential Equations*, Vestnik Kievskogo Universiteta. Matematika i Mekhanika, 24, 3-7, 1982.

[8] M. U. AKHMETOV and N. A. PERESTYUK, *On Comparison Method for Differential Systems with Pulse Effect*, Differential Equations, 26(9), 1079-1081, 1990.

[9] M. U. AKHMETOV and N. A. PERESTYUK, *Periodic and Almost Periodic Solutions of Strongly Nonlinear Impulsive Systems*, Prikladn. Matem. Mekhan., 56(6), 926-934, 1992.

[10] M. U. AKHMETOV and N. A. PERESTYUK, *Almost Periodic Solutions of Nonlinear Sampled Data Systems*, Ukranian Math. J., 41(3), 259-263, 1989.

[11] M. U. AKHMETOV, *Nonlinear Oscillations of Impulsive Differential Equations*, Post Doctoral Dissertation, Kiev, 1993.

[12] A. HALANAY, *Differential Equations*, Academic Press Inc., 1966.

[13] D. BAINOV and V. COVACHEV, *Impulsive Differential Equations with a Small Parameter*, World Scientific, 1994.

[14] R. D. DRIVER, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1977.

[15] L. E. EL'SGOL'TS and S. B. NORKIN, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, Inc., New York, 1973.

[16] E. KREYSZIG, *Introductory Functional Analysis with Applications*, Wiley Classic Library, 1989.

[17] A. M. SAMOILENKO and N. A. PERESTYUK, *Impulsive Differential Equations*, World Scientific, 1995.

[18] D. D. BAINOV and P. S. SIMEONOV, *Impulsive Differential Equations*, Asymptotic Properties of the solution, World Scientific, 1995.

[19] A. E. TAYLOR, *Intoduction to Functional Analysis*, J. Wiley and Sons, Inc., 1957.

[20] J. K. HALE, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[21] V. LAKSHMIKANTHAM, D. D. BAINOV and P. S. SIMEONOV, *Theory of Impulsive Differential Equations*, World Scientific, Singapor, 1989.

[22] Y. KUANG, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.

[23] R. BELLMAN and K. L. COOKE, *Differential-Difference Equations*, Academic Press Inc., 1963.

[24] D. D. BAINOV and P. S. SIMEONOV, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, Harlow, 1993.

[25] L. E. EL'SGOL'TS, *A Note on Some Periodic Solutions of Equations with Deviating Argument*, Ibid. 7, 189-193, 1969.

[26] R. G. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Academic Press, Inc., 1972.

[27] L. E. EL'SGOL'TS, *Introduction to the Theory of Differential Equations with Deviating Arguments*, Holden-Day Inc,. 1966.

[28] 0. ARINO and M. PITUK, *More on Linear Differential Systems with Small Delays*, J. Diff. Eqns. 170.2, 381-407, 2001.

[29] T. A. BURTON, *Stability and Periodic of Ordinary and Functional Differential Equations*, Academic Press Inc., 1985.

[30] D. D. BAINOV and P. S. SIMEONOV, *Periodic Solutions of Linear Impulsive Differential Equations with Delay*, Commun. App. Anal. 7, no.1, 7-29, 2003.

[31] M. A. NAIMARK, *Linear Differential Operators*, Fredrick Ungar Publishing Co. Inc., 1967.

[32] A. ANOKHIN, L. BEREZANSKY and E. BRAVERMAN, *Stability of Linear Delay Impulsive Differential Equations*, Dyn. Syst. Applic. 4, 173-188, 1995.

[33] D. BAINOV and P. SIMEONOV, *Oscillation Theory of Impulsive Differential Eqautions*, International Publications, 1998.

[34] M. RAMA MAHANA RAO, K. SRIVASTAVA SANJAY and S. SIVASUNDARAM, *Stability of Linaer Delay Impulsive Differential Equations with Impulse Effect*, J. Math. Anal. Appl. 163, 47-59, 1992.

[35] O. ARINO, I. GYORI and M. PITUK, *Asymptotically Diagonal Delay Differential Systems*, J. Math. Anal. Appl. 204, 701-728, 1996.

[36] J. S. YU, *Stability for Nonlinear Differential Equations of Unstable Type Under Impulsive Perturbations*, App. Math. Lett. 14.7, 849-857, 2001.

[37] D. YUJUN, *Periodic Boundary Value Problems for Functional Differential Equations with Impulses*, J. Math. Anal. Appl. 210.1, 170-181, 1997.

[38] D. D. BAINOV and P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, 1992.

[39] X. H. TANG, Z. HE and J. S. YU, *Stability Theorem for Delay Differential Equations with Impulses*, App. Math. Comput. 131.2-3, 373-381, 2002.

[40] J. S. YU, *Explicit Conditions for Stability of Nonlinear Scalar Delay Differential Equations with Impulses*, Nonlinear Anal. 46.1, 53-67, 2001.

[41] J. S. YU and B. G. ZHANG, *Stability Theorem for Delay Differential Eqautions with Impulses*, J. Math. Anal. Appl. 199.1, 162-175, 1996.

[42] J. KLOCH, *An Illustrative Example for the Perron Condition*. Ann. Polon. Math. 35.1, 11-14, 1977/78.

[43] H. AKÇA and V. COVACHEV, *Periodic Solutions of Impulsive Systems with Delay*, Fun. Diff. Eqns. 5.3-4, 275-286, 1998.

[44] O. PERRON, DieStabilitatsfrage bei Differentialgleichungen Math.Z.32, 703-728, 1930.

[45] L. BEREZANSKY and E. BRAVEMAN, *Stability of Impulsive Functional Differential Equations*, Fun. Diff. Eqns. 2, 11-22, 1995.

[46] H. E. BRAY, *Elementary Properties of the Stieltjes Integral*, Ann. of Math. Series II 20, 177-186, 1918-1919.

[47] I. KIGURADZE and B. PUZA, *On Periodic Solutions of Systems of Differential Equations with Deviating Arguments*, Nonlinear Anal. 42, 229-242, 2000.

[48] D. FRANCO and JUAN J. NIETO, *First-Order Impulsive Ordinary Differential Equations with Anti-Periodic and Nonlinear Boundary Conditions*, Nonlinear Anal.42, 163-173, 2000.

[49] J. CRONIN, *Differential Equations: Introduction and Qualitative Theory*, Mracel Dekker Inc., 1994.

[50] R. K. MILLER and A. N. MICHAL, *Ordinary Differential Equations*, Academic Press, 1982.

[51] S. J. HUA and Y. J. SHE, *Nonlinear Delay Differential Equations with Impulsive Perturbations*, Math. Appl. 9.3, 272-277, 1996.

[52] R. DATKO, *The Perron Condition for Differential- Difference Equations in a Hilbert Space*. SIAM J. Math. Anal. 12.6, 842-852, 1981.

[53] D. D. BAINOV and P. S. SIMEONOV, *Periodic Solutions of Quasilinear Impulsive Differential Equations with Delay in the Noncritical Case*, Int. J. Differ. Eqn. App. 5.4, 313-324, 2002.

[54] S. G. HRISTOVA and D. D. BAINOV, *Existence of Periodic Solutions of Nonlinear Systems of Differential Equations with Impulse Effect*, J. Math. Anal. App. 125.1, 192-202, 1987.

[55] S. L. ROSS, *Differential Equations*, John Wiley and Sons Inc., 1984.

[56] H. HOCHSTADT, *Integral Eqautions*, John Wiley and Sons Inc., 1973.

[57] M. A. BABRAM, K. EZZINBI and R. BENKHALTI, *Periodic Solutions of Functional Differential Equations of Neutral Type*, J. Math. Anal. App. 204, 898-909, 1996.

[58] I. BAJO and E. LIZ, *Periodic Boundary Value Problem for First Order Differential Equations with Impulses at Variable Times*, J. Math. Anal. App. 204,65-73, 1996.

[59] S. M. S. GODOY and J. G. DOS REIS, *Stability and Existence of Periodic Solutions of a Functional Differential Equations*, J. Math. Anal. App. 198, 381-398, 1996.

[60] V. LAKSHMIKANTHAM and X. LIU, *Stability Criteria for Impulsive Differential Equations in Terms of Two Measures*, J. Math. Anal. App. 137, 591-604, 1989.

[61] S. HU, V. LAKSHMIKANTHAM and S. LEELA, *Impulsive Differential Systems and the Pulse Phenomena*, J. Math. Anal. App. 137, 605-612, 1989.

[62] F. RIESZ and B. S. NAGY, *Functional Analysis*, Newyork, Ungar, 1955.

[63] T. A. BURTON and L. HATVANI, *On the Existence of Periodic Solutions of Some Nonlinear Differential Equations with Unbounded Delay*, Nonlinear Anal. 16, 389-398, 1991.

[64] S. N. CHOW, *Remarks on One Dimensional Delay Differential Equations*, J. Math. Anal. App. 41, 426-429, 1973.

[65] J. K. HALE, *Functional Differential Eqautions with Infinite Delays*, J. Math. Anal. App. 48, 276-283, 1974.

[66] D. BAINOV, S. HRISTOVA, S. HU and V. LAKSHMIKANTHAM, *Periodic Boundary Value Problems for Systems of First order Impulsive Differential Equations*, Differential Integral Equations 2, 37-43, 1989.

[67] S. HU and V. LAKSHMIKANTHAM, *Periodic Boundary Value Problems for Second Order impulsive Differential Systems*, Nonlinear Anal. 13, 75-85, 1989.

[68] G. POTCOVARU, *The Perron Condition for a Class of Linear Functional Systems*. (Romanian) Stud. Cerc. Mat. 50.3-4, 237-242, 1998.

[69] V. LAKSHMIKANTHAM and S. LEELA, *Existence and Monotone Method for Periodic Solutions of First Order Differential Equations*, J. Math. Anal. App. 91, 237-243, 1983.

[70] C. BUŞE, *Perron Conditions and its Connection with Exponential Stability of Solutions for Integro-Differential Equations on the Half Line*. Int. J. Differ. Equ. Appl. 1 A.1,23-30, 2000.

[71] R. D. NUSSBAUM, *Periodic Solutions of Some Nonlinear Autonomous Functional Differential Equations*, Ann. Mat. Pura Appl. 101, 263-306, 1974.

[72] D. D. BAINOV and P. S. SIMEONOV, *Stability with Respect to Part of the Variables in Systems with Impulse Effect*, J. Math. Anal. App. 117, 247-263, 1986.

[73] T. YONEYAMA, *On the $\frac{3}{2}$ Stability Theorem for One Dimensional Delay Differential Equations*, J. Math. Anal. App. 125, 161-173, 1987.

[74] M. P. CHEN, J. S. YU and J. H. SHEN, *The Presistence of Nonoscillatory Solutions of Delay Differential Equtations Under Impulsive Perturbations*, Comput. Math, Appl. 27.8, 1-6, 1994.

[75] O. ARINO and M. L. HBID, *Periodic Solutions for Retarded Differential Systems Close to Ordinary Ones*, Nonlinear Anal. 14.1, 23-34, 1990.

[76] A. HALANAY, *The Perron Condition in the Theory of General Systems with Retardation*. (Russian) Mathematica(Cluj)2 (25), 257-267, 1960.

[77] J. J. NIETO, *Periodic Boundary Value Problems for First Order Impulsive Ordinary Differential Equations*, Nonlinear Anal. 51, 1223-1232, 2002.

[78] E. A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations*, New York, McGraw-Hill, 1955.

[79] S. SCHWABIK, M. TVRDY and O. VEJVODA, *Differential and Integral Equations*, D. Reidel Publishing Co. 1979.

[80] V. LAKSHMIKANTHAM and S. LEELA, *Differential and Integral Inequalities*, Academic Press, 1969.

[81] S. KESAVAN, *Topics in Functional Analysis and Applications*, John Wiley and Sons, 1989.

[82] G. DAJUN, *Multiple Positive Solutions of Impulsive Nonlinear Fredholm Integral Equations and Applications*, J. Math. Anal. App. 173, 318-324, 1993.

[83] K. GOPALSAMY and B. G. ZHANG, *On Delay Differential Equations with Impulses*, J. Math. Anal. App. 139, 110-122, 1989.

[84] J. J. NIETO, *Periodic Boundary Value problems for First Order Impulsive Ordinary Differential Equations*, Nonlinear Anal. 51, 1223-1232, 2002.

[85] L. V. KANTOROVICH and G. P. AKILOV, *Functional Analysis*, Pergamon Press Ltd. 1982.

VITA

Jehad Alzabut was born in Jeddah, Saudi Arabia on May 18, 1972. The elementary and secondary education has been completed in Elandalus and Elgazali College respectively. He has finished the high school at Qurdoba College with honor degree. He received his B.S. degree in Mathematics from Middle East Technical University in February 1996. He worked as a mathematics teacher in The College of Saudi Arabian Embassy in Ankara from 1996 to 1999. During that period, he got his M.S. degree in Applied Mathematics at Middle East Technical University, August 1999. Theme of the thesis: 0scillation Criteria for First and Second Impulsive Delay Differential Equations. The degree of Doctor of Philosophy has been received in April 2004 with a thesis entitled "Periodic Solutions and Stability of Linear Impulsive Delay Differential Equations ". Currently, he is an instructor in the department of mathematics and computer science at Çankaya University. His main fields of interest are Impulsive and Functional Differential Equations; Periodicity, Stability and Oscillations.