RARE Z DECAYS AND NONCOMMUTATIVE THEORIES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

CEM YÜCE

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF

DOCTOR OF PHILOSOPHY

IN

THE DEPARTMENT OF PHYSICS

JUNE 2004

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Sinan Bilikmen
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Tahmasib Aliev
Supervisor

Examining Committee Members

Prof. Dr. Hüseyin Koru (Gazi University)

Prof. Dr. Tahmasib Aliev (METU)

Prof. Dr. Erhan Onur İltan (METU)

Prof. Dr. Mustafa Savcı(METU)

Prof. Dr. Sibel Başkal (METU)

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and result that are not original to this work

Name, Last name:

Signature :

ABSTRACT

RARE Z DECAYS AND NONCOMMUTATIVE THEORIES

Yüce, Cem

Ph.D., Department of Physics

Supervisor: Prof. Dr. Tahmasib Aliev

June 2004, 80 pages.

Leptonic decay modes of Z-boson constitute one of the important class of the decays for checking predictions and improving parameters of the standard model. In next generation of the accelerators, it will be produced more than 10^8 Z-boson pear year. Therefore, It appears real possibility to analyze the rare decays of Z, which are absent at tree level in standard model. Moreover, the rare decays are quite sensitive to the existence of new physics beyond the standard model. One of the possible source for the new physics is noncommutative theories (NC).

Noncommutative theories have rich phenomenological implications due to the appearance of new interactions, which are forbidden in standard model.

In this thesis, we examine the $Z \to \nu\bar{\nu}\gamma$ decay in noncommutative standard model. We study the sensitivity of the decay width on the noncommutative scale parameter Λ and parameters C_{0i} and C_{ij}, which defines the direction of background electric and magnetic fields.

Keywords: Noncommutative theories, Z Decays

ÖZ

NADİR Z BOZUNUMLARI VE KOMUTATİF OLMAYAN TEORİ LER

Yüce, Cem
Doktora, Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Tahmasib Aliev

Haziran 2004, 80 sayfa.

Z bozonlarının leptonik bozunumları standard model parametrelerinin tahminleri ve geliştirilmesi açısından bozunumların önemli bir sınıfını oluşturur. Yeni nesil hızlandırıcılar, yılda 10^8 den fazla Z bozonu üretecektir. Bununla birlikte, standard modelin ağaç seviyesinde olmayan Z bozonlarının nadir bozunumlarinin analizi mümkün gözükmektedir. Üstelik, bu nadir bozunumlar standard model ötesi yeni fizik için oldukça hassastır. Yeni fizik için muhtemel bir kaynak komutatif olmayan teorilerdir.

Komutatif olmayan teoriler standard modelde yasaklanan yeni etkileşimler sebebiyle zengin fenomolojik bit yapiya sahiptir.

Bu tezde, $Z \to \nu\bar{\nu}\gamma$ bozunumunu komutatif olmayan standard modelde inceledik. Bozunum sabitinin komutatif olmayan büyüklük parametresi Λ ve elektrik ve manyetik alanlarini tarifleyen C_{0i} and C_{ij} parametrelerine olan hasasiyetini çaliştık.

Anahtar Kelimeler: Komutatif olmayan teoriler, Z bozunumlari

... TO MY PARENTS

ACKNOWLEDGMENTS

I thank my supervisor T. Aliev for his constant support, the advise during the preparation of this thesis and his merciful, gentle attitudes in my Ph.D years. I am also grateful to him for having accepted me as a Ph.D student, when I was in a bad moody. A thank goes also to all of my sincere friends in the department and in the dormitory. I especially thank Hayrish for not only being a close friend but also for reminding me the important things.

My special thanks are for the most influential and the most perfect people in my life; my Dad: Veli and my mom: Nergiz. It is not enough to show my gratitude how much I thank for these exalted characters.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Standard model describe all known interactions except for the gravitational force. At present, the standard model successfully describes all the experimental results in the energy range available in experiments.

The underlying gauge group of the standard model is $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. The particle spectrum of the standard model consists of the eight gluons, which are gauge bosons of $SU(3)_C$ and mediate the strong interaction, the photon γ which is responsible for the electromagnetic interaction and the three weak bosons $W^{\mp}$, Z, which are the intermediate vector bosons.

Despite the success of standard model, it is obvious that the standard model is the effective theory of more fundamental theory and it has many unsolved problems, such as the number of fermion family, the origin of CP violation and mass of fermions etc.

There are various models proposed to solve at least part of the problems of standard model. One of the promising extension of SM is the noncommutative theories. Noncommutative theories has the potential to provide an attractive and motivated theory beyond the standard model. The possibility of noncommutative space-time is an intriguing one which arises naturally in string theory and give rise to a rich phenomenology. One of the distinguished property of noncommutative theories is the existence of the new interactions absent in standard model. This give us the possibility to calculate the decay rate and cross section for some processes which are forbidden in standard model. The particle spectrum of the noncommutative theory is assumed to be the same as the standard model.

In this thesis, we study the $Z \to \nu\bar{\nu}\gamma$ decay in noncommutative standard model, which is strictly forbidden in SM at tree level. The thesis is structured as follows:

In the second chapter, we give the basic formulas for the noncommutative theories. The $\star$-product formalism which plays vital role in noncommutative space is explained. Then, the noncommutative quantum mechanics and non-Abelian gauge theory is studied via $\star$-product. Lastly, the action in noncommutative space is computed.

In the third chapter, noncommutative field theory is studied deeply by using the Seiberg-Witten map. The contribution to the fields coming from the noncommutativity of space-time is calculated. Then, the actions of the noncommutative electro-weak and noncommutative quantum chromodynamics are computed up to the first order in noncommutative parameter $\Theta^{\mu\nu}$. The Feynmann rules for noncommutative quantum chromodynamics are given. As an application of noncommutative quantum electrodynamics, Moller scattering is studied.

In the final chapter of this thesis, we study the rare $Z \to \nu\bar{\nu}\gamma$ decay in noncommutative theories. Firstly, we find the relevant Feynman rules for this decay as well as the other triple and quarter gauge boson vertices in above mentioned theories, which are absent in standard model. Moreover, the analytic form of the $Z \to \nu\bar{\nu}\gamma$ decay rate is calculated. At the end of this chapter, we present our numerical analysis of the decay with.

CHAPTER 2

NONCOMMUTATIVITY

2.1 Introduction

In quantum mechanics, the phase space is defined by replacing the canonical position and momentum variables with the Hermitian operators which obeys the well-known Heisenberg commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. Later, inspired by the quantum mechanics, it was suggested that one could use the idea of space-time noncommutativity at very small length scales to introduce an effective ultraviolet cutoff.

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}, \tag{2.1}$$

where $\Theta^{\mu\nu}$ is an antisymmetric tensor describing the strength of the noncommutative effects and plays an analogous role to $\hbar$ in usual quantum mechanics. This idea replaced the space-time point by the the Planck cell of dimension given by the Planck area.

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\Theta^{\mu\nu}|. \tag{2.2}$$

Later on, the ideas noncommutative geometry were revieved in 1980's by the Mathematician A. Connes, who generalized the notion of differential structure to the noncommutative setting [1, 2]. One concrete example of physics in noncommutative space-time is Yang-Mills theory on a noncommutative torus [3]. The quantized motion of a particle in magnetic field is described by the noncommuting coordinates on the plane perpendicular to the magnetic field. We will now illustrate how noncommutativity emerges in a simple quantum mechanical example, the Landau-level problem. The Lagrangian of a particle of mass m moving in the plane in the presence of constant perpendicular magnetic field B is given as

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - \vec{x}.\vec{A}, \tag{2.3}$$

where $A_i = -\frac{B}{2}\epsilon_{ij}x^j$ is the vector potential. The Hamiltonian is $H = \frac{\vec{\pi}^2}{2}$, where $\vec{\pi} = \vec{p} + \vec{A}$. From the canonical commutation relations, it follows that the momentum operators have the non-vanishing quantum commutators

$$[\hat{\pi}, \hat{\pi}^j] = iB\epsilon_{ij}, \tag{2.4}$$

so the momentum space in the presence of a background magnetic field becomes noncommutative.

Spatial noncommutativity arises when $m \to 0$. The Landau Lagrangian becomes

$$L = -\frac{B}{2} x^i \epsilon_{ij} x^j. \tag{2.5}$$

It is a first order Lagrangian which is already expressed in phase space with the spatial coordinates x^1, x^2 being canonically conjugate variables, so that

$$[\hat{x}^i, \hat{x}^j] = \frac{i}{B}\epsilon_{ij}. \tag{2.6}$$

More concrete example comes from the string theory, at present the best candidate for the quantum theory of gravity. Noncommutative geometry have been extensively studied in connection with string theory [4, 5, 6, 7] and M theory [8].

Although the noncommutativity of space-time was suggested in 1947 by Synder [9], the physical theories with space-time noncommutativity have not been studied seriously until recently. Perhaps the main reason for this is that postulating an uncertainty relation between position measurements will lead to nonlocal theory. The other reason is that space-time noncommutativity breaks the Lorentz invariance.

One of the motivation for considering space-time commutativity seriously is the belief that in the quantum theories of gravity, space-time should change it's nature at distances comparable to Planck scale. In the quantum theory of gravity, one can not measure the position better accuracies than Planck length.

Now, the physicist have constructed many noncommutative theories. Those theories are made from the standard theories just by replacing the usual multiplication with the $\star$ product.

$$(\Psi \star \Phi)(x) = e^{\frac{i}{2}\frac{\partial}{\partial x^\mu}\Theta^{\mu\nu}\frac{\partial}{\partial y^\nu}} \ \Psi(x)\ \Phi(y)|_{y\to x} \tag{2.7}$$

Noncommutative quantum field theories (NCQFT) [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36], gauge theories [37, 38, 39, 40, 41, 42, 43, 44] and noncommutative quantum mechanics [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57] have recently received great interests. The solitonic structures [58, 59], and the experimental points [64, 65] of the noncommutative field theories have also been studied. The noncommutative field theory from the phenomenological point of view have been also discussed by various authors [70, 71, 72, 73, 74, 75, 76, 77].

Noncommutative field theories are also constructed from the standard field theories just by replacing the usual multiplication with the $\star$ product in the Lagrangian density. Such a replacement in field theory leads to the number of unusual phenomena. The field theory with the space-time noncommutativity are nonlocal and contain infinitely many time derivative interactions which would appear to lead to non-renormalizability problems in a full quantum theory. Unfortunately, answering such questions for a noncommutative theory is complicated by the mixing of low and high momentum modes in loop diagrams which ruin the conventional Wilsonian renormalization scheme that requires distinct separation of energy scales. This effect is commonly known as UV/IR mixing. It appears to make the renormalization of these theories a complete disaster.

The nonlocality of noncommutative field theories leads to many interesting phenomena which makes these models interesting in their own right as potentially well defined, nonlocal quantum field theories.

There is a problem about the unitarity of the theory. For example, the scalar noncommutative field theory at one loop level unitary if $\Theta^{0i} = 0$ and not unitary $\Theta^{0i} \neq 0$. The other problem is that the noncommutativity of coordinates breaks the Lorentz invariance of the quantum field theory.

There are many reason which motivate the search for models in noncommutative space.

2.2 Theoretical Background for the Noncommutative Theories

In this section, we will give the theoretical background for the noncommutative theories. Firstly, the $\star$-product will be introduced and then it will be applied to the quantum mechanics and non-Abelian gauge theory.

2.2.1 Star Product

As it was noted before that the noncommutative theories are constructed from the usual theories just by replacing the usual product with the star product. In this section, we will study the properties of star product.

The star product plays a vital role in the theory, because it gives us, together with the Seiberg-Witten map, a possibility to express the noncommutative fields in terms of the well-known commutative fields. In other words, with the help of associative star product, the study of noncommutative theories can be mapped into that of ordinary theories where ordinary product is replaced by the star product.

Let $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$ be two noncommutative fields. Hence, in general, fields $\hat{f}$ and $\hat{g}$ don't commute, not because of canonical commutation relations between the fields and canonical momentum densities, but because of the $\hat{x}$ themselves. We can relate the fields $\hat{f}(\hat{x})$ to the ordinary function $f(x)$ of ordinary variables.

$$\hat{f}(\hat{x}) = \int \frac{d^4p}{2\pi^4} e^{-ip\hat{x}} f(p) \ , \tag{2.8}$$

where $f(p)$ is an ordinary function. The multiplication of the two fields is given as

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \int \frac{d^4p}{2\pi^4} e^{-ip\hat{x}} f(p) \int \frac{d^4k}{2\pi^4} e^{-ik\hat{x}} g(k) \ . \tag{2.9}$$

The two exponentials don't commute. So we use the Baker-Campbell-Hausdorff theorem.

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \ .$$

For $A = -ip\hat{x}$, $B = -ik\hat{x}$ and if $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$, the series terminates and it is found that

$$\begin{aligned}
\hat{f} \star \hat{g} &= \int \frac{d^4p}{2\pi^4} \frac{d^4k}{2\pi^4} e^{-i(p+k)\hat{x}} e^{[-\frac{i}{2} p_\mu \theta^{\mu\nu} k_\nu]} f(p) g(k) \\
&= \int \frac{d^4p}{2\pi^4} \frac{d^4k}{2\pi^4} \frac{d^4x}{2\pi^4} \frac{d^4y}{2\pi^4} e^{-i(p+k)\hat{x}} e^{ipx} e^{iky} e^{[\frac{i}{2} \partial^x_\mu \theta^{\mu\nu} \partial^y_\nu]} f(x) g(y) \\
&= \int \frac{d^4p}{2\pi^4} \frac{d^4q}{2\pi^4} \frac{d^4x}{2\pi^4} \frac{d^4y}{2\pi^4} e^{-iq\hat{x}} e^{ipx} e^{i(q-p)y} e^{[\frac{i}{2} \partial^x_\mu \theta^{\mu\nu} \partial^y_\nu]} f(x) g(y) \\
&= \int \frac{d^4q}{2\pi^4} e^{-iq\hat{x}} \int \frac{d^4x}{2\pi^4} e^{iqx} e^{[\frac{i}{2} \partial^x_\mu \theta^{\mu\nu} \partial^y_\nu]} f(x) g(y) \ .
\end{aligned} \tag{2.10}$$

Looking at the Fourier transform in the last equation, the star product of the functions are defined as follows

$$(f \star g)(x) = e^{\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{ij} \frac{\partial}{\partial y^j}} f(x) g(y)|_{y \to x} \tag{2.11}$$

From now on, we omit the hat sign. The star product is also known as Moyal product. We can write some theorems about the star product.

1-)$f \star g \star h$ is associative

2-)$\int dx f \star g = \int dx f g$

3-)$\int dx f \star g \star h = \int dx f \star gh = \int dx f g \star h$

2.2.2 Application of Star product to Quantum Mechanics

In this section, the star product is applied to the usual quantum mechanics. The Schrodinger equation on for a particle under the influence of a potential $V(\vec{x})$ in noncommutative two dimensions is given just by replacing the ordinary product with the star product.

$$i\frac{\partial \Psi}{\partial t} = \left[\frac{p^2}{2m} + V(\mathbf{x})\right] \star \Psi. \tag{2.12}$$

As it is well-known, under the star operation the terms containing time derivative and p^2 are unchanged, however the potential term changes

$$V(\mathbf{x}) \star \Psi = V(\mathbf{x}) + \sum \frac{1}{n!}(\frac{i}{2})^n \partial_{i_1}...\partial_{i_n} V(\mathbf{x}) \Theta^{i_1 j_1}...\Theta^{i_n j_n} \partial_{j_1}...\partial_{j_n} \Psi. \tag{2.13}$$

Now replace ∂_{j_k} by p_{j_k} and introduce $\tilde{p_{i_k}} = \Theta^{i_k j_k} p_{j_k}$

$$\partial_{i_1}...\partial_{i_n} V(\mathbf{x}) \tilde{p}^{i_1}...\tilde{p}^{i_n} \Psi = i^n \int dk e^{i\mathbf{kx}} V(\mathbf{k})(\mathbf{k}\tilde{\mathbf{p}})^n \Psi. \tag{2.14}$$

Summing over n gives

$$V(\mathbf{x}) \star \Psi = \int dk e^{i\mathbf{kx}} e^{\frac{i}{2}\tilde{\mathbf{p}}\mathbf{k}} V(\mathbf{k})\Psi. \tag{2.15}$$

If we use $\mathbf{k}\tilde{\mathbf{k}} = 0$, we find

$$V(\mathbf{x}) \star \Psi = V(\mathbf{x} - \frac{\tilde{\mathbf{p}}}{2})\Psi. \tag{2.16}$$

It is concluded that the noncommutativity replaces the potential $V(\mathbf{x})$ to the potential $V(\mathbf{x} - \tilde{\mathbf{p}}/2)$ on the Schrodinger equation.

2.2.3 Application of Star Product to the Gauge Theory

Gauge theories are very important for the understanding of the fundamental forces of nature. The standard model made a unification electromagnetism, the weak, the strong force possible. Therefore a generalization of the gauge invariant principle to the noncommutative space is of particular interest. The star product formalism plays a crucial role in this theory because it gives us a possibility to express the noncommutative fields entering the theory in terms of the well known commutative fields. It makes it possible to read off explicitly the corrections to the noncommutative theory predicted by the noncommutative one. In the following subsections, the non-Abelian gauge theory will be studied as an application of the star product.

2.2.4 Non-Abelian Gauge Theory on Noncommutative Space

Noncommutative spaces, especially in the case of canonical noncommutativity, have been intensively studied in recent years. For example, a gauge theory has been developed on such a noncommutative space [37, 38].

Let us recall the non-Abelian gauge theory on the commutative space. A non-Abelian gauge theory is based on a Lie algebra

$$[T^a, T^b] = i f^{ab}{}_c T^c, \tag{2.17}$$

where T^a are hermitian traceless matrices generating the unitary group and $f^{ab}{}_c$ are real totally antisymmetric coefficients called the structure constants of the algebra. Some celebrated examples are

$$T^a = 1/2\sigma^a \ ; \qquad T^a = 1/2\lambda^a. \tag{2.18}$$

where σ^a, λ^a are the Pauli spin SU(2) and Gell-Mann matrices, respectively. Note that the successive transformations matters for non-Abelian theories. Whenever the order of the transformations matters, they are called non-Abelian transformations.

Infinitesimal transformation for matter field is given by

$$\delta_\alpha \psi^0(x) = i\alpha(x)\psi^0(x), \qquad \alpha(x) = \alpha_a(x)T^a. \tag{2.19}$$

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi^0(x) = i\alpha_a(x)\beta_b(x)f^{ab}{}_c T^c\psi^0(x) \equiv \delta_{\alpha\times\beta}\psi^0(x), \tag{2.20}$$

and for the gauge field

$$\begin{aligned} a_i(x) &= a_{i,a}(x)T^a \qquad (2.21) \\ \delta_\alpha a_i(x) &= \partial_i\alpha(x) + i[\alpha(x), a_i(x)]. \end{aligned}$$

Here, ψ^0, a_i denote the field and the gauge potential in the commutative case, respectively. In a gauge theory on noncommutative coordinates, (1.19) is replaced by

$$\delta_\alpha\psi(x) = i\Lambda_\alpha[a] \star \psi(x). \tag{2.22}$$

The variations δ_α

$$i\delta_\alpha\Lambda_\beta[a] - i\delta_\beta\Lambda_\alpha[a] + \Lambda_\alpha[a] \star \Lambda_\beta[a] - \Lambda_\beta[a] \star \Lambda_\alpha[a] = i\Lambda_{\alpha\times\beta}[a]. \tag{2.23}$$

The variation $\delta_\beta\Lambda_\alpha[a]$ refers to the a_i-dependence of $\Lambda_\alpha[a]$ and the transformation property of a_i.

It is natural to expand the star product in its "noncommutativity" and to solve the above equation in a power series expansion. For this purpose we introduce a parameter h:

$$
\begin{aligned}
(f \star g)(x) &= e^{\frac{i}{2}h\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x)g(y)|_{y\to x} \qquad (2.24)\\
&= f(x)g(x) + \frac{i}{2}h\theta^{ij}\partial_i f(x)\partial_j g(x) - \frac{1}{8}h^2\theta^{ij}\theta^{kl}\partial_i\partial_k f(x)\partial_j\partial_l g(x) + \ldots \quad .
\end{aligned}
$$

θ is used as an expansion parameter.

It is assumed that $\Lambda_\alpha[a]$ can be expanded in the parameter h:

$$
\Lambda_\alpha[a] = \alpha + h\Lambda^1_\alpha[a] + h^2\Lambda^2_\alpha[a] + \cdots \quad . \qquad (2.25)
$$

Now, the equation (2.22) is expanded in h. To first order we obtain

$$
i\delta_\alpha\Lambda^1_\beta[a] - i\delta_\beta\Lambda^1_\alpha[a] + [\alpha, \Lambda^1_\beta[a]] - [\beta, \Lambda^1_\alpha[a]] - i\Lambda^1_{\alpha\times\beta}[a] = -\frac{i}{2}\theta^{ij}\{\partial_i\alpha, \partial_j\beta\}. \qquad (2.26)
$$

The equation (2.26) can be solved with an ansatz linear in θ, because the inhomogeneous part is linear in θ. For dimensional reasons there is only a finite number of terms that can be used in such an ansatz. The proper combination of such terms is

$$
\Lambda^1_\alpha[a] = \frac{1}{4}\theta^{ij}\{\partial_i\alpha, a_j\} = \frac{1}{2}\theta^{ij}\partial_i\alpha_a a_{j,b} : T^a T^b : . \qquad (2.27)
$$

Similarly, to second order in h, Λ^2 gives

$$
\begin{aligned}
\Lambda^2_\alpha[a] &= \frac{1}{32}\theta^{ij}\theta^{kl}\Big(-4\{\partial_i\alpha, \{a_k, \partial_l a_j\}\} - i\{\partial_i\alpha, \{a_k, [a_j, a_l]\}\} - i\{a_j, \{a_l, [\partial_i\alpha, a_k]\}\}\\
&\quad +2i[\partial_i\partial_k\alpha, \partial_j a_l] - 2[\partial_j a_l, [\partial_i\alpha, a_k]] + 2i[[a_j, a_l], [\partial_i\alpha, a_k]]\Big). \qquad (2.28)
\end{aligned}
$$

The solution (2.27) and (2.28) are such that they are first and second order in θ, respectively. Having obtained the $\Lambda_\alpha[a]$, let us study the first order and second order contributions to the matter field and gauge field on noncommutative space.

2.2.5 Fields

In the standard non-Abelian gauge theory, fields have the transformation property (2.19). The noncommutative non-Abelian gauge theory is supposed to transform as in (2.22). The $\star$-product plays vital role in such a transformation. With the expansion property of any two fields under $\star$-product (2.24), we can write the noncommutative field ψ in terms of the usual field ψ^0 and gauge field a_i.

In the similar way, the fields can be expanded in powers of h

$$\psi[a] = \psi^0 + h\psi^1[a] + h^2\psi^2[a] + \cdots \quad . \tag{2.29}$$

To first order in h:

$$\delta_\alpha \psi^1[a] = i\alpha\psi^1[a] + i\Lambda^1_\alpha[a]\psi^0 - \frac{1}{2}\theta^{ij}\partial_i\alpha\partial_j\psi^0. \tag{2.30}$$

If the solution for $\Lambda^1_\alpha[a]$ (1.27) is substituted, it is found that

$$\psi^1[a] = -\frac{1}{2}\theta^{ij}a_i\partial_j\psi^0 + \frac{i}{4}\theta^{ij}a_ia_j\psi^0. \tag{2.31}$$

One can proceed to the next order,

$$\delta_\alpha\psi^2[a] = i\alpha\psi^2[a] + i\Lambda^1_\alpha[a]\psi^1[a] + i\Lambda^2_\alpha[a]\psi^0 - \frac{1}{2}\theta^{ij}\partial_i\Lambda^1_\alpha[a]\partial_j\psi^0 \tag{2.32}$$
$$-\frac{1}{2}\theta^{ij}\partial_i\alpha\partial_j\psi^1[a] - \frac{i}{8}\theta^{ij}\theta^{kl}\partial_i\partial_k\alpha\partial_j\partial_l\psi^0,$$

substituting for $\Lambda^2_\alpha[a]$ gives us $\psi^2[a]$

$$\begin{aligned}\psi^2[a] &= \frac{1}{32}\theta^{ij}\theta^{kl}\Big(-4i\partial_i a_k\partial_j\partial_l\psi^0 + 4a_i a_k\partial_j\partial_l\psi^0 + 8a_i\partial_j a_k\partial_l\psi^0 \\ &-4a_i\partial_k a_j\partial_l\psi^0 - 4ia_i a_j a_k\partial_l\psi^0 + 4ia_k a_j a_i\partial_l\psi^0 - 4ia_j a_k a_i\partial_l\psi^0 \\ &+4\partial_j a_k a_i\partial_l\psi^0 - 2\partial_i a_k\partial_j a_l\psi^0 + 4ia_i a_l\partial_k a_j\psi^0 + 4ia_i\partial_k a_j a_l\psi^0 \\ &-4ia_i\partial_j a_k a_l\psi^0 + 3a_i a_j a_l a_k\psi^0 + 4a_i a_k a_j a_l\psi^0 + 2a_i a_l a_k a_j\psi^0\Big).\end{aligned} \tag{2.33}$$

2.2.6 Gauge potentials and field strengths

Finally, the $\star$-product (2.24) expansion enables us to obtain the contributions to the gauge field. The transformation property of the noncommutative gauge field A_i is given by

$$\delta_\alpha A_i = \partial_i\Lambda_\alpha[a] + i[\Lambda_\alpha[a],^* A_i]. \tag{2.34}$$

In the similar way, we expand A_i in h.

$$A_i[a] = a_i + hA_i^1[a] + h^2A_i^2[a] + \cdots \quad . \tag{2.35}$$

To first order we obtain the following

$$\delta_\alpha A_i^1[a] = \partial_i\Lambda^1_\alpha[a] + i[\Lambda^1_\alpha[a], a_i] + i[\alpha, A_i^1[a]] - \frac{1}{2}\theta^{kl}\{\partial_k\alpha, \partial_l a_i\}. \tag{2.36}$$

If we substitute the solution for $\Lambda^1_\alpha[a]$, we find the first order contribution to the gauge field as follows

$$A_i^1[a] = -\frac{1}{4}\theta^{kl}\{a_k, \partial_l a_i + F^0_{li}\}, \tag{2.37}$$

where F^0_{ij} is the field strength of the ordinary Lie algebra-valued gauge theory

$$F^0_{ij} = \partial_i a_j - \partial_j a_i - i[a_i, a_j]. \tag{2.38}$$

To second order in h

$$\begin{aligned}\delta_\alpha A_i^2[a] &= \partial_i\Lambda_\alpha^2[a] + i[\alpha, A_i^2[a]] + i[\Lambda_\alpha^1[a], A_i^1[a]] + i[\Lambda_\alpha^2[a], a_i] - \\ &-\frac{1}{2}\theta^{kl}\{\partial_k\alpha, \partial_l A_i^1[a]\} - \frac{1}{2}\theta^{kl}\{\partial_k\Lambda_\alpha^1[a], \partial_l a_i\} - \frac{i}{8}\theta^{kl}\theta^{mn}[\partial_k\partial_m\alpha, \partial_l\partial_n a_i].\end{aligned} \tag{2.39}$$

If we use $\Lambda_\alpha^2[a]$, it has the following solution

$$\begin{aligned}A_i^2[a] &= \frac{1}{32}\theta^{kl}\theta^{mn}\Big(4i[\partial_k\partial_m a_i, \partial_l a_n] - 2i[\partial_k\partial_i a_m, \partial_l a_n] + 4\{a_k, \{a_m, \partial_n F_{li}^0\}\} \\ &+2[[\partial_k a_m, a_i], \partial_l a_n] - 4\{\partial_l a_i, \{\partial_m a_k, a_n\}\} + 4\{a_k, \{F_{lm}^0, F_{ni}^0\}\} \\ &-i\{\partial_i a_n, \{a_l, [a_m, a_k]\}\} - i\{a_m, \{a_k, [\partial_i a_n, a_l]\}\} \\ &+4i[[a_m, a_l], [a_k, \partial_n a_i]] - 2i[[a_m, a_l], [a_k, \partial_i a_n]] - \{a_m, \{a_k, [a_l, [a_n, a_i]]\}\} \\ &+\{a_k, \{[a_l, a_m], [a_n, a_i]\}\} + [[a_m, a_l], [a_k, [a_n, a_i]]]\Big).\end{aligned} \tag{2.40}$$

The $\star$-product enables us to write the first order and second order contribution to the matter field and gauge field in noncommutative space. One can find the higher order contribution to these fields. Note that the calculations get length when trying to find those in higher order.

2.2.7 Actions

In this subsection, we shall study the action in $\star$-product. It was shown before that the integral has the trace property for the $\star$-product:

$$\int f \star g \;\, \mathrm{d}x = \int g \star f \;\, \mathrm{d}x = \int fg \;\, \mathrm{d}x. \tag{2.41}$$

Thus we find an invariant action for the gauge potential

$$S = -\frac{1}{4}\mathrm{Tr}\int F_{ij} \star F^{ij} \;\, \mathrm{d}x, \tag{2.42}$$

as well as for the matter fields

$$S = \int \bar{\psi} \star (\gamma^i \mathcal{D}_i - m)\psi \ \mathrm{d}x. \tag{2.43}$$

Our aim is to expand these actions in the fields a_i and ψ^0 and to treat them as conventional field theories depending on a coupling constant θ. We only do this here to first order in h and construct the Lagrangian from our previous results:

$$\begin{aligned}
m\bar{\psi} \star \psi &= m\bar{\psi}^0\psi^0 + \frac{i}{2}h\theta^{kl} m\mathcal{D}_k\bar{\psi}^0\mathcal{D}_l\psi^0 \\
\bar{\psi} \star \gamma^i\mathcal{D}_i\psi &= \bar{\psi}^0\gamma^i\mathcal{D}_i\psi^0 + \frac{i}{2}h\theta^{kl}\mathcal{D}_k\bar{\psi}^0\gamma^i\mathcal{D}_l\mathcal{D}_i\psi^0 - \frac{1}{2}h\theta^{kl}\bar{\psi}^0\gamma^i F^0_{ik}\mathcal{D}_l\psi^0 \\
F_{ij} \star F^{ij} &= F^0_{ij}F^{0ij} + \frac{i}{2}h\theta^{kl}\mathcal{D}_k F^0_{ij}\mathcal{D}_l F^{0ij} + \frac{1}{2}h\theta^{kl}\{\{F^0_{ik}, F^0_{jl}\}, F^{0ij}\} \\
&\quad -\frac{1}{4}h\theta^{kl}\{F^0_{kl}, F^0_{ij}F^{0ij}\} - \frac{i}{4}h\theta^{kl}[a_k, \{a_l, F^0_{ij}F^{0ij}\}]
\end{aligned}$$

For the action we use partial integration and the cyclicity of the trace and obtain to first order in h:

$$\begin{aligned}
\int \bar{\psi} \star (\gamma^i\mathcal{D}_i - m)\psi \,\mathrm{d}x &= \int \bar{\psi}^0(\gamma^i\mathcal{D}_i - m)\psi^0 \,\mathrm{d}x - \frac{1}{4}h\theta^{kl}\int \bar{\psi}^0 F^0_{kl}(\gamma^i\mathcal{D}_i - m)\psi^0 \,\mathrm{d}x \\
&\quad -\frac{1}{2}h\theta^{kl}\int \bar{\psi}^0\gamma^i F^0_{ik}\mathcal{D}_l\psi^0 \,\mathrm{d}x \qquad (2.44) \\
-\frac{1}{4}\mathrm{Tr}\int F_{ij} \star F^{ij} \,\mathrm{d}x &= -\frac{1}{4}\mathrm{Tr}\int F^0_{ij}F^{0ij} \,\mathrm{d}x + \frac{1}{8}h\theta^{kl}\mathrm{Tr}\int F^0_{kl}F^0_{ij}F^{0ij} \,\mathrm{d}x \\
&\quad -\frac{1}{2}h\theta^{kl}\mathrm{Tr}\int F^0_{ik}F^0_{jl}F^{0ij} \,\mathrm{d}x \qquad (2.45)
\end{aligned}$$

CHAPTER 3

NONCOMMUTATIVE FIELD THEORY

3.1 The Standard Model on Non-Commutative Space-Time

In the previous section, the non-Abelian gauge transformation on the non-commutative space was studied by using the $\star$-product. Now we want to apply this method to construct the Standard model in noncommutative space. Calmet, Jurco, Schupp, Wess, Wohlgenannt [45] constructed the Standard Model on a non-commutative space up to first order in the non-commutativity parameter $\Theta^{\mu\nu}$. The symmetry group is $SU(3) \times SU(2) \times U(1)$. Obviously, at zeroth order the action must be coincides with the ordinary Standard Model. At the first order in $\Theta^{\mu\nu}$ it is found new vertices which are absent in the Standard Model on commutative space-time.

3.1.1 Gauge fields on non-commutative space-time

In an analogous way of the Standard model, we can hope to infer the structure of the noncommutative Standard model from local gauge invariance. The present belief is that all particle interactions may be dictated by so-called local gauge symmetries. This is connected with the idea that the conserved physical quantities such as electric charge, color, etc. are conserved in local regions of space, and not just globally. It is assumed that this is also true in noncommutative space. So, our aim is to construct the local gauge invariance for the noncommutative field theory.

In this subsection, we will briefly study the gauge fields on noncommutative space and expand the action over the noncommutative parameter $\Theta^{\mu\nu}$. We will derive the formulas up to the first order of $\Theta^{\mu\nu}$. At zeroth order, the fields and the action coincide with those of the Standard model. There is no new particles introduced in this theory. We will show that there exists some new vertices.

Before going further, let us recall the transformation of the field Ψ in commutative space

$$\delta\Psi = i\alpha(x)\Psi,$$

where $\alpha(x)$ is a gauge parameter. As we said earlier, the ordinary product is replaced with the $\star$-product when doing transition to the noncommutative space. But, here a little care should be taken. The gauge parameter α is also replaced with a new parameter Λ. This is because the gauge parameter Λ in noncommutative theories does not satisfy the Lie algebra (2.17). Λ depends in general on $\alpha(x)$ and the gauge field $A_\mu(x)$.

In an analogous way, we can write an infinitesimal non-commutative local gauge transformation $\hat{\delta}$ of a fundamental matter field as follows

$$\hat{\delta}\widehat{\Psi} = i\widehat{\Lambda} \star \widehat{\Psi}. \tag{3.1}$$

In the non-Abelian case $\widehat{\Psi}$ is a vector, $\widehat{\Lambda}$ a matrix whose entries are functions on non-commutative space-time and $\star$ includes matrix multiplication, i.e., $[\widehat{\Lambda}\star$

$\widehat{\Psi}]_a \equiv \sum_b [\Lambda]_{ab} \star \widehat{\Psi}_b$.

The transformation properties of the products of a field and a coordinate, $\widehat{\Psi} \star x^\mu$ and $x^\mu \star \widehat{\Psi}$ are equal. If we use the equation (2.24), we see that

$$\widehat{\Psi} \star x^\mu = \Psi x^\mu + \frac{i}{2}\Theta^{ij}\partial_i \Psi \partial_j x^\mu + ...$$
$$x^\mu \star \widehat{\Psi} = \Psi x^\mu - \frac{i}{2}\Theta^{ij}\partial_i \Psi \partial_j x^\mu + ...$$

In the last relation, we used the antisymmetry of Θ^{ij} ($\Theta^{ij} = -\Theta^{ji}$).

In an analogy to the covariant derivatives of ordinary gauge theory, the covariant coordinates $X^\mu = x^\mu + \Theta^{\mu\nu}\widehat{A}_\nu$ is introduced, where $\widehat{A}_\nu$ is a noncommutative analog of the gauge potential with the following transformation property:

$$\hat{\delta}\widehat{A}_\mu = \partial\mu\widehat{\Lambda} + i[\widehat{\Lambda}, \widehat{A}_\mu]. \tag{3.2}$$

In the similar way, we can define the noncommutative field strength as in the following way

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu, \widehat{A}_\nu], \qquad \hat{\delta}\widehat{F}_{\mu\nu} = i[\widehat{\Lambda}, \widehat{F}_{\mu\nu}] \tag{3.3}$$

Further, we can write down the covariant derivative as follows

$$\widehat{D}_\mu \widehat{\Psi} = \partial_\mu \widehat{\Psi} - i\widehat{A}_\mu \star \widehat{\Psi}. \tag{3.4}$$

Up to now, what we have done is just to write the local gauge transformations of the fields, covariant derivative and field strength in noncommutative space with the help of $\star$-product. The only changes comparing to the usual gauge theory come from the replacement of the ordinary product with the $\star$-product and from the new gauge parameter Λ instead of α. Now, the next questions arise. How do we deal with $\star$-product? Can we somehow find some transformations relating the noncommutative fields and gauge parameters Λ to the usual fields and gauge parameters α with which we are familiar? In the followings, we will answer these questions.

3.1.2 Seiberg-Witten map

Here, we will introduce so-called Seiberg-Witten map which enables us to express the noncommutative variables in terms of the commutative variables. Actually, there are two ways to do this. The first one is the $\star$-product. We studied the $\star$-product in the previous chapter. Star product of f, g is defined,

$$f \star g = f \cdot g + \frac{i}{2}\theta^{\mu\nu}(x)\partial\mu f \cdot \partial\nu g + \mathbf{O}(\Theta^2)$$

with higher order terms chosen in such a way as to yield an associative product. The star product is a local function of f, g, meaning that it is a formal series that at each order in Θ depends on f, g and a *finite* number of derivatives of f and g.

Secondly, the non-commutative fields $\widehat{A}$, $\widehat{\Psi}$ and non-commutative gauge parameter $\widehat{\Lambda}$ can be expressed in a similar fashion by so-called Seiberg-Witten maps in terms of the corresponding ordinary fields A, Ψ and ordinary gauge parameter Λ.

$$\widehat{A}_\xi[A] \;=\; A_\xi + \frac{1}{4}\Theta^{\mu\nu}\{A_\nu, \partial_\mu A_\xi\} + \frac{1}{4}\Theta^{\mu\nu}\{F_{\mu\xi}, A_\nu\} + \mathbf{O}(\Theta^2), \quad (3.5)$$

$$\widehat{\Psi}[\Psi, A] = \Psi + \frac{1}{2}\Theta^{\mu\nu}A_\nu\partial_\mu\Psi + \frac{i}{8}\Theta^{\mu\nu}[A_\mu, A_\nu]\Psi + \mathbf{O}(\Theta^2), \tag{3.6}$$

$$\widehat{\Lambda}[\Lambda, A] = \Lambda + \frac{1}{4}\Theta^{\mu\nu}\{A_\nu, \partial\mu\Lambda\} + \mathbf{O}(\Theta^2), \tag{3.7}$$

where $F_{\mu\nu} = \partial\mu A_\nu - \partial\nu A_\mu - i[A_\mu, A_\nu]$ is the ordinary field strength.

We can choose one of the method. If we can use the Seiberg-Witten map, then we should forget about the $\star$-product in all of the formulas written for the noncommutative theory. Instead of it, we should directly write the transformations (3.5-3.7), whenever the noncommutative variables with *hat* is required. The Seiberg-Witten map is easy to handle, since it enable us to make transition from the noncommutative variable to the commutative variables. So, we don't need to deal with the derivative terms in $\star$-product.

We will henceforth omit the explicit dependence of the non-commutative fields and parameters on their ordinary counterparts with the understanding, that the hat denotes non-commutative quantities that can be expanded as local functions of their classical counterparts via Seiberg-Witten maps.

The Seiberg-Witten maps have the remarkable property that ordinary gauge transformations $\delta A_\mu = \partial\mu\Lambda + i[\Lambda, A_\mu]$ and $\delta\Psi = i\Lambda\cdot\Psi$ induce non-commutative gauge transformations of the fields $\widehat{A}$, $\widehat{\Psi}$ with gauge parameter $\widehat{\Lambda}$ as given above:

$$\delta\widehat{A}_\mu = \hat{\delta}\widehat{A}_\mu, \qquad \delta\widehat{\Psi} = \hat{\delta}\widehat{\Psi}. \tag{3.8}$$

For consistency we have to require that any pair of non-commutative gauge parameters $\widehat{\Lambda}$, $\widehat{\Sigma}$ satisfy

$$[\widehat{\Lambda}, \widehat{\Sigma}] + i\delta_\Lambda\widehat{\Sigma} - i\delta_\Sigma\widehat{\Lambda} = \widehat{[\Lambda, \Sigma]}. \tag{3.9}$$

Since only the gauge parameters are involved for the the consistency condition, it is convenient to construct the Seiberg-Witten map on it. Then, the remaining Seiberg-Witten maps can be calculated from the gauge equivalence condition.

Before writing the action in noncommutative space, let us mention about the algebra of the non-Abelian gauge theory in noncommutative space, since there is a difference between the commutative and noncommutative theories.

3.1.3 Non-Abelian gauge groups

The gauge parameter in Standard non-Abelian gauge theory is based on a Lie group $(T^a, T^b = if^{ab}_c T^c)$, where f^{ab}_c are the antisymmetric structure constants. However, the gauge parameter Λ in noncommutative Standard model is not based on a Lie group. The commutator

$$[\widehat{\Lambda} \stackrel{\star}{,} \widehat{\Lambda}'] = \frac{1}{2}\{\Lambda_a(x) \stackrel{\star}{,} \Lambda'_b(x)\}[T^a, T^b] + \frac{1}{2}[\Lambda_a(x) \stackrel{\star}{,} \Lambda'_b(x)]\{T^a, T^b\} \tag{3.10}$$

of two Lie algebra-valued non-commutative gauge parameters $\widehat{\Lambda} = \Lambda_a(x)T^a$ and $\widehat{\Lambda}' = \Lambda'_a(x)T^a$ does not close in the Lie algebra. It is in general enveloping algebra-valued [38]. If we try, to construct non-commutative $SU(N)$ with Lie algebra-valued gauge parameters, we immediately face the problem that a tracelessness condition is incompatible with (3.10). We thus have to consider enveloping algebra-valued noncommutative gauge parameters

$$\widehat{\Lambda} = \Lambda^0_a(x)T^a + \Lambda^1_{ab}(x) : T^a T^b : + \Lambda^2_{abc}(x) : T^a T^b T^c : + \dots \tag{3.11}$$

and fields. (The symbol $: \;\; := \frac{1}{2}\{ \;\; \}$.) We see that there are infinite number of parameters $\Lambda^0_a(x)$, $\Lambda^1_{ab}(x)$, $\Lambda^2_{abc}(x)$, $\ldots$; however, these are not independent. In fact, they can be expressed in terms of the parameter $\alpha(x)$ and field $A_\mu(x)$ via the Seiberg-Witten map.

3.2 The Standard Model on The Noncommutative Space

One of most profound insights in high energy physics is that the interactions in nature are governed by the symmetry principles. Einstein, Salam-Weinberg made use of this idea. Requiring the local gauge invariance, they were lead to the general theory of relativity and electro-weak theory. We assume that the interactions in noncommutative space is also governed by the local gauge invariance. By Noether's theorem, The Lagrangian plays vital role for the gauge theories. In this section, after giving the necessary transformations for the noncommutative fields, parameters and field strength, we will obtain the action in noncommutative space.

Now, our aim is to construct the standard model on noncommutative space. The structure group of the Standard Model is $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$. There are several ways to do this in the noncommutative case since there exists freedom in the choice of Seiberg-Witten map. We will follow the method introduced by Calmet-Jurco-Schupp-Wess-Wohlgenannt [45]. In [45], the whole gauge field V_μ of G_{SM} as defined by

$$V_\nu = g'\mathcal{A}_\nu(x)Y + g\sum_{a=1}^{3} B_{\nu a}(x)T^a_L + g_S\sum_{b=1}^{8} G_{\nu b}(x)T^b_S \tag{3.12}$$

and the commutative gauge parameter Λ by

$$\Lambda = g'\alpha(x)Y + g\sum_{a=1}^{3}\alpha_a^L(x)T_L^a + g_S\sum_{b=1}^{8}\alpha_b^S(x)T_S^b, \tag{3.13}$$

where Y, T_L^a, T_S^b are the generators of $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ respectively. The non-commutative gauge parameter $\widehat{\Lambda}$ is then given via the Seiberg-Witten map by

$$\widehat{\Lambda} = \Lambda + \frac{1}{4}\Theta^{\mu\nu}\{V_\nu, \partial_\mu\Lambda\}. \tag{3.14}$$

It is seen that the noncommutative gauge parameter $\widehat{\Lambda}$ depends on the gauge potential V_ν in addition of the commutative gauge parameter Λ.

Before giving the Seiberg-Witten map for the field $\widehat{\Psi}$, let us briefly study the particle spectrum. Note that there is no change of the particle spectrum of the commutative and noncommutative theories. In other words, no new particles are introduced in the noncommutative theories. So, we can copy of the particle spectrum of the standard model. The particle spectrum is given as follows:

$$\Psi_L^{(i)} = \begin{pmatrix} L_L^{(i)} \\ Q_L^{(i)} \end{pmatrix}, \qquad \Psi_R^{(i)} = \left(e_R^{(i)};\ u_R^{(i)};\ d_R^{(i)}\right), \qquad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{3.15}$$

where $i = \{1, 2, 3\}$ is the generation index and ϕ^+ and ϕ^0 are the complex scalar Higgs fields. For more information about the particle spectrum can be found in Table 3.1

Table 3.1: The Standard Model fields. The electric charge is given by the Gell-Mann-Nishijima relation $Q = (T_3 + Y)$.

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_Q$
e_R	$\mathbf{1}$	$\mathbf{1}$	-1	-1
$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
u_R	$\mathbf{3}$	$\mathbf{1}$	$2/3$	$2/3$
d_R	$\mathbf{3}$	$\mathbf{1}$	$-1/3$	$-1/3$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$1/6$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$1/2$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
B^i	$\mathbf{1}$	$\mathbf{3}$	0	$(\pm 1, 0)$
A	$\mathbf{1}$	$\mathbf{1}$	0	0
G^a	$\mathbf{8}$	$\mathbf{1}$	0	0

The noncommutative fermion fields $\widehat{\Psi}^{(n)}$ corresponding to particles labelled by (n) up to first order is given as in the equation (3.6)

$$\widehat{\Psi}^{(n)} = \Psi^{(n)} + \frac{1}{2}\Theta^{\mu\nu}\rho_{(n)}(V_\nu)\partial_\mu\Psi^{(n)} + \frac{i}{8}\Theta^{\mu\nu}[\rho_{(n)}(V_\mu), \rho_{(n)}(V_\nu)]\Psi^{(n)}, \quad (3.16)$$

where $\rho_{(n)}(V_\nu)$ is defined Table 3.2. This formula is written in general case. We can write it specifically just by looking at the Table 3.2. As an example, if we take $\widehat{\Psi} = \widehat{L}_L$, then $\rho_{(n)}(V_\mu) = -\frac{g'}{2}\mathcal{A}_\nu(x) + gB_\nu(x)$, we find

$$\widehat{L}_L = \begin{pmatrix} \widehat{\nu}_L \\ \\ \widehat{e}_L \end{pmatrix} = L_L + \frac{\Theta^{\mu\nu}}{2}\left(gB_\nu - \frac{g'}{2}\mathcal{A}_\nu\right)\partial_\mu L_L + \frac{ig}{4}\Theta^{\mu\nu}B_\mu B_\nu L_L \quad (3.17)$$

Note that $\Theta^{\mu\nu}A_\mu A_\nu = \dfrac{\Theta^{\mu\nu}}{2}[A_\mu, A_\nu] = 0.$

Now, let us write the explicit form of the transformation for the gauge potential. The Seiberg-Witten map for the noncommutative gauge potential $\widehat{V}_\mu$ yields

Table 3.2: The gauge fields. (The symbols T_L^a and T_S^b are here the Pauli and Gell-Mann matrices respectively.)

$\Psi^{(n)}$	$\rho_{(n)}(V_\nu)$
e_R	$-g'\mathcal{A}_\nu(x)$
$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$-\frac{1}{2}g'\mathcal{A}_\nu(x) + gB_{\nu a}(x)T_L^a$
u_R	$\frac{2}{3}g'\mathcal{A}_\nu(x) + g_S G_{\nu b}(x)T_S^b$
d_R	$-\frac{1}{3}g'\mathcal{A}_\nu(x) + g_S G_{\nu b}(x)T_S^b$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\frac{1}{6}g'\mathcal{A}_\nu(x) + gB_{\nu a}(x)T_L^a + g_S G_{\nu b}(x)T_S^b$

$$\widehat{V}_\xi = V_\xi + \frac{1}{4}\Theta^{\mu\nu}\{V_\nu, \partial_\mu V_\xi\} + \frac{1}{4}\Theta^{\mu\nu}\{F_{\mu\xi}, V_\nu\} + \mathbf{O}(\Theta^2), \tag{3.18}$$

where the ordinary field strength $F^{\mu\nu} \equiv \partial^\mu V^\nu - \partial^\nu V^\mu - i[V^\mu, V^\nu]$. The noncommutative field strength is

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu - i[\widehat{V}_\mu \stackrel{*}{,} \widehat{V}_\nu]. \tag{3.19}$$

Having written the necessary definitions and the transformations, now we can proceed. In studying gauge theories, the physicist use the Lagrangian for their computations instead od writing the relativistic wave equation. This is mainly because of the Noether's theorem which dictates that an invariance under a transformation leads to the conservation of a physical quantity. Another reason to study with the Lagrangian is that to each Lagrangian, there corresponds a set of Feynman rules. Interactions are computed by evaluating a perturbation series in iL_{int}, the interaction terms in iL. So, writing the action in noncommutative space is of great importance.

Now, we can write the action of the noncommutative Standard Model in a very compact way just by replacing the ordinary product to the $\star$-product.

$$
\begin{aligned}
S_{NCSM} \;=\; & \int d^4x \sum_{i=1}^{3} \overline{\widehat{\Psi}}^{(i)}_L \star i\gamma^\mu \widehat{D_\mu} \widehat{\Psi}^{(i)}_L + \int d^4x \sum_{i=1}^{3} \overline{\widehat{\Psi}}^{(i)}_R \star i\gamma^\mu \widehat{D_\mu} \widehat{\Psi}^{(i)}_R \qquad (3.20) \\
& - \int d^4x \frac{1}{2g'} \mathbf{tr_1} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g} \mathbf{tr_2} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} \\
& - \int d^4x \frac{1}{2g_S} \mathbf{tr_3} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} + \int d^4x \Big(\rho_0(\widehat{D}_\mu \widehat{\Phi})^\dagger \star \rho_0(\widehat{D}^\mu \widehat{\Phi}) \\
& - \mu^2 \rho_0(\widehat{\Phi})^\dagger \star \rho_0(\widehat{\Phi}) - \lambda \rho_0(\widehat{\Phi})^\dagger \star \rho_0(\widehat{\Phi}) \star \rho_0(\widehat{\Phi})^\dagger \star \rho_0(\widehat{\Phi}) \Big) \\
& + \int d^4x \Big(- \sum_{i,j=1}^{3} W^{ij} \Big((\bar{\widehat{L}}^{(i)}_L \star \rho_L(\widehat{\Phi})) \star \widehat{e}^{(j)}_R + \bar{\widehat{e}}^{(i)}_R \star (\rho_L(\widehat{\Phi})^\dagger \star \widehat{L}^{(j)}_L) \Big) \\
& - \sum_{i,j=1}^{3} G^{ij}_u \Big((\bar{\widehat{Q}}^{(i)}_L \star \rho_{\bar{Q}}(\bar{\widehat{\Phi}})) \star \widehat{u}^{(j)}_R + \bar{\widehat{u}}^{(i)}_R \star (\rho_{\bar{Q}}(\bar{\widehat{\Phi}})^\dagger \star \widehat{Q}^{(j)}_L) \Big) \\
& - \sum_{i,j=1}^{3} G^{ij}_d \Big((\bar{\widehat{Q}}^{(i)}_L \star \rho_Q(\widehat{\Phi})) \star \widehat{d}^{(j)}_R + \bar{\widehat{d}}^{(i)}_R \star (\rho_Q(\widehat{\Phi})^\dagger \star \widehat{Q}^{(j)}_L) \Big) \Big),
\end{aligned}
$$

with $\bar{\Phi} = i\tau_2 \Phi^*$. The matrices W^{ij}, G^{ij}_u and G^{ij}_d are the Yukawa couplings. New vertices can be found from this action by expanding the $\star$-product and using the Seiberg-Witten map.

The gauge fields in the Seiberg-Witten map are also summarized in Table 3.2.

The representation used in the trace of the kinetic terms for the gauge bosons is not uniquely determined by gauge invariance of the action. The simplest choice of a sum of traces over the $U(1)$, $SU(2)$ and $SU(3)$ sectors is taken into account, since we want to find the Standard Model on noncommutative space-time with minimal modifications. In this spirit Y is chosen

$$
Y = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (3.21)
$$

in the definition of $\mathbf{tr_1}$. The traces $\mathbf{tr_2}$ and trace $\mathbf{tr_3}$ are the usual $SU(2)$, respectively $SU(3)$ traces. The representations ρ_L, ρ_Q, $\rho_{\bar{Q}}$ of the gauge potentials V_μ, V'_μ that appear in the Seiberg-Witten map of the Higgs are those of the fermions on the left and right of the Higgs in the Yukawa couplings,

$$\begin{aligned}
\rho_{\bar{Q}}(\hat{\Phi}[\phi, V_\mu, V'_\nu]) &= \hat{\Phi}[\phi,\ \frac{1}{6}g'\mathcal{A}_\mu + gB^a_\mu T^a_L + g_S G^a_\mu T^a_S,\ -\frac{2}{3}g'\mathcal{A}_\nu - g_S G^a_\nu T^a_S], \\
\rho_Q(\hat{\Phi}[\phi, V_\mu, V'_\nu]) &= \hat{\Phi}[\phi,\ \frac{1}{6}g'\mathcal{A}_\mu + gB^a_\mu T^a_L + g_S G^a_\mu T^a_S,\ \frac{1}{3}g'\mathcal{A}_\nu - g_S G^a_\nu T^a_S], \\
\rho_L(\hat{\Phi}[\phi, V_\mu, V'_\nu]) &= \hat{\Phi}[\phi,\ -\frac{1}{2}g'\mathcal{A}_\mu + gB^a_\mu T^a_L,\ g'\mathcal{A}_\nu]. \qquad (3.22)
\end{aligned}$$

The representation ρ_0 of these gauge potentials in the kinetic term of the Higgs and in the Higgs potential is the simplest possible one

$$\rho_0(\hat{\Phi}[\phi, V_\mu, V'_\nu]) = \hat{\Phi}[\phi,\ \frac{1}{2}g'\mathcal{A}_\mu + gB^a_\mu T^a_L,\ 0]. \qquad (3.23)$$

3.3 The Electro-weak Noncommutative Standard Model

In this section the Seiberg-Witten map will be applied to the electro-weak non-commutative Standard Model. The gauge group of the model is $SU(3)_C \times SU(2)_L \times U(1)_Y$. As before, there are no new particles introduced. That is, the particle content is the same as the Standard Model. The matter fields and gauge fields content is summarized in Table 3.1.

In the following, we will work in the leading order of the expansion in Θ. Fields with a hat mean noncommutative whereas those without a hat mean ordinary fields. In particular, the following definitions will be used: $\mathcal{A}_\mu$ is the ordinary $U(1)_Y$ field, $B_\mu = B^i_\mu T^i_L$ are the ordinary $SU(2)_L$ fields and $G_\mu = G^i_\mu T^i_S$ are the ordinary $SU(3)_C$ fields. For the lepton field $L^{(i)}_L$ of the ith generation which is in the fundamental representation of $SU(2)_L$ and in the Y representation of $U(1)_Y$, we have the following expansion

$$\widehat{L}_L^{(i)}[\mathcal{A}, B] = L_L^{(i)} + L_L^{(i)1}[\mathcal{A}, B] + \mathbf{O}(\Theta^2). \tag{3.24}$$

If we use the equation (3.6) and Table 3.2, the first order contribution is computed as

$$\begin{aligned} L_L^{(i)1}[\mathcal{A}, B] &= -\frac{1}{2}g'\Theta^{\mu\nu}\mathcal{A}_\mu\partial_\nu L_L - \frac{1}{2}g\Theta^{\mu\nu}B_\mu\partial_\nu L_L \\ &\quad + \frac{i}{4}\Theta^{\mu\nu}\left(g'\mathcal{A}_\mu + gB_\mu\right)\left(g'\mathcal{A}_\nu + gB_\nu\right)L_L. \end{aligned} \tag{3.25}$$

We can do the same calculations for the right handed lepton field of the ith generation. We get

$$\widehat{e}_R^{(i)}[\mathcal{A}] = e_R^{(i)} + e_R^{(i)1}[\mathcal{A}] + \mathbf{O}(\Theta^2), \tag{3.26}$$

with

$$e_R^{(i)1}[\mathcal{A}] = -\frac{1}{2}g'\Theta^{\mu\nu}\mathcal{A}_\mu\partial_\nu e_R^{(i)}. \tag{3.27}$$

We found the first order contribution for the leptonic parts. The quarks are also fell the electro-weak interaction. The expansion for a left-handed quark doublet $\widehat{Q}_L^{(i)}$ of the ith generation is written as

$$\widehat{Q}_L^{(i)}[\mathcal{A}, B, G] = Q_L^{(i)} + Q_L^{(i)1}[\mathcal{A}, B, G] + \mathbf{O}(\Theta^2) \tag{3.28}$$

Then, the first order contribution is found by using the Seiberg-Witten map (3.6) and Table 3.2.

$$Q_L^{(i)1}[\mathcal{A}, B, G] = -\frac{1}{2}g'\Theta^{\mu\nu}\mathcal{A}_\mu\partial_\nu Q_L - \frac{1}{2}g\Theta^{\mu\nu}B_\mu\partial_\nu Q_L \tag{3.29}$$

$$-\frac{1}{2}g_S\Theta^{\mu\nu}G_\mu\partial_\nu Q_L + \frac{i}{4}\Theta^{\mu\nu}\left(g'\mathcal{A}_\mu + gB_\mu + g_SG_\mu\right)\times \left(g'\mathcal{A}_\nu + gB_\nu + g_SG_\nu\right)Q_L.$$

For a right-handed quark e.g., $\widehat{u}_R^{(i)}$, we have

$$\widehat{u}_R^{(i)}[\mathcal{A},G] = u_R^{(i)} + u_R^{(i)1}[\mathcal{A},G] + \mathbf{O}(\Theta^2), \tag{3.30}$$

In the similar way, we obtain

$$\begin{aligned} u_R^{(i)1}[\mathcal{A},G] &= -\frac{1}{2}g'\Theta^{\mu\nu}\mathcal{A}_\mu\partial_\nu u_R - \frac{1}{2}g_S\Theta^{\mu\nu}G_\mu\partial_\nu u_R \\ &\quad + \frac{i}{4}\Theta^{\mu\nu}\left(g'\mathcal{A}_\mu + g_SG_\mu\right)\left(g'\mathcal{A}_\nu + g_SG_\nu\right)u_R. \end{aligned} \tag{3.31}$$

The same expansion is obtained for a right-handed down type quark $d_R^{(i)}$.

In constructing the noncommutative electro-weak Standard model, we have obtained the first order contribution to the fields. The action (3.20) also includes the field strength $F_{\mu\nu}$. To finish our work, we should also calculate the contribution to the field strength due to the noncommutativity of space-time. The field strength $\widehat{F}_{\mu\nu} = \partial_\mu\widehat{V}_\nu - \partial_\nu\widehat{V}_\mu - i[\widehat{V}_\mu \stackrel{*}{,} \widehat{V}_\nu]$ has the following expansion:

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + F^1_{\mu\nu} + \mathbf{O}(\Theta^2), \tag{3.32}$$

with

$$F_{\mu\nu} = g'f_{\mu\nu} + gF^L_{\mu\nu} + g_SF^S_{\mu\nu}, \tag{3.33}$$

where $f_{\mu\nu}$ is the field strength corresponding to the group $U(1)_Y$, $F^L_{\mu\nu}$ that to $SU(2)_L$ and $F^S_{\mu\nu}$ that to $SU(3)_C$. The coupling constants of the gauge groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ are respectively denoted by g', g and g_S. The leading order in Θ is given by

$$F^1_{\mu\nu} = \frac{1}{2}\Theta^{\alpha\beta}\{F_{\mu\alpha}, F_{\nu\beta}\} - \frac{1}{4}\Theta^{\alpha\beta}\{V_\alpha, (\partial_\beta + D_\beta)F_{\mu\nu}\}, \quad (3.34)$$

with

$$D_\beta F_{\mu\nu} = \partial_\beta F_{\mu\nu} - i[V_\beta, F_{\mu\nu}]. \quad (3.35)$$

As a final step, we will compute the noncommutative gauge potential in terms of the commutative gauge potential, since the action (3.20) includes it in covariant derivative D_μ.

The leading order expansion for the mathematical vector field V is given by

$$\widehat{V}_\mu = V_\mu + i\Gamma_\mu + \mathbf{O}(\Theta^2). \quad (3.36)$$

If we look at the equation (3.5), we see that

$$\Gamma_\mu = \frac{1}{4}\Theta^{\alpha\beta}\{V_\beta \partial_\alpha V_\mu\} + \frac{1}{4}\Theta^{\alpha\beta}\{F_{\alpha\mu} V_\beta\}, \quad (3.37)$$

where $V_\mu = g'\mathcal{A}_\alpha + gB_\alpha + g_S G_\alpha$. Then, it is found

$$\begin{aligned}\Gamma_\mu = {} & i\frac{1}{4}\Theta^{\alpha\beta}\{g'\mathcal{A}_\alpha + gB_\alpha + g_S G_\alpha, g'\partial_\beta\mathcal{A}_\mu + g\partial_\beta B_\mu + g_S\partial_\beta G_\mu \quad (3.38)\\ & + g' f_{\beta\mu} + gF^L_{\beta\mu} + g_S F^S_{\beta\mu}\}.\end{aligned}$$

Having found the necessary transformations of the field from the noncommutative space to the commutative space, now we can write the action in noncommutative space. The action of the noncommutative electro-weak Standard Model reads

$$S_{NCSM} = S_{Matter,leptons} + S_{Matter,quarks} + S_{Gauge} + S_{Higgs} + S_{Yuk.} \quad (3.39)$$

The action is written separately for simplicity. Firstly, we will consider the fermions (leptons and quarks). The fermionic matter part is

$$S_{Matter,ferm.} = \int d^4x \left(\sum_f \overline{\widehat{\Psi}}_{fL} \star i\gamma^\mu D_\mu \widehat{\Psi}_{fL} + \sum_f \overline{\widehat{\Psi}}_{fR} \star i\gamma^\mu D_\mu \widehat{\Psi}_{fR} \right) \quad (3.40)$$

where $\widehat{\Psi}^{(f)}_L$ denotes the left-handed $SU(2)$ doublets $\widehat{\Psi}^{(f)}_R$ the right-handed $SU(2)$ singlets and the index f runs over the three flavors. We thus have:

$$\Psi^{(1)}_L = \begin{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ \begin{pmatrix} u^r_L \\ d^r_L \end{pmatrix} \\ \begin{pmatrix} u^y_L \\ d^y_L \end{pmatrix} \\ \begin{pmatrix} u^b_L \\ d^b_L \end{pmatrix} \end{pmatrix}, \Psi^{(1)}_R = \left(e_R;\ u^r_R;\ d^r_R;\ u^y_R;\ d^y_R;\ u^b_R;\ d^b_R \right) \quad (3.41)$$

for the first generation.

If we rewrite the equation (3.40) in terms of the left-handed and right-handed fields up to the leading order, we get

$$\begin{aligned} S_{Matter,ferm.} &= \int d^4x \Big(\sum_i \left(\bar{L}^{(i)}_L + \bar{L}^{(i)1}_L \right) \star i \left(\gamma^\mu (D^{SM}_\mu + \Gamma_\mu) \right) \star \left(L^{(i)}_L + L^{(i)1}_L \right) \\ &\quad + \sum_i \left(\bar{e}^{(i)}_R + \bar{e}^{(i)1}_R \right) \star i \left(\gamma^\mu (D^{SM}_\mu + \Gamma_\mu) \right) \star \left(e^{(i)}_R + e^{(i)1}_R \right) \Big) \quad (3.42) \end{aligned}$$

Now, we can make use of the equations (3.25), (3.28) and (3.37) to evaluate the action. The above action becomes

$$S_{Matter,ferm.} = \int d^4x \sum_i \bar{L}^{(i)}_L i\gamma^\mu D^{SM}_\mu L^{(i)}_L + \int d^4x \sum_i \bar{e}^{(i)}_R i\gamma^\mu D^{SM}_\mu e^{(i)}_R$$

$$
\begin{aligned}
&-\frac{1}{4}\Theta^{\mu\nu}\int d^4x\sum_i \bar{L}_L^{(i)}(g'f_{\mu\nu}+gF^L_{\mu\nu})i\gamma^\mu D_\mu^{SM}L_L^{(i)}\\
&-\frac{1}{2}\Theta^{\mu\nu}\int d^4x\sum_i \bar{L}_L^{(i)}\gamma^\alpha(g'f_{\alpha\mu}+gF^L_{\alpha\mu})iD_\nu^{SM}L_L^{(i)}\\
&-\frac{1}{4}\Theta^{\mu\nu}\int d^4x\sum_i \bar{e}_R^{(i)}g'f_{\mu\nu}i\gamma^\mu D_\mu^{SM}e_R^{(i)}\\
&-\frac{1}{2}\Theta^{\mu\nu}\int d^4x\sum_i \bar{e}_R^{(i)}\gamma^\alpha g'f_{\alpha\mu}iD_\nu^{SM}e_R^{(i)}+\mathbf{O}(\Theta^2). \qquad (3.43)
\end{aligned}
$$

Because of the quark contamination of the electro-weak theory, let us write the matter-quark interaction part of the theory. It is given by

$$
\begin{aligned}
S_{Matter,quarks} &= \int d^4x\Big(\sum_i\left(\bar{Q}_L^{(i)}+\bar{Q}_L^{(i)1}\right)\star i\left(\gamma^\mu(D_\mu^{SM}+\Gamma_\mu)\right)\star\left(Q_L^{(i)}+Q_L^{(i)1}\right)\\
&+\sum_i\left(\bar{u}_R^{(i)}+\bar{u}_R^{(i)1}\right)\star i\left(\gamma^\mu(D_\mu^{SM}+\Gamma_\mu)\right)\star\left(u_R^{(i)}+u_R^{(i)1}\right)\Big)\\
&+\sum_i\left(\bar{d}_R^{(i)}+\bar{d}_R^{(i)1}\right)\star i\left(\gamma^\mu(D_\mu^{SM}+\Gamma_\mu)\right)\star\left(d_R^{(i)}+d_R^{(i)1}\right)+\mathbf{O}(\Theta^2)\\
&= \int d^4x\sum_i \bar{Q}_L^{(i)}i\gamma^\mu D_\mu^{SM}Q_L^{(i)}\\
&-\frac{1}{4}\Theta^{\mu\nu}\int d^4x\sum_i \bar{Q}_L^{(i)}(g'f_{\mu\nu}+gF^L_{\mu\nu}+g_SF^S_{\mu\nu})i\gamma^\alpha D_\alpha^{SM}Q_L^{(i)}\\
&-\frac{1}{2}\Theta^{\mu\nu}\int d^4x\sum_i \bar{Q}_L^{(i)}\gamma^\alpha(g'f_{\alpha\mu}+gF^L_{\alpha\mu}+g_SF^S_{\alpha\mu})iD_\nu^{SM}Q_L^{(i)}\\
&+\int d^4x\sum_i \bar{u}_R^{(i)}i\gamma^\mu D_\mu^{SM}u_R^{(i)}\\
&-\frac{1}{4}\Theta^{\mu\nu}\int d^4x\sum_i \bar{u}_R^{(i)}\left(g'f_{\mu\nu}+g_SF^S_{\mu\nu}\right)i\gamma^\alpha D_\alpha^{SM}u_R^{(i)}\\
&-\frac{1}{2}\Theta^{\mu\nu}\int d^4x\sum_i \bar{u}_R^{(i)}\gamma^\alpha\left(g'f_{\alpha\mu}+g_SF^S_{\mu\nu}\right)iD_\nu^{SM}u_R^{(i)}\\
&+\int d^4x\sum_i \bar{d}_R^{(i)}i\gamma^\mu D_\mu^{SM}d_R^{(i)}\\
&-\frac{1}{4}\Theta^{\mu\nu}\int d^4x\sum_i \bar{d}_R^{(i)}\left(g'f_{\mu\nu}+g_SF^S_{\mu\nu}\right)i\gamma^\alpha D_\alpha^{SM}d_R^{(i)}\\
&-\frac{1}{2}\Theta^{\mu\nu}\int d^4x\sum_i \bar{d}_R^{(i)}\gamma^\alpha\left(g'f_{\alpha\mu}+g_SF^S_{\mu\nu}\right)iD_\nu^{SM}d_R^{(i)}\\
&+\mathbf{O}(\Theta^2). \qquad (3.44)
\end{aligned}
$$

In the last step, we used the equations (3.29), (3.31) and (3.37).

The commutative Standard Model is recovered at zeroth order, but some new interactions appear in the theory.

The most striking feature comes from the gauge part. In the standard theory, nothing contributes to the Feynman rules in the gauge part of the action. But, this is not the case in noncommutative Standard theory as can be seen in the following. There are point-like interactions between gluons, electro-weak bosons and quarks. The gauge part of the action reads

$$\begin{aligned}
S_{gauge} &= -\int d^4x \frac{1}{2g'} \mathbf{tr_1} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} \\
&\quad - \int d^4x \frac{1}{2g} \mathbf{tr_2} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} - \int d^4x \frac{1}{2g_S} \mathbf{tr_3} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} \\
&= -\frac{1}{4} \int d^4x \; f_{\mu\nu} f^{\mu\nu} \\
&\quad -\frac{1}{2} \operatorname{Tr} \int d^4x \; F^L_{\mu\nu} F^{L\mu\nu} - g \; \Theta^{\mu\nu} \operatorname{Tr} \int d^4x \; F^L_{\mu\rho} F^L_{\nu\sigma} F^{L\rho\sigma} \\
&\quad -\frac{1}{2} \operatorname{Tr} \int d^4x \; F^S_{\mu\nu} F^{S\mu\nu} + \frac{1}{4} g_S \; \Theta^{\mu\nu} \operatorname{Tr} \int d^4x \; F^S_{\mu\nu} F^S_{\rho\sigma} F^{S\rho\sigma} \\
&\quad -g_S \; \Theta^{\mu\nu} \operatorname{Tr} \int d^4x \; F^S_{\mu\rho} F^S_{\nu\sigma} F^{S\rho\sigma} + \mathbf{O}(\Theta^2).
\end{aligned} \tag{3.45}$$

The coefficients of the triple vertex in the $U(1)$ sector are also different from plain NCQED with a single electron. These coefficients depend on the representation we are choosing for the Y in the kinetic terms. For the simple choice that we have taken $\mathbf{tr_1} Y^3 = 0$ and this coefficient is zero. Note that a term

$$+\frac{1}{4} g \; \Theta^{\mu\nu} \operatorname{Tr} \int d^4x \; F^L_{\mu\nu} F^L_{\rho\sigma} F^{L\rho\sigma} \tag{3.46}$$

vanishes, the trace over the three Pauli matrices yields $2i\epsilon^{abc}$ and the sum $\epsilon^{abc} F^{bL}_{\rho\sigma} F^{cL\rho\sigma}$ vanishes. Note that because the trace over $(\tau^3)^3$ vanishes, there is also no cubic self-interaction term for the electromagnetic photon coming from the $SU(2)$ sector. Limits on noncommutative QED found from triple photon self-interactions do therefore not apply for the minimal noncommutative Standard Model.

As in the usual commutative Standard Model, the Higgs mechanism can be applied to break the $SU(2)_L \times U(1)_Y$ gauge symmetry and thus to generate masses for the electro-weak gauge bosons. The noncommutative action for a scalar field ϕ in the fundamental representation of $SU(2)_L$ and with the hypercharge $Y = 1/2$ reads:

$$\begin{aligned} S_{Higgs} &= \int d^4x \Big(\rho_0 \left(D_\mu \widehat{\Phi} \right)^\dagger \star \rho_0 \left(D^\mu \widehat{\Phi} \right) - \mu^2 \\ &\quad \rho_0(\widehat{\Phi})^\dagger \star \rho_0(\widehat{\Phi}) - \lambda(\rho_0(\widehat{\Phi})^\dagger \star \rho_0(\widehat{\Phi})) \star (\rho_0(\widehat{\Phi})^\dagger \star \rho_0(\widehat{\Phi})) \Big) \end{aligned} \tag{3.47}$$

In the leading order of the expansion in Θ, we obtain:

$$\begin{aligned} S_{Higgs} &= \int d^4x \Big((D_\mu^{SM}\phi)^\dagger D^{SM\mu}\phi - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)(\phi^\dagger\phi) \Big) \\ &\quad + \int d^4x \Big((D_\mu^{SM}\phi)^\dagger \left(D^{SM\mu}\rho_0(\phi^1) + \frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha V^\mu \partial_\beta \phi + \Gamma^\mu \phi \right) \\ &\quad + \left(D_\mu^{SM}\rho_0(\phi^1) + \frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha V_\mu \partial_\beta\phi + \Gamma_\mu\phi \right)^\dagger D^{SM\mu}\phi \\ &\quad + \frac{1}{4}\mu^2\Theta^{\mu\nu}\phi^\dagger(g' f_{\mu\nu} + gF^L_{\mu\nu})\phi - \lambda i \Theta^{\alpha\beta}\phi^\dagger\phi(D_\alpha^{SM}\phi)^\dagger(D_\beta^{SM}\phi) \Big) \\ &\quad + \mathbf{O}(\Theta^2), \end{aligned} \tag{3.48}$$

with

$$\Gamma_\mu = -iV_\mu^1 = i\frac{1}{4}\Theta^{\alpha\beta}\{g'\mathcal{A}_\alpha + gB_\alpha, g'\partial_\beta\mathcal{A}_\mu + g\partial_\beta B_\mu + g' f_{\beta\mu} + gF^L_{\beta\mu}\} \tag{3.49}$$

and

$$\rho_0(\hat{\Phi}) = \phi + \rho_0(\phi^1) + \mathcal{O}(\times^\in), \tag{3.50}$$

where

$$\rho_0(\phi^1) = -\frac{\Theta^{\alpha\beta}}{2}(g'\mathcal{A}_\alpha + gB_\alpha)\partial_\beta\phi + \frac{i\Theta^{\alpha\beta}}{4}(g'\mathcal{A}_\alpha + gB_\alpha)(g'\mathcal{A}_\beta + gB_\beta)\phi \quad (3.51)$$

The Yukawa couplings can then generate masses for the fermions, one has:

$$\begin{aligned} S_{Yukawa} &= \int d^4x \Big(- \sum_{i,j=1}^{3} W^{ij} \Big((\bar{\widehat{L}}_L^{(i)} \star \rho_L(\widehat{\Phi})) \star \widehat{e}_R^{(j)} + \bar{\widehat{e}}_R^{(i)} \star (\rho_L(\widehat{\Phi})^\dagger \star \widehat{L}_L^{(j)}) \Big) \\ &- \sum_{i,j=1}^{3} G_u^{ij} \Big((\bar{\widehat{Q}}_L^{(i)} \star \rho_{\bar{Q}}(\bar{\widehat{\Phi}})) \star \widehat{u}_R^{(j)} + \bar{\widehat{u}}_R^{(i)} \star (\rho_{\bar{Q}}(\bar{\widehat{\Phi}})^\dagger \star \widehat{Q}_L^{(j)}) \Big) \\ &- \sum_{i,j=1}^{3} G_d^{ij} \Big((\bar{\widehat{Q}}_L^{(i)} \star \rho_Q(\widehat{\Phi})) \star \widehat{d}_R^{(j)} + \bar{\widehat{d}}_R^{(i)} \star (\rho_Q(\widehat{\Phi})^\dagger \star \widehat{Q}_L^{(j)}) \Big) \Big). \end{aligned} \quad (3.52)$$

The sum runs over the different generations. The leading order expansion is

$$\begin{aligned} S_{Yukawa} &= S_{Yukawa}^{SM} - \int d^4x \Big(\sum_{i,j=1}^{3} W^{ij} \Big((\bar{L}_L^i \phi) e_R^{1j} + (\bar{L}_L^i \rho_L(\phi^1)) e_R^j \\ &+ (\bar{L}_L^{1i}\phi) e_R^j + i\frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha L_L^i \partial_\beta \phi e_R^j + \bar{e}_R^i(\phi^\dagger L_L^{1j}) \\ &+ \bar{e}_R^i(\rho_L(\phi^1)^\dagger L_L^j) + \bar{e}_R^{1i}(\phi^\dagger L_L^j) + i\frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha e_R^i \partial_\beta \phi^\dagger L_L^j \Big) \\ &- \sum_{i,j=1}^{3} G_u^{ij} \Big((\bar{Q}_L^i \bar{\phi}) u_R^{1j} + (\bar{Q}_L^i \rho_{\bar{Q}}(\bar{\phi}^1)) u_R^j + (\bar{Q}_L^{1i}\bar{\phi}) u_R^j \\ &+ i\frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha Q_L^i \partial_\beta \bar{\phi} u_R^j + \bar{u}_R^i(\bar{\phi}^\dagger Q_L^{1j}) + \bar{u}_R^i(\rho_{\bar{Q}}(\bar{\phi}^1)^\dagger Q_L^j) \\ &+ \bar{u}_R^{1i}(\bar{\phi}^\dagger Q_L^j) + i\frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha u_R^i \partial_\beta \bar{\phi}^\dagger Q_L^j \Big) \\ &- \sum_{i,j=1}^{3} G_d^{ij} \Big((\bar{Q}_L^i \phi) d_R^{1j} + (\bar{Q}_L^i \rho_Q(\phi^1)) d_R^j + (\bar{Q}_L^{1i}\phi) d_R^j \\ &+ i\frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha Q_L^i \partial_\beta \phi d_R^j + \bar{d}_R^i(\phi^\dagger Q_L^{1j}) + \bar{d}_R^i(\rho_Q(\phi^1)^\dagger Q_L^j) \\ &+ \bar{d}_R^{1i}(\phi^\dagger Q_L^j) + i\frac{1}{2}\Theta^{\alpha\beta}\partial_\alpha \bar{d}_R^i \partial_\beta \phi^\dagger Q_L^j \Big) \Big) + \mathbf{O}(\Theta^2), \end{aligned} \quad (3.53)$$

where L_L^i stands for a left-handed leptonic doublet of the ith generation, e_R^i for a leptonic singlet of the ith generation, Q_L^i for a left-handed quark doublet of the ith generation, u_R^i for a right-handed up-type quark singlet of the ith and d_R^i stands for a right-handed down-type quark singlet of the ith generation. We used

$$\rho(\Phi) = \phi + \rho(\phi^1) + \mathcal{O}(\times^{\in}), \tag{3.54}$$

where ρ stands for ρ_L, ρ_Q and $\rho_{\bar{Q}}$, respectively. Once again we recover the Standard Model, but some new interactions arise. The Yukawa coupling matrices can be diagonalized using biunitary transformations. We thus obtain a Cabibbo Kobayashi Maskawa matrix in the charged currents, as in the Standard Model and as long as right-handed neutrinos are absent, we do not predict lepton flavor changing currents. In the next paragraph, we will present the Lagrangian for the charged and the neutral currents. Clearly, flavor physics is much richer than in the Standard Model on a commutative space.

3.4 Currents

In this section, the electro-weak currents in the leading order of the expansion in Θ is studied.

3.4.1 Charged Currents

Firstly, let us study the electro-weak charged currents. Let

$$L_1 = \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L \qquad L_2 = \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L .$$

The Lagrangian density

$$\mathcal{L} = \bar{L_1} \quad V_{CKM} J_1 \quad L2 + \bar{L_2} \quad V_{CKM}^{\dagger} J_2 \quad L1, \tag{3.55}$$

with

$$\begin{aligned} J_1 = {} & \frac{1}{\sqrt{2}} g W^+ + (\frac{1}{2}\Theta^{\mu\nu}\gamma^\alpha + \Theta^{\nu\alpha}\gamma^\mu) \qquad (3.56) \\ & \Bigg(\Big(-\frac{\sqrt{2}}{4} Y g' g (\cos\Theta_W \partial_\mu A_\nu - \cos\Theta_W \partial_\nu A_\mu - \sin\Theta_W \partial_\mu Z_\nu + \sin\Theta_W \partial_\nu Z_\mu) W_\alpha^+ \Big) \\ & + g\frac{\sqrt{2}}{8} \Big(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ \\ & -2ig \left(\cos\Theta_W Z_\mu W_\nu^+ + \sin\Theta_W A_\mu W_\nu^+ - \cos\Theta_W W_\mu^+ Z_\nu - \sin\Theta_W W_\mu^+ A_\nu \right) \Big) \cdot \\ & (-2i\partial_\alpha + 2Yg' \sin\Theta_W Z_\alpha - 2Yg' \cos\Theta_W A_\alpha + g\cos\Theta_W Z_\alpha + g\sin\Theta_W A_\alpha) \\ & -\frac{\sqrt{2}}{8} g^2 \Big(\cos\Theta_W \partial_\mu Z_\nu - \cos\Theta_W \partial_\nu Z_\mu + \sin\Theta_W \partial_\mu A_\nu - \sin\Theta_W \partial_\nu A_\mu \\ & -2ig(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) \Big) W_\alpha^+ \Bigg) \end{aligned}$$

and

$$\begin{aligned} J_2 = {} & \frac{1}{\sqrt{2}} g W^- + (\frac{1}{2}\Theta^{\mu\nu}\gamma^\alpha + \Theta^{\nu\alpha}\gamma^\mu) \qquad (3.57) \\ & \Bigg(\Big(-\frac{\sqrt{2}}{4} Y g' g (\cos\Theta_W \partial_\mu A_\nu - \cos\Theta_W \partial_\nu A_\mu - \sin\Theta_W \partial_\mu Z_\nu + \sin\Theta_W \partial_\nu Z_\mu) W_\alpha^- \Big) \\ & + g\frac{\sqrt{2}}{8} \Big(\partial_\mu W_\nu^- - \partial_\nu W_\mu^- \\ & -2ig \left(\cos\Theta_W W_\mu^- Z_\nu + \sin\Theta_W W_\mu^- A_\nu - \cos\Theta_W Z_\mu W_\nu^- - \sin\Theta_W A_\mu W_\nu^- \right) \Big) \cdot \\ & (-2i\partial_\alpha + 2Yg' \sin\Theta_W Z_\alpha - 2Yg' \cos\Theta_W A_\alpha - g\cos\Theta_W Z_\alpha - g\sin\Theta_W A_\alpha) \\ & -\frac{\sqrt{2}}{8} g^2 \Big(\cos\Theta_W \partial_\mu Z_\nu - \cos\Theta_W \partial_\nu Z_\mu + \sin\Theta_W \partial_\mu A_\nu - \sin\Theta_W \partial_\nu A_\mu \\ & -2ig(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-) \Big) W_\alpha^- \Bigg). \end{aligned}$$

Note that $g' \cos\Theta_W = g \sin\Theta_W$.

3.4.2 Neutral currents

In this subsection, the neutral current in the leading order of the expansion in Θ is studied.

$$
\begin{aligned}
\mathcal{L}_{nc} \;=\;& \mathcal{L}_{nc}^{SM} - i\frac{1}{2}\sum_i \bar{u}_L^{(i)}\left(\frac{1}{2}\Theta^{\mu\nu}\gamma^\alpha + \Theta^{\nu\alpha}\gamma^\mu\right) \qquad (3.58)\\
& \bigg(\Big(\cos\Theta_W\partial_\mu A_\nu - \cos\Theta_W\partial_\nu A_\mu - \sin\Theta_W\partial_\mu Z_\nu + \sin\Theta_W\partial_\nu Z_\mu\Big)\\
& \Big(g'Y\partial_\alpha - iY^2g'^2\cos\Theta_W A_\alpha + iY^2g'^2\sin\Theta_W Z_\alpha - \frac{iYg'g}{2}\cos\Theta_W Z_\alpha\\
& \frac{iYg'g}{2}\sin\Theta_W A_\alpha\Big) + \frac{1}{2}\Big(\cos\Theta_W\partial_\mu Z_\nu - \cos\Theta_W\partial_\nu Z_\mu + \sin\Theta_W\partial_\mu A_\nu\\
& - \sin\Theta_W\partial_\nu A_\mu - 2ig(W_\mu^+W_\nu^- - W_\nu^+W_\mu^-)\Big)\Big(g\partial_\alpha - iYg'g\cos\Theta_W A_\alpha\\
& + iYg'g\cos\Theta_W Z_\alpha - \frac{1}{2}ig^2\cos\Theta_W Z_\alpha - \frac{1}{2}ig^2\sin\Theta_W A_\alpha\Big)\\
& -\frac{i}{2}g^2\Big(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ - 2ig(\cos\Theta_W Z_\mu W\\
& +_\nu + \sin\Theta_W A_\mu W_\nu^+ - W_\mu^+\cos\Theta_W Z_\nu - W_\mu^+\sin\Theta_W A_\nu)\Big)\\
& W_\alpha^-\bigg)u_L^{(i)} - i\frac{1}{2}\sum_i \bar{u}_R^{(i)}\left(\frac{1}{2}\Theta^{\mu\nu}\gamma^\alpha + \Theta^{\nu\alpha}\gamma^\mu\right)\\
& \bigg(\Big(\cos\Theta_W\partial_\mu A_\nu - \cos\Theta_W\partial_\nu A_\mu - \sin\Theta_W\partial_\mu Z_\nu + \sin\Theta_W\partial_\nu Z_\mu\Big)\\
& \Big(g'Y\partial_\alpha - iY^2g'^2\cos\Theta_W A_\alpha + iY^2g'^2\sin\Theta_W Z_\alpha\Big)\bigg)u_R^{(i)}\\
& -i\frac{1}{2}\sum_i \bar{d}_L^{(i)}\left(\frac{1}{2}\Theta^{\mu\nu}\gamma^\alpha + \Theta^{\nu\alpha}\gamma^\mu\right)\\
& \bigg(\Big(\cos\Theta_W\partial_\mu A_\nu - \cos\Theta_W\partial_\nu A_\mu - \sin\Theta_W\partial_\mu Z_\nu + \sin\Theta_W\partial_\nu Z_\mu\Big)\\
& \Big(g'Y\partial_\alpha - iY^2g'^2\cos\Theta_W A_\alpha + iY^2g'^2\sin\Theta_W Z_\alpha - i\frac{1}{2}Yg'g\cos\Theta_W Z_\alpha\\
& -i\frac{1}{2}Yg'g\sin\Theta_W A_\alpha\Big) - \frac{1}{2}\Big(\cos\Theta_W\partial_\mu Z_\nu - \cos\Theta_W\partial_\nu Z_\mu + \sin\Theta_W\partial_\mu A_\nu\\
& - \sin\Theta_W\partial_\nu A_\mu - 2ig(W_\mu^+W_\nu^- - W_\nu^+W_\mu^-)\Big)\Big(g\partial_\alpha - iYg'g\cos\Theta_W A_\alpha\\
& + iYg'g\cos\Theta_W Z_\alpha + \frac{1}{2}ig^2\cos\Theta_W Z_\alpha + \frac{1}{2}ig^2\sin\Theta_W A_\alpha\Big)
\end{aligned}
$$

$$-\frac{i}{2}g^2\Big(\partial_\mu W_\nu^- - \partial_\nu W_\mu^- + 2ig$$

$$\big(\cos\Theta_W Z_\mu W_\nu^- + \sin\Theta_W A_\mu W_\nu^- - W_\mu^- \cos\Theta_W Z_\nu - W_\mu^- \sin\Theta_W A_\nu\big)\Big)$$

$$W_\alpha^+\Big) d_L^{(i)} - i\frac{1}{2}\sum_i \bar{d}_R^{(i)}\left(\frac{1}{2}\Theta^{\mu\nu}\gamma^\alpha + \Theta^{\nu\alpha}\gamma^\mu\right)$$

$$\Big(\Big(\cos\Theta_W \partial_\mu A_\nu - \cos\Theta_W \partial_\nu A_\mu - \sin\Theta_W \partial_\mu Z_\nu + \sin\Theta_W \partial_\nu Z_\mu\Big)$$

$$\Big(g' Y \partial_\alpha - iY^2 g'^2 \cos\Theta_W A_\alpha + iY^2 g'^2 \sin\Theta_W Z_\alpha\Big)\Big) d_R^{(i)}.$$

3.5 Noncommutative Quantum Chromodynamics

In this section, we will investigate the noncommutative quantum chrodynamics, specifically. Here, we derive the new Feynman rules which absent in the standard model. We will follow the method introduced by Carlson-Carone-Lebed [36]. The noncommutative SU (N) gauge transformation is defined as

$$\delta_\alpha \psi = i\Lambda_\alpha \star \psi, \delta_\alpha \tag{3.59}$$

$$A_\mu = \partial_\mu \Lambda_\alpha + i[\Lambda_\alpha \overset{\star}{,} A_\mu]. \tag{3.60}$$

Here, Λ_α is a U(N) matrix function that is associated with an element of SU(N) corresponding to the gauge parameter α. The appropriate consistency condition is

$$(\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi(x) = \delta_{\alpha\times\beta}\psi(x), \tag{3.61}$$

where $\alpha \times \beta$ represents $\alpha_a \beta_b f^{abc} T^c$, with f^{abc} and T^a the structure constants and generators of SU(N), respectively. The above equations yield

$$\Lambda_\alpha[A^0] = \alpha + \frac{1}{4}\Theta^{\mu\nu}\{\partial_\mu \alpha\ ,\ A_\nu^0\} + O(\Theta^2). \tag{3.62}$$

Then the gauge field transform as

$$A^\mu = A^{0\mu} - \frac{1}{4}\Theta_{\rho\nu}\{A^{0\rho},\, \partial^\nu A^{0\mu} + F^{0\nu\mu}\} \tag{3.63}$$

and the matter field transform as

$$\psi = \psi^0 - \frac{1}{2}\Theta^{\mu\nu}A^0_\mu \partial_\nu \psi^0 + \frac{i}{4}\Theta^{\mu\nu}A^0_\mu A^0_\nu \psi^0, \tag{3.64}$$

to linear order in Θ. While A^0 and ψ^0 have the usual transformation properties of fields in an SU(N) gauge theory, the Lagrangian expressed in terms of these fields is different. The action

$$S = \int d^4x \left[\bar\psi \star (i\not{\mathcal{D}} - m)\psi - \frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}\star F^{\mu\nu}\right], \tag{3.65}$$

in which

$$\mathcal{D}_\mu \psi \equiv \partial_\mu \psi - iA_\mu \star \psi \ ,\ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu], \tag{3.66}$$

one may expand the action in terms of ψ^0, A^0_μ, and Θ:

$$S = \int d^4x \Big[\bar\psi^0(i\not{\mathcal{D}} - m)\psi^0 - \frac{1}{4}\ \Theta^{\mu\nu}\bar\psi^0 F^0_{\mu\nu}(i\not{\mathcal{D}} - m)\psi^0 - \frac{1}{2}\ \Theta^{\mu\nu}\bar\psi^0\gamma^\rho F^0_{\rho\mu}\ i\mathcal{D}_\nu\psi^0$$

$$-\frac{1}{2g^2}\ \mathrm{Tr}\ F^0_{\mu\nu}F^{0\mu\nu} + \frac{1}{4g^2}\ \Theta^{\mu\nu}\ \mathrm{Tr}\ F^0_{\mu\nu}F^0_{\rho\sigma}F^{0\rho\sigma} - \frac{1}{g^2}\ \Theta^{\mu\nu}\ \mathrm{Tr}\ F^0_{\mu\rho}F^0_{\nu\sigma}F^{0\rho\sigma}\Big]. \tag{3.67}$$

This action is written up to order Θ. D_μ and $F_{\mu\nu}$ are given as usual.

$$\mathcal{D}_\mu\psi^0 \equiv \partial_\mu\psi^0 - iA^0_\mu\psi^0 \quad , \quad F^0_{\mu\nu} \equiv \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - i[A^0_\mu, A^0_\nu]. \tag{3.68}$$

Feynman rules may be extracted from the above action. The structure constants for SU(N) are defined by

$$[T^a, T^b] = if^{abc}\ T^c \qquad \text{and} \qquad \{T^a, T^b\} = d^{abc}T^c + \frac{1}{N}\delta^{ab}. \tag{3.69}$$

In addition, we will use the definition for the contractions: $\Theta^{\mu} \cdot p \equiv \Theta^{\mu\nu} p_{\nu}$ and $p \cdot \Theta \cdot q \equiv \Theta^{\mu\nu} p_{\mu} q_{\nu}$, for any four-vectors p, q. Finally, the totally antisymmetric tensor is introduced

$$\Theta^{\mu\nu\rho} \equiv \Theta^{\mu\nu}\gamma^{\rho} + \Theta^{\nu\rho}\gamma^{\mu} + \Theta^{\rho\mu}\gamma^{\nu}. \tag{3.70}$$

The Feynman rules for the $O(\Theta^1)$ contributions are found as [36]

qqg vertex (i):

$$\frac{g}{2} T^a \left[\Theta^{\mu} \cdot p \; (\not{p}' - \; m) - \Theta^{\mu} \cdot p' (\not{p} + \; m) - p' \cdot \Theta \cdot p \; \gamma^{\mu} \right] \tag{3.71}$$

$qqgg$ vertex (ii):

$$\frac{g^2}{2} \left\{ T^a T^b [m \; \Theta^{\mu\nu} + \Theta^{\mu\nu\rho}(p+q)_{\rho}] - T^b T^a [m \; \Theta^{\mu\nu} + \Theta^{\mu\nu\rho}(p+r)_{\rho}] \right\} \tag{3.72}$$

ggg vertex (iii):

$$\begin{aligned}
-\frac{1}{2} g d_{abc} \{ r \cdot \Theta \cdot q \, [(q-r)^{\mu} g^{\nu\rho} + (p-q)^{\rho} g^{\mu\nu} + (r-p)^{\nu} g^{\mu\rho}] \, . \\
+(q^2 g^{\rho\nu} - q^{\rho} q^{\nu}) \Theta^{\mu} \cdot r + (r^2 g^{\rho\nu} - r^{\rho} r^{\nu}) \Theta^{\mu} \cdot q + (q^2 g^{\mu\nu} - q^{\mu} q^{\nu}) \Theta^{\rho} \cdot p \\
+(p^2 g^{\mu\nu} - p^{\mu} p^{\nu}) \Theta^{\rho} \cdot q + (r^2 g^{\mu\rho} - r^{\mu} r^{\rho}) \Theta^{\nu} \cdot p + (p^2 g^{\mu\rho} - p^{\mu} p^{\rho}) \Theta^{\nu} \cdot r \\
+(q \cdot p \; r^{\nu} - r \cdot q \; p^{\nu}) \Theta^{\mu\rho} + (r \cdot q \; p^{\rho} - p \cdot r \; q^{\rho}) \Theta^{\nu\mu} + \\
(p \cdot r \; q^{\mu} - q \cdot p \; r^{\mu}) \Theta^{\rho\nu} \}
\end{aligned} \tag{3.73}$$

$gggg$ vertex (iv):

$$\begin{aligned}
-i \frac{g^2}{2} f^{abe} d^{cde} \{ \Theta^{\mu\nu} (g^{\rho\sigma} r \cdot s - r^{\sigma} s^{\rho}) + \Theta^{\rho\sigma} (r^{\nu} s^{\mu} - r^{\mu} s^{\nu}) \\
-\Theta^{\mu\rho} (g^{\nu\sigma} r \cdot s - r^{\sigma} s^{\nu}) - \Theta^{\mu\sigma} (g^{\nu\rho} r \cdot s - r^{\nu} s^{\rho}) + \Theta^{\nu\rho} (g^{\mu\sigma} r \cdot s - r^{\sigma} s^{\mu}) \\
+\Theta^{\nu\sigma} (g^{\mu\rho} r \cdot s - r^{\mu} s^{\rho}) + \Theta^{\mu} \cdot r \; (s^{\nu} g^{\rho\sigma} - s^{\rho} g^{\nu\sigma}) - \Theta^{\mu} \cdot s \; (r^{\nu} g^{\rho\sigma} - r^{\sigma} g^{\nu\rho})
\end{aligned}$$

$$+\Theta^{\nu}\cdot r\ (s^{\mu}g^{\rho\sigma}-s^{\rho}g^{\mu\sigma})+\Theta^{\nu}\cdot s\ (r^{\mu}g^{\rho\sigma}-r^{\sigma}g^{\mu\rho})+\Theta^{\rho}\cdot r\ (s^{\nu}g^{\mu\sigma}$$

$$-s^{\mu}g^{\nu\sigma})+\Theta^{\rho}\cdot s\ (r^{\mu}g^{\nu\sigma}-r^{\nu}g^{\mu\sigma})+\Theta^{\sigma}\cdot r\ (s^{\mu}g^{\nu\rho}-s^{\nu}g^{\mu\rho})$$

$$+\Theta^{\sigma}\cdot s\ (r^{\nu}g^{\mu\rho}-r^{\mu}g^{\nu\rho})+r\cdot\Theta\cdot s\ (g^{\nu\rho}g^{\mu\sigma}-g^{\mu\rho}g^{\nu\sigma})\}+$$

$$[(\mu,p,a)\leftrightarrow(\sigma,s,d)]+[(\rho,r,c)\leftrightarrow(\mu,p,a)]+[(\sigma,s,d)\leftrightarrow(\nu,q,b)]$$

$$+[(\rho,r,c)\leftrightarrow(\nu,q,b)]+[(\rho,r,c)\leftrightarrow(\nu,q,b)]\,[(\sigma,s,d)\leftrightarrow(\mu,p,a)] \tag{3.74}$$

It is immediately seen that there exists a new quark-quark-gluon-gluon interaction in the noncommutative QCD.

3.6 Application of Noncommutative QED

In terms of ordinary products, the noncommutative quantum electromagnetic presents the following action from which the Feynman rules can be derived

$$S=\int d^4x(-\frac{1}{4\pi}F^{\mu\nu}F_{\mu\nu}+i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi-e\exp(ip_1\Theta p_2/2)\gamma^{\mu}A_{\mu}-m\bar{\Psi}\Psi), \tag{3.75}$$

where $F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}+2e\sin(p_1\Theta p_2/2)A^{\mu}A^{\nu}$.

The Feynman Rules of the noncommutative QED are as follows:

for the $ee\gamma$ vertex;

$$\Gamma_{\mu}=ie\gamma_{\mu}\exp\left(ip_1\wedge p_2/2\right). \tag{3.76}$$

and $\gamma\gamma\gamma$ vertex factor is given by

$$\Gamma_{\mu\nu\rho}(\gamma\gamma\gamma)=$$

$$2e\sin(p_1\wedge p_2/2)[(p_1-p_2)_{\rho}\ g_{\mu\nu}+(p_2-p_3)_{\mu}\ g_{\nu\rho}+(p_3-p_1)_{\nu}\ g_{\mu\rho}]. \tag{3.77}$$

In the following; as an application, we will apply the Feynman rules in non-commutative quantum electrodynamics for Moller scattering.

3.6.1 Moller Scattering

We will use the same definition used by Hewett-Petriello-Rizzo [64] for the momenta of the incoming, represented by $p_{1,2}$, and outgoing, corresponding to $k_{1,2}$, particles in terms of the coordinates fixed in the laboratory as

$$\begin{aligned} p_1^\mu &= \tfrac{\sqrt{s}}{2}(1,-1,0,0) & p_2^\mu &= \tfrac{\sqrt{s}}{2}(1,1,0,0) \\ k_1^\mu &= \tfrac{\sqrt{s}}{2}(1,-c_\Theta,-s_\Theta c_\phi,-s_\Theta s_\phi) & k_2^\mu &= \tfrac{\sqrt{s}}{2}(1,c_\Theta,s_\Theta c_\phi,s_\Theta s_\phi). \end{aligned} \tag{3.78}$$

Note that the ordering of the co-ordinates is given by (t, z, x, y), so that the z-axis is along the beam direction as usual. $\Theta_{\mu\nu} = \frac{1}{\Lambda_{NC}^2} C_{\mu\nu}$

$$C_{\mu\nu} = \begin{pmatrix} 0 & \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \\ -\sin\alpha\cos\beta & 0 & \cos\gamma & -\sin\gamma\sin\beta \\ -\sin\alpha\sin\beta & -\cos\gamma & 0 & -\sin\gamma\cos\beta \\ -\cos\alpha & \sin\gamma\sin\beta & \sin\gamma\cos\beta & 0 \end{pmatrix}, \tag{3.79}$$

Note that the matrix $C_{\mu\nu}$ is not a tensor since its elements are identical in all reference frames. It leads to the Lorentz violation. How α, β and γ are chosen in the matrix $C_{\mu\nu}$ determine the ways in which Lorentz violation may be manifested in experiment.

The matrix elements C_{0i} are related to the NC space–time components and are defined by the direction of the background electric field $\vec{E}$. The remaining elements C_{ij} are related to the NC space–space components and are defined the direction of the background magnetic field $\vec{B}$.

Using these definitions, the bilinear products of these momenta with the matrix $C_{\mu\nu}$ can be evaluated as

$$\begin{aligned}
p_1.C.p_2 &= \frac{s}{2}C_{01} \\
k_1.C.k_2 &= \frac{s}{2}[C_{01}c_\Theta + C_{02}s_\Theta c_\phi + C_{03}s_\Theta s_\phi] \\
p_1.C.k_1 &= \frac{s}{4}[C_{01}(1-c_\Theta) + (C_{12}-C_{02})s_\Theta c_\phi - (C_{03}+C_{31})s_\Theta s_\phi] \\
p_1.C.k_2 &= \frac{s}{4}[C_{01}(1+c_\Theta) - (C_{12}-C_{02})s_\Theta c_\phi + (C_{03}+C_{31})s_\Theta s_\phi] \\
p_2.C.k_1 &= \frac{s}{4}[-C_{01}(1+c_\Theta) - (C_{12}+C_{02})s_\Theta c_\phi - (C_{03}-C_{31})s_\Theta s_\phi] \\
p_2.C.k_2 &= \frac{s}{4}[-C_{01}(1-c_\Theta) + (C_{12}+C_{02})s_\Theta c_\phi + (C_{03}-C_{31})s_\Theta s_\phi]
\end{aligned} \tag{3.80}$$

Note that the term C_{23} vanishes in the above expressions since the z-axis is defined to be along the direction of the initial beams and there is no $\mathbf{B}$ field associated non-commutative asymmetry relative to this direction.

Addition to it, there is also possibilities of the Z-boson exchange. Interestingly, if we find the photon self energy, we should take care of the noncommutative $Z\gamma\gamma$ coupling which does not exist in the standard model. This coupling is discussed in the last chapter.

Following the Feynman rules (3.76) and (3.77), we see that the t- and u-channel exchange graphs now pick up kinematic phases given by

$$\begin{aligned}
\phi_t &= \frac{1}{2}[p_1 \cdot \Theta \cdot k_1 + p_2 \cdot \Theta \cdot k_2] \\
\phi_u &= \frac{1}{2}[p_1 \cdot \Theta \cdot k_2 + p_2 \cdot \Theta \cdot k_1].
\end{aligned} \tag{3.81}$$

Clearly, only the interference terms between the t- and u-channel diagrams pick up a relative phase when the full amplitude is squared. The phase is defined as Δ_{Moller} and we find it to be given by

$$\Delta_{Moller} = \phi_u - \phi_t = \frac{-\sqrt{ut}}{\Lambda_{NC}^2}[c_{12}c_\phi - c_{31}s_\phi]. \tag{3.82}$$

The Mandelstam variables are defined as usual: $t, u = -s(1 \mp \cos\Theta)/2$. Hence the resulting differential distributions for this process appear exactly as in the SM except that the t, u-channel interference terms should be multiplied by $\cos\Delta_{Moller}$. In the limit $\Lambda_{NC} \to \infty$, $\cos\Delta \to 1$ then the standard model is recovered.

Hewett-Petriello-Rizzo take the case $c_{12} \neq 0$ for simplicity in the numerical calculation. If instead c_{31} is non-zero, the results will be similar except for the phase of the ϕ dependence. Since it is only consider one non-vanishing value of c_{ij} at a time, we set its magnitude to unity when obtaining our results.

The differential cross section for Moller scattering in the laboratory center of mass frame can be written as [64]

$$\frac{d\sigma}{dz\ d\phi} = \frac{\alpha^2}{4s}\Big[(e_{ij}+f_{ij})(P_{ij}^{uu}+P_{ij}^{tt}+2P_{ij}^{ut}\cos\Delta_{Moller})+(e_{ij}-f_{ij})(\frac{t^2}{s^2}P_{ij}^{uu}+\frac{u^2}{s^2}P_{ij}^{tt})\Big], \tag{3.83}$$

where $z = \cos\Theta$, a sum over the gauge boson indices is implied, $e_{ij} = (v_i v_j + a_i a_j)^2$ and $f_{ij} = (v_i a_j + a_i v_j)^2$ are combinations of the electron's vector and axial vector couplings and

$$P_{ij}^{qr} = s^2 \frac{(q - m_i^2)(r - m_j^2) + \Gamma_i \Gamma_j m_i m_j}{[(q - m_i^2)^2 + (\Gamma_i m_i)^2][(r - m_j^2)^2 + (\Gamma_j m_j)^2]}, \tag{3.84}$$

with $m_i(\Gamma_i)$ being the mass (width) of the i^{th} gauge boson, where i=1(2) corresponds to the photon(Z). The expression for the differential Left-Right

Polarization asymmetry, $A_{LR}(z,\phi)$, can be easily obtained from the above by forming the ratio

$$A_{LR}(z,\phi) = N(z,\phi)/D(z,\phi), \tag{3.85}$$

where $D(z,\phi)$ is the differential cross section expression above and $N(z,\phi)$ can be obtained from $D(z,\phi)$ by the redefinition of the coupling combinations e_{ij} and f_{ij} as

$$e_{ij} = f_{ij} = (v_i v_j + a_i a_j)(v_i a_j + a_i v_j). \tag{3.86}$$

Note that the cross section is not actually invariant due to the presence of Δ_{Moller}, though it is expressed in an apparently covariant form using Mandelstam variables.

We now examine how the Moller cross section behaves as $\sqrt{s}$ grows beyond Λ_{NC}. In the SM for large s we expect the scaled cross section, ie, the product $s \cdot \sigma_{Moll}$, to be roughly constant after a cut on $|\cos\Theta|$ cut is performed. Ordinarily when new operators are introduced, the modified scaled cross section is expected to grow rapidly near the appropriate scale beyond which the contact interaction limit no longer applies. However, in the present case, the theory above the scale Λ_{NC} *is* a well-defined theory since it is not a low energy limit. We would thus anticipate that the $\cos\Delta_{Moller}$ factor leads to a modulation of the scaled cross section that averages out rapidly with a period that depends on the hardness of the $|\cos\Theta|$ cut as the value of $\sqrt{s}$ increases.

CHAPTER 4

THE $Z \to \nu\bar{\nu}\gamma$ DECAY IN THE NONCOMMUTATIVE STANDARD MODEL

Leptonic decay modes of Z-boson constitute one of the important class of the decays for checking predictions and improving parameters of the standard model. For example, one of the essential results of LEP experiments is determination number of light neutrinos from $Z \to \nu\bar{\nu}$ decay. With the Giga-Z option of the Tesla project, it is possible to produce more Z bosons [60]. This circumstance allows to determine the parameters of the standard model in more refined way. At the same time, withe the increasing Z-boson, it appears real possibility to analyze the rare decays of Z, which are absent in tree level at standard model. Moreover, the rare decays are also quite sensitive to the existence of new physics beyond the standard model. One of the possible source for the new physics is noncommutative theories.

In this chapter, we analyze the possibility of testing the noncommutative effects in rare $Z \to \nu\bar{\nu}\gamma$ decay which is forbidden at tree level in the standard model.

First of all, we derive the required Feynman rules for the rare $Z \to \nu\bar{\nu}\gamma$ decay as well as for the other three and four gauge boson interactions such as

ZZZ, $Z\gamma\gamma$, $ZZZZ$ and $ZZ\gamma\gamma$ etc. Then obtained Feynman rules are used for doing calculation of the decay $Z \to \nu\bar{\nu}\gamma$ and the amplitude for this decay is found. Then, the numerical calculations are performed to find the decay rate of this decay.

As we already noted that a simple way to introduce a noncommutative structure into space-time is to promote the usual space-time coordinates x to noncommutative coordinates $\hat{x}$ with

$$[\hat{x}_\mu, \hat{x}_\nu] = i\Theta_{\mu\nu} = \frac{i}{\Lambda_{NC}^2} C_{\mu\nu}, \tag{4.1}$$

where $\theta_{\mu\nu}$ is the real antisymmetric matrix. Note that, $\Theta^{\mu\nu}$ plays the same role as $\hbar$ does in quantum mechanics.

In the last equality, we have parameterized the effect in terms of an overall scale Λ_{NC}, which characterizes the threshold where noncommutative effects become relevant a real constant antisymmetric matrix $C_{\mu\nu}$, whose dimensionless elements are presumably of order unity. One might expect the scale Λ_{NC} to be of the order of Planck scale. However in the large extra dimension theory, where gravity becomes strong at scales of order a TeV, it is possible that NC effects could be of order a TeV. For this reason in the present work we consider the possibility that Λ_{NC} may lie not too far above the TeV scale [61, 62, 63].

In the present work we adopt Hewett–Petriello–Rizzo parametrization [64] for the matrix $C_{\mu\nu}$. The matrix $C_{\mu\nu}$ is parameterized as [64]

$$C_{\mu\nu} = \begin{pmatrix} 0 & \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \\ -\sin\alpha\cos\beta & 0 & \cos\gamma & -\sin\gamma\sin\beta \\ -\sin\alpha\sin\beta & -\cos\gamma & 0 & -\sin\gamma\cos\beta \\ -\cos\alpha & \sin\gamma\sin\beta & \sin\gamma\cos\beta & 0. \end{pmatrix}, \qquad (4.2)$$

Note that the matrix $C_{\mu\nu}$ is not a tensor since its elements are identical in all reference frames. It leads to the Lorentz violation. How α, β and γ are chosen in the matrix $C_{\mu\nu}$ determine the ways in which Lorentz violation may be manifested in experiment.

The matrix elements C_{0i} are related to the NC space–time components and are defined by the direction of the background electric field $\vec{E}$. The remaining elements C_{ij} are related to the NC space–space components and are defined the direction of the background magnetic field $\vec{B}$.

NCQFT has rich phenomenological implications due to the appearance of new interactions.

Experimental signatures of noncommutativity have been discussed by various authors [64, 65, 66, 67]. The next–generation linear colliders (NLC) are planned to operate in e^+e^-, $\gamma\gamma$ and γe modes.

Here, we consider the possibility of testing the NC effects at NLC in the $Z \to \nu\bar{\nu}\gamma$ decay which is forbidden in standard model at tree level.

This chapter is structured as follows. Firstly, we will derive the Feynman rules by starting with the action. Then, we will apply the required Feynman rules for $Z \to \nu\bar{\nu}\gamma$ decay. Having obtained the square of the amplitude, $|M^2|$

we will make numerical analysis to obtain the decay rate.

4.1 Feynman Rules

In this section, we present necessary theoretical background for the $Z \to \nu\bar{\nu}\gamma$ decay in noncommutative standard model. For calculating the matrix element, we need relevant Feynman rules. Before giving details of calculation of Feynman rules, few words are in order.

The decay processes which involve more than two vector particles are particularly interesting from the theoretical point of view in noncommutative theories. It is the place where different models show the greatest difference. In particular, there are models that do not require any triple gauge boson interaction. This depends on a choice of representation. There are, however, some models which include triple boson interaction [68, 69]. We will, in particular, follow the models introduced by Mocioiu, Pospelov and Roiban [69].

The action of the noncommutative electro-weak standard model reads

$$S^{NC} = S^{NC}_{Matter,leptons} + S^{NC}_{Gauge}, \tag{4.3}$$

where

$$S^{NC}_{Matter,leptons} = \int d^4x \ \bar{\hat{\Psi}} \star i\gamma^\mu \hat{D}_\mu \hat{\Psi} \ , \quad S^{NC}_{Gauge} = -\frac{1}{4} \int d^4x \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}.$$

The action is written separately for simplicity. Actually, S^{NC}_{Gauge} part is the kinetic term. In Standard model, no Feynman rules are derived from the kinetic terms. However; in noncommutative space, this is not the case as can be seen later. The neutral gauge boson interactions such as ZZZ, $Z\gamma\gamma$ and ZZ comes from the kinetic term whose order is determined by the noncommutative parameter $\Theta^{\mu\nu}$.

$S^{NC}_{Matter,quarks}$, the action which includes the quark field, terms are excluded because these terms do not contribute the $Z \to \nu\bar{\nu}\gamma$ decay.

We will first consider leptonic part of the action.

$$S^{NC}_{Matter,leptons} = \int d^4x \ \bar{\hat{\Psi}} \star i\gamma^\mu \hat{D}_\mu \hat{\Psi}. \tag{4.4}$$

Here $\hat{\Psi} = \Psi + e\theta^{\rho\sigma} A_\sigma \partial_\rho \Psi$ and $\hat{A_\mu} = A_\mu + e\theta^{\rho\sigma} A_\sigma(\partial_\rho A_\mu - \frac{1}{2}\partial_\mu A_\rho)$ is the Abelian noncommutative gauge potential expanded by the Seiberg-Witten map.

$$S^{NC}_{Matter,leptons} = S + ie\theta^{\rho\sigma} \int d^4x \left[\partial_\rho \bar{\Psi} A_\sigma \gamma^\mu \partial_\mu \Psi - \partial_\mu \bar{\Psi} A_\sigma \gamma^\mu \partial_\rho \Psi + \bar{\Psi} \partial_\rho A_\mu \gamma^\mu \partial_\sigma \Psi\right] \tag{4.5}$$

If we integrate the above equation by parts, we get

$$S^{NC}_{Matter,leptons} = S - ie\theta^{\mu\rho\sigma} \int d^4x \ \bar{\Psi} F_{\mu\rho} \partial_\sigma \Psi, \tag{4.6}$$

where $\theta^{\mu\rho\sigma} = \theta^{\mu\rho}\gamma^\sigma + \theta^{\rho\sigma}\gamma^\mu + \theta^{\sigma\mu}\gamma^\rho$ and $F_{\mu\rho} = \partial_\mu A_\rho - \partial_\rho A_\mu$ is the field strength. From this equation, we extract the following Feynman rule for the gauge invariant $\gamma(k_1)\bar{\nu}(k_2)\nu(k_3)$ vertex in momentum space.

$$\Gamma^\mu = \frac{ie}{2}\theta^{\mu\rho\sigma} k_{1\sigma} k_{2\rho}(1 - \gamma^5) \tag{4.7}$$

We obtained $\gamma\nu\bar{\nu}$ vertex factor. Let us find the Feynman rules coming from the kinetic term of the Lagrangian. S^{NC}_{Gauge} is given by

$$S^{NC}_{Gauge} = -\frac{1}{4} \int d^4x \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}. \tag{4.8}$$

Here $\hat{F}_{\mu\nu}$ denotes the noncommutative field given by

$$\hat{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - g'\theta^{\rho\sigma}\partial_\rho B_\sigma \partial_\sigma B_\rho, \tag{4.9}$$

where $(g' = \dfrac{e}{\sin\theta_W})$. Just as the electromagnetic current is coupled to the photon A_μ, the weak hypercharge current is coupled to the vector boson B_μ in Standard electro-weak theory [79].

Expanding the action (4.8) to first order in $\theta_{\mu\nu}$, we get

$$S^{NC}_{Gauge} = S + \frac{g'}{2}\int d^4x \theta^{\rho\sigma}(\partial_\mu B_\nu - \partial_\nu B_\mu)\partial_\rho B^\mu \partial_\sigma B^\nu. \tag{4.10}$$

The relation between B_μ and A_μ, Z_μ is given by

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu \tag{4.11}$$

Expanding the action to first order gives the triple gauge boson interaction. Going to the physical basis, we obtain the following interaction terms in the noncommutative space.

the ZZZ interaction term

$$S^{NC}_{ZZZ} = \frac{g'}{2}\sin^3\theta_W \int d^4x\ \theta^{\rho\sigma}\left[(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\partial_\rho Z^\mu \partial_\sigma Z^\nu\right]. \tag{4.12}$$

the $\gamma\gamma\gamma$ interaction term

$$S^{NC}_{\gamma\gamma\gamma} = \frac{g'}{2}\cos^3\theta_W \int d^4x\ \theta^{\rho\sigma}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\rho A^\mu \partial_\sigma A^\nu\right]. \tag{4.13}$$

the $ZZ\gamma$ interaction term

$$\begin{aligned} S^{NC}_{ZZ\gamma} = & \quad g'\sin^2\theta_W\cos\theta_W \times \\ & \int d^4x\ \theta^{\rho\sigma}[(\partial_\mu Z_\nu - \partial_\nu Z_\mu)\partial_\rho A^\mu \partial_\sigma Z^\nu + \partial_\mu A^\nu \partial_\rho Z^\mu \partial_\sigma Z^\nu]. \end{aligned} \tag{4.14}$$

the $Z\gamma\gamma$ interaction term

$$S^{NC}_{Z\gamma\gamma} = g' \sin\theta_W \cos^2\theta_W \times \int d^4x\ \theta^{\rho\sigma} \left[\partial_\mu Z^\nu \partial_\rho A^\mu \partial_\sigma A^\nu + (\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\rho Z^\mu \partial_\sigma A^\nu\right]. \quad (4.15)$$

If we expand the action equation (4.8) by using the equation (4.11), we also obtain the four point interaction terms between the Z boson and γ. These are given in the followings:

the $\gamma\gamma\gamma\gamma$ interaction term

$$S^{NC}_{\gamma\gamma\gamma\gamma} = \frac{g'}{2}\cos^4\theta_W \theta^{\alpha\beta} \int d^4x\ \theta^{\rho\sigma} \left[\partial_\alpha A_\mu \partial_\beta A_\nu \partial_\rho A^\mu \partial_\sigma A^\nu\right]. \quad (4.16)$$

the $ZZZZ$ interaction term

$$S^{NC}_{ZZZZ} = \frac{g'}{2}\sin^4\theta_W \theta^{\alpha\beta} \int d^4x\ \theta^{\rho\sigma} \left[\partial_\alpha Z_\mu \partial_\beta Z_\nu \partial_\rho Z^\mu \partial_\sigma Z^\nu\right]. \quad (4.17)$$

the $Z\gamma\gamma\gamma$ interaction term

$$S^{NC}_{Z\gamma\gamma\gamma} = 2g' \cos^3\theta_W \sin\theta_W \theta^{\alpha\beta} \int d^4x\ \theta^{\rho\sigma} \left[\partial_\alpha Z_\mu \partial_\beta A_\nu \partial_\rho A^\mu \partial_\sigma A^\nu\right]. \quad (4.18)$$

the $ZZZ\gamma$ interaction term

$$S^{NC}_{ZZZ\gamma} = 2g' \cos\theta_W \sin^3\theta_W \theta^{\alpha\beta} \int d^4x\ \theta^{\rho\sigma} \left[\partial_\alpha A_\mu \partial_\beta Z_\nu \partial_\rho Z^\mu \partial_\sigma Z^\nu\right]. \quad (4.19)$$

the $ZZ\gamma\gamma$ interaction term

$$S^{NC}_{ZZ\gamma\gamma} = g' \cos^2\theta_W \sin^2\theta_W \theta^{\alpha\beta}\theta^{\rho\sigma} \int d^4x\ [\partial_\alpha Z_\mu \partial_\beta Z_\nu \partial_\rho A^\mu \partial_\sigma A^\nu + \partial_\alpha A_\mu \partial_\beta Z_\nu (\partial_\rho A^\mu \partial_\sigma Z^\nu + \partial_\rho Z^\mu \partial_\sigma A^\nu)]. \quad (4.20)$$

Now, we want to find the Feynman rules from those actions. In general, Feynman rules are obtained by varying the corresponding action in the momentum space [78].

Note that the interactions are computed by evaluating a perturbation series in L_{int}, where $S_{int} = \int d^4 x L_{int}$.

Firstly, let us find the Feynman rule for $ZZ\gamma$ interaction. From now on, the constants g' and the functions $\sin\theta_W$, $\cos\theta_W$ will be written at the end for simplicity.

Let us vary the $ZZ\gamma$ part. To do this, we firstly take the derivative of the fields in the Lagrangian $[\partial_\mu A^\nu(p) = -ip_\mu A^\nu(p)]$. The variation formula is given by

$$\frac{\delta^3\ \theta^{\rho\sigma}}{\delta A_b(k_1)\delta Z_c(k_2)\delta Z_d(k_3)}\ [\ p_{3_\mu} p_{1_\rho} p_{2_\sigma} A^\nu(p_3) Z^\mu(p_1) Z^\nu(p_2) + p_{3_\rho} p_{2_\sigma} A^\mu(p_3) Z^\nu(p_2)$$
$$(\ p_{1_\mu} Z_\nu(p_1) - p_{1_\nu} Z_\mu(p_1)\ )\ ].$$

If we start to perform the calculation with respect to the photon field $A_b(k_1)$, the above equation is simplified

$$\frac{\delta^2\ \theta^{\rho\sigma}}{\delta Z_c(k_2)\delta Z_d(k_3)} \left[k_{1_\mu} p_{1_\rho} p_{2_\sigma} Z^\mu(p_1) Z^b(p_2) + (p_{1_b} Z_\nu(p_1) - p_{1_\nu} Z_b(p_1)) k_{1_\rho} p_{2_\sigma} Z^\nu(p_2)\right],$$

Then, varying this equation with respect to one of the Z boson field yields:

$$\frac{\delta\ \theta^{\rho\sigma}}{\delta Z_d(k_3)}\ [\ k_{1_c} k_{2_\rho} p_{2_\sigma} Z^b(p_2) + k_{1_\mu} p_{1_\rho} k_{2_\sigma} Z^\mu(p_1) g_{bc} + (k_{2_b} g_{\nu c} - k_{2_\nu} g_{bc}) k_{1_\rho} p_{2_\sigma} Z^\nu(p_2)$$
$$+ (p_{1_b} Z_c(p_1) - p_{1_c} Z_b(p_1)) k_{1_\rho} k_{2_\sigma}\ ].$$

Finally, we end up by varying the above equation with respect to the last field $Z_d(k_3)$

$$\theta^{\rho\sigma}\left[k_{1_c} k_{2_\rho} k_{3_\sigma} g_{bd} + k_{1_d} k_{3_\rho} k_{2_\sigma} g_{bc} + (k_{2_b} g_{dc} - k_{2_d} g_{bc}) k_{1_\rho} k_{3_\sigma} + (k_{3_b} g_{cd} - k_{3_c} g_{bd}) k_{1_\rho} k_{2_\sigma}\right]$$

Assuming $k_1+k_2+k_3=0$ and using the antisymmetry of $\theta^{\rho\sigma}$, we can simplify the above equation

$$k_2 \wedge k_3 \;\; \left[(k_1-k_3)_c g_{bd} + (k_2-k_1)_d g_{bc} + (k_3-k_2)_b g_{cd}\right].$$

Note that wedge product is defined as

$$k_2 \wedge k_3 = k_{2\rho}\theta^{\rho\sigma}k_{3\sigma} \tag{4.21}$$

Writing the constants in the $ZZ\gamma$ action Eqn. (4.14) yields the $ZZ\gamma$ vertex factor

$$\Gamma_{cbd}(ZZ\gamma) =$$

$$g' \sin^2\theta_W \cos\theta_W k_2 \wedge k_3[(k_1-k_3)_c\; g_{bd} + (k_2-k_1)_d\; g_{bc} + (k_3-k_2)_b\; g_{cd}]. \tag{4.22}$$

Secondly, let us find the Feynman rule for the ZZZ interaction. The calculations are the same as before. The corresponding variation formula is given by

$$\frac{\delta^3}{\delta Z_b(k_1)\delta Z_c(k_2)\delta Z_d(k_3)}\theta^{\rho\sigma}\left[(p_{1\mu}Z_\nu(p_1) - p_{1\nu}Z_\mu(p_1))p_{2\rho}p_{3\sigma}Z^\mu(p_2)Z^\nu(p_3)\right].$$

If we firstly take the variation with respect to the field $Z_b(k_1)$, we find

$$\frac{\delta^2}{\delta Z_c(k_2)\delta Z_d(k_3)}\theta^{\rho\sigma}\;[\;(k_{1\mu}g_{\nu b} - k_{1\nu}g_{\mu b})p_{2\rho}p_{3\sigma}Z^\mu(p_2)Z^\nu(p_3) + k_{1\rho}p_{3\sigma}Z^\nu(p_3)\times$$
$$(p_{1b}Z_\nu(p_1) - p_{1\nu}Z_b(p_1)) + (p_{1\mu}Z_b(p_1) - p_{1b}Z_\mu(p_1))p_{2\rho}k_{1\sigma}Z^\mu(p_2)\;].$$

Let us perform the calculation for the field $Z_c(k_2)$. Then, this equation is reduced to the following equation with one field to be varyied.

$$\frac{\delta}{\delta Z_d(k_3)}\theta^{\rho\sigma}\;[\;(k_{1c}g_{\nu b} - k_{1\nu}g_{cb})k_{2\rho}p_{3\sigma}Z^\nu(p_3) + (k_{1\mu}g_{cb} - k_{1c}g_{\mu b})p_{2\rho}k_{2\sigma}Z^\mu(p_2) +$$

$$(k_{2b}g_{\nu c} - k_{2\nu}g_{bc})k_{1\rho}p_{3\sigma}Z^{\nu}(p_3) + (p_{1b}Z_c(p_1) - p_{1c}Z_b(p_1))k_{1\rho}k_{2\sigma} +$$
$$(k_{2\mu}g_{bc} - k_{2b}g_{\mu c})p_{2\rho}k_{1\sigma}Z^{\mu}(p_2) + (p_{1c}Z_b(p_1) - p_{1b}Z_c(p_1))k_{2\rho}k_{1\sigma} \quad].$$

Now, there is only one field to be varied. If we vary the above equation to the field $Z_d(k_3)$, we obtain

$$\theta^{\rho\sigma} \ [\ (k_{1c}g_{db} - k_{1d}g_{cb})k_{2\rho}k_{3\sigma} + (k_{1d}g_{cb} - k_{1c}g_{db})k_{3\rho}k_{2\sigma} +$$
$$(k_{2b}g_{dc} - k_{2d}g_{bc})k_{1\rho}k_{3\sigma} + (k_{3b}g_{cd} - k_{3c}g_{bd})k_{1\rho}k_{2\sigma} +$$
$$(k_{2d}g_{bc} - k_{2b}g_{dc})k_{3\rho}k_{1\sigma} + (k_{3c}g_{bd} - k_{3b}g_{cd})k_{2\rho}k_{1\sigma} \quad].$$

If we assume $k_1 + k_2 + k_3 = 0$ and use the antisymmetry of $\theta^{\rho\sigma}$ as before, it is simplified

$$2k_1 \wedge k_2 \ \left[(k_1 - k_3)_c \ g_{bd} + (k_2 - k_1)_d \ g_{bc} + (k_3 - k_2)_b \ g_{cd}\right].$$

As a last step, if we write the constants in the ZZZ action (4.12) we find the ZZZ vertex factor

$$\Gamma_{cbd}(ZZZ) =$$
$$g' \sin^3\theta_W k_1 \wedge k_2 \ [\ (k_1 - k_3)_c \ g_{bd} + (k_2 - k_1)_d \ g_{bc} + (k_3 - k_2)_b \ g_{cd}\ ] \quad (4.23)$$

In the similar way, we can calculate the Feynman rules for the $\gamma\gamma\gamma$ and $Z\gamma\gamma$ just by replacing $Z^{\mu} \leftrightarrow A^{\mu}$. The corresponding Feynman rules are found for $\gamma\gamma\gamma$ vertex

$$\Gamma_{cbd}(\gamma\gamma\gamma) =$$
$$g' \cos^3\theta_W k_1 \wedge k_2 \ [\ (k_1 - k_3)_c \ g_{bd} + (k_2 - k_1)_d \ g_{bc} + (k_3 - k_2)_b \ g_{cd}\ ], \quad (4.24)$$

and for $Z\gamma\gamma$ vertex

$$\Gamma_{cbd}(Z\gamma\gamma) =$$

$$g' \sin\theta_W \cos^2\theta_W k_1 \wedge k_2 \ \left[(k_1 - k_3)_c \ g_{bd} + (k_2 - k_1)_d \ g_{bc} + (k_3 - k_2)_b \ g_{cd}\right] \quad (4.25)$$

Note that the only change comes from the Weinberg angle.

Having obtained the Feynman rules for the triple gauge boson interaction, let us find those for the four point interactions in noncommutative space. Let us study firstly the four-gamma vertex. The Feynman rule for this vertex also can be determined by varying the corresponding action in momentum space as we did for the triple gauge boson interactions.

The equation which will be varied is given by

$$\frac{-i\delta^4}{\delta A_b(k_1)\delta A_c(k_2)\delta A_d(k_3)\delta A_e(k_4)}\theta^{\alpha\beta}\theta^{\rho\sigma}\left[p_{1\alpha}p_{2\beta}p_{3\rho}p_{4\sigma}A_\mu(p_1)A_\nu(p_2)A^\mu(p_3)A^\nu(p_4)\right]$$

Note that there are two antisymmetric tensors here. This means that four point interaction gives the second order contribution on the noncommutative parameter. Let us perform the variation step by step. Doing the variation firstly for the fields $A_e(k_4)$ yields

$$-i\frac{\delta^3}{\delta A_b(k_1)\delta A_c(k_2)\delta A_d(k_3)}\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ k_{4\alpha}p_{2\beta}p_{3\rho}p_{4\sigma}A_\nu(p_2)A_e(p_3)A^\nu(p_4) +$$

$$p_{1\alpha}k_{4\beta}p_{3\rho}p_{4\sigma}A_\mu(p_1)A^\mu(p_3)A_e(p_4) + p_{1\alpha}p_{2\beta}k_{4\rho}p_{4\sigma}A_e(p_1)A_\nu(p_2)A^\nu(p_4) +$$

$$p_{1\alpha}p_{2\beta}p_{3\rho}k_{4\sigma}A_\mu(p_1)A_e(p_2)A^\mu(p_3)\ ].$$

As a second step, let us vary the above equation for the field $A_d(k_3)$. Then it is reduced to the following equation

$$-i\frac{\delta^2}{\delta A_b(k_1)\delta A_c(k_2)}$$

$$\begin{aligned}\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ & k_{4\alpha}k_{3\beta}p_{3\rho}p_{4\sigma}A_e(p_3)A_d(p_4) \ + \ k_{4\alpha}p_{2\beta}k_{3\rho}p_{4\sigma}g_{ed}A_\nu(p_2)A^\nu(p_4) + \\ & k_{4\alpha}p_{2\beta}p_{3\rho}k_{3\sigma}A_d(p_2)A_e(p_3) \ + \ k_{3\alpha}k_{4\beta}p_{3\rho}p_{4\sigma}A_d(p_3)A_e(p_4) \ + \\ & p_{1\alpha}k_{4\beta}k_{3\rho}p_{4\sigma}A_d(p_1)A_e(p_4) \ + \ p_{1\alpha}k_{4\beta}p_{3\rho}k_{3\sigma}g_{ed}A_\mu(p_1)A^\mu(p_3) + \\ & p_{1\alpha}k_{3\beta}k_{4\rho}p_{4\sigma}A_e(p_1)A_d(p_4) \ + \ k_{3\alpha}p_{2\beta}k_{4\rho}p_{4\sigma}g_{ed}A_\nu(p_2)A^\nu(p_4) + \\ & p_{1\alpha}p_{2\beta}k_{4\rho}k_{3\sigma}A_e(p_1)A_d(p_2) \ + \ k_{3\alpha}p_{2\beta}p_{3\rho}k_{4\sigma}A_e(p_2)A_d(p_3) \ + \\ & p_{1\alpha}p_{2\beta}k_{3\rho}k_{4\sigma}A_d(p_1)A_e(p_2) \ + \ p_{1\alpha}k_{3\beta}p_{3\rho}k_{4\sigma}g_{ed}A_\mu(p_1)A^\mu(p_3)\ ].\end{aligned}$$

Since, all of the fields are photon field, the number of the terms gets bigger when performing variation. If we vary the last equation with respect to the field $A_c(k_2)$. There are now 24 terms.

$$-i\frac{\delta}{\delta A_b(k_1)}\ \theta^{\alpha\beta}\theta^{\rho\sigma}$$

$$\begin{aligned}[\ & k_{4\alpha}k_{3\beta}k_{2\rho}p_{4\sigma}g_{ec}A_d(p_4) + k_{4\alpha}k_{3\beta}p_{3\rho}k_{2\sigma}g_{dc}A_e(p_3) + k_{4\alpha}k_{2\beta}k_{3\rho}p_{4\sigma}g_{ed}A_c(p_4) + \\ & k_{4\alpha}p_{2\beta}k_{3\rho}k_{2\sigma}g_{ed}A_c(p_2) + k_{4\alpha}k_{2\beta}p_{3\rho}k_{3\sigma}g_{dc}A_e(p_3) + k_{4\alpha}p_{2\beta}k_{2\rho}k_{3\sigma}g_{ec}A_d(p_2) + \\ & k_{3\alpha}k_{4\beta}k_{2\rho}p_{4\sigma}g_{dc}A_e(p_4) + k_{3\alpha}k_{4\beta}p_{3\rho}k_{2\sigma}g_{ec}A_d(p_3) + k_{2\alpha}k_{4\beta}k_{3\rho}p_{4\sigma}g_{dc}A_e(p_4) + \\ & p_{1\alpha}k_{4\beta}k_{3\rho}k_{2\sigma}g_{ec}A_d(p_1) + k_{2\alpha}k_{4\beta}p_{3\rho}k_{3\sigma}g_{ed}A_c(p_3) + p_{1\alpha}k_{4\beta}k_{2\rho}k_{3\sigma}g_{ed}A_c(p_1) + \\ & k_{2\alpha}k_{3\beta}k_{4\rho}p_{4\sigma}g_{ec}A_d(p_4) + p_{1\alpha}k_{3\beta}k_{4\rho}k_{2\sigma}g_{dc}A_e(p_1) + k_{3\alpha}k_{2\beta}k_{4\rho}p_{4\sigma}g_{ed}A_c(p_4) + \\ & k_{3\alpha}p_{2\beta}k_{4\rho}k_{2\sigma}g_{ed}A_c(p_2) + k_{2\alpha}p_{2\beta}k_{4\rho}k_{3\sigma}g_{ec}A_d(p_2) + p_{1\alpha}k_{2\beta}k_{4\rho}k_{3\sigma}g_{dc}A_e(p_1) + \\ & k_{3\alpha}k_{2\beta}p_{3\rho}k_{4\sigma}g_{ec}A_d(p_3) + k_{3\alpha}p_{2\beta}k_{2\rho}k_{4\sigma}g_{dc}A_e(p_2) + k_{2\alpha}p_{2\beta}k_{3\rho}k_{4\sigma}g_{dc}A_e(p_2) + \\ & p_{1\alpha}k_{2\beta}k_{3\rho}k_{4\sigma}g_{ec}A_d(p_1) + k_{2\alpha}k_{3\beta}p_{3\rho}k_{4\sigma}g_{ed}A_c(p_3) + p_{1\alpha}k_{3\beta}k_{2\rho}k_{4\sigma}g_{ed}A_c(p_1)\ ].\end{aligned}$$

Finally, let us vary for the last photon field $A_b(k_1)$. Note that there does not exist the momentum p_i at the end. In other words, the representations of the momentums should be k_i.

$$-i\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ k_{4\alpha}k_{3\beta}k_{2\rho}k_{1\sigma}g_{ec}g_{db} + k_{4\alpha}k_{3\beta}k_{1\rho}k_{2\sigma}g_{dc}g_{eb} + k_{4\alpha}k_{2\beta}k_{3\rho}k_{1\sigma}g_{ed}g_{bc} +$$

$$k_{4\alpha}k_{1\beta}k_{3\rho}k_{2\sigma}g_{ed}g_{bc} + k_{4\alpha}k_{2\beta}k_{1\rho}k_{3\sigma}g_{dc}g_{eb} + k_{4\alpha}k_{1\beta}k_{2\rho}k_{3\sigma}g_{ec}g_{db} +$$

$$k_{3\alpha}k_{4\beta}k_{2\rho}k_{1\sigma}g_{dc}g_{eb} + k_{3\alpha}k_{4\beta}k_{1\rho}k_{2\sigma}g_{ec}g_{db} + k_{2\alpha}k_{4\beta}k_{3\rho}k_{1\sigma}g_{dc}g_{eb} +$$

$$k_{1\alpha}k_{4\beta}k_{3\rho}k_{2\sigma}g_{ec}g_{db} + k_{2\alpha}k_{4\beta}k_{1\rho}k_{3\sigma}g_{ed}g_{bc} + k_{1\alpha}k_{4\beta}k_{2\rho}k_{3\sigma}g_{ed}g_{bc} +$$

$$k_{2\alpha}k_{3\beta}k_{4\rho}k_{1\sigma}g_{ec}g_{db} + k_{1\alpha}k_{3\beta}k_{4\rho}k_{2\sigma}g_{dc}g_{eb} + k_{3\alpha}k_{2\beta}k_{4\rho}k_{1\sigma}g_{ed}g_{bc} +$$

$$k_{3\alpha}k_{1\beta}k_{4\rho}k_{2\sigma}g_{ed}g_{bc} + k_{2\alpha}k_{1\beta}k_{4\rho}k_{3\sigma}g_{ec}g_{db} + k_{1\alpha}k_{2\beta}k_{4\rho}k_{3\sigma}g_{dc}g_{eb} +$$

$$k_{3\alpha}k_{2\beta}k_{1\rho}k_{4\sigma}g_{ec}g_{db} + k_{3\alpha}k_{1\beta}k_{2\rho}k_{4\sigma}g_{dc}g_{eb} + k_{2\alpha}k_{1\beta}k_{3\rho}k_{4\sigma}g_{dc}g_{eb} +$$

$$k_{1\alpha}k_{2\beta}k_{3\rho}k_{4\sigma}g_{ec}g_{db} + k_{2\alpha}k_{3\beta}k_{1\rho}k_{4\sigma}g_{ed}g_{bc} + k_{1\alpha}k_{3\beta}k_{2\rho}k_{4\sigma}g_{ed}g_{cb}\quad].$$

This equation seems to be very long. However, it can be rewritten in more compact way by using the antisymmetry of the noncommutative parameter $\theta^{\mu\nu}$.

$$-4i[\ k_1\wedge k_2\ k_3\wedge k_4(g_{ec}g_{db} - g_{dc}g_{eb}) + k_1\wedge k_3\ k_2\wedge k_4(g_{ed}g_{bc} - g_{dc}g_{eb}) +$$

$$k_1\wedge k_4\ k_2\wedge k_3(g_{ed}g_{bc} - g_{ec}g_{db})\ ].$$

Then, the Feynman rules for $\gamma\gamma\gamma\gamma$ interaction is obtained if we write the constants in the action (4.16).

$$\Gamma_{bcde}(\gamma\gamma\gamma\gamma) = 2ig'\cos^4\theta_W[\ k_1\wedge k_2\ k_3\wedge k_4(g_{dc}g_{eb} - g_{ec}g_{db}) +$$

$$k_1\wedge k_3\ k_2\wedge k_4(g_{dc}g_{eb} - g_{ed}g_{bc}) + k_1\wedge k_4\ k_2\wedge k_3(g_{ec}g_{db} - g_{ed}g_{bc})\ ]. \tag{4.26}$$

The Feynman rules for $ZZZZ$ interaction can also be obtained just by replacing $Z^\mu \to A^\mu$. It is given by;

$$\Gamma_{bcde}(ZZZZ) = 2ig' \sin^4\theta_W [\ k_1 \wedge k_2\ k_3 \wedge k_4 (g_{dc}g_{eb} \quad - \quad g_{ec}g_{db}) +$$

$$k_1 \wedge k_3\ k_2 \wedge k_4 (g_{dc}g_{eb} - g_{ed}g_{bc}) + k_1 \wedge k_4\ k_2 \wedge k_3 (g_{ec}g_{db} \quad - \quad g_{ed}g_{bc})\ ]. \tag{4.27}$$

It is interesting to note that $\Gamma_{bcde}(\gamma\gamma\gamma\gamma)$ includes $\cos^4\theta_W$ and $\Gamma_{bcde}(ZZZZ)$ includes $\sin^4\theta_W$, while the rest are the same for the two.

Now, let us calculate the Feynman rule for $Z\gamma\gamma\gamma$ interaction similarly. Since one of the field is Z boson, our task gets easier. The variational formula with which we will deal is given by;

$$-i\frac{\delta^4}{\delta A_b(k_1)\delta A_c(k_2)\delta A_d(k_3)\delta Z_e(k_4)}\theta^{\alpha\beta}\theta^{\rho\sigma}\left[p_{1\beta}p_{2\rho}p_{3\sigma}p_{4\alpha}A_\nu(p_1)A^\mu(p_2)A^\nu(p_3)Z_\mu(p_4)\right].$$

To get rid of the Z field from the equation, we should firstly vary with respect to that field $Z_e(k_4)$. This equation is reduced to the following equation without Z field

$$-i\frac{\delta^3}{\delta A_b(k_1)\delta A_c(k_2)\delta A_d(k_3)}\theta^{\alpha\beta}\theta^{\rho\sigma}\left[p_{1\beta}p_{2\rho}p_{3\sigma}k_{4\alpha}A_\nu(p_1)A_e(p_2)A^\nu(p_3)\right].$$

It is time to deal with the photon field after getting rid of the Z field. If we perform the variation for one of the field $\delta A_d(k_3)$, we obtain

$$\frac{-i\delta^2}{\delta A_b(k_1)\delta A_c(k_2)}\theta^{\alpha\beta}\theta^{\rho\sigma}\ [p_{1\beta}p_{2\rho}k_{3\sigma}k_{4\alpha}A_d(p_1)A_e(p_2) + k_{3\beta}p_{2\rho}p_{3\sigma}k_{4\alpha}A_e(p_2)A_d(p_3)$$

$$+p_{1\beta}k_{3\rho}p_{3\sigma}k_{4\alpha}g_{ed}A_\nu(p_1)A^\nu(p_3)\ ].$$

Note that, it doesn't matter which of the fields is used firstly. As a second step, let us use the field $A_c(k_2)$. Then;

$$-i\frac{\delta}{\delta A_b(k_1)}\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ k_{2\beta}p_{2\rho}k_{3\sigma}k_{4\alpha}g_{dc}A_e(p_2) + p_{1\beta}k_{2\rho}k_{3\sigma}k_{4\alpha}g_{ec}A_d(p_1) +$$

$$p_{1\beta}k_{3\rho}k_{2\sigma}k_{4\alpha}g_{ed}A_c(p_1) + k_{2\beta}k_{3\rho}p_{3\sigma}k_{4\alpha}g_{ed}A_c(p_3) +$$

$$k_{3\beta}p_{2\rho}k_{2\sigma}k_{4\alpha}g_{dc}A_e(p_2) + k_{3\beta}k_{2\rho}p_{3\sigma}k_{4\alpha}g_{ec}A_d(p_3) \ ].$$

Finally, the required formulation is obtained by varying the last equation with respect to last photon field $A_b(k_1)$. Then, it becomes

$$-i\theta^{\alpha\beta}\theta^{\rho\sigma} \ [\ k_{2\beta}k_{1\rho}k_{3\sigma}k_{4\alpha}g_{dc}g_{eb} + k_{1\beta}k_{2\rho}k_{3\sigma}k_{4\alpha}g_{ec}g_{db} +$$

$$k_{1\beta}k_{3\rho}k_{2\sigma}k_{4\alpha}g_{ed}g_{bc} + k_{2\beta}k_{3\rho}k_{1\sigma}k_{4\alpha}g_{ed}g_{bc} +$$

$$k_{3\beta}k_{1\rho}k_{2\sigma}k_{4\alpha}g_{dc}g_{be} + k_{3\beta}k_{2\rho}k_{1\sigma}k_{4\alpha}g_{ec}g_{bd} \ ].$$

It can be rewritten in more compact way as before.

$$-i[\ k_1 \wedge k_4 \ \ k_2 \wedge k_3(g_{ed}g_{bc} - g_{ec}g_{db}) + k_1 \wedge k_3 \ k_2 \wedge k_4(g_{ed}g_{bc} - g_{dc}g_{eb}) +$$

$$k_1 \wedge k_2 \ k_3 \wedge k_4(g_{ec}g_{bd} - g_{dc}g_{be}) \ ].$$

Putting the constants gives the Feynman rule for $Z\gamma\gamma\gamma$ interaction.

$$\Gamma_{bcde}(Z\gamma\gamma\gamma) = 2ig' \cos^3\theta_W \sin\theta_W [\ k_1 \wedge k_4 \ \ k_2 \wedge k_3(g_{ec}g_{db} - g_{ed}g_{bc}) +$$

$$k_1 \wedge k_3 \ k_2 \wedge k_4(g_{dc}g_{eb} - g_{ed}g_{bc}) + k_1 \wedge k_2 \ k_3 \wedge k_4(g_{dc}g_{be} - g_{ec}g_{bd}) \ ]. \quad (4.28)$$

The Feynman rule for $ZZZ\gamma$ interaction can be found by replacing $A^\mu \rightarrow Z^\mu$. It is given by:

$$\Gamma_{bcde}(ZZZ\gamma) = 2ig' \cos\theta_W \sin^3\theta_W [\ k_1 \wedge k_4 \ \ k_2 \wedge k_3(g_{ec}g_{db} - g_{ed}g_{bc}) +$$

$$k_1 \wedge k_3 \ k_2 \wedge k_4(g_{dc}g_{eb} - g_{ed}g_{bc}) + k_1 \wedge k_2 \ k_3 \wedge k_4(g_{dc}g_{be} - g_{ec}g_{bd}) \ ]. \quad (4.29)$$

Let us study the last four point interaction $ZZ\gamma\gamma$. The equation which we vary is given by;

$$\frac{-i\delta^4}{\delta Z_b(k_1)\delta Z_c(k_2)\delta A_d(k_3)\delta A_e(k_4)}\theta^{\alpha\beta}\theta^{\rho\sigma}[\ p_{1\alpha}p_{2\beta}p_{3\rho}p_{4\sigma}Z_\mu(p_1)Z_\nu(p_2)A^\mu(p_3)A^\nu(p_4)$$

$$+p_{1_\rho}p_{2_\beta}p_{3_\alpha}p_{4_\sigma}A_\mu(p_3)Z_\nu(p_2)(A^\mu(p_4)Z^\nu - Z^\mu(p_1)A^\nu(p_4))(p_1)\ ].$$

We will vary firstly with respect to the photon fields. The variation to $A_e(k_4)$ gives

$$\frac{-i\theta^{\alpha\beta}\theta^{\rho\sigma}\ \delta^3}{\delta Z_b(k_1)\delta Z_c(k_2)\delta A_d(k_3)}[\ p_{1_\rho}p_{2_\beta}k_{4\alpha}p_{4_\sigma}Z_\nu(p_2)(\ A_e(p_4)Z^\nu(p_1) - Z_e(p_1)A^\nu(p_4)\ ) +$$
$$p_{1_\rho}p_{2_\beta}p_{3_\alpha}k_{4\sigma}(\ Z_\nu(p_2)Z^\nu(p_1)A_e(p_3) - Z^\mu(p_1)Z_e(p_2)A_\mu(p_3)\ ) +$$
$$p_{1_\alpha}p_{2_\beta}p_{3_\rho}k_{4_\sigma}\ Z_\mu(p_1)Z_e(p_2)A^\mu(p_3)\ + p_{1_\alpha}p_{2_\beta}k_{4_\rho}p_{4_\sigma}Z_e(p_1)Z_\nu(p_2)A^\nu(p_4)\ ]$$

There is only one photon field now. The variation to this second photon field $A_d(k_3)$ yields

$$-i\frac{\delta^2}{\delta Z_b(k_1)\delta Z_c(k_2)}\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ p_{1_\rho}p_{2_\beta}k_{4\alpha}k_{3\sigma}(\ g_{ed}Z_\nu(p_2)Z^\nu(p_1) - Z_e(p_1)Z_d(p_2)\ ) +$$
$$p_{1_\rho}p_{2_\beta}k_{3\alpha}k_{4\sigma}(\ g_{ed}Z_\nu(p_2)Z^\nu(p_1) - Z_d(p_1)Z_e(p_2)\ ) +$$
$$p_{1_\alpha}p_{2_\beta}k_{3_\rho}k_{4_\sigma}\ Z_d(p_1)\ Z_e(p_2) + p_{1_\alpha}p_{2_\beta}k_{4_\rho}k_{3_\sigma}Z_e(p_1)Z_d(p_2)\ ]$$

Now, let us use the Z field. As we did for the photon field, if we vary to the one of the Z field $Z_c(k_2)$, we obtain

$$\begin{aligned}-i\frac{\delta}{\delta Z_b(k_1)}\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ & p_{1_\rho}k_{2\beta}k_{4\alpha}k_{3_\sigma}\ g_{ed}Z_c(p_1) \ + \ k_{2\rho}p_{2_\beta}k_{4\alpha}k_{3_\sigma}\ g_{ed}Z_c(p_2) - \\ & k_{2\rho}p_{2_\beta}k_{4\alpha}k_{3_\sigma}\ g_{ec}Z_d(p_2) \ - \ p_{1_\rho}k_{2\beta}k_{4\alpha}k_{3_\sigma}\ g_{dc}Z_e(p_1) + \\ & p_{1_\rho}k_{2\beta}k_{3\alpha}k_{4_\sigma}\ g_{ed}Z_c(p_1) \ + \ k_{2\rho}p_{2_\beta}k_{3\alpha}k_{4_\sigma}\ g_{ed}Z_c(p_2) - \\ & p_{1_\rho}k_{2\beta}k_{3\alpha}k_{4_\sigma}\ g_{ec}Z_d(p_1) \ - \ k_{2\rho}p_{2_\beta}k_{3\alpha}k_{4_\sigma}\ g_{dc}Z_e(p_2) + \\ & p_{1_\alpha}k_{2\beta}k_{3_\rho}k_{4_\sigma}\ g_{ec}Z_d(p_1) \ + \ k_{2\alpha}p_{2_\beta}k_{3_\rho}k_{4_\sigma}\ g_{dc}Z_e(p_2) + \\ & p_{1_\alpha}k_{2\beta}k_{4_\rho}k_{3_\sigma}\ g_{dc}Z_e(p_1) \ + \ k_{2\alpha}p_{2_\beta}k_{4_\rho}k_{3_\sigma}\ g_{ec}Z_d(p_2)\ \ ]\end{aligned}$$

The calculations end when performing the variation to the last field $Z_b(k_1)$. The Feynman rules without the constants is obtained.

$$
\begin{aligned}
-i\theta^{\alpha\beta}\theta^{\rho\sigma}\ [\ & k_{1\rho}k_{2\beta}k_{4\alpha}k_{3\sigma}\ g_{ed}g_{cb} \ + \ k_{2\rho}k_{1\beta}k_{4\alpha}k_{3\sigma}\ g_{ed}g_{bc} - \\
& k_{2\rho}k_{1\beta}k_{4\alpha}k_{3\sigma}\ g_{ec}g_{bd} \ - \ k_{1\rho}k_{2\beta}k_{4\alpha}k_{3\sigma}\ g_{dc}g_{be} + \\
& k_{1\rho}k_{2\beta}k_{3\alpha}k_{4\sigma}\ g_{ed}g_{bc} \ + \ k_{2\rho}k_{1\beta}k_{3\alpha}k_{4\sigma}\ g_{ed}g_{bc} - \\
& k_{1\rho}k_{2\beta}k_{3\alpha}k_{4\sigma}\ g_{ec}g_{bd} \ - \ k_{2\rho}k_{1\beta}k_{3\alpha}k_{4\sigma}\ g_{dc}g_{be} + \\
& k_{1\alpha}k_{2\beta}k_{3\rho}k_{4\sigma}\ g_{ec}g_{bd} \ + \ k_{2\alpha}k_{1\beta}k_{3\rho}k_{4\sigma}\ g_{dc}g_{be} + \\
& k_{1\alpha}k_{2\beta}k_{4\rho}k_{3\sigma}\ g_{dc}g_{be} \ + \ k_{2\alpha}k_{1\beta}k_{4\rho}k_{3\sigma}\ g_{ec}g_{bd}\ \ ].
\end{aligned}
$$

This equation looks very complicated. We can rewrite it by using the anti-symmetry property of the noncommutative parameter as before.

$$
\begin{aligned}
2i[\ & k_1 \wedge k_4\ \ k_2 \wedge k_3(g_{ed}g_{bc} - g_{ec}g_{db}) + k_1 \wedge k_3\ k_2 \wedge k_4(g_{ed}g_{bc} - g_{dc}g_{eb}) + \\
& k_1 \wedge k_2\ k_3 \wedge k_4(g_{dc}g_{be} - g_{ec}g_{bd})\ ].
\end{aligned}
$$

If we substitute the constants in the corresponding Lagrangian, we get

$$
\begin{aligned}
\Gamma_{bcde}(ZZ\gamma\gamma) = 2ig' \cos^2\theta_W \sin^2\theta_W [\ & k_1 \wedge k_4\ \ k_2 \wedge k_3(g_{ed}g_{bc} - g_{ec}g_{db}) + \\
& k_1 \wedge k_3\ k_2 \wedge k_4(g_{ed}g_{bc} - g_{dc}g_{eb}) + +k_1 \wedge k_2\ k_3 \wedge k_4(g_{dc}g_{be} - g_{ec}g_{bd})\ ]. \quad (4.30)
\end{aligned}
$$

Up to now, we have obtained the Feynman rules after length calculations. As a result, we can take these Feynman rules and proceed to investigate some scattering and decay processes. Some decay processes are of great importance because these decays are absent at least tree level in Standard model. One of the such decays is $Z \to \nu\bar{\nu}\gamma$ decay. In the following section, we will apply

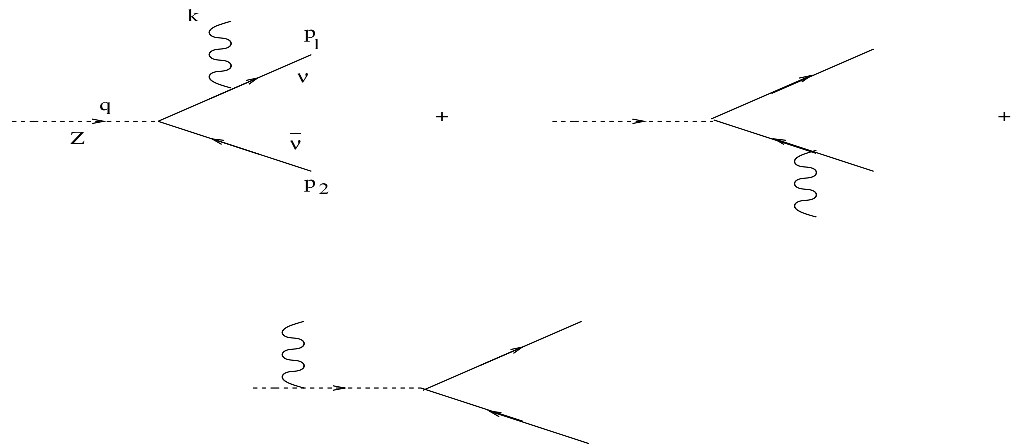

Figure 4.1: Feynman diagrams for $Z \to \nu\bar{\nu}\gamma$.

the necessary Feynman rules obtained before for this decay and find the decay rate.

4.2 $Z \to \nu\bar{\nu}\gamma$ Decay

Having obtained the Feynman rules in noncommutative space, let us apply them for the decay $Z \to \nu\bar{\nu}\gamma$. The Feynman diagram for this decay is given in figure (4.1). As can be seen from the relevant Feynman rules, such a decay is possible at tree level. If we use equations (4.7) and (4.22), the amplitude for this decay is found as follows:

$$M =$$

$$\frac{eg_z}{8} \, [\ \bar{u}(p_1)(1-\gamma^5)\gamma^\nu \epsilon_\nu(q) \ \frac{\not{p}_2 + \not{k}}{(p_2+k)^2} \ \epsilon^\star_\mu(k)\theta^{\mu\rho\sigma} p_{2\rho} k_\sigma (1-\gamma^5)\nu(p_2) -$$

$$\bar{u}(p_1)\theta^{\mu\rho\sigma}p_{1\rho}k_\sigma(1-\gamma^5)\epsilon^\star_\mu(k)\ \frac{\not{p}_1+\not{k}}{(p_1+k)^2}\ \epsilon_\nu(q)\gamma^\nu(1-\gamma^5)\nu(p_2)\ ]$$

$$-\frac{ie^2(q\wedge k)}{4((q-k)^2-M_Z^2)}\bar{u}(p_1)\gamma^\alpha(1-\gamma^5)\nu(p_2)[g_{\alpha\nu}-((q-k)_\alpha(q-k)_\nu)/M_Z^2]$$

$$\epsilon^\star_\mu(k)[(k+q)^\nu g^{\rho\mu}+(q-2k)^\rho g^{\mu\nu}+(k-2q)^\mu g^{\nu\rho}]\epsilon_\rho(q),\quad(4.31)$$

where $g_z = \dfrac{e}{\sin\theta_W\cos\theta_W}$. Here, q, k, p_1 and p_2 are the for momentum vectors of Z-boson, photon, neutrino and anti-neutrino, respectively. This can be seen in figure 4.1.

After performing the summation over spins of final particles from the equation (4.31), we get

$$|M|^2 = \frac{1}{2}(\frac{eg_z}{M_Z k.p_2\ p_1.k})^2$$

$$\{k.p_1(2\ q.p_2(k.p_1\ p_1.q\ \eta^2 - q\wedge p_1\ k.p_1\ p_1\wedge k) - (k.q+p_1.q)\ (p_1\wedge k)^2) +$$

$$(k.p_1\ p_1.p_2\ \eta^2 - p_2\wedge p_1\ k.p_1\ p_1\wedge k - (k.p_2+p_1.p_2)\ (p_1\wedge k)^2)M_Z^2)$$

$$(k.p_2)^2 + k.p_2\ (k.p_2\ \kappa^2(2p_1.q\ q.p_2+p_1.p_2M_Z^2) - k.p_2\ (2q\wedge p_2\ p_1.q +$$

$$p_1\wedge p_2M_Z^2)\ p_2\wedge k - (2p_1.q\ (k.q+q.p_2)+(k.p_1+p_1.p_2)M_Z^2)\ (p_2\wedge k)^2)$$

$$(p_1.k)^2 + k.p_2\ p_1.kM_Z^2(-2\eta.\kappa\ k.p_1\ k.p_2\ p_1.p_2 - p_1\wedge p_2\ k.p_2(k.p_2+2p_1.p_2)$$

$$p_1\wedge k + p_1\wedge p_2\ k.p_1(k.p_1+2p_1.p_2)\ p_2\wedge k - p_1.p_2(3(k.p_1+k.p_2)+4p_1.p_2)$$

$$p_1\wedge k\ p_2\wedge k) + 2\ p_2\wedge k\ p_1.q\ (k.p_1\ k.p_2\ q\wedge p_1 + (p_1\wedge p_2\ k.p_1 -$$

$$2p_1.p_2\ p_1\wedge k)\ k.q + 2\ k.p_2\ p_1.q\ p_1\wedge k) + 2(k.p_1\ k.p_2\ q\wedge p_2\ p_1\wedge k -$$

$$p_1\wedge k\ (p_1\wedge p_2\ k.p_2 + 2\ p_1.p_2\ p_2\wedge k)\ k.q - (2\eta.\kappa\ k.p_1\ k.p_2 + 2k.p_2\ p_1\wedge p_2$$

$$p_1\wedge k + (-2p_1\wedge p_2\ k.p_1 + (k.p_1+k.p_2+4\ p_1.p_2)\ p_1\wedge k)\ p_2\wedge k)p_1.q)$$

$$q.p_2 + 4\ k.p_1\ p_1\wedge k\ p_2\wedge k\ (q.p_2)^2\} + \frac{1}{32}\left(\frac{e^2\ q\wedge k}{M_Z^2(q.k)^2}\right)^2\{4(k.q)^3\ p_1.p_2 +$$

$$(-2k.p_2\ p_1.q + 2p_1.q\ p_2.q + p_1.p_2M_Z^2) - 2k.qM_Z^2(3k.p_2\ p_1.q + 2p_1.q\ q.p_2 +$$

$$2p_1.p_2M_Z^2) + (k.q)^2(-4p_2.k\ p_1.q - 4p_1.q\ p_2.q + 6p_1.p_2M_Z^2) + 2k.p_1\ (k.q^2$$

$$(6k.p_2 - 2q.p_2) - k.q\ (2k.p_2 + 3q.p_2)M_Z^2 - (4k.p_2 + q.p_2)M_Z^4)\}, \qquad (4.32)$$

where $\eta^\mu = \theta^{\mu\nu}p_{1\nu}$, $\kappa^\mu = \theta^{\mu\nu}p_{2\nu}$ and $\eta^2 = \eta^\mu\eta_\mu = \theta_{\mu\rho}\theta^{\mu\nu}p_{1\nu}p_1{}^\rho$, $\kappa^2 = \kappa^\mu\kappa_\mu = \theta_{\mu\rho}\theta^{\mu\nu}p_{2\nu}p_2{}^\rho$

It is very difficult to work with $|M|^2$, since it includes wedge product between the four momentum vectors. In the next calculation, we will set $\beta = \pi/2$ in equation (4.2), then α and γ become the background electric and magnetic fields relative to the z-axis. In this case, the matrix element become

$$C_{02} = \sin\alpha; \quad C_{03} = \cos\alpha,$$

$$C_{12} = \cos\gamma; \quad C_{13} = \sin\gamma.$$

To find the decay rate, we should specify the four vectors p_1^μ, q^μ, k^μ. In the rest frame of Z boson, these are given as:

$$\begin{aligned} p_1^\mu &= (p_1, p_1\sin\theta\sin\phi, p_1\sin\theta\cos\phi, p_1\cos\theta) \\ k^\mu &= (k, 0, 0, k) \\ q^\mu &= (M_Z, 0, 0, 0), \end{aligned} \qquad (4.33)$$

where p_1 is the momentum of neutrino and k, q are that of the photon and Z-boson, respectively. Here, we set the z-axis along the photon momentum. The invariant quantity in $|M|^2$ is found as

$$\begin{aligned} p_1.k &= M_Z(-M_Z/2 + p_1 + k), \quad p_1.p_2 = (-M_Z^2/2 + M_Zk), \\ p_2.k &= M_Z(-M_Z/2 + p_1), \qquad p_1.q = M_Zp_1, \\ p_2.q &= M_Z(-M_Z + p_1 + k), \qquad k.q = M_Zk. \end{aligned} \qquad (4.34)$$

Finding the decay rate includes the integration. So, we need the numerical analysis from now on.

In the next section, we will perform the numerical analysis to obtain the decay rate.

4.3 Numerical Analysis

In this section, we present the numerical analysis for the decay width of $Z \rightarrow \nu\bar{\nu}\gamma$ decay in the noncommutative standard model. The values of input parameters are:

$$e = \sqrt{\frac{4\pi}{137}}, \qquad M_Z = 91GeV$$

The decay rate is calculated in two cases. In the first case, it is found when the spatial noncommutativity is taken into account. Then, it is calculated for the temporal noncommutativity.

The decay rate is given by the Golden rule

$$d\Gamma = \frac{|M|^2}{3M_Z}\left(\frac{d^3p_1}{(2\pi)^3 2E_1}\right)\left(\frac{d^3p_2}{(2\pi)^3 2E_2}\right)\left(\frac{d^3k}{(2\pi)^3 2E_3}\right)(2\pi)^4\delta^4(q-p_1-p_2-k) \quad (4.35)$$

Now, let us perform the integrations. We will write the constants at the end for simplicity.

For the spatial noncommutatvity, we assume that $C_{02} = C_{03} = 0$. Then, $|M|^2$ is reduced to the following

$$\frac{1}{2}\left(\frac{eg_z}{M_Z}\right)^2 8C_{13}^2 k^2 (M_Z^2 + M_Z(4p_1 - 2k) - 4p_1 k) \qquad (4.36)$$

If we integrate the above equation with respect to p_1 from $M_Z/2-k$ to $M_Z/2$, it is found

$$8C_{13}^2k^2(2M^2k-4Mk^2+\frac{2}{3}k^3) \tag{4.37}$$

Finally, performing integration from 0 to $M_Z/2$ over k yields

$$\frac{23M_ZC_{13}^2}{360} \tag{4.38}$$

Now, let us do the same calculations for the temporal noncommutativity $C_{12}=C_{13}=0$. Then, $|M|^2$ is reduced to the following under this assumption.

$$\begin{aligned}
&\frac{e^2C_{03}^2}{M_Z^2}\ (\ -8M_Zp_1(-M_Z^2/2+M_Zp_1)-24p_1k\ (-M_Z^2/2+M_Zp_1)- \\
&16\ p_1(-M_Z^2/2+M_Zp_1)k^2/M_Z+4M_Z^2\ (-M_Z2/2+M_Zk)- \\
&16M_Zk(-M_Z^2/2+M_Zk)+24k^2(-M_Z^2/2+M_Zk)+ \\
&16k^3(-M_Z^2/2+M_Zk)/M_Z+8M_Zp_1\ (-M_Z^2+M_Zp_1+M_Zk)- \\
&16p_1k(M_Zp_1+M_Zk-M_Z^2)-16p_1k^2\ (M_Zp_1+M_Zk-M_Z^2)/M_Z- \\
&32(M_Zp_1-M_Z^2/2)(M_Zp_1+M_Zk-M_Z^2/2)-16k(M_Zp_1-M_Z^2/2) \\
&(M_Zp_1+M_Zk-M_Z^2/2)/M_Z+48k^2(-M_Z^2/2+M_Zp_1+M_Zk)/M_Z^2 \\
&(M_Zp_1-M_Z^2/2)-8(M_Zp_1+M_Zk-M_Z^2)\ (-M_Z^2/2+M_Zp_1+M_Zk) \\
&-24k(M_Zp_1+M_Zk-M_Z^2)\ (M_Zp_1+M_Zk-M_Z^2/2)/M_Z-16k^2 \\
&(-M_Z^2+M_Zp_1+M_Zk)\ (-M_Z^2/2+M_Zp_1+M_Zk)/M_Z^2)+\frac{1}{2}(\frac{eg_z}{M_Z})^2 \\
&\frac{1}{M_Z(M_Z-2p_1)}\ 8\ (\ C_{02}^2M_Z(M_Z-2p_1)\ (M_Z-y)^2\ (M_Z^2+M_Z \\
&(4p_1-2k)-4p_1k)+C_{03}^2(2M_Z^5(5k-p_1)-M_Z^6+16p_1k^3\ (p_1+k) \\
&(3p_1+k)+2M_Z^4\ (6p_1^2+2p_1k-17k^2)+4M_Z^2k\ (12p_1^3+31p_1^2k-
\end{aligned}$$

$$11p_1k^2 - 8k^3) + 8M_Zk^2\ (-9p_1^3 - 15p_1^2k + k^3) + M_Z^3(-8p_1^3 -$$

$$72p_1^2k + 24p_1k^2 + 50k^3))) \quad (4.39)$$

Let us perform the integration. Since there are massless particles in considering the problem, we have an infrared divergence in lower bound on the integration over the photon energy. More consistent way to remove the infrared divergence is to consider bremstrahlung and radiative $O(\alpha)$ correction diagrams together. Here, we consider more simple way for removing the infrared divergence, namely we take the lower bound of a photon energy not zeo but some finite value E_{min}, where E_{min} is the minimum energy measured in a detector. Because of the infrared divergence, we shift the border of the integration from $(\frac{M_Z}{2} - k - E_{min})$ to $(\frac{M_Z}{2} - E_{min})$ over p_1. Then, the integration gives

$$\frac{4eg_z}{3M_Z^3}(6C_{02}^2\ M_Z^5k - 6C_{03}^2\ M_Z^5k - 12C_{02}^2\ M_Z^3E_{min}^2k + 12C_{03}^2\ M_Z^3E_{min}^2k -$$

$$24C_{02}^2\ M_Z^4k^2 + 48C_{03}^2M_Z^4k^2 + 12C_{03}^2M_Z^3E_{min}k^2 + 24C_{02}^2M_Z^2E_{min}^2k^2 -$$

$$72C_{03}^2M_Z^2E_{min}^2k^2 + 32C_{02}^2M_Z^3k^3 - 137C_{03}^2M_Z^3k^3 - 48C_{03}^2M_Z^2E_{min}k^3 -$$

$$2C_{02}^2M_Zk^5 - 96C_{03}^2M_Zk^5 + 108C_{03}^2M_ZE_{min}^2k^3 - 16C_{02}^2M_Z^2k^4 +$$

$$+180C_{03}^2M_Z^2k^4 + 36C_{03}^2M_ZE_{min}k^4 + 12C_{02}^2M_ZE_{min}^2k^3 - 72C_{03}^2E_{min}^2k^4 +$$

$$24C_{03}^2E_{min}k^5 - 24C_{03}^2M_Z^2k^4Log[2E_{min}] - 24C_{03}^2M_Zk^5Log[2E_{min}] +$$

$$6C_{03}^2M_Z^3k^3Log[2E_{min}] + 6C_{03}^2M_Z^3k^3Log[2E_{min} + 2k] +$$

$$24C_{03}^2M_Zk^5Log[2E_{min} + 2k]) - 24C_{03}^2M_Z^2k^4Log[2E_{min} + 2k] +$$

$$\frac{g_e^2C_{03}^2}{M_Z^2}(-4M_Z^4y + 24M_Z^3k^2 - (64M_Z^2k^3)/3 + (16M_Zk^4)/3). \quad (4.40)$$

Finally, integrating this equation from 0 to $M_Z/2$ over the photon energy k gives

$$\frac{eg_z M_Z^6(1490C_{02}^2 - 187C_{03}^2 + 60C_{03}^2 Log[M_Z/(2E_{min})])}{3600} + \frac{e^2 C_{03}^2 M_Z^6}{5}. \quad (4.41)$$

In the calculation, we take the minimum energy of the photon $E_{min} = 0.05GeV$. Using this value and $M_Z = 91GeV$, we get

$$\frac{eg_z M_Z^6(1490C_{02}^2 + 297C_{03}^2)}{3600}. \quad (4.42)$$

Writing the constants in the differential decay rate and using the equations give the total decay rates

$$\Gamma_{SpatialNC} = \left(\frac{(eg_z)^2 M_Z^5}{32(8\pi)^3\Lambda^4}\right) \frac{23C_{13}^2}{360} \quad (4.43)$$

$$\Gamma_{TemporalNC} = \left(\frac{(eg_z)^2 M_Z^5}{32(8\pi)^3\Lambda^4}\right) \frac{1490C_{02}^2 + 297C_{03}^2}{3600} \quad (4.44)$$

It is interesting to note that for the spatial noncommutativity, the decay rate depends only on the parameter C_{13}. However, in the case of the temporal noncommutativity, the decay rate depends on C_{02} and C_{03}. Figure 4.2 shows the graph of the decay rate versus noncommutative scale Λ_{NC} and the parameter C_{13}.

The dependence of the decay rate on Λ_{NC} and on the noncommutative parameters C_{02}, C_{03} and C_{13} have the same form, but the magnitude of the decay width in C_{02} case is approximately six time larger than C_{13} and C_{03} cases due to the factor in front of C_{02}. The decay widths corresponding C_{13} and C_{03} are nearly equal to each other. Branching ratio is predicted to be in order $Br(Z \to \nu\bar{\nu}\gamma) \sim 10^{-11}$.

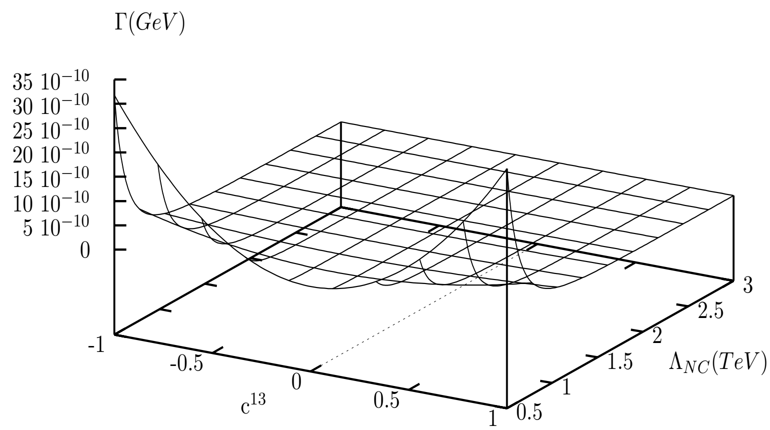

Figure 4.2: The decay rate versus noncommutative scale Λ_{NC} and the matrix element C_{13}

CHAPTER 5

CONCLUSION

In this work, we study the rare $Z \to \nu\bar{\nu}\gamma$ decay in the noncommutative Standard model. This decay is not allowed at tree level in Standard model, whereas it is possible in noncommutative theories.

It is shown that for the temporal noncommutativity, the parameter C_{02} is dominant in the decay rate. The contribution of the square of the parameter C_{03} to the decay rate is 5 times less than that of the square of C_{02}.

As for the spatial noncommutativity, it is shown that the decay rate just depend on the parameter C_{13}.

It is also observed that the new physics enters into the theory by the noncommutative scale parameter Λ. The decay rate depends inversely on the forth power of it. The range of Λ is obtained from low energy experiments $0.5TeV \leq \Lambda \leq 3TeV$.

The noncommutative extension of the Standard model is of great importance from the phenomenological point of view. Many other scattering and decay processes of Z bosons which are absent at least tree level in Standard model naturally come into the theory. One of such a scattering is $ZZ \to \gamma\gamma$. Note that there is no coupling of Z boson and γ photon in Standard model.

REFERENCES

[1] A. Connes, Inst. Hautes Etudes Sci. Publ. Math. **62** (1985) 257.

[2] A. Connes, M. A. Rieffel Contemp. Math. **62** (1987) 237.

[3] A. Connes, M. R. Douglas, A. Schwarz, JHEP **02** (1998) 003.

[4] N. Seiberg and E. Witten , JHEP. **9909**, (1999) 032.

[5] N. Seiberg, JHEP. **0306**, (2003) 010.

[6] J. de Boer, P. A. Grassi, P. Van Nieuwenhuizen, Phys. Lett. B **574**(2003) 98.

[7] A. Bilal, Chong-Sun Chu, R, Russo, Nucl. Phys. B **582** (2000) 65.

[8] A. Konechny, A. Schwarz, Phys. Rept. **360** (2002) 353.

[9] H.S. Synder, Phys. Rev. 71 (1947) 38.

[10] R. J. Szabo, Phys. Rept. **378** (2003) 207.

[11] M. R. Douglas, N. A. Nekrasov, Rev. Mod. Phys. **73** (2001) 977.

[12] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane, T. Okamoto, Phys. Rev. Lett. **87** (2001) 141601.

[13] S. Hellerman, M. V. Raamsdonk, JHEP **0110** (2001) 039.

[14] S. S. Gubser, S. L. Sondhi, Nucl. Phys. B **605** (2001) 395.

[15] H.O. Girotti, M. Gomes, V.O. Rivelles, A.J. da Silva, Nucl. Phys .B **587** (2000) 299.

[16] J. Gomis, T. Mehen, Nucl. Phys. B **591** (2000) 265.

[17] H. Grosse, C. Klimcik, P. Presnajder, Int. J. Theor. Phys. **35** (1996) 231.

[18] M. Chaichian, A. Demichev, P. Presnajder, Nucl. Phys. B **567** (2000) 360.

[19] A. Das, J. Frenkel, S.H. Pereira, J.C. Taylor, Phys. Lett. B **577** (2003) 76.

[20] O. Bertolami, L. Guisado, JHEP **0312** (2003) 013.

[21] N. Mahajan, Phys. Lett. B **569** (2003) 85.

[22] L. Alvarez-Gaume, M.A. Vazquez-Mozo, Nucl. Phys. B **668** (2003) 293.

[23] C. Becchi, S. Giusto, C. Imbimbo, Nucl. Phys. B **664** (2003) 371.

[24] S. Ghosh, Phys. Lett. B **571** (2003) 97.

[25] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda, R. Wulkenhaar, Eur. Phys. J. C **29** (2003) 133.

[26] E. Langmann, Nucl. Phys. B **654** (2003) 404 .

[27] H.O. Girotti, M. Gomes, A.Yu. Petrov, V.O. Rivelles, A.J. da Silva, Phys. Rev. D **67** (2003) 125003.

[28] C. Acatrinei, Phys. Rev. D **67** (2003) 045020.

[29] F. Ruiz, Nucl. Phys. B **637** (2002) 143.

[30] A.Y. Dymarsky, Phys. Lett. B **527** (2002) 125.

[31] Wung-Hong Huang, Phys. Lett. B **496** (2000) 206.

[32] N. Chair, M.M. Sheikh-Jabbari, Phys. Lett. B **504** (2001) 141.

[33] D. Bak, S. Ku Kim, K. Soh, J. H. Yee, Phys. Rev. Lett. **85** (2000) 3087.

[34] Frederic Zamora, JHEP **0005** (2000) 002.

[35] B. A. Campbell, K. Kaminsky, Nucl. Phys. B **581** (2000) 240.

[36] C. E. Carlson, C. D. Carone, R. F. Lebed, Phys. Lett. B **518** (2001) 201.

[37] B. Jurco, L. Moller , S. Schraml, P.Schupp, J. Wess, Eur. Phys. J. C **21** (2001) 383.

[38] B. Jurco, S. Schraml, P. Schupp, J. Wess, Eur. Phys. J. C **17** (2000) 521.

[39] L. Bonora, M. Salizzoni, Phys. Lett. B **504** (2001) 80.

[40] Y. Okawa, H. Ooguri, Phys. Rev. D **64** (2001) 046009.

[41] J. Ambjorn, Y.M. Makeenko, J. Nishimura, R.J. Szabo, Phys. Lett. B **480** (2000) 399.

[42] A. Matusis, L. Susskind, N. Toumbas, JHEP **0012** (2000) 002.

[43] J. Madore, S. Schraml, P. Schupp, Julius Wess, Eur. Phys. J. C **16** (2000) 161.

[44] M.M. Sheikh-Jabbari, Phys. Rev. Lett. **84** (2000) 5265.

[45] X. Calmet, B. Jurco, P. Schupp, J. Wess, M. Wohlgenannt, Eur. Phys. J. C **23** (2002) 363.

[46] D. Bigatti and L. Susskind, Phys. Rev. **D 62** (2000) 066004.

[47] J. Gamboa, M. Loewe and J. C. Rojas, Phys. Rev. **D 64** (2001) 067901.

[48] H.R. Christiansen and F.A. Schaposnik, Phys. Rev. **D 65** (2002) 086005

[49] J. Gamboa, M. Loewe, F. Mendez J. C. Rojas, Mod. Phys. Lett. **A 16** (2001) 2075.

[50] S. Bellucci, A. Nersessian and C. Sochichiu, Phys. Lett. **B 522** (2001) 345.

[51] P. M. Ho and H. C. Kao, Phys. Rev. Lett. **88** (2002) 151602.

[52] C. Acatrinei, JHEP **0109** (2001) 007.

[53] V.P. Nair and A.P. Polychronakos, Phys. Lett. **B 505** (2001) 267.

[54] A. Smailagic and E. Spallucci , Phys. Rev. **D 65** (2002) 107701.

[55] A. Bramon, A. Grau and G. Pancheri, J. Phys. **A 35** (2002) L363.

[56] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, Ann. Phys **260** (1997) 224.

[57] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. **86** (2001) 2716.

[58] D. J. Gross, N. A. Nekrasov, JHEP **0103** (2001) 044.

[59] J. A. Harvey, P. Kraus, F. Larsen, JHEP **0012**(2000) 024.

[60] R. Hawkings and K. Monig, Eur. Phys. J. direct **C 1** (1999) 8.

[61] N. Arkani-Hamed, S. Dimopoulos and Gia Dvali, Phys. Lett. B **429** (1998) 263.

[62] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and Gia Dvali, Phys. Lett. B **463** (1998) 257.

[63] L. Randall, R. Sundrum, Phys. Rev. Lett. **83**(1999) 3370.

[64] J. L. Hewett, F. J. Petriello, T. G. Rizzo, Phys. Rev. D **64** (2001) 075012.

[65] OPAL Collaboration, Phys. Lett. B **568** (2003) 181.

[66] P. Mathews, Phys. Rev. **D 63** (2001) 075007.

[67] S. Godfrey and M. A. Doncheski, Phys. Rev. **D 65** (2002) 015005.

[68] W. Behr, N.G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic, J. Wess, Eur. Phys. J. **C 29** (2003) 441.

[69] I. Mocioiu, M. Pospelov and R. Roiban, Phys. Lett. B **489** (2000) 390.

[70] A. Anisimov, T. Banks, M. Dine, M. Graesser, Phys. Rev. D **65** (2002) 085032.

[71] T.M. Aliev, O. Ozcan, M. Savci, Eur. Phys. J. C **27** (2003) 447.

[72] E.O. Iltan, New J. Phys. **4** (2002) 54.

[73] E.O. Iltan, JHEP **0211** (2002) 029.

[74] H. Grosse, Y. Liao, Phys. Lett. B **520** (2001) 63.

[75] P. Mathews, Phys. Rev. D **63** (2001) 075007.

[76] J. M. Conroy, H. J. Kwee, V. Nazaryan, Phys. Rev. D **68** (2003) 054004.

[77] I. Hinchliffe, N. Kersting, Int. J. Mod. Phys. A **19** (2004) 179.

[78] W. Greiner and A. Schafer, Quantum Chromodynamics, Springer (1994).

[79] F. Halzen, A. D. Martin, Quarks and leptons: an introductory course in modern particle physics, New York Wiley (1984).

VITA

Cem Yüce was born in Eskisehir in 1977. After having graduated from Atatürk High School, he joined to Physics department of Pamukkale University as a B.Sc. student and received his bachelor's degree in 1998. Just after completing master's program at METU in the Department of Physics, he started for Ph.D. studies at Middle East Technical University. He worked as a research assistant in METU.