EXPERIMENTAL DESIGN WITH SHORT-TAILED AND LONG-TAILED SYMMETRIC ERROR DISTRIBUTIONS

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ABSTRACT

EXPERIMENTAL DESIGN WITH SHORT-TAILED AND LONG-TAILED SYMMETRIC ERROR DISTRIBUTIONS

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One-way and two-way classification models in experimental design for both balanced and unbalanced cases are considered when the errors have Generalized Secant Hyperbolic distribution. Efficient and robust estimators for main and interaction effects are obtained by using the modified maximum likelihood estimation (MML) technique. The test statistics analogous to the normal-theory F statistics are defined to test main and interaction effects and a test statistic for testing linear contrasts is defined. It is shown that test statistics based on MML estimators are efficient and robust. The methodology obtained is also generalized to situations where the error distributions from block to block are non-identical.

Keywords: Experimental design, Non-normality, Generalized Secant Hyperbolic, Modified Maximum Likelihood, Robustness.

ÖZ

KISA VE UZUN KUYRUKLU SİMETRİK DAĞILIMA SAHİP HATA TERİMİ İLE DENEYSEL TASARIM

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Tek ve çift yönlü bölümlendirilmiş deneysel tasarım modellerinde dengeli ve dengesiz durumlar için hata terimlerinin Genelleştirilmiş Sekant Hiperbolik dağılımına sahip olması durumu düşünülmüştür. Uyarlanmış en çok olabilirlik metodu ile ana etkiler ve etkileşimler için etkin ve sağlam tahmin ediciler elde edilmiştir. Ana etkiler veya etkileşimler arasında fark olup olmadığını test eden, normal teorideki F istatistiklerine benzer test istatistikleri ve işlemlerin doğrusal bağıntılarını test eden istatistikler tanımlanmıştır. Uyarlanmış en çok olabilirlik metodu ile bulunan tahmin edicilere dayanan test istatistiklerinin etkin ve sağlam oldukları gösterilmiştir. Elde edilen yöntembilim hata terimlerinin özdeş olmadığı duruma genelleştirilmiştir.

Anahtar Kelimeler: Deneysel tasarım, Normal olmayan dağılımlar, Genelleştirilmis Sekant Hiperbolik, Uyarlanmış en çok olabilirlik, Güçlülük.

To My Parents

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CHAPTER 1

INTRODUCTION

In experimental design, most of the statistical procedures are based on the assumption of normality. In practice, however, non-normal distributions occur so frequently. It is, therefore, very important to develop statistical procedures which are appropriate and efficient for non-normal distributions. A number of studies have been carried out to investigate the effect of non-normality on the test statistics used in the analysis of variance. The effect of non-normality on Type I error was studied by Pearson (1931), Geary (1947), Gayen (1950), Box and Andersen (1955), Hack (1958), Box and Watson (1962), Tiku (1964) and the effect of non-normality on Type II error was studied by David and Johnson (1951), Srivastava (1959), Donaldson (1968) and Tiku (1971). The effect of moderate non-normality on the level of significance is known to be not very serious but the power is considerably lower. Şenoğlu and Tiku (2001, 2002) studied the one-way and two-way classification experimental design models with skewed (Generalized Logistic and Weibull) error distributions. Their work does not include symmetric error distributions except the logistic one. To complete this study we consider the model with error terms having a distribution from Generalized Secant Hyperbolic (GSH) family. This family consists of symmetric distributions, with kurtosis ranging from 1.8 to infinity i.e., both short- and long-tailed, and includes the logistic as a special case, the uniform as a limiting case, and closely approximates normal and Student t with corresponding kurtosis (Vaughan, 2002). Hence it can be considered as a more general and flexible one within the symmetric distributions families.

The aim of this thesis is to estimate the parameters of the one-way classification and two-way classification with interaction balanced and unbalanced experimental design models when the error components are independently and identically distributed and have a distribution in the GSH family, examine the statistical properties of the estimators and develop procedures for testing the main and interaction effects and linear contrasts which are of enormous interest from a theoretical as well as a practical point of view. Furthermore, the distributions of these test statistics are developed and the robustness of these statistics is discussed. Finally, the assumption is generalized such that the error distributions from block to block are not necessarily identical.

The outline of this thesis is as follows: Chapter 1 briefly presents the Fisher solution of the problem of estimating and testing the main and interaction effects under the assumption of normality, what has been done in the literature under the assumption of non-normality, a brief theoretical background of the technique employed and the properties of the Generalized Secant Hyperbolic family. In Chapter 2, the estimators of the parameters in one-way classification model, the distribution of F statistic, and the test statistic for testing linear contrasts are found. Two-way classification model with interaction is given in Chapter 3. In Chapter 4, the methodology used in Chapter 2 is generalized to situations where the error distributions from block to block are not identical. Finally, real life applications and conclusions are presented in Chapter 5.

1.1 Historical Perspective

1.1.1 Model Description and Test Procedures Under Normality

i) One-way Classification Model

Consider the one-way classification fixed-effects model

$$y_{ij} = \mu + \tau_i + e_{ij} \qquad (1 \le i \le k, 1 \le j \le n), \tag{1.1.1.1}$$

having k blocks with n observations in each block; y_{ij} is the j^{th} observation in the i^{th} block, μ is the parameter common to all blocks called the overall mean, τ_i is the parameter unique to the i^{th} block called the i^{th} block effect, and e_{ij} is the random error component. In the fixed effects model, the block effects τ_i are usually defined as deviations from the overall mean, so that

$$\sum_{i=1}^{k} \tau_i = 0 . \tag{1.1.1.2}$$

Assume that e_{ij}'s are normally and independently distributed with mean zero and variance σ^2. Then, the observations y_{ij}'s are also normally and independently distributed with mean $\mu + \tau_i$ and variance σ^2.

The classical maximum likelihood estimators are

$$\tilde{\mu} = \bar{y}_{..} , \tag{1.1.1.3}$$

$$\tilde{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} \tag{1.1.1.4}$$

and

$$\tilde{\sigma}^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2}{N-k} \tag{1.1.1.5}$$

where $\bar{y}_{i.} = \frac{1}{n}\sum_{j=1}^{n} y_{ij}$, $\bar{y}_{..} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n} y_{ij}$, $N = kn$.

In testing equality of the k block means

$$H_0 : \mu_1 = \mu_2 = ... = \mu_k$$

$$H_1 : \mu_i \neq \mu_j \text{ for at least one pair (i,j) } (\mu_i = \mu + \tau_i, 1 \le i \le k), \qquad (1.1.1.6)$$

if H_0 is true, all blocks have a common mean μ. An equivalent way to state the above hypotheses is in terms of the block effects τ_i

$$H_0 : \tau_1 = \tau_2 = \ldots = \tau_k = 0$$
$$H_1 : \tau_i \neq 0 \text{ for at least one i } (1 \le i \le k) \qquad (1.1.1.7)$$

The appropriate procedure for testing the equality of k block means is the analysis of variance. The name analysis of variance is derived from the partitioning of total variability into its component parts. The total sum of squares

$$S = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 \qquad (1.1.1.8)$$

is a measure of overall variability in the data and it can be decomposed as

$$\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{..})^2 = n\sum_{i=1}^{k}(\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2$$
$$S \quad = \quad S_1 \quad + \quad S_2 \qquad (1.1.1.9)$$

Under the null hypothesis given in (1.1.1.7), S/σ^2 is distributed as chi-square with N-1 degrees of freedom, S_1/σ^2 is distributed as chi-square with k-1 degrees of freedom, S_2/σ^2 is distributed as chi-square with N-k degrees of freedom irrespective of whether H_0 is true or not. Since the degrees of freedom for S_1 and S_2 add to N-1, the total number of degrees of freedom, Cochran's theorem implies that S_1/σ^2 and S_2/σ^2 are independently distributed chi-square random variables. Therefore, if the null hypothesis is true, the Neyman-Pearson likelihood ratio statistic, also called the Fisher statistic, is

$$F = \frac{S_1/(k-1)}{S_2/(N-k)}, \tag{1.1.1.10}$$

and distributed as central F with k-1 and N-k degrees of freedom.

The null hypothesis H_0 is rejected if the value of F is found to be greater than the tabulated value F_α for a preassigned level of significance α. Thus, Type I error of the test is

$$P(F \geq F_\alpha(k-1, N-k) | H_0) = \alpha. \tag{1.1.1.11}$$

Under the alternative hypothesis, S_1/σ^2 is distributed as noncentral chi-square with k-1 degrees of freedom and noncentrality parameter λ_F^2, $\lambda_F^2 = n\sum_{i=1}^{k}\left(\frac{\tau_i}{\sigma}\right)^2$, and S_2/σ^2 is distributed as central chi-square with N-k degrees of freedom. Thus, if the alternative hypothesis is true, the Fisher statistic is distributed as noncentral F with k-1 and N-k degrees of freedom and noncentrality parameter λ_F^2. The power of the test is given by

$$P(F \geq F_\alpha(k-1, N-k) | H_1) = 1-\beta. \tag{1.1.1.12}$$

ii) Two-way Classification Model

Consider the two-way classification fixed-effects model

$$y_{ijl} = \mu + \tau_i + \delta_j + \gamma_{ij} + e_{ijl} \qquad (1 \leq i \leq k, 1 \leq j \leq c, 1 \leq l \leq n) \tag{1.1.1.13}$$

where y_{ijl} is the l^{th} observation in the i^{th} block and j^{th} column, μ is the overall mean effect, τ_i is the effect of the i^{th} block, δ_j is the effect of the j^{th} column, γ_{ij} is the effect of interaction between the i^{th} block and the j^{th} column, and e_{ijl} is a random error component. Since both the block effects and the column effects are assumed to be fixed, they are defined as deviations from the overall mean. Therefore, $\sum_{i=1}^{k}\tau_i = 0$ and $\sum_{j=1}^{c}\delta_j = 0$. Similarly, the interaction effects are fixed and are defined such that $\sum_{i=1}^{k}\gamma_{ij} = \sum_{j=1}^{c}\gamma_{ij} = 0$.

Assume that e_{ijl}'s are normally and independently distributed with mean zero and variance σ^2. Then, the observations y_{ijl}'s are also normally and independently distributed with mean $\mu + \tau_i + \delta_j + \gamma_{ij}$ and variance σ^2.

The maximum likelihood estimators of the parameters in model (1.1.1.13) are

$$\tilde{\mu} = \bar{y}_{...}, \tag{1.1.1.14}$$

$$\tilde{\tau}_i = \bar{y}_{i..} - \tilde{\mu}, \tag{1.1.1.15}$$

$$\tilde{\delta}_j = \bar{y}_{.j.} - \tilde{\mu}, \tag{1.1.1.16}$$

$$\tilde{\gamma}_{ij} = \bar{y}_{ij.} - \tilde{\mu} - \tilde{\tau}_i - \tilde{\delta}_j \tag{1.1.1.17}$$

and

$$\tilde{\sigma}^2 = \frac{\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n}(y_{ijl} - \tilde{\mu} - \tilde{\tau}_i - \tilde{\delta}_j - \tilde{\gamma}_{ij})^2}{N - kc} \tag{1.1.1.18}$$

where $\bar{y}_{..} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} y_{ijl}$, $\bar{y}_{i..} = \frac{1}{cn}\sum_{j=1}^{c}\sum_{l=1}^{n} y_{ijl}$, $\bar{y}_{.j.} = \frac{1}{kn}\sum_{i=1}^{k}\sum_{l=1}^{n} y_{ijl}$ and $\bar{y}_{ij.} = \frac{1}{n}\sum_{l=1}^{n} y_{ijl}$.

In testing equality of the main and interaction effects

$$H_{01} : \tau_1 = \tau_2 = ... = \tau_k = 0,$$

$$H_{02} : \delta_1 = \delta_2 = ... = \delta_c = 0$$

and

$$H_{03} : \gamma_{ij} = 0 \text{ for all } i = 1, 2,...,k \text{ and } j = 1, 2,...,c, \tag{1.1.1.19}$$

the appropriate procedure is the analysis of variance.

The F statistics based on the LS estimators of the parameters in (1.1.1.14) - (1.1.1.18) for testing H_{01}, H_{02} and H_{03}, respectively, are given by

$$F_1 = \frac{cn\sum_{i=1}^{k} \tilde{\tau}_i^2}{(k-1)\tilde{\sigma}^2} , \tag{1.1.1.20}$$

$$F_2 = \frac{kn\sum_{j=1}^{c} \tilde{\delta}_i^2}{(c-1)\tilde{\sigma}^2} \tag{1.1.1.21}$$

and

$$F_3 = \frac{n\sum_{i=1}^{k}\sum_{j=1}^{c} \tilde{\gamma}_{ij}^2}{(k-1)(c-1)\tilde{\sigma}^2} . \tag{1.1.1.22}$$

Under the null hypotheses, the distributions of F_1, F_2, F_3 are central F with degrees of freedom (k-1, N-kc), (c-1, N-kc) and ((k-1)(c-1), N-kc), respectively. Under the alternative hypotheses, the distributions of F_1, F_2, F_3 are noncentral F with the same degrees of freedom and noncentrality parameters

$$\lambda_1^2 = \frac{cn\sum_{i=1}^{k} \tau_i^2}{\sigma^2}, \ \lambda_2^2 = \frac{kn\sum_{j=1}^{c} \delta_j^2}{\sigma^2} \text{ and } \lambda_3^2 = \frac{n\sum_{i=1}^{k}\sum_{j=1}^{c} \gamma_{ij}^2}{\sigma^2}, \tag{1.1.1.23}$$

respectively.

1.1.2 Robustness

A test is desirable while it is powerful, i.e., sensitive to changes in the specified factors under test, it is also robust, i.e., insensitive to changes in extraneous factors. Specifically, a test is called robust when its significance level is fairly insensitive to departures from the distributional assumption but it maintains high power (Ito, 1980). A statistical hypothesis in the classical analysis of variance is usually tested on the assumption that the observations are (i) independently and (ii) normally distributed (iii) with a common variance. In this study, the observations are assumed to be independently and non-normally distributed and the usual test criteria F is denoted by W when it is used for non-normal distributions.

1.1.3 Non-normality

Analysis of variance procedures have traditionally been based on the assumption of normality. In practice, however, non-normal distributions occur so frequently. Therefore, a number of studies have been made to investigate the effect of non-normality on the test statistics used in the analysis of variance.

1.1.4 Type I Error of the F-test Under Non-normality

The effect of non-normality on the frequency distributions of the variance ratios used for testing the equality of a set of means in one-way classification analysis of variance was first studied by Pearson (1931) by way of sampling experiments. He showed that 'between' and 'within' mean squares still continue to provide unbiased estimates of the population variance, but they are no longer independently distributed; in fact, their variances and covariances contain a term in λ_4 (standardized fourth cumulant). However, in view of the fact that the expressions for the first two moments of their ratio W are, up to certain approximations, independent of the population skewness and kurtosis, he inferred that the normal theory test will not be seriously invalidated, provided the total number of samples is not too small. In fact, he established that the effect of moderate non-normality on the level of significance is not very serious, i.e., the difference between α and the level of significance of the W-test α^*,

$$\alpha^* = P(W > F_\alpha(k-1, N-k) | H_0) \qquad (1.1.4.1)$$

is not considerable.

Considering the effect of kurtosis only, Geary (1947) gave an approximate formula for the probability correction for W, based on the large sample assumption.

Gayen (1950) derived the mathematical form of the distribution of the test statistic W under the null hypothesis for populations characterized by the a priori values of the universal λ's and expressed by the first four terms of the Edgeworth series:

$$f(x_{ij}) = \phi(x_{ij}) - \frac{\lambda_3}{3!}\phi^{(3)}(x_{ij}) + \frac{\lambda_4}{4!}\phi^{(4)}(x_{ij}) + \frac{\lambda_3^2}{72}\phi^{(6)}(x_{ij}) \quad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i) \qquad (1.1.4.2)$$

where $x_{ij} = \dfrac{y_{ij} - \mu}{\sigma_i}$, $\phi(x_{ij})$ is the density function of the standardized normal distribution N(0, 1), $\phi^{(r)}(x_{ij})$ its r^{th} derivative, and $\lambda_3 (= \sqrt{\beta_1})$, $\lambda_4 (= \beta_2 - 3)$ are the measures of skewness and excess of the universe, respectively. Gayen (1950) derived the density function of W in the form:

$$P(W \mid H_0) = P_0(W \mid H_0) - \lambda_4 a(W) + \lambda_3^2 b(W) \qquad (1.1.4.3)$$

where $P_0(W \mid H_0)$ is the density function of the central F distribution with k-1 and N-k degrees of freedom when the null hypothesis is true, and a(W) and b(W) are corrective factors due to nonzero λ_4 and λ_3^2, respectively. He remarked that (1.1.4.3) gives the density function of W for samples of any size drawn from the Edgeworth series (1.1.4.2) when the terms in cumulants other than λ_3, λ_4 and λ_3^2 are negligible and also that it approximates fairly closely to the actual distribution of W for samples drawn from any population with finite cumulants provided the samples are so large that terms in N^{-3} can be neglected. He evaluated the value of the level of significance of the W-test as follows:

$$\alpha^* = \alpha - \lambda_4 A + \lambda_3^2 B \qquad (1.1.4.4)$$

where A and B are certain functions of $F_\alpha(k\text{-}1,N\text{-}k)$ $(N = \sum_{i=1}^{k} n_i)$, k-1, N-k, $n_1, n_2, \ldots, n_k$ and k involving incomplete beta function ratios.

Gayen (1950) restricted the population density function to the first four terms of an Edgeworth series and he did not consider the case when the error distribution from block to block is not identical. Tiku (1964) considered the situation when the distributions of x_{ij}'s are not necessarily identical from block to block, and assumed that the error terms e_{ij} have standard cumulants

$$\lambda_{ri} = \frac{\kappa_{ri}}{\sigma^r}, \qquad r = 3, 4, \ldots \qquad (1.1.4.5)$$

while they are assumed to have the same variance σ^2. Using Laguerre orthogonal polynomials, he expanded the distributions of 'between' and 'within' sum of the squares and derived the distribution of W when the null hypothesis is true, and evaluated the value of the significance level of the W-test as follows:

$$\alpha^* = \alpha - \lambda_4 A + \lambda_3^2 B + \Lambda_3^2 C + \lambda_6 D - \lambda_4^2 E - \Lambda_4^2 H \qquad (1.1.4.6)$$

where $\lambda_r = \frac{1}{k}\sum_{i=}^{k}\lambda_{ri}$, $\Lambda_r\Lambda_s = \frac{1}{k}\sum_{i=1}^{k}\lambda_{ri}\lambda_{si} - \lambda_r\lambda_s$, and A, B, etc. are certain functions of $F_\alpha(k\text{-}1,N\text{-}k)$, k-1, N-k, $n_1, n_2, \ldots, n_k$ and k involving incomplete beta function ratios. If the cumulants vary from block to block, λ_r and $\Lambda_r\Lambda_s$ represent their first and second moments.

Tiku did not restrict the population to any special form and the terms in the sixth and the square of the fourth cumulant are included in addition to the terms in the fourth and the square of the third cumulant in Gayen's formula (1.1.4.3). To compare his results with Gayen's, Tiku provided the corrective terms other than A and B in (1.1.4.6), and remarked that larger effects due to nonnormality appear if the skewness is in different directions in different groups and that they are appreciable unless the degrees of freedom for error, N-k, are fairly large.

1.1.5 Power Function of the F-test Under Non-normality

There have been fewer attempts to investigate the effect of non-normality on the power of the F-test. The power function of the W-test under the general non-normal situation is

$$P(W > F_\alpha(k-1, N-k) | H_1) = 1 - \beta^* \tag{1.1.5.1}$$

where β^* is the actual Type II error probability.

David and Johnson (1951) studied the distribution of F when the expectations, the variances, and also the higher cumulants of the distributions of the error terms may vary from block to block. They obtained the product moments of 'between' and 'within' sum of squares under the most general assumptions of non-normality. However, they did not make direct use of the distribution of a non-central variance ratio as is done in the case of the normal theory power function.

Srivastava (1959) studied the effect of non-normality on the power of the analysis of variance by investigating the non-central distribution of the variance ratio on the assumption that the distribution of the error term is represented by the first four terms of the Edgeworth series. He considered the case of k groups of n observations each, and derived an expression for β^* in terms of confluent hypergeometric functions as follows:

$$1-\beta^* = (1-\beta) + \lambda_3 P + \lambda_4 Q + \lambda_3^2 R \qquad (1.1.5.2)$$

where 1- β is the normal-theory power of the F-test, and P, Q and R are certain functions of F_α(k-1,N-k), k, n, N and three noncentrality parameters

$$\delta_2 = \frac{n\sum_{i=}^{k}\tau_i^2}{\sigma^2}, \quad \delta_3 = \frac{n\sum_{i=}^{k}\tau_i^3}{\sigma^3} \text{ and } \delta_4 = \frac{n\sum_{i=}^{k}\tau_i^4}{\sigma^4}. \qquad (1.1.5.3)$$

However, this formulae is useful only in determining the effects of non-normality which is not of very serious type because Barton and Dennis (1952) showed that only the values of λ_3 and λ_4 within certain limits ($\lambda_3^2 \le 0.2$, $0 \le \lambda_4 \le 2.4$) can be permitted if the Edgeworth series is to represent a positive definite and unimodal frequency function. Therefore, the first four terms of an Edgeworth series only represent near-normal populations (Durand and Greenwood, 1957). Considering Barton and Dennis's limits, the effect of skewness is not much on the power of the W-test and the presence of a fair degree of kurtosis leads to a noticeable change in the power curve particularly in the case of small samples. But a small departure from normality in respect of kurtosis again does not cause any significant deviation in the power. The effect of non-normality on the power diminishes with increasing sample size. As a result, the effect of non-normality on the power will not be of much consequence in the case of near-normal populations.

Donaldson (1968) obtained values of the power for exponential and lognormal distributions through Monte Carlo simulations.

Tiku (1971) generalized his earlier results to obtain the power function of the W-test under the general non-normal situation. Hence he obtained the power function of the F-test from Laguerre series expansion of 'between' and 'within' sum of squares under non-normal situations. His result is as follows:

$$1-\beta^{*} = (1-\beta) - \lambda_3\delta_3 A + \lambda_4(B + B_1\delta_4) - \lambda_3^2 C + \lambda_5\delta_3 D - \lambda_6 E + \lambda_4^2 H \qquad (1.1.5.4)$$

where 1-β is the power of the normal-theory F-test, A, B, B_1, C, D, E and H are corrective functions of F_α(k-1,N-k), k, n, N, δ_2, δ_3 and δ_4 due to non-normality. Note that (1.1.5.4) includes corrections only due to the first few population standard cumulants. Srivastava's equation (1.1.5.2) is similar to (1.1.5.4) but he did not work out the corrective terms due to $\lambda_5, \lambda_6, \lambda_4^2$. For moderately non-normal populations it is expected that the contributions due to higher order standard cumulants will not be important especially for large N. This might not be true for extremely non-normal populations in which case these cumulants could be very large in magnitude and approximations like (1.1.5.4) which ignore these cumulants might not be very useful.

Şenoğlu (2000) showed that the power of the W-test above is considerably lower than the tests constructed by using the Modified Maximum Likelihood (MML) estimators of the parameters. He considered one-way classification and two-way classification models, the latter with interaction, under Weibull and Generalized Logistic error distributions. He derived the MML estimators of the parameters, the distributions of F statistics, and test statistics for testing linear contrasts.

1.2 Theoretical Backround

1.2.1 Generalized Secant Hyperbolic (GSH) Distribution

The properties of a family of distributions generalizing the secant hyperbolic were developed by Vaughan (2002). This family consists of symmetric distributions, with kurtosis ranging from 1.8 to infinity, and includes the logistic as a special case, the uniform as a limiting case, and closely approximates the normal and Student t distributions with corresponding kurtosis. A significant difference between this family and Student t is that for any member of the Generalized Secant Hyperbolic family, all moments are finite. Thus, technical difficulties associated with evaluating moments of Student t are not present with this family. Moreover, the Student t distribution represents only long-tailed symmetric distributions, i.e. its kurtosis $\beta_2 = \mu_4 / \mu_2^2$ is greater than 3. However, short-tailed symmetric distributions with $\beta_2 < 3$ do also occur in practice. For example, Vaughan (2002) showed that an important data set (ages of 100 randomly chosen patients in a coronary heart disease study) is modeled by a short-tailed symmetric distribution with kurtosis $\beta_2 = 2$. Kendall and Stuart (1968, p. 407) give a number of data sets in the context of time series and state that they come from symmetric short-tailed distributions. To have a unified approach to symmetric non-normal distributions, we need a family of distributions which represents both short- and long-tailed distributions. GSH distribution is considered as such a family.

1.2.2 Estimation of Parameters for Location-Scale GSH Distribution

Consider the model (Vaughan, 2002)

$$y_i = \mu + e_i \quad (1 \le i \le n) \qquad (1.2.2.1)$$

where e_i are assumed to be iid and have one of the distributions in the family of Generalized Secant Hyperbolic $(-\pi < t < \infty)$

$$\text{GSH}(0, \sigma; t):\ f(e_i) = \frac{c_1}{\sigma} \frac{\exp(c_2 e_i / \sigma)}{\exp(2c_2 e_i / \sigma) + 2a \exp(c_2 e_i / \sigma) + 1} \quad (-\infty < e_i < \infty) \qquad (1.2.2.2)$$

where for $-\pi < t \le 0$:

$$a = \cos(t)\,,\ c_2 = \sqrt{(\pi^2 - t^2)/3} \text{ and } c_1 = \frac{\sin(t)}{t} c_2$$

and for $t > 0$:

$$a = \cosh(t)\,,\ c_2 = \sqrt{(\pi^2 + t^2)/3} \text{ and } c_1 = \frac{\sinh(t)}{t} c_2\,.$$

For t > π, t < π and t = π, GSH(0, σ; t) represents short-tailed, long-tailed and approximately normal distributions, respectively.

The coefficient of kurtosis, $\beta_2 = \mu_4 / \mu_2^2$, for a few representative values of the shape parameter t is given below:

$t =$	$-\pi\sqrt{2/3}$	$-\pi/2$	0	π	$\pi\sqrt{11}$	∞
$\beta_2 =$	9.0	5.0	4.2	3.0	2.0	1.8

i) Maximum Likelihood Estimation

Given a random sample $y_1, y_2, \ldots, y_n$ of size n from GSH distribution, the Fisher likelihood function is

$$L = c_1^n \frac{\exp\left(c_2 \sum_{i=1}^{n} z_i\right)}{\prod_{i=1}^{n} \left[\exp(2c_2 z_i) + 2a\exp(c_2 z_i) + 1\right]} \tag{1.2.2.3}$$

where $z_i = \dfrac{y_i - \mu}{\sigma}$ $(1 \le i \le n)$.

The likelihood equations for estimating μ and σ are

$$\frac{\partial \ln L}{\partial \mu} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{n} g(z_i) = 0 \qquad (1.2.2.4)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -n\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{n} z_i + 2\frac{c_2}{\sigma}\sum_{i=1}^{n} z_i g(z_i) = 0 \qquad (1.2.2.5)$$

where $g(z_i) = \dfrac{\exp(2c_2 z_i) + a\exp(c_2 z_i)}{\exp(2c_2 z_i) + 2a\exp(c_2 z_i) + 1}$.

Equations (1.2.2.4) and (1.2.2.5) have no explicit solutions since the terms involve the nonlinear function $g(z_i)$. An iterative process can be used to solve these equations, but without extensive simulations, the properties of the resulting maximum likelihood estimates are difficult to determine, especially for small samples. An alternative estimation procedure called the modified maximum likelihood overcomes the difficulties mentioned above.

ii) Modified Maximum Likelihood Estimation

Tiku and Suresh (1992) introduced modified maximum likelihood estimation for location-scale models, with the following properties:

1. The estimates are explicit functions of sample observations and are easier to compute than the maximum likelihood estimates.
2. It is aymptotically equivalent to maximum likelihood when regularity conditions hold (Tiku et al., 1986; Vaughan and Tiku, 2000 and Bhattacharyya, 1985).
3. The estimates are almost fully efficient in terms of the Minimum Variance Bounds (MVBs) even for small samples.
4. The estimates have little or no bias.

5. The method is essentially self-censoring, since it assigns small weights to extremes.

For these reasons, Vaughan (2002) used modified maximum likelihood estimation technique in his analysis.

Tiku's Modified Maximum Likelihood methodology proceeds in three steps as follows:

1. Express the likelihood equations in terms of ordered variates $z_{(i)} = \frac{y_{(i)} - \mu}{\sigma}$ $(1 \leq i \leq n)$,
2. linearize the intractable terms in the likelihood equations by using the first two terms of the Taylor series expansion and
3. solve the resulting equations to get the modified maximum likelihood estimators.

Since complete sums are invariant to ordering, the likelihood equations can be written as follows:

$$\frac{\partial \ln L}{\partial \mu} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{n} g(z_{(i)}) = 0 \qquad (1.2.2.6)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -n\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{n} z_{(i)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{n} z_{(i)} g(z_{(i)}) = 0\,. \qquad (1.2.2.7)$$

Let $t_{(i)} = E(z_{(i)})$, $t_{(i)}$ $(1 \leq i \leq n)$ are the expected values of the standardized ordered variates. For large n, the values of $t_{(i)}$ can be found as follows:

$$t_{(i)} = \frac{1}{c_2}\ln\left[\frac{\sin\left(t\frac{i}{n+1}\right)}{\sin\left(t\left(1 - i/(n+1)\right)\right)}\right], \quad -\pi < t < 0$$

$$= \frac{\sqrt{3}}{\pi} \ln\left(\frac{i/(n+1)}{1 - i/(n+1)}\right), \qquad t = 0$$

$$= \frac{1}{c_2} \ln\left[\frac{\sinh\left(t \frac{i}{n+1}\right)}{\sinh\left(t\left(1 - i/(n+1)\right)\right)}\right], \quad t > 0. \qquad (1.2.2.8)$$

For small n, the values of $t_{(i)}$ can be found by using the formula of $E(y_{(i)})$ given by Vaughan (2002).

Since $z_{(i)}$ is located in the vicinity of $t_{(i)}$, it is approximated by the Taylor series expansion

$$g(z_{(i)}) \cong g(t_{(i)}) + (z_{(i)} - t_{(i)}) g'(t_{(i)})$$

$$= \alpha_{(i)} + \beta_{(i)} z_{(i)} \qquad (1.2.2.9)$$

where

$$\alpha_{(i)} = \frac{\exp(2c_2 t_{(i)}) + a \exp(c_2 t_{(i)})}{\exp(2c_2 t_{(i)}) + 2a \exp(c_2 t_{(i)}) + 1} - \beta_{(i)} t_{(i)}$$

and

$$\beta_{(i)} = \frac{ac_2 \exp(3c_2 t_{(i)}) + 2c_2 \exp(2c_2 t_{(i)}) + ac_2 \exp(c_2 t_{(i)})}{\left[\exp(2c_2 t_{(i)}) + 2a \exp(c_2 t_{(i)}) + 1\right]^2}.$$

Since $t_{(i)} = -t_{(n-i+1)}$ from symmetry, $\sum_{i=1}^{n} \alpha_{(i)} = \frac{n}{2}$ and $\sum_{i=1}^{n} \beta_{(i)} t_{(i)} = 0$.

The following modified likelihood equations are obtained by incorporating (1.2.2.9) in (1.2.2.6) and (1.2.2.7):

$$\frac{\partial \ln L}{\partial \mu} \cong \frac{\partial \ln L^*}{\partial \mu} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{n}\left[\alpha_{(i)} + \beta_{(i)} z_{(i)}\right] = 0 \tag{1.2.2.10}$$

and

$$\frac{\partial \ln L}{\partial \sigma} \cong \frac{\partial \ln L^*}{\partial \sigma} = -n\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{n} z_{(i)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{n} z_{(i)}\left[\alpha_{(i)} + \beta_{(i)} z_{(i)}\right] = 0. \tag{1.2.2.11}$$

The simultaneous solutions of the equations (1.2.2.10) and (1.2.2.11) are the MML estimators:

$$\hat{\mu} = \frac{\sum_{i=1}^{n}\beta_{(i)} y_{(i)}}{m} \tag{1.2.2.12}$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4nC}}{2\sqrt{n(n-1)}} \tag{1.2.2.13}$$

where $m = \sum_{i=1}^{n}\beta_{(i)}$, $B = nc_2(\bar{y} - \bar{y}_a)$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_{(i)}$, $\bar{y}_a = \frac{2}{n}\sum_{i=1}^{n}\alpha_{(i)} y_{(i)}$ and

$$C = 2c_2\sum_{i=1}^{n}\beta_{(i)}(y_{(i)} - \hat{\mu})^2 = \sum_{i=1}^{n}\beta_{(i)} y_{(i)}^2 - m\hat{\mu}^2 .$$

The divisor n in the original expression for $\hat{\sigma}$ is replaced by $\sqrt{n(n-1)}$ to reduce the bias.

It may be noted that (1.2.2.9) is an asymptotically strict equality. Moreover, in the limit when n tends to infinity

$$\lim_{n\to\infty}\frac{1}{n}\frac{\partial \ln L}{\partial \mu} \equiv \lim_{n\to\infty}\frac{1}{n}\frac{\partial \ln L^*}{\partial \mu} \equiv 0 \tag{1.2.2.14}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\frac{\partial \ln L}{\partial\sigma} \equiv \lim_{n\to\infty}\frac{1}{n}\frac{\partial \ln L^{*}}{\partial\sigma} \equiv 0 \qquad (1.2.2.15)$$

Consequently, the MML estimators $\hat{\mu}$ and $\hat{\sigma}$ above are asymptotically equivalent to ML estimators and, thus, $\hat{\mu}$ and $\hat{\sigma}$ are asymptotically unbiased and efficient, at least heuristically. Note, however, that $\hat{\mu}$ is unbiased for all n due to symmetry.

CHAPTER 2

ONE-WAY CLASSIFICATION

In this chapter parameters of the one-way classification model for balanced and unbalanced designs are estimated under the assumption of Generalized Secant Hyperbolic (GSH) ditributed error terms. Statistical properties of the estimators are studied, the test statistics for testing the block effects and linear contrasts are developed and the robustness of the test statistics are examined.

2.1 Balanced Design

Consider the one-way classification fixed-effects model

$$y_{ij} = \mu + \tau_i + e_{ij} \qquad (i = 1, 2,\ldots, k;\ j = 1, 2,\ldots, n), \tag{2.1.1}$$

having k blocks with n observations in each block. Without loss of generality assume that

$$\sum_{i=1}^{k} \tau_i = 0. \tag{2.1.2}$$

In (2.1.1), suppose that the errors e_{ij} are iid and have one of the distributions in the family of GSH distribution given in (1.2.2.2).

2.1.1 Maximum Likelihood Estimation

The Fisher likelihood function is

$$L = \frac{c_1^N}{\sigma^N} \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{\exp(c_2 z_{ij})}{\exp(2c_2 z_{ij}) + 2a\exp(c_2 z_{ij}) + 1} \qquad (2.1.1.1)$$

where $N = kn$, $z_{ij} = \dfrac{y_{ij} - \mu - \tau_i}{\sigma}$ $(1 \le i \le k, 1 \le j \le n)$.

The likelihood equations for estimating μ, τ_i $(1 \le i \le k)$ and σ are

$$\frac{\partial \ln L}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} g(z_{ij}) = 0, \qquad (2.1.1.2)$$

$$\frac{\partial \ln L}{\partial \tau_i} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{n} g(z_{ij}) = 0 \qquad (2.1.1.3)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} z_{ij} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} z_{ij}\, g(z_{ij}) = 0 \qquad (2.1.1.4)$$

where $g(z_{ij}) = \dfrac{\exp(2c_2 z_{ij}) + a\exp(c_2 z_{ij})}{\exp(2c_2 z_{ij}) + 2a\exp(c_2 z_{ij}) + 1}$.

The Fisher information matrix is

$$I = \begin{bmatrix} -E\left(\dfrac{\partial^2 \ln L}{\partial \mu_i^2}\right) & -E\left(\dfrac{\partial^2 \ln L}{\partial \mu_i \partial \sigma}\right) \\ -E\left(\dfrac{\partial^2 \ln L}{\partial \mu_i \partial \sigma}\right) & -E\left(\dfrac{\partial^2 \ln L}{\partial \sigma^2}\right) \end{bmatrix} \qquad (2.1.1.5)$$

where $E\left(\frac{\partial^2 \ln L}{\partial \mu_i \partial \sigma}\right) = 0$,

for $-\pi < t < 0$, $$E\left(\frac{\partial^2 \ln L}{\partial \mu_i^2}\right) = -\frac{c_2^2 n(t - \sin t \cos t)}{2\sigma^2 t \sin^2 t}$$

$$E\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right) = -\frac{N}{6\sigma^2}\left(\frac{\pi^2 - t^2}{\sin^2 t} - \frac{(\pi^2 - 3t^2)\cos t}{t \sin t}\right)$$

for $t \geq 0$, $$E\left(\frac{\partial^2 \ln L}{\partial \mu_i^2}\right) = -\frac{c_2^2 n(\sinh t \cosh t - t)}{2\sigma^2 t \sinh^2 t}$$

$$E\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right) = -\frac{N}{6\sigma^2}\left(\frac{(\pi^2 + 3t^2)\cosh t}{t \sinh t} - \frac{\pi^2 + t^2}{\sinh^2 t}\right).$$

The variance-covariance matrix is $V = I^{-1} = (V_{ij})$, where

$$V_{11} = -\frac{1}{E\left(\frac{\partial^2 \ln L}{\partial \mu_i^2}\right)}, \quad V_{12} = V_{21} = 0 \text{ and } V_{22} = -\frac{1}{E\left(\frac{\partial^2 \ln L}{\partial \sigma^2}\right)}.$$

Equations (2.1.1.2)-(2.1.1.4) do not admit explicit solutions because of the terms involving the nonlinear function g(z). Solving these equations by iteration is difficult and time consuming since there are k+1 equations to solve simultaneously. Even if these equations can be solved by iteration, without extensive simulations the properties of the resulting maximum likelihood estimates are difficult to determine, especially for small samples (Vaughan, 2002). To alleviate these difficulties, the method of modified maximum likelihood is used (Tiku, 1967, 1968; Tiku and Suresh, 1992). This method gives explicit and highly efficient estimators (Smith et al., 1973; Tan, 1985; Vaughan, 1992, 2002).

2.1.2 Modified Maximum Likelihood Estimation

Let

$$y_{i(1)} \le y_{i(2)} \le \ldots \le y_{i(n)} \qquad (1 \le i \le k) \tag{2.1.2.1}$$

be the order statistics of the n observations y_{ij} $(1 \le j \le n)$ in the i^{th} block. Then

$$z_{i(j)} = \frac{y_{i(j)} - \mu - \tau_i}{\sigma} \qquad (1 \le i \le k) \tag{2.1.2.2}$$

are the ordered z_{ij} $(1 \le j \le n)$ variates. Since complete sums are invariant to ordering, the likelihood equations are obtained by replacing z_{ij} by $z_{i(j)}$. Hence the likelihood equations become

$$\frac{\partial \ln L}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} g(z_{i(j)}) = 0, \tag{2.1.2.3}$$

$$\frac{\partial \ln L}{\partial \tau_i} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{n} g(z_{i(j)}) = 0 \tag{2.1.2.4}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} z_{i(j)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} z_{i(j)} g(z_{i(j)}) = 0. \tag{2.1.2.5}$$

Since the function g(z) is almost linear in small intervals $a < z < b$ (Tiku, 1967, 1968) and $z_{i(j)}$ is located in the vicinity of $t_{i(j)} = E(z_{i(j)})$ at any rate for large n, an appropriate linear approximation for $g(z_{i(j)})$ $(1 \le i \le k)$ is obtained by using the first two terms of a Taylor series expansion, namely

$$g(z_{i(j)}) \cong g(t_{i(j)}) + g'(t_{i(j)})(z_{i(j)} - t_{i(j)})$$

$$= \alpha_{i(j)} + \beta_{i(j)} z_{i(j)} \qquad (1 \le j \le n) \tag{2.1.2.6}$$

where $t_{i(j)} = E(z_{i(j)})$ is the expected value of the j^{th} order statistic $z_{i(j)}$ in the i^{th} block,

$$\alpha_{i(j)} = g(t_{i(j)}) - \beta_{i(j)} t_{i(j)} \text{ and } \beta_{i(j)} = g'(t_{i(j)}).$$

Here,

$$t_{1(j)} = t_{2(j)} = \ldots = t_{k(j)} = t_{(j)}$$

$$\alpha_{1(j)} = \alpha_{2(j)} = \ldots = \alpha_{k(j)} = \alpha_{(j)}$$

$$\beta_{1(j)} = \beta_{2(j)} = \ldots = \beta_{k(j)} = \beta_{(j)} \quad \text{for all } j = 1,2,\ldots,n \tag{2.1.2.7}$$

and

$$\alpha_{(j)} = \frac{\exp(2c_2 t_{(j)}) + a\exp(c_2 t_{(j)})}{\exp(2c_2 t_{(j)}) + 2a\exp(c_2 t_{(j)}) + 1} - \beta_j t_{(j)}, \tag{2.1.2.8}$$

$$\beta_{(j)} = \frac{ac_2\exp(3c_2 t_{(j)}) + 2c_2\exp(2c_2 t_{(j)}) + ac_2\exp(c_2 t_{(j)})}{\left[\exp(2c_2 t_{(j)}) + 2a\exp(c_2 t_{(j)}) + 1\right]^2}. \tag{2.1.2.9}$$

When $\beta_{(j)} < 0$, we set $\beta_{(j)} = 0$ (Vaughan, 2002). Thus, $\hat{\sigma}$ is always real and positive. Here $\sum_{j=1}^{n} \alpha_{(j)} = \frac{n}{2}$ and $\sum_{j=1}^{n} \beta_{(j)} t_{(j)} = 0$.

Remark: It may be noted that the coefficients $\beta_{(j)}$ have inverted umbrella ordering, i.e., they decrease until the middle value and then increase in a symmetric fashion when the GSH(0, σ; t) represents short-tailed distributions. The coefficients $\beta_{(j)}$ have umbrella ordering when the GSH(0, σ; t) represents long-tailed and approximately normal distributions. This gives MML estimators excellent robustness properties.

Although the formulation to find the exact values of expected values $t_{(j)}$, $1 \le j \le n$, is available (Vaughan, 2002), it is not practical. Thus, we can safely use their approximate values for $n \ge 10$ such that

$$t_{(j)} = \frac{1}{c_2} \ln\left[\sin(tq_j)/\sin(t(1-q_j))\right], \qquad -\pi < t < 0$$

$$= \frac{\sqrt{3}}{\pi} \ln(q_j/(1-q_j)), \qquad t = 0$$

$$= \frac{1}{c_2} \ln\left[\sinh(tq_j)/\sinh(t(1-q_j))\right], \qquad t > 0 \qquad (2.1.2.10)$$

where $q_j = j/(n+1)$.

The use of these approximate values in place of the exact values does not affect the efficiency of the MML estimators in any substantial way.

Incorporating (2.1.2.6) into (2.1.2.3)-(2.1.2.5), the following modified likelihood equations are obtained

$$\frac{\partial \ln L}{\partial \mu} \cong \frac{\partial \ln L^*}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left[\alpha_{(j)} + \beta_{(j)} z_{i(j)}\right] = 0, \qquad (2.1.2.11)$$

$$\frac{\partial \ln L}{\partial \tau_i} \cong \frac{\partial \ln L^*}{\partial \tau_i} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{n}\left[\alpha_{(j)} + \beta_{(j)} z_{i(j)}\right] = 0 \qquad (2.1.2.12)$$

and

$$\frac{\partial \ln L}{\partial \sigma} \cong \frac{\partial \ln L^*}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n} z_{i(j)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n}\left[\alpha_{(j)} + \beta_{(j)} z_{i(j)}\right] z_{i(j)} = 0. \qquad (2.1.2.13)$$

These equations are asymptotically equivalent to the corresponding likelihood equations (2.1.2.3)-(2.1.2.5) and their solutions yield the following MML estimators:

$$\hat{\mu} = \hat{\mu}_{..}, \tag{2.1.2.14}$$

$$\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu} \tag{2.1.2.15}$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N-k)}} \tag{2.1.2.16}$$

where

$$\hat{\mu}_{..} = \frac{1}{km}\sum_{i=1}^{k}\sum_{j=1}^{n}\beta_{(j)}y_{i(j)}, \; \hat{\mu}_{i.} = \frac{1}{m}\sum_{j=1}^{n}\beta_{(j)}y_{i(j)}, \; m = \sum_{j=1}^{n}\beta_{(j)},$$

$$B = Nc_2(\bar{y}_{..} - \bar{y}_a), \; C = 2c_2\sum_{i=1}^{k}\sum_{j=1}^{n}\beta_{(j)}\left(y_{i(j)} - \hat{\mu} - \hat{\tau}_i\right)^2,$$

$$\bar{y}_{..} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n}y_{i(j)} \text{ and } \bar{y}_a = \frac{2}{N}\sum_{i=1}^{k}\sum_{j=1}^{n}\alpha_{(j)}y_{i(j)}.$$

Note that $\hat{\sigma}$ is the bias-corrected estimator of σ. The estimators are explicit functions of sample observations and, therefore easy to compute.

Lemma 2.1: Asymptotically, the estimator $\hat{\mu}_i = \hat{\mu}_{i.}$ is the MVB estimator of μ_i $(= \mu + \tau_i)$ and is normally distributed with variance

$$V(\hat{\mu}_i) \cong \frac{\sigma^2}{2mc_2}. \tag{2.1.2.17}$$

Proof: Since $\partial \ln L^* / \partial \mu_i$ is asymptotically equivalent to $\partial \ln L / \partial \mu_i$ and assumes the form

$$\frac{\partial \ln L^*}{\partial \mu_i} = \frac{2mc_2}{\sigma^2}(\hat{\mu}_i - \mu_i), \qquad (2.1.2.18)$$

(2.1.2.17) is obtained. By dividing both sides of (2.1.2.18) by n, we can apply central limit theorem and since $E\left[\frac{\partial^r \ln L^*}{\partial \mu_i{}^r}\right] = 0$ for all $r \geq 3$, $\hat{\mu}_i$ is asymptotically normally distributed.

Corollary 2.1: Asymptotically, the estimator $\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu}$ is the MVB estimator of τ_i and is normally distributed with variance

$$V(\hat{\tau}_i) \cong \frac{\sigma^2}{2mc_2}. \qquad (2.1.2.19)$$

Proof: As in Lemma 2.1, since $\partial \ln L^* / \partial \tau_i$ is asymptotically equivalent to $\partial \ln L / \partial \tau_i$ and assumes the form

$$\frac{\partial \ln L^*}{\partial \tau_i} = \frac{2mc_2}{\sigma^2}(\hat{\tau}_i - \tau_i), \qquad (2.1.2.20)$$

(2.1.2.19) is obtained and since $E\left[\frac{\partial^r \ln L^*}{\partial \tau_i{}^r}\right] = 0$ for all $r \geq 3$, $\hat{\tau}_i$ is asymptotically normally distributed.

Corollary 2.2: Asymptotically, the estimator $\hat{\mu} = \hat{\mu}_{..}$ is the MVB estimator of μ and is normally distributed with variance

$$V(\hat{\mu}) \cong \frac{\sigma^2}{2kmc_2}. \tag{2.1.2.21}$$

Proof: As in Lemma 2.1, since $\partial \ln L^* / \partial\mu$ is asymptotically equivalent to $\partial \ln L / \partial\mu$ and assumes the form

$$\frac{\partial \ln L^*}{\partial\mu} = \frac{2kmc_2}{\sigma^2}(\hat{\mu} - \mu), \tag{2.1.2.22}$$

(2.1.2.21) is obtained and since $E\left[\frac{\partial^r \ln L^*}{\partial\mu^r}\right] = 0$ for all $r \geq 3$, $\hat{\mu}$ is asymptotically normally distributed.

Corollary 2.3: Since $\hat{\mu}_i$ $(1 \leq i \leq k)$ are independent of each other and $\hat{\mu} = \frac{1}{k}\sum_{i=1}^{k}\hat{\mu}_i$,

$$V(\hat{\tau}_i) \cong \frac{(k-1)\sigma^2}{2kmc_2}. \tag{2.1.2.23}$$

Remark: The estimators $\hat{\tau}_i$ and $\hat{\sigma}$ are uncorrelated and since $E\left(\frac{\partial^{r+s} \ln L^*}{\partial\tau_i^{\,r}\partial\sigma^s}\right) = 0$ for all $r \geq 1$ and $s \geq 1$, asymptotically, they are independent of each other.

Lemma 2.2: Asymptotically, $\frac{N\hat{\sigma}^2(\mu_i)}{\sigma^2}$ is conditionally ($\mu_i = \mu + \tau_i$) distributed as chi-square with N degrees of freedom.

Proof: For large n, $\frac{B}{\sqrt{nC}} \cong 0$ where $C = 2c_2 \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \mu_i)^2$. Therefore, it can be shown that

$$\frac{\partial \ln L^*}{\partial \sigma} \cong \frac{N}{\sigma^3}\left(\frac{C}{N} - \sigma^2\right). \tag{2.1.2.24}$$

Asymptotically, $\frac{C}{N}$ is the MVB estimator of σ^2. Evaluation of the cumulants of $\frac{\partial \ln L^*}{\partial \sigma}$ in terms of the expected values of the derivatives of $\frac{\partial \ln L^*}{\partial \sigma}$ immediately leads to the result that $\frac{N\hat{\sigma}^2(\mu_i)}{\sigma^2}$ is distributed as chi-square with N degrees of freedom (Bartlett, 1953).

Corollary 2.4: Asymptotically, $\frac{N\hat{\sigma}^2}{\sigma^2}$ is distributed as chi-square with N-k degrees of freedom.

2.1.3 Efficiency Properties

The estimator $\hat{\mu}_i$ is unbiased, in fact, it is asymptotically minimum variance bound (MVB) estimator of μ_i, and is normally distributed. Therefore, $\hat{\mu}_i$ is BAN estimator. The MVB for estimating μ_i is as follows:

for $-\pi < t \leq 0$

$$MVB(\mu_i) = \frac{2\sigma^2 t \sin^2 t}{nc_2^2 (t - \sin t \cos t)}, \tag{2.1.3.1}$$

for $t \geq 0$

$$\mathrm{MVB}(\mu_i) = \frac{2\sigma^2 t \sinh^2 t}{n c_2^2 (\sinh t \cosh t - t)}. \qquad (2.1.3.2)$$

The estimator $\hat{\sigma}^2$ is asymptotically the MVB estimator of σ^2 and is distributed as a multiple of chi-square; see Lemma 2.2. The MVB for estimating σ^2 is as follows:

for $-\pi < t \le 0$

$$\mathrm{MVB}(\sigma) = \left(\frac{6\sigma^2}{N}\right) \Bigg/ \left(\frac{\pi^2 - t^2}{\sin^2 t} - \frac{(\pi^2 - 3t^2)\cos t}{t \sin t}\right), \qquad (2.1.3.3)$$

for $t \ge 0$

$$\mathrm{MVB}(\sigma) = \left(\frac{6\sigma^2}{N}\right) \Bigg/ \left(\frac{(\pi^2 + 3t^2)\cosh t}{t \sinh t} - \frac{\pi^2 + t^2}{\sinh^2 t}\right). \qquad (2.1.3.4)$$

Given in Table 2.1 are the simulated values (based on N=100,000/n Monte Carlo runs) of the variances of the MML and LS estimators of μ_i $(1 \le i \le k)$, relative efficiency (RE) of the LS estimator $\tilde{\mu}_i = \bar{y}_{i.}$, the MVB of μ_i and the efficiency (E) of $\hat{\mu}_i$.

It can be seen that $\hat{\mu}_i$ is considerably more efficient than $\tilde{\mu}_i$ even for small sample sizes other than approximately normal distribution ($\beta_2 = 3.0$). Actually, for approximately normal distribution $\hat{\mu}_i$ is as efficient as $\tilde{\mu}_i$. A disconcerting feature of $\tilde{\mu}_i$ is that its relative efficiency decreases as sample size n increases. Realize that both $\hat{\mu}_i$ and $\tilde{\mu}_i$ are unbiased estimators of μ_i. The parameter σ is of much less importance than μ in the model (2.1.1). It must be said, however, that the MML estimator $\hat{\sigma}$ can sometimes have larger bias than $\tilde{\sigma}$ for small sample sizes. Thus, deficiency of MML and LS estimators are calculated through simulations.

Table 2.1 Variances of the LS and MML estimators of μ_i $(1 \le i \le k)$;
(1) $V(\tilde{\mu}_i)/\sigma^2$ (2) $V(\hat{\mu}_i)/\sigma^2$ (3) $RE(\tilde{\mu}_i) = [V(\hat{\mu}_i)/V(\tilde{\mu}_i)]*100$
(4) $MVB(\mu_i)/\sigma^2$ (5) $E(\hat{\mu}_i) = [MVB(\mu_i)/V(\hat{\mu}_i)]*100$

k = 4	$\beta_2 =$	2.0	3.0	4.2	5.0	9.0
n = 6	(1)	0.165	0.165	0.165	0.164	0.166
	(2)	0.131	0.170	0.158	0.146	0.103
	(3)	79.82	102.56	95.62	88.88	61.96
	(4)	0.088	0.162	0.152	0.135	0.076
	(5)	66.93	76.20	96.30	92.65	74.29
n = 10	(1)	0.099	0.100	0.100	0.101	0.980
	(2)	0.070	0.101	0.094	0.086	0.055
	(3)	70.60	100.44	94.46	85.66	56.59
	(4)	0.053	0.097	0.091	0.081	0.046
	(5)	75.54	96.68	96.77	93.69	82.63
n = 15	(1)	0.066	0.067	0.066	0.065	0.064
	(2)	0.043	0.068	0.061	0.055	0.033
	(3)	64.65	100.44	92.54	83.67	52.23
	(4)	0.035	0.065	0.061	0.054	0.031
	(5)	82.31	96.09	98.97	98.70	91.34
n = 20	(1)	0.049	0.050	0.050	0.050	0.049
	(2)	0.030	0.049	0.046	0.041	0.025
	(3)	61.14	99.13	92.31	82.43	50.96
	(4)	0.026	0.049	0.046	0.041	0.023
	(5)	87.29	98.97	98.85	98.93	91.76

Given in Table 2.2 are the simulated values of the deficiencies (Def) of the least square estimators $\tilde{\mu}_i = \bar{y}_{i.}$ and $\tilde{\sigma}^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2/(N-k)$ and the MML estimators $\hat{\mu}_i$ and $\hat{\sigma}^2$. Note that since $\hat{\mu}_i$ and $\hat{\sigma}^2$ are uncorrelated with one another and so are the LS estimators $\tilde{\mu}_i$ and $\tilde{\sigma}^2$ (this follows from symmetry), the joint deficiencies can be calculated as follows:

$$\mathrm{Def}(\tilde{\mu}_i, \tilde{\sigma}) = \mathrm{MSE}(\tilde{\mu}_i) + \mathrm{MSE}(\tilde{\sigma})$$
$$\mathrm{Def}(\hat{\mu}_i, \hat{\sigma}) = \mathrm{MSE}(\hat{\mu}_i) + \mathrm{MSE}(\hat{\sigma}). \qquad (2.1.3.5)$$

Table 2.2 Deficiencies of the LS and MML estimators of μ_i $(1 \le i \le k)$ and σ
(1) Def$(\tilde{\mu}_i, \tilde{\sigma})$ (2) Def$(\hat{\mu}_i, \hat{\sigma})$

k = 4	$\beta_2 =$	2.0	2.5	3.0	4.2	5.0	7.0	9.0
n = 7	(1)	0.153	0.160	0.163	0.175	0.182	0.198	0.209
	(2)	0.146	0.154	0.167	0.177	0.176	0.183	0.223
n = 10	(1)	0.109	0.114	0.114	0.120	0.125	0.138	0.149
	(2)	0.084	0.109	0.117	0.119	0.114	0.115	0.135
n = 15	(1)	0.072	0.074	0.076	0.081	0.085	0.090	0.100
	(2)	0.048	0.069	0.077	0.077	0.074	0.069	0.079
n = 20	(1)	0.055	0.056	0.057	0.060	0.064	0.069	0.075
	(2)	0.035	0.052	0.057	0.057	0.054	0.051	0.054

Deficiency of MML estimators are considerably smaller than the defficiency of LS estimators even for sample size n = 7 other than approximately normal ($\beta_2 = 3.0$), near normal (logistic, $\beta_2 = 4.2$) and very long-tailed ($\beta_2 = 9.0$) distributions. However, for n $\ge$ 9 defficiency of MML estimators becomes smaller than that of LS estimators for near normal and long-tailed distributions.

2.1.4 Testing Block Effects

To test the equality of block effects, that is, to test the null hypothesis

$$H_0 : \tau_1 = \tau_2 = ... = \tau_k = 0$$

against the alternative hypothesis

$$H_1 : \tau_i \neq 0 \text{ for at least one i =1,2,...,k,}$$

the following decomposition of sum of squares which is structurally the same as that based on the normal samples is obtained:

Under H_0, the MML estimator of σ is

$$\hat{\sigma}_0 = \frac{-B + \sqrt{B^2 + 4NC_0}}{2N} \tag{2.1.4.1}$$

where $C_0 = 2c_2 \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{..})^2$.

Since for large n, $\frac{B}{\sqrt{nC_0}} \cong 0$, we have

$$N\hat{\sigma}_0^2 \cong 2c_2 \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{..})^2 . \tag{2.1.4.2}$$

Under H_1, the MML estimator of σ is

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2N} . \tag{2.1.4.3}$$

Since for large n, $\frac{B}{\sqrt{nC}} \cong 0$, we have

$$N\hat{\sigma}^2 \cong 2c_2 \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{i.})^2 . \tag{2.1.4.4}$$

Now, the total sum of squares can be written as

$$\sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{..})^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{i.})^2 + m \sum_{i=1}^{k} (\hat{\mu}_{i.} - \hat{\mu})^2 . \tag{2.1.4.5}$$

Hence, we have the decomposition of the total sum of squares such that

$$S_T = S_b + S_e \tag{2.1.4.6}$$

where $S_T = 2c_2 \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{..})^2$,

$$S_b = 2c_2 m \sum_{i=1}^{k} (\hat{\mu}_{i.} - \hat{\mu})^2$$

and

$$S_e = 2c_2 \sum_{i=1}^{k} \sum_{j=1}^{n} \beta_j (y_{i(j)} - \hat{\mu}_{i.})^2 \ .$$

Asymptotically, $\frac{S_T}{\sigma^2}$, $\frac{S_b}{\sigma^2}$ and $\frac{S_e}{\sigma^2}$ are distributed as chi-squares with N-1, k-1 and N-k degrees of freedom. Since the degrees of freedom for S_b and S_e add to N-1, the total number of degrees of freedom, Cochran's theorem implies that $\frac{S_b}{\sigma^2}$ and $\frac{S_e}{\sigma^2}$ are independently distributed chi-square random variables. Therefore, if the null hypothesis of no difference in block means is true, the ratio

$$W = \frac{S_b / (k-1)}{S_e / (N-k)} \cong \frac{2c_2 m \sum_{i=1}^{k} \hat{\tau}_i^2}{(k-1)\hat{\sigma}^2} \tag{2.1.4.7}$$

is distributed as central F with (k-1, N-k) degrees of freedom for large n. The distribution of W under H_1 is noncentral F with (k-1, N-k) degrees of freedom and noncentrality parameter

$$\lambda_W^2 = 2c_2 m \sum_{i=1}^{k} \left(\frac{\tau_i}{\sigma} \right)^2 \tag{2.1.4.8}$$

for large n. Large values of W lead to the rejection of H_0 in favour of H_1.

Since $\lambda_W^2 > \lambda_F^2$, the W-test above is more powerful than F-test. This is expected since more efficient estimators are used in W-test. The simulated values of the power of the W and F-tests are given in Table 2.3 for various values of d. For d = 0, the power reduces to the type I error. The presumed value of the type I error is 0.05.

The W-test is clearly more powerful than the traditional F-test (even for approximately normal distribution) and it has considerably higher power when the GSH family represents short- and long-tailed distributions.

Table 2.3 Values of the power of F and W-tests, $d = \tau_1/\sigma$, $\tau_2 = -d\sigma$, $\tau_3 = \tau_4 = 0$, k = 4, n = 10

β_2		d = 0.00	0.25	0.50	0.75	1.00
2.0	F	0.051	0.131	0.397	0.764	0.963
	W	0.062	0.168	0.545	0.917	0.995
3.0	F	0.051	0.126	0.393	0.769	0.958
	W	0.052	0.129	0.401	0.772	0.960
4.2	F	0.050	0.129	0.411	0.773	0.954
	W	0.050	0.132	0.423	0.794	0.962
5.0	F	0.051	0.128	0.420	0.766	0.952
	W	0.054	0.147	0.479	0.833	0.976
9.0	F	0.046	0.133	0.446	0.781	0.944
	W	0.052	0.210	0.668	0.948	0.996

2.1.5 Testing Linear Contrasts

The W-test gives an overall assessment whether block differences exist or not. If W statistic is not significantly large, that does not necessarily imply that no block differences exist. It is, therefore, always advisable to construct linear contrasts to assesss the block effects (Tiku and Akkaya, 2004). A contrast, L, is a comparison involving two or more block means. It is defined as a linear combination of the block means μ_i where the coefficients l_i sum up to zero:

$$L = \sum_{i=1}^{k} l_i \tau_i = \sum_{i=1}^{k} l_i \mu_i \,. \tag{2.1.5.1}$$

Without loss of generality, assume that $\sum_{i=1}^{k} l_i^2 = 1$ in which case L is called a standardized linear contrast. Two contrasts

$$L_1 = \sum_{i=1}^{k} l_{1i} \mu_i \text{ and } L_2 = \sum_{i=1}^{k} l_{2i} \mu_i \tag{2.1.5.2}$$

are called orthogonal if $\sum_{i=1}^{k} l_{1i} l_{2i} = 0$. A convenient way of constructing standardized orthogonal linear contrasts is through Helmert transformation:

$$L_1 = (\mu_1 - \mu_2)/\sqrt{2}$$

$$L_2 = (\mu_1 + \mu_2 - 2\mu_3)/\sqrt{6}$$

.

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$$L_{k-1} = (\mu_1 + \ldots + \mu_{k-1} - (k-1)\mu_k)/\sqrt{k(k-1)} \,. \tag{2.1.5.3}$$

Each of these is orthogonal to the mean vector

$$\mu = (\mu_1 + \ldots + \mu_k)/k \,. \tag{2.1.5.4}$$

Now, the MML estimator of the linear contrast $L = \sum_{i=1}^{k} l_i \mu_i$ is $\sum_{i=1}^{k} l_i \hat{\mu}_i$ with variance

$$\frac{\sigma^2}{2mc_2} \sum_{i=1}^{k} l_i^2 , \tag{2.1.5.5}$$

for large n. Since $\hat{\mu}_i$ are asymptotically normally distributed and $\hat{\sigma}$ converges to σ as n becomes large, then the distribution of the statistic

$$T = \frac{\sqrt{2mc_2} \sum_{i=1}^{k} l_i \hat{\mu}_i}{\hat{\sigma} \sqrt{\sum_{i=1}^{k} l_i^2}} \tag{2.1.5.6}$$

is asymptotically normal N(0,1) under the null hypothesis

$$H_0 : \sum_{i=1}^{k} l_i \mu_i = 0 . \tag{2.1.5.7}$$

Large values of $|T|$ lead to the rejection of H_0. The asymptotic power function of the test is (with Type I error α)

$$1 - \beta \cong P\left(|Z| \geq z_{\alpha/2} - |\lambda_T|\right) \tag{2.1.5.8}$$

where Z is a standard normal variate and

$$\lambda_T^2 = \frac{2mc_2 \left(\sum_{i=1}^{k} l_i \mu_i \right)^2}{\sigma^2 \sum_{i=1}^{k} l_i^2} \tag{2.1.5.9}$$

is the noncentrality parameter.

Now, suppose there are three blocks (k = 3) and consider the first two orthogonal contrasts in (2.1.5.3),

$$L_1 = (\mu_1 - \mu_2)/\sqrt{2} \quad \text{and} \quad L_2 = (\mu_1 + \mu_2 - 2\mu_3)/\sqrt{6}\,.$$

In testing

$$H_{01}\colon L_1 = 0 \ \text{ and } \ H_{02}\colon L_2 = 0 \tag{2.1.5.10}$$

the MML estimators of L_1 and L_2 are

$$\hat{L}_1 = (\hat{\mu}_1 - \hat{\mu}_2)/\sqrt{2} \quad \text{and} \quad \hat{L}_2 = (\hat{\mu}_1 + \hat{\mu}_2 - 2\hat{\mu}_3)/\sqrt{6}\,. \tag{2.1.5.11}$$

To test H_{01} and H_{02}, the test statistics become

$$T_1 = \frac{\sqrt{2mc_2}\,\hat{L}_1}{\hat{\sigma}} \quad \text{and} \quad T_2 = \frac{\sqrt{2mc_2}\,\hat{L}_2}{\hat{\sigma}}\,. \tag{2.1.5.12}$$

Large values of $|T_1|$ and $|T_2|$ lead to the rejection of H_{01} and H_{02}. The asymptotic power function of the tests are (with Type I error α)

$$1-\beta_1 \cong P\left(|Z_1| \geq z_{\alpha/2} - |\lambda_{T1}|\right)$$

and

$$1-\beta_2 \cong P\left(|Z_2| \geq z_{\alpha/2} - |\lambda_{T2}|\right) \tag{2.1.5.13}$$

with the noncentrality parameters,

$$\lambda_{T1}{}^2 = \frac{2mc_2L_1{}^2}{\sigma^2} \quad \text{and} \quad \lambda_{T2}{}^2 = \frac{2mc_2L_2{}^2}{\sigma^2}. \tag{2.1.5.14}$$

If the errors e_{ij} are iid normal $N(0, \sigma^2)$, then the MLE of L_1 and L_2 are

$$\tilde{L}_1 = (\bar{y}_{1.} - \bar{y}_{2.})/\sqrt{2} \quad \text{and} \quad \tilde{L}_2 = (\bar{y}_{1.} + \bar{y}_{2.} - 2\bar{y}_{3.})/\sqrt{6} \tag{2.1.5.15}$$

with $V(\tilde{L}_1) = V(\tilde{L}_2) = \frac{\sigma^2}{n}$. Here, σ^2 is estimated by $\tilde{\sigma}^2$ in (1.1.1.5). Now, the test statistics are

$$t_1 = \frac{\sqrt{n}\tilde{L}_1}{\tilde{\sigma}} \quad \text{and} \quad t_2 = \frac{\sqrt{n}\tilde{L}_2}{\tilde{\sigma}}. \tag{2.1.5.16}$$

The null distributions of the above statistics are Student t with 3(n-1) degrees of freedom and their distributions under the alternative hypothesis is noncentral t with 3(n-1) degrees of freedom and noncentrality parameter

$$\lambda_{t1}^2 = \frac{nL_1^2}{\sigma^2} \quad \text{and} \quad \lambda_{t2}^2 = \frac{nL_2^2}{\sigma^2}. \tag{2.1.5.17}$$

Since $\lambda_T^2 > \lambda_t^2$, the T-test above is more powerful than t-test.

For illustration, the simulated values of the power of the t and T-tests when $l_1 = 1/\sqrt{6}$, $l_2 = 1/\sqrt{6}$, and $l_3 = -2/\sqrt{6}$ are given in Table 2.4. For $\tau = 0$, the power reduces to the Type I error. The presumed value of the type I error is 0.050. Without loss of generality, σ was taken to be equal to 1.0.

Table 2.4 Power values of t and T-tests; $\tau_1 = \tau_3 = \tau$, $\tau_2 = -2\tau$; k = 3, n = 10

B_2	$\tau =$	0.0	0.2	0.4	0.6
2.0	t	0.057	0.277	0.758	0.981
	T	0.058	0.360	0.896	0.997
3.0	t	0.057	0.293	0.763	0.981
	T	0.057	0.297	0.767	0.981
4.2	t	0.058	0.290	0.768	0.974
	T	0.056	0.298	0.789	0.981
5.0	t	0.055	0.289	0.722	0.971
	T	0.056	0.327	0.829	0.987
9.0	t	0.054	0.322	0.782	0.961
	T	0.058	0.469	0.940	0.997

It can be seen that the T-test maintains higher power.

2.1.6 Robustness of Estimators and Tests

In experimental design it is very important to obtain estimators and hypothesis testing procedures which have certain optimal properties with respect to an assumed error distribution. In spite of our best efforts to identify the underlying distribution through graphical techniques (Q-Q plots, for example) or goodness-of-fit tests, in practice, the shape parameters might be misspecified or the data might contain outliers (inliers) or be contaminated. Thus deviations from an assumed distribution occur. That brings the issue of robustness in focus. An estimator is called robust if it is fully efficient (or nearly so) for an assumed distribution but maintains high efficiency for plausible alternatives. Also, a test is said to have criterion robustness if its Type I error is not substantially higher than a pre-specified level and is said to have efficiency robustness if its power is high, at any rate for plausible alternatives to an assumed distribution (Tiku et al., 1986, Preface).

To show the robustness of both MML estimators and the test procedures (based on MMLE) we consider, for illustration, the following plausible alternatives (1)-(4) to the assumed distribution GSH in (1.2.2.2) with $t = -\pi / 2$:

(1) Misspecification of the distribution: GSH(μ, σ, $-\pi / 4$)

(2) Dixon's outlier model: (n-1) observations come from GSH(μ, σ, $-\pi / 2$) but one observation (we do not know which one) comes from GSH(μ, 4σ, $-\pi / 2$)

(3) Mixture model: 0.90 GSH(μ, σ, $-\pi / 2$) + 0.10 GSH(μ, 4σ, $-\pi / 2$)

(4) Contamination model: 0.90 GSH(μ, σ, $-\pi / 2$) + 0.10 Uniform(-1/2, 1/2)

Note that the coefficients α_j and β_j in (1)-(4) are always computed from (2.1.2.8) and (2.1.2.9) with $t = -\pi / 2$.

The simulated variances of $\tilde{\mu}_i$ and $\hat{\mu}_i$, the simulated means of $\tilde{\sigma}$ and $\hat{\sigma}$ are given in Table 2.5. Also given in this table are the values of the relative efficiency of the LS estimators of μ_i and σ.

Table 2.5 Means, variances and relative efficiencies; n = 10, σ = 1

	Variance		Mean		RE	
Model	$\tilde{\mu}_i$	$\hat{\mu}_i$	$\tilde{\sigma}$	$\hat{\sigma}$	$\tilde{\mu}_i$	$\tilde{\sigma}$
(1)	0.059	0.041	0.748	0.774	69.01	81.47
(2)	0.250	0.115	1.514	1.448	46.02	63.71
(3)	0.631	0.240	2.399	2.212	38.09	79.21
(4)	0.089	0.071	0.938	0.997	79.58	84.76

It is obvious that the MML estimators $\hat{\mu}_i$ and $\hat{\sigma}$ are remarkably efficient and robust.

To show the robustness property of W-test, the simulated values of the type I error and the power of the W and F-tests are given in Table 2.6. It may be noted that the W-test has a double advantage: it has not only smaller type I error but has also higher power than the F-test.

Table 2.6 Values of the type I error and power for the W and F-tests; $d = \tau_1/\sigma$, $\tau_2 = -d\sigma$, $\tau_3 = \tau_4 = 0$; σ = 1, k = 4, n = 10

Alternative models											
(1)			(2)			(3)			(4)		
d	F	W	d	F	W	d	F	W	d	F	W
0.0	0.047	0.037	0.0	0.031	0.022	0.0	0.039	0.027	0.0	0.047	0.044
0.1	0.06	0.06	0.2	0.06	0.05	0.2	0.06	0.05	0.1	0.06	0.06
0.2	0.14	0.15	0.4	0.14	0.15	0.4	0.15	0.15	0.2	0.10	0.11
0.3	0.27	0.32	0.6	0.29	0.33	0.6	0.32	0.37	0.3	0.18	0.20
0.4	0.47	0.55	0.8	0.49	0.60	0.8	0.52	0.61	0.4	0.30	0.35
0.5	0.65	0.75	1.0	0.67	0.81	1.0	0.68	0.80	0.5	0.45	0.53
0.6	0.81	0.90	1.2	0.80	0.93	1.2	0.81	0.92	0.6	0.61	0.70
0.7	0.90	0.96	1.4	0.89	0.98	1.4	0.88	0.97	0.7	0.75	0.83
0.8	0.99	1.00	1.6	0.94	1.00	1.6	0.93	0.99	0.8	0.86	0.92

The T-test based on the MMLE for testing linear contrasts has efficiency and robustness properties exactly similar to those in Table 2.6. The simulated values of the type I error and the power of the T and t-tests are given in Table 2.7.

Table 2.7 Values of the type I error and power for the T and t-tests; $\tau_1 = \tau_3 = \tau$, $\tau_2 = -2\tau$; σ = 1, k = 3, n = 10

Alternative models											
(1)			(2)			(3)			(4)		
d	t	T	d	t	T	d	t	T	d	t	T
0.00	0.060	0.046	0.0	0.054	0.038	0.0	0.055	0.039	0.0	0.048	0.046
0.05	0.09	0.09	0.1	0.09	0.08	0.1	0.10	0.09	0.1	0.10	0.11
0.10	0.20	0.22	0.2	0.22	0.24	0.2	0.24	0.26	0.2	0.28	0.32
0.15	0.37	0.43	0.3	0.41	0.49	0.3	0.41	0.49	0.3	0.54	0.62
0.20	0.56	0.66	0.4	0.59	0.72	0.4	0.61	0.73	0.4	0.79	0.87
0.25	0.73	0.83	0.5	0.74	0.88	0.5	0.75	0.87	0.5	0.93	0.97
0.30	0.85	0.93	0.6	0.84	0.96	0.6	0.84	0.95	0.6	0.98	0.99

2.2 Unbalanced Design

Consider now a more general form of the model

$$y_{ij} = \mu + \tau_i + e_{ij} \qquad (i = 1,2,\ldots,k;\ j = 1,2,\ldots,n_i), \tag{2.2.1}$$

with unequal number of observations in the blocks. Without loss of generality, assume that

$$\sum_{i=1}^{k} m_i \tau_i = 0 \tag{2.2.2}$$

where the constants m_i $(1 \le i \le k)$ will be defined later.

The Fisher likelihood function is

$$L = \frac{c_1^N}{\sigma^N} \prod_{i=1}^{k} \prod_{j=1}^{n_i} \frac{\exp(c_2 z_{ij})}{\exp(2c_2 z_{ij}) + 2a\exp(c_2 z_{ij}) + 1} \tag{2.2.3}$$

where $N = \sum_{i=1}^{k} n_i$, $z_{ij} = \dfrac{y_{ij} - \mu - \tau_i}{\sigma}$ $(1 \le i \le k, 1 \le j \le n_i)$.

The likelihood equations can be written as follows:

$$\frac{\partial \ln L}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i} g(z_{i(j)}) = 0, \tag{2.2.4}$$

$$\frac{\partial \ln L}{\partial \tau_i} = -n_i\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{n_i} g(z_{i(j)}) = 0 \tag{2.2.5}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i} z_{i(j)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i} z_{i(j)} g(z_{i(j)}) = 0. \tag{2.2.6}$$

The linear approximations for $g(z_{i(j)})$ $(1 \le i \le k,\ 1 \le j \le n_i)$ are

$$\begin{aligned} g(z_{i(j)}) &\cong g(t_{i(j)}) + g'(t_{i(j)})(z_{i(j)} - t_{i(j)}) \\ &= \alpha_{i(j)} + \beta_{i(j)} z_{i(j)} \qquad (1 \le j \le n_i) \end{aligned} \tag{2.2.7}$$

where $t_{i(j)} = E(z_{i(j)})$ is the expected value of the j^{th} order statistic $z_{i(j)}$ in the i^{th} block,

$\alpha_{i(j)} = g(t_{i(j)}) + \beta_{i(j)} t_{i(j)}$ and $\beta_{i(j)} = g'(t_{i(j)})$.

Hence, the coefficients $\alpha_{i(j)}$ and $\beta_{i(j)}$ are different for i = 1, 2,..., k.

The modified likelihood equations for estimating μ, τ_i $(1 \le i \le k)$ and σ are

$$\frac{\partial \ln L}{\partial \mu} \cong \frac{\partial \ln L^*}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left[\alpha_{i(j)} + \beta_{i(j)} z_{i(j)}\right] = 0, \tag{2.2.8}$$

$$\frac{\partial \ln L}{\partial \tau_i} \cong \frac{\partial \ln L^*}{\partial \tau_i} = -n_i \frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{n_i}\left[\alpha_{i(j)} + \beta_{i(j)} z_{i(j)}\right] = 0 \qquad (2.2.9)$$

and

$$\frac{\partial \ln L}{\partial \sigma} \cong \frac{\partial \ln L^*}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i} z_{i(j)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left[\alpha_{i(j)} + \beta_{i(j)} z_{i(j)}\right] z_{i(j)} = 0\,. \qquad (2.2.10)$$

The solutions of (2.2.8)-(2.2.10) are the MML estimators:

$$\hat{\mu} = \hat{\mu}_{..}\,, \qquad (2.2.11)$$

$$\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu} \qquad (2.2.12)$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N-k)}} \qquad (2.2.13)$$

where

$$\hat{\mu}_{..} = \frac{1}{M}\sum_{i=1}^{k} m_i \hat{\mu}_{i.}\,,\;\; \hat{\mu}_{i.} = \frac{1}{m_i}\sum_{j=1}^{n_i}\beta_{i(j)} y_{i(j)}\,,\;\; m_i = \sum_{j=1}^{n_i}\beta_{i(j)}\,,\;\; M = \sum_{i=1}^{k} m_i\,,$$

$$B = Nc_2(\bar{y} - \bar{y}_a)\,,\;\; C = 2c_2\sum_{i=1}^{k}\sum_{j=1}^{n_i}\beta_{i(j)}(y_{i(j)} - \hat{\mu} - \hat{\tau}_i)^2\,,$$

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{i(j)} \;\text{ and }\; \bar{y}_a = \frac{2}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha_{i(j)} y_{i(j)}\,.$$

In testing the equality of block effects, the variance ratio statistic based on the estimators (2.2.11)-(2.2.13) is given as

$$W = \frac{2c_2 \sum_{i=1}^{k} m_i \hat{\tau}_i^2}{(k-1)\hat{\sigma}^2}. \qquad (2.2.14)$$

The null distribution of W for large n_i ($1 \le i \le k$) is central F with (k-1, N-k) degrees of freedom and the distribution of W under H_1 is for large n_i ($1 \le i \le k$) noncentral F with (k-1, N-k) degrees of freedom and noncentrality parameter

$$\lambda_W^2 = 2c_2 \sum_{i=1}^{k} \left(\frac{m_i \tau_i}{\sigma} \right)^2. \qquad (2.2.15)$$

In testing a linear contrast, the T-test based on the estimators (2.2.11)-(2.2.13) is given as

$$T = \frac{\sqrt{2c_2} \sum_{i=1}^{k} l_i \hat{\mu}_i}{\hat{\sigma} \sqrt{\sum_{i=1}^{k} \frac{l_i^2}{m_i}}}. \qquad (2.2.16)$$

The null distribution of T is asymptotically normal N(0, 1). Large values of $|T|$ lead to the rejection of H_0. The asymptotic power function of the test is (with Type I error α)

$$1 - \beta \cong P\left(|Z| \ge z_{\alpha/2} - |\lambda_T| \right) \qquad (2.2.17)$$

where Z is a standard normal variate and

$$\lambda_T^2 = \frac{2c_2 \left(\sum_{i=1}^{k} l_i \mu_i \right)^2}{\sigma^2 \sum_{i=1}^{k} \frac{l_i^2}{m_i}} \qquad (2.2.18)$$

is the noncentrality parameter.

The MML estimators obtained for unbalanced designs and the test statistics W and T based on them have efficiency and robustness properties similar to those reported in the previous sections. Therefore, the details are not given for conciseness.

CHAPTER 3

TWO-WAY CLASSIFICATION WITH INTERACTION

In this chapter parameters of two-way classification model with interaction for balanced and unbalanced designs are estimated under the assumption of Generalized Secant Hyperbolic (GSH) ditributed error terms. Statistical properties of the estimators are studied and the test statistics analogous to the normal-theory F statistics are defined to test block, column and interaction effects.

3.1 Balanced Design

Consider the two-way classification fixed-effects model

$$y_{ijl} = \mu + \tau_i + \delta_j + \gamma_{ij} + e_{ijl} \qquad (1 \le i \le k, 1 \le j \le c, 1 \le l \le n) \tag{3.1.1}$$

having k blocks, c columns with n obsevations in each cell. Without loss of generality assume that

$$\sum_{i=1}^{k} \tau_i = \sum_{j=1}^{c} \delta_j = \sum_{i=1}^{k} \gamma_{ij} = \sum_{j=1}^{c} \gamma_{ij} = 0. \tag{3.1.2}$$

3.1.1 Maximum Likelihood Estimation

The Fisher likelihood function is

$$L = \frac{c_1^N}{\sigma^N} \prod_{i=1}^{k} \prod_{j=1}^{c} \prod_{l=1}^{n} \frac{\exp(c_2 z_{ijl})}{\exp(2c_2 z_{ijl}) + 2a \exp(c_2 z_{ijl}) + 1} \tag{3.1.1.1}$$

where $N = nck$, $z_{ijl} = \dfrac{y_{ij} - \mu - \tau_i - \delta_j - \gamma_{ij}}{\sigma}$ $(1 \le i \le k, 1 \le j \le c, 1 \le l \le n)$.

The likelihood equations for estimating μ, τ_i, δ_j, γ_{ij} ($1 \le i \le k$, $1 \le j \le c$) and σ are

$$\frac{\partial \ln L}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} g(z_{ijl}) = 0, \tag{3.1.1.2}$$

$$\frac{\partial \ln L}{\partial \tau_i} = -nc\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{c}\sum_{l=1}^{n} g(z_{ijl}) = 0, \tag{3.1.1.3}$$

$$\frac{\partial \ln L}{\partial \delta_j} = -kn\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{l=1}^{n} g(z_{ijl}) = 0, \tag{3.1.1.4}$$

$$\frac{\partial \ln L}{\partial \gamma_{ij}} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{l=1}^{n} g(z_{ijl}) = 0 \tag{3.1.1.5}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} z_{ijl} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} z_{ijl}\, g(z_{ijl}) = 0 \tag{3.1.1.6}$$

where $g(z_{ijl}) = \dfrac{\exp(2c_2 z_{ijl}) + a\exp(c_2 z_{ijl})}{\exp(2c_2 z_{ijl}) + 2a\exp(c_2 z_{ijl}) + 1}$.

3.1.2 Modified Maximum Likelihood Estimation

Let

$$y_{ij(1)} \le y_{ij(2)} \le ... \le y_{ij(n)} \qquad (1 \le i \le k, 1 \le j \le c) \tag{3.1.2.1}$$

be the order statistics of the n observations y_{ijl} $(1 \le l \le n)$ in the $(i, j)^{th}$ cell. Then

$$z_{ij(l)} = \frac{y_{ij(l)} - \mu - \tau_i - \delta_j - \gamma_{ij}}{\sigma} \qquad (1 \le i \le k, 1 \le j \le c) \tag{3.1.2.2}$$

are the ordered z_{ijl} $(1 \le l \le n)$ variates. The following likelihood equations are obtained by replacing z_{ijl} by $z_{ij(l)}$:

$$\frac{\partial \ln L}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} g(z_{ij(l)}) = 0, \tag{3.1.2.3}$$

$$\frac{\partial \ln L}{\partial \tau_i} = -nc\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{j=1}^{c}\sum_{l=1}^{n} g(z_{ij(l)}) = 0, \tag{3.1.2.4}$$

$$\frac{\partial \ln L}{\partial \delta_j} = -kn\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{l=1}^{n} g(z_{ij(l)}) = 0, \tag{3.1.2.5}$$

$$\frac{\partial \ln L}{\partial \gamma_{ij}} = -n\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{l=1}^{n} g(z_{ij(l)}) = 0 \tag{3.1.2.6}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} z_{ij(l)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} z_{ij(l)} g(z_{ij(l)}) = 0. \tag{3.1.2.7}$$

From the linear approximations $(1 \le i \le k, 1 \le j \le c)$

$$\begin{aligned} g(z_{ij(l)}) &\cong g(t_{ij(l)}) + g'(t_{ij(l)})(z_{ij(l)} - t_{ij(l)}) \\ &= \alpha_{ij(l)} + \beta_{ij(l)} z_{ij(l)} \qquad (1 \le l \le n) \end{aligned} \tag{3.1.2.8}$$

where $t_{ij(l)} = E(z_{ij(l)})$ is the expected value of the l^{th} order statistic $z_{ij(l)}$ in the i^{th} block and j^{th} column, we can obtain

$$\alpha_{ij(l)} = g(t_{ij(l)}) - \beta_{ij(l)} t_{ij(l)} \text{ and } \beta_{ij(l)} = g'(t_{ij(l)}).$$

Here,

$$t_{ij(l)} = t_{(l)}$$

$$\alpha_{ij(l)} = \alpha_{(l)}$$

$$\beta_{ij(l)} = \beta_{(l)} \qquad \text{for all } i = 1,2,..., k \text{ and } j = 1, 2,\ldots, c \tag{3.1.2.9}$$

and

$$\alpha_{(l)} = \frac{\exp(2c_2 t_{(l)}) + a\exp(c_2 t_{(l)})}{\exp(2c_2 t_{(l)}) + 2a\exp(c_2 t_{(l)}) + 1} - \beta_{(l)} t_{(l)}, \tag{3.1.2.10}$$

$$\beta_{(l)} = \frac{ac_2 \exp(3c_2 t_{(l)}) + 2c_2 \exp(2c_2 t_{(l)}) + ac_2 \exp(c_2 t_{(l)})}{\left[\exp(2c_2 t_{(l)}) + 2a\exp(c_2 t_{(l)}) + 1\right]^2}. \tag{3.1.2.11}$$

Following the same steps explained in Chapter 2, the MML estimators are obtained as follows:

$$\hat{\mu} = \hat{\mu}_{...}, \tag{3.1.2.12}$$

$$\hat{\tau}_i = \hat{\mu}_{i..} - \hat{\mu}, \tag{3.1.2.13}$$

$$\hat{\delta}_j = \hat{\mu}_{.j.} - \hat{\mu}, \tag{3.1.2.14}$$

$$\hat{\gamma}_{ij} = \hat{\mu}_{ij.} - \hat{\mu} - \hat{\tau}_i - \hat{\delta}_j \tag{3.1.2.15}$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N-kc)}} \tag{3.1.2.16}$$

where

$$\hat{\mu}_{...} = \frac{1}{kcm}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n}\beta_{(l)} y_{ij(l)}\,,\;\; \hat{\mu}_{i..} = \frac{1}{cm}\sum_{j=1}^{c}\sum_{l=1}^{n}\beta_{(l)} y_{ij(l)}\,,\;\; \hat{\mu}_{.j.} = \frac{1}{km}\sum_{i=1}^{k}\sum_{l=1}^{n}\beta_{(l)} y_{ij(l)}\,,$$

$$\hat{\mu}_{ij.} = \frac{1}{m}\sum_{l=1}^{n}\beta_{(l)} y_{ij(l)}\,,\;\; m = \sum_{l=1}^{n}\beta_{(l)}\,,$$

$$B = Nc_2(\bar{y}_{...} - \bar{y}_a)\,,\; C = 2c_2\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n}\left[\beta_{(l)}(y_{ij(l)} - \hat{\mu} - \hat{\tau}_i - \hat{\delta}_j - \hat{\gamma}_{ij})^2\right],$$

$$\bar{y}_{...} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n} y_{ij(l)} \text{ and } \bar{y}_a = \frac{2}{N}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n}\alpha_{(l)} y_{ij(l)}\,.$$

Lemma 3.1: Asymptotically, the estimator $\hat{\mu}$ is the MVB estimator of μ and is normally distributed with variance

$$V(\hat{\mu}) \cong \frac{\sigma^2}{2mkcc_2}\,. \tag{3.1.2.17}$$

Corollary 3.1: Asymptotically, the estimator $\hat{\tau}_i$ is the MVB estimator of τ_i and is normally distributed with variance

$$V(\hat{\tau}_i) \cong \frac{\sigma^2}{2mcc_2}\,. \tag{3.1.2.18}$$

Corollary 3.2: Asymptotically, the estimator $\hat{\delta}_j$ is the MVB estimator of δ_j and is normally distributed with variance

$$V(\hat{\delta}_j) \cong \frac{\sigma^2}{2mkc_2}. \tag{3.1.2.19}$$

Corollary 3.3: Asymptotically, the estimator $\hat{\gamma}_{ij}$ is the MVB estimator of γ_{ij} and is normally distributed with variance

$$V(\hat{\gamma}_{ij}) \cong \frac{\sigma^2}{2mc_2}. \tag{3.1.2.20}$$

Lemma 3.2: Asymptotically, $\frac{N\hat{\sigma}^2}{\sigma^2}$ is distributed as chi-square with N-kc degrees of freedom.

3.1.3 Efficiency Properties

Since the estimators $\hat{\tau}_i$, $\hat{\delta}_j$ and $\hat{\gamma}_{ij}$ are linear contrasts of $\hat{\mu}_{ij.}$, they are unbiased and uncorrelated (asymptotically independent) with $\hat{\sigma}^2$. In fact, they are asymptotically the MVB estimators and are normally distributed. In other words, they are the BAN estimators. The estimator $\hat{\sigma}^2$ is also asymptotically the MVB estimator of σ^2 and is distributed as a multiple of chi-square; see Lemma 3.2. Since the estimators of $\mu_i = \mu + \tau_i$ and σ have efficiency properties similar to those given in Chapter 2, it suffices here to consider only the relative efficiencies of the LS estimators $\tilde{\delta}_j$ and $\tilde{\gamma}_{ij}$. The simulated values of RE($\tilde{\delta}_j$) and RE($\tilde{\gamma}_{ij}$) are given in Table 3.1.

Table 3.1 Values of (1) RE($\tilde{\delta}_j$) (2) RE($\tilde{\gamma}_{ij}$); k = 3, c = 3

	$\beta_2 =$	2.0	3.0	4.0	5.0	9.0
n = 4	(1)	87.19	101.29	97.46	92.05	70.40
	(2)	88.09	101.34	97.62	91.98	69.51
n = 5	(1)	82.36	101.53	96.20	90.07	66.66
	(2)	82.35	101.48	96.35	90.32	65.61
n = 6	(1)	79.48	101.62	95.66	88.92	63.33
	(2)	78.71	102.14	95.46	88.50	63.37
n = 10	(1)	68.85	101.33	93.44	85.68	55.58
	(2)	69.53	101.22	93.56	84.79	54.83

The MML estimators $\hat{\delta}_j$ and $\hat{\gamma}_{ij}$ are considerably more efficient than LS estimators $\tilde{\delta}_j$ and $\tilde{\gamma}_{ij}$ even for small sample sizes other than approximately normal distribution ($\beta_2 = 3.0$). Note that for approximately normal distribution $\hat{\delta}_j$ and $\hat{\gamma}_{ij}$ are as efficient as $\tilde{\delta}_j$ and $\tilde{\gamma}_{ij}$. For short-tailed ($\beta_2 = 2.0$) and very long-tailed ($\beta_2 = 9.0$) distributions, MML estimators are enourmously more efficient than LS estimators. The relative efficiencies of $\tilde{\delta}_j$ and $\tilde{\gamma}_{ij}$ decreases as sample size n increases.

3.1.4 Testing Block Effects

It is of great practical interest to test the null hypotheses

$$H_{01} : \tau_1 = \tau_2 = ... = \tau_k = 0,$$

$$H_{02} : \delta_1 = \delta_2 = ... = \delta_c = 0$$

and

$$H_{03} : \gamma_{ij} = 0 \text{ for all i = 1, 2,..., k and j = 1, 2,..., c.} \qquad (3.1.4.1)$$

To test these null hypotheses, the corresponding W statistics are

$$W_1 = \frac{2cmc_2 \sum_{i=1}^{k} \hat{\tau}_i^2}{(k-1)\hat{\sigma}^2}, \qquad (3.1.4.2)$$

$$W_2 = \frac{2kmc_2 \sum_{j=1}^{c} \hat{\delta}_j^2}{(c-1)\hat{\sigma}^2} \qquad (3.1.4.3)$$

and

$$W_3 = \frac{2mc_2 \sum_{i=1}^{k} \sum_{j=1}^{c} \hat{\gamma}_{ij}^2}{(k-1)(c-1)\hat{\sigma}^2}, \qquad (3.1.4.4)$$

respectively. For large n, their null distributions are central F with degrees of freedom (ν_1,ν_4), (ν_2,ν_4) and (ν_3,ν_4), respectively;

$$\nu_1 = k\text{-}1,\ \nu_2 = c\text{-}1,\ \ \nu_3 = (k\text{-}1)(c\text{-}1)\ \text{ and }\ \nu_4 = kc(n\text{-}1).$$

Their non-null distributions are noncentral F with degrees of freedom (ν_1,ν_4), (ν_2,ν_4) and (ν_3,ν_4) and noncentrality parameters,

$$\lambda_{W_1}^2 = \frac{2cmc_2 \sum_{i=1}^{k} \tau_i^2}{\sigma^2},\ \ \lambda_{W_2}^2 = \frac{2kmc_2 \sum_{j=1}^{c} \delta_j^2}{\sigma^2}\ \text{ and }\ \lambda_{W_3}^2 = \frac{2mc_2 \sum_{i=1}^{k} \sum_{j=1}^{c} \gamma_{ij}^2}{\sigma^2}, \qquad (3.1.4.5)$$

respectively, for large n.

For small sample sizes, the simulated values of the probabilities

$$P\{W_i \ge F_{0.05}(\nu_i, \nu_4) | H_{0i}\} \quad \text{and} \quad P\{F_i \ge F_{0.05}(\nu_i, \nu_4) | H_{0i}\} \quad (i = 1, 2, 3)$$

are given in Table 3.2.

Table 3.2 The values of Type I error for F_i and W_i – tests (i =1, 2, 3); k = 3, c = 3

	$\beta_2 =$	2.0	3.0	4.2	5.0	9.0
n = 4	F_1	0.050	0.050	0.049	0.048	0.043
	W_1	0.034	0.053	0.046	0.047	0.059
	F_2	0.052	0.049	0.049	0.048	0.043
	W_2	0.035	0.053	0.047	0.046	0.058
	F_3	0.051	0.048	0.049	0.048	0.041
	W_3	0.032	0.054	0.046	0.046	0.062
n = 5	F_1	0.050	0.048	0.051	0.047	0.045
	W_1	0.037	0.051	0.049	0.047	0.057
	F_2	0.052	0.048	0.047	0.046	0.048
	W_2	0.035	0.052	0.047	0.049	0.059
	F_3	0.048	0.051	0.049	0.048	0.040
	W_3	0.032	0.054	0.047	0.049	0.059
n = 6	F_1	0.050	0.049	0.048	0.049	0.047
	W_1	0.037	0.052	0.045	0.051	0.053
	F_2	0.051	0.050	0.046	0.045	0.044
	W_2	0.039	0.053	0.044	0.048	0.053
	F_3	0.054	0.049	0.047	0.046	0.044
	W_3	0.041	0.053	0.047	0.050	0.055
n = 10	F_1	0.050	0.050	0.049	0.048	0.045
	W_1	0.055	0.052	0.046	0.050	0.055
	F_2	0.052	0.046	0.047	0.047	0.047
	W_2	0.052	0.049	0.050	0.049	0.054
	F_3	0.051	0.050	0.049	0.049	0.043
	W_3	0.057	0.051	0.045	0.055	0.057

Simulation errors being of the order $\pm$ 0.01 for $n \leq 10$, the results in Table 3.2 agree with the theory. As in one-way classification model, the W_i-tests are more powerful than the traditional F_i-tests (i = 1, 2, 3).

3.2 Unbalanced Design

Consider the more general form of the model

$$y_{ijl} = \mu + \tau_i + \delta_j + \gamma_{ij} + e_{ijl} \qquad (1 \leq i \leq k, 1 \leq j \leq c, 1 \leq l \leq n_{ij}) \qquad (3.2.1)$$

with unequal number of observations in the cells. Without loss of generality, assume that

$$\sum_{i=1}^{k} m_{ij}\tau_i = \sum_{j=1}^{c} m_{ij}\delta_j = \sum_{i=1}^{k} m_{ij}\gamma_{ij} = \sum_{j=1}^{c} m_{ij}\gamma_{ij} = 0 \qquad (3.2.2)$$

where the constants m_{ij} ($1 \le i \le k$, $1 \le j \le c$) will be defined later.

The Fisher likelihood function is

$$L = \frac{c_1^N}{\sigma^N} \prod_{i=1}^{k} \prod_{j=1}^{c} \prod_{l=1}^{n_{ij}} \frac{\exp(c_2 z_{ijl})}{\exp(2c_2 z_{ijl}) + 2a\exp(c_2 z_{ijl}) + 1} \qquad (3.2.3)$$

where $N = \sum_{i=1}^{k}\sum_{j=1}^{c} n_{ij}$, $z_{ijl} = \dfrac{y_{ijl} - \mu - \tau_i - \delta_j - \gamma_{ij}}{\sigma}$ $\quad (1 \le i \le k, 1 \le j \le c, 1 \le l \le n_{ij})$.

The likelihood equations can be written as follows:

$$\frac{\partial \ln L}{\partial \mu} = -N\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}} g(z_{ij(l)}) = 0, \qquad (3.2.4)$$

$$\frac{\partial \ln L}{\partial \tau_i} = -\frac{c_2}{\sigma}\sum_{j=1}^{c} n_{ij} + 2\frac{c_2}{\sigma}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}} g(z_{ij(l)}) = 0, \qquad (3.2.5)$$

$$\frac{\partial \ln L}{\partial \delta_j} = -\frac{c_2}{\sigma}\sum_{i=1}^{k} n_{ij} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{l=1}^{n_{ij}} g(z_{ij(l)}) = 0, \qquad (3.2.6)$$

$$\frac{\partial \ln L}{\partial \gamma_{ij}} = -n_{ij}\frac{c_2}{\sigma} + 2\frac{c_2}{\sigma}\sum_{l=1}^{n_{ij}} g(z_{ij(l)}) = 0, \qquad (3.2.7)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -N\frac{1}{\sigma} - \frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}} z_{ij(l)} + 2\frac{c_2}{\sigma}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}} z_{ij(l)} g(z_{ij(l)}) = 0\,. \quad (3.2.8)$$

The linear approximations for $g(z_{ij(l)})$ $(1 \le i \le k,\ 1 \le j \le c)$ are

$$\begin{aligned} g(z_{ij(l)}) &\cong g(t_{ij(l)}) + g'(t_{ij(l)})(z_{ij(l)} - t_{ij(l)}) \\ &= \alpha_{ij(l)} + \beta_{ij(l)} z_{ij(l)} \qquad (1 \le l \le n_{ij}) \end{aligned} \quad (3.2.9)$$

where $t_{ij(l)} = E(z_{ij(l)})$ is the expected value of the l^{th} order statistic $z_{ij(l)}$ in the $(i, j)^{th}$ cell,

$\alpha_{ij(l)} = g(t_{ij(l)}) + \beta_{ij(l)} t_{ij(l)}$ and $\beta_{ij(l)} = g'(t_{ij(l)})$.

Hence, the coefficients $\alpha_{ij(l)}$ and $\beta_{ij(l)}$ are different for i = 1, 2,..., k and j = 1, 2,..., c.

Replacing $g(z_{ij(l)})$ in equations (3.2.4) to (3.2.8) by its linear approximation (3.2.9), the following MML estimators are obtained:

$$\hat{\mu} = \hat{\mu}_{...}\,, \quad (3.2.10)$$

$$\hat{\tau}_i = \hat{\mu}_{i..} - \hat{\mu}\,, \quad (3.2.11)$$

$$\hat{\delta}_j = \hat{\mu}_{.j.} - \hat{\mu}\,, \quad (3.2.12)$$

$$\hat{\gamma}_{ij} = \hat{\mu}_{ij.} - \hat{\mu} - \hat{\tau}_i - \hat{\delta}_j \quad (3.2.13)$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N - kc)}} \quad (3.2.14)$$

where

$$\hat{\mu}_{...} = \frac{1}{M}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}}\beta_{ij(l)}y_{ij(l)}\,,\quad \hat{\mu}_{i..} = \frac{1}{\sum_{j=1}^{c}m_{ij}}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}}\beta_{ij(l)}y_{ij(l)}\,,$$

$$\hat{\mu}_{.j.} = \frac{1}{\sum_{i=1}^{k}m_{ij}}\sum_{i=1}^{k}\sum_{l=1}^{n_{ij}}\beta_{ij(l)}y_{ij(l)}\,,\quad \hat{\mu}_{ij.} = \frac{1}{m_{ij}}\sum_{l=1}^{n_{ij}}\beta_{ij(l)}y_{ij(l)}\,,$$

$$m_{ij} = \sum_{l=1}^{n_{ij}}\beta_{ij(l)}\,,\quad M = \sum_{i=1}^{k}\sum_{j=1}^{c}m_{ij}\,,$$

$$B = Nc_2(\bar{y} - \bar{y}_a)\,,\quad C = 2c_2\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}}\beta_{ij(l)}(y_{ij(l)} - \hat{\mu} - \hat{\tau}_i - \hat{\delta}_j - \hat{\gamma}_{ij})^2\,,$$

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}}y_{ij(l)} \text{ and } \bar{y}_a = \frac{2}{N}\sum_{i=1}^{k}\sum_{j=1}^{c}\sum_{l=1}^{n_{ij}}\alpha_{ij(l)}y_{ij(l)}\,.$$

In testing the null hypotheses in (3.1.4.1), the W statistics based on the estimators (3.2.10)-(3.2.14) are given as

$$W_1 = \frac{2c_2\sum_{i=1}^{k}\sum_{j=1}^{c}m_{ij}\hat{\tau}_i^2}{(k-1)\hat{\sigma}^2}\,, \tag{3.2.15}$$

$$W_2 = \frac{2c_2\sum_{i=1}^{k}\sum_{j=1}^{c}m_{ij}\hat{\delta}_j^2}{(c-1)\hat{\sigma}^2} \tag{3.2.16}$$

and

$$W_3 = \frac{2c_2\sum_{i=1}^{k}\sum_{j=1}^{c}m_{ij}\hat{\gamma}_{ij}^2}{(k-1)(c-1)\hat{\sigma}^2}\,, \tag{3.2.17}$$

respectively. For large n, their null distributions are central F with degrees of freedom (ν_1,ν_4), (ν_2,ν_4) and (ν_3,ν_4), respectively;

$$\nu_1 = k-1,\ \nu_2 = c-1,\ \ \nu_3 = (k-1)(c-1)\ \text{ and }\ \nu_4 = kc(n-1).$$

Their non-null distributions are noncentral F with degrees of freedom (ν_1,ν_4), (ν_2,ν_4) and (ν_3,ν_4) and noncentrality parameters,

$$\lambda_{W_1}^2 = \frac{2c_2\sum_{i=1}^{k}\sum_{j=1}^{c} m_{ij}\tau_i^2}{\sigma^2},\quad \lambda_{W_2}^2 = \frac{2c_2\sum_{i=1}^{k}\sum_{j=1}^{c} m_{ij}\delta_j^2}{\sigma^2}\quad \text{and}\quad \lambda_{W_3}^2 = \frac{2c_2\sum_{i=1}^{k}\sum_{j=1}^{c} m_{ij}\gamma_{ij}^2}{\sigma^2}, \qquad (3.2.18)$$

respectively, for large n.

The MML estimators obtained for unbalanced designs and the test statistics W_i (i = 1, 2, 3) based on them have efficiency and robustness properties similar to those reported in the previous sections. Therefore, the details are not given for conciseness.

CHAPTER 4

ONE-WAY CLASSIFICATION WITH NON-IDENTICAL ERROR DISTRIBUTIONS

In this chapter, parameters of the one-way classification model are estimated under the assumption of non-identical error distributions. The test statistics for testing the block effects and linear contrasts are defined.

4.1 GSH Distributions with Different Shape Parameters When the Variances Are Equal

Consider the one-way classification fixed-effects model in (2.1.1) and suppose the distribution of e_{ij} ($1 \le i \le k$, $1 \le j \le n$) is the Generalized Secant Hyporbolic

$$\text{GSH}(0, \sigma; t_i):\ f(e_{ij}) = \frac{c_{1i}}{\sigma} \frac{\exp(c_{2i} e_{ij} / \sigma)}{\exp(2c_{2i} e_{ij} / \sigma) + 2a_i \exp(c_{2i} e_{ij} / \sigma) + 1} \qquad (-\infty < e_{ij} < \infty) \qquad (4.1.1)$$

where for $-\pi < t_i \le 0$:

$$a_i = \cos(t_i),\ c_{2i} = \sqrt{(\pi^2 - t_i^2)/3}\ \text{ and }\ c_{1i} = \frac{\sin(t_i)}{t_i} c_{2i}$$

and for $t_i > 0$:

$$a_i = \cosh(t_i),\ c_{2i} = \sqrt{(\pi^2 + t_i^2)/3}\ \text{ and }\ c_{1i} = \frac{\sinh(t_i)}{t_i} c_{2i}.$$

Without loss of generality assume that

$$\sum_{i=1}^{k} c_{2i} m_i \tau_i = 0 \tag{4.1.2}$$

where the values of m_i will be determined. Following exactly the same steps given in the previous chapters, the MML estimators are obtained as follows:

$$\hat{\mu} = \hat{\mu}_{..}, \tag{4.1.3}$$

$$\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu} \tag{4.1.4}$$

and

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2\sqrt{N(N-k)}} \tag{4.1.5}$$

where

$$\hat{\mu}_{..} = \frac{\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} y_{i(j)}}{\sum_{i=1}^{k} c_{2i} m_i}, \quad \hat{\mu}_{i.} = \frac{1}{m_i} \sum_{j=1}^{n} \beta_{i(j)} y_{i(j)}, \quad m_i = \sum_{j=1}^{n} \beta_{i(j)},$$

$$B = \sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} (1 - 2\alpha_{i(j)}) y_{i(j)}, \quad C = 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu} - \hat{\tau}_i)^2,$$

$$\alpha_{i(j)} = \frac{\exp(2c_{2i} t_{i(j)}) + a_i \exp(c_{2i} t_{i(j)})}{\exp(2c_{2i} t_{i(j)}) + 2a_i \exp(c_{2i} t_{i(j)}) + 1} - \beta_{i(j)} t_{i(j)},$$

$$\beta_{i(j)} = \frac{a_i c_{2i} \exp(3c_{2i} t_{i(j)}) + 2c_{2i} \exp(2c_{2i} t_{i(j)}) + a_i c_{2i} \exp(c_{2i} t_{i(j)})}{\left[\exp(2c_{2i} t_{i(j)}) + 2a_i \exp(c_{2i} t_{i(j)}) + 1\right]^2}$$

and

$$t_{i(j)} = \frac{1}{c_{2i}} \ln\left[\sin(t_i q_j) / \sin(t_i (1 - q_j))\right], \qquad -\pi < t_i < 0$$

$$= \frac{\sqrt{3}}{\pi} \ln(q_j/(1-q_j)), \qquad t_i = 0$$

$$= \frac{1}{c_{2i}} \ln\left[\sinh(t_i q_j)/\sinh(t_i(1-q_j))\right], \qquad t_i > 0.$$

Lemma 4.1.1: Asymptotically, the estimator $\hat{\mu}_i = \hat{\mu}_{i.}$ is the MVB estimator of μ_i and is normally distributed with variance

$$V(\hat{\mu}_i) \cong \frac{\sigma^2}{2c_{2i}m_i}. \tag{4.1.6}$$

Corollary 4.1.1: Asymptotically, the estimator $\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu}$ is the MVB estimator of τ_i and is normally distributed with variance

$$V(\hat{\tau}_i) \cong \frac{\sigma^2}{2c_{2i}m_i}. \tag{4.1.7}$$

Corollary 4.1.2: Asymptotically, the estimator $\hat{\mu} = \hat{\mu}_{..}$ is the MVB estimator of μ and is normally distributed with variance

$$V(\hat{\mu}) \cong \frac{\sigma^2}{2\sum_{i=1}^{k} c_{2i}m_i}. \tag{4.1.8}$$

Corollary 4.1.3: Since $\hat{\mu}_i$ $(1 \le i \le k)$ are independent of each other and $\hat{\mu} = \frac{1}{k}\sum_{i=1}^{k}\hat{\mu}_i$,

$$V(\hat{\tau}_i) \cong \frac{\sigma^2}{2}\left(\frac{1}{c_{2i}m_i} - \frac{1}{\sum_{i=1}^{k} c_{2i}m_i}\right). \tag{4.1.9}$$

Lemma 4.1.2: Asymptotically, $\frac{N\hat{\sigma}^2(\mu_i)}{\sigma^2}$ is conditionally $(\mu_i = \mu + \tau_i)$ distributed as chi-square with N degrees of freedom.

Corollary 4.1.4: Asymptotically, $\frac{N\hat{\sigma}^2}{\sigma^2}$ is distributed as chi-square with N-k degrees of freedom.

4.1.1 Testing Block Effects

When testing the equality of block effects, the following decomposition of sum of squares is obtained:

Under H_0, the MML estimator of σ is

$$\hat{\sigma}_0 = \frac{-B + \sqrt{B^2 + 4NC_0}}{2N} \tag{4.1.1.1}$$

where $C_0 = 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{..})^2$.

Since for large n, $\frac{B}{\sqrt{nC_0}} \cong 0$, we have

$$N\hat{\sigma}_0^2 \cong 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{..})^2 . \qquad (4.1.1.2)$$

Under H_1, the MML estimator of σ is

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 + 4NC}}{2N} . \qquad (4.1.1.3)$$

Since for large n, $\frac{B}{\sqrt{nC}} \cong 0$, we have

$$N\hat{\sigma}^2 \cong 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{i.})^2 . \qquad (4.1.1.4)$$

Now, the total sum of squares can be written as

$$2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{..})^2 = 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{i.})^2 + 2\sum_{i=1}^{k} c_{2i} m_i \hat{\tau}_i^2 . \quad (4.1.1.5)$$

Hence, we have the decomposition of the total sum of squares such that

$$S_T = S_b + S_e \qquad (4.1.1.6)$$

where $S_T = 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{..})^2$,

$$S_b = 2\sum_{i=1}^{k} c_{2i} m_i \hat{\tau}_i^2$$

and

$$S_e = 2\sum_{i=1}^{k} c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu}_{i.})^2 .$$

Asymptotically, $\frac{S_T}{\sigma^2}$, $\frac{S_b}{\sigma^2}$ and $\frac{S_e}{\sigma^2}$ are distributed as chi-squares with N-1, k-1 and N-k degrees of freedom. Since the degrees of freedom for S_b and S_e add to N-1, the total number of degrees of freedom, Cochran's theorem implies that $\frac{S_b}{\sigma^2}$ and $\frac{S_e}{\sigma^2}$ are independently distributed chi-square random variables. Therefore, if the null hypothesis of no difference in block means is true, the ratio

$$W = \frac{S_b/(k-1)}{S_e/(N-k)} \cong \frac{2\sum_{i=1}^{k} c_{2i} m_i \hat{\tau}_i^2}{(k-1)\hat{\sigma}^2} \tag{4.1.1.7}$$

is distributed as central F with (k-1, N-k) degrees of freedom for large n. The distribution of W under H_1 is noncentral F with (k-1, N-k) degrees of freedom and noncentrality parameter

$$\lambda_W^2 = 2\sum_{i=1}^{k} c_{2i} m_i \left(\frac{\tau_i}{\sigma}\right)^2 \tag{4.1.1.8}$$

for large n. Large values of W lead to the rejection of H_0 in favour of H_1.

4.1.2 Testing Linear Contrasts

To test the linear contrasts, the statistic

$$T = \frac{\sqrt{2}\sum_{i=1}^{k} l_i \hat{\mu}_i}{\hat{\sigma}\sqrt{\sum_{i=1}^{k} \frac{l_i^2}{c_{2i} m_i}}} \tag{4.1.2.1}$$

is defined. Under the null hypothesis (2.1.5.7), it is asymptotically normally N(0,1) distributed. Large values of $|T|$ lead to the rejection of H_0. The asymptotic power function of the test is (with Type I error α)

$$1-\beta \cong P\left(|Z| \geq z_{\alpha/2} - |\lambda_T|\right) \tag{4.1.2.2}$$

where Z is a standard normal variate and

$$\lambda_T^2 = \frac{2\left(\sum_{i=1}^{k} l_i \mu_i\right)^2}{\sigma^2 \sum_{i=1}^{k} \frac{l_i^2}{c_{2i} m_i}} \tag{4.1.2.3}$$

is the noncentrality parameter.

Note that when the error components e_{ij} are non-identically distributed as in (4.1.1), the formulations for unbalanced design are similar to the balanced design.

4.2 GSH Distributions with Different Shape Parameters When the Variances Are Not Equal

Now, suppose the distribution of e_{ij} ($1 \leq i \leq k$, $1 \leq j \leq n$) is the Generalized Secant Hyporbolic

GSH(0, σ_i ; t_i): $$f(e_{ij}) = \frac{c_{1i}}{\sigma_i} \frac{\exp(c_{2i} e_{ij} / \sigma_i)}{\exp(2c_{2i} e_{ij} / \sigma_i) + 2a_i \exp(c_{2i} e_{ij} / \sigma_i) + 1} \quad (-\infty < e_{ij} < \infty) \tag{4.2.1}$$

where for $-\pi < t_i \leq 0$:

$$a_i = \cos(t_i),\ c_{2i} = \sqrt{(\pi^2 - t_i^2)/3}\ \text{ and }\ c_{1i} = \frac{\sin(t_i)}{t_i} c_{2i}$$

and for $t_i > 0$:

$$a_i = \cosh(t_i),\ c_{2i} = \sqrt{(\pi^2 + t_i^2)/3}\ \text{ and }\ c_{1i} = \frac{\sinh(t_i)}{t_i} c_{2i}.$$

Without loss of generality assume that

$$\sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i^2} m_i \tau_i = 0 \tag{4.2.2}$$

where the values of m_i will be determined.

The Fisher likelihood function is

$$L = \prod_{i=1}^{k} \frac{c_{1i}^n}{\sigma_i^n} \prod_{j=1}^{n} \frac{\exp(c_{2i} z_{ij})}{\exp(2c_{2i} z_{ij}) + 2a_i \exp(c_{2i} z_{ij}) + 1} \tag{4.2.3}$$

where $N = nk$, $z_{ij} = \dfrac{y_{ij} - \mu - \tau_i}{\sigma_i}$ $(1 \le i \le k, 1 \le j \le n)$.

The likelihood equations for estimating μ, τ_i $(1 \le i \le k)$ and σ_i are

$$\frac{\partial \ln L}{\partial \mu} = -n \sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i} + 2 \sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i} \sum_{j=1}^{n} g(z_{ij}) = 0, \tag{4.2.4}$$

$$\frac{\partial \ln L}{\partial \tau_i} = -n \frac{c_{2i}}{\sigma_i} + 2 \frac{c_{2i}}{\sigma_i} \sum_{j=1}^{n} g(z_{ij}) = 0 \tag{4.2.5}$$

and

$$\frac{\partial \ln L}{\partial \sigma_i} = -n\frac{1}{\sigma_i} - \frac{c_{2i}}{\sigma_i}\sum_{j=1}^{n} z_{ij} + 2\frac{c_{2i}}{\sigma_i}\sum_{j=1}^{n} z_{ij} g(z_{ij}) = 0 \qquad (4.2.6)$$

where $g(z_{ij}) = \dfrac{\exp(2c_{2i}z_{ij}) + a_i \exp(c_{2i}z_{ij})}{\exp(2c_{2i}z_{ij}) + 2a_i \exp(c_{2i}z_{ij}) + 1}$.

Equations (4.2.4)-(4.2.6) do not admit explicit solutions because of the terms involving the nonlinear function $g(z_{ij})$. Thus, the MML method is used here. Let

$$y_{i(1)} \le y_{i(2)} \le \ldots \le y_{i(n)} \qquad (1 \le i \le k) \qquad (4.2.7)$$

be the order statistics of the n observations y_{ij} $(1 \le j \le n)$ in the i^{th} block. Then

$$z_{i(j)} = \frac{y_{i(j)} - \mu - \tau_i}{\sigma_i} \qquad (1 \le i \le k) \qquad (4.2.8)$$

are the ordered z_{ij} $(1 \le j \le n)$ variates. Since complete sums are invariant to ordering, the likelihood equations are obtained by replacing z_{ij} by $z_{i(j)}$. Hence the likelihood equations become

$$\frac{\partial \ln L}{\partial \mu} = -n\sum_{i=1}^{k}\frac{c_{2i}}{\sigma_i} + 2\sum_{i=1}^{k}\frac{c_{2i}}{\sigma_i}\sum_{j=1}^{n} g(z_{i(j)}) = 0, \qquad (4.2.9)$$

$$\frac{\partial \ln L}{\partial \tau_i} = -n\frac{c_{2i}}{\sigma_i} + 2\frac{c_{2i}}{\sigma_i}\sum_{j=1}^{n} g(z_{i(j)}) = 0 \qquad (4.2.10)$$

and

$$\frac{\partial \ln L}{\partial \sigma_i} = -n\frac{1}{\sigma_i} - \frac{c_{2i}}{\sigma_i}\sum_{j=1}^{n} z_{i(j)} + 2\frac{c_{2i}}{\sigma_i}\sum_{j=1}^{n} z_{i(j)} g(z_{i(j)}) = 0. \qquad (4.2.11)$$

The linear approximations for $g(z_{i(j)})$ $(1 \le i \le k)$ are given as follows

$$g(z_{i(j)}) \cong g(t_{i(j)}) + g'(t_{i(j)})(z_{i(j)} - t_{i(j)})$$
$$= \alpha_{i(j)} + \beta_{i(j)} z_{i(j)} \qquad (1 \le j \le n) \tag{4.2.12}$$

where $\alpha_{i(j)} = \dfrac{\exp(2c_{2i}t_{i(j)}) + a_i \exp(c_{2i}t_{i(j)})}{\exp(2c_{2i}t_{i(j)}) + 2a_i \exp(c_{2i}t_{i(j)}) + 1} - \beta_{i(j)} t_{i(j)}$,

$$\beta_{i(j)} = \frac{a_i c_{2i} \exp(3c_{2i}t_{i(j)}) + 2c_{2i} \exp(2c_{2i}t_{i(j)}) + a_i c_{2i} \exp(c_{2i}t_{i(j)})}{\left[\exp(2c_{2i}t_{i(j)}) + 2a_i \exp(c_{2i}t_{i(j)}) + 1\right]^2}.$$

Therefore, incorporating (4.2.12) into (4.2.9)-(4.2.11), the following MML estimators are obtained:

$$\hat{\mu} = \hat{\mu}_{..}, \tag{4.2.13}$$

$$\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu} \tag{4.2.14}$$

and

$$\hat{\sigma}_i = \frac{-B + \sqrt{B^2 + 4nC}}{2(n-1)} \tag{4.2.15}$$

where

$$\hat{\mu}_{..} = \frac{\sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i^2} \sum_{j=1}^{n} \beta_{i(j)} y_{i(j)}}{\sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i^2} m_i}, \quad \hat{\mu}_{i.} = \frac{1}{m_i} \sum_{j=1}^{n} \beta_{i(j)} y_{i(j)}, \quad m_i = \sum_{j=1}^{n} \beta_{i(j)},$$

$$B = c_{2i} \sum_{j=1}^{n} (1 - 2\alpha_{i(j)}) y_{i(j)} \quad \text{and} \quad C = 2c_{2i} \sum_{j=1}^{n} \beta_{i(j)} (y_{i(j)} - \hat{\mu} - \hat{\tau}_i)^2.$$

Lemma 4.2.1: Asymptotically, the estimator $\hat{\mu}_i = \hat{\mu}_{i.}$ is the MVB estimator of μ_i and is normally distributed with variance

$$V(\hat{\mu}_i) \cong \frac{\sigma_i^2}{2c_{2i} m_i}. \tag{4.2.16}$$

Corollary 4.2.1: Asymptotically, the estimator $\hat{\tau}_i = \hat{\mu}_{i.} - \hat{\mu}$ is the MVB estimator of τ_i and is normally distributed with variance

$$V(\hat{\tau}_i) \cong \frac{\sigma_i^2}{2c_{2i} m_i}. \tag{4.2.17}$$

Corollary 4.2.2: Asymptotically, the estimator $\hat{\mu} = \hat{\mu}_{..}$ is the MVB estimator of μ and is normally distributed with variance

$$V(\hat{\mu}) \cong \frac{1}{2\sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i^2} m_i}. \tag{4.2.18}$$

Corollary 4.2.3: Since $\hat{\mu}_i$ $(1 \le i \le k)$ are independent of each other and $\hat{\mu} = \frac{1}{k}\sum_{i=1}^{k} \hat{\mu}_i$,

$$V(\hat{\tau}_i) \cong \frac{\sigma_i^2}{2c_{2i} m_i} - \frac{1}{2\sum_{i=1}^{k} \frac{c_{2i}}{\sigma_i^2} m_i}. \tag{4.2.19}$$

Lemma 4.2.2: Asymptotically, $\frac{n\hat{\sigma}_i^2(\mu_i)}{\sigma^2}$ is conditionally $(\mu_i = \mu + \tau_i)$ distributed as chi-square with n degrees of freedom.

Corollary 4.2.4: Asymptotically, $\frac{n\hat{\sigma}_i^2}{\sigma^2}$ is distributed as chi-square with n-1 degrees of freedom.

4.2.1 Testing Block Effects

When testing the equality of block effects, the following decomposition of sum of squares is obtained:

Under H_0, the MML estimator of σ_i is

$$\hat{\sigma}_{i0} = \frac{-B + \sqrt{B^2 + 4nC_0}}{2n} \tag{4.2.1.1}$$

where $C_0 = 2c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{..})^2$.

Since for large n, $\frac{B}{\sqrt{nC_0}} \cong 0$, we have

$$n\hat{\sigma}_{i0}^2 \cong 2c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{..})^2. \tag{4.2.1.2}$$

Under H_1, the MML estimator of σ_i is

$$\hat{\sigma}_i = \frac{-B + \sqrt{B^2 + 4nC}}{2n}. \tag{4.2.1.3}$$

Since for large n, $\frac{B}{\sqrt{nC}} \cong 0$, we have

$$n\hat{\sigma}_i^2 \cong 2c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{i.})^2 . \qquad (4.2.1.4)$$

Since the following equation can be written for each block (i = 1, 2,..., k)

$$2c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{..})^2 = 2c_{2i}\sum_{j=1}^{n}\beta_{ij}(y_{i(j)} - \hat{\mu}_{i.})^2 + 2c_{2i}m_i\hat{\tau}_i^2 , \qquad (4.2.1.5)$$

the total sum of squares can be written as

$$2\sum_{i=1}^{k}c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{..})^2 = 2\sum_{i=1}^{k}c_{2i}\sum_{j=1}^{n}\beta_{ij}(y_{i(j)} - \hat{\mu}_{i.})^2 + 2\sum_{i=1}^{k}c_{2i}m_i\hat{\tau}_i^2 \qquad (4.2.1.6)$$

Hence, we have the decomposition of the total sum of squares such that

$$S_T = S_b + S_e \qquad (4.2.1.7)$$

where $S_T = 2\sum_{i=1}^{k}c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{..})^2$,

$$S_b = 2\sum_{i=1}^{k}c_{2i}m_i\hat{\tau}_i^2$$

and

$$S_e = 2\sum_{i=1}^{k}c_{2i}\sum_{j=1}^{n}\beta_{i(j)}(y_{i(j)} - \hat{\mu}_{i.})^2 .$$

Asymptotically, $\frac{S_T}{\sigma^2}$, $\frac{S_b}{\sigma^2}$ and $\frac{S_e}{\sigma^2}$ are distributed as chi-squares with N-1, k-1 and N-k degrees of freedom. Since the degrees of freedom for S_b and S_e add to N-1, the total number of degrees of freedom, Cochran's theorem implies that $\frac{S_b}{\sigma^2}$ and $\frac{S_e}{\sigma^2}$ are independently distributed chi-square random variables. Therefore, if the null hypothesis of no difference in block means is true, the ratio

$$W = \frac{S_b / (k-1)}{S_e / (N-k)} \cong \frac{2\sum_{i=1}^{k} c_{2i} m_i \hat{\tau}_i^2}{(k-1)\sum_{i=1}^{k} \hat{\sigma}_i^2} \tag{4.2.1.8}$$

is distributed as central F with (k-1, N-k) degrees of freedom for large n. The distribution of W under H_1 is noncentral F with (k-1, N-k) degrees of freedom and noncentrality parameter

$$\lambda_W^2 = \frac{2\sum_{i=1}^{k} c_{2i} m_i \tau_i^2}{\sum_{i=1}^{k} \sigma_i^2} \tag{4.2.1.9}$$

for large n. Large values of W lead to the rejection of H_0 in favour of H_1.

4.2.2 Testing Linear Contrasts

To test the linear contrasts, the statistic

$$T = \frac{\sqrt{2}\sum_{i=1}^{k} l_i \hat{\mu}_i}{\sqrt{\sum_{i=1}^{k} \frac{l_i^2 \hat{\sigma}_i^2}{c_{2i} m_i}}} \tag{4.2.2.1}$$

is defined. Under the null hypothesis (2.1.5.7), it is asymptotically normally N(0,1) distributed. Large values of $|T|$ lead to the rejection of H_0. The asymptotic power function of the test is (with Type I error α)

$$1-\beta \cong P\left(|Z| \geq z_{\alpha/2} - |\lambda_T|\right) \tag{4.2.2.2}$$

where Z is a standard normal variate and

$$\lambda_T^2 = \frac{2\left(\sum_{i=1}^{k} l_i \mu_i\right)^2}{\sum_{i=1}^{k} \frac{l_i^2 \sigma_i^2}{c_{2i} m_i}} \tag{4.2.2.3}$$

is the noncentrality parameter.

CHAPTER 5

APPLICATIONS AND CONCLUSIONS

5.1 Applications

Example 5.1: Brian Everitt (private communication) gives weights, in pounds, of young girls receiving two different treatments, cognitive behavioral treatment (I) and family therapy (II) for anorexia over a fixed period of time with the control group (III) receiving the standard treatment. This data is reproduced in Hand et al. (1994, p. 229). From the given data, the differences ‘after’ minus ‘before’ weight measurements are calculated:

I:	1.7	0.7	-0.1	-0.7	-3.5	17.1	-7.6	1.6	11.7	6.1
	1.1	-4.0	20.9	-9.1	2.1	-1.4	1.4	-0.3	-3.7	-0.8
	2.4	12.6	1.9	3.9	0.1	15.4	-0.7			
II:	11.4	11.0	5.5	9.4	13.6	-2.9	-0.1	7.4	21.5	-5.3
	-3.8	13.4	13.1	9.0	3.9	5.7	10.7			
III:	-0.5	-9.3	-5.4	12.3	-2.0	-10.2	-12.2	11.6	-7.1	6.2
	-9.2	8.3	3.3	11.3	0.0	-1.0	-10.6	-4.6	-6.7	2.8
	0.3	1.8	3.7	15.9	-10.2					

Clearly the main problem is to compare the methods of treatment. To locate the plausible distribution, the Q-Q plot is used and is given in Figure 5.1.

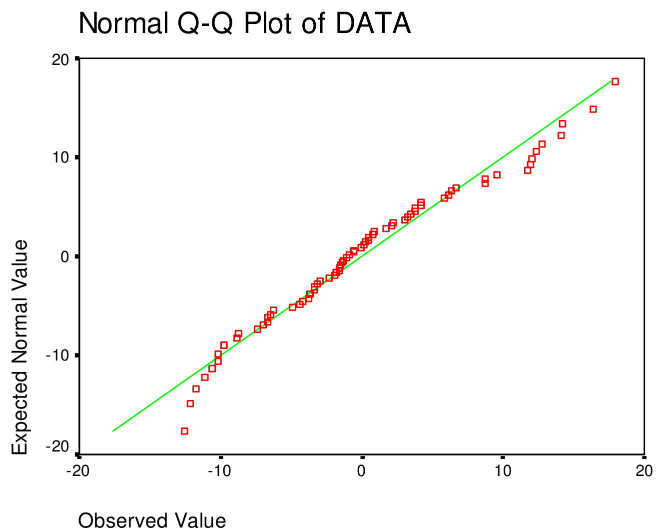

Figure 5.1 Weight Gains of Anorexia Patients

The Q-Q plot of data indicates an approximately symmetric distribution with short tails.

To locate the most plausible value of t, the MML estimators of μ, τ_i (i = 1, 2, 3) and σ are calculated from (2.2.11)-(2.2.13). Then, the values of

$$\ln \hat{L} = N \ln c_1 - N \ln \hat{\sigma} + c_2 \sum_{i=1}^{k} \sum_{j=1}^{n_i} \hat{z}_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \ln\left[\exp(2c_2 \hat{z}_{ij}) + 2a \exp(c_2 \hat{z}_{ij}) + 1\right]$$

where $\hat{z}_{ij} = \dfrac{y_{ij} - \hat{\mu} - \hat{\tau}_i}{\hat{\sigma}} \quad (1 \le i \le 3)$

are calculated. The value that maximizes $\ln \hat{L}$ is the most appropriate choice. For the given data, we have the following values:

$t =$	$\pi\sqrt{7/5}$	$\pi\sqrt{17/7}$	$\pi\sqrt{19/5}$	$\pi\sqrt{5}$	$\pi\sqrt{7}$	$\pi\sqrt{11}$
$\ln \hat{L} =$	-246.16	- 245.82	-245.62	-245.56	-245.57	-245.66

Therefore, the information provided by Q-Q plot supplemented by the determination of the shape parameter (Tiku and Akkaya, 2004) indicates the GSH distribution with t = $\pi\sqrt{5}$ is the most plausible distribution.

The estimates and their standard errors are calculated as follows:

	LSE	MMLE
	$\tilde{\mu} = 2.764$	$\hat{\mu} = 3.807$
	$SE(\tilde{\mu}) = \pm 0.887$	$SE(\hat{\mu}) = \pm 0.780$
I:	$\tilde{\mu}_1 = 3.007$	$\hat{\mu}_1 = 5.011$
	$SE(\tilde{\mu}_1) = \pm 1.398$	$SE(\hat{\mu}_1) = \pm 1.221$
II:	$\tilde{\mu}_2 = 7.265$	$\hat{\mu}_2 = 6.630$
	$SE(\tilde{\mu}_2) = \pm 1.476$	$SE(\hat{\mu}_2) = \pm 1.627$
III:	$\tilde{\mu}_3 = -0.450$	$\hat{\mu}_3 = 0.676$
	$SE(\tilde{\mu}_3) = \pm 1.476$	$SE(\hat{\mu}_3) = \pm 1.293$.

The MMLE is clearly more precise.

To test the null hypothesis

$$H_0 : \mu_1 = \mu_2 = \mu_3$$
$$H_1 : \mu_i \neq \mu_j \text{ for at least one pair (i, j)} \qquad (i, j = 1, 2, 3; i \neq j),$$

the test statistics based on the LSE and MMLE are F = 5.422 and W = 18.089, respectively. Since the tabulated F value is 3.130 for 0.05 significance level, both F and W-tests reject the null hypothesis. The W statistic, however, gives a smaller probability for H_0 to be true.

In testing two orthogonal linear contrasts

$$H_{10}: \mu_1 - \mu_2 = 0 \quad \text{and} \quad H_{02}: \mu_1 + \mu_2 - 2\mu_3 = 0,$$

the statistics for H_{10} based on the LSE and MMLE are $|t_1| = 1.700$ and $|T_1| = 2.438$, respectively. Since the tabulated Z value is 1.96 for 0.05 significance level, the statistic based on the LSE, t_1, does not reject H_{10}. However, the statistic based on the MMLE, T_1, rejects H_{10}. For H_{02}, the statistics based on the LSE and MMLE are $|t_2| = 2.864$ and $|T_2| = 2.042$, respectively. Therefore, both t and T-tests reject the null hypothesis H_{02}.

Example 5.2: Snedecor and Cochran (1967) gives the following data coming from an experiment to study the gain in weight of rats fed on four different diets, distinguished by amount of protein (low and high) and by source of protein (beef and cereal):

Weight gains of rats:

	Protein source									
	Beef					Cereal				
Low	90	76	90	64	86	107	95	97	80	98
	51	72	90	95	78	74	74	67	89	58
High	73	102	118	104	81	98	74	56	111	95
	107	100	87	117	111	88	82	77	86	92

The normal Q-Q plot of weight gains of rats is given in Figure 5.2.

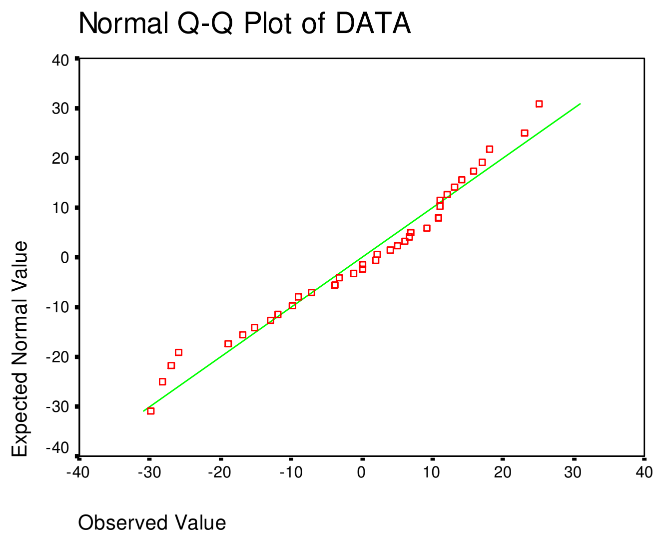

Figure 5.2 Weight Gains of Rats

The Q-Q plot of data indicates an approximately symmetric distribution with short tails. The determination of the shape parameter (Tiku and Akkaya, 2004) indicates the GSH distribution with $t = 3\pi$ beautifully models the data.

The estimates and their standard errors are calculated as follows:

LSE	MMLE
$\tilde{\mu} = 87.250$	$\hat{\mu} = 84.219$
$SE(\tilde{\mu}) = \pm 2.364$	$SE(\hat{\mu}) = \pm 2.117$
$\tilde{\tau}_1 = -\tilde{\tau}_2 = 2.350$	$\hat{\tau}_1 = -\hat{\tau}_2 = 0.904$
$SE(\tilde{\tau}_i) = \pm 3.344$	$SE(\hat{\tau}_i) = \pm 2.993$
$\tilde{\delta}_1 = -\tilde{\delta}_2 = -5.700$	$\hat{\delta}_1 = -\hat{\delta}_2 = -5.920$
$SE(\tilde{\delta}_j) = \pm 3.344$	$SE(\hat{\delta}_j) = \pm 2.993$
$\tilde{\gamma}_{11} = -\tilde{\gamma}_{12} = -\tilde{\gamma}_{21} = \tilde{\gamma}_{22} = -4.700$	$\hat{\gamma}_{11} = -\hat{\gamma}_{12} = -\hat{\gamma}_{21} = \hat{\gamma}_{22} = -5.211$
$SE(\tilde{\gamma}_{ij}) = \pm 4.729$	$SE(\hat{\gamma}_{ij}) = \pm 4.233$

where i = 1, 2 and j = 1, 2.

Again MMLE are more efficient. To test the hypothesis

$$H_{01} : \tau_1 = \tau_2 = 0,$$

the test statistics based on the LSE and MMLE are F_1 = 0.988 and W_1 = 0.182, respectively. Since the tabulated F value is 4.113 for 0.05 significance level, both F and W-tests fail to reject the null hypothesis. Furthermore, to test the hypothesis

$$H_{02} : \delta_1 = \delta_2 = 0,$$

the test statistics based on the LSE and MMLE are F_2 = 5.812 and W_2 = 7.822, respectively. Therefore, both F and W-tests reject the null hypothesis. The W statistic gives a smaller probability for H_{02} to be true. However, in testing the hypothesis

$$H_{03} : \gamma_{ij} = 0 \text{ for all i = 1, 2 and j = 1, 2,}$$

the test statistics based on the LSE and MMLE are F_3 = 3.952 and W_3 = 6.062, respectively. Hence F-test fails to reject the null hypothesis but W-test rejects the null hypothesis.

5.2 Summary and Conclusions

In the framework of one-way and two-way classification models for both balanced and unbalanced cases in experimental design under the assumption of GSH distributed error terms, the model parameters are estimated by using the MML estimation method. MML method is theoretically and computationally straightforward besides being flexible in the sense that it can be used for any location-scale distributions, symmetric or skew. It also provides explicit solutions for the likelihood equations when Fisher method of maximum likelihood becomes intractable.

The W statistics for testing the main and interaction effects and the T statistics for testing the linear contrasts in both balanced and unbalanced cases are developed. To analyze the efficiency and robustness of the estimators as well as test statistics, the simulation study is conducted.

The estimation and test procedures developed for the one-way classification with identical error distributions is generalized to the non-identical error distributions, i.e., the error terms are assummed to have GSH distribution with different shape parameters and variances in each block.

On the basis of this research the following conclusions could be stated:

1. The MML estimators, $\hat{\mu}$, $\hat{\tau}_i$, $\hat{\delta}_j$ and $\hat{\gamma}_{ij}$ are found to be more efficient than the corresponding LS estimators even for small sample sizes other than approximately normal distribution ($\beta_2 = 3.0$). The LS estimators have a disconcerting feature, i.e., their relative efficiency decreases as the sample size, n increases. The same efficiency properties hold for $\hat{\sigma}$ although for small sample sizes it has larger bias than $\tilde{\sigma}$. Thus, defficiency of MML and LS estimators are calculated through simulations. Deficiency of MML estimators are found to be considerably smaller than the defficiency of LS estimators even for small sample size other than approximately normal ($\beta_2 = 3.0$), near normal (logistic, $\beta_2 = 4.2$) and very long-tailed ($\beta_2 = 9.0$) distributions. However, for n $\geq$ 9 defficiency of MML estimators becomes smaller than that of LS estimators for near normal and long-tailed distributions.

2. Since $\frac{\partial \ln L}{\partial \theta}$ is asymptotically equivalent to $\frac{\partial \ln L^*}{\partial \theta}$ (θ is any parameter), the MML estimators are asymptotically MVB estimators (in fact, BAN).

3. The MML estimators are robust.

4. Under the null hypotheses in (3.1.4.1), the distributions of W-tests used for testing the main and interaction effects of the model are found to be central F with (k-1, kc(n-1)), (c-1, kc(n-1)) and ((k-1)(c-1), kc(n-1)) degrees of freedoms for large sample sizes, respectively. Under the alternative hypotheses, their distributions are found to be noncentral F with the same degrees of freedoms and the noncentrality parameters given in (3.1.4.5).

5. Although type I error of the W-test is larger than that of F-test for short-tailed distributions, the W-test is clearly more powerful than the traditional F-test (even for approximately normal). W-test has considerably higher power when the GSH family represents short- and long-tailed distributions.

6. Under the null hypothesis in (2.1.5.7), the distribution of T-test used for testing the linear contrasts of the model is found to be normal for large sample sizes.

7. The power of T-test is shown to be considerably higher than that of t-test.

8. The W and T-tests have both criterion and inference robustness.

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APPENDIX

LISTING OF COMPUTER PROGRAMS DEVELOPED IN THE STUDY

1. SIMULATION OF ONE-WAY CLASSIFICATION MODEL

```
PROGRAM ONE_WAY

USE NUMERICAL_LIBRARIES

REAL B2,T,A,C1,C2,PI,MU,GAM(10),SIGMA,T1(100),ALFA(100)
REAL BET(100),M,G(100),E(10,100),Y(10,100),U1(100)
REAL U2(100),X(10,100),MLMU,MLMI(10),XBAR,XABAR,B,C
REAL MLSIGMA,LSMU,LSMI(10),LSSIGMA,LSMUMEAN,MLMUMEAN
REAL LSMIMEAN(10),MLMIMEAN(10),LSSIGMAMEAN
REAL MLSIGMAMEAN,LSMUVAR,LSMIVAR(10),LSSIGMAVAR
REAL MLMUVAR,MLMIVAR(10),MLSIGMAVAR
REAL FML,FLS,POWERML,POWERLS,F,W
REAL REMU,REMI(10),RESIGMA,I11,I22,MVBMI,MVBSIGMA
REAL EMLMI(10),EMLSIGMA,L1ML,TML1,L2ML,TML2,S,L1LS,TLS1
REAL L2LS,TLS2,POWERML1,POWERML2,POWERLS1,POWERLS2
REALLSSIGMAMSE,LSMIMSE(10),MLMIMSE(10),MLSIGMAMSE
REAL DEFLS(10),DEFML(10)
INTEGER K,N,NSUM,NN

OPEN (unit=1,file='c:\Concon\Documents\output.txt')

PI=22.0/7.0
PRINT *,'ENTER THE KURTOSIS'
READ *,B2
IF (B2.GT.4.2) THEN
  T=-PI*SQRT((5.0*B2-21.0)/(5.0*B2-9.0))
ELSE IF (B2.EQ.4.2) THEN
  T=0.0
ELSE IF (B2.LT.4.2.AND.B2.GT.1.8) THEN
  T=PI*SQRT((21.0-5.0*B2)/(5.0*B2-9.0))
ENDIF
WRITE(1,*)'SHAPE PARAMETER=',T

PRINT *,'INPUT THE NUMBER OF TREATMENTS'
```

```
      READ *,K
      WRITE(1,*) 'NUMBER OF TREATMENTS=',K

      PRINT*,'INPUT THE SAMPLE SIZE IN THE BLOCK'
      READ*,N
      WRITE(1,*) 'SAMPLE SIZE IN THE BLOCK=',N
      NSUM=N*K

      DFN=K*1.0-1.0
      DFD=NSUM*1.0-K*1.0
      F=FIN(0.95,DFN,DFD)
      S=ANORIN(0.975)

      PRINT*,'INPUT THE OVERALL MEAN'
      READ*,MU
      WRITE(1,*) 'OVERALL MEAN=',MU

      PRINT*,'INPUT THE BLOCK EFFECTS'
      DO 2 I=1,K
        READ*,GAM(I)
        WRITE(1,*) I,'TH BLOCK EFFECT=',GAM(I)
 2    CONTINUE

      SIGMA=1.0

      IF (T.GT.(-1.0*PI).AND.T.LE.0.0) THEN
        A=COS(T)
        C2=SQRT((PI*PI-T*T)/3.0)
        C1=(SIN(T)/T)*C2
      ELSE IF (T.GT.0.0) THEN
        A=COSH(T)
        C2=SQRT((PI*PI+T*T)/3.0)
        C1=(SINH(T)/T)*C2
      ENDIF

      IF (T.GT.(-PI).AND.T.LT.0.0) THEN
        DO 10 J=1,N
          V2=J
          V1=V2/(N*1.0+1.0)
          T1(J)=LOG(SIN(T*V1)/SIN(T*(1.0-V1)))/C2
10      CONTINUE
      ELSE IF (T.EQ.0.0) THEN
        DO 12 J=1,N
          V2=J
          V1=V2/(N*1.0+1.0)
```

```
        W=(V1)/(1.0-V1)
        T1(J)=(SQRT(3.0)/PI)*LOG(W)
12    CONTINUE
    ELSE IF (T.GT.0.0) THEN
      DO 14 J=1,N
        V2=J
        V1=V2/(N*1.0+1.0)
        T1(J)=LOG(SINH(T*V1)/SINH(T*(1.0-V1)))/C2
14    CONTINUE
    ENDIF

    DO 16 J=1,N
      BET(J)=2.0*C2*EXP(2.0*C2*T1(J))+A*C2*EXP(3.0*C2*T1(J))
      BET(J)=BET(J)+A*C2*EXP(C2*T1(J))
      BET(J)=BET(J)/(EXP(2.0*C2*T1(J))+2.0*A*EXP(C2*T1(J))+1.0)**2
      ALFA(J)=EXP(2.0*C2*T1(J))+A*EXP(C2*T1(J))
      ALFA(J)=ALFA(J)/(EXP(2.0*C2*T1(J))+2.0*A*EXP(C2*T1(J))+1.0)
      ALFA(J)=ALFA(J)-BET(J)*T1(J)
16  CONTINUE

    DO 17 J=1,N
      IF (BET(J).LT.0.0) THEN
        BET(J)=0.0
        ALFA(J)=EXP(2.0*C2*T1(J))+A*EXP(C2*T1(J))
        ALFA(J)=ALFA(J)/(EXP(2.0*C2*T1(J))+2.0*A*EXP(C2*T1(J))+1.0)
      ENDIF
17  CONTINUE

    M=0.0
    DO 18 J=1,N
      M=M+BET(J)
18  CONTINUE

    LSMUMEAN=0.0
    LSMUVAR=0.0
    MLMUMEAN=0.0
    MLMUVAR=0.0
    DO 19 I=1,K
      LSMIMEAN(I)=0.0
      LSMIVAR(I)=0.0
      MLMIMEAN(I)=0.0
      MLMIVAR(I)=0.0
19  CONTINUE
    LSSIGMAMEAN=0.0
    MLSIGMAMEAN=0.0
```

```
      LSSIGMAVAR=0.0
      MLSIGMAVAR=0.0

      POWERLS=0.0
      POWERML=0.0
      POWERML1=0.0
      POWERML2=0.0
      POWERLS1=0.0
      POWERLS2=0.0

      NN=100000/N

      DO 100 L=1,NN

      DO 20 I=1,K
        CALL RNUN(N,G)
        IF (T.GT.(-PI).AND.T.LT.0.0) THEN
          DO 21 J=1,N
            E(I,J)=LOG(SIN(T*G(J))/SIN(T*(1.0-G(J))))/C2
21        CONTINUE
        ELSE IF (T.EQ.0.0) THEN
          DO 22 J=1,N
            E(I,J)=(SQRT(3.0)/PI)*LOG(G(J)/(1.0-G(J)))
22        CONTINUE
       ELSE IF (T.GT.0.0) THEN
          DO 23 J=1,N
            E(I,J)=LOG(SINH(T*G(J))/SINH(T*(1.0-G(J))))/C2
23        CONTINUE
       ENDIF
20    CONTINUE

      DO 25 I=1,K
        DO 26 J=1,N
          Y(I,J)=E(I,J)+MU+GAM(I)
26      CONTINUE
25    CONTINUE


C FINDING MMLE

      DO 30 I=1,K
        DO 31 J=1,N
          U1(J)=Y(I,J)
31      CONTINUE
        CALL SVRGN(N,U1,U2)
```

```
          DO 32 J=1,N
            X(I,J)=U2(J)
32        CONTINUE
30      CONTINUE

        MLMU=0.0
        DO 35 I=1,K
          DO 36 J=1,N
             MLMU=MLMU+BET(J)*X(I,J)
36        CONTINUE
35      CONTINUE
        MLMU=MLMU/(K*M*1.0)
        MLMUMEAN=MLMUMEAN+MLMU
        MLMUVAR=MLMUVAR+MLMU**2


        DO 40 I=1,K
          MLMI(I)=0.0
          DO 41 J=1,N
             MLMI(I)=MLMI(I)+BET(J)*X(I,J)
41        CONTINUE
          MLMI(I)=MLMI(I)/M
          MLMIMEAN(I)=MLMIMEAN(I)+MLMI(I)
          MLMIVAR(I)=MLMIVAR(I)+MLMI(I)**2
40      CONTINUE

        XBAR=0.0
        XABAR=0.0
        DO 45 I=1,K
          DO 46 J=1,N
            XBAR=XBAR+X(I,J)
            XABAR=XABAR+X(I,J)*ALFA(J)
46        CONTINUE
45      CONTINUE
        XBAR=XBAR/(NSUM*1.0)
        XABAR=XABAR*2.0/(1.0*NSUM)
        B=C2*NSUM*(XBAR-XABAR)
        C=0.0
        DO 50 I=1,K
          DO 51 J=1,N
              C=C+BET(J)*(X(I,J)-MLMI(I))**2
51        CONTINUE
50      CONTINUE
        C=C*2.0*C2
        MLSIGMA=(-B+SQRT(B**2+4.0*NSUM*C))
```

```
      MLSIGMA=MLSIGMA/(2.0*SQRT(NSUM*(NSUM-K)*1.0))
      MLSIGMAMEAN=MLSIGMAMEAN+MLSIGMA
      MLSIGMAVAR=MLSIGMAVAR+MLSIGMA**2

      FML=0.0
      DO 55 I=1,K
        FML=FML+(MLMI(I)-MLMU)**2
 55   CONTINUE
      FML=FML*2.0*C2*M/((K*1.0-1.0)*MLSIGMA**2)
      IF (FML.GT.F) THEN
        POWERML=POWERML+1.0
      ENDIF

      L1ML=(MLMI(1)-MLMI(2))/SQRT(2.0)
      TML1=SQRT(2.0*M*C2)*L1ML/MLSIGMA
      L2ML=(MLMI(1)+MLMI(2)-2.0*MLMI(3))/SQRT(6.0)
      TML2=SQRT(2.0*M*C2)*L2ML/MLSIGMA
      IF (ABS(TML1).GT.S) THEN
        POWERML1=POWERML1+1.0
      ENDIF
      IF (ABS(TML2).GT.S) THEN
        POWERML2=POWERML2+1.0
      ENDIF

C FINDING LSE

      LSMU=XBAR
      LSMUMEAN=LSMUMEAN+LSMU
      LSMUVAR=LSMUVAR+LSMU**2

      DO 60 I=1,K
        LSMI(I)=0.0
        DO 61 J=1,N
      LSMI(I)=LSMI(I)+Y(I,J)
 61     CONTINUE
        LSMI(I)=LSMI(I)/(N*1.0)
        LSMIMEAN(I)=LSMIMEAN(I)+LSMI(I)
        LSMIVAR(I)=LSMIVAR(I)+LSMI(I)**2
 60   CONTINUE

      LSSIGMA=0.0
      DO 65 I=1,K
        DO 66 J=1,N
          LSSIGMA=LSSIGMA+(Y(I,J)-LSMI(I))**2
 66     CONTINUE
```

```
 65   CONTINUE
      LSSIGMA=LSSIGMA/((NSUM-K)*1.0)
      LSSIGMA=SQRT(LSSIGMA)
      LSSIGMAMEAN=LSSIGMAMEAN+LSSIGMA
      LSSIGMAVAR=LSSIGMAVAR+LSSIGMA**2

      FLS=0.0
      DO 70 I=1,K
        FLS=FLS+N*(LSMI(I)-LSMU)**2
 70   CONTINUE
       FLS=FLS/((LSSIGMA**2)*(K*1.0-1.0))
       IF (FLS.GT.F) THEN
        POWERLS=POWERLS+1.0
       ENDIF

       L1LS=(LSMI(1)-LSMI(2))/SQRT(2.0)
       TLS1=SQRT(N*1.0)*L1LS/LSSIGMA
       L2LS=(LSMI(1)+LSMI(2)-2.0*LSMI(3))/SQRT(6.0)
       TLS2=SQRT(N*1.0)*L2LS/LSSIGMA
       IF (ABS(TLS1).GT.S) THEN
        POWERLS1=POWERLS1+1.0
       ENDIF
       IF (ABS(TLS2).GT.S) THEN
        POWERLS2=POWERLS2+1.0
       ENDIF

100   CONTINUE

      LSMUMEAN=LSMUMEAN/(NN*1.0)
      LSMUVAR=LSMUVAR/(NN*1.0)-LSMUMEAN**2
      MLMUMEAN=MLMUMEAN/(NN*1.0)
      MLMUVAR=MLMUVAR/(NN*1.0)-MLMUMEAN**2
      DO 80 I=1,K
        LSMIMEAN(I)=LSMIMEAN(I)/(NN*1.0)
        LSMIVAR(I)=LSMIVAR(I)/(NN*1.0)-LSMIMEAN(I)**2
        MLMIMEAN(I)=MLMIMEAN(I)/(NN*1.0)
        MLMIVAR(I)=MLMIVAR(I)/(NN*1.0)-MLMIMEAN(I)**2
 80   CONTINUE
      LSSIGMAMEAN=LSSIGMAMEAN/(NN*1.0)
      LSSIGMAVAR=LSSIGMAVAR/(NN*1.0)-LSSIGMAMEAN**2
      MLSIGMAMEAN=MLSIGMAMEAN/(NN*1.0)
      MLSIGMAVAR=MLSIGMAVAR/(NN*1.0)-MLSIGMAMEAN**2

      WRITE(1,*)'LSE OF M=',LSMUMEAN
      DO 90 I=1,K
```

```
      WRITE(1,*)'LSE OF M',I,'=',LSMIMEAN(I)
90    CONTINUE
      WRITE(1,*)'LSE OF SIGMA=',LSSIGMAMEAN
      WRITE(1,*)'MMLE OF M=',MLMUMEAN
      DO 91 I=1,K
        WRITE(1,*)'MMLE OF M',I,'=',MLMIMEAN(I)
91    CONTINUE
      WRITE(1,*)'MMLE OF SIGMA=',MLSIGMAMEAN

      WRITE(1,*)'VAR OF LSE OF M=',LSMUVAR
      DO 92 I=1,K
        WRITE(1,*)'VAR OF LSE OF M',I,'=',LSMIVAR(I)
92    CONTINUE
      WRITE(1,*)'VAR OF LSE OF SIGMA=',LSSIGMAVAR

      WRITE(1,*)'VAR OF MML OF M=',MLMUVAR
      DO 93 I=1,K
        WRITE(1,*)'VAR OF MML OF M',I,'=',MLMIVAR(I)
93    CONTINUE
      WRITE(1,*)'VAR OF MML OF SIGMA=',MLSIGMAVAR

      LSSIGMAMSE=LSSIGMAVAR+(LSSIGMAMEAN-1.0)**2
      MLSIGMAMSE=MLSIGMAVAR+(MLSIGMAMEAN-1.0)**2
      DO 94 I=1,K
        LSMIMSE(I)=LSMIVAR(I)+LSMIMEAN(I)**2
        MLMIMSE(I)=MLMIVAR(I)+MLMIMEAN(I)**2
        DEFLS(I)=LSMIMSE(I)+LSSIGMAMSE
        DEFML(I)=MLMIMSE(I)+MLSIGMAMSE
        WRITE(1,*)'DEFLS=',DEFLS(I)
        WRITE(1,*)'DEFML=',DEFML(I)
94   CONTINUE

      REMU=(MLMUVAR/LSMUVAR)*100.0
      DO 95 I=1,K
        REMI(I)=(MLMIVAR(I)/LSMIVAR(I))*100.0
95    CONTINUE
      RESIGMA=(MLSIGMAMSE/LSSIGMAMSE)*100

      WRITE(1,*)'RELATIVE EFFICIENCY OF M=',REMU
      DO 96 I=1,K
       WRITE(1,*)''RELATIVE EFFICIENCY OF M',I,'=',REMI(I)
96    CONTINUE
      WRITE(1,*)''RELATIVE EFFICIENCY OF SIGMA=',RESIGMA

      POWERLS=POWERLS/(NN*1.0)
```

```fortran
      POWERML=POWERML/(NN*1.0)

      WRITE(1,*) 'POWER OF F-TEST=',POWERLS
      WRITE(1,*) 'POWER OF W-TEST=',POWERML

      IF (T.GT.-PI.AND.T.LT.0.0) THEN
        I11=-N*(C2**2)*(T-SIN(T)*COS(T))/(2.0*(SIGMA**2)*T*SIN(T)**2)
        I22=(PI**2-T**2)/(SIN(T)**2)
        I22=I22-((PI**2-3.0*T**2)*COS(T)/(T*SIN(T)))
        I22=-NSUM*I22/(6.0*SIGMA**2)
      ELSE IF (T.GT.0.0) THEN
        I11=-N*(C2**2)*(SINH(T)*COSH(T)-T)/(2.0*SIGMA**2*T*SINH(T)**2)
        I22=(PI**2+3.0*T**2)*COSH(T)/(T*SINH(T))
        I22=I22-((PI**2+T**2)/(SINH(T)**2))
        I22=-NSUM*I22/(6.0*SIGMA**2)
      ELSE IF (T.EQ.0.0) THEN
        I11=-N*(C2**2)/(3.0*SIGMA**2)
        I22=-NSUM*(PI**2+3.0)/(9.0*SIGMA**2)
      ENDIF

      MVBMI=-1.0/I11
      MVBSIGMA=-1.0/I22

      WRITE(1,*)'MVB OF MI=',MVBMI
      WRITE(1,*)'MVB OF SIGMA=',MVBSIGMA

      DO 300 I=1,K
        EMLMI(I)=(MVBMI/MLMIVAR(I))*100.0
        WRITE(1,*)'EFFICIENCY OF M',I,'=',EMLMI(I)
300   CONTINUE

      EMLSIGMA=(MVBSIGMA/MLSIGMAVAR)*100.0
      WRITE(1,*)'EFFICIENCY OF SIGMA=',EMLSIGMA

      POWERLS1=POWERLS1/(NN*1.0)
      POWERML1=POWERML1/(NN*1.0)
      POWERLS2=POWERLS2/(NN*1.0)
      POWERML2=POWERML2/(NN*1.0)

      WRITE(1,*)'POWER OF t1-TEST=’,POWERLS1
      WRITE(1,*)'POWER OF t2-TEST=',POWERLS2
      WRITE(1,*)'POWER OF T1-TEST=',POWERML1
      WRITE(1,*)'POWER OF T2-TEST=',POWERML2
      END
```

2. SIMULATION OF TWO-WAY CLASSIFICATION MODEL

```
PROGRAM TWO_WAY

USE NUMERICAL_LIBRARIES

REAL B2,T,A,C1,C2,PI,MU,TA(10),GAM(10),IN(10,10),SIGMA,T1(100)
REAL ALFA(100),BET(100),M,G(100),E(10,10,100),Y(10,10,100)
REAL U1(100),U2(100),X(10,10,100),MLMU,MLTA(10),MLGAM(10)
REAL MLIN(10,10),XBAR,XABAR,B,MLSIGMA,LSMU,LSTA(10)
REAL LSGAM(10),LSIN(10,10),LSSIGMA,LSMUMEAN,MLMUMEAN
REAL LSTAMEAN(10),MLTAMEAN(10),LSGAMMEAN(10)
REAL MLGAMMEAN(10),LSINMEAN(10,10),MLINMEAN(10,10)
REAL LSSIGMAMEAN,MLSIGMAMEAN,LSMUVAR,LSTAVAR(10)
REAL LSGAMVAR(10),LSINVAR(10,10),LSSIGMAVAR
REAL MLMUVAR,MLTAVAR(10),MLGAMVAR(10),MLINVAR(10,10)
REAL MLSIGMAVAR,REMU,RETA(10),REGAM(10),REIN(10,10)
REAL RESIGMA,W,POWERLS1,POWERLS2,POWERLS3,POWERML1
REAL POWERML2,POWERML3,DFN1,DFN2,DFN3,DFD
REAL F1,F2,F3,FML1,FML2,FML3,FLS1,FLS2,FLS3
INTEGER K,C,N,NSUM,NN

OPEN (unit=1,file='c:\Concon\Documents\sonuc.txt')

PI=22.0/7.0

PRINT *,'ENTER THE KURTOSIS'
READ *,B2
IF (B2.GT.4.2) THEN
  T=-PI*SQRT((5.0*B2-21.0)/(5.0*B2-9.0))
ELSE IF (B2.EQ.4.2) THEN
  T=0.0
ELSE IF (B2.LT.4.2.AND.B2.GT.1.8) THEN
  T=PI*SQRT((21.0-5.0*B2)/(5.0*B2-9.0))
ENDIF
WRITE(1,*) 'SHAPE PARAMETER=',T

PRINT *,'INPUT THE NUMBER OF BLOCKS'
READ *,K
WRITE(1,*) 'NUMBER OF BLOCKS=',K

PRINT *,'INPUT THE NUMBER OF ROWS'
READ *,C
WRITE(1,*) 'NUMBER OF ROWS=',C
```

```
      PRINT*,'INPUT THE SAMPLE SIZE IN THE CELL'
      READ*,N
      WRITE(1,*) 'SAMPLE SIZE IN THE CELL=',N

      NSUM=N*K*C

      DFN1=K*1.0-1.0
      DFN2=C*1.0-1.0
      DFN3=(K*1.0-1.0)*(C*1.0-1.0)
      DFD=(NSUM-K*C)*1.0
      F1=FIN(0.95,DFN1,DFD)
      F2=FIN(0.95,DFN2,DFD)
      F3=FIN(0.95,DFN3,DFD)

      PRINT*,'INPUT THE OVERALL MEAN'
      READ*,MU
      WRITE(1,*) 'OVERALL MEAN=',MU

      PRINT*,'INPUT THE BLOCK EFFECTS'
      DO 1 I=1,K
        READ*,TA(I)
        WRITE(1,*) I,'TH BLOCK EFFECT=',TA(I)
1     CONTINUE

      PRINT*,'INPUT THE ROW EFFECTS'
      DO 2 J=1,C
        READ*,GAM(I)
        WRITE(1,*) I,'TH ROW EFFECT=',GAM(I)
2     CONTINUE

      PRINT*,'INPUT THE INTERACTION EFFECTS'
      DO 5 I=1,K
        DO 6 J=1,C
          READ*,IN(I,J)
          WRITE(1,*) I,J,'TH INTERACTION EFFECT=',IN(I,J)
6       CONTINUE
5     CONTINUE

      SIGMA=1.0

      IF (T.GT.(-1.0*PI).AND.T.LE.0.0) THEN
        A=COS(T)
        C2=SQRT((PI*PI-T*T)/3.0)
        C1=(SIN(T)/T)*C2
      ELSE IF (T.GT.0.0) THEN
```

```
      A=COSH(T)
      C2=SQRT((PI*PI+T*T)/3.0)
      C1=(SINH(T)/T)*C2
    ENDIF

    IF (T.GT.(-PI).AND.T.LT.0.0) THEN
      DO 10 L=1,N
        V2=L
        V1=V2/(N*1.0+1.0)
        T1(L)=LOG(SIN(T*V1)/SIN(T*(1.0-V1)))/C2
10    CONTINUE
    ELSE IF (T.EQ.0.0) THEN
      DO 12 L=1,N
        V2=L
        V1=V2/(N*1.0+1.0)
        W=V1/(1.0-V1)
        T1(L)=(SQRT(3.0)/PI)*LOG(W)
12    CONTINUE
    ELSE IF (T.GT.0.0) THEN
      DO 14 L=1,N
        V2=L
        V1=V2/(N*1.0+1.0)
        T1(L)=LOG(SINH(T*V1)/SINH(T*(1.0-V1)))/C2
14    CONTINUE
    ENDIF

    DO 15 L=1,N
      BET(L)=2.0*C2*EXP(2.0*C2*T1(L))+A*C2*EXP(3.0*C2*T1(L))
      BET(L)=BET(L)+A*C2*EXP(C2*T1(L))
      BET(L)=BET(L)/(EXP(2.0*C2*T1(L))+2.0*A*EXP(C2*T1(L))+1.0)**2
      ALFA(L)=EXP(2.0*C2*T1(L))+A*EXP(C2*T1(L))
      ALFA(L)=ALFA(L)/(EXP(2.0*C2*T1(L))+2.0*A*EXP(C2*T1(L))+1.0)
      ALFA(L)=ALFA(L)-BET(L)*T1(L)
15  CONTINUE

    DO 16 L=1,N
       IF (BET(L).LT.0.0) THEN
        BET(L)=0.0
        ALFA(L)=EXP(2.0*C2*T1(L))+A*EXP(C2*T1(L))
        ALFA(L)=ALFA(L)/(EXP(2.0*C2*T1(L))+2.0*A*EXP(C2*T1(L))+1.0)
      ENDIF
16  CONTINUE

    M=0.0
    DO 17 L=1,N
```

```
      M=M+BET(L)
17    CONTINUE

      LSMUMEAN=0.0
      LSMUVAR=0.0
      MLMUMEAN=0.0
      MLMUVAR=0.0
      DO 18 I=1,K
        LSTAMEAN(I)=0.0
        LSTAVAR(I)=0.0
        MLTAMEAN(I)=0.0
        MLTAVAR(I)=0.0
18    CONTINUE
      DO 19 J=1,C
        LSGAMMEAN(J)=0.0
        LSGAMVAR(J)=0.0
        MLGAMMEAN(J)=0.0
        MLGAMVAR(J)=0.0
19    CONTINUE
      DO 20 I=1,K
        DO 21 J=1,C
          LSINMEAN(I,J)=0.0
          LSINVAR(I,J)=0.0
          MLINMEAN(I,J)=0.0
          MLINVAR(I,J)=0.0
21      CONTINUE
20    CONTINUE
      LSSIGMAMEAN=0.0
      MLSIGMAMEAN=0.0
      LSSIGMAVAR=0.0
      MLSIGMAVAR=0.0

      POWERLS1=0.0
      POWERLS2=0.0
      POWERLS3=0.0
      POWERML1=0.0
      POWERML2=0.0
      POWERML3=0.0

      NN=100000/N

      DO 300 L1=1,NN

      DO 25 I=1,K
        DO 26 J=1,C
```

```
              CALL RNUN(N,G)
              IF (T.GT.(-PI).AND.T.LT.0.0) THEN
                DO 27 L=1,N
                  E(I,J,L)=LOG(SIN(T*G(L))/SIN(T*(1.0-G(L))))/C2
   27           CONTINUE
              ELSE IF (T.EQ.0.0) THEN
                DO 28 L=1,N
                  E(I,J,L)=(SQRT(3.0)/PI)*LOG(G(L)/(1.0-G(L)))
   28           CONTINUE
              ELSE IF (T.GT.0.0) THEN
                DO 29 L=1,N
                  E(I,J,L)=LOG(SINH(T*G(L))/SINH(T*(1.0-G(L))))/C2
   29           CONTINUE
              ENDIF
   26       CONTINUE
   25     CONTINUE

          DO 30 I=1,K
            DO 31 J=1,C
              DO 32 L=1,N
               Y(I,J,L)=E(I,J,L)+MU+TA(I)+GAM(J)+IN(I,J)
   32         CONTINUE
   31       CONTINUE
   30     CONTINUE

C FINDING MMLE

          DO 33 I=1,K
            DO 34 J=1,C
              DO 35 L=1,N
                U1(L)=Y(I,J,L)
   35         CONTINUE
              CALL SVRGN(N,U1,U2)
              DO 36 L=1,N
                X(I,J,L)=U2(L)
   36         CONTINUE
   34       CONTINUE
   33     CONTINUE

          MLMU=0.0
          DO 37 I=1,K
            DO 38 J=1,C
              DO 39 L=1,N
                MLMU=MLMU+BET(L)*X(I,J,L)
   39         CONTINUE
```

```
38      CONTINUE
37    CONTINUE
      MLMU=MLMU/(K*C*M*1.0)
      MLMUMEAN=MLMUMEAN+MLMU
      MLMUVAR=MLMUVAR+MLMU**2

      DO 40 I=1,K
        MLTA(I)=0.0
        DO 41 J=1,C
          DO 42 L=1,N
             MLTA(I)=MLTA(I)+BET(L)*X(I,J,L)
42        CONTINUE
41      CONTINUE
        MLTA(I)=MLTA(I)/(C*M*1.0)-MLMU
        MLTAMEAN(I)=MLTAMEAN(I)+MLTA(I)
        MLTAVAR(I)=MLTAVAR(I)+MLTA(I)**2
40    CONTINUE

      DO 43 J=1,C
        MLGAM(J)=0.0
        DO 44 I=1,K
          DO 45 L=1,N
             MLGAM(J)=MLGAM(J)+BET(L)*X(I,J,L)
45        CONTINUE
44      CONTINUE
        MLGAM(J)=MLGAM(J)/(K*M*1.0)-MLMU
        MLGAMMEAN(J)=MLGAMMEAN(J)+MLGAM(J)
        MLGAMVAR(J)=MLGAMVAR(J)+MLGAM(J)**2
43    CONTINUE

      DO 46 I=1,K
        DO 47 J=1,C
          MLIN(I,J)=0.0
          DO 48 L=1,N
            MLIN(I,J)=MLIN(I,J)+BET(L)*X(I,J,L)
48        CONTINUE
          MLIN(I,J)=(MLIN(I,J)/M)-MLMU-MLTA(I)-MLGAM(J)
          MLINMEAN(I,J)=MLINMEAN(I,J)+MLIN(I,J)
          MLINVAR(I,J)=MLINVAR(I,J)+MLIN(I,J)**2
47      CONTINUE
46    CONTINUE

      XBAR=0.0
      XABAR=0.0
      DO 49 I=1,K
```

```
    DO 50 J=1,C
      DO 51 L=1,N
        XBAR=XBAR+X(I,J,L)
        XABAR=XABAR+X(I,J,L)*ALFA(L)
51      CONTINUE
50    CONTINUE
49  CONTINUE
    XBAR=XBAR/(NSUM*1.0)
    XABAR=XABAR*2.0/(1.0*NSUM)
    B=C2*NSUM*(XBAR-XABAR)
    CC=0.0
    DO 52 I=1,K
      DO 53 J=1,C
        DO 54 L=1,N
          CC=CC+BET(L)*(X(I,J,L)-MLMU-MLTA(I)-MLGAM(J)-MLIN(I,J))**2
54      CONTINUE
53    CONTINUE
52  CONTINUE
    CC=CC*2.0*C2
    MLSIGMA=(-B+SQRT(B*B+4.0*NSUM*CC))
    MLSIGMA=MLSIGMA/(2.0*SQRT(1.0*NSUM*(NSUM-K*C)))
    MLSIGMAMEAN=MLSIGMAMEAN+MLSIGMA
    MLSIGMAVAR=MLSIGMAVAR+MLSIGMA**2

    FML1=0.0
    DO 55 I=1,K
      FML1=FML1+MLTA(I)**2
55  CONTINUE
    FML1=FML1*2.0*C2*C*M/((K*1.0-1.0)*MLSIGMA**2)
    IF (FML1.GT.F1) THEN
      POWERML1=POWERML1+1.0
    ENDIF

    FML2=0.0
    DO 56 J=1,C
      FML2=FML2+MLGAM(J)**2
56  CONTINUE
    FML2=FML2*2.0*C2*K*M/((C*1.0-1.0)*MLSIGMA**2)
    IF (FML2.GT.F2) THEN
      POWERML2=POWERML2+1.0
    ENDIF

    FML3=0.0
    DO 57 I=1,K
      DO 58 J=1,C
```

```
          FML3=FML3+MLIN(I,J)**2
 58     CONTINUE
 57   CONTINUE
      FML3=FML3*2.0*C2*M/((K*1.0-1.0)*(C*1.0-1.0)*MLSIGMA**2)
      IF (FML3.GT.F3) THEN
        POWERML3=POWERML3+1.0
      ENDIF

C FINDING LSE

      LSMU=XBAR
      LSMUMEAN=LSMUMEAN+LSMU
      LSMUVAR=LSMUVAR+LSMU**2

      DO 60 I=1,K
        LSTA(I)=0.0
        DO 61 J=1,C
          DO 62 L=1,N
            LSTA(I)=LSTA(I)+Y(I,J,L)
 62       CONTINUE
 61     CONTINUE
        LSTA(I)=LSTA(I)/(N*C*1.0)-LSMU
        LSTAMEAN(I)=LSTAMEAN(I)+LSTA(I)
        LSTAVAR(I)=LSTAVAR(I)+LSTA(I)**2
 60   CONTINUE

      DO 63 J=1,C
        LSGAM(J)=0.0
        DO 64 I=1,K
          DO 65 L=1,N
            LSGAM(J)=LSGAM(J)+Y(I,J,L)
 65       CONTINUE
 64     CONTINUE
        LSGAM(J)=LSGAM(J)/(N*K*1.0)-LSMU
        LSGAMMEAN(J)=LSGAMMEAN(J)+LSGAM(J)
        LSGAMVAR(J)=LSGAMVAR(J)+LSGAM(J)**2
 63   CONTINUE

      DO 66 I=1,K
        DO 67 J=1,C
          LSIN(I,J)=0.0
          DO 68 L=1,N
            LSIN(I,J)=LSIN(I,J)+Y(I,J,L)
 68       CONTINUE
          LSIN(I,J)=LSIN(I,J)/(N*1.0)-LSMU-LSTA(I)-LSGAM(J)
```

```
        LSINMEAN(I,J)=LSINMEAN(I,J)+LSIN(I,J)
        LSINVAR(I,J)=LSINVAR(I,J)+LSIN(I,J)**2
 67   CONTINUE
 66  CONTINUE

     LSSIGMA=0.0
     DO 69 I=1,K
       DO 70 J=1,C
         DO 71 L=1,N
           LSSIGMA=LSSIGMA+(Y(I,J,L)-LSMU-LSTA(I)-LSGAM(J)-
LSIN(I,J))**2
 71        CONTINUE
 70    CONTINUE
 69  CONTINUE
     LSSIGMA=LSSIGMA/((NSUM-K*C)*1.0)
     LSSIGMA=SQRT(LSSIGMA)
     LSSIGMAMEAN=LSSIGMAMEAN+LSSIGMA
     LSSIGMAVAR=LSSIGMAVAR+LSSIGMA**2

     FLS1=0.0
     DO 73 I=1,K
       FLS1=FLS1+LSTA(I)**2
 73  CONTINUE
     FLS1=FLS1*C*N/((K*1.0-1.0)*LSSIGMA**2)
     IF (FLS1.GT.F1) THEN
       POWERLS1=POWERLS1+1.0
     ENDIF

     FLS2=0.0
     DO 74 J=1,C
       FLS2=FLS2+LSGAM(J)**2
 74  CONTINUE
     FLS2=FLS2*K*N/((C*1.0-1.0)*LSSIGMA**2)
     IF (FLS2.GT.F2) THEN
       POWERLS2=POWERLS2+1.0
     ENDIF

     FLS3=0.0
     DO 75 I=1,K
       DO 76 J=1,C
         FLS3=FLS3+LSIN(I,J)**2
 76    CONTINUE
 75  CONTINUE
     FLS3=FLS3*N/((K*1.0-1.0)*(C*1.0-1.0)*LSSIGMA**2)
     IF (FLS3.GT.F3) THEN
```

```
        POWERLS3=POWERLS3+1.0
      ENDIF

300 CONTINUE

      LSMUMEAN=LSMUMEAN/(NN*1.0)
      LSMUVAR=LSMUVAR/(NN*1.0)-LSMUMEAN**2
      MLMUMEAN=MLMUMEAN/(NN*1.0)
      MLMUVAR=MLMUVAR/(NN*1.0)-MLMUMEAN**2

      DO 80 I=1,K
        LSTAMEAN(I)=LSTAMEAN(I)/(NN*1.0)
        LSTAVAR(I)=LSTAVAR(I)/(NN*1.0)-LSTAMEAN(I)**2
        MLTAMEAN(I)=MLTAMEAN(I)/(NN*1.0)
        MLTAVAR(I)=MLTAVAR(I)/(NN*1.0)-MLTAMEAN(I)**2
 80   CONTINUE

      DO 81 J=1,C
        LSGAMMEAN(J)=LSGAMMEAN(J)/(NN*1.0)
        LSGAMVAR(J)=LSGAMVAR(J)/(NN*1.0)-LSGAMMEAN(J)**2
        MLGAMMEAN(J)=MLGAMMEAN(J)/(NN*1.0)
        MLGAMVAR(J)=MLGAMVAR(J)/(NN*1.0)-MLGAMMEAN(J)**2
 81   CONTINUE

      DO 82 I=1,K
        DO 83 J=1,C
          LSINMEAN(I,J)=LSINMEAN(I,J)/(NN*1.0)
          LSINVAR(I,J)=LSINVAR(I,J)/(NN*1.0)-LSINMEAN(I,J)**2
          MLINMEAN(I,J)=MLINMEAN(I,J)/(NN*1.0)
          MLINVAR(I,J)=MLINVAR(I,J)/(NN*1.0)-MLINMEAN(I,J)**2
 83     CONTINUE
 82   CONTINUE

      LSSIGMAMEAN=LSSIGMAMEAN/(NN*1.0)
      LSSIGMAVAR=LSSIGMAVAR/(NN*1.0)-LSSIGMAMEAN**2
      MLSIGMAMEAN=MLSIGMAMEAN/(NN*1.0)
      MLSIGMAVAR=MLSIGMAVAR/(NN*1.0)-MLSIGMAMEAN**2

      WRITE(1,*)'LSE OF M=',LSMUMEAN
      DO 90 I=1,K
        WRITE(1,*)'LSE OF t',I,'=',LSTAMEAN(I)
 90   CONTINUE
      DO 91 J=1,C
        WRITE(1,*)'LSE OF g',J,'=',LSGAMMEAN(J)
 91   CONTINUE
```

```
      DO 92 I=1,K
        DO 93 J=1,C
          WRITE(1,*)'LSE OF int',I,J,'=',LSINMEAN(I,J)
93      CONTINUE
92    CONTINUE
      WRITE(1,*)'LSE OF SIGMA=',LSSIGMAMEAN
      WRITE(1,*)' '
      WRITE(1,*)'MMLE OF M=',MLMUMEAN
      DO 94 I=1,K
        WRITE(1,*)'MMLE OF t',I,'=',MLTAMEAN(I)
94    CONTINUE
      DO 95 J=1,C
        WRITE(1,*)'MMLE OF g',J,'=',MLGAMMEAN(J)
95    CONTINUE
      DO 96 I=1,K
        DO 97 J=1,C
          WRITE(1,*)'MMLE OF int',I,J,'=',MLINMEAN(I,J)
97      CONTINUE
96    CONTINUE
      WRITE(1,*)'MMLE OF SIGMA=',MLSIGMAMEAN
      WRITE(1,*)' '

      WRITE(1,*)'VAR OF LSE OF M=',LSMUVAR
      DO 98 I=1,K
        WRITE(1,*)'VAR OF LSE OF t',I,'=',LSTAVAR(I)
98    CONTINUE
      DO 99 J=1,C
        WRITE(1,*)'VAR OF LSE OF g',J,'=',LSGAMVAR(J)
99    CONTINUE
      DO 100 I=1,K
        DO 101 J=1,C
          WRITE(1,*)'VAR OF LSE OF int',I,J,'=',LSINVAR(I,J)
101     CONTINUE
100   CONTINUE
      WRITE(1,*)'VAR OF LSE OF SIGMA=',LSSIGMAVAR
      WRITE(1,*)' '

      WRITE(1,*)'VAR OF MML OF M=',MLMUVAR
      DO 102 I=1,K
        WRITE(1,*)'VAR OF MML OF t',I,'=',MLTAVAR(I)
102   CONTINUE
      DO 103 J=1,C
        WRITE(1,*)'VAR OF MML OF g',J,'=',MLGAMVAR(J)
103   CONTINUE
      DO 104 I=1,K
```

```
      DO 105 J=1,C
        WRITE(1,*)'VAR OF MML OF int',I,J,'=',MLINVAR(I,J)
105   CONTINUE
104   CONTINUE
      WRITE(1,*)'VAR OF MML OF SIGMA=',MLSIGMAVAR

      REMU=(MLMUVAR/LSMUVAR)*100.0
      RETA(1)=(MLTAVAR(1)/LSTAVAR(1))*100.0
      REGAM(1)=(MLGAMVAR(1)/LSGAMVAR(1))*100.0
      REIN(1,1)=(MLINVAR(1,1)/LSINVAR(1,1))*100.0
      RESIGMA=(MLSIGMAVAR/LSSIGMAVAR)*100

      WRITE(1,*)'RELATIVE EFFICIENCY OF M=',REMU
      WRITE(1,*)'RELATIVE EFFICIENCY OF t=',RETA(1)
      WRITE(1,*)'RELATIVE EFFICIENCY OF g=',REGAM(1)
      WRITE(1,*)'RELATIVE EFFICIENCY OF int=',REIN(1,1)
      WRITE(1,*)'RELATIVE EFFICIENCY OF SIGMA=',RESIGMA

      POWERLS1=POWERLS1/(NN*1.0)
      POWERML1=POWERML1/(NN*1.0)
      POWERLS2=POWERLS2/(NN*1.0)
      POWERML2=POWERML2/(NN*1.0)
      POWERLS3=POWERLS3/(NN*1.0)
      POWERML3=POWERML3/(NN*1.0)

      WRITE(1,*)'POWER OF F1-TEST=',POWERLS1
      WRITE(1,*)'POWER OF F2-TEST=',POWERLS2
      WRITE(1,*)'POWER OF F3-TEST=',POWERLS3

      WRITE(1,*)'POWER OF W1-TEST=',POWERML1
      WRITE(1,*)'POWER OF W2-TEST=',POWERML2
      WRITE(1,*)'POWER OF W3-TEST=',POWERML3
      END
```