“FREE FLEXURAL (or BENDING) VIBRATIONS ANALYSIS OF COMPOSITE, ORTHOTROPIC PLATE AND/OR PANELS WITH VARIOUS BONDED JOINTS”
(- - -IN AERO-STRUCTURAL SYSTEMS - - -)

VOLUME I

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULLFILMENT OF THE REQUIREMENTS
FOR
THE DEGREE OFMASTER OF SCIENCE
IN
AEROSPACE ENGINEERING

SEPTEMBER 2004

Approval of the Graduate School of Natural and Applied Sciences

____

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ABSTRACT

“FREE FLEXURAL (or BENDING) VIBRATIONS ANALYSIS OF COMPOSITE ORTHOTROPIC PLATE AND/OR PANELS WITH VARIOUS BONDED JOINTS”
(- - - IN AERO-STRUCTURAL SYSTEMS - - -)

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(M.S.) in Department of Aerospace Engineering

Supervisor: Prof. Dr. Umur YÜCEOĞLU

September 2004, 491 Pages

In this “Thesis”, the problems of the “Free Flexural (or Bending) Vibrations of Composite, Orthotropic Plates and/or Panels with Various Bonded Joints” are formulated and investigated in detail. The composite bonded plate system is composed of “Plate Adherends” adhesively bonded by relatively very thin adhesive layers. The general problem is considered in terms of the three “Main PROBLEMS”, namely “Main PROBLEM I”, “Main PROBLEM II” and “Main PROBLEM III”. The theoretical formulation of the “Main PROBLEMS” is based on “Mindlin Plate Theory” which is a “First Order Shear Deformation Plate Theory (FSDPT)”. Thus, the transverse shear deformations, the transverse and the rotary moments of inertia of the plates are included in the formulation. Very thin, elastic deformable adhesive layers are considered as continua with transverse normal and shear stresses. The damping effects in the plates and the adhesive layers are neglected.

The entire composite bonded joint assembly is assumed to be simple supported along the two opposite edges, so that the “Classical Levy’s Solutions” can be applied in this direction. The dynamic equations of the “Bonded Joint System” which combines together the “Mindlin Plate” dynamic equations with the adhesive layer equations are reduced to a system of “First Order Ordinary Differential Equations” in the “state vector” form. This “special form” of the “Governing System of the First Order Ordinary Differential Equations” are numerically integrated by means of the “Modified Transfer Matrix Method” which is a combination of the “Classical Levy’s Method”, the “Transfer Matrix Method” and the “Integrating Matrix Method (with Interpolation Polynomials and/or Chebyshev Polynomials)”.

The “Main PROBLEMS” are investigated and presented in terms of the mode shapes and the corresponding natural frequencies for various sets of boundary conditions. The significant effects of the “hard” and the “soft” adhesive layer elastic constants on the mode shapes and on the natural frequencies are demonstrated. Some important parametric studies such as the influences of the “Joint Length Ratio”, the “Joint Position Ratio”, the “Bending Stiffness Ratio”, etc. on the natural frequencies are computed and plotted for the “hard” and “soft” adhesive cases for several support conditions.

Keywords: Composite Orthotropic Plate Vibrations, Bonded Plates, Lap Joint, Symmetric Single Lap Joint, Symmetric Double Lap Joint.

ÖZ

“ÇEŞİTLİ YAPIŞTIRICILARLA BİRLEŞTİRİLMİŞ, KOMPOZİT, ORTOTROPİK LEVHA (PLAKA) VE/VEYA PANELLERİN SERBEST EĞİLME TİTREŞİMLERİNİN ANALİZİ” (HAVA-ARACI YAPISAL SİSTEMLERİNDE)

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Eylül 2004, 491 Sayfa

Bu tezde, “Çeşitli Yapıştırıcılarla Birleştirilmiş, Ortotropik, Kompozit Levha (Plaka) ve/veya Panellerin Serbest Titreşim Analizi” problemleri incelenmiştir. Kompozit plaka ve/veya panel sistemi, üst ve alt plakalardan ve nispeten çok ince olan yapışkan yüzeylerden oluşmaktadır. Tezde incelenen genel problemler “Ana PROBLEM I”, “Ana PROBLEM II”, ve “Ana PROBLEM III” olarak üçe ayrılmıştır. “Ana PROBLEMLER” in teorik formülleştirilmesi bir çeşit “Birinci Derece Kayma Deformasyonu Plaka Teorisi” olan “Mindlin Plaka Teorisi”ne dayanmaktadır. Yani enine kayma deformasyonu ve plakalardaki dönme atalet momenti denklemlerde dahil edilmiştir. Çok ince, elastik olarak deforme olabilen yapıştırıcı tabakaların enine dik streslerle ve kayma stresleri ile süreklilik gösterdiği düşünülmüştür. Plakalardaki ve yapıştırıcı tabakalardaki titreşim sönümleyici özellikler ihmal edilmiştir.

Yapıştırıcıyla birleştirilmiş kompozit sistemin karşılıklı iki kenarının basit mesnetli sınır koşullarına sahip olduğu kabul edilmiştir, bu nedenle bu yönde “Klasik Lévy Çözümü” uygulanmaktadır. ”Ortotropik Mindlin Plaka Teorisi” denklemleri ile yapıştırıcı yüzeylere ait olan denklemleri birleştiren dinamik denklemler, “durum vektörü” halindeki “Esas Birinci Derece Basit Diferansiyel Denklemler Sistemi” ne indirgenmiştir. “Birinci Derece Basit Diferansiyel Denklemler”in özel hali olan bu denklemlerin integrali numerik olarak “Değiştirilmiş Transfer Matris Metodu (Interpolasyon Polinomları ve/veya Chebyshev Polinomlarıyla)” ile alınmıştır. Bu teknik, “Klasik Lévy Çözümü”, “Transfer Matris Metodu” ve “Integral Alma Matris Metodu”nu birleştirmektedir.

Her “Ana PROBLEM” için çeşitli sınır koşullarında detaylı olarak mod şekilleri ve ilgili frekans değerleri verilmiştir. “Sert” ve”Yumuşak” yapıştırıcıların mode şekilleri ve frekans değerleri üzerindeki etkileri gösterilmiştir. “Birleştirici Uzunluk Oranı”, “Birleştirici Pozisyon Oranı” ve “Eğilme Şiddeti Oranı” gibi çeşitli parametrik çalışmalar hazırlanmıştır ve “Sert” ve “Yumuşak” sınır koşulları için çizilmiştir.

Anahtar Kelimeler: Kompozit Ortotropik Plakaların Titreşimleri, Yapıştırılmış Plakalar, Üst Üste Binmiş Ekler, Simetrik Üst Üste Binmiş Ekler, Simetrik Çift Taraftan Üst Üste Binmiş Ekler

ACKNOWLEDGMENTS

I wish to thank my supervisor, Prof. Dr. Umur YUCEOGLU and also to Dr. Varlık ÖZERCİYES who gave me immeasurable support in my “Thesis” studies. Also I want to express my gratitude to my parents, to Serkan Selcuk ERSOY, and to my friends who encourage me to complete my M.S. education.

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CHAPTER 1

INTRODUCTION

1.1 Introductory Remarks

In recent years, the “Advanced Composites” and the “Advanced Metal Alloys” technologies have been developing so rapidly with the “Adhesive Bonding” techniques and the epoxy-based adhesives, the. (Lubin [I.1], Hoskins [I.2], Marshall [I.3], ASM International Handbook[I.4], Schwartz [I.5], Baker [I.6]). These systems are more and more being used as very efficient, and light-weight primary and secondary structural systems and components in air and space vehicle, hydrodynamic and other vehicle structures. The “joining and/or extension” (and also “stiffening and/or repairing”) of these aero-structural systems and components are usually constructed in the forms of various types of “Adhesive (or Bonded) Joints”. The main reasons for using “Adhesive (or Bonded) Joints” are their smoothness, light-weight, damping and the crack-retarding, and ease of manufacture characteristics.

In general, Composites are materials that are combinations of two or more components or phases. One material serves as a “resin” or "matrix," which is the material that holds everything together, while the other material serves as a “reinforcement”, in the form of fibers embedded in the matrix.

In the analysis and design of flight vehicle structures, the stress analysis and the dynamic response of “Adhesive (or Bonded) Joints” are extremely important. This is because of the dynamic stress concentrations, dynamic crack propagation, fatigue and fracture, sound transmission, sonic fatigue, etc. and the subsequent complex failure modes that occur in joints under operational conditions.

Therefore, there are considerable number of studies and investigations in stress analysis of “Adhesive Joints” in open engineering and scientific literature all over the world. These are not the concern of the present study and will not be reviewed here. However, the studies on the dynamic response of “Adhesive (or Bonded) Joints”, are relatively few and far between. Here, in the present Thesis, the various types of bonded joints in plates are to be considered.

The “Adhesive (or Bonded) Joints” are, in general, employed for;

- “joining and/or extension” of and,
- “stiffening and/or repairing” of the aircraft and spacecraft primary and/or secondary structural systems, panels and components.

Some major applications of composites in aircraft structures are given in Table 1.1. (Hoskins and Baker[I.2], ASM International Handbook [I.4])

1.2 Literature Survey and Brief Review

Some research studies about the "joining and extension of plates and/or panels" can be found in [I.7], and investigations related to "stiffening and/or repairing of plates and/or panels" are available in [I.6]. In particular, the free vibrations of the "Bonded Lap Joints" in beams or beam-like plate strips are considered in [VIII.4, VIII.6, VIII.7, VIII.11]. The free vibrations of the orthotropic rectangular plates with "Bonded Single Lap Joints" are analyzed in Yuceoglu et al [IV.2, VIII.13].For the free vibration problems of the second group mentioned above (i.e. in (2)), mainly for the stiffening of and, in some cases for repairing of the composite plates or panels by bonded plate strips, Yuceoglu and Özerciyes [IV.2, IV.13] can be mentioned.

The "Single and Double Lap Joints" are, excluding Yuceoglu et al [IV.2, IV.13], analyzed as beams and/or beam-like plate strips and the plate action and the plate dynamic response of the entire "Bonded Lap Joint Assembly" are completely neglected.

Therefore, the main purpose of this study is to investigate the "Free Flexural (or Bending) Vibrations of Composite, Orthotropic Mindlin Plates or Panels with various Bonded Joints". This study is an extension of a previous work by Yuceoglu et al [IV.2, IV.13]. In the present theoretical analysis, the transverse shear deformations and the rotatory moments of inertia are to be taken into account in the manner of the "Mindlin Plate Theory" [V.3] which is a " First Order shear Deformation Plate Theory (FSDPT)"[V.2, V.3]. The important effect of the transverse shear deformations in multi-layer plates, even if the individual layers are very thin, are pointed out by Whitney [VI.2], and by Whitney and Pagano [VI.3]. This last point is important for instance, in the "overlap regions" or the two-layer or three-layer regions of the all types of "Bonded Joint".

Another important consideration is that Khdeir and Librescu [VI.8] showed that the natural frequencies of the multi-layer composite plates obtained by the (FSDPT) and by the (HSDPT) are not significantly different. Therefore, in the percent study a (FSDPT) such as “Mindlin Plate Theory” [V.3] is used. In this connection, one can refer to reviews, up to 1994 of the “First Order Shear Deformation Plate Theories (FSDPT)” and the “Higher Order Shear Deformation Plate Theories (HSDPT)” which can be found in Kapania and Raciti [II.4], and in Reddy [28] and Reddy and Robbins [III.2], respectively.

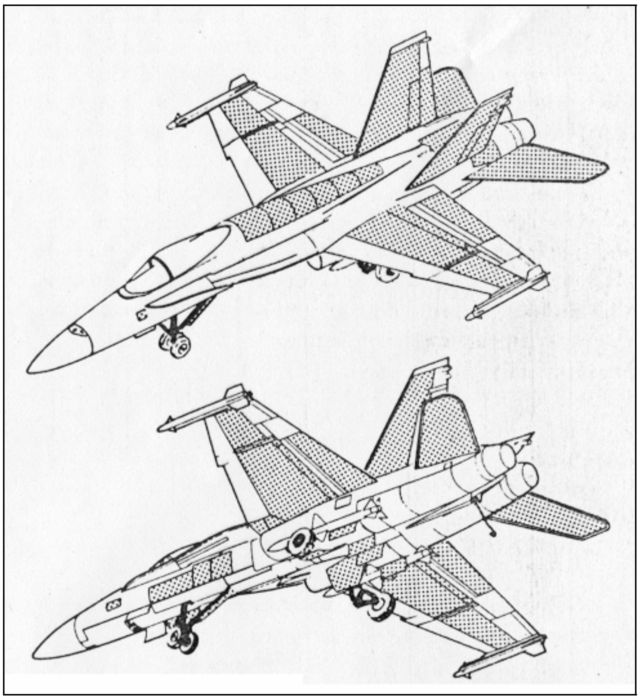
Table 1.1. "Advanced Composites" and "Composite Components" in Aircraft Structures for Various Types of Aircraft (most of the data below taken from Ref. [I.2] and [I.4])"

Composite Component	F-14	F -15	F-16	F-18 Hornet	B-1	AV-8B	DC-10 Demo	L-1011	B-737 Demo	B-727	B-757	B-767	Lear Fan	JAS-39 Gripen	F-18 Super Hornet
Doors	√			√	√	√					√	√	√	√	√
Rudder		√				√	√				√	√	√	√	√
Elevator										√	√	√	√	√	√
Vertical tail		√	√	√	√	√	√	√						√	√
Horizontal tail	√	√	√	√	√	√			√				√		√
Aileron						√		√			√	√		√	√
Spoiler									√		√	√		√	√
Flap					√	√					√		√	√	√
Wing box				√		√							√	√	√
Body						√							√		
Miscellaneous	Fairings	Speed brake		Speed brake, fairings	Slats, inlets	Fairings		Fairings			Fairings	Fairings	Fairings	Fairings, speed brake, slats, canard	Fairin gs, speed brake, slats

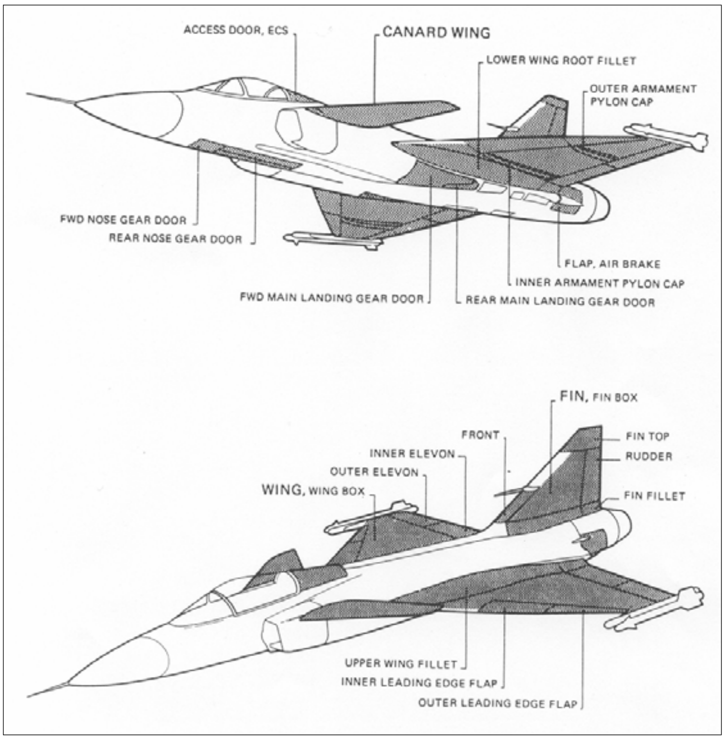
Note:

(1): √ means that composite is used for that component.

(2): Some of the composite applications in the above table are given according to the information available in 1997 or so.



a) Mc-Donell Douglas “F/A-18 (Hornet)”



b) “Saab JAS 39 Gripen”

Figure 1.1 Some Examples of Composites in Aircraft Systems

(Shaded areas are made of composites)

CHAPTER 2

MAIN PURPOSE AND SCOPE

2.1 Introductory Remarks and Motivation for Present Study

In this Section, certain group or class of the “Free Flexural (or Bending) Vibration Problems of Mindlin Plates with Various Bonded Lap Joints” will be considered.

The knowledge about the free and forced vibrations of the “Single Layer and/or Multi-Layer Composite Plates or Panels” is extremely important in the studies of panel flutter, fatigue and fracture, dynamic crack propagation, sonic fatigue and sound transmission, dynamic structural stability, etc in complex air and space vehicle structures and systems.

From the brief review and studies available in the scientific and the engineering literature on plates bonded with lap joint, and also from the practical applications in aerospace vehicles structures (or aero-structures), following free vibrations problems shall be considered. The “Free Dynamic Response” of “Bonded Joints”composite orthotropic and/or isotropic base plates or panels have great importance from the point of view of the “structural integrity and safety”. The “Bonded Joints” are mostly, in flight vehicla structures, used in

- The "joining and/or extension" of “Single Layer and/or Multi-layer Composite Structural Systems” in flight vehicle structures in the form of various types of "Adhesively Bonded Joints”.

- The "Bonded Repairing and/or Strengthening" of relatively slender and already cracked "Advanced Composite" or "Advanced Metal Alloy" plate or panel systems in aero-structures or (aero-structural systems).

The motivation and justification for the present Thesis are based on the above mentioned general free flexural (or bending) vibration problems encountered in aero-structures. In order to give a general idea about the practical applications and importance of these problems, one may refer to the three basic cases presented in this Thesis. These cases are properly defined and referred to as the " Main PROBLEMS". The "Main PROBLEMS" are properly defined in Chapter 3 in detail. Some general ideas, however, are given in this chapter.

2.2 Main Purpose, Scope and Objectives

The "main purpose" of this "Thesis" is to analyze the "Free Flexural (or Bending) Vibrations of Orthotropic Composite Mindlin Plates or Panels with Various Bonded Joints" specially, in "Aero-Structural Systems".

The "scope" of the present study (or "Thesis") is concantrated on the linear free dynamic response with all types of damping effects being neglected. The "Bonded Joints" types are limited to the joints defined in the "Main PROBLEMS".

The "main objectives" of this study is firstly to determine the free dynamic response characteristics in terms of their natural frequencies and corresponding mode shapes, secondly, to perform some important parametric studies in order to help the aerospace vehicle designers. Finally, based on the free dynamic response and the parametric studies, the important conclusions and recommandations will be stated.

2.3 Statements of “Main PROBLEMS” and General Configurations

The “Main PROBLEMS” are considered as follows:

- **The “Main PROBLEM I” is defined as the “Free Vibrations of Orthotropic, Composite Mindlin Plates or Panels with a Bonded Lap Joint”.**

The general configuration, the geometry, the coordinate system and the longitudinal cross section of this type of the bonded plate system (i.e. “Main PROBLEM I”) are shown in Figure 3.2.a, Figure 3.2.b, Figure 3.3.a and Figure 3.3.b, respectively. The analysis will be based on Mindlin Plate Theory [VI.3,VaI.4], and thus the transverse shear deformations, and the transverse and the rotatory inertias in the plate layers will be included in the formulation.

- **The “Main PROBLEM II” is defined as the “Free Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Bonded Symmetric Single Lap Joint (Symmetric Single Doubler Joint)”.**

The general configuration, the geometry, the coordnate system, and the longitudinal cross section of this type of the bonded plate system (i.e. “ Main PROBLEM II”) are presented in Figure 3.4.a, Figure 3.4.b, Figure 3.5.a and Figure 3.5.b, respectively. The analysis will be based on Mindlin Plate Theory [VI.3,VI.4] taking into account the transverse shear deformations, and the transverse, and the rotatory inertias in the plate layers.

- **The “Main PROBLEM III” is defined as the “Free Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Bonded Symmetric Doubler Lap Joint (Symmetric Double Doubler Joint)”.**

Again, the general configuration, the geometry, the coordnate system and the longitudinal cross section of this type of the bonded plate system (i.e. “ Main PROBLEM III”) are given in Figure 3.6.a, Figure 3.6.b, Figure 3.7.a and Figure

3.7.b,respectively. The analysis will be based on the Mindlin Plate Theory [VI.3,VI.4] taking into account the transverse shear deformations, and the transverse and the rotatory inertias in the plate layers.

2.4 Original Contributions

- Papers published on the dynamic response of “Adhesive (or Bonded) Joints”, are relatively few and far between. The detailed study of the free dynamic response of the “Bonded Joints” systems in “Mindlin Plates” is an important contribution of this “Thesis”.

- “Free Flexural Vibrations Response of the Orthotropic Composite Mindlin Plates with Various Bonded Joints” are obtained as a result of the present “Thesis” which will significantly affect the design of aero-structures or aero-structural systems and components. (These systems are not yet investigated and their dynamic response is not available in the open literature).

CHAPTER 3

MAIN PROBLEMS

3.1 Introductory Remarks

In this Chapter, "Higher Order" and the "First Order" shear deformation theories for the multi-layer plates will be briefly reviewed. Also the complete set of dynamic equations of the Mindlin Plate Theory will be developed for easy reference.

After then, proper definitions of the "Main PROBLEMS (I, II, III)" will be stated. Corresponding theoretical formulations of these problems will be given in the following chapters.

3.2 Brief Remarks on "Higher Order Shear Deformation Plate Theories (HSDPT)" used in Multi-Layer Plates

The "Classical Plate Theory (CLPT)" does not include the effect of the transverse shear deformations, the rotatory moments of inertia, and the transverse normal stresses. New and improved theories, such as "First Order Shear Deformation Plate Theories (FSDPT)" have been proposed by research workers in order to improve the "Classical Plate Theory (CLPT)". Improved methods showed that the results obtained by using "Classical Plate Theory (CLPT)" are not accurate enough for a lot of practical problems, such as vibration problems, elastic wave propagation problems, the analysis of anistropic plates, the stresses concentration due to holes or cut-outs, etc.

The simplest ones of all the improved plate theories were produced by Reissner [V.1] and Mindlin [V.3, V.4]. The Reissner's theory includes the effect of shear deformations and results in the following displacements of the form;

$$\left| \begin{array}{l} u(x,y) = u_0(x,y) + z\psi_x(x,y) \\ v(x,y) = v_0(x,y) + z\psi_y(x,y) \\ w(x,y) = w_0(x,y) \end{array} \right. \tag{3.1}$$

where z is the coordinate that is normal to the middle or reference plane, u_0, v_0, and w_0 are dependent to the in-plane coordinates x and y, and ψ_x, ψ_y and w_0 are actually weighted averages. Reissner [V.1,V.2] assumed bending stresses are linearly distributed over the thickness of the plates as in the standard theory of thin plates.

$$\begin{aligned} \sigma_x &= \frac{M_x}{h^2/6}\frac{z}{h/2} \\ \sigma_y &= \frac{M_y}{h^2/6}\frac{z}{h/2} \\ \tau_{xy} &= \frac{M_{xy}}{h^2/6}\frac{z}{h/2} \end{aligned} \tag{3.2}$$

The equilibrium equations in terms of resultants and stress-strain relations are obtained by using Castigliano's theorem combined with the Lagrangian multiplier method of the calculus of variations accounting the energy of transverse shear stresses.

In the same order of approximation, Mindlin [V.3,V.4] assumed displacements of the form in (3.1) and obtained governing equations by predicting a uniform shear stress through the thickness with a "Shear Correction Factor". Mindlin used shear correction factor "κ^2" which is evaluated by comparison with an exact elasticity solution in wave propagation. However, this "κ^2" (which is difficult to evaluate) depends on the frequencies and the material characteristics.

The terms in the displacement function (3.1) are the first terms in a power series in z. So both Mindlin and Reissner theories can be considered as the "First Order Shear Deformation Plate Theories (FSDPT)".

In the "Higher Order Shear Deformation Plate Theories (HSDPT)", in-plane displacement functions have a non-linear dependence on z. One of these theories is based upon the following displacement forms:

$$\begin{aligned} u(x,y) &= u_0(x,y) + z\psi_x(x,y) \\ v(x,y) &= v_0(x,y) + z\psi_y(x,y) \\ w(x,y) &= w_0(x,y) + z\psi_z(x,y) + z^2\xi_z(x,y) \end{aligned} \tag{3.3}$$

This form of displacements has been used to derive a general "Higher Order Shear Deformation Plate Theory (HSDPT)" and also to derive the corresponding "Higher Order Shear Deformation Shell Theory (HSDST)".and "Theory of Laminated Cylindrical Shells". (see the review by Kapania and Raciti [II.3,II.4])

Another "Higher Order Shear Deformation Plate Theory (HSDPT)" is based upon the following displacement forms:

$$\left| \begin{aligned} u(x,y) &= u_0(x,y) + z\psi_x(x,y) + z^2\xi_x(x,y) \\ v(x,y) &= v_0(x,y) + z\psi_y(x,y) + z^2\xi_y(x,y) \\ w(x,y) &= w_0(x,y) + z\psi_z(x,y) + z^2\xi_z(x,y) \end{aligned} \right. \tag{3.4}$$

This form of displacement field has been used for multi-layer plates, and it was seen that the second order terms in z do not provide significant advantages over first order theories such as Reissner and Mindlin theories [V.3,V.4].

Reissner [V.5] has another theory for bending as,

$$
\left| \begin{array}{l}
u(x,y) = z\psi_x(x,y) + z^3\phi_x(x,y) \\
v(x,y) = z\psi_y(x,y) + z^3\phi_y(x,y) \\
w(x,y) = w_0(x,y) + z^2\xi_z(x,y)
\end{array} \right. \tag{3.5}
$$

Reissner [V.1, V.6] used the above displacement field for pure bending of an infinite plate with a circular hole and obtained very accurate results than the ones obtained by using the other theories. The disadvantage of this theory is that it considers only out-of-plane effects, but it does not account for the effect of in-plane modes of deformation.

Another researcher, Reddy [III.2] proposed a theory which includes in-plane modes of deformation and out-of-plane modes of deformation by using the displacement field of the following form:

$$
\left| \begin{array}{l}
u(x,y) = u_0(x,y) + z\psi_x(x,y) + z^2\xi_x(x,y) + z^3\zeta_x(x,y) \\
v(x,y) = v_0(x,y) + z\psi_y(x,y) + z^2\xi_y(x,y) + z^3\zeta_y(x,y) \\
w(x,y) = w_0(x,y) + z\psi_z(x,y) + z^2\xi_z(x,y)
\end{array} \right. \tag{3.6}
$$

In this theory, “Shear Correction Factors” are not needed.

In general, the governing equations and the natural boundary conditions of a laminated plate system based upon (3.6) can be derived from the “Hamilton’s Principle” in terms of “functionals” such that,

$$
\int_0^T (\delta U + \delta V - \delta K)dt = 0 \tag{3.7}
$$

where δ stands for the variations of the strain energy U, the work done by applied forces V and the kinetic energy K of the entire system. In this way, the governing partial differential equations and the natural (or consistent) boundary conditions are obtained.

Reddy’s theory [III.2] has been used for some class of laminated plate problems. This theory can also be applied to other classes of plate problems such as, cutouts, loaded holes, surface cracks, and also problems involving impact caused localized stress gradients that are essentially three-dimensional in nature.

In the stress analysis, “Higher Order Shear Deformation Plate Theories (HSDPT)” proved to be more accurate than “First Order Shear Deformation Plate Theories (FSDPT). There are significant differences in the stresses obtained from (HSDPT) and (FSDPT) in vibration and buckling problems. However, the natural frequencies obtained from both theories were very close as was shown by Reddy [V.6] and Khdeir and Librescu [VI.8] (with differences about 0.01). On the other hand, the differences in natural frequencies obtained by using the (CLPT)’s and the (HSDPT)’s are approximately 0.1. Orthotropic plates under uniformly distributed transverse loads were also analyzed and it was seen that there are no differences in deflection shapes but some differences in stresses. It can be concluded that for moderately thick plates and multi-layer plates, it is more practical to use a “First Order Shear Deformation Plate Theory (FSDPT)” such as “Mindlin’s Plate Theory” [V.3] instead of a complicated and cumbersome “Higher Order Shear Deformation Plate Theory (HSDPT)”, to obtain natural frequencies and corresponding mode shapes.

The higher order moment and transverse shear force resultants, which are due to the particular form of the proposed displacement field, is another disadvantage of higher order plate theories. They can not easily be seen to have physical meaning and they may create a complicated, unusual and formidable image for the analysis of composite plate systems.

Some of the significant differences between “Classical Plate Theory (CLPT)”, “First Order Shear Deformation Plate Theories (FSDPT)” and “Higher Order Shear Deformation Plate Theories (HSDPT)” are very briefly as follows:

- In the “Classical Plate Theory (CLPT)”, normals to the mid-plane before deformation remain straight and normal to the mid-plane after deformation. Also, the rotatory moments of inertia, are neglected.

- In “First Order Shear Deformation Plate Theories (FSDPT)”, plane sections originally perpendicular to the mid-plane of the plate remain plane after deformation, but not necessarily perpendicular to the mid-plane. The “Shear Correction Factors κ_x, κ_y” are used to account for the transverse shear stresses and also the rotatory moments of inertia. Also, the extensional, transverse and the rotary moments of inertia are included. The transverse shear deformations in the plate are accounted for in the theory.

- In “Higher Order Shear Deformation Plate Theories (HSDPT)”, the distorsion (of third order) of normals to the mid-plane of the undeformed plate is allowed. This eliminates the “Shear Correction Factors” from the equations. However, complicated inertia and stress resultant terms which does not have the customary or the usual physical meanings, appear in the governing equations.

3.3 “First Order Shear Deformation Theories of Plates (FSDPT)” and Mindlin’s Plate Theory

There is no difference in the natural frequencies obtained by the (FSDPT)’s and the (HSDPT)’s .as mentioned in Section 3.1. Therefore, in the present study, Mindlin Plate Theory” which is a “First Order Shear Deformation Plate Theory (FSDPT)” is used for the theoretical formulations of the “Main PROBLEMS.

The general coordinate system and the positive sign convention for the displacements, stress resultants and external surface loads or stresses for the “Main PROBLEMS” are given in Figure 3.1. The basic assumptions of the Mindlin Plate Theory [V.3] are that the in-plane displacement u and v are proportional to z and transverse displacement w is independent of z. Then,

$$\left| \begin{aligned} u &= z\psi_x(x,y,t), \\ v &= z\psi_y(x,y,t), \\ w &= w(x,y,t) \end{aligned} \right. \tag{3.8}$$

where angle of rotations ψ_x and ψ_y are negative of the rotations in xz and yz-planes, respectively. (see also Figures 4.1 and 4.2 in Chapter 4)

Hooke's law or strain-stress relations for orthotropic materials,

$$\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z \\
\varepsilon_y &= a_{21}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z \\
\varepsilon_z &= a_{31}\sigma_x + a_{32}\sigma_y + a_{33}\sigma_z \\
\gamma_{yz} &= a_{44}\tau_{yz} \\
\gamma_{xz} &= a_{55}\tau_{xz} \\
\gamma_{xy} &= a_{66}\tau_{xy}
\end{aligned} \tag{3.9}$$

The equation containing the strain "ε_z" which is normal to the faces of the plate (planes at z=±h/2) is ignored. The remaining equations are then solved for σ_x, σ_y, τ_{xy}, τ_{xz} and τ_{yz} in terms of ε_x, ε_y, γ_{xy}, γ_{xz}, γ_{yz} and σ_z .

$$\begin{aligned}
\sigma_x &= B_{11}\varepsilon_x + B_{12}\varepsilon_y - C_1\sigma_z \\
\sigma_y &= B_{21}\varepsilon_x + B_{22}\varepsilon_y - C_2\sigma_z \\
\tau_{yz} &= B_{44}\gamma_{yz} \\
\tau_{xz} &= B_{55}\gamma_{xz} \\
\tau_{xy} &= B_{66}\gamma_{xy}
\end{aligned} \tag{3.10}$$

In the above equations, the coefficients B_{ij} can be expressed for orthotropic materials in terms of the engineering material constants as,

$$\begin{aligned}
B_{11} &= \frac{E_1}{1-\nu_{12}\nu_{21}}, \qquad & B_{44} &= G_{23} \\
B_{22} &= \frac{E_2}{1-\nu_{12}\nu_{21}}, \qquad & B_{55} &= G_{13} \\
B_{12} &= B_{21} = \nu_{21}B_{11} = \nu_{12}B_{22}, \qquad & B_{66} &= G_{12}
\end{aligned} \tag{3.11}$$

where E and G are moduli of elasticity and rigidity respectively, ν is Poisson's ratio and subscripts denote the directions parallel to the coordinate axes of the plate.

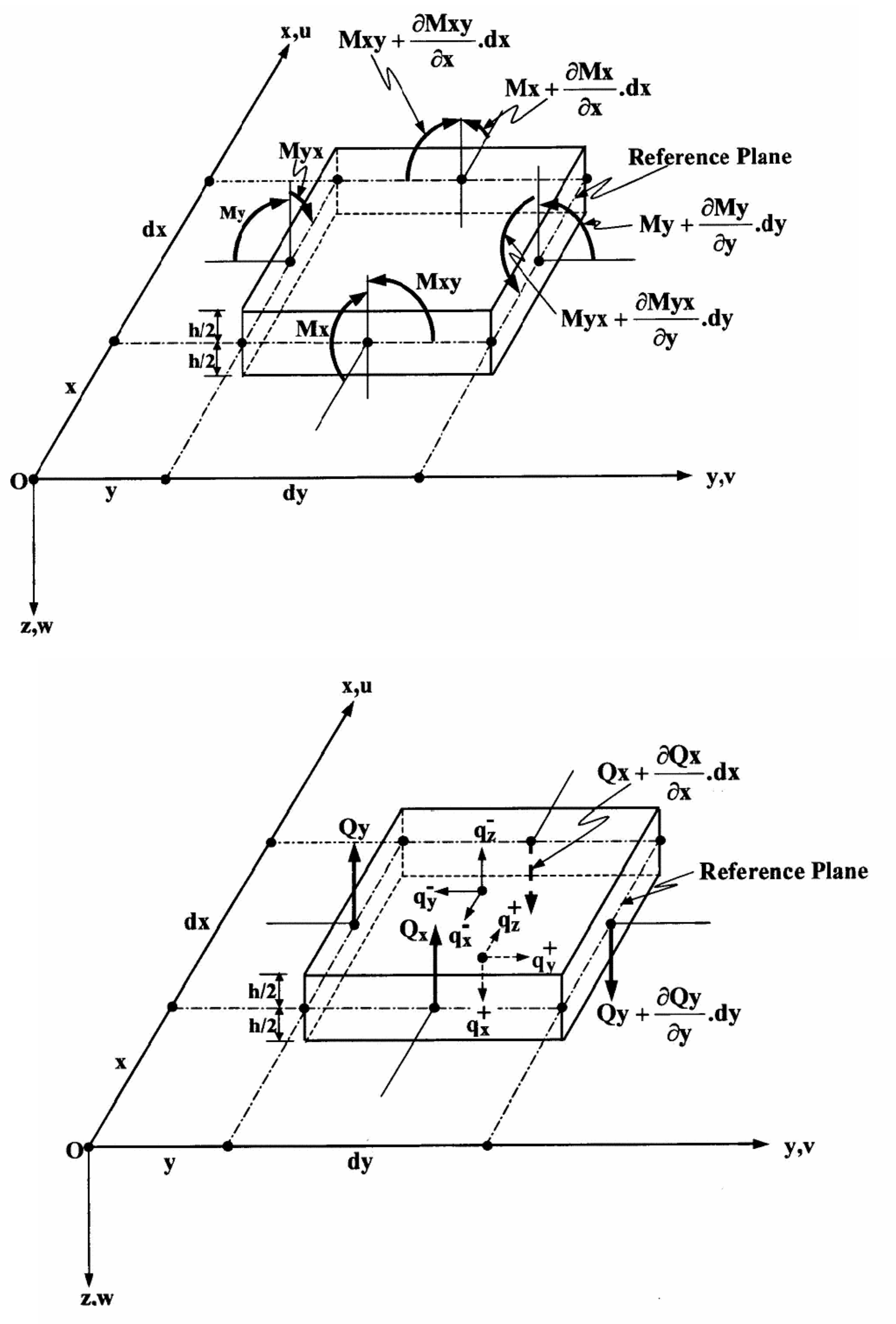


Figure 3.1 Coordinate System and Sign Convention for Displacements and Stress Resultants in Mindlin Plates in Bending (or Flexure)

The bending and twisting moments per unit length and the transverse shear resultants per unit length, are written in the integral form as,

$$\begin{aligned} \left(M_x, M_y, M_{yx}\right) &= \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \tau_{xy}\right) z\,dz \\ \left(Q_x, Q_y\right) &= \int_{-h/2}^{h/2} \left(\tau_{xz}, \tau_{yz}\right) dz \end{aligned} \tag{3.12}$$

In order to obtain the plate stress resultants – strain relations, the integrations are performed across the thickness of the plate. The results are then altered in two respects:

(1) The weighted average of σ_z is ignored, since this stress is very small compared with the bending and twisting stresses. Therefore, the integrals containing σ_z are dropped. This assumption is also made in other "First Order Shear Deformation Plate Theories (FSDPT)" and the "Classical Plate Theory (CLPT)".

(2) The coefficients of the integrals containing γ_{xz} and γ_{yz} are replaced by constants whose magnitudes are to be determined later.

After integrating, the following relations between stress resultants and strains are obtained,

Stress Resultants – Strain Relations

$$\begin{aligned} M_x &= \left[B_{11}\Gamma_x + B_{12}\Gamma_y\right] \\ M_y &= \left[B_{12}\Gamma_x + B_{22}\Gamma_y\right] \\ M_{yx} &= B_{66}\Gamma_{xy} \\ Q_x &= \kappa_x^2 B_{55}\Gamma_{xz} \\ Q_y &= \kappa_y^2 B_{44}\Gamma_{yz} \end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
\left(\Gamma_x , \Gamma_y , \Gamma_{xy}\right) &= \int_{-h/2}^{h/2} \left(\varepsilon_x , \varepsilon_y , \gamma_{xy}\right) z dz \\
\left(\Gamma_{xz} , \Gamma_{yz}\right) &= \int_{-h/2}^{h/2} \left(\gamma_{xz} , \gamma_{yz}\right) dz
\end{aligned}
\tag{3.14}
$$

In the present study, κ^2, is taken as used in Mindlin's Plate Theory. In Mindlin's Plate Theory the "Shear Correction Factor" is determined by equating the exact results obtained from three dimensional elasticity equation and the result based on the "Mindlin's Plate Theory" for the case of straight-crested flexural waves in an infinite plate. These values are found to be $\pi^2/12$ (or, in Reissner's Plate Theory, is 5/6). [V.3] in isotropic plates. In orthotropic plates, the "Shear Correction Factors" are defined as κ_x^2,κ_y^2, respectively and these are obtained from a study by Wittrick [V.11].

By considering the strain-displacement equations in three dimensional elasticity and inserting the assumed form of the displacement functions, strain displacement relations can be obtained as follows,

Strain Displacement Equations in 3D Elasticity ,

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\
\varepsilon_z &= \frac{\partial w}{\partial z} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{aligned}
\tag{3.15}
$$

Inserting (3.8) in to (3.15),

Strain Displacements Relations,

$$\varepsilon_x = z\frac{\partial \psi_x}{\partial x} \qquad \gamma_{xy} = z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)$$
$$\varepsilon_y = z\frac{\partial \psi_y}{\partial y} \qquad \gamma_{xz} = \left(\psi_x + \frac{\partial w}{\partial x}\right) \tag{3.16}$$
$$\varepsilon_z = 0 \qquad \gamma_{yz} = \left(\psi_y + \frac{\partial w}{\partial y}\right)$$

Therefore, the following relations between stress resultants and plate displacements functions are obtained:

Stress Resultants – Displacement Relation

$$M_x = \frac{h^3}{12}\left(B_{11}\frac{\partial \psi_x}{\partial x} + B_{12}\frac{\partial \psi_y}{\partial y}\right)$$
$$M_y = \frac{h^3}{12}\left(B_{12}\frac{\partial \psi_x}{\partial x} + B_{22}\frac{\partial \psi_y}{\partial y}\right)$$
$$M_{yx} = \frac{h^3}{12}B_{66}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \tag{3.17.a}$$
$$Q_x = \kappa_x^2 h B_{55}\left(\psi_x + \frac{\partial w}{\partial x}\right)$$
$$Q_y = \kappa_y^2 h B_{44}\left(\psi_y + \frac{\partial w}{\partial y}\right)$$

Also, in terms of the ‘‘Orthotropic Plate Stiffnesses’’,

$$
\begin{aligned}
M_x &= D_{11}\frac{\partial \psi_x}{\partial x} + D_{12}\frac{\partial \psi_y}{\partial y} \\
M_y &= D_{12}\frac{\partial \psi_x}{\partial x} + D_{22}\frac{\partial \psi_y}{\partial y} \\
M_{yx} &= D_{66}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \\
Q_x &= A_{55}\left(\psi_x + \frac{\partial w}{\partial x}\right) \\
Q_y &= A_{44}\left(\psi_y + \frac{\partial w}{\partial y}\right)
\end{aligned}
\qquad (3.17.b)
$$

where the "Bending Stiffness D's" and the "Shear Stiffness A's" of the Orthotropic Mindlin Plate are given as,

$$
\begin{aligned}
D_{ik} &= \frac{h^3 B_{ik}}{12} \quad (i,k = 1,2) \\
D_{66} &= \frac{h^3 B_{66}}{12} \\
A_{44} &= \kappa_y^2 h B_{44} \\
A_{55} &= \kappa_x^2 h B_{44}
\end{aligned}
\qquad (3.17.c)
$$

The relation between stress resultants and the plate displacements are obtained. .Now, the equations of motion of the three dimensional elasticity theory are needed. These equations are,

Equations of Motion (3D Elasticity)

$$
\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \rho\frac{\partial^2 u}{\partial t^2} \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \rho\frac{\partial^2 v}{\partial t^2} \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= \rho\frac{\partial^2 w}{\partial t^2}
\end{aligned}
\qquad (3.18)
$$

The first two governing equations of the Mindlin Plate Theory for orthotropic plates are obtained by multiplying the first two equations by "z" and integrating across the thickness (between –h/2 and +h/2) as,

$$\left|\begin{array}{l}\dfrac{\partial M_x}{\partial x}+\dfrac{\partial M_{yx}}{\partial y}-Q_x+\dfrac{h}{2}\left(q_{zx}^{(+)}+q_{zx}^{(-)}\right)=\dfrac{\rho h^3}{12}\dfrac{\partial^2\psi_x}{\partial t^2}\\ \dfrac{\partial M_{yx}}{\partial x}+\dfrac{\partial M_y}{\partial y}-Q_y+\dfrac{h}{2}\left(q_{zy}^{(+)}+q_{zy}^{(-)}\right)=\dfrac{\rho h^3}{12}\dfrac{\partial^2\psi_y}{\partial t^2}\end{array}\right. \tag{3.19}$$

where $q_{zx}^{(+)}$, $q_{zy}^{(+)}$ and $q_{zx}^{(-)}$, $q_{zy}^{(-)}$ are the surface stresses at z=+h/2 and z=-h/2, respectively.) The coefficient terms in the right hand sides of (3.19) are for the rotatory moments of inertia.

Additionally, the third equation of (3.18) is integrated over the plate thickness and, using (3.12) yields;

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\left(q_z^{(+)}-q_z^{(-)}\right)=\rho h\frac{\partial^2 w}{\partial t^2} \tag{3.20}$$

where $q_z^{(+)}$ and $q_z^{(-)}$ are the surface stresses at z=+h/2 and z=-h/2, respectively. Equations (3.19) and (3.20) are the "Dynamic Equations of the Mindlin Plate Theory".

In the present study M_y, M_{yx}, Q_y, ψ_x, ψ_y and w are chosen as intrinsic variables whereas M_x, Q_x are chosen as auxiliary variables. For reduction of the differential equations to the "First Order Systems of the ordinary Differential Equations" (see the preceding chapters), the partial derivatives of the intrinsic variables with respect to "y" should be written in terms of other intrinsic variables. These equations can be obtained from (3.17), (3.19) and (3.20) as,

$$
\left|
\begin{aligned}
&\frac{\partial \psi_y}{\partial y} = \frac{1}{B_{22}}\left(\frac{12}{h^3}M_y - B_{12}\frac{\partial \psi_x}{\partial x}\right) \\
&\frac{\partial \psi_x}{\partial y} = \frac{12}{h^3 B_{66}}M_{yx} - \frac{\partial \psi_y}{\partial x} \\
&\frac{\partial w}{\partial y} = \frac{1}{\kappa_y^2 h B_{44}}Q_y - \psi_y \\
&\frac{\partial M_{yx}}{\partial y} = \frac{\rho h^3}{12}\frac{\partial^2 \psi_x}{\partial t^2} - \frac{\partial M_x}{\partial x} + Q_x - \frac{h}{2}\left(q_{zx}^{(+)} + q_{zx}^{(-)}\right) \\
&\frac{\partial M_y}{\partial y} = \frac{\rho h^3}{12}\frac{\partial^2 \psi_y}{\partial t^2} - \frac{\partial M_{yx}}{\partial x} + Q_y - \frac{h}{2}\left(q_{zy}^{(+)} + q_{zy}^{(-)}\right) \\
&\frac{\partial Q_y}{\partial y} = \rho h\frac{\partial^2 w}{\partial t^2} - \frac{\partial Q_x}{\partial x} - \left(q_z^{(+)} - q_z^{(-)}\right)
\end{aligned}
\right. \qquad (3.21)
$$

(3.21) will be used in the theoretical formulation of the “Main PROBLEMS” in Chapter 4, 5 and 6.

In order to solve the dynamic equations of the “Mindlin Plates” boundary conditions (support conditions) along the edges of the plate shall be prescribed. “Mindlin Plate Theory” requires prescription of three Boundary conditions along each edge,

<u>Boundary Conditions (Support Condition)</u>

(F) (free)	$M_{nt}=M_n=Q_n=0$	
(S) (simply supported)	$w=\psi_t=M_n=0$	(3.22)
(C) (clamped)	$w=\psi_n=\psi_t=0$	

where n and t are normal and tangential coordinates of the edges.

3.4 Definition of “Main PROBLEMS” and Some “Special CASES”, and Limitations

In general, three “Main PROBLEMS” are considered in the present “Thesis”. The appropriate definitions of the “Main PROBLEMS” are given as in the following:

- **“Main PROBLEM I”**

  “Free Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Bonded Single Lap Joint”.

  I.a) “Free Flexural (Or Bending) Vibrations Of Orthotropic Composite Mindlin Plates With a Centrally Bonded Single Lap Joint” (Main PROBLEM I.a)

  I.b) “Free Flexural (Or Bending) Vibrations Of Orthotropic Composite Mindlin Plates With a Non-Centrally Bonded (Eccentrically) Single Lap Joint” (Main PROBLEM I.b)

- **“Main PROBLEM II”**

  “Free Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Bonded Symmetric Single Lap Joint (Symmetric Doubler Joint)”.

  II.a) “Free Flexural (or Bending) Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Centrally Bonded Symmetric Single Lap Joint (Symmetric Doubler Joint)” (Main PROBLEM II a)

  II.b) “Free Flexural (or Bending) Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (Symmetric Doubler Joint)” (Main PROBLEM II b)

- **“Main PROBLEM III”**

  “Free Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)”.

  III.a) “Free Flexural (or Bending) Vibrations of Orthotropic Composite Mindlin Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)” (Main PROBLEM III a)

  III.b) “Free Flexural (or Bending) Vibrations of Orthotropic Composite Mindlin Plates or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)” (Main PROBLEM III b)

The general configurations, the geometries and the coordinate systems of the above defined “Main PROBLEMS” are given in Figures 3.2.a and 3.2.b, 3.3.a and 3.3.b, and 3.4.a and 3.4.b, respectively.

Also some “Special CASES” to be considered in this study are defined next;

- “Special CASE of Main PROBLEM II”
- “Special CASE of Main PROBLEM III”

In the analytical formulation of the above “Main PROBLEMS”, the “Mindlin Plate Theory” is used taking into account the influences of the thickness shear deformations and the rotatory and the transverse moments of inertia of plates in the dynamic equations. The basic equations of the theory are systematically derived from the fundamental equations of three-dimensional elasticity in the preceding section. The transverse normal and shear strains and the corresponding stresses in the relatively very thin adhesive layers are included in the formulation.

In all three “Main PROBLEMS”, the entire system are assumed to have simple support boundary conditions at the two opposite edges (in the x-direction) while the other two opposite edges (in the y-direction) may have arbitrary support conditions in the sense of the “Mindlin Plate Theory”. This allows to use the “Classical Lévy’s Solution” in the x-direction.

The “Method of Solution” employed in this work is the “Modified Transfer Matrix Method" which is very effective and accurate in handling certain class of plate and shell free vibrations problems. This is a semi-analytical and numerical technique which combines, the ‘‘Classical Levy’s Method”, the “Integrating Matrix Method with Interpolation Polynomials and/or Chebyshev Polynomials)” and the “Transfer Matrix Method” for continuous systems. This solution method is used successfully in Yuceoglu and Özerciyes [IV.4-IV.13]. This semi-analytical and numerical solution tecnique is a considerable extension, modification and further development of the method employed earlier in Yuceoglu et al [VII.2].

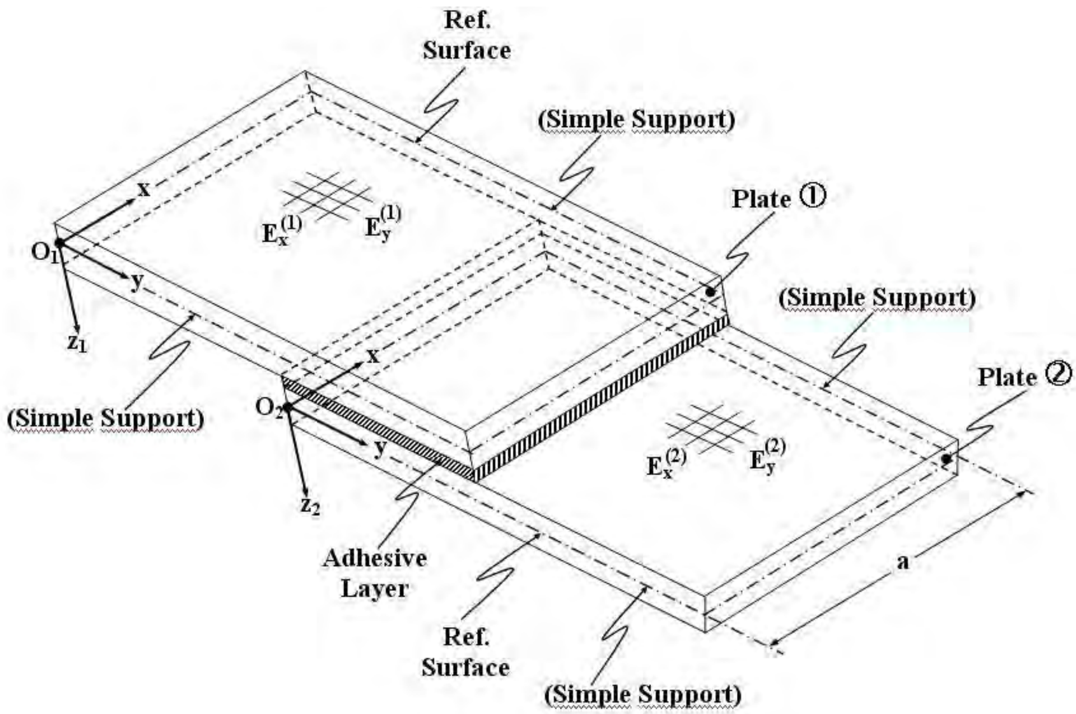


Figure 3.2.a General Configuration and Coordinate System of “Composite , Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint”

(‘‘Main PROBLEM Ia’’)

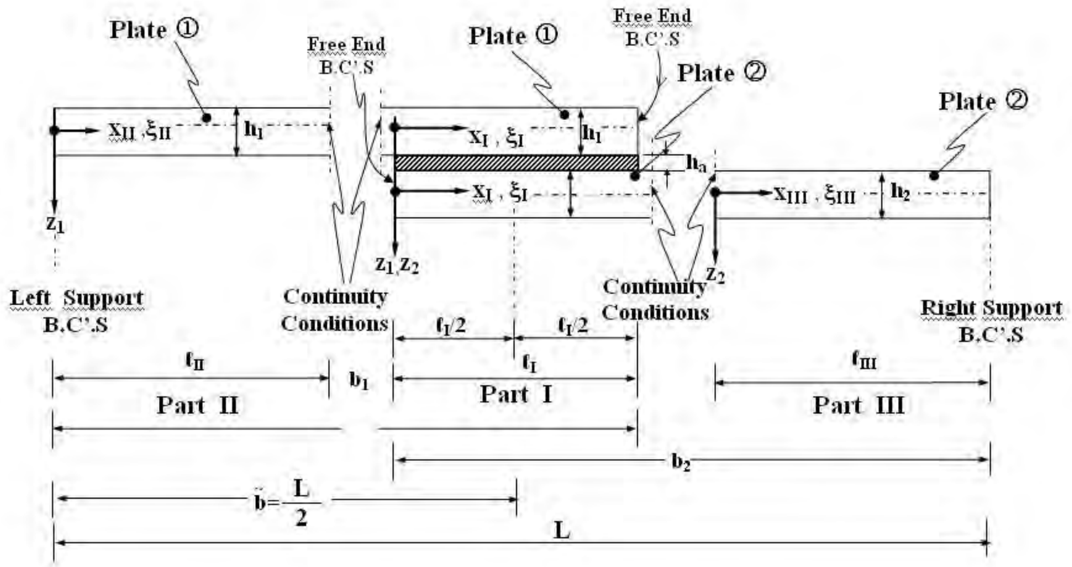


Figure 3.2.b Longitudinal Cross-Section of “Composite , Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint”

(‘‘Main PROBLEM Ia’’)

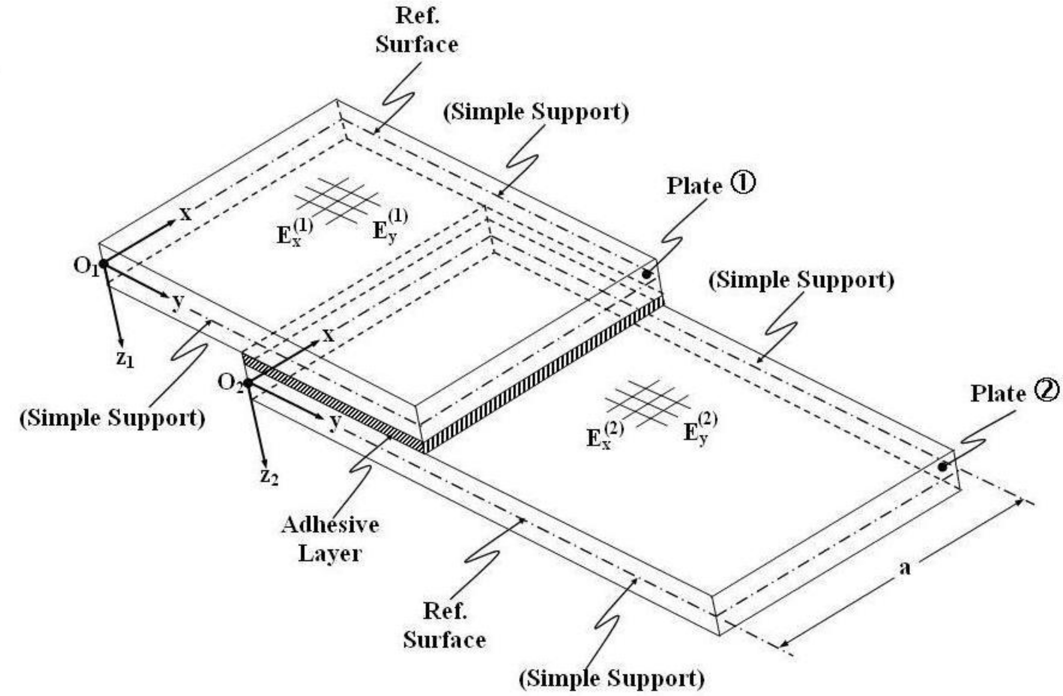


Figure 3.3.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally (or Eccentrically) Bonded Single Lap Joint"

(''<u>Main PROBLEM I b</u>'')

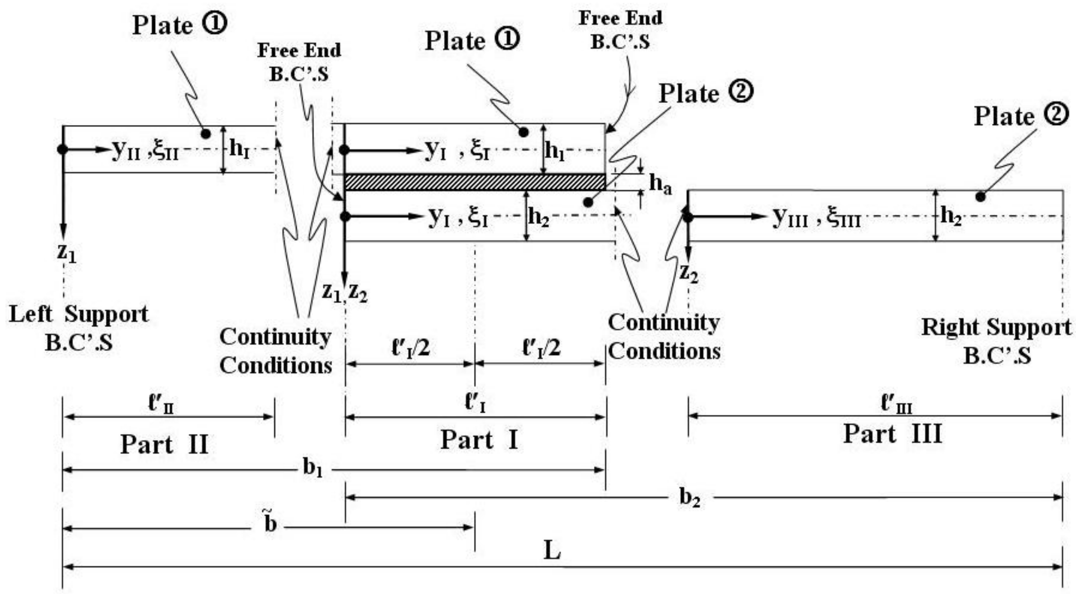


Figure 3.3.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally (or Eccentrically) Bonded Single Lap Joint"

(''<u>Main PROBLEM Ib</u>'')

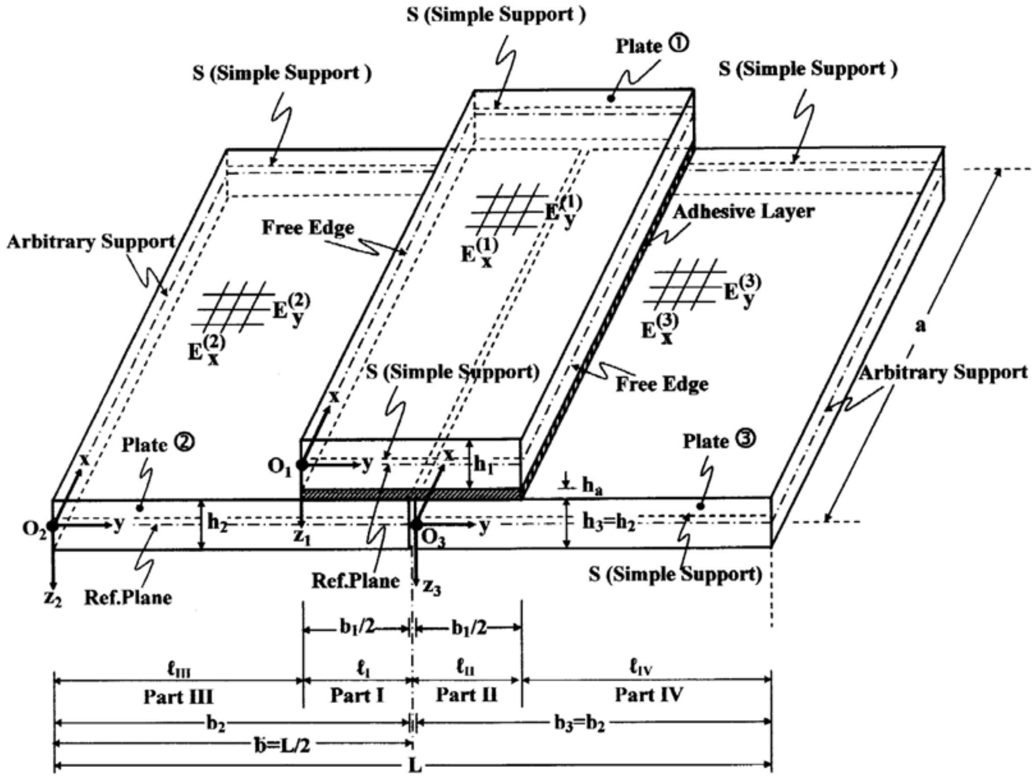


Figure 3.4.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(''Main PROBLEM II a'')

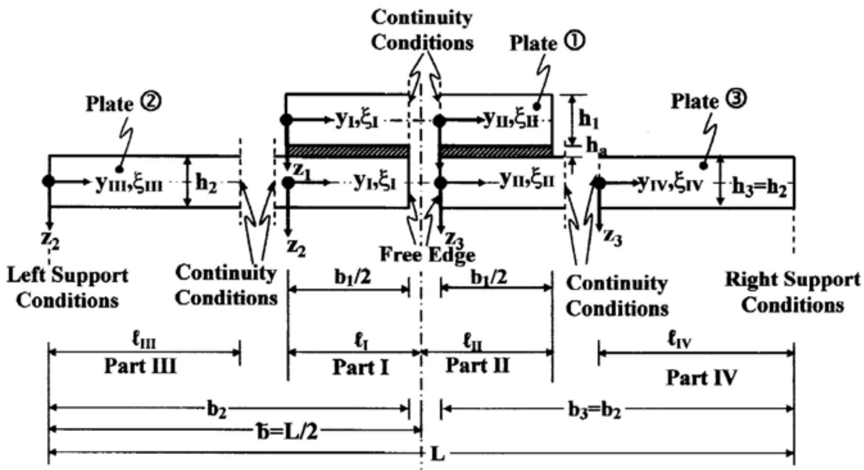


Figure 3.4.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(''Main PROBLEM II a'')

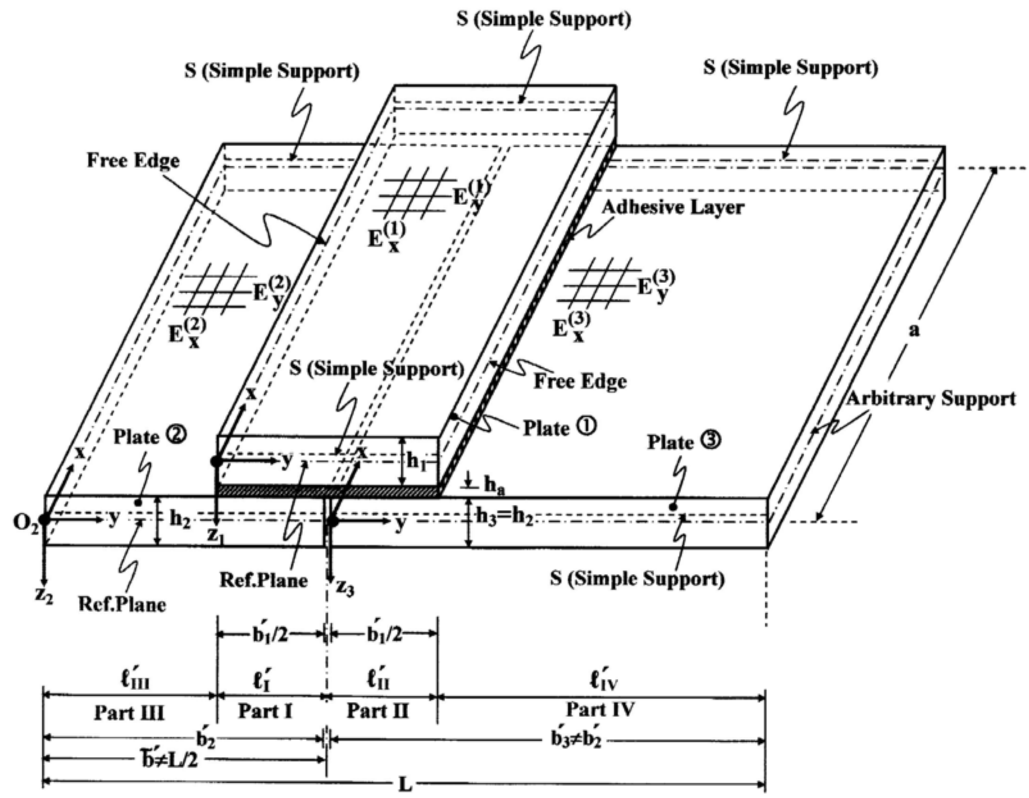


Figure 3.5.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(''Main PROBLEM II b'')

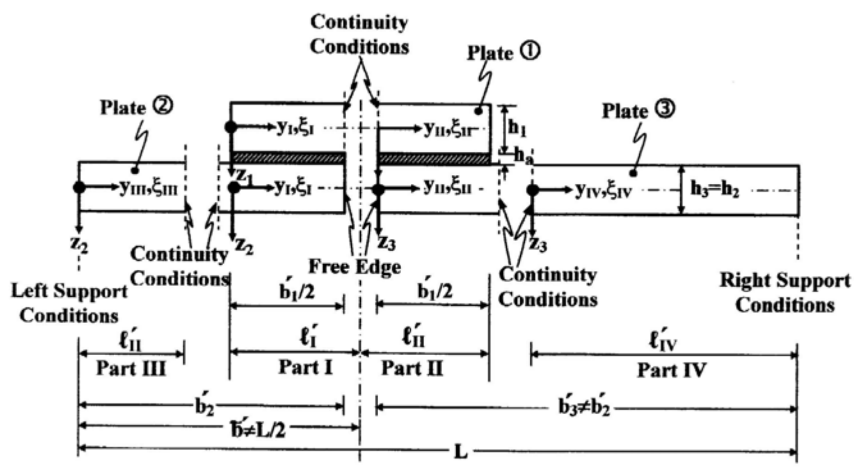


Figure 3.5.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(''Main PROBLEM II b'')

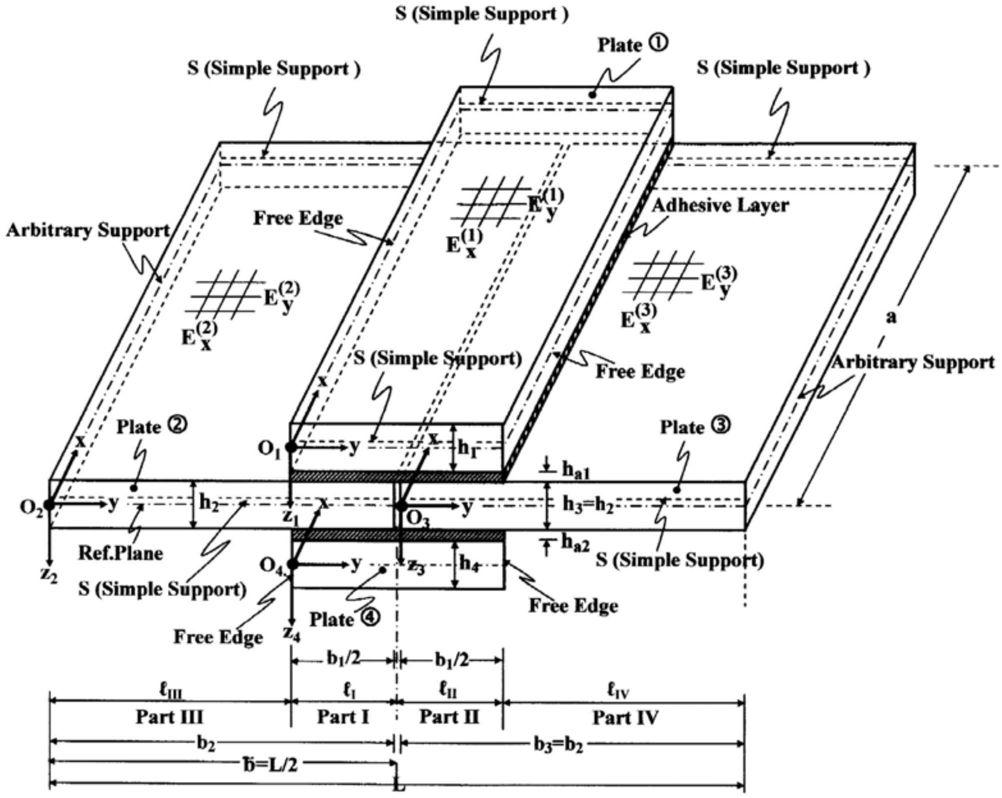


Figure 3.6.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(''Main PROBLEM III a'')

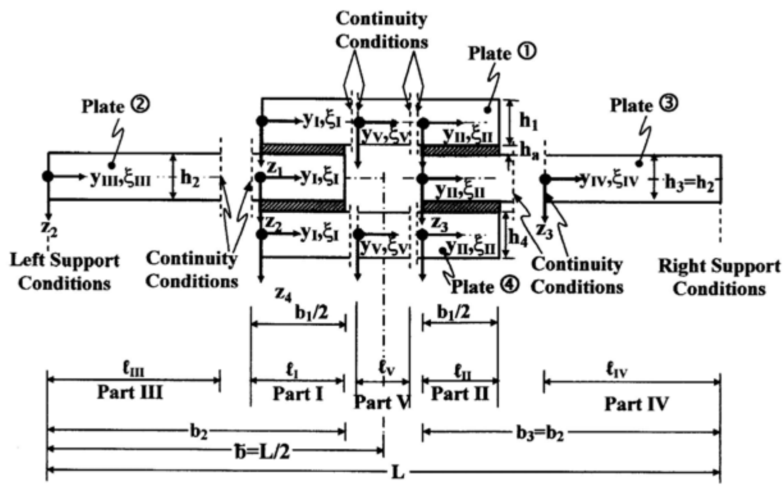


Figure 3.6.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(''Main PROBLEM III a'')

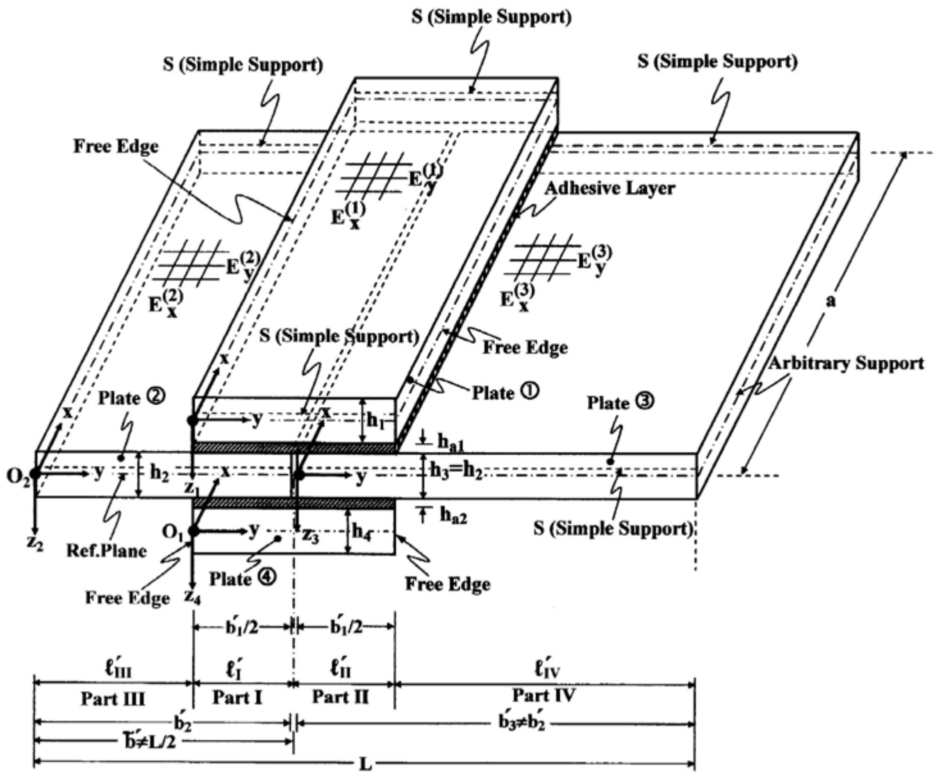


Figure 3.7.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(''<u>Main PROBLEM III b</u>'')

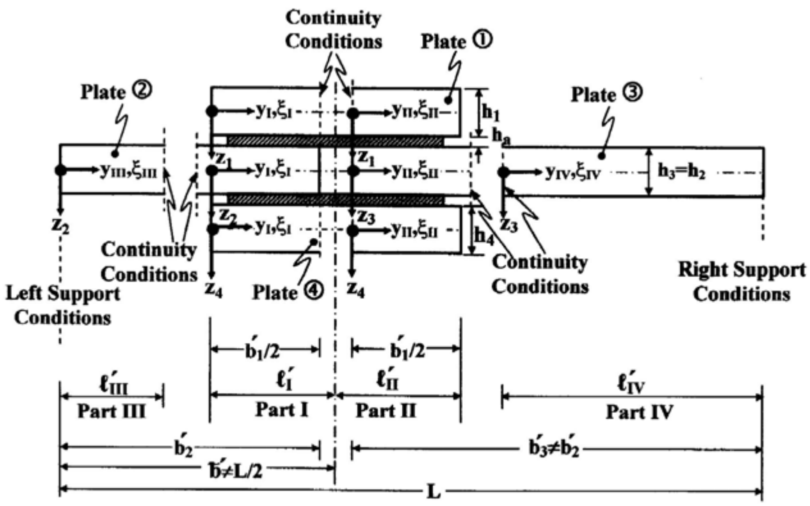


Figure 3.7.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(''<u>Main PROBLEM III b</u>'')

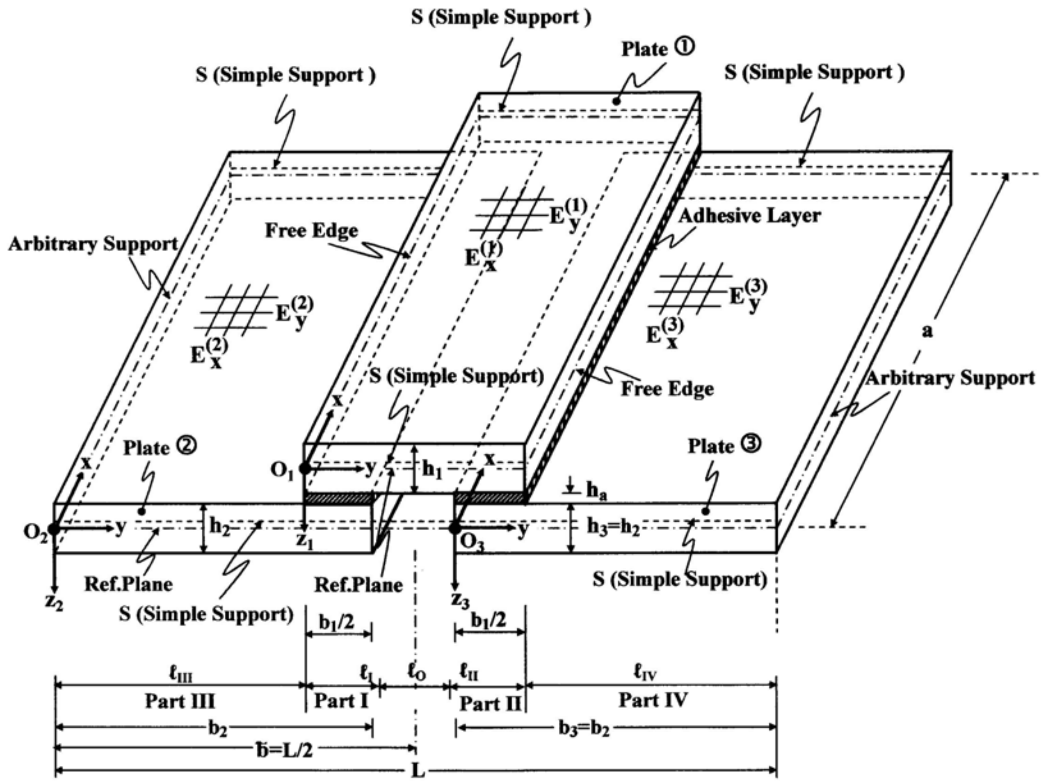


Figure 3.8.a General Configuration and Coordinate System of “Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap”

(“Special Case of Main PROBLEM II a’’)

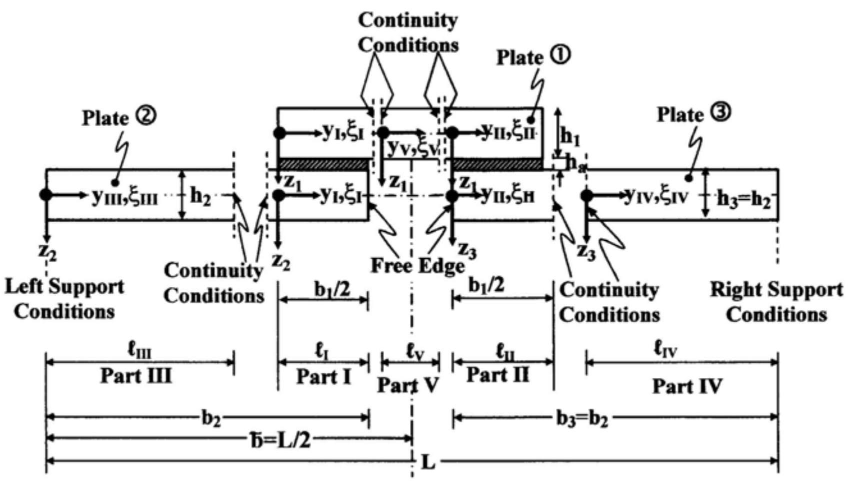


Figure 3.8.b Longitudinal Cross-Section of “Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap”

(“Special Case of ‘Main PROBLEM II a’’)

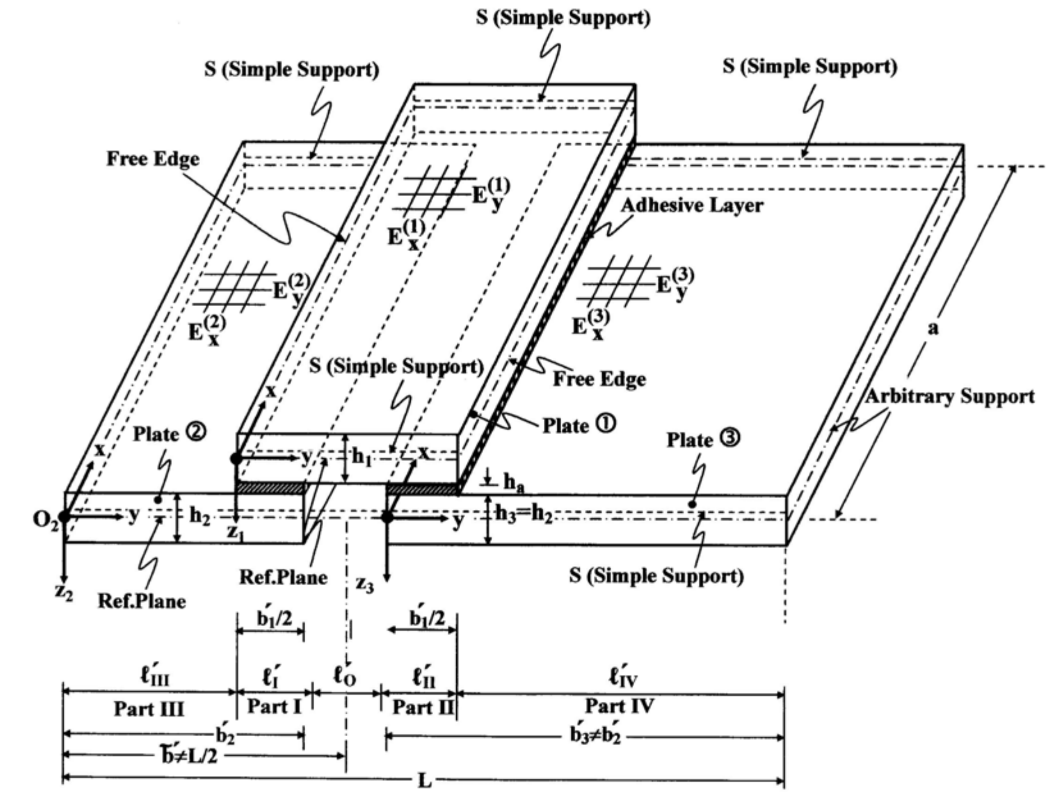


Figure 3.9.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

(''Special Case of Main PROBLEM II b''

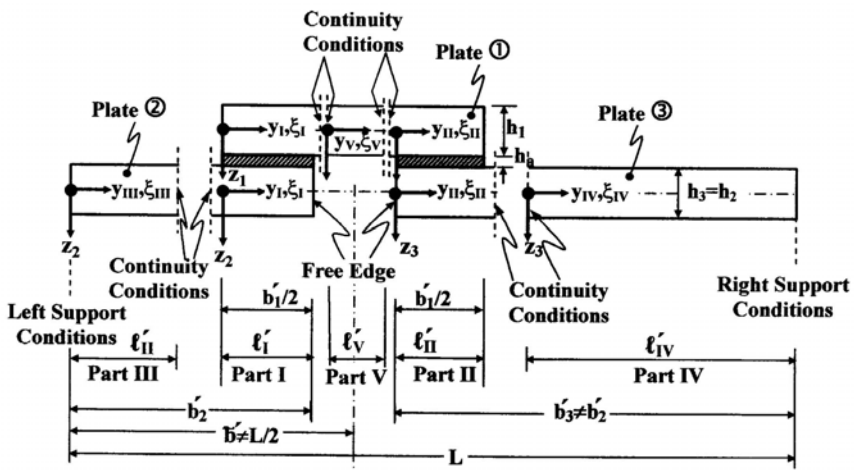


Figure 3.9.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

(''Special Case of Main PROBLEM II b'')

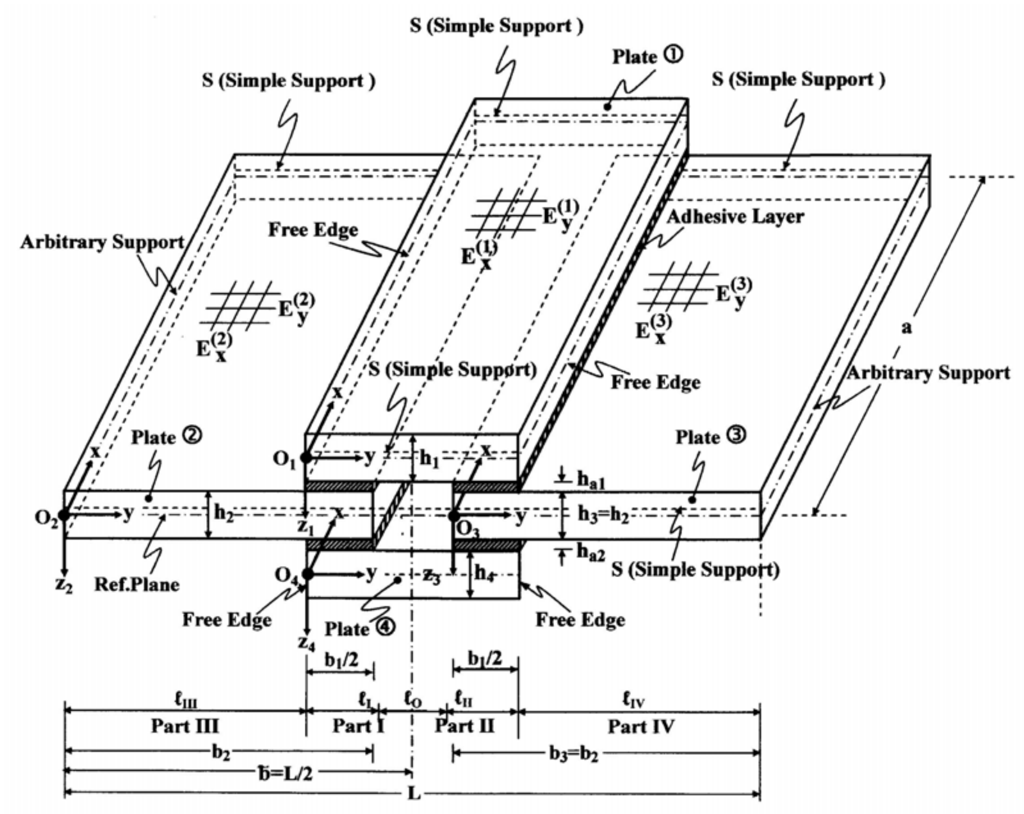


Figure 3.10.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

(''Special Case of Main PROBLEM III a'')

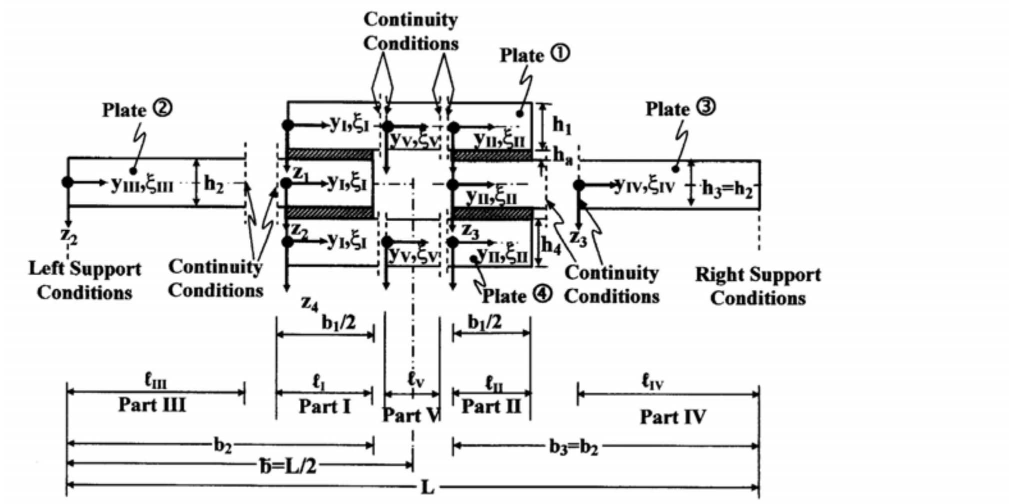


Figure 3.10.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

(''Special Case of Main PROBLEM III a'')

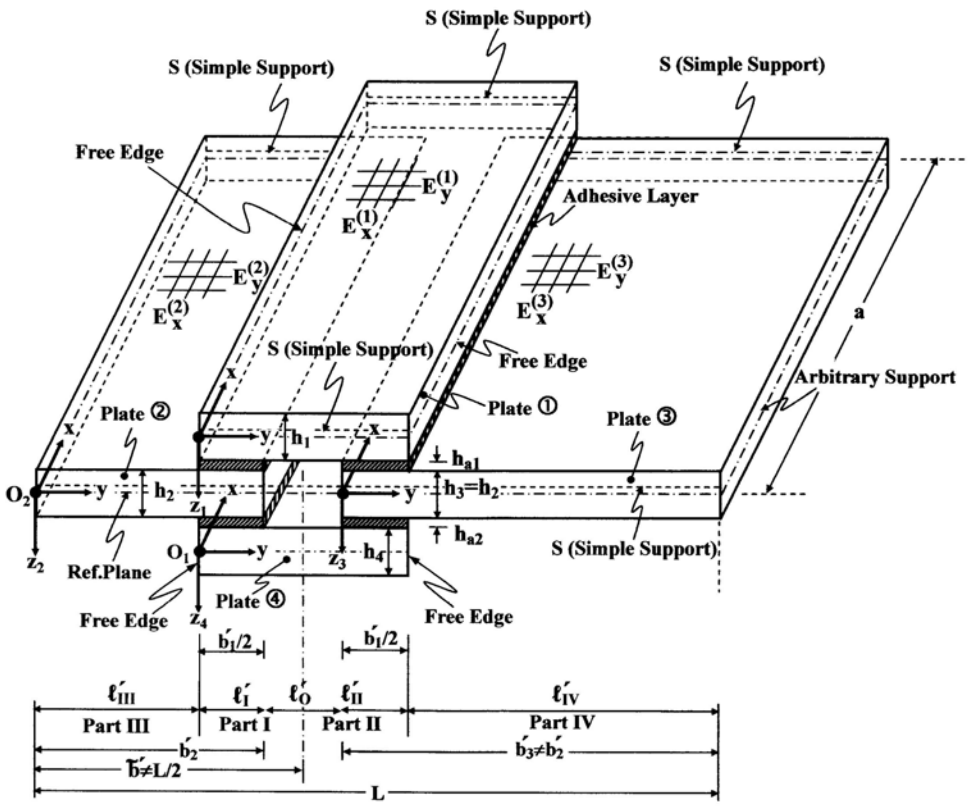


Figure 3.11.a General Configuration and Coordinate System of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

("Special Case of Main PROBLEM III b")

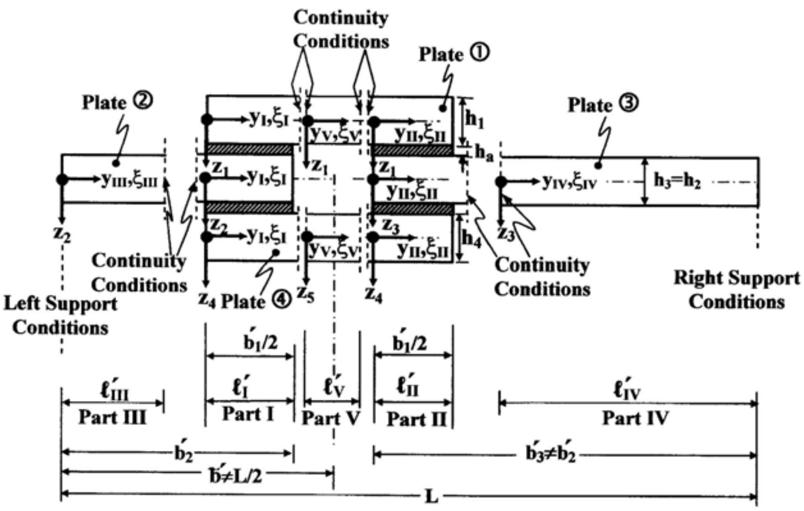


Figure 3.11.b Longitudinal Cross-Section of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

("Special Case of Main PROBLEM III b")

CHAPTER 4

("Main PROBLEM I")__FREE FLEXURAL (or BENDING) VIBRATIONS of COMPOSITE ORTHOTROPIC or ISOTROPIC PLATES with a BONDED SINGLE LAP JOINT

In this section, the "Governing System of Coupled Ordinary Differential Equations" will be presented in the coupled matrix or the "state vector" form for the "Overlap Region" (or Part I region), and for "Single Layer Regions" (or Part II and Part III regions) for the "Main PROBLEM I" without making any distinction for the "Main PROBLEM I a " and the "Main PROBLEM I b" as described below.

4.1 Statement of "Main PROBLEM Ia"

Figure 3.2.a shows the general configuration, geometry and the coordinate system of the "Composite Orthotropic or Isotropic Plate System with a Centrally Bonded Single Lap Joint". This system is composed of two dissimilar "Orthotropic Plates or Adherends" lap-jointed centrally over a certain length. In this problem, adherends are bonded by a relatively very thin elastic adhesive layer.

4.2 Statement of "Main PROBLEM Ib"

Figure 3.2.b shows the general configuration, geometry and the coordinate system of the "Composite Orthotropic or Isotropic Plate System with Non-Centrally (or eccentrically) Bonded Single Lap Joint". This system is composed of two dissimilar orthotropic plates or adherends lap-jointed non-centrally over a certain length. In this problem adherends are bonded by a relatively very thin elastic adhesive layer.

4.3 Main assumptions and Analytical Modeling

The analytical formulation of “Main PROBLEM I” is based on the following assumptions;

(i) The analysis is carried out only for the free flexural (or bending) vibrations of the composite plate system. The in-plane or extensional moments of inertia are neglected, but the rotatory and the transverse moments of inertia of the plates are included in the formulation.

(ii) The rotatory and transverse moments of inertia of the adherends as well as their transverse shear deformations are taken into account in the sense of the “Mindlin Plate Theory”.

(iii) There is no slip and separation on the interfaces of the adhesive layer and plates or adherends.

(iv) Since the thickness of the adhesive layer is very small relative to the thickness of the plate adherends, the inertias and the masses of the adhesive layers are neglected and both adhesive normal and shear stresses are to be constant across its thickness.

(v) The damping effects in Mindlin Plates and in the adhesive layers are neglected.

(vi) The plates are assumed to be simply supported along edges x=0 and x=a while arbitrary support conditions may be specified in the y-direction.

(vii) The coordinate system of each plate is attached to its medium plane or the reference plane.

(viii) The principal directions of orthotropy in plates are parallel to the edges and to the coordinates as shown in Figure 3.2.a and in Figure 3.3.a.

(ix) It is assumed that the following relations exist between the deformations of the plate adherends in z-direction.

$$\left| w^{(2)} > w^{(1)} \right.$$

For the general formulation of the problem, the entire "Composite Bonded Plate or Panel System" is divided into three parts, namely, Part I, Part II, Part and III in the y-direction as shown in Figure 3.2.a and Figure 3.2.b Part I corresponds to the "Overlap Region" which contains two plates 1,2, Part II correspond the continuation of the upper plate 1 and Part III correspond the continuation of the lower plate 2 as a single plates in the y-direction.

4.4 Theoretical Formulation of "Main PROBLEM I" (Theoretical Analysis)

4.4.1 Analysis of Adhesive Layer in the "Overlap Region"

The system in "Overlap Region" (or Part I) is composed of two plates which are adhesively bonded by very thin elastic adhesive layers. The stresses at the upper and lower faces of the plates due to the adhesive layers are considered as external surface stresses or loads on the plates. The adhesive stresses should be related to the unknown displacement functions and angle of rotations of the adherends in the "Overlap Region". Figures 4.1 and Figure 4.2 show an exaggerated view of the deformation of an infinitesimal element of two-layer plate system. The positive sign convention for displacements, stress resultants and angles of rotation are also shown on the same figures.(see also Figures 3.1, 4.1 and 4.2)

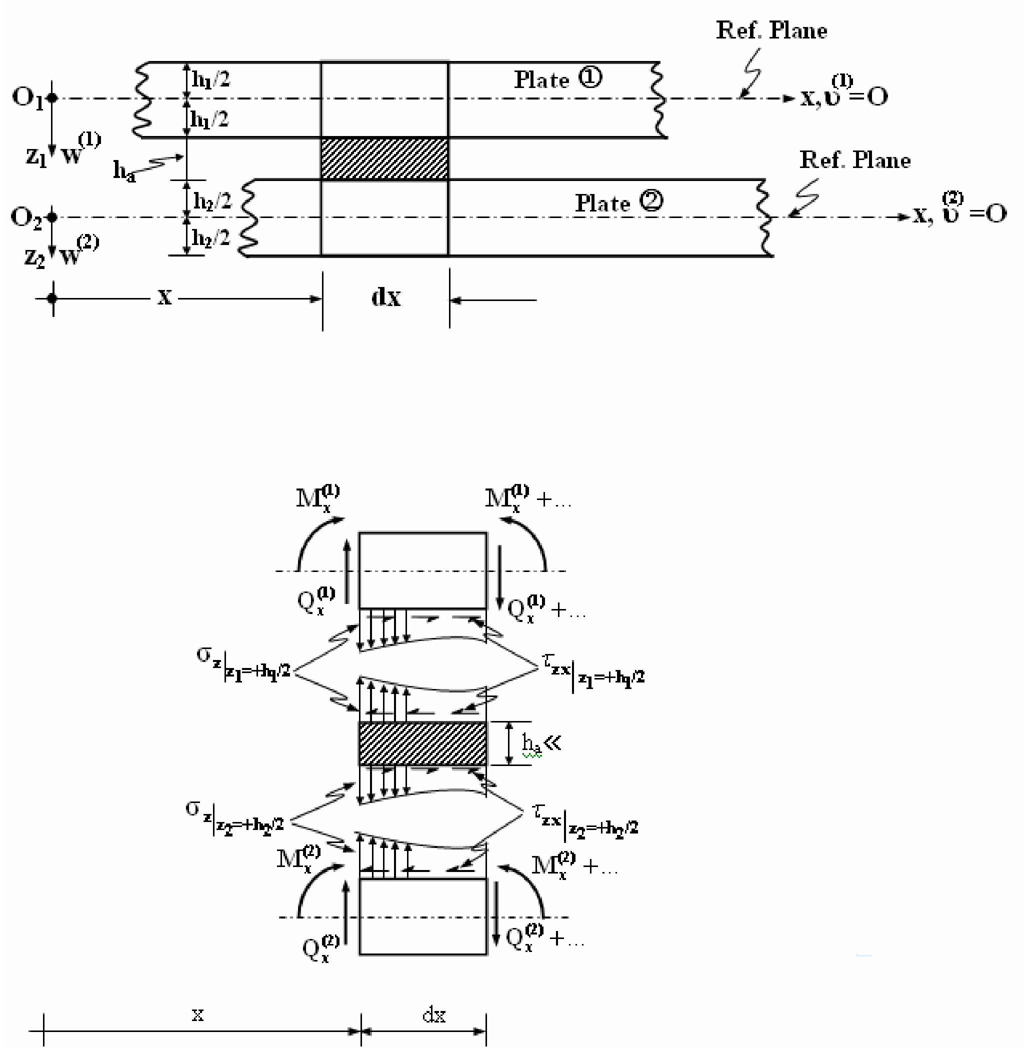


Figure 4.1 Stress Distributions at Plate Adhesive Layer Interfaces in the "Overlap Region"

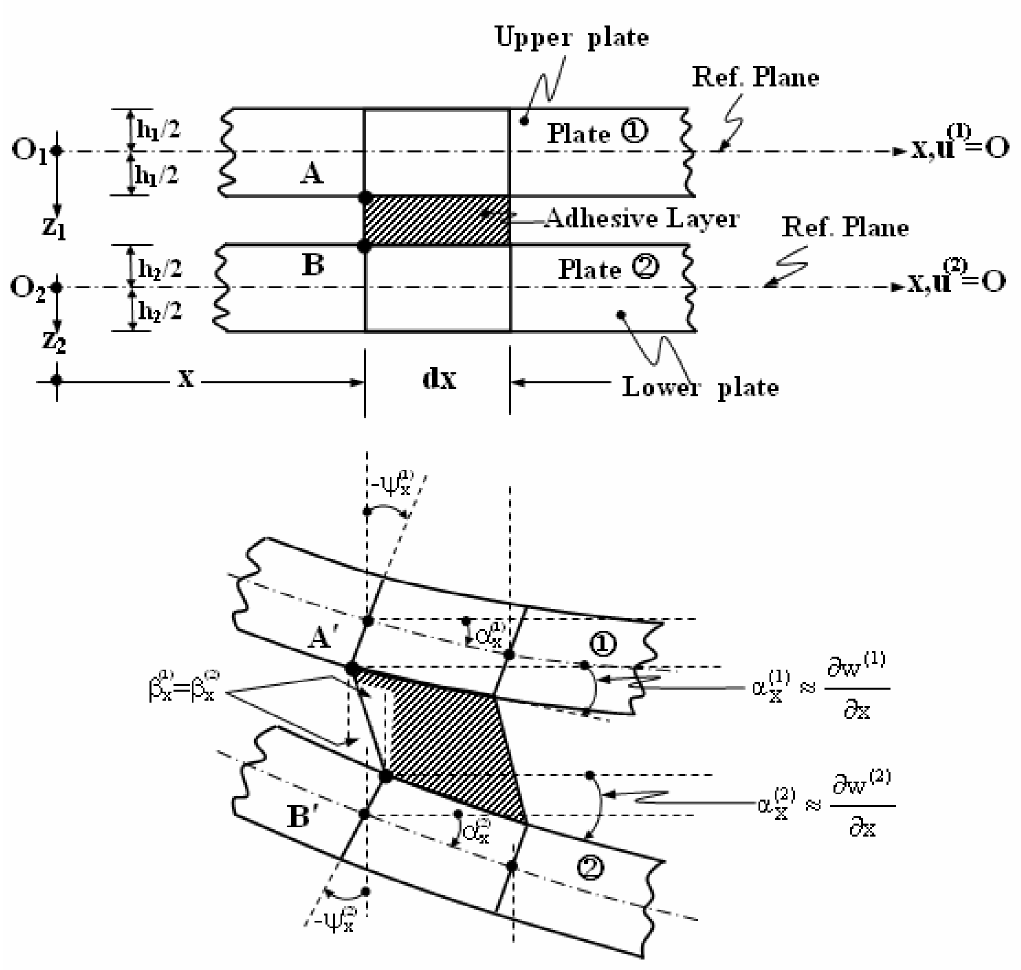


Figure 4.2 Deformations of Adherends (Mindlin Plates) and in-between Adhesive Layers in the "Overlap Region"

The purpose of this section is to obtain the relations between the stresses in adhesive layer and the plate interfaces and displacements of the adherends. The shear strains $\gamma_{xz}^{(j)}$ and $\gamma_{yz}^{(j)}$ at interfaces is found by considering the distortion in the right angles of the adhesive infinitesimal element after deformation as,

$$\gamma_{xz}^{(j)} = \alpha_x^{(j)} + \beta_x^{(j)} \qquad j = 1,2, \qquad (4.1)$$

with the assumption of small deformations, $\alpha^{(j)}$ and $\beta^{(j)}$ can be expressed as the slopes of the adherends and the displacements in x and y directions, respectively. That is,

Between Plates 1 and 2 for Part I,

$$\tan\left(\alpha_s^{(j)}\right) = \frac{\partial w^{(j)}}{\partial s} \approx \alpha_s^{(j)}$$
$$\tan\left(\beta_s^{(j)}\right) = \frac{\left(u_{B'}\right)_s + \left(u_{A'}\right)_s}{h_a} \approx \beta_s^{(j)} \qquad j=1,2 \qquad (4.2)$$

where h_a is the thicknesses of the adhesive layer, and $(u_A)_s$, $(u_B)_s$ are axial deformation of points A, B in s direction respectively. Axial deformations $(u_A)_s$, $(u_B)_s$ are caused only by the bending of the plates, then the displacement components can be written as,

For Part I;

$$(u_{A'})_s = -\frac{h_1}{2}\psi_s^{(1)}$$
$$\left(u_{B'}\right)_s = -\frac{h2}{2}\psi_s^{(2)} \qquad (4.3)$$

The shear strains in the interfaces can be expressed in terms of displacements and angles of rotation of the adherends by using (4.1) through (4.3)

Adhesive Strains Between Plates 1 and 2 for Part I

$$\gamma_{xz}^{(j)} = -\frac{1}{2h_a}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(2)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial x}}}$$

$$\gamma_{yz}^{(j)} = -\frac{1}{2h_a}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(2)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial y}}} \qquad j=1,2 \qquad (4.4)$$

$$\varepsilon_z^{(j)} = w^{(2)} - w^{(1)}$$

It should be mentioned here that in some previous studies by Yuceoglu and Özerciyes [IV.4-IV.13], the double underlined terms (=) are neglected. In this present study these terms are included in the theoretical formulation.

The interface shear stresses in the adhesive layers can be expressed by using the interface shear strains as,

Adhesive Stresses Between Plates 1 and 2 for Part I,

$$\tau_{xz}^{(j)} = G_a\gamma_{xz}^{(j)} = -\frac{G_a}{2h_a}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(2)}\right) - \underline{\underline{G_a\frac{\partial w^{(j)}}{\partial x}}}$$

$$\tau_{yz}^{(j)} = G_a\gamma_{yz}^{(j)} = -\frac{G_a}{2h_a}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(2)}\right) - \underline{\underline{G_a\frac{\partial w^{(j)}}{\partial y}}} \qquad j=1,2 \qquad (4.5)$$

$$\sigma_z^{(j)} = \frac{E_a}{h_a}(w^{(2)} - w^{(1)})$$

The thicknesses h_a of the adhesive layers are assumed to be much smaller than the thicknesses of the plates and $\psi_x^{(j)}, \psi_y^{(j)}$ have the same order of magnitudes when compared with $\frac{\partial w^{(j)}}{\partial x}$ and $\frac{\partial w^{(j)}}{\partial y}$. Therefore, one may assume that the variation of the transverse displacements in the adhesive layers is linear that is equivalent to assuming that the normal strain ε_z is constant across the thickness. E_a is the modulus of elasticity (or Young's modulus) and G_a is the modulus of rigidity (or shear modulus) of the adhesive layer.

4.4.2 Analysis of Part I (or the "<u>Overlap Region</u>") of Composite Plate System

4.4.2.1 Implementation of the Adhesive Layer Equations to Governing Equations (equation of motions) of Plate Adherends

Adhesive stresses are related to unknown displacement functions and angle of rotations of the adherends in the "<u>Overlap Region</u>" since they are considered as surface loads or external stresses acting on the upper and lower faces of the plate adherends. Normal and tangential stresses at the interface may be considered as the ''compatibility or coupling conditions'' of the two plates in the "<u>Overlap Region</u>".

The governing equations (or plate equations of motion) given in (3.19) and (3.20) can be written by using the adhesive stresses as load terms q_{zx}'s and q_{zy}'s as,

<u>For Plate 1;</u>

$$\left|\begin{array}{l} \dfrac{\partial M_x^{(1)}}{\partial x}+\dfrac{\partial M_{yx}^{(1)}}{\partial y}-Q_x^{(1)}-\underline{\dfrac{h_1G_a}{4h_a}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)}-\underline{\underline{\dfrac{h_1G_a}{2}\dfrac{\partial w^{(1)}}{\partial x}}}=\dfrac{\rho_1h_1^3}{12}\dfrac{\partial^2\psi_x^{(1)}}{\partial t^2} \\ \dfrac{\partial M_{yx}^{(1)}}{\partial x}+\dfrac{\partial M_{y}^{(1)}}{\partial y}-Q_y^{(1)}-\underline{\dfrac{h_1G_a}{4h_a}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)}-\underline{\underline{\dfrac{h_1G_a}{2}\dfrac{\partial w^{(1)}}{\partial y}}}=\dfrac{\rho_1h_1^3}{12}\dfrac{\partial^2\psi_y^{(1)}}{\partial t^2} \\ \dfrac{\partial Q_x^{(1)}}{\partial x}+\dfrac{\partial Q_y^{(1)}}{\partial y}+\underline{\dfrac{E_a}{h_a}\left(w^{(2)}-w^{(1)}\right)}=\rho_1h_1\dfrac{\partial^2w^{(1)}}{\partial t^2} \end{array}\right. \qquad (4.6.a)$$

<u>For Plate 2 in Part I;</u>

$$\left|\begin{array}{l} \dfrac{\partial M_x^{(2)}}{\partial x}+\dfrac{\partial M_{yx}^{(2)}}{\partial y}-Q_x^{(2)}-\underline{\dfrac{h_2G_a}{4h_a}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)}-\dfrac{h_2G_a}{2}\dfrac{\partial w^{(2)}}{\partial x}=\dfrac{\rho_2h_2^3}{12}\dfrac{\partial^2\psi_x^{(2)}}{\partial t^2} \\ \dfrac{\partial M_{yx}^{(2)}}{\partial x}+\dfrac{\partial M_{y}^{(2)}}{\partial y}-Q_y^{(2)}-\underline{\dfrac{h_2G_a}{4h_a}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)}-\dfrac{h_2G_a}{2}\dfrac{\partial w^{(2)}}{\partial y}=\dfrac{\rho_2h_2^3}{12}\dfrac{\partial^2\psi_y^{(2)}}{\partial t^2} \\ \dfrac{\partial Q_x^{(2)}}{\partial x}+\dfrac{\partial Q_y^{(2)}}{\partial y}-\underline{\dfrac{E_a}{h_a}\left(w^{(2)}-w^{(1)}\right)}=\rho_2h_2\dfrac{\partial^2w^{(2)}}{\partial t^2} \end{array}\right. \qquad (4.6\ b)$$

In the above equations, the “underlined terms” are the “coupling terms” between the adherends.

At this stage, the “Classical Levy’s Method” with the trigonometric series expansions is to be considered. The “Classical Lévy’s Method” is restricted to rectangular plates with any two opposite edges simply supported. However, the other two edges may have arbitrary boundary conditions.

In the present study, the edges at x=0 and x=a are simply supported “Boundary Conditions” which have to be satisfied along these edges are as follows,

$$\text{at } x=0,a \rightarrow \quad w^{(j)}=0,\ M_x^{(j)}=0,\ \psi_y^{(j)}=0 \quad (j=1,2) \qquad (4.7)$$

By using the “Classical Lévy’s Method” angles of rotations and displacements of the adherends can be expressed as,

Displacements and Angles of Rotation,

$$w^{(j)}(\eta,\xi_I,t)=h_I\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{W}_{mn}^{(j)}(\xi_I)\,sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$\psi_x^{(j)}(\eta,\xi_I,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Psi}_{mnx}^{(j)}(\xi_I)\cos(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad (j=1,2) \qquad (4.8)$$

$$\psi_y^{(j)}(\eta,\xi_I,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Psi}_{mny}^{(j)}(\xi_I)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

where "m" is the number of half-waves in the x-direction, superscript (j) denotes upper plate adherend for j=1 and, lower plate adherend for j=2. “i” is defined as $\sqrt{-1}$ and the “barred” (—) quantities are the dimensionless transverse displacements and angles of rotation. The nondimensional independent space variables η, and ξ_I are defined as x/a, and y_I/ℓ_I, respectively. "$\overline{\omega}_{mn}$" is the dimensionless circular frequency or the natural frequency of the flexural (or bending) vibrations of the entire composite plate or panel system.

"Stress Resultants" can also be expressed in trigonometric series in the x-direction as,

Stress Resultants for Part I;

$$M_x^{(j)}(\eta,\xi_I,t)=\frac{h_I^5 B_{11}^{(I)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnx}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$M_{yx}^{(j)}(\eta,\xi_I,t)=\frac{h_I^5 B_{11}^{(I)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnyx}^{(j)}(\xi_I)cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$M_y^{(j)}(\eta,\xi_I,t)=\frac{h_I^5 B_{11}^{(I)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mny}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t} \quad (j=1,2) \qquad (4.9)$$

$$Q_y^{(j)}(\eta,\xi_I,t)=\frac{h_I^4 B_{11}^{(I)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mny}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$Q_x^{(j)}(\eta,\xi_I,t)=\frac{h_I^4 B_{11}^{(I)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mnx}^{(j)}(\xi_I)cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

where the "barred" $(-)$ $\overline{W}_{mn}^{(j)},\overline{\Phi}_{mnx}^{(j)},\overline{\Phi}_{mny}^{(j)},\overline{M}_{mnx}^{(j)},\overline{M}_{mnyx}^{(j)},\overline{M}_{mny}^{(j)},\overline{Q}_{mny}^{(j)},\overline{Q}_{mnx}^{(j)}$ are "dimensionless fundamental dependent variables".

For non-dimensionalization of equations, parameters in the governing differential equations shall be nondimensionalized with respect to main or reference quantities which are chosen as $"B_{11}^{(1)}"$, $"h_1"$, $"\rho_1"$ and "a".

The dimensionless coordinates or independent space variables are,

η=x/a,

$\xi_I=y_I/l_I$, $\quad\xi_{II}=y_{II}/l_{II}$, $\quad\xi_{III}=y_{III}/l_{III}$ (4.10)

The dimensionless parameters related to orthotropic elastic constant and the adhesive layers are,

$$\overline{B}_{ik}^{(j)}=B_{ik}^{(j)}\big/B_{11}^{(I)} \qquad (j=1,2\ \ and\, i,k=1,2,3),$$

$$\overline{B}_{\ell\ell}^{(j)}=B_{\ell\ell}^{(j)}\big/B_{11}^{(I)} \qquad (j=1,2\ \ and\ \ell=4,5,6) \qquad (4.11)$$

$$\overline{G}_a=G_a\big/B_{11}^{(I)}, \quad \overline{E}_a=E_a\big/B_{11}^{(I)}$$

The dimensionless parameters related to the densities and the geometry of the plates and adhesive layers are,

$$\begin{aligned}
&\overline{\rho} = \rho_2 / \rho_1 , \\
&\overline{h}_a = h_a / h_1 , \ \overline{h} = h_2 / h_1 , \\
&\overline{L}_I = \ell_I / a , \quad \overline{L}_{II} = \ell_{II} / a , \quad \overline{L}_{III} = \ell_{III} / a ,
\end{aligned} \tag{4.12}$$

The dimensionless frequency parameter $\overline{\omega}_{mn}$ of the entire doubly stiffened, composite plate or panel system is;

$$\begin{aligned}
&\overline{\omega}_{mn} = \rho_1 a^4 \omega_{mn}^2 / h_1^2 B_{11}^{(1)} \\
&\overline{\Omega} = \overline{\omega}_{mn}
\end{aligned} \qquad (m,n=1,2,3...) \tag{4.13}$$

where the dimensionless natural frequency parameter $\overline{\omega}_{mn}$ is ordered in terms of its magnitudes as $\overline{\Omega}_1 < \overline{\Omega}_2 < \overline{\Omega}_3 < ....$ with the subscript indicating the first, second, third, dimensionless natural frequencies depending on the given m, n values.

4.4.2.2 Reduction to "Governing Systems of First Order Ordinary Differential Equations" for "Main PROBLEM I",

Here, M_y, M_{xy},Q_y, ψ_x, ψ_y and w are chosen as intrinsic variables and M_x,Q_x are chosen as auxiliary variables, Then, from the Stress Resultants and Moment Resultant Equations given in (3.2a), first order partial differential equations can be written with respect to the dimensionless independent variables ξ and η as,

For the "<u>Overlap Region</u>" or Part I;

$$
\begin{aligned}
&\frac{1}{l_I}\frac{\partial \psi_y^{(j)}}{\partial \xi_I} = \frac{1}{B_{22}^{(j)}}\left(\frac{12}{h_j^3}M_y^{(j)} - B_{12}^{(j)}\frac{1}{a}\frac{\partial \psi_x^{(j)}}{\partial \eta}\right)\\
&\frac{1}{l_I}\frac{\partial \psi_x^{(j)}}{\partial \xi_I} = \frac{12}{h_j^{(3)}B_{66}^{(j)}}M_{yx}^{(j)} - \frac{1}{a}\frac{\partial \psi_y^{(j)}}{\partial \eta}\\
&\frac{1}{l_I}\frac{\partial w^{(j)}}{\partial \xi_I} = \frac{1}{\kappa_y^2 h_j B_{44}^{(j)}}Q_y^{(j)} - \psi_y^{(j)} \qquad\qquad (j=1,2) \qquad (4.14)\\
&\frac{1}{l_I}\frac{\partial M_{yx}^{(j)}}{\partial \xi_I} = \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_x^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_x^{(j)}}{\partial \eta} + Q_x^{(j)} - \frac{h_j}{2}\left(q_{zx}^{(+)} + q_{zx}^{(-)}\right)\\
&\frac{1}{l_\iota}\frac{\partial M_y^{(j)}}{\partial \xi_I} = \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_y^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_{yx}^{(j)}}{\partial \eta} + Q_y^{(j)} - \frac{h_j}{2}\left(q_{zy}^{(+)} + q_{zy}^{(-)}\right)\\
&\frac{1}{l_I}\frac{\partial Q_y^{(j)}}{\partial \xi_I} = \rho_j h_j\frac{\partial^2 w^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial Q_x^{(j)}}{\partial \eta} - \left(q_z^{(+)} - q_z^{(-)}\right)
\end{aligned}
$$

where q's are surface loads and the stresses on the upper (-) and lower (+) surfaces of the plates. Also, note that superscript and subscript (j) denotes the upper plate adherend for j=1 and lower plate adherend for j=2.

By substituting (4.8) and (4.9) in to (4.14) and making the necessary non-dimensionalizations with respect to (4.10), (4.11) and (4.12), "Governing System of First Order Ordinary Differential Equations" for the bonded plate or panel system are developed as,

For Plate 1, in Part I (Overlap Region);

$$
\begin{aligned}
\frac{d\overline{\Psi}^{(1)}_{mnx}}{d\xi_I} &= \frac{12\overline{L}_I}{\overline{B}^{(1)}_{66}}\left(\frac{h_1}{a}\right)^2 \overline{M}^{(1)}_{mnyx} - \overline{L}_I m\pi\overline{\Psi}^{(1)}_{mny} \\
\frac{d\overline{\Psi}^{(1)}_{mny}}{d\xi_I} &= \frac{12\overline{L}_I}{\overline{B}^{(1)}_{22}}\left(\frac{h_1}{a}\right)^2 \overline{M}^{(1)}_{mny} + \overline{L}_I \frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}} m\pi\overline{\Psi}^{(1)}_{mnx} \\
\frac{d\overline{W}^{(1)}_{mn}}{d\xi_I} &= \overline{L}_I\left(\frac{h_1}{a}\right)^2 \frac{1}{\kappa_y^2\overline{B}^{(1)}_{44}}\overline{Q}^{(1)}_{mny} - \overline{L}_I\frac{a}{h_1}\overline{\Psi}^{(1)}_{mny} \\
\frac{d\overline{M}^{(1)}_{mnyx}}{d\xi_I} &= \left\{\begin{array}{l} -\dfrac{\overline{L}_I}{12}\dfrac{a^4\rho_1\omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \dfrac{\overline{L}_I}{12}\left(\dfrac{am\pi}{h_1}\right)^2\left(1-\dfrac{\left(\overline{B}^{(1)}_{12}\right)^2}{\overline{B}^{(1)}_{22}}\right) \\ +\overline{L}_I\left(\dfrac{a}{h_1}\right)^4\kappa_x^2\overline{B}^{(1)}_{55} + \overline{L}_I\dfrac{\overline{G}_a}{4\overline{h}_a}\left(\dfrac{a}{h_1}\right)^4 \end{array}\right\}\overline{\Psi}^{(1)}_{mnx} \\
&- \overline{L}_I\frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}} m\pi\overline{M}^{(1)}_{mny} + \left\{\overline{L}_I\kappa_x^2\overline{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\frac{\overline{L}_I\overline{G}_a m\pi}{2}\left(\frac{a}{h_1}\right)^3}}\right\}\overline{W}^{(1)}_{mn} \\
&+ \overline{L}_I\frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4\overline{h}\overline{\Psi}^{(2)}_{mnx} \\
\frac{d\overline{M}^{(1)}_{mny}}{d\xi_I} &= \left\{-\frac{\overline{L}_I}{12}\frac{\rho_1 a^4\omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \overline{L}_I\left(\frac{a}{h_1}\right)^4\frac{\overline{G}_a}{4\overline{h}_a} - \underline{\underline{\frac{\overline{L}_I}{2}\left(\frac{a}{h_1}\right)^4\overline{G}_a}}\right\}\overline{\Psi}^{(1)}_{mny} + \overline{L}_I m\pi\overline{M}^{(1)}_{mnyx} \\
&+ \left\{\overline{L}_I\frac{a}{h_1} + \underline{\underline{\frac{\overline{L}_I}{2\kappa_y^2\overline{B}^{(1)}_{44}}\overline{G}_a\frac{a}{h_1}}}\right\}\overline{Q}^{(1)}_{mny} + \overline{L}_I\frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4\overline{h}\overline{\Psi}^{(2)}_{mny} \qquad (4.15) \\
\frac{d\overline{Q}^{(j)}_{mny}}{d\xi_I} &= \left\{-\overline{L}_I\frac{\rho_1 a^4\omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \overline{L}_I\kappa_x^2\overline{B}^{(1)}_{55}\left(\frac{am\pi}{h_1}\right)^2 + \overline{L}_I\left(\frac{a}{h_1}\right)^4\frac{\overline{E}_a}{\overline{h}_a}\right\}\overline{W}^{(1)}_{mn} \\
&+ \overline{L}_I\kappa_x^2\overline{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi\overline{\Psi}^{(1)}_{mnx} - \overline{L}_I\left(\frac{a}{h_1}\right)^4\frac{\overline{E}_a}{\overline{h}_a}\overline{W}^{(2)}_{mn}
\end{aligned}
$$

where double underlined terms come from $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ terms or "doubly underlined terms").

For Plate 2, in Part I (Overlap Region);

$$\frac{d\overline{\Psi}_{mnx}^{(2)}}{\partial \xi_I} = \overline{L}_I \frac{12}{\overline{h}^3 \overline{B}_{66}^{(2)}}\left(\frac{h_1}{a}\right)^2 \overline{M}_{mnyx}^{(2)} - \overline{L}_I m\pi \overline{\Psi}_{mny}^{(2)}$$

$$\frac{d\overline{\Psi}_{mny}^{(2)}}{\partial \xi_I} = \overline{L}_I \left(\frac{h_1}{a}\right)^2 \frac{12}{\overline{B}_{22}^{(2)} \overline{h}^3} \overline{M}_{mny}^{(2)} + \overline{L}_I \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}} m\pi \overline{\Psi}_{mnx}^{(2)}$$

$$\frac{d\overline{W}_{mn}^{(2)}}{d\xi_I} = \overline{L}_I \left(\frac{h_1}{a}\right)^2 \frac{1}{\kappa_y^2 \overline{h}\overline{B}_{44}^{(2)}} \overline{Q}_{mny}^{(2)} - \overline{L}_I \overline{\Psi}_{mny}^{(2)}$$

$$\frac{d\overline{M}_{mnyx}^{(2)}}{d\xi_I} = \left\{ \begin{array}{l} -\overline{L}_I \overline{\rho}\overline{h}^3 \dfrac{\rho_1 a^4 \omega_{mn}^2}{12 h_1^2 B_{11}^{(1)}} + \overline{L}_I \kappa_x^2 \overline{h}\overline{B}_{55}^{(2)} \left(\dfrac{a}{h_1}\right)^4 \\ + \overline{L}_I \dfrac{\overline{h}^3}{12}\left(\dfrac{am\pi}{h_1}\right)^2 \left(\overline{B}_{11}^{(2)} - \dfrac{\left(\overline{B}_{12}^{(2)}\right)^2}{\overline{B}_{22}^{(2)}}\right) + \overline{L}_I \left(\dfrac{a}{h_1}\right)^4 \dfrac{\overline{G}_a \overline{h}^2}{4\overline{h}_a} \end{array} \right\} \overline{\Psi}_{mnx}^{(2)}$$

$$- \overline{L}_I \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}} m\pi \overline{M}_{mny}^{(2)} + \left\{ \overline{L}_I \kappa_x^2 \overline{h} m\pi \left(\frac{a}{h_1}\right)^3 \overline{B}_{55}^{(2)} + \underline{\underline{\overline{L}_I \frac{\overline{h}m\pi}{2}\left(\frac{a}{h_1}\right)^3 \overline{G}_a}} \right\} \overline{W}_{mn}^{(2)}$$

$$+ \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{G}_a \overline{h}}{4\overline{h}_a} \overline{\Psi}_{mnx}^{(1)}$$

$$\frac{d\overline{M}_{mny}^{(2)}}{d\xi_I} = \left\{ -\overline{L}_I \overline{\rho}\overline{h}^3 \frac{a^4 \rho_1 \omega_{mn}^2}{12 h_1^2 B_{11}^{(1)}} + \overline{L}_I \frac{\overline{h}^2 \overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\overline{L}_I \frac{\overline{h}}{2}\left(\frac{a}{h_1}\right)^4 \overline{G}_a}} \right\} \overline{\Psi}_{mny}^{(2)}$$

$$+ \overline{L}_I (m\pi) \overline{M}_{mnyx}^{(2)} + \overline{L}_I \frac{\overline{h}\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \overline{\Psi}_{mny}^{(1)}$$

$$+ \left\{ \overline{L}_I \frac{a}{h_1} + \underline{\underline{\overline{L}_I \frac{a}{h_1} \frac{\overline{G}_a}{2\kappa_y^2 \overline{B}_{44}^{(2)}}}} \right\} \overline{Q}_{mny}^{(2)}$$

$$\frac{d\overline{Q}_{mny}^{(2)}}{d\xi_I} = \left\{ \begin{array}{l} -\overline{L}_I \overline{\rho}\overline{h} \dfrac{\rho_1 a^4 \omega_{mn}^2}{h_1^2 B_{11}^{(1)}} + \overline{L}_I \kappa_x^2 \overline{h}\overline{B}_{55}^{(2)} \left(\dfrac{am\pi}{h_1}\right)^2 \\ + \overline{L}_I \left(\dfrac{a}{h_1}\right)^4 \dfrac{\overline{E}_a}{\overline{h}_a} \end{array} \right\} \overline{W}_{mn}^{(2)} +$$

$$\overline{L}_I \kappa_x^2 \overline{h}\overline{B}_{55}^{(2)} \left(\frac{a}{h_1}\right)^3 m\pi \overline{\Psi}_{mnx}^{(2)} - \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a} \overline{W}_{mn}^{(1)} \qquad (4.16)$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

The quantities having the subscript "mn" are "dimensionless fundamental dependent variables" of the problem in Part I Region (or "Overlap Region").A ''Two-Point Boundary Value Problem" is created in the ''Overlap Regions'' (or Part I region) by reducing the system of partial differential equations to a ''Governing System of First Order Ordinary Differential Equations'' in ξ_I or y_I direction.

Thus "Governing System of First Order Ordinary Differential Equations" in the compact matrix or ''state vector'' form for the ''Overlap Region'' (or Part I region) can be written as,

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{Y}^{(1)}_{mn}\\ \overline{Y}^{(2)}_{mn}\end{Bmatrix}=\begin{bmatrix}\overline{\mathcal{C}}_{1,1} & \overline{\mathcal{C}}_{1,2}\\ \overline{\mathcal{C}}_{2,1} & \overline{\mathcal{C}}_{2,2}\end{bmatrix}\begin{Bmatrix}\overline{Y}^{(1)}_{mn}\\ \overline{Y}^{(2)}_{mn}\end{Bmatrix},\quad \begin{matrix}(0<\xi_I<1) & \text{(Upper Plate)}\\ (0<\xi_I<1) & \text{(Lower Plate)}\end{matrix} \tag{4.18}$$

with the ''Arbitrary Boundary Conditions'' and the ''Continuity Conditions''.

where ξ_I is defined as y_I/ℓ_I. The superscripts show the related plate layer, the sub-matrices [$\overline{\mathcal{C}}_{i,j}$] are partitioned square matrices of dimension (6x6) which explicitly include the nondimensional geometric and material characteristics of the plate adherends, and of the adhesive layers and dimensionless natural frequency parameter $\overline{\omega}_{mn}$ of the entire composite system. $\overline{Y}^{(j)}_{mn}$ (j=1,2) are the ''state vectors' corresponding to the "state variables" or ''dimensionless fundamental dependent variables'' of the problem under study as,

$$\left\{\overline{Y}^{(j)}_{mn}\right\}=\left\{\overline{\Psi}^{(j)}_{mnx},\overline{\Psi}^{(j)}_{mny},\overline{W}^{(j)}_{mn};\overline{M}^{(j)}_{mnyx},\overline{M}^{(j)}_{mny},\overline{Q}^{(j)}_{mny}\right\}^T,\quad (j=1,2) \tag{4.19}$$

4.4.3 Analysis of Part II (or Single Layer) of Composite Plate System

The "Governing System of Ordinary Differential Equations" can be obtained for the plate adherend in Part II region by using the same procedure in the previous section. There is no adhesive layer in Part II. Therefore, coupling terms in (4.15) including the adhesive layer elastic constants are dropped. The "Governing System of First Order Ordinary Differential Equations” in the ‘‘state vector’’ form, for Part II region,

$$\frac{d}{d\xi_{II}}\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(1)}\right\}=\left[\overline{\mathcal{D}}\right]\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(1)}\right\} \qquad (0<\xi_{II}<1) \qquad \text{(Upper Plate)} \qquad (4.20)$$

with the ‘‘Arbitrary Boundary Conditions’’ at ξ_{II}=0 and the ‘‘Continuity Conditions’’ at ξ_{II} =1 for the orthotropic plate adherend.

where ξ_{II} is defined as y_{II}/ℓ_{II} and [$\overline{\mathcal{D}}$] is the “Coefficient Matrix” of dimension (6x6) which explicitly includes dimensionless geometric and material characteristics of the upper plate adherend as well as the dimensionless natural frequencies $\overline{\boldsymbol{\omega}}_{\boldsymbol{mn}}$ of the entire composite plate system. The column matrix or the “state vector” $\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(1)}\right\}$ is defined as,

$$\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(1)}\right\}=\left\{\overline{\Psi}_{mnx}^{(1)},\overline{\Psi}_{mny}^{(1)},\overline{W}_{mn}^{(1)};\overline{M}_{mnyx}^{(1)},\overline{M}_{mny}^{(1)},\overline{Q}_{mny}^{(1)}\right\}^{T}, \qquad (4.21)$$

4.4.4 Analysis of Part III (or Single Layer) of Composite Plate System

The “Governing System of Ordinary Differential Equations” can be obtained for the plate adherend in Part III region by using the same procedure in the previous section. There is no adhesive layer in Part III. Therefore, coupling terms in (4.16) including the adhesive layer elastic constants are dropped. The “Governing System of First Order Ordinary Differential Equations” in the ‘‘state vector’’ form, for Part III region,

$$\frac{d}{d\xi_{III}}\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(2)}\right\}=\left[\overline{\boldsymbol{\mathcal{E}}}\right]\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(2)}\right\}, \qquad (0<\xi_{III}<1) \qquad \text{(Lower Plate)} \qquad (4.22)$$

with the ‘‘Arbitrary Boundary Conditions’’ at ξ_{III} =1 and the ‘‘Continuity Conditions’’ at ξ_{III} =0 for the orthotropic plate adherend.

where ξ_{III} is defined as y_{III}/ℓ_{III} and [$\overline{\boldsymbol{\mathcal{E}}}$] is the “Coefficient Matrix” of dimension (6x6) which explicitly includes dimensionless geometric and material characteristics of the lower plate adherend as well as the dimensionless natural frequencies $\overline{\boldsymbol{\omega}}_{\boldsymbol{mn}}$ of the entire composite plate system. The column matrix or the "state vector" $\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(2)}\right\}$ is defined as,

$$\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(2)}\right\} = \left\{\overline{\Psi}_{mnx}^{(2)}, \overline{\Psi}_{mny}^{(2)}, \overline{W}_{mn}^{(2)}; \overline{M}_{mnyx}^{(2)}, \overline{M}_{mny}^{(2)}, \overline{Q}_{mny}^{(2)}\right\}^T, \tag{4.23}$$

4.4.5 System of Governing Ordinary Differential Equations for ("Main PROBLEM I")

In the previous sections, the "Governing System of Coupled Ordinary Differential Equations" are obtained in the matrix or "state vector" form for the "Overlap Region" (or Part I region), and for "Single Layer Regions" (or Part II and Part III regions). These equations can be written in "open matrix form" as,

For Part I region or the "Overlap Region" (or Two-Layer Composite Plate Region),

$$\frac{d}{d\xi_I}\left\{\begin{array}{c} \bar{\psi}_{mnx}^{(1)} \\ \bar{\psi}_{mny}^{(1)} \\ \bar{W}_{mn}^{(1)} \\ \bar{M}_{mnyx}^{(1)} \\ \bar{M}_{mny}^{(1)} \\ \bar{Q}_{mny}^{(1)} \\ \bar{\psi}_{mnx}^{(2)} \\ \bar{\psi}_{mny}^{(2)} \\ \bar{W}_{mn}^{(2)} \\ \bar{M}_{mnyx}^{(2)} \\ \bar{M}_{mny}^{(2)} \\ \bar{Q}_{mny}^{(2)} \end{array}\right\} = \left[\begin{array}{cccccc|cccccc} 0 & c_{1,2} & 0 & c_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{2,1} & 0 & 0 & 0 & c_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{3,2} & 0 & 0 & 0 & c_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{4,1} & 0 & c_{4,3} & 0 & c_{4,5} & 0 & c_{4,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{5,2} & 0 & c_{5,4} & 0 & c_{5,6} & 0 & c_{5,8} & 0 & 0 & 0 & 0 \\ c_{6,1} & 0 & c_{6,3} & 0 & 0 & 0 & 0 & 0 & c_{6,9} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{7,8} & 0 & c_{7,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{8,7} & 0 & 0 & 0 & c_{8,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{9,8} & 0 & 0 & 0 & c_{9,12} \\ c_{10,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{10,9} & 0 & c_{10,11} & 0 \\ 0 & c_{11,2} & 0 & 0 & 0 & 0 & 0 & c_{11,8} & 0 & c_{11,10} & 0 & c_{11,12} \\ 0 & 0 & c_{12,3} & 0 & 0 & 0 & c_{12,7} & 0 & c_{12,9} & 0 & 0 & 0 \end{array}\right] \left\{\begin{array}{c} \bar{\psi}_{mnx}^{(1)} \\ \bar{\psi}_{mny}^{(1)} \\ \bar{W}_{mn}^{(1)} \\ \bar{M}_{mnyx}^{(1)} \\ \bar{M}_{mny}^{(1)} \\ \bar{Q}_{mny}^{(1)} \\ \bar{\psi}_{mnx}^{(2)} \\ \bar{\psi}_{mny}^{(2)} \\ \bar{W}_{mn}^{(2)} \\ \bar{M}_{mnyx}^{(2)} \\ \bar{M}_{mny}^{(2)} \\ \bar{Q}_{mny}^{(2)} \end{array}\right\}$$

$(0<\xi_I<1)$ (Upper Plate)

$(0<\xi_I<1)$ (Lower Plate) (4.24 a)

The elements, in the "open matrix form", of the ''Coefficient Sub-Matrix related to the plate layers are, For plate 1 (Upper Plate Adherend),

$$c_{1,2} = -\overline{L}_1 m\pi \qquad c_{1,4} = \frac{12\overline{L}_1}{\overline{B}_{66}^{(1)}}\left(\frac{h_1}{a}\right)^2 \qquad c_{2,1} = \overline{L}_1 m\pi \frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}$$

$$c_{2,5} = \frac{12\overline{L}_1}{\overline{B}_{22}^{(1)}}\left(\frac{h_1}{a}\right)^2 \qquad c_{3,2} = -\overline{L}_1\left(\frac{a}{h_1}\right) \qquad c_{3,6} = \frac{\overline{L}_1}{\kappa_y^2 \overline{B}_{44}^{(1)}}\left(\frac{h_1}{a}\right)^2$$

$$c_{4,1} = -\frac{\overline{L}_1}{12}\overline{\Omega} + \overline{L}_1 \kappa_x^2 \overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_1}{12}\left(\frac{am\pi}{h_1}\right)^2\left(1 - \frac{\overline{B}_{12}^{(1)^2}}{\overline{B}_{22}^{(1)}}\right) + \frac{\overline{G}_a \overline{L}_1}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4$$

$$c_{4,3} = \overline{L}_1 \kappa_x^2 \overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\overline{L}_1 \frac{m\pi}{2}\left(\frac{a}{h_1}\right)^3 \overline{G}_a}} \qquad c_{4,5} = -\overline{L}_1 m\pi \frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}$$

$$c_{5,2} = -\frac{\overline{L}_1}{12}\overline{\Omega} + \frac{\overline{G}_a \overline{L}_1}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\frac{\overline{L}_1 \overline{G}_a}{2}\left(\frac{a}{h_1}\right)^4}} \qquad c_{5,4} = \overline{L}_1 m\pi$$

$$c_{5,6} = \overline{L}_1\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_1\left(\frac{a}{h_1}\right)\frac{\overline{G}_a}{2\kappa_y^2 \overline{B}_{44}^{(1)}}}} \qquad c_{6,1} = \kappa_x^2 \overline{L}_1 \overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$c_{6,3} = -\overline{L}_1\overline{\Omega} + \overline{L}_1 \kappa_x^2\left(\frac{am\pi}{h_1}\right)^2 \overline{B}_{55}^{(1)} + \frac{\overline{L}_1 \overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c_{4,7} = \frac{\overline{L}_1 \overline{G}_a}{4\overline{h}_a}\overline{h}\left(\frac{a}{h_1}\right)^4$$

$$c_{5,8} = \frac{\overline{L}_1 \overline{G}_a \overline{h}}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^2 \qquad c_{6,9} = -\frac{\overline{L}_1 \overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad (4.24$$

b)

For Plate 2 (Lower Adherend Plate),

$$c_{7,8} = -\overline{L}_1 m\pi \qquad c_{7,10} = \frac{12\overline{L}_1}{\overline{B}_{66}^{(2)}\overline{h}^3}\left(\frac{h_1}{a}\right)^2 \qquad c_{8,7} = \overline{L}_1 m\pi \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}$$

$$c_{8,11} = \frac{12\overline{L}_1}{\overline{B}_{22}^{(2)}\overline{h}^3}\left(\frac{h_1}{a}\right)^2 \qquad c_{9,8} = -\overline{L}_1\left(\frac{a}{h_1}\right) \qquad c_{9,12} = \frac{\overline{L}_1}{\kappa_y^2 \overline{B}_{44}^{(2)}\overline{h}}\left(\frac{h_1}{a}\right)^2$$

$$c_{10,7} = -\frac{\overline{L}_1}{12}\overline{\rho}\overline{h}^3\overline{\Omega} + \overline{L}_1 \kappa_x^2 \overline{h}\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_1 \overline{h}^3}{12}\left(\frac{am\pi}{h_1}\right)^2\left(\overline{B}_{11}^{(2)} - \frac{\overline{B}_{12}^{(2)^2}}{\overline{B}_{22}^{(2)}}\right) + \frac{\overline{G}_a \overline{L}_1 \overline{h}^2}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4$$

$$c_{10,9} = \overline{L}_1 \kappa_x^2 \overline{h}\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\overline{L}_1 \overline{h}\frac{m\pi}{2}\overline{G}_a}}\left(\frac{a}{h_1}\right)^3 \qquad c_{10,11} = -\overline{L}_1 m\pi \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}$$

$$c_{11,8} = -\frac{\overline{L}_1}{12}\overline{\rho}\overline{h}^3\overline{\Omega} + \frac{\overline{L}_1\overline{G}_a\overline{h}^2}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\frac{\overline{L}_1\overline{G}_a\overline{h}}{2}}}\left(\frac{a}{h_1}\right)^4 \qquad c_{11,10} = \overline{L}_1 m\pi$$

$$c_{11,12} = \overline{L}_1\left(\frac{a}{h_1}\right) + \overline{L}_1\left(\frac{a}{h_1}\right)\underline{\underline{\frac{\overline{G}_a}{2\kappa_y^2\overline{B}_{44}^{(2)}}}} \qquad c_{12,7} = \overline{L}_1\overline{h}\kappa_x^2\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$c_{12,9} = -\overline{L}_1\overline{\rho}\overline{h}\overline{\Omega} + \overline{L}_1\kappa_x^2\left(\frac{m\pi a}{h_1}\right)^2\overline{h}\overline{B}_{55}^{(2)} + \frac{\overline{L}_1\overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c_{10,1} = \frac{\overline{L}_1\overline{G}_a}{4\overline{h}_a}\overline{h}\left(\frac{a}{h_1}\right)^4$$

$$c_{11,2} = \frac{\overline{L}_1\overline{G}_a\overline{h}}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c_{12,3} = -\frac{\overline{L}_1\overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad (4.24\ \text{c})$$

For Part II region (or Single Layer Orthotropic Plate Adherend),

$$\frac{d}{d\xi_{II}}\begin{Bmatrix}\overline{\Psi}_{mnx}^{(1)}\\ \overline{\Psi}_{mny}^{(1)}\\ \overline{W}_{mn}^{(1)}\\ \overline{M}_{mnyx}^{(1)}\\ \overline{M}_{mny}^{(1)}\\ \overline{Q}_{mny}^{(1)}\end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0 & d_{1,2} & 0 & d_{1,4} & 0 & 0\\ d_{2,1} & 0 & 0 & 0 & d_{2,5} & 0\\ 0 & d_{3,2} & 0 & 0 & 0 & d_{3,6}\\ \hline d_{4,1} & 0 & d_{4,3} & 0 & d_{4,5} & 0\\ 0 & d_{5,2} & 0 & d_{5,4} & 0 & d_{5,6}\\ d_{6,1} & 0 & d_{6,3} & 0 & 0 & 0\end{array}\right]\begin{Bmatrix}\overline{\Psi}_{mnx}^{(1)}\\ \overline{\Psi}_{mny}^{(1)}\\ \overline{W}_{mn}^{(1)}\\ \overline{M}_{mnyx}^{(1)}\\ \overline{M}_{mny}^{(1)}\\ \overline{Q}_{mny}^{(1)}\end{Bmatrix},$$

$(0<\xi_{II}<1)$ (Upper Plate) (4.25 a)

where the elements of the above ‘‘Coefficient Matrix $\lfloor\mathcal{D}\rfloor$’’ are,

$$d_{1,2} = -\overline{L}_{II}m\pi \qquad d_{1,4} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(1)}}\left(\frac{h_1}{a}\right)^2 \qquad d_{2,1} = \overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}$$

$$d_{2,5} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(1)}}\left(\frac{h_1}{a}\right)^2 \qquad d_{3,2} = -\overline{L}_{II}\left(\frac{a}{h_1}\right) \qquad d_{3,6} = \frac{\overline{L}_{II}}{\kappa_y^2\overline{B}_{44}^{(1)}}\left(\frac{h_1}{a}\right)^2$$

$$d_{4,1} = -\frac{\overline{L}_{II}}{12}\overline{\Omega} + \overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_{II}}{12}\left(\frac{am\pi}{h_1}\right)^2\left(1 - \frac{\overline{B}_{12}^{(1)2}}{\overline{B}_{22}^{(1)}}\right)$$

$$d_{4,3} = \overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi \qquad d_{4,5} = -\overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}} \qquad d_{5,2} = -\frac{\overline{L}_{II}}{12}\overline{\Omega}$$

$$d_{5,4} = \overline{L}_{II}m\pi \qquad d_{5,6} = \overline{L}_{II}\left(\frac{a}{h_1}\right) \qquad d_{6,1} = \kappa_x^2\overline{L}_{II}\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$d_{6,3} = -\overline{L}_{II}\overline{\Omega} + \overline{L}_{II}\kappa_x^2\left(\frac{m\pi a}{h_1}\right)^2 \overline{B}_{55}^{(1)} \qquad (4.25\text{ b})$$

<u>For Part III region (or Single Layer Plate Adherend)</u>,

$$\frac{d}{d\xi_{III}}\begin{Bmatrix} \overline{\psi}_{mnx}^{(2)} \\ \overline{\psi}_{mny}^{(2)} \\ \overline{W}_{mn}^{(2)} \\ \overline{M}_{mnyx}^{(2)} \\ \overline{M}_{mny}^{(2)} \\ \overline{Q}_{mny}^{(2)} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0 & e_{1,2} & 0 & e_{1,4} & 0 & 0 \\ e_{2,1} & 0 & 0 & 0 & e_{2,5} & 0 \\ 0 & e_{3,2} & 0 & 0 & 0 & e_{3,6} \\ \hline e_{4,1} & 0 & e_{4,3} & 0 & e_{4,5} & 0 \\ 0 & e_{5,2} & 0 & e_{5,4} & 0 & e_{5,6} \\ e_{6,1} & 0 & e_{6,3} & 0 & 0 & 0 \end{array}\right]\begin{Bmatrix} \overline{\psi}_{mnx}^{(2)} \\ \overline{\psi}_{mny}^{(2)} \\ \overline{W}_{mn}^{(2)} \\ \overline{M}_{mnyx}^{(2)} \\ \overline{M}_{mny}^{(2)} \\ \overline{Q}_{mny}^{(2)} \end{Bmatrix},$$

$(0<\xi_{III}<1)$ (Lower Plate) (4.26.a)

where the elements of the above ''Coefficient Matrix $\lfloor\mathcal{E}\rfloor$'' are,

$$e_{1,2} = -\overline{L}_{III}m\pi \qquad e_{1,4} = \frac{12\overline{L}_{III}}{\overline{B}_{66}^{(2)}\overline{h}^3}\left(\frac{h_1}{a}\right)^2 \qquad e_{2,1} = \overline{L}_{III}m\pi\frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}$$

$$e_{2,5} = \frac{12\overline{L}_{III}}{\overline{B}_{22}^{(2)}\overline{h}^3}\left(\frac{h_1}{a}\right)^2 \qquad e_{3,2} = -\overline{L}_{III}\left(\frac{a}{h_1}\right) \qquad e_{3,6} = \frac{\overline{L}_{III}}{\kappa_y^2\overline{B}_{44}^{(2)}\overline{h}}\left(\frac{h_1}{a}\right)^2$$

$$e_{4,1} = -\frac{\overline{L}_{III}}{12}\overline{\rho}_2\overline{h}_2^3\overline{\Omega} + \overline{L}_{III}\kappa_x^2\overline{h}\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_{III}\overline{h}_2^3}{12}\left(\frac{am\pi}{h_1}\right)^2\left(\overline{B}_{11}^{(2)} - \frac{\overline{B}_{12}^{(2)^2}}{\overline{B}_{22}^{(2)}}\right)$$

$$e_{4,3} = \overline{L}_{III}\kappa_x^2\overline{h}\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^3 m\pi \qquad e_{4,5} = -\overline{L}_{III}m\pi\frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}} \qquad e_{5,2} = -\frac{\overline{L}_{III}}{12}\overline{\rho}_2\overline{h}_2^3\overline{\Omega}$$

$$e_{5,4} = \overline{L}_{III}m\pi \qquad c_{5,6} = \overline{L}_{III}\left(\frac{a}{h_1}\right) \qquad e_{6,1} = \overline{L}_{III}\overline{h}\kappa_x^2\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$c_{6,3} = -\overline{L}_{III}\overline{\rho}_2\overline{h}_2\overline{\Omega} + \overline{L}_{III}\kappa_x^2\left(\frac{m\pi a}{h_1}\right)^2\overline{h}\overline{B}_{55}^{(2)} \qquad (4.26.\text{b})$$

The "Boundary Conditions" at x=0 and x=a are already satisfied by trigonometric expansion in ''Classical Lévy's Type Solution''. The ''Appropriate Boundary Conditions'' and the ''Continuity Conditions'' are needed to solve the ''Governing System of First Order Ordinary Differential Equations''. Then,

The ''Boundary Conditions'' along the edges in the y-direction,

F (Free): $\overline{M}_{yx}^{(j)} = \overline{M}_{y}^{(j)} = \overline{Q}_{y}^{(j)} = 0$

C (Clamped): $\overline{w}^{(j)} = \overline{\psi}_{x}^{(j)} = \overline{\psi}_{y}^{(j)} = 0 \qquad (j=1,2) \qquad (4.27)$

S (Simply Supported): $\overline{w}^{(j)} = \overline{\psi}_{x}^{(j)} = \overline{M}_{y}^{(j)} = 0$

The ''Continuity Conditions'' between Part I and Part II,

$$\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=1}\right\} = \left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{I}=0}\right\} \tag{4.28}$$

The ''Continuity Conditions'' between Part I and Part III,

$$\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{III}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{I}=1}\right\} \tag{4.29}$$

Finally, as a summary, the entire set of the "Governing System of First Order Ordinary Differential Equations" for "Main PROBLEM I" is given as,

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{mn}\\ \overline{\boldsymbol{Y}}^{(2)}_{mn}\end{Bmatrix} = \begin{bmatrix}\overline{\mathcal{C}}_{1,1} & \overline{\mathcal{C}}_{1,2}\\ \mathcal{C}_{2,1} & \mathcal{C}_{2,2}\end{bmatrix}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{mn}\\ \overline{\boldsymbol{Y}}^{(2)}_{mn}\end{Bmatrix}, \quad \begin{matrix}(0<\xi_I<1)\ (\text{UpperPlate})\\ (0<\xi_I<1)\ (\text{LowerPlate})\end{matrix} \quad (\text{in Part I})$$

$$\frac{d}{d\xi_{II}}\left\{\overline{\boldsymbol{Y}}^{(1)}_{mn}\right\} = \left[\mathcal{D}\right]\left\{\overline{\boldsymbol{Y}}^{(1)}_{mn}\right\}, \qquad (0<\xi_{II}<1) \qquad (\text{Upper Plate}) \qquad (\text{in Part II})$$

$$\frac{d}{d\xi_{III}}\left\{\overline{\boldsymbol{Y}}^{(2)}_{mn}\right\} = \left[\mathcal{E}\right]\left\{\overline{\boldsymbol{Y}}^{(2)}_{mn}\right\}, \qquad (0<\xi_{III}<1) \qquad (\text{Lower Plate}) \qquad (\text{in Part III})$$

(4.30 a,b,c)

with the "Appropriate Boundary Conditions" and the "Continuity Conditions" in each Part I, Part II Regions respectively.

The above entire system of equations forms a "Two-Point Boundary Value Problem" for the "Main PROBLEM I" between the left and the right supports in the y-direction. It is obvious that, once the natural frequencies are obtained, then, the Equations (4.30.a,b,c) can be integrated numerically for a given particular geometry, materials and the support conditions by making use of “Modified Transfer Matrix Method (with Interpolation Polynomials and/or Chebyshev Polynomials)”.

CHAPTER 5

("Main PROBLEM II")__FREE FLEXURAL (or BENDING) VIBRATIONS of COMPOSITE ORTHOTROPIC or ISOTROPIC PLATES with a BONDED SYMMETRIC SINGLE LAP JOINT (or SYMMETRIC DOUBLER JOINT)

In this section, the "Governing System of Coupled Ordinary Differential Equations" will be presented in the compact matrix or the "state vector" form for the "Overlap Region" (or Part I and Part II regions), and for "Single Layer Regions" (or Part III and Part IV regions) for "Main PROBLEM II" without making any distinction for "Main PROBLEM II a " and "Main PROBLEM II b".

5.1 Statement of "Main PROBLEM II a"

Figure 3.4.a shows the general configuration, the geometry and the coordinate system of the "Composite Orthotropic or Isotropic Plate System with a Centrally Bonded Symmetric Single Lap Joint". This system composed of an "Orthotropic Doubler" and "Orthotropic or Isotropic Adherends" bonded centrally by a relatively very thin elastic adhesive layer.

5.2 Statement of "Main PROBLEM IIb"

Figure 3.5.a shows the general configuration, the geometry and the coordinate system of the "Composite Orthotropic or Isotropic Plate System with Non-Centrally (or eccentrically) Bonded Symmetric Single Lap Joint". This system composed of a "Orthotropic Doubler" and "Orthotropic or Isotropic Adherends" bonded non-centrally by a relatively very thin elastic adhesive layer.

5.3 Main assumptions and Analytical Modeling

The analytical formulation of "Main PROBLEM II" is based on the following assumptions;

(i) The analysis is carried out only for the free flexural (or bending) vibrations of the plate system. The in-plane or extensional moments of inertia are neglected, but the rotatory and the transverse moments of inertia of the plates are included in the formulation.

(ii) The rotatory and the transverse moments of inertia of the adherends as well as their transverse shear deformations are taken into account in the sense of the "Mindlin Plate Theory".

(iii) There is no slip and separation on the interfaces of the adhesive layers and plates.

(iv) Since the thickness of the adhesive layers are very small relative to the thickness of the plates, the inertias and the masses of the adhesive layers are neglected, and both adhesive normal and shear stresses are to be constant across the thickness.

(v) The damping effects in Mindlin Plates and in the adhesive layers are neglected.

(vi) The plates are assumed to be simply supported along edges x=0 and x=a while arbitrary support conditions may be specified in the y-direction.

(vii) The coordinate system of each plate is attached to its medium plane or the reference plane.

(viii) The principal directions of orthotropy in plates are parallel to the edges and to the coordinates as shown in Figure 3.4.a and in Figure 3.5.a.

(ix) It is assumed that the following relations exist between the deformations of the plate adherends and doublers in z-direction.

For Part I;	For Part II;
$w^{(2)} > w^{(1)}$	$w^{(3)} > w^{(1)}$

For the general formulation of the problem, the entire "Composite Bonded Plate or Panel System" is divided into four parts, namely, Part I, Part II, Part III and Part IV in the y-direction as shown in Figure 3.4.b and in Figure 3.5.b. Part I corresponds to the "Overlap Region" which contains two plates 1,2, Part II corresponds to the "Overlap Region" which contains two plates 1,3 and Part II and Part III correspond the continuation of the lower plates 2 and 3 respectively, as a single plate in the y-direction.

5.4 Theoretical Formulation of "Main PROBLEM II" (Theoretical Analysis)

5.4.1 Analysis of Adhesive Layer in the "Overlap Region"

The system in "Overlap Region" (or Part I and Part II region) is composed of two plates which are adhesively bonded by very thin and elastic adhesive layers. The stresses at the upper and lower faces of the plates due to the adhesive layers are considered as external surface stresses or loads on the plates. The adhesive stresses should be related to the unknown displacement functions and angles of rotations of the adherends in the "Overlap Region". Figures 5.1 and Figure 5.2 show an exaggerated view of the deformations of an infinitesimal element of two-layer plate system. The positive sign convention for displacements, stress resultants and angles of rotation are also shown in the same figures. (see also Figures 3.1 and 5.1 and 5.2).

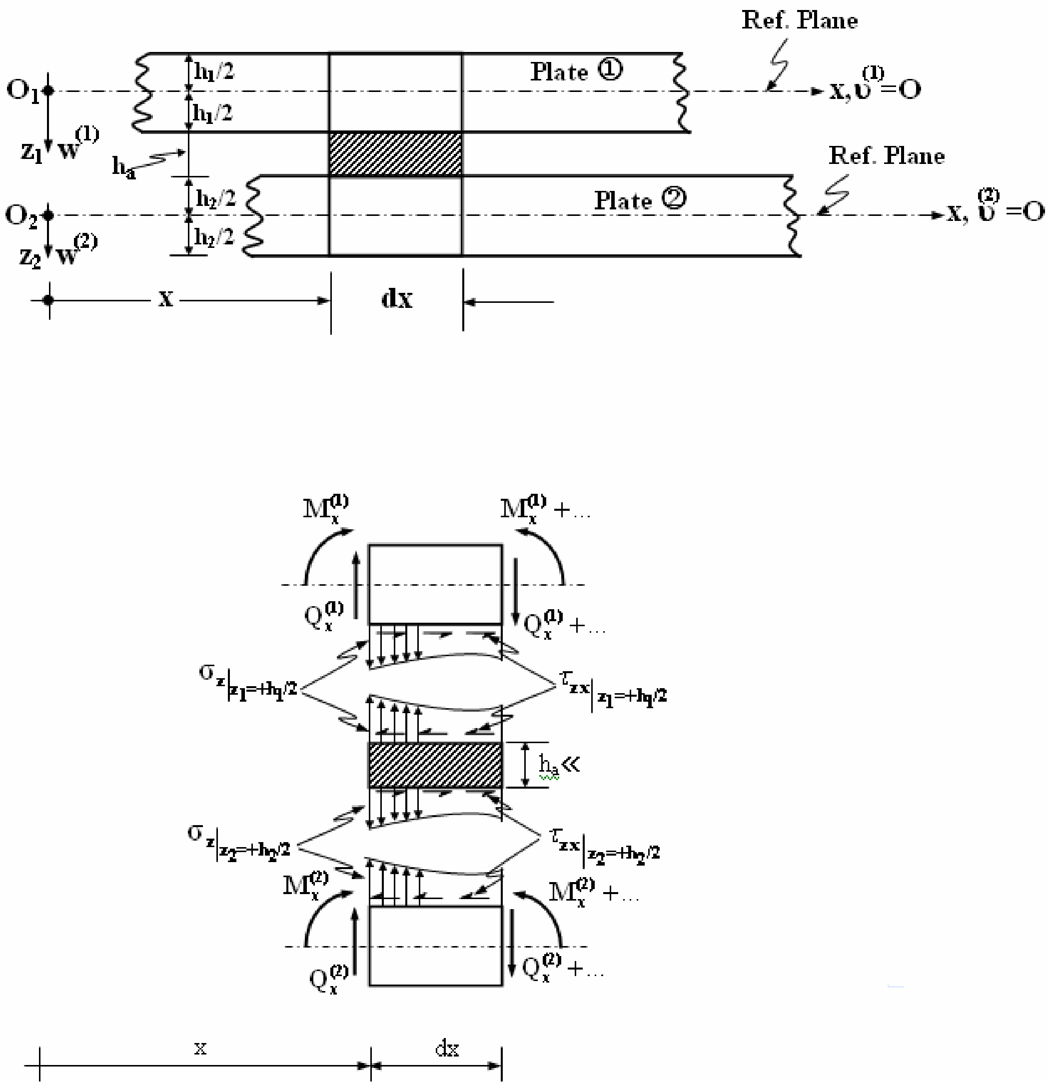


Figure 5.1 Stress Distributions at Plate Adhesive Layer Interfaces in the “Overlap Region” for Part I

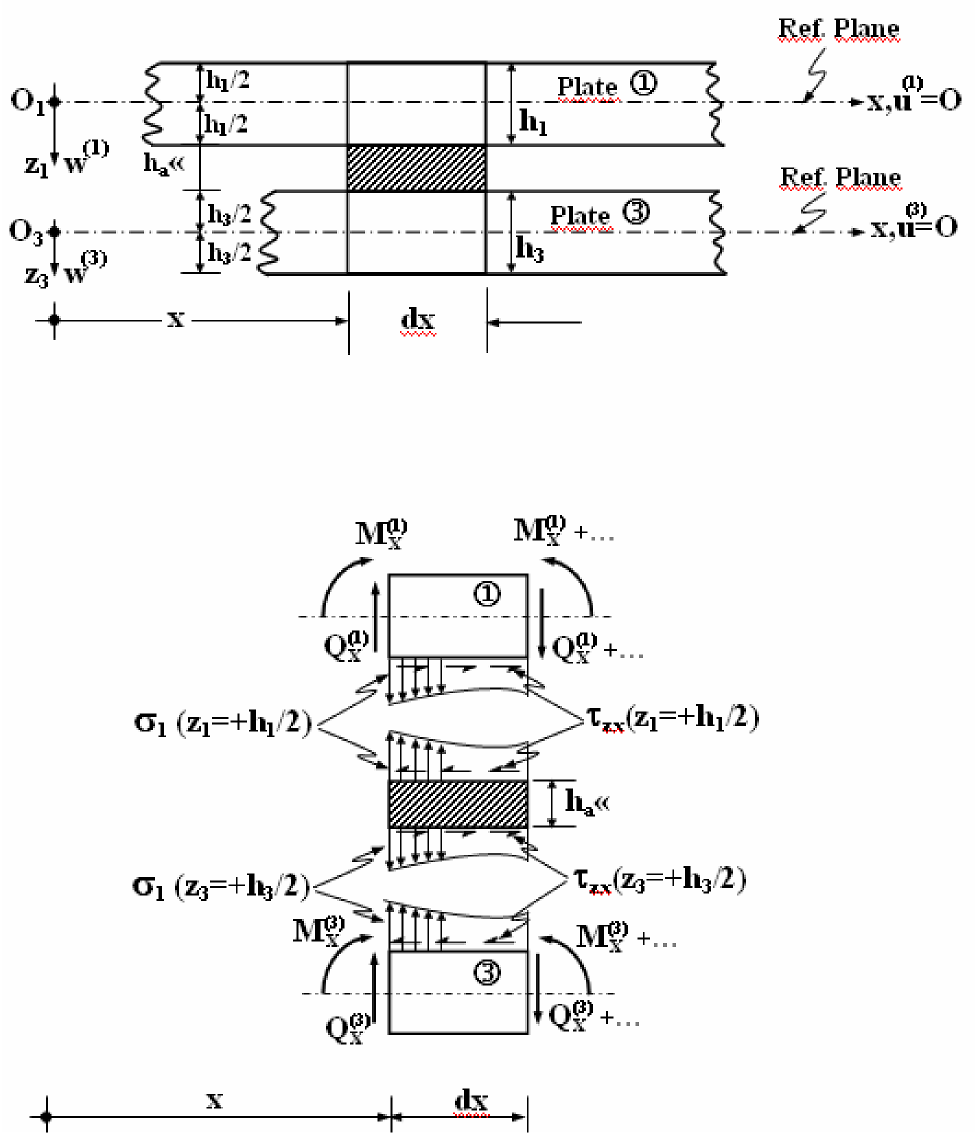


Figure 5.2 Stress Distributions at Plate Adhesive Layer Interfaces in the "Overlap Region" for Part II

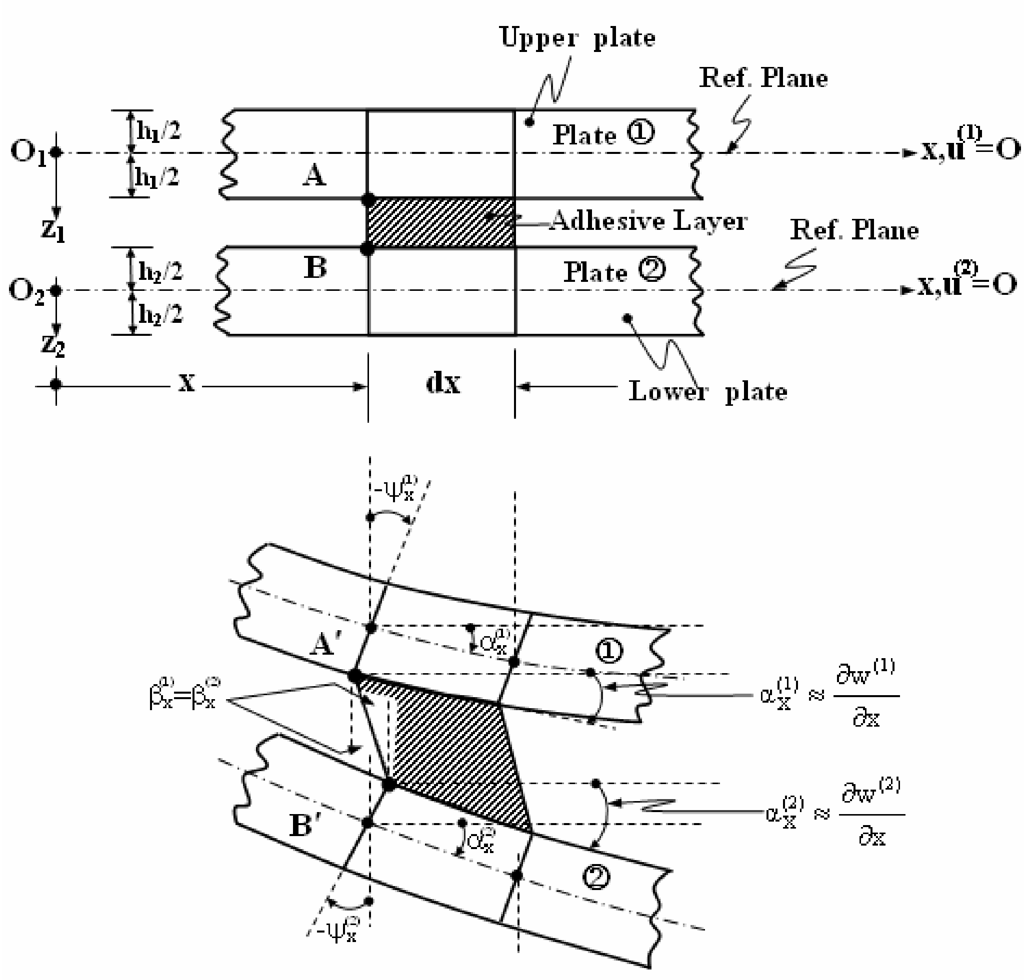


Figure 5.3 Deformations of Adherends (Mindlin Plates) and in-between Adhesive Layers in the "Overlap Region" for Part I

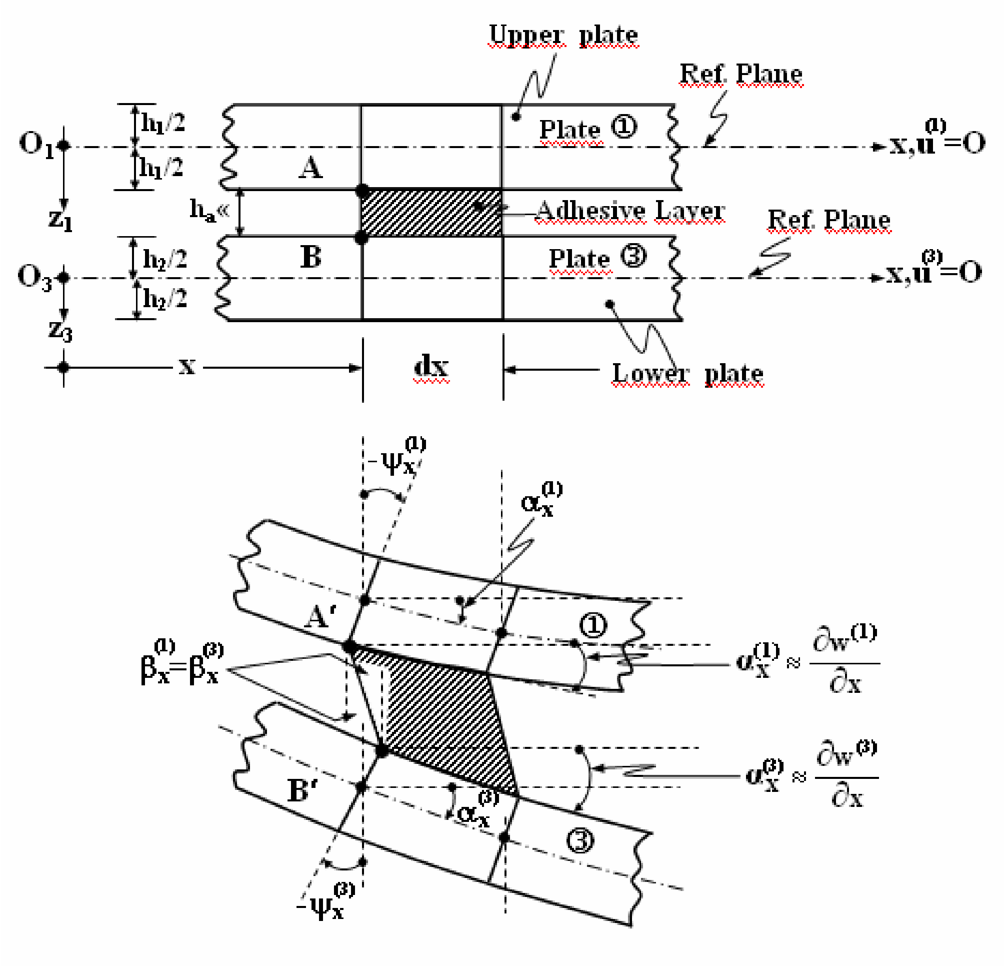


Figure 5.4 Deformations of Adherends (Mindlin Plates) and in-between Adhesive Layers in the "Overlap Region" for Part II

The purpose of this section is to obtain the relations between the stresses in the adhesive layers and the plate interfaces and displacements of the adherends. The shear strains $\gamma_{xz}^{(j)}$ and $\gamma_{yz}^{(j)}$ at interfaces are found by considering the distortion in the right angles of the adhesive infinitesimal element after deformation as,

$$\left| \begin{array}{ll} \gamma_{xz}^{(j)} = \alpha_x^{(j)} + \beta_x^{(j)} & (\mathrm{j}=1,2 \quad \text{for Part I}) \\ \gamma_{yz}^{(j)} = \alpha_y^{(j)} + \beta_y^{(j)} & (\mathrm{j}=1,3 \quad \text{for Part II}) \end{array} \right. \qquad (5.1)$$

with the assumption of small deformations, $\alpha^{(j)}$ and $\beta^{(j)}$ can be expressed as the slopes of the adherends and the displacements in x and y directions, respectively. That is,

<u>Between Plates 1 and 2 for Part I,</u>

$$\tan\left(\alpha_s^{(j)}\right) = \frac{\partial w^{(j)}}{\partial s} \approx \alpha_s^{(j)}$$

$$\tan\left(\beta_s^{(j)}\right) = \frac{\left(u_{B'}\right)_s + \left(u_{A'}\right)_s}{h_a} \approx \beta_s^{(j)} \qquad (\mathrm{j}=1,2) \qquad (5.2.a)$$

<u>Between Plates 1 and 3 for Part II;</u>

$$\tan\left(\alpha_s^{(j)}\right) = \frac{\partial w^{(j)}}{\partial s} \approx \alpha_s^{(j)}$$

$$\tan\left(\beta_s^{(j)}\right) = \frac{\left(u_{B'}\right)_s + \left(u_{A'}\right)_s}{h_a} \approx \beta_s^{(j)} \qquad (\mathrm{j}=1,3) \qquad (5.2.b)$$

where h_a is the thicknesses of the upper and lower adhesive layers, respectively and $(u_A)_s$, $(u_B)_s$ are axial deformation of points A, B in s direction respectively. Axial deformations $(u_A)_s$, $(u_B)_s$ are caused only by the bending of the plates, then the displacement components can be written as,

<u>For Part I;</u>

$$\left(u_{A'}\right)_s = -\frac{h_1}{2}\psi_s^{(1)}$$

$$\left(u_{B'}\right)_s = -\frac{h2}{2}\psi_s^{(2)}$$

<u>For Part II;</u>

$$\left(u_{A'}\right)_s = -\frac{h_1}{2}\psi_s^{(1)}$$

$$\left(u_{B'}\right)_s = -\frac{h3}{2}\psi_s^{(3)} \qquad (5.3)$$

The shear strains in the interfaces can be expressed in terms of displacements and angles of rotation of the adherends by using (5.1) through (5.3)

Adhesive Strains Between Plates 1 and 2 for Part I

For Part I;

$$\gamma_{xz}^{(j)} = -\frac{1}{2h_a}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(2)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial x}}}$$

$$\gamma_{yz}^{(j)} = -\frac{1}{2h_a}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(2)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial y}}} \qquad (j=1,2) \qquad (5.4.a)$$

$$\varepsilon_z^{(j)} = w^{(2)} - w^{(1)}$$

Adhesive Strains Between Plates 1 and 3 for Part II

For Part II;

$$\gamma_{xz}^{(j)} = -\frac{1}{2h_a}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(3)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial x}}}$$

$$\gamma_{yz}^{(j)} = -\frac{1}{2h_a}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(3)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial y}}} \qquad (j=1,3) \qquad (5.4\ b)$$

$$\varepsilon_z^{(j)} = w^{(3)} - w^{(1)}$$

It should be mentioned here that in some previous studies by Yuceoglu and Özerciyes [IV.4-IV.13], the double underlined terms (=) are neglected. In this present study these terms are not neglected in the theoretical formulation.

The interface shear stresses in the adhesive layers can be expressed by using the interface shear strains as,

Adhesive Stresses Between Plates 1 and 2 for Part I,

For Part I;

$$\tau_{xz}^{(j)} = G_a \gamma_{xz}^{(j)} = -\frac{G_a}{2h_a}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(2)}\right) - G_a \frac{\partial w^{(j)}}{\partial x}$$

$$\tau_{yz}^{(j)} = G_a \gamma_{yz}^{(j)} = -\frac{G_a}{2h_a}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(2)}\right) - G_a \frac{\partial w^{(j)}}{\partial y} \qquad (j=1,2) \qquad (5.5.a)$$

$$\sigma_z^{(j)} = \frac{E_a}{h_a}(w^{(2)} - w^{(1)})$$

Adhesive Stresses Between Plates 1 and 3 for Part II

For Part II;

$$\tau_{xz}^{(j)} = G_a \gamma_{xz}^{(j)} = -\frac{G_a}{2h_a}\left(h_1\psi_x^{(1)} + h_3\psi_x^{(3)}\right) - G_a \frac{\partial w^{(j)}}{\partial x}$$

$$\tau_{yz}^{(j)} = G_a \gamma_{yz}^{(j)} = -\frac{G_a}{2h_a}\left(h_1\psi_y^{(1)} + h_3\psi_y^{(3)}\right) - G_a \frac{\partial w^{(j)}}{\partial y} \qquad (j=1,3) \qquad (5.5.b)$$

$$\sigma_z^{(j)} = \frac{E_a}{h_a}(w^{(3)} - w^{(1)})$$

The thicknesses h_a of the adhesive layers are assumed to be much smaller than the thicknesses of the plates and $\psi_x^{(j)}, \psi_y^{(j)}$ have the same order of magnitudes when compared with $\frac{\partial w^{(j)}}{\partial x}$ and $\frac{\partial w^{(j)}}{\partial y}$. Therefore, one may assume that the variation of the transverse displacements in the adhesive layers is linear that is equivalent to assuming that the normal strain ε_z is constant across the thickness. E_a is the modulus of elasticity (or Young's modulus) and G_a is the modulus of rigidity (or shear modulus) of the adhesive layer.

5.4.2 Analysis of Part I and Part II (or the "Overlap Region") of Composite Plate System

5.4.2.1 Implementation of the Adhesive Layer Equations to Governing Equations (equation of motions) of Plate Adherends.

The adhesive stresses are related to unknown displacement functions and angle of rotations of the adherends in the "Overlap Region" since they are considered as surface loads or external stresses acting on the upper and lower faces of the plates. The normal and tangential stresses at the interface may be considered as the ‘‘compatibility or coupling conditions’’ of the two plates in the "Overlap Region" of Part and Part II.

The governing equations (or plate equations of motion) given in (3.19) and (3.20) can be written by using the adhesive stresses as load terms q_{zx}'s and q_{zy}'s as,

<u>For Plate 1 in Part I;</u>

$$\left|\begin{aligned}
&\frac{\partial M_x^{(1)}}{\partial x}+\frac{\partial M_{yx}^{(1)}}{\partial y}-Q_x^{(1)}-\underline{\frac{h_1G_a}{4h_a}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)}-\underline{\underline{\frac{h_1G_a}{2}\frac{\partial w^{(1)}}{\partial x}}}=\frac{\rho_1h_1^3}{12}\frac{\partial^2\psi_x^{(1)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(1)}}{\partial x}+\frac{\partial M_y^{(1)}}{\partial y}-Q_y^{(1)}-\underline{\frac{h_1G_a}{4h_a}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)}-\underline{\underline{\frac{h_1G_a}{2}\frac{\partial w^{(1)}}{\partial y}}}=\frac{\rho_1h_1^3}{12}\frac{\partial^2\psi_y^{(1)}}{\partial t^2}\\
&\frac{\partial Q_x^{(1)}}{\partial x}+\frac{\partial Q_y^{(1)}}{\partial y}+\underline{\frac{E_a}{h_a}\left(w^{(2)}-w^{(1)}\right)}=\rho_1h_1\frac{\partial^2w^{(1)}}{\partial t^2}
\end{aligned}\right. \qquad (5.6.a)$$

<u>For Plate 1 in Part II;</u>

$$\left|\begin{aligned}
&\frac{\partial M_x^{(1)}}{\partial x}+\frac{\partial M_{yx}^{(1)}}{\partial y}-Q_x^{(1)}-\underline{\frac{h_1G_a}{4h_a}\left(h_1\psi_x^{(1)}+h_3\psi_x^{(3)}\right)}-\underline{\underline{\frac{h_1G_a}{2}\frac{\partial w^{(1)}}{\partial x}}}=\frac{\rho_1h_1^3}{12}\frac{\partial^2\psi_x^{(1)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(1)}}{\partial x}+\frac{\partial M_y^{(1)}}{\partial y}-Q_y^{(1)}-\underline{\frac{h_1G_a}{4h_a}\left(h_1\psi_y^{(1)}+h_3\psi_y^{(3)}\right)}-\underline{\underline{\frac{h_1G_a}{2}\frac{\partial w^{(1)}}{\partial y}}}=\frac{\rho_1h_1^3}{12}\frac{\partial^2\psi_y^{(1)}}{\partial t^2}\\
&\frac{\partial Q_x^{(1)}}{\partial x}+\frac{\partial Q_y^{(1)}}{\partial y}+\underline{\frac{E_a}{h_a}\left(w^{(3)}-w^{(1)}\right)}=\rho_1h_1\frac{\partial^2w^{(1)}}{\partial t^2}
\end{aligned}\right. \qquad (5.6.b)$$

For Plate 2 in Part I;

$$\left|\begin{array}{l}
\dfrac{\partial M_x^{(2)}}{\partial x}+\dfrac{\partial M_{yx}^{(2)}}{\partial y}-Q_x^{(2)}-\underline{\dfrac{h_2 G_a}{4h_a}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)}-\underline{\underline{\dfrac{h_2 G_a}{2}\dfrac{\partial w^{(2)}}{\partial x}}}=\dfrac{\rho_2 h_2^3}{12}\dfrac{\partial^2\psi_x^{(2)}}{\partial t^2} \\
\dfrac{\partial M_{yx}^{(2)}}{\partial x}+\dfrac{\partial M_y^{(2)}}{\partial y}-Q_y^{(2)}-\underline{\dfrac{h_2 G_a}{4h_a}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)}-\underline{\underline{\dfrac{h_2 G_a}{2}\dfrac{\partial w^{(2)}}{\partial y}}}=\dfrac{\rho_2 h_2^3}{12}\dfrac{\partial^2\psi_y^{(2)}}{\partial t^2} \\
\dfrac{\partial Q_x^{(2)}}{\partial x}+\dfrac{\partial Q_y^{(2)}}{\partial y}-\underline{\dfrac{E_a}{h_a}\left(w^{(2)}-w^{(1)}\right)}=\rho_2 h_2\dfrac{\partial^2 w^{(2)}}{\partial t^2}
\end{array}\right. \qquad (5.6\text{ c})$$

For Plate 3 in Part II;

$$\left|\begin{array}{l}
\dfrac{\partial M_x^{(3)}}{\partial x}+\dfrac{\partial M_{yx}^{(3)}}{\partial y}-Q_x^{(3)}-\underline{\dfrac{h_3 G_a}{4h_a}\left(h_1\psi_x^{(1)}+h_3\psi_x^{(3)}\right)}-\underline{\underline{\dfrac{h_3 G_a}{2}\dfrac{\partial w^{(3)}}{\partial x}}}=\dfrac{\rho_3 h_3^3}{12}\dfrac{\partial^2\psi_x^{(3)}}{\partial t^2} \\
\dfrac{\partial M_{yx}^{(3)}}{\partial x}+\dfrac{\partial M_y^{(3)}}{\partial y}-Q_y^{(3)}-\underline{\dfrac{h_3 G_a}{4h_a}\left(h_1\psi_y^{(1)}+h_3\psi_y^{(3)}\right)}-\underline{\underline{\dfrac{h_3 G_a}{2}\dfrac{\partial w^{(3)}}{\partial y}}}=\dfrac{\rho_3 h_3^3}{12}\dfrac{\partial^2\psi_y^{(3)}}{\partial t^2} \\
\dfrac{\partial Q_x^{(3)}}{\partial x}+\dfrac{\partial Q_y^{(3)}}{\partial y}-\underline{\dfrac{E_a}{h_a}\left(w^{(3)}-w^{(1)}\right)}=\rho_3 h_3\dfrac{\partial^2 w^{(3)}}{\partial t^2}
\end{array}\right. \qquad (5.6\text{ d})$$

In the above equations, the “underlined terms” are the “coupling terms” between the adherends and the doubler due to in-between adhesive layers.

At this stage, the “Classical Levy’s Method” with the trigonometric series expansions is to be considered. The “Classical Lévy’s Method” is restricted to rectangular plates with any two opposite edges simply supported. However, the other two edges may have arbitrary boundary conditions.

In the present study, the edges at x=0 and x=a are simply supported. The “Boundary Conditions” which have to be satisfied along these edges are as follows,

$$\text{at } x=0,a \;\rightarrow\; w^{(j)}=0,\; M_x^{(j)}=0,\; \psi_y^{(j)}=0 \qquad (j=1,2,3) \qquad (5.7)$$

In the “Classical Lévy’s Method”, the angles of rotations and the displacements of the adherends can be expressed as, Displacements and Angles of Rotation for Part I,

$$w^{(j)}(\eta,\xi_I,t)=h_I\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{W}_{mn}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$\psi_x^{(j)}(\eta,\xi_I,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Psi}_{mnx}^{(j)}(\xi_I)\cos(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad \text{(j=1,2)} \qquad (5.8)$$

$$\psi_y^{(j)}(\eta,\xi_I,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Psi}_{mny}^{(j)}(\xi_I)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

Displacements and Angles of Rotation, for Part II,

$$w^{(j)}(\eta,\xi_{II},t)=h_I\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{W}_{mn}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$\psi_x^{(j)}(\eta,\xi_{II},t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Phi}_{mnx}^{(j)}(\xi_{II})cos(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad \text{(j=1,3)} \qquad (5.9)$$

$$\psi_y^{(j)}(\eta,\xi_{II},t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Phi}_{mny}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

where "m" is the number of half-waves in the x-direction, the superscript (j) denotes adherend plates for j=2 and j=3, respectively and the doubler for j=1. In the above equations, “i” is defined as $\sqrt{-1}$ and the “barred” (—) quantities are the dimensionless transverse displacements and angles of rotation. The nondimensional independent space variables η, ξ_I and ξ_{II} are defined as x/a, y_I/ℓ_I, and y_{II}/ℓ_{II} respectively. And, "$\overline{\omega}_{mn}$" is the dimensionless circular frequency or the natural frequency of the flexural (or bending) vibrations of the entire composite bonded plate or panel system.

The “Stress Resultants” can also be expressed in trigonometric series in the x-direction as, Stress Resultants for Part I;

$$M_x^{(j)}(\eta,\xi_1,t)=\frac{h_1^5 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnx}^{(j)}(\xi_1)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$M_{yx}^{(j)}(\eta,\xi_1,t)=\frac{h_1^5 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnyx}^{(j)}(\xi_1)\cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$M_y^{(j)}(\eta,\xi_1,t)=\frac{h_1^5 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mny}^{(j)}(\xi_1)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}\qquad (j=1,2)\qquad (5.10)$$

$$Q_y^{(j)}(\eta,\xi_1,t)=\frac{h_1^4 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mny}^{(j)}(\xi_1)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$Q_x^{(j)}(\eta,\xi_1,t)=\frac{h_1^4 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mnx}^{(j)}(\xi_1)\cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

Stress Resultants for Part II;

$$M_x^{(j)}(\eta,\xi_1,t)=\frac{h_1^5 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnx}^{(j)}(\xi_1)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$M_{yx}^{(j)}(\eta,\xi_1,t)=\frac{h_1^5 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnyx}^{(j)}(\xi_1)\cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$M_y^{(j)}(\eta,\xi_1,t)=\frac{h_1^5 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mny}^{(j)}(\xi_1)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}\qquad (j=1,3)\qquad (5.11)$$

$$Q_y^{(j)}(\eta,\xi_1,t)=\frac{h_1^4 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mny}^{(j)}(\xi_1)\sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$Q_x^{(j)}(\eta,\xi_1,t)=\frac{h_1^4 B_{11}^{(1)}}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mnx}^{(j)}(\xi_1)\cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

where the "barred" (—) $\overline{W}_{mn}^{(j)},\overline{\Psi}_{mnx}^{(j)},\overline{\Psi}_{mny}^{(j)},\overline{M}_{mnx}^{(j)},\overline{M}_{mnyx}^{(j)},\overline{M}_{mny}^{(j)},\overline{Q}_{mny}^{(j)},\overline{Q}_{mnx}^{(j)}$ are the "dimensionless fundamental dependent variables" which will appear in the "state vectors" of the problem under consideration.

For non-dimensionalization of equations, the parameters in the governing differential equations shall be nondimensionalized with respect to the main or reference quantities which are chosen as $"B_{11}^{(1)}"$, $"h_1"$, $"\rho_1"$ and "a".

The dimensionless coordinates or independent space variables are,

$\eta = x/a$,

$\xi_I = y_I/l_I$, $\xi_{II} = y_{II}/l_{II}$, $\xi_{III} = y_{III}/l_{III}$ $\xi_{IV} = y_{IV}/l_{IV}$ (5.12)

The dimensionless parameters related to orthotropic elastic constants of the plates and the adhesive layers are,

$$\overline{B}_{ik}^{(j)} = B_{ik}^{(j)} \Big/ B_{11}^{(1)} \qquad (j = 1,2,3 \;\; and \; i,k = 1,2,3),$$
$$\overline{B}_{\ell\ell}^{(j)} = B_{\ell\ell}^{(j)} \Big/ B_{11}^{(1)} \qquad (j = 1,2,3 \;\; and \; \ell = 4,5,6) \qquad (5.13)$$
$$\overline{G}_a = G_a \Big/ B_{11}^{(1)}, \quad \overline{E}_a = E_a \Big/ B_{11}^{(1)}$$

The dimensionless parameters related to the densities and the geometry of the plates and adhesive layers are,

$$\overline{\rho}_2 = \rho_2 \big/ \rho_1, \quad \overline{\rho}_3 = \rho_3 \big/ \rho_1,$$
$$\overline{h}_a = h_a \big/ h_1, \quad \overline{h}_2 = h_2 \big/ h_1, \quad \overline{h}_3 = h_3 \big/ h_1, \qquad (5.14)$$
$$\overline{L}_I = \ell_I \big/ a, \quad \overline{L}_{II} = \ell_{II} \big/ a, \quad \overline{L}_{III} = \ell_{III} \big/ a, \quad \overline{L}_{IV} = \ell_{IV} \big/ a$$

The dimensionless frequency parameter $\overline{\omega}_{mn}$ of the entire doubly stiffened, composite bonded plate or panel system is;

$$\overline{\omega}_{mn} = \rho_1 a^4 \omega_{mn}^2 \big/ h_1^2 B_{11}^{(1)} \qquad (m,n=1,2,3...) \qquad (5.15)$$
$$\overline{\Omega} = \overline{\omega}_{mn}$$

where the dimensionless natural frequency parameter $\overline{\omega}_{mn}$ depending on the given m, n values is ordered in terms of its magnitudes as $\overline{\Omega}_1 < \overline{\Omega}_2 < \overline{\Omega}_3 <$.... with the subscript indicating the first, second, third, dimensionless natural frequencies.

5.4.2.2 Reduction to "Governing Systems of First Order Ordinary Differential Equations" for "Main PROBLEM I",

Here, M_y, M_{xy},Q_y, ψ_x, ψ_y and w are chosen as intrinsic variables and M_x,Q_x are chosen as auxiliary variables, Then, from the Stress Resultants and Moment Resultant Equations given in (3.20), first order partial differential equations can be written with respect to the dimensionless independent variables ξ and η as,

For the "Overlap Region" or Part I;

$$\begin{aligned}
\frac{1}{l_I}\frac{\partial \psi_y^{(j)}}{\partial \xi_I} &= \frac{1}{B_{22}^{(j)}}\left(\frac{12}{h_j^3}M_y^{(j)} - B_{12}^{(j)}\frac{1}{a}\frac{\partial \psi_x^{(j)}}{\partial \eta}\right)\\
\frac{1}{l_I}\frac{\partial \psi_x^{(j)}}{\partial \xi_I} &= \frac{12}{h_j^{(3)}B_{66}^{(j)}}M_{yx}^{(j)} - \frac{1}{a}\frac{\partial \psi_y^{(j)}}{\partial \eta}\\
\frac{1}{l_I}\frac{\partial w^{(j)}}{\partial \xi_I} &= \frac{1}{\kappa_y^2 h_j B_{44}^{(j)}}Q_y^{(j)} - \psi_y^{(j)}\\
\frac{1}{l_I}\frac{\partial M_{yx}^{(j)}}{\partial \xi_I} &= \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_x^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_x^{(j)}}{\partial \eta} + Q_x^{(j)} - \frac{h_j}{2}\left(q_{zx}^{(+)} + q_{zx}^{(-)}\right)\\
\frac{1}{l_\iota}\frac{\partial M_y^{(j)}}{\partial \xi_I} &= \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_y^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_{yx}^{(j)}}{\partial \eta} + Q_y^{(j)} - \frac{h_j}{2}\left(q_{zy}^{(+)} + q_{zy}^{(-)}\right)\\
\frac{1}{l_I}\frac{\partial Q_y^{(j)}}{\partial \xi_I} &= \rho_j h_j\frac{\partial^2 w^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial Q_x^{(j)}}{\partial \eta} - \left(q_z^{(+)} - q_z^{(-)}\right)
\end{aligned} \qquad (j=1,2) \quad (5.16)$$

For the "Overlap Region" or Part II;

$$\begin{aligned}
\frac{1}{l_{II}}\frac{\partial \psi_y^{(j)}}{\partial \xi_{II}} &= \frac{1}{B_{22}^{(j)}}\left(\frac{12}{h_j^3}M_y^{(j)} - B_{12}^{(j)}\frac{1}{a}\frac{\partial \psi_x^{(j)}}{\partial \eta}\right)\\
\frac{1}{l_{II}}\frac{\partial \psi_x^{(j)}}{\partial \xi_{II}} &= \frac{12}{h_j^{(3)}B_{66}^{(j)}}M_{yx}^{(j)} - \frac{1}{a}\frac{\partial \psi_y^{(j)}}{\partial \eta}\\
\frac{1}{l_{II}}\frac{\partial w^{(j)}}{\partial \xi_{II}} &= \frac{1}{\kappa_y^2 h_j B_{44}^{(j)}}Q_y^{(j)} - \psi_y^{(j)}\\
\frac{1}{l_{II}}\frac{\partial M_{yx}^{(j)}}{\partial \xi_{II}} &= \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_x^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_x^{(j)}}{\partial \eta} + Q_x^{(j)} - \frac{h_j}{2}\left(q_{zx}^{(+)} + q_{zx}^{(-)}\right)\\
\frac{1}{l_{II}}\frac{\partial M_y^{(j)}}{\partial \xi_{II}} &= \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_y^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_{yx}^{(j)}}{\partial \eta} + Q_y^{(j)} - \frac{h_j}{2}\left(q_{zy}^{(+)} + q_{zy}^{(-)}\right)\\
\frac{1}{l_{II}}\frac{\partial Q_y^{(j)}}{\partial \xi_{II}} &= \rho_j h_j\frac{\partial^2 w^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial Q_x^{(j)}}{\partial \eta} - \left(q_z^{(+)} - q_z^{(-)}\right)
\end{aligned} \qquad (j=1,3) \quad (5.17)$$

where q's are surface loads and the stresses on the upper (-) and lower (+) surfaces of the plates. Also, note that the superscript and the subscript "j" denotes the doubler for j=1, and the plate adherends for j=2 and j=3.

By substituting (5.10) and (5.11) in to (5.17) and making the necessary non-dimensionalizaitons with respect to (5.13), (5.14) and (5.15), "Governing System of First Order Ordinary Differential Equations" for the bonded plate or panel system are obtained as,

For Plate 1, in Part I (Overlap Region);

$$
\begin{aligned}
&\frac{d\overline{\Psi}^{(1)}_{mnx}}{d\xi_I} = \frac{12\overline{L}_I}{\overline{B}^{(1)}_{66}}\left(\frac{h_1}{a}\right)^2 \overline{M}^{(1)}_{mnyx} - \overline{L}_I m\pi \overline{\Psi}^{(1)}_{mny} \\
&\frac{d\overline{\Psi}^{(1)}_{y}}{d\xi_I} = \frac{12\overline{L}_I}{\overline{B}^{(1)}_{22}}\left(\frac{h_1}{a}\right)^2 \overline{M}^{(1)}_{mny} + \overline{L}_I \frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}} m\pi \overline{\Psi}^{(1)}_{mnx} \\
&\frac{d\overline{W}_{mn}{}^{(1)}}{d\xi_I} = \overline{L}_I \left(\frac{h_1}{a}\right)^2 \frac{1}{\kappa_y^2 \overline{B}^{(1)}_{44}} \overline{Q}^{(1)}_{mny} - \overline{L}_I \frac{a}{h_1} \overline{\Psi}^{(1)}_{mny} \\
&\frac{d\overline{M}^{(1)}_{mnyx}}{d\xi_I} = \left\{ \begin{array}{l} -\dfrac{\overline{L}_I}{12} \dfrac{a^4 \rho_1 \omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \dfrac{\overline{L}_I}{12}\left(\dfrac{am\pi}{h_1}\right)^2 \left(1 - \dfrac{\left(\overline{B}^{(1)}_{12}\right)^2}{\overline{B}^{(1)}_{22}}\right) \\ + \overline{L}_I \left(\dfrac{a}{h_1}\right)^4 \kappa_x^2 \overline{B}^{(1)}_{55} + \overline{L}_I \dfrac{\overline{G}_a}{4\overline{h}_a}\left(\dfrac{a}{h_1}\right)^4 \end{array} \right\} \overline{\Psi}^{(1)}_{mnx} \\
&- \overline{L}_I \frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}} m\pi \overline{M}^{(1)}_{mny} + \left\{ \overline{L}_I \kappa_x^2 \overline{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\frac{\overline{L}_I \overline{G}_a m\pi}{2}\left(\frac{a}{h_1}\right)^3}} \right\} \overline{W}^{(1)}_{mn} \\
&+ \overline{L}_I \frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \overline{h}_2 \overline{\Psi}^{(2)}_{mnx} \\
&\frac{d\overline{M}^{(1)}_{mny}}{d\xi_I} = \left\{ -\frac{\overline{L}_I}{12} \frac{\rho_1 a^4 \omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{G}_a}{4\overline{h}_a} - \underline{\underline{\frac{\overline{L}_I}{2}\left(\frac{a}{h_1}\right)^4}} \overline{G}_a \right\} \overline{\Psi}^{(1)}_{mny} + \overline{L}_I m\pi \overline{M}^{(1)}_{mnyx} \\
&+ \left\{ \overline{L}_I \frac{a}{h_1} + \underline{\underline{\frac{\overline{L}_I}{2\kappa_y^2 \overline{B}^{(1)}_{44}}}} \overline{G}_a \frac{a}{h_1} \right\} \overline{Q}_{mny}{}^{(1)} + \overline{L}_I \frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \overline{h}_2 \overline{\Psi}^{(2)}_{mny} \\
&\frac{d\overline{Q}^{(j)}_{mny}}{d\xi_I} = \left\{ -\overline{L}_I \frac{\rho_1 a^4 \omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \overline{L}_I \kappa_x^2 \overline{B}^{(1)}_{55}\left(\frac{am\pi}{h_1}\right)^2 + \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a} \right\} \overline{W}^{(1)}_{mn} \\
&+ \overline{L}_I \kappa_x^2 \overline{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi \overline{\Psi}^{(1)}_{mnx} - \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a} \overline{W}^{(2)}_{mn}
\end{aligned}
\tag{5.18}
$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

For Plate 1, in Part II (Overlap Region);

$$\frac{d\overline{\Psi}_{mnx}^{(1)}}{d\xi_{II}} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(1)}}\left(\frac{h_1}{a}\right)^2 \overline{M}_{mnyx}^{(1)} - \overline{L}_{II} m\pi \overline{\Psi}_{mny}^{(1)}$$

$$\frac{d\overline{\Psi}_{mny}^{(1)}}{d\xi_{II}} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(1)}}\left(\frac{h_1}{a}\right)^2 \overline{M}_{mny}^{(1)} + \overline{L}_{II}\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}} m\pi \overline{\Psi}_{mnx}^{(1)}$$

$$\frac{d\overline{W}_{mn}^{(1)}}{d\xi_{II}} = \overline{L}_{II}\left(\frac{h_1}{a}\right)^2 \frac{1}{\kappa_y^2 \overline{B}_{44}^{(1)}}\overline{Q}_{mny}^{(1)} - \overline{L}_{II}\frac{a}{h_1}\overline{\Psi}_{mny}^{(1)}$$

$$\frac{d\overline{M}_{mnyx}^{(1)}}{d\xi_{II}} = \left\{ \begin{aligned} &-\frac{\overline{L}_{II}}{12}\frac{a^4 \rho_1 \omega_{mn}^2}{h_1^2 B_{11}^{(1)}} + \frac{\overline{L}_{II}}{12}\left(\frac{am\pi}{h_1}\right)^2 \left(1 - \frac{\left(\overline{B}_{12}^{(1)}\right)^2}{\overline{B}_{22}^{(1)}}\right) \\ &+ \overline{L}_{II}\left(\frac{a}{h_1}\right)^4 \kappa_x^2 \overline{B}_{55}^{(1)} + \overline{L}_{II}\frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \end{aligned} \right\} \overline{\Psi}_{mnx}^{(1)}$$

$$- \overline{L}_{II}\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}} m\pi \overline{M}_{mny}^{(1)} + \left\{ \overline{L}_{II}\kappa_x^2 \overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\frac{\overline{L}_{II}\overline{G}_a m\pi}{2}\left(\frac{a}{h_1}\right)^3}} \right\} \overline{W}_{mn}^{(1)}$$

$$+ \overline{L}_{II}\frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \overline{h}_3 \overline{\Psi}_{mnx}^{(3)}$$

$$\frac{d\overline{M}_{mny}^{(1)}}{d\xi_{II}} = \left\{ -\frac{\overline{L}_{II}}{12}\frac{\rho_1 a^4 \omega_{mn}^2}{h_1^2 B_{11}^{(1)}} + \overline{L}_{II}\left(\frac{a}{h_1}\right)^4 \frac{\overline{G}_a}{4\overline{h}_a} - \underline{\underline{\frac{\overline{L}_{II}}{2}\left(\frac{a}{h_1}\right)^4 \overline{G}_a}} \right\} \overline{\Psi}_{mny}^{(1)} + \overline{L}_{II} m\pi \overline{M}_{mnyx}^{(1)}$$

$$+ \left\{ \overline{L}_{II}\frac{a}{h_1} + \underline{\underline{\frac{\overline{L}_{II}}{2\kappa_y^2 \overline{B}_{44}^{(1)}}\overline{G}_a \frac{a}{h_1}}} \right\} \overline{Q}_{mny}^{(1)} + \overline{L}_{II}\frac{\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \overline{h}_3 \overline{\Psi}_{mny}^{(3)} \tag{5.19}$$

$$\frac{d\overline{Q}_{mny}^{(j)}}{d\xi_{II}} = \left\{ -\overline{L}_{II}\frac{\rho_1 a^4 \omega_{mn}^2}{h_1^2 B_{11}^{(1)}} + \overline{L}_{II}\kappa_x^2 \overline{B}_{55}^{(1)}\left(\frac{am\pi}{h_1}\right)^2 + \overline{L}_{II}\left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a} \right\} \overline{W}_{mn}^{(1)}$$

$$+ \overline{L}_{II}\kappa_x^2 \overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi \overline{\Psi}_{mnx}^{(1)} - \overline{L}_{II}\left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a}\overline{W}_{mn}^{(3)}$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or “doubly underlined terms”).

For Plate 2, in Part I (Overlap Region);

$$
\begin{aligned}
\frac{d\overline{\Psi}^{(2)}_{mnx}}{d\xi_I} &= \overline{L}_I \frac{12}{\overline{h}_2^3 \overline{B}^{(2)}_{66}} \left(\frac{h_1}{a}\right)^2 \overline{M}^{(2)}_{mnyx} - \overline{L}_I m\pi \overline{\Psi}^{(2)}_{mny} \\
\frac{d\overline{\Psi}^{(2)}_{mny}}{d\xi_I} &= \overline{L}_I \left(\frac{h_1}{a}\right)^2 \frac{12}{\overline{B}^{(2)}_{22} \overline{h}_2^3} \overline{M}^{(2)}_{mny} + \overline{L}_I \frac{\overline{B}^{(2)}_{12}}{\overline{B}^{(2)}_{22}} m\pi \overline{\Psi}^{(2)}_{mnx} \\
\frac{d\overline{W}^{(2)}_{mn}}{d\xi_I} &= \overline{L}_I \left(\frac{h_1}{a}\right)^2 \frac{1}{\kappa_y^2 \overline{h}_2 \overline{B}^{(2)}_{44}} \overline{Q}^{(2)}_{mny} - \overline{L}_I \overline{\Psi}^{(2)}_{mny} \\
\frac{d\overline{M}^{(2)}_{mnyx}}{d\xi_I} &= \left\{ \begin{aligned} & -\overline{L}_I \overline{\rho}_2 \overline{h}_2^3 \frac{\rho_1 a^4 \omega^2_{mn}}{12 h_1^2 B^{(1)}_{11}} + \overline{L}_I \kappa_x^2 \overline{h}_2 \overline{B}^{(2)}_{55} \left(\frac{a}{h_1}\right)^4 \\ & + \overline{L}_I \frac{\overline{h}_2^3}{12} \left(\frac{am\pi}{h_1}\right)^2 \left(\overline{B}^{(2)}_{11} - \frac{\left(\overline{B}^{(2)}_{12}\right)^2}{\overline{B}^{(2)}_{22}} \right) + \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{G}_a \overline{h}_2^2}{4\overline{h}_a} \end{aligned} \right\} \overline{\Psi}^{(2)}_{mnx} \\
& - \overline{L}_I \frac{\overline{B}^{(2)}_{12}}{\overline{B}^{(2)}_{22}} m\pi \overline{M}^{(2)}_{mny} + \left\{ \overline{L}_I \kappa_x^2 \overline{h}_2 m\pi \left(\frac{a}{h_1}\right)^3 \overline{B}^{(2)}_{55} + \underline{\underline{\overline{L}_I \frac{\overline{h}_2 m\pi}{2} \left(\frac{a}{h_1}\right)^3 \overline{G}_a}} \right\} \overline{W}^{(2)}_{mn} \\
& + \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{G}_a \overline{h}_2}{4\overline{h}_a} \overline{\Psi}^{(1)}_{mnx} \\
\frac{d\overline{M}^{(2)}_{mny}}{d\xi_I} &= \left\{ -\overline{L}_I \overline{\rho}_2 \overline{h}_2^3 \frac{a^4 \rho_1 \omega^2_{mn}}{12 h_1^2 B^{(1)}_{11}} + \overline{L}_I \frac{\overline{h}_2^2 \overline{G}_a}{4\overline{h}_a} \left(\frac{a}{h_1}\right)^4 - \underline{\underline{\overline{L}_I \frac{\overline{h}_2}{2} \overline{G}_a}} \right\} \overline{\Psi}^{(2)}_{mny} \\
& + \overline{L}_I (m\pi) \overline{M}^{(2)}_{mnyx} + \overline{L}_I \frac{\overline{h}_2 \overline{G}_a}{4\overline{h}_a} \left(\frac{a}{h_1}\right)^4 \overline{\Psi}^{(1)}_{mny} \\
& + \left\{ \overline{L}_I \frac{a}{h_1} + \underline{\underline{\overline{L}_I \frac{a}{h_1} \frac{\overline{G}_a}{2\kappa_y^2 \overline{B}^{(2)}_{44}}}} \right\} \overline{Q}^{(12)}_{mny} \\
\frac{d\overline{Q}^{(2)}_{mny}}{d\xi_I} &= \left\{ -\overline{L}_I \overline{\rho}_2 \overline{h}_2 \frac{\rho_1 a^4 \omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \overline{L}_I \kappa_x^2 \overline{h}_2 \overline{B}^{(2)}_{55} \left(\frac{am\pi}{h_1}\right)^2 + \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a} \right\} \overline{W}^{(2)}_{mn} + \\
& \overline{L}_I \kappa_x^2 \overline{h}_2 \overline{B}^{(2)}_{55} \left(\frac{a}{h_1}\right)^3 m\pi \overline{\Psi}^{(2)}_{mnx} - \overline{L}_I \left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a} \overline{W}^{(1)}_{mn}
\end{aligned}
\tag{5.20}
$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

For Plate 3, in Part II (Overlap Region);

$$\frac{d\overline{\Psi}^{(3)}_{mnx}}{d\xi_{II}} = \overline{L}_{II}\,\frac{12}{\overline{h}_3^{3}\,\overline{B}^{(3)}_{66}}\left(\frac{h_1}{a}\right)^2 \overline{M}^{(3)}_{mnyx} - \overline{L}_{II}\, m\pi\, \overline{\Psi}^{(3)}_{mny}$$

$$\frac{d\overline{\Psi}^{(3)}_{mny}}{d\xi_{II}} = \overline{L}_{II}\left(\frac{h_1}{a}\right)^2 \frac{12}{\overline{B}^{(3)}_{22}\,\overline{h}_3^{3}}\,\overline{M}^{(3)}_{mny} + \overline{L}_{II}\,\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}}\, m\pi\, \overline{\Psi}^{(3)}_{mnx}$$

$$\frac{dw^{(3)}_{mn}}{d\xi_{II}} = \overline{L}_{II}\left(\frac{h_1}{a}\right)^2 \frac{1}{\kappa_y^2\,\overline{h}_3\,\overline{B}^{(3)}_{44}}\,\overline{Q}^{(3)}_{mny} - \overline{L}_{II}\,\overline{\Psi}^{(3)}_{mny}$$

$$\frac{d\overline{M}^{(3)}_{mnyx}}{d\xi_{II}} = \left\{\begin{array}{l} -\overline{L}_{II}\,\overline{\rho}_3\,\overline{h}_3^{3}\,\dfrac{\rho_1 a^4\omega^2_{mn}}{12 h_1^2 B^{(1)}_{11}} + \overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\,\overline{B}^{(3)}_{55}\left(\dfrac{a}{h_1}\right)^4 \\ +\overline{L}_{II}\,\dfrac{\overline{h}_3^{3}}{12}\left(\dfrac{am\pi}{h_1}\right)^2\left(\overline{B}^{(3)}_{11} - \dfrac{\left(\overline{B}^{(3)}_{12}\right)^2}{\overline{B}^{(3)}_{22}}\right) + \overline{L}_{II}\left(\dfrac{a}{h_1}\right)^4 \dfrac{\overline{G}_a\,\overline{h}_3^{2}}{4\overline{h}_a} \end{array}\right\}\overline{\Psi}^{(3)}_{mnx}$$

$$-\overline{L}_{II}\,\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}}\, m\pi\,\overline{M}^{(3)}_{mny} + \left\{\overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\, m\pi\left(\frac{a}{h_1}\right)^3 \overline{B}^{(3)}_{55} + \underline{\underline{\overline{L}_{II}\,\frac{\overline{h}_3 m\pi}{2}\left(\frac{a}{h_1}\right)^3 \overline{G}_a}}\right\}\overline{W}^{(3)}_{mn}$$

$$+\overline{L}_{II}\left(\frac{a}{h_1}\right)^4 \frac{\overline{G}_a\,\overline{h}_3}{4\overline{h}_a}\,\overline{\Psi}^{(1)}_{mnx}$$

$$\frac{d\overline{M}^{(3)}_{mny}}{d\xi_{II}} = \left\{-\overline{L}_{I}\,\overline{\rho}_3\,\overline{h}_3^{3}\,\frac{a^4\rho_1\omega^2_{mn}}{12 h_1^2 B^{(1)}_{11}} + \overline{L}_{II}\,\frac{\overline{h}_3^{2}\,\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\overline{L}_{II}\,\frac{\overline{h}_3}{2}\,\overline{G}_a}}\right\}\overline{\Psi}^{(3)}_{mny}$$

$$+\overline{L}_{II}\,(m\pi)\,\overline{M}^{(3)}_{mnyx} + \overline{L}_{II}\,\frac{\overline{h}_3\,\overline{G}_a}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \overline{\Psi}^{(1)}_{mny}$$

$$+\left\{\overline{L}_{II}\,\frac{a}{h_1} + \underline{\underline{\overline{L}_{II}\,\frac{a}{h_1}\,\frac{\overline{G}_a}{2\kappa_y^2\,\overline{B}^{(3)}_{44}}}}\right\}\overline{Q}^{(3)}_{mny}$$

$$\frac{d\overline{Q}^{(3)}_{mny}}{d\xi_{II}} = \left\{\begin{array}{l} -\overline{L}_{II}\,\overline{\rho}_3\,\overline{h}_3\,\dfrac{\rho_1 a^4\omega^2_{mn}}{h_1^2 B^{(1)}_{11}} + \overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\,\overline{B}^{(3)}_{55}\left(\dfrac{am\pi}{h_1}\right)^2 \\ +\overline{L}_{I}\left(\dfrac{a}{h_1}\right)^4 \dfrac{\overline{E}_a}{\overline{h}_a} \end{array}\right\}\overline{W}^{(3)}_{mn} +$$

$$\overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\,\overline{B}^{(3)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi\,\overline{\Psi}^{(3)}_{mnx} - \overline{L}_{II}\left(\frac{a}{h_1}\right)^4 \frac{\overline{E}_a}{\overline{h}_a}\,\overline{W}^{(1)}_{mn} \qquad (5.21)$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

The quantities having the subscript "mn" are "dimensionless fundamental dependent variables" of the problem in Part I and Part II Region (or "Overlap Region"). A ''Two-Point Boundary Value Problem" is created in the ''Overlap Regions'' or (Part I and Part II regions) by reducing the system of partial differential equations to a ''Governing System of First Order Ordinary Differential Equations'' in ξ_I or y_I and ξ_{II} or y_{II} direction.

Thus, "Governing System of First Order Ordinary Differential Equations" in the compact matrix or ''state vector'' form for the ''Overlap Regions'' (or Part I and Part II regions) can be written as,

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(2)}\end{Bmatrix}=\begin{bmatrix}\overline{C}_{1,1} & \overline{C}_{1,2}\\ \overline{C}_{2,1} & \overline{C}_{2,2}\end{bmatrix}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(2)}\end{Bmatrix},\quad (0<\xi_I<1)\qquad \text{(in Part I)}\qquad (5.22)$$

with the ''Arbitrary Boundary Conditions'' and the ''Continuity Conditions''.

$$\frac{d}{d\xi_{II}}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(3)}\end{Bmatrix}=\begin{bmatrix}\overline{C}'_{1,1} & \overline{C}'_{1,2}\\ \overline{C}'_{2,1} & \overline{C}'_{2,2}\end{bmatrix}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(3)}\end{Bmatrix},\quad \left(0<\xi_{II}<1\right)\qquad \text{(in Part II)}\qquad (5.23)$$

with the ''Arbitrary Boundary Conditions'' and the ''Continuity Conditions''.

where ξ_I and ξ_{II} are defined as y_I/ℓ_I, y_{II}/ℓ_I respectively. The superscripts show the related plate layer, the sub-matrices $[\overline{C}_{i,j}]$ and $[\overline{C}'_{i,j}]$ are partitioned square matrices of dimension (6x6) which explicitly include the nondimensional geometric and material characteristics of the plate adherends, doubler plate and of the adhesive layer and dimensionless natural frequency parameter $\overline{\omega}_{mn}$ of the entire composite system. $\overline{Y}_{mn}^{(j)}$ (j=1,2,3) are the ''state vectors' corresponding to the "state variables" or the ''dimensionless fundamental dependent variables'' of the problem under study as;

$$\left\{\overline{Y}_{mn}^{(j)}\right\}=\left\{\overline{\Psi}_{mnx}^{(j)},\overline{\Psi}_{mny}^{(j)},\overline{W}_{mn}^{(j)};\overline{M}_{mnyx}^{(j)},\overline{M}_{mny}^{(j)},\overline{Q}_{mny}^{(j)}\right\}^T,\quad \text{(j=1,2,3)}\qquad (5.24)$$

5.4.3 Analysis of Part III (or Single Layer) of Composite Plate System

The "Governing System of the Ordinary Differential Equations" can be obtained for the plate adherend in Part III region by using the same procedure in the previous section. There is no adhesive layer in Part III. Therefore, coupling terms in (5.20) including the adhesive layer elastic constants are dropped. The "Governing System of First Order Ordinary Differential Equations" in the ''state vector'' form, for Part III region,

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}_{mn}^{(2)}\right\}=\left[\overline{\mathcal{D}}\right]\left\{\overline{Y}_{mn}^{(2)}\right\} \qquad (0<\xi_{III}<1) \qquad \text{(in Part III)} \qquad (5.25)$$

with the ''Arbitrary Boundary Conditions'' at ξ_{III}=0 and the ''Continuity Conditions'' at ξ_{III} =1 for the orthotropic plate adherend.

where ξ_{III} is defined as y_{III}/ℓ_{III} and [$\overline{\mathcal{D}}$] is the "Coefficient Matrix" of dimension (6x6) which explicitly includes dimensionless geometric and material characteristics of the plate adherend in the Part III region as well as the dimensionless natural frequencies $\overline{\omega}_{mn}$ of the entire composite bonded plate system. The column matrix or the "state vector" $\left\{\overline{Y}_{mn}^{(2)}\right\}$ is defined as,

$$\left\{\overline{Y}_{mn}^{(2)}\right\}=\left\{\overline{\Psi}_{mnx}^{(2)},\overline{\Psi}_{mny}^{(2)},\overline{W}_{mn}^{(2)};\overline{M}_{mnyx}^{(2)},\overline{M}_{mny}^{(2)},\overline{Q}_{mny}^{(2)}\right\}^{T}, \qquad (5.26)$$

5.4.4 Analysis of Part IV (or Single Layer) of Composite Plate System

The "Governing System of First Order Ordinary Differential Equations" can be obtained for the plate adherend in Part IV region by using the same procedure in the previous section. There is no adhesive layer in Part IV. Therefore, coupling terms in (5.21) including the adhesive layer elastic constants are dropped. The "Governing System of First Order Ordinary Differential Equations" in the ''state vector'' form, for Part IV region,

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}_{mn}^{(3)}\right\}=\left[\overline{\mathcal{E}}\right]\left\{\overline{Y}_{mn}^{(3)}\right\}, \qquad (0<\xi_{IV}<1) \qquad \text{(in Part IV)} \qquad (5.27)$$

with the ‘‘Arbitrary Boundary Conditions’’ at ξ_{IV} =1 and the ‘‘Continuity Conditions’’ at ξ_{IV} =0 for the orthotropic plate adherend.

where ξ_{IV} is defined as y_{IV}/ℓ_{IV} and $[\overline{\mathcal{E}}]$ is the “Coefficient Matrix” of dimension (6x6) which explicitly includes dimensionless geometric and material characteristics of the plate adherend in the Part IV region as well as the dimensionless natural frequencies $\overline{\omega}_{mn}$ of the entire composite bonded plate system. The dimensionless column matrix or the “state vector” $\left\{\overline{Y}^{(3)}_{mn}\right\}$ is defined as,

$$\left\{\overline{Y}^{(3)}_{mn}\right\} = \left\{\overline{\Psi}^{(3)}_{mnx}, \overline{\Psi}^{(3)}_{mny}, \overline{W}^{(3)}_{mn}; \overline{M}^{(3)}_{mnyx}, \overline{M}^{(3)}_{mny}, \overline{Q}^{(3)}_{mny}\right\}^{T}, \qquad (5.28)$$

5.4.5 System of Governing Ordinary Differential Equations for (“Main PROBLEM II”)

In the previous sections, the “Governing System of First Order Ordinary Differential Equations” are obtained in the matrix or the “state vector” form for the “Overlap Region” (or Part I and Part II regions), and for the “Single Layer Regions” (or Part III and Part IV regions). These equations can be written in “open matrix form” as,

For Part I region or the “Overlap Region” (or Two-Layer Composite Plate Region),

$$\frac{d}{d\xi_I}\begin{Bmatrix} \overline{\psi}^{(1)}_{mnx} \\ \overline{\psi}^{(1)}_{mny} \\ \overline{W}^{(1)}_{mn} \\ \overline{M}^{(1)}_{mnyx} \\ \overline{M}^{(1)}_{mny} \\ \overline{Q}^{(1)}_{mny} \\ \overline{\psi}^{(2)}_{mnx} \\ \overline{\psi}^{(2)}_{mny} \\ \overline{W}^{(2)}_{mn} \\ \overline{M}^{(2)}_{mnyx} \\ \overline{M}^{(2)}_{mny} \\ \overline{Q}^{(2)}_{mny} \end{Bmatrix} = \left[\begin{array}{cccccc|cccccc} 0 & c_{1,2} & 0 & c_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{2,1} & 0 & 0 & 0 & c_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{3,2} & 0 & 0 & 0 & c_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{4,1} & 0 & c_{4,3} & 0 & c_{4,5} & 0 & c_{4,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{5,2} & 0 & c_{5,4} & 0 & c_{5,6} & 0 & c_{5,8} & 0 & 0 & 0 & 0 \\ c_{6,1} & 0 & c_{6,3} & 0 & 0 & 0 & 0 & 0 & c_{6,9} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{7,8} & 0 & c_{7,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{8,7} & 0 & 0 & 0 & c_{8,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{9,8} & 0 & 0 & 0 & c_{9,12} \\ c_{10,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{10,9} & 0 & c_{10,11} & 0 \\ 0 & c_{11,2} & 0 & 0 & 0 & 0 & 0 & c_{11,8} & 0 & c_{11,10} & 0 & c_{11,12} \\ 0 & 0 & c_{12,3} & 0 & 0 & 0 & c_{12,7} & 0 & c_{12,9} & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} \overline{\psi}^{(1)}_{mnx} \\ \overline{\psi}^{(1)}_{mny} \\ \overline{W}^{(1)}_{mn} \\ \overline{M}^{(1)}_{mnyx} \\ \overline{M}^{(1)}_{mny} \\ \overline{Q}^{(1)}_{mny} \\ \overline{\psi}^{(2)}_{mnx} \\ \overline{\psi}^{(2)}_{mny} \\ \overline{W}^{(2)}_{mn} \\ \overline{M}^{(2)}_{mnyx} \\ \overline{M}^{(2)}_{mny} \\ \overline{Q}^{(2)}_{mny} \end{Bmatrix}$$

$(0<\xi_I<1)$ (in Part I) (5.29 a)

The elements, in the “open matrix form”, of the ‘‘Coefficient Sub-Matrix” related to the plate layers are,

For plate 1 (Doubler),

$$c_{1,2} = -\bar{L}_1 m\pi \qquad c_{1,4} = \frac{12\bar{L}_1}{\bar{B}^{(1)}_{66}}\left(\frac{h_1}{a}\right)^2 \qquad c_{2,1} = \bar{L}_1 m\pi \frac{\bar{B}^{(1)}_{12}}{\bar{B}^{(1)}_{22}}$$

$$c_{2,5} = \frac{12\bar{L}_1}{\bar{B}^{(1)}_{22}}\left(\frac{h_1}{a}\right)^2 \qquad c_{3,2} = -\bar{L}_1\left(\frac{a}{h_1}\right) \qquad c_{3,6} = \frac{\bar{L}_1}{\kappa_y^2 \bar{B}^{(1)}_{44}}\left(\frac{h_1}{a}\right)^2$$

$$c_{4,1} = -\frac{\bar{L}_1}{12}\bar{\Omega} + \bar{L}_1 \kappa_x^2 \bar{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^4 + \frac{\bar{L}_1}{12}\left(\frac{am\pi}{h_1}\right)^2\left(1 - \frac{\bar{B}^{(1)^2}_{12}}{\bar{B}^{(1)}_{22}}\right) + \frac{\bar{G}_a \bar{L}_1}{4\bar{h}_a}\left(\frac{a}{h_1}\right)^4$$

$$c_{4,3} = \bar{L}_1 \kappa_x^2 \bar{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\bar{L}_1 \frac{m\pi}{2}\left(\frac{a}{h_1}\right)^3 \bar{G}_a}} \qquad c_{4,5} = -\bar{L}_1 m\pi \frac{\bar{B}^{(1)}_{12}}{\bar{B}^{(1)}_{22}}$$

$$c_{5,2} = -\frac{\bar{L}_1}{12}\bar{\Omega} + \frac{\bar{G}_a \bar{L}_1}{4\bar{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\frac{\bar{L}_1 \bar{G}_a}{2}\left(\frac{a}{h_1}\right)^4}} \qquad c_{5,4} = \bar{L}_1 m\pi$$

$$c_{5,6} = \bar{L}_1\left(\frac{a}{h_1}\right) + \underline{\underline{\bar{L}_1\left(\frac{a}{h_1}\right)\frac{\bar{G}_a}{2\kappa_y^2 \bar{B}^{(1)}_{44}}}} \qquad c_{6,1} = \kappa_x^2 \bar{L}_1 \bar{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$c_{6,3} = -\bar{L}_1\bar{\Omega} + \bar{L}_1 \kappa_x^2\left(\frac{am\pi}{h_1}\right)^2 \bar{B}^{(1)}_{55} + \frac{\bar{L}_1 \bar{E}_a}{\bar{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c_{4,7} = \frac{\bar{L}_1 \bar{G}_a}{4\bar{h}_a}\bar{h}_2\left(\frac{a}{h_1}\right)^4$$

$$c_{5,8} = \frac{\bar{L}_1 \bar{G}_a \bar{h}_2}{4\bar{h}_a}\left(\frac{a}{h_1}\right)^2 \qquad c_{6,9} = -\frac{\bar{L}_1 \bar{E}_a}{\bar{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad (5.29\text{ b})$$

For Plate 2 (Adherend Plate),

$$c_{7,8} = -\bar{L}_1 m\pi \qquad c_{7,10} = \frac{12\bar{L}_1}{\bar{B}^{(2)}_{66}\bar{h}_2^3}\left(\frac{h_1}{a}\right)^2 \qquad c_{8,7} = \bar{L}_1 m\pi \frac{\bar{B}^{(2)}_{12}}{\bar{B}^{(2)}_{22}}$$

$$c_{8,11} = \frac{12\bar{L}_1}{\bar{B}^{(2)}_{22}\bar{h}_2^3}\left(\frac{h_1}{a}\right)^2 \qquad c_{9,8} = -\bar{L}_1\left(\frac{a}{h_1}\right) \qquad c_{9,12} = \frac{\bar{L}_1}{\kappa_y^2 \bar{B}^{(2)}_{44}\bar{h}_2}\left(\frac{h_1}{a}\right)^2$$

$$c_{10,7} = -\frac{\bar{L}_1}{12}\bar{\rho}_2 \bar{h}_2^3 \bar{\Omega} + \bar{L}_1 \kappa_x^2 \bar{h}_2 \bar{B}^{(2)}_{55}\left(\frac{a}{h_1}\right)^4 + \frac{\bar{L}_1 \bar{h}_2^3}{12}\left(\frac{am\pi}{h_1}\right)^2\left(\bar{B}^{(2)}_{11} - \frac{\bar{B}^{(2)^2}_{12}}{\bar{B}^{(2)}_{22}}\right) + \frac{\bar{G}_a \bar{L}_1 \bar{h}_2^2}{4\bar{h}_a}\left(\frac{a}{h_1}\right)^4$$

$$c_{10,9} = \overline{L}_1 \kappa_x^2 \overline{h}_2 \overline{B}_{55}^{(2)} \left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\overline{L}_1 \overline{h}_2 \frac{m\pi}{2} \overline{G}_a}} \left(\frac{a}{h_1}\right)^3 \qquad c_{10,11} = -\overline{L}_1 m\pi \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}$$

$$c_{11,8} = -\frac{\overline{L}_1}{12} \overline{\rho}_2 \overline{h}_2^3 \overline{\Omega} + \frac{\overline{L}_1 \overline{G}_a \overline{h}_2^2}{4\overline{h}_a} \left(\frac{a}{h_1}\right)^4 - \underline{\underline{\frac{\overline{L}_1 \overline{G}_a \overline{h}_2}{2}}} \left(\frac{a}{h_1}\right)^4 \qquad c_{11,10} = \overline{L}_1 m\pi$$

$$c_{11,12} = \overline{L}_1 \left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_1 \left(\frac{a}{h_1}\right) \frac{\overline{G}_a}{2\kappa_y^2 \overline{B}_{44}^{(2)}}}} \qquad c_{12,7} = \overline{L}_1 \overline{h}_2 \kappa_x^2 \overline{B}_{55}^{(2)} \left(\frac{a}{h_1}\right)^3 m\pi$$

$$c_{12,9} = -\overline{L}_1 \overline{\rho}_2 \overline{h}_2 \overline{\Omega} + \overline{L}_1 \kappa_x^2 \left(\frac{m\pi a}{h_1}\right)^2 \overline{h}_2 \overline{B}_{55}^{(2)} + \frac{\overline{L}_1 \overline{E}_a}{\overline{h}_a} \left(\frac{a}{h_1}\right)^4 \qquad c_{10,1} = \frac{\overline{L}_1 \overline{G}_a}{4\overline{h}_a} \overline{h}_2 \left(\frac{a}{h_1}\right)^4$$

$$c_{11,2} = \frac{\overline{L}_1 \overline{G}_a \overline{h}_2}{4\overline{h}_a} \left(\frac{a}{h_1}\right)^4 \qquad c_{12,3} = -\frac{\overline{L}_1 \overline{E}_a}{\overline{h}_a} \left(\frac{a}{h_1}\right)^4 \qquad (5.29\ c)$$

<u>For Part II region or the "Overlap Region" (or Two-Layer Composite Plate Region),</u>

$$\frac{d}{d\xi_{II}} \begin{Bmatrix} \overline{\psi}^{(1)}_{mnx} \\ \overline{\psi}^{(1)}_{mny} \\ \overline{W}^{(1)}_{mn} \\ \overline{M}^{(1)}_{mnyx} \\ \overline{M}^{(1)}_{mny} \\ \overline{Q}^{(1)}_{mny} \\ \overline{\psi}^{(3)}_{mnx} \\ \overline{\psi}^{(3)}_{mny} \\ \overline{W}^{(3)}_{mn} \\ \overline{M}^{(3)}_{mnyx} \\ \overline{M}^{(3)}_{mny} \\ \overline{Q}^{(3)}_{mny} \end{Bmatrix} = \left[\begin{array}{cccccc|cccccc} 0 & c'_{1,2} & 0 & c'_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c'_{2,1} & 0 & 0 & 0 & c'_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c'_{3,2} & 0 & 0 & 0 & c'_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 \\ c'_{4,1} & 0 & c'_{4,3} & 0 & c'_{4,5} & 0 & c'_{4,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{5,2} & 0 & c'_{5,4} & 0 & c'_{5,6} & 0 & c'_{5,8} & 0 & 0 & 0 & 0 \\ c'_{6,1} & 0 & c'_{6,3} & 0 & 0 & 0 & 0 & 0 & c'_{6,9} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{7,8} & 0 & c'_{7,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c'_{8,7} & 0 & 0 & 0 & c'_{8,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{9,8} & 0 & 0 & 0 & c'_{9,12} \\ c'_{10,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{10,9} & 0 & c'_{10,11} & 0 \\ 0 & c'_{11,2} & 0 & 0 & 0 & 0 & 0 & c'_{11,8} & 0 & c'_{11,10} & 0 & c'_{11,12} \\ 0 & 0 & c'_{12,3} & 0 & 0 & 0 & c'_{12,7} & 0 & c'_{12,9} & 0 & 0 & 0 \end{array} \right] \begin{Bmatrix} \overline{\psi}^{(1)}_{mnx} \\ \overline{\psi}^{(1)}_{mny} \\ \overline{W}^{(1)}_{mn} \\ \overline{M}^{(1)}_{mnyx} \\ \overline{M}^{(1)}_{mny} \\ \overline{Q}^{(1)}_{mny} \\ \overline{\psi}^{(3)}_{mnx} \\ \overline{\psi}^{(3)}_{mny} \\ \overline{W}^{(3)}_{mn} \\ \overline{M}^{(3)}_{mnyx} \\ \overline{M}^{(3)}_{mny} \\ \overline{Q}^{(3)}_{mny} \end{Bmatrix}$$

$(0<\xi_{II}<1)$ (in Part II) (5.30 a)

The elements, in the open form, of the ''Coefficient Sub-Matrix related to the plate layers are,

<u>For plate 1 (Doubler),</u>

$$c'_{1,2} = -\overline{L}_{II} m\pi \qquad c'_{1,4} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(1)}} \left(\frac{h_1}{a}\right)^2 \qquad c'_{2,1} = \overline{L}_{II} m\pi \frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}$$

$$c'_{2,5} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(1)}}\left(\frac{h_1}{a}\right)^2 \qquad c'_{3,2} = -\overline{L}_{II}\left(\frac{a}{h_1}\right) \qquad c'_{3,6} = \frac{\overline{L}_{II}}{\kappa_y^2\overline{B}_{44}^{(1)}}\left(\frac{h_1}{a}\right)^2$$

$$c'_{4,1} = -\frac{\overline{L}_{II}}{12}\overline{\Omega} + \overline{L}_I\kappa_x^2\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_{II}}{12}\left(\frac{am\pi}{h_1}\right)^2\left(1-\frac{\overline{B}_{12}^{(1)^2}}{\overline{B}_{22}^{(1)}}\right) + \frac{\overline{G}_a\overline{L}_{II}}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4$$

$$c'_{4,3} = \overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\overline{L}_{II}\frac{m\pi}{2}\left(\frac{a}{h_1}\right)^3\overline{G}_a}} \qquad c'_{4,5} = -\overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}$$

$$c'_{5,2} = -\frac{\overline{L}_{II}}{12}\overline{\Omega} + \frac{\overline{G}_a\overline{L}_{II}}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\frac{\overline{L}_{II}\overline{G}_a}{2}\left(\frac{a}{h_1}\right)^4}} \qquad c'_{5,4} = \overline{L}_{II}m\pi$$

$$c'_{5,6} = \overline{L}_{II}\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_{II}\left(\frac{a}{h_1}\right)\frac{\overline{G}_a}{2\kappa_y^2\overline{B}_{44}^{(1)}}}} \qquad c'_{6,1} = \kappa_x^2\overline{L}_{II}\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$c'_{6,3} = -\overline{L}_{II}\overline{\Omega} + \overline{L}_{II}\kappa_x^2\left(\frac{am\pi}{h_1}\right)^2\overline{B}_{55}^{(1)} + \frac{\overline{L}_{II}\overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c'_{4,7} = \frac{\overline{L}_{II}\overline{G}_a}{4\overline{h}_a}\overline{h}_3\left(\frac{a}{h_1}\right)^4$$

$$c'_{5,8} = \frac{\overline{L}_{II}\overline{G}_a\overline{h}_3}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^2 \qquad c'_{6,9} = -\frac{\overline{L}_{II}\overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad (5.30\text{ b})$$

<u>For Plate 3 (Adherend Plate),</u>

$$c'_{7,8} = -\overline{L}_{II}m\pi \qquad c'_{7,10} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(3)}\overline{h}_3^3}\left(\frac{h_1}{a}\right)^2 \qquad c'_{8,7} = \overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(3)}}{\overline{B}_{22}^{(3)}}$$

$$c'_{8,11} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(3)}\overline{h}_3^3}\left(\frac{h_1}{a}\right)^2 \qquad c'_{9,8} = -\overline{L}_{II}\left(\frac{a}{h_1}\right) \qquad c'_{9,12} = \frac{\overline{L}_{II}}{\kappa_y^2\overline{B}_{44}^{(3)}\overline{h}_3}\left(\frac{h_1}{a}\right)^2$$

$$c'_{10,7} = -\frac{\overline{L}_{II}}{12}\overline{\rho}_3\overline{h}_3^3\overline{\Omega} + \overline{L}_{II}\kappa_x^2\overline{h}_3\overline{B}_{55}^{(3)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_{II}\overline{h}_3^3}{12}\left(\frac{am\pi}{h_1}\right)^2\left(\overline{B}_{11}^{(3)} - \frac{\overline{B}_{12}^{(3)^2}}{\overline{B}_{22}^{(3)}}\right) + \frac{\overline{G}_a\overline{L}_{II}\overline{h}_3^2}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4$$

$$c'_{10,9} = \overline{L}_{II}\kappa_x^2\overline{h}_3\overline{B}_{55}^{(3)}\left(\frac{a}{h_1}\right)^3 m\pi + \underline{\underline{\overline{L}_{II}\overline{h}_3\frac{m\pi}{2}\overline{G}_a}}\left(\frac{a}{h_1}\right)^3 \qquad c'_{10,11} = -\overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(3)}}{\overline{B}_{22}^{(3)}}$$

$$c'_{11,8} = -\frac{\overline{L}_{II}}{12}\overline{\rho}_3\overline{h}_3^3\overline{\Omega} + \frac{\overline{L}_{II}\overline{G}_a\overline{h}_3^2}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 - \underline{\underline{\frac{\overline{L}_{II}\overline{G}_a\overline{h}_3}{2}}}\left(\frac{a}{h_1}\right)^4 \qquad c'_{11,10} = \overline{L}_{II}m\pi$$

$$c'_{11,12} = \overline{L}_{II}\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_{II}\left(\frac{a}{h_1}\right)\frac{\overline{G}_a}{2\kappa_y^2\overline{B}_{44}^{(3)}}}} \qquad c'_{12,7} = \overline{L}_{II}\overline{h}_3\kappa_x^2\overline{B}_{55}^{(3)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$c'_{12,9} = -\overline{L}_{II}\overline{\rho}_3\overline{h}_3\overline{\Omega} + \overline{L}_I\kappa_x^2\left(\frac{m\pi a}{h_1}\right)^2\overline{h}_3\overline{B}_{55}^{(3)} + \frac{\overline{L}_{II}\overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c'_{10,1} = \frac{\overline{L}_{II}\overline{G}_a}{4\overline{h}_a}\overline{h}_3\left(\frac{a}{h_1}\right)^4$$

$$c'_{11,2} = \frac{\overline{L}_{II}\overline{G}_a\overline{h}_3}{4\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad c'_{12,3} = -\frac{\overline{L}_{II}\overline{E}_a}{\overline{h}_a}\left(\frac{a}{h_1}\right)^4 \qquad (5.30\text{ c})$$

<u>For Part III region (or Single Layer Orthotropic Plate Adherend),</u>

$$\frac{d}{d\xi_{III}}\begin{Bmatrix}\overline{\psi}_{mnx}^{(2)}\\ \overline{\psi}_{mny}^{(2)}\\ \overline{W}_{mn}^{(2)}\\ \overline{M}_{mnyx}^{(2)}\\ \overline{M}_{mny}^{(2)}\\ \overline{Q}_{mny}^{(2)}\end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0 & d_{1,2} & 0 & d_{1,4} & 0 & 0\\ d_{2,1} & 0 & 0 & 0 & d_{2,5} & 0\\ 0 & d_{3,2} & 0 & 0 & 0 & d_{3,6}\\ \hline d_{4,1} & 0 & d_{4,3} & 0 & d_{4,5} & 0\\ 0 & d_{5,2} & 0 & d_{5,4} & 0 & d_{5,6}\\ d_{6,1} & 0 & d_{6,3} & 0 & 0 & 0\end{array}\right]\begin{Bmatrix}\overline{\psi}_{mnx}^{(2)}\\ \overline{\psi}_{mny}^{(2)}\\ \overline{W}_{mn}^{(2)}\\ \overline{M}_{mnyx}^{(2)}\\ \overline{M}_{mny}^{(2)}\\ \overline{Q}_{mny}^{(2)}\end{Bmatrix},$$

$(0<\xi_{III}<1)$ (in Part III) (5.31 a)

where the elements of the above ''Coefficient Matrix $\lfloor\mathcal{D}\rfloor$'' are,

$$d_{1,2} = -\overline{L}_{III}m\pi \qquad d_{1,4} = \frac{12\overline{L}_{III}}{\overline{B}_{66}^{(2)}}\left(\frac{h_1}{a}\right)^2 \qquad d_{2,1} = \overline{L}_{III}m\pi\frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}$$

$$d_{2,5} = \frac{12\overline{L}_{III}}{\overline{B}_{22}^{(2)}}\left(\frac{h_1}{a}\right)^2 \qquad d_{3,2} = -\overline{L}_{III}\left(\frac{a}{h_1}\right) \qquad d_{3,6} = \frac{\overline{L}_{III}}{\kappa_y^2\overline{B}_{44}^{(2)}}\left(\frac{h_1}{a}\right)^2$$

$$d_{4,1} = -\frac{\overline{L}_{III}}{12}\overline{\Omega} + \overline{L}_{III}\kappa_x^2\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_{III}}{12}\left(\frac{am\pi}{h_1}\right)^2\left(1-\frac{\overline{B}_{12}^{(2)^2}}{\overline{B}_{22}^{(2)}}\right)$$

$$d_{4,3} = \overline{L}_{III}\kappa_x^2\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^3 m\pi \qquad d_{4,5} = -\overline{L}_{III}m\pi\frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}} \qquad d_{5,2} = -\frac{\overline{L}_{III}}{12}\overline{\Omega}$$

$$d_{5,4} = \overline{L}_{IIII}m\pi \qquad d_{5,6} = \overline{L}_{III}\left(\frac{a}{h_1}\right) \qquad d_{6,1} = \kappa_x^2\overline{L}_{III}\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^3 m\pi$$

$$d_{6,3} = -\overline{L}_{III}\overline{\Omega} + \overline{L}_{III}\kappa_x^2\left(\frac{m\pi a}{h_1}\right)^2\overline{B}_{55}^{(2)} \qquad (5.31\text{ b})$$

<u>For Part IV region (or Single Layer Plate Adherend),</u>

$$\frac{d}{d\xi_{IV}}\begin{Bmatrix}\overline{\psi}^{(3)}_{mnx}\\ \overline{\psi}^{(3)}_{mny}\\ \overline{W}^{(3)}_{mn}\\ \overline{M}^{(3)}_{mnyx}\\ \overline{M}^{(3)}_{mny}\\ \overline{Q}^{(3)}_{mny}\end{Bmatrix}=\left[\begin{array}{ccc|ccc}0 & e_{1,2} & 0 & e_{1,4} & 0 & 0\\ e_{2,1} & 0 & 0 & 0 & e_{2,5} & 0\\ 0 & e_{3,2} & 0 & 0 & 0 & e_{3,6}\\ \hline e_{4,1} & 0 & e_{4,3} & 0 & e_{4,5} & 0\\ 0 & e_{5,2} & 0 & e_{5,4} & 0 & e_{5,6}\\ e_{6,1} & 0 & e_{6,3} & 0 & 0 & 0\end{array}\right]\begin{Bmatrix}\overline{\psi}^{(3)}_{mnx}\\ \overline{\psi}^{(3)}_{mny}\\ \overline{W}^{(3)}_{mn}\\ \overline{M}^{(3)}_{mnyx}\\ \overline{M}^{(3)}_{mny}\\ \overline{Q}^{(3)}_{mny}\end{Bmatrix},$$

$(0<\xi_{IV}<1)$ (in Part IV) (5.32.a)

where the elements of the above ''Coefficient Matrix $\lfloor \mathcal{E} \rfloor$'' are,

$$e_{1,2} = -\overline{L}_{IV} m\pi \qquad e_{1,4} = \frac{12\overline{L}_{IV}}{\overline{B}^{(3)}_{66}\overline{h}_3^3}\left(\frac{h_1}{a}\right)^2 \qquad e_{2,1} = \overline{L}_{IV} m\pi \frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}}$$

$$e_{2,5} = \frac{12\overline{L}_{IV}}{\overline{B}^{(3)}_{22}\overline{h}_3^3}\left(\frac{h_1}{a}\right)^2 \qquad e_{3,2} = -\overline{L}_{IV}\left(\frac{a}{h_1}\right) \qquad e_{3,6} = \frac{\overline{L}_{IV}}{\kappa_y^2\overline{B}^{(3)}_{44}\overline{h}_3}\left(\frac{h_1}{a}\right)^2$$

$$e_{4,1} = -\frac{\overline{L}_{IV}}{12}\overline{\rho}_3\overline{h}_3^3\overline{\Omega} + \overline{L}_{IV}\kappa_x^2\overline{B}^{(3)}_{55}\overline{h}_3\left(\frac{a}{h_1}\right)^4 + \frac{\overline{L}_{IV}\overline{h}_3^3}{12}\left(\frac{am\pi}{h_1}\right)^2\left(\overline{B}^{(3)}_{11} - \frac{\overline{B}^{(3)^2}_{12}}{\overline{B}^{(3)}_{22}}\right)$$

$$e_{4,3} = \overline{L}_{IV}\kappa_x^2\overline{B}^{(3)}_{55}\overline{h}_3\left(\frac{a}{h_1}\right)^3 m\pi \qquad e_{4,5} = -\overline{L}_{IV} m\pi\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}} \qquad e_{5,2} = -\frac{\overline{L}_{IV}}{12}\overline{\rho}_3\overline{h}_3^3\overline{\Omega}$$

$$e_{5,4} = \overline{L}_{IV} m\pi \qquad e_{5,6} = \overline{L}_{IV}\left(\frac{a}{h_1}\right) \qquad e_{6,1} = \kappa_x^2\overline{L}_{IV} B^{(3)}_{55}\overline{h}_3\left(\frac{a}{h_1}\right)^3 m\pi$$

$$\boldsymbol{e}_{6,3} = -\overline{L}_{IV}\overline{\rho}_3\overline{h}_3\overline{\Omega} + \overline{L}_{IV}\kappa_x^2\overline{B}^{(3)}_{55}\overline{h}_3\left(\frac{am\pi}{h_1}\right)^2 \qquad (5.32.b)$$

The "Boundary Conditions" at x=0 and x=a are already satisfied by trigonometric expansion in ''Classical Lévy's Type Solution''. The ''Appropriate Boundary Conditions'' and the ''Continuity Conditions'' are needed to solve the ''Governing System of First Order Ordinary Differential Equations''.

The ''Boundary Conditions'' along the edges in the y-direction,

F (Free): $\overline{M}_{yx}^{(j)} = \overline{M}_{y}^{(j)} = \overline{Q}_{y}^{(j)} = 0$

C (Clamped): $\overline{w}^{(j)} = \overline{\psi}_{x}^{(j)} = \overline{\psi}_{y}^{(j)} = 0$ (j=1,2,3) (5.33)

S (Simply Supported): $\overline{w}^{(j)} = \overline{\psi}_{x}^{(j)} = \overline{M}_{y}^{(j)} = 0$

The ''Continuity Conditions'' between Part I and Part II,

$$\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{I}=1}\right\} \tag{5.34}$$

The ''Continuity Conditions'' between Part I and Part III,

$$\left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_{I}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1}\right\} \tag{5.34}$$

The ''Continuity Conditions'' between Part II and Part IV,

$$\left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{VI}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1}\right\} \tag{5.35}$$

Finally, as a summary, the entire set of the "Governing System of First Order Ordinary Differential Equations" for the "Main PROBLEM II" is given as;

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{mn}\\ \overline{\boldsymbol{Y}}^{(2)}_{mn}\end{Bmatrix} = \begin{bmatrix}\overline{\mathbf{C}}_{1,1} & \overline{\mathbf{C}}_{1,2}\\ \overline{\mathbf{C}}_{2,1} & \overline{\mathbf{C}}_{2,2}\end{bmatrix}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{mn}\\ \overline{\boldsymbol{Y}}^{(2)}_{mn}\end{Bmatrix}, \quad (0<\xi_I<1) \quad \text{(in Part I)}$$

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{mn}\\ \overline{\boldsymbol{Y}}^{(3)}_{mn}\end{Bmatrix} = \begin{bmatrix}\overline{\mathbf{C}'}_{1,1} & \overline{\mathbf{C}'}_{1,2}\\ \overline{\mathbf{C}'}_{2,1} & \overline{\mathbf{C}'}_{2,2}\end{bmatrix}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{mn}\\ \overline{\boldsymbol{Y}}^{(3)}_{mn}\end{Bmatrix}, \quad (0<\xi_{II}<1) \quad \text{(in Part II)}$$

$$\frac{d}{d\xi_{III}}\left\{\overline{\boldsymbol{Y}}^{(2)}_{mn}\right\} = \left[\overline{\mathcal{D}}\right]\left\{\overline{\boldsymbol{Y}}^{(2)}_{mn}\right\}, \quad (0<\xi_{III}<1) \quad \text{(in Part III)}$$

$$\frac{d}{d\xi_{IV}}\left\{\overline{\boldsymbol{Y}}^{(3)}_{mn}\right\} = \left[\overline{\mathcal{E}}\right]\left\{\overline{\boldsymbol{Y}}^{(3)}_{mn}\right\}, \quad (0<\xi_{IV}<1) \quad \text{(in Part IV)}$$

(5.36 a,b,c,d)

with the “Appropriate Boundary Conditions” and the “Continuity Conditions” in each Part I, Part II, and Part III Regions respectively.

The above system of equations forms a “Two-Point Boundary Value Problem” for the “Main PROBLEM II” between the left and right supports in y-direction. It is obvious that, once the natural frequencies are obtained, then, the Equations (5.36.a,b,c,d) can be integrated numerically for a given particular geometry, materials and the support conditions by making use of the “Modified Transfer Matrix Method (with Interpolation Polynomials and/or Interpolation Polynomials)”.

CHAPTER 6

("Main PROBLEM III")__FREE FLEXURAL (or BENDING) VIBRATIONS of COMPOSITE ORTHOTROPIC or ISOTROPIC PLATES with a BONDED SYMMETRIC DOUBLE LAP JOINT (or SYMMETRIC DOUBLE DOUBLER JOINT)

In this section, the "Governing System of Coupled Ordinary Differential Equations" will be presented in the compact matrix or the "state vector" form for the "Bonded Region" (or Part I and Part II regions), and for "Single Layer Regions" (or Part III and Part IV regions) for the "Main PROBLEM III" without making any distinction for the "Main PROBLEM III a " and the "Main PROBLEM III b" as described below.

6.1 Statement of "Main PROBLEM IIIa"

Figure 3.6.a shows the general configuration, geometry and coordinate system of the "Composite Orthotropic or Isotropic Plate System with Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)". This system is composed of two "Orthotropic Doublers" and the "Orthotropic or Isotropic Adherends" joined centrally by a relatively very thin elastic adhesive layer.

6.2 Statement of "Main PROBLEM IIIb"

Figure 3.7.a shows the general configuration, geometry and coordinate system of the "Composite Orthotropic or Isotropic Plate System with Non-Centrally Bonded Symmetric Double Lap Joint". This system composed of two "Orthotropic Doublers" and the "Orthotropic or Isotropic Adherends" joined non-centrally by a relatively very thin elastic adhesive layer.

6.3 Main assumptions and Analytical Modeling

The analytical formulation of "Main PROBLEM III" is based on the following assumptions;

(i) The analysis is carried out only for the free flexural (or bending) vibrations of the composite plate system. The in-plane or extensional moments of inertia are neglected, but the rotatory and the transverse moments of inertia of the plates are included in the formulation.

(ii) The rotatory and the transverse moments of inertia of the adherends as well as their transverse shear deformations are taken into account in the sense of the "Mindlin Plate Theory".

(iii) There is no slip and separation on the interfaces of the adhesive layers and plates or adherends.

(iv) Since the thickness of the adhesive layers are very small relative to the thickness of the plates, the inertia and the mass of the adhesive layers are neglected, and both adhesive normal and shear stresses are to be constant across their thicknesses.

(v) The damping effects in Mindlin Plates and in the adhesive layers are neglected.

(vi) The plates are assumed to be simply supported along edges x=0 and x=a while arbitrary support conditions may be specified in the y-direction.

(vii) The coordinate system of each plate is attached to its medium plane or the reference plane.

(viii) The principal directions of orthotropy in plates are parallel to the edges and to the coordinates as shown in Figure 3.6.a and Figure 3.7.a.

(ix) It is assumed that the following relations exist between the deformations of the plate adherends and doublers in z-direction.

For Part I;	For Part II;
$w^{(4)} > w^{(2)} > w^{(1)}$	$w^{(4)} > w^{(3)} > w^{(1)}$

For the general formulation of the problem, the entire “Composite Bonded Plate or Panel System” is divided into four parts, namely, Part I, Part II, Part III and Part IV in the y-direction as shown in Figure 3.6.b and Figure 3.7.b Part I corresponds to the “Bonded Region” which contains three plates 1,2,4, Part II corresponds to the “Bonded Region” which contains three plates 1,3,4 and Part II and Part III correspond the continuation of the middle plates 2 and 3 respectively, as a single plates in the y-direction.

6.4 Theoretical Formulation of “Main PROBLEM III” (Theoretical Analysis)

6.4.1 Analysis of Adhesive Layer in the “Bonded Region”

The system in “Bonded Region” (or Part I and Part II region) is composed of three plates which are adhesively bonded by very thin elastic adhesive layers. The stresses at the upper and lower faces of the plates due to the adhesive layers are considered as external surface stresses or loads on the plates. The adhesive stresses could be related to the unknown displacement functions and angle of rotations of the adherends in the “Bonded Region”. Figures 6.1 and Figure 6.2 show an exaggerated view of the deformation of an infinitesimal element of three-layer plate system. The positive sign convention for displacements, stress resultants and angles of rotation are also shown on the same figures (see also Figures 3.1 and 6.1 and 6.2).

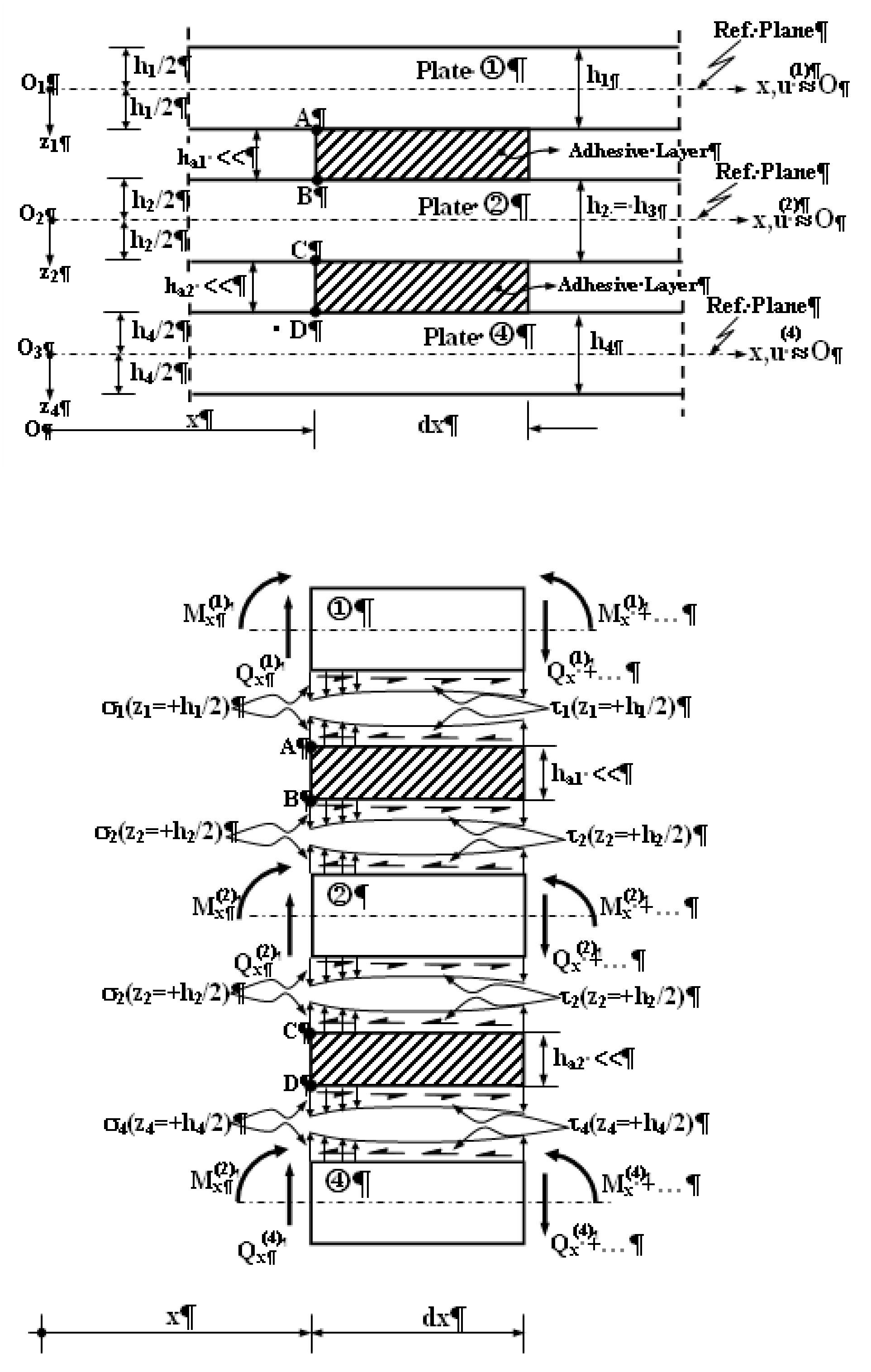


Figure 6.1 Stress Distributions at Plate Adhesive Layer Interfaces in the “Bonded Region” for Part I

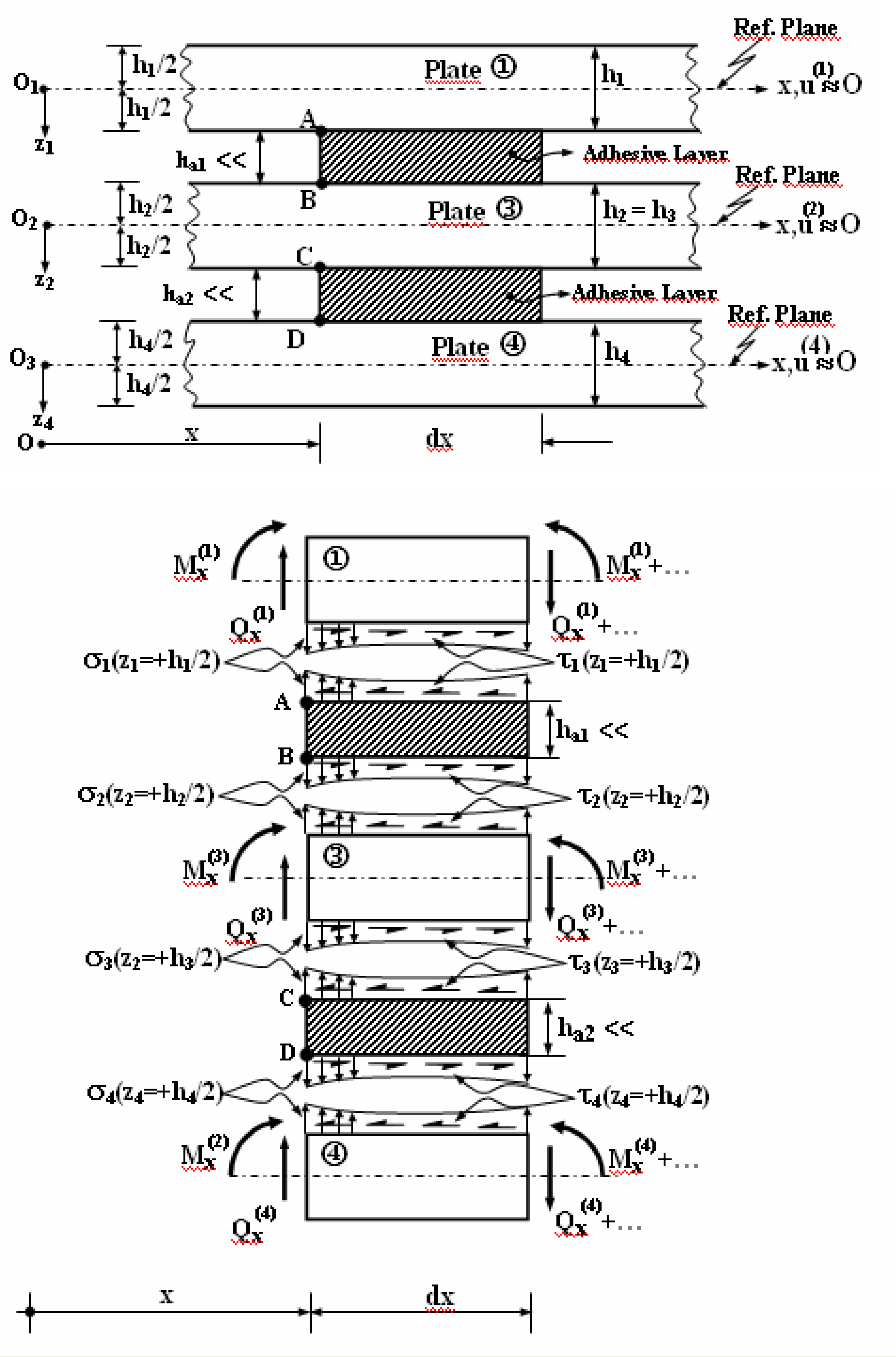


Figure 6.2 Stress Distributions at Plate Adhesive Layer Interfaces in the "Bonded Region" for Part II

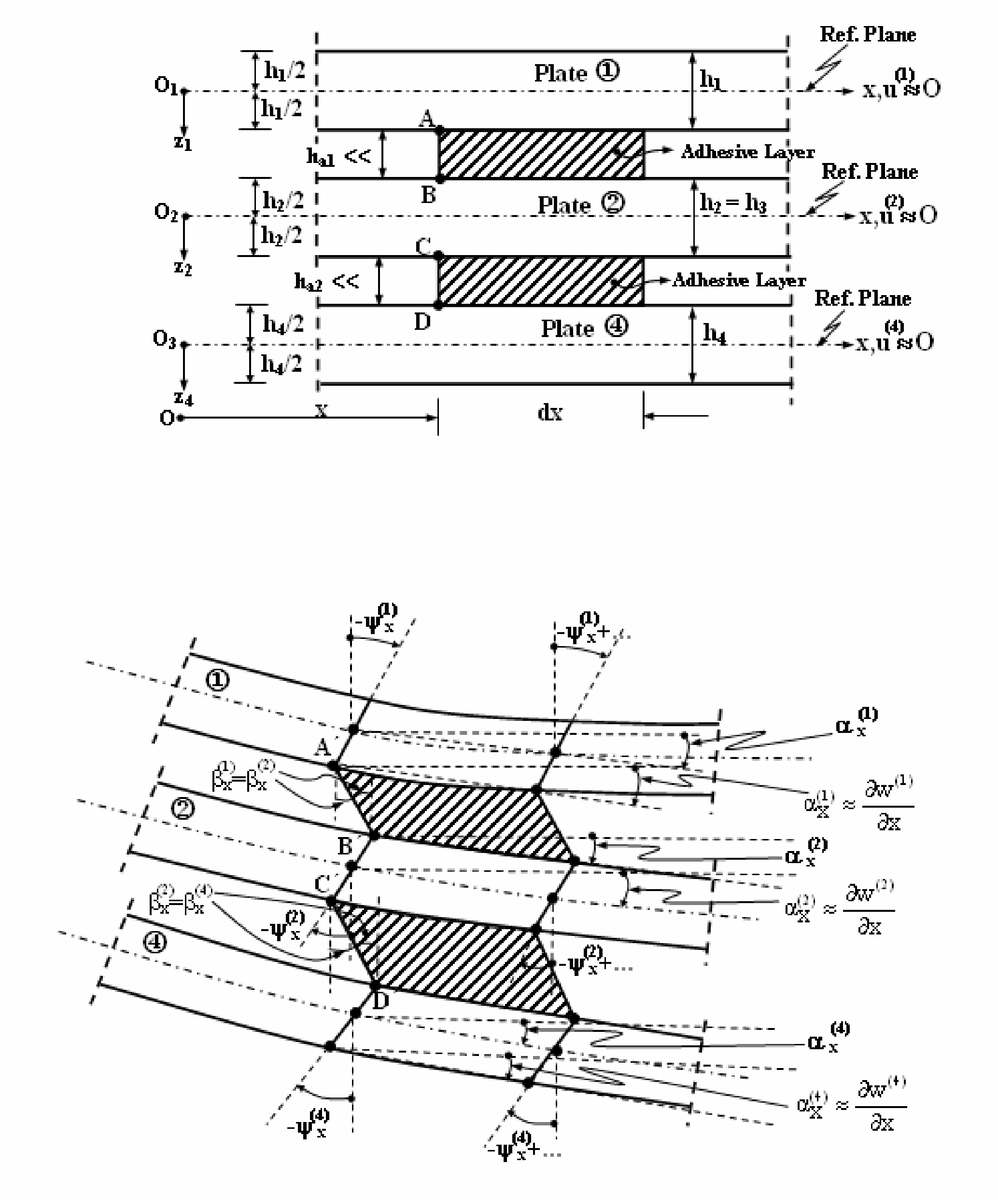


Figure 6.3 Deformations of Adherends (Mindlin Plates) and in-between Adhesive Layers in the “Bonded Region” for Part I

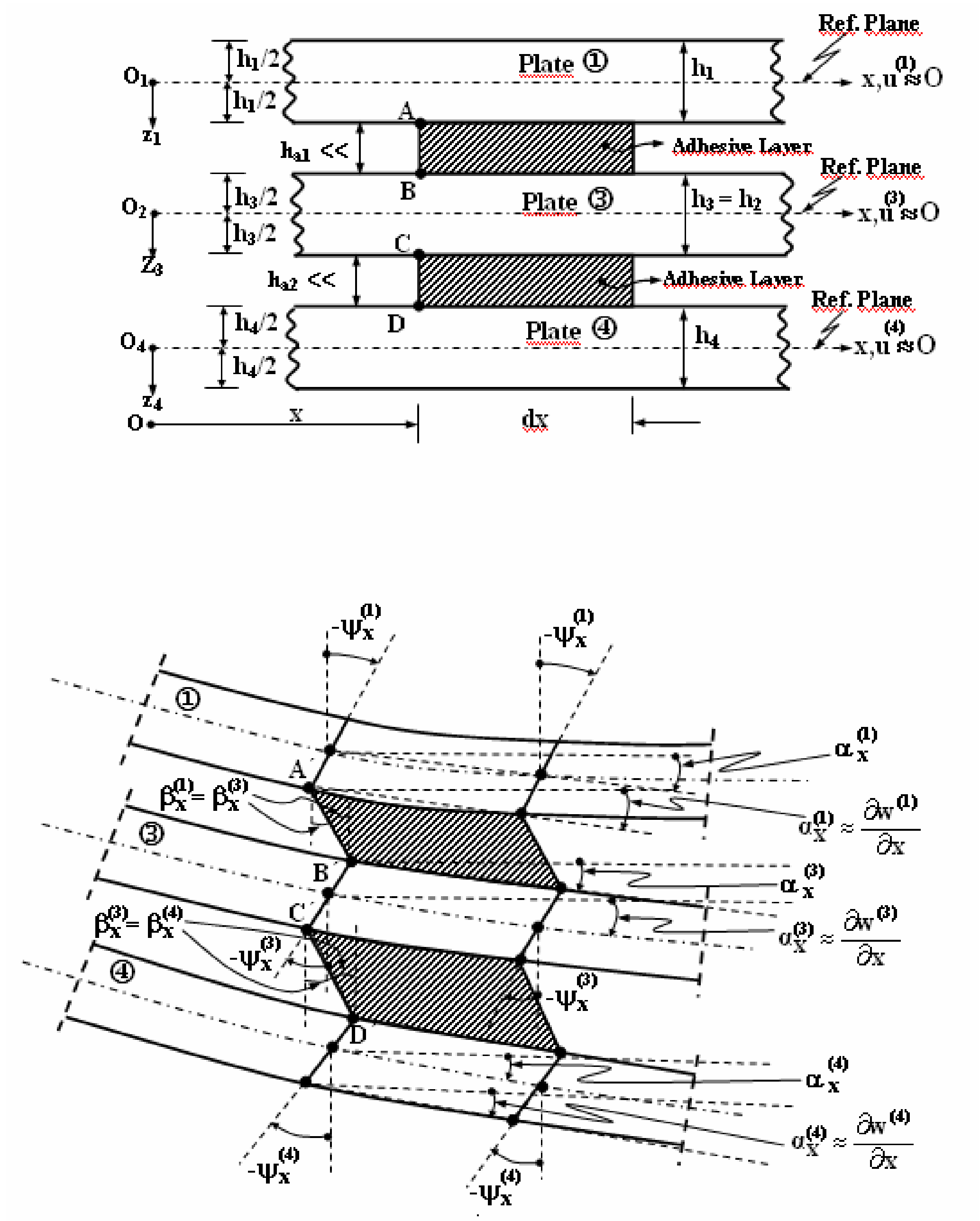


Figure 6.4 Deformations of Adherends (Mindlin Plates) and in-between Adhesive Layers in the "Bonded Region" for Part II

The purpose of this section is to obtain the relations between the stresses in the upper and lower adhesive layers and the plate interfaces and displacements of the adherends. The shear strains $\gamma_{xz}^{(j)}$ and $\gamma_{yz}^{(j)}$ at interfaces are found by considering the distortion in the right angles of the adhesive infinitesimal element after deformation as,

$$\left| \begin{array}{ll} \gamma_{xz}^{(j)} = \alpha_x^{(j)} + \beta_x^{(j)} & (\mathrm{j}=1,2,4 \ \text{ for Part I}) \\ \gamma_{yz}^{(j)} = \alpha_y^{(j)} + \beta_y^{(j)} & (\mathrm{j}=1,3,4 \ \text{ for Part II}) \end{array} \right. \tag{6.1}$$

with the assumption of small deformations, $\alpha^{(j)}$ and $\beta^{(j)}$ can be expressed as the slopes of the adherends and the displacements in x and y directions, respectively. That is,

<u>Between Plates 1 and 2 for Part I,</u>

$$\tan\left(\alpha_s^{(j)}\right) = \frac{\partial w^{(j)}}{\partial s} \approx \alpha_s^{(j)}$$
$$\tan\left(\beta_s^{(j)}\right) = \frac{\left(u_{B'}\right)_s + \left(u_{A'}\right)_s}{h_{a1}} \approx \beta_s^{(j)} \qquad (\mathrm{j}=1,2) \tag{6.2.a}$$

<u>Between Plates 1 and 3 for Part II;</u>

$$\tan\left(\alpha_s^{(j)}\right) = \frac{\partial w^{(j)}}{\partial s} \approx \alpha_s^{(j)}$$
$$\tan\left(\beta_s^{(j)}\right) = \frac{\left(u_{B'}\right)_s + \left(u_{A'}\right)_s}{h_{a1}} \approx \beta_s^{(j)} \qquad (\mathrm{j}=1,3) \tag{6.2.b}$$

<u>Between Plates 2 and 4 for Part I,</u>

$$\tan\left(\alpha_s^{(j)}\right) = \frac{\partial w^{(j)}}{\partial s} \approx \alpha_s^{(j)}$$
$$\tan\left(\beta_s^{(j)}\right) = \frac{\left(u_{D'}\right)_s + \left(u_{C'}\right)_s}{h_{a4}} \approx \beta_s^{(j)} \qquad (\mathrm{j}=2,4) \tag{6.2.c}$$

Between Plates 3 and 4 for Part II;

$$\tan\left(\alpha_s^{(j)}\right)=\frac{\partial w^{(j)}}{\partial s}\approx\alpha_s^{(j)}$$

$$\tan\left(\beta_s^{(j)}\right)=\frac{\left(u_{D'}\right)_s+\left(u_{C'}\right)_s}{h_{a4}}\approx\beta_s^{(j)} \qquad (j=3,4) \qquad (6.2.d)$$

where h_{a1} and h_{a4} are the thicknesses of the upper and lower adhesive layers, respectively and $(u_A)_s$, $(u_B)_s$ $(u_C)_s$ and $(u_D)_s$ are axial deformation of points A, B C and D in s direction respectively. Axial deformations $(u_A)_s$, $(u_B)_s$ $(u_C)_s$ and $(u_D)_s$ are caused only by the bending of the plates, then the displacement components can be written as,

For Part I;

$$(u_{A'})_s=-\frac{h_1}{2}\psi_s^{(1)}$$

$$\left(u_{B'}\right)_s=-\frac{h2}{2}\psi_s^{(2)}$$

$$(u_{C'})_s=-\frac{h2}{2}\psi_s^{(2)}$$

$$(u_{C'})_s=-\frac{h_4}{2}\psi_s^{(4)}$$

For Part II;

$$(u_{A'})_s=-\frac{h_1}{2}\psi_s^{(1)}$$

$$\left(u_{B'}\right)_s=-\frac{h3}{2}\psi_s^{(3)}$$

$$(u_{C'})_s=-\frac{h3}{2}\psi_s^{(3)}$$

$$(u_{C'})_s=-\frac{h_4}{2}\psi_s^{(4)} \qquad (6.3)$$

The shear strains in the interfaces can be expressed in terms of displacements and angles of rotation of the adherends by using (6.1) through (6.3)

Adhesive Strains Between Plates 1 and 2 for Part I

For Part I;

$$\gamma_{xz}^{(j)}=-\frac{1}{2h_{a1}}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)-\frac{\partial w^{(j)}}{\partial x}$$

$$\gamma_{yz}^{(j)}=-\frac{1}{2h_{a1}}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)-\frac{\partial w^{(j)}}{\partial y} \qquad (j=1,2) \qquad (6.4.a)$$

$$\varepsilon_z^{(j)}=w^{(2)}-w^{(1)}$$

Adhesive Strains Between Plates 1 and 3 for Part II

For Part II;

$$\gamma_{xz}^{(j)} = -\frac{1}{2h_{a1}}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(3)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial x}}}$$

$$\gamma_{yz}^{(j)} = -\frac{1}{2h_{a1}}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(3)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial y}}} \qquad (j=1,3) \qquad (6.4\ b)$$

$$\varepsilon_z^{(j)} = w^{(3)} - w^{(1)}$$

Adhesive Strains Between Plates 2 and 4 for Part I,

For Part I;

$$\gamma_{xz}^{(j)} = -\frac{1}{2h_{a4}}\left(h_2\psi_x^{(2)} + h_4\psi_x^{(4)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial x}}}$$

$$\gamma_{yz}^{(j)} = -\frac{1}{2h_{a4}}\left(h_2\psi_y^{(2)} + h_4\psi_y^{(4)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial y}}} \qquad (j=2,4) \qquad (6.4.c)$$

$$\varepsilon_z^{(j)} = w^{(4)} - w^{(2)}$$

Adhesive Strains Between Plates 2 and 4 for Part I, and Between Plates 3 and 4 for Part II

For Part II;

$$\gamma_{xz}^{(j)} = -\frac{1}{2h_{a4}}\left(h_3\psi_x^{(3)} + h_4\psi_x^{(4)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial x}}}$$

$$\gamma_{yz}^{(j)} = -\frac{1}{2h_{a4}}\left(h_3\psi_y^{(3)} + h_4\psi_y^{(4)}\right) - \underline{\underline{\frac{\partial w^{(j)}}{\partial y}}} \qquad (j=3,4) \qquad (6.4.d)$$

$$\varepsilon_z^{(j)} = w^{(4)} - w^{(3)}$$

It should be mentioned here that in some previous studies by Yuceoglu and Özerciyes [IV.4-IV.13], the double underlined terms (=) are neglected. In this present study these terms are not neglected in the theoretical formulation.

Interface shear stresses in the adhesive layers can be expressed by using the interface shear strains as,

Adhesive Stresses Between Plates 1 and 2 for Part I,

For Part I;

$$\tau_{xz}^{(j)} = G_{a1}\gamma_{xz}^{(j)} = -\frac{G_{a1}}{2h_{a1}}\left(h_1\psi_x^{(1)} + h_2\psi_x^{(2)}\right) - \underline{\underline{G_{a1}\frac{\partial w^{(j)}}{\partial x}}}$$

$$\tau_{yz}^{(j)} = G_{a1}\gamma_{yz}^{(j)} = -\frac{G_{a1}}{2h_{a1}}\left(h_1\psi_y^{(1)} + h_2\psi_y^{(2)}\right) - \underline{\underline{G_{a1}\frac{\partial w^{(j)}}{\partial y}}} \qquad \text{(j=1,2)} \quad (6.5.a)$$

$$\sigma_z^{(j)} = \frac{E_{a1}}{h_{a1}}(w^{(2)} - w^{(1)})$$

Adhesive Stresses Between Plates 1 and 3 for Part II

For Part II;

$$\tau_{xz}^{(j)} = G_{a1}\gamma_{xz}^{(j)} = -\frac{G_{a1}}{2h_{a1}}\left(h_1\psi_x^{(1)} + h_3\psi_x^{(3)}\right) - \underline{\underline{G_{a1}\frac{\partial w^{(j)}}{\partial x}}}$$

$$\tau_{yz}^{(j)} = G_{a1}\gamma_{yz}^{(j)} = -\frac{G_{a1}}{2h_{a1}}\left(h_1\psi_y^{(1)} + h_3\psi_y^{(3)}\right) - \underline{\underline{G_{a1}\frac{\partial w^{(j)}}{\partial y}}} \qquad \text{(j=1,3)} \quad (6.5.b)$$

$$\sigma_z^{(j)} = \frac{E_{a1}}{h_{a1}}(w^{(3)} - w^{(1)})$$

Adhesive Stresses Between Plates 2 and 4 for Part I,

For Part I;

$$\tau_{xz}^{(j)} = G_{a4}\gamma_{xz}^{(j)} = -\frac{G_{a4}}{2h_{a4}}\left(h_2\psi_x^{(2)} + h_4\psi_x^{(4)}\right) - \underline{\underline{G_{a4}\frac{\partial w^{(j)}}{\partial x}}}$$

$$\tau_{yz}^{(j)} = G_{a4}\gamma_{yz}^{(j)} = -\frac{G_{a4}}{2h_{a4}}\left(h_2\psi_y^{(2)} + h_4\psi_y^{(4)}\right) - \underline{\underline{G_{a4}\frac{\partial w^{(j)}}{\partial y}}} \qquad \text{(j=2,4)} \quad (6.5.c)$$

$$\sigma_z^{(j)} = \frac{E_{a4}}{h_{a4}}(w^{(4)} - w^{(2)})$$

Adhesive Stresses Between Plates 3 and 4 for Part II,

For Part II;

$$\tau_{xz}^{(j)} = G_{a4}\gamma_{xz}^{(j)} = -\frac{G_{a4}}{2h_{a4}}\left(h_3\psi_x^{(3)} + h_4\psi_x^{(4)}\right) - \underline{\underline{G_{a4}\frac{\partial w^{(j)}}{\partial x}}}$$

$$\tau_{yz}^{(j)} = G_{a4}\gamma_{yz}^{(j)} = -\frac{G_{a4}}{2h_{a4}}\left(h_3\psi_y^{(3)} + h_4\psi_y^{(4)}\right) - \underline{\underline{G_{a4}\frac{\partial w^{(j)}}{\partial y}}} \qquad (j=3,4) \quad (6.5.d)$$

$$\sigma_z^{(j)} = \frac{E_{a4}}{h_{a4}}(w^{(4)} - w^{(3)})$$

The thicknesses h_{a1} and h_{a4} of the adhesive layers are assumed to be much smaller than the thicknesses of the plates and $\psi_x^{(j)}, \psi_y^{(j)}$ have the same order of magnitudes when compared with $\frac{\partial w^{(j)}}{\partial x}$ and $\frac{\partial w^{(j)}}{\partial y}$. Therefore, one may assume that the variation of the transverse displacements in the adhesive layers is linear. This is equivalent to assume that the normal strain ε_z is constant across the thickness. E_{a1} and E_{a4} are the modulai of elasticity (or Young's modulus) and G_{a1} and G_{a4} are the modulus of rigidity (or shear modulus) of the upper and lower adhesive layers, respectively.

6.4.2 Analysis of Part I and Part II (or the "Bonded Region") of Composite Plate System

6.4.2.1 Implementation of the Adhesive Layer Equations to Governing Equations (equation of motions) of Plate Adherends.

The adhesive stresses are related to unknown displacement functions and angle of rotations of the adherends in the "Bonded Region" since they are considered as surface loads or external stresses acting on the upper and lower faces of the plates. The normal and tangential stresses at the interface may be considered as the ''compatibility or coupling conditions'' of the three plates in the "Bonded Region" in of Part I and Part II.

The governing equations (or plate equations of motion) given in (3.19) and (3.20) can be written by using the adhesive stresses as load terms q_{zx}'s and q_{zy}'s as,

For Plate 1 in Part I;

$$\left|\begin{aligned}
&\frac{\partial M_x^{(1)}}{\partial x}+\frac{\partial M_{yx}^{(1)}}{\partial y}-Q_x^{(1)}-\underline{\frac{h_1 G_{a1}}{4h_{a1}}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)}-\underline{\underline{\frac{h_1 G_{a1}}{2}\frac{\partial w^{(1)}}{\partial x}}}=\frac{\rho_1 h_1^3}{12}\frac{\partial^2\psi_x^{(1)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(1)}}{\partial x}+\frac{\partial M_{y}^{(1)}}{\partial y}-Q_y^{(1)}-\underline{\frac{h_1 G_{a1}}{4h_{a1}}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)}-\underline{\underline{\frac{h_1 G_{a1}}{2}\frac{\partial w^{(1)}}{\partial y}}}=\frac{\rho_1 h_1^3}{12}\frac{\partial^2\psi_y^{(1)}}{\partial t^2}\\
&\frac{\partial Q_x^{(1)}}{\partial x}+\frac{\partial Q_y^{(1)}}{\partial y}+\underline{\frac{E_{a1}}{h_{a1}}\left(w^{(2)}-w^{(1)}\right)}=\rho_1 h_1\frac{\partial^2 w^{(1)}}{\partial t^2}
\end{aligned}\right. \qquad (6.6.a)$$

For Plate 1 in Part II;

$$\left|\begin{aligned}
&\frac{\partial M_x^{(1)}}{\partial x}+\frac{\partial M_{yx}^{(1)}}{\partial y}-Q_x^{(1)}-\underline{\frac{h_1 G_{a1}}{4h_{a1}}\left(h_1\psi_x^{(1)}+h_3\psi_x^{(3)}\right)}-\underline{\underline{\frac{h_1 G_{a1}}{2}\frac{\partial w^{(1)}}{\partial x}}}=\frac{\rho_1 h_1^3}{12}\frac{\partial^2\psi_x^{(1)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(1)}}{\partial x}+\frac{\partial M_{y}^{(1)}}{\partial y}-Q_y^{(1)}-\underline{\frac{h_1 G_{a1}}{4h_{a1}}\left(h_1\psi_y^{(1)}+h_3\psi_y^{(3)}\right)}-\underline{\underline{\frac{h_1 G_{a1}}{2}\frac{\partial w^{(1)}}{\partial y}}}=\frac{\rho_1 h_1^3}{12}\frac{\partial^2\psi_y^{(1)}}{\partial t^2}\\
&\frac{\partial Q_x^{(1)}}{\partial x}+\frac{\partial Q_y^{(1)}}{\partial y}+\underline{\frac{E_{a1}}{h_{a1}}\left(w^{(3)}-w^{(1)}\right)}=\rho_1 h_1\frac{\partial^2 w^{(1)}}{\partial t^2}
\end{aligned}\right. \qquad (6.6.b)$$

For Plate 2 in Part I;

$$\left|\begin{aligned}
&\frac{\partial M_x^{(2)}}{\partial x}+\frac{\partial M_{yx}^{(2)}}{\partial y}-Q_x^{(2)}-\underline{\frac{h_2 G_{a1}}{4h_{a1}}\left(h_1\psi_x^{(1)}+h_2\psi_x^{(2)}\right)-\frac{h_2 G_{a4}}{4h_{a4}}\left(h_2\psi_x^{(2)}+h_4\psi_x^{(4)}\right)}\\
&-\left(\underline{\underline{\frac{h_2 G_{a1}}{2}}}+\underline{\underline{\frac{h_2 G_{a4}}{2}}}\right)\frac{\partial w^{(2)}}{\partial x}=\frac{\rho_2 h_2^3}{12}\frac{\partial^2\psi_x^{(2)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(2)}}{\partial x}+\frac{\partial M_{y}^{(2)}}{\partial y}-Q_y^{(2)}-\underline{\frac{h_2 G_{a1}}{4h_{a1}}\left(h_1\psi_y^{(1)}+h_2\psi_y^{(2)}\right)}-\underline{\frac{h_2 G_{a4}}{4h_{a4}}\left(h_2\psi_y^{(2)}+h_4\psi_y^{(4)}\right)}\\
&-\left(\underline{\underline{\frac{h_2 G_{a1}}{2}}}+\underline{\underline{\frac{h_2 G_{a4}}{2}}}\right)\frac{\partial w^{(2)}}{\partial y}=\frac{\rho_2 h_2^3}{12}\frac{\partial^2\psi_y^{(2)}}{\partial t^2}\\
&\frac{\partial Q_x^{(2)}}{\partial x}+\frac{\partial Q_y^{(2)}}{\partial y}-\underline{\frac{E_{a1}}{h_{a1}}\left(w^{(2)}-w^{(1)}\right)}+\underline{\frac{E_{a4}}{h_{a4}}\left(w^{(4)}-w^{(2)}\right)}=\rho_2 h_2\frac{\partial^2 w^{(2)}}{\partial t^2}
\end{aligned}\right. \qquad (6.6\ c)$$

For Plate 3 in Part II;

$$
\left|\begin{aligned}
&\frac{\partial M_x^{(3)}}{\partial x}+\frac{\partial M_{yx}^{(3)}}{\partial y}-Q_x^{(3)}-\underline{\frac{h_3G_{a1}}{4h_{a1}}\left(h_1\psi_x^{(1)}+h_3\psi_x^{(3)}\right)-\frac{h_3G_{a4}}{4h_{a4}}\left(h_3\psi_x^{(3)}+h_4\psi_x^{(4)}\right)-}\\
&-\left(\underline{\underline{\frac{h_3G_{a1}}{2}}}+\underline{\underline{\frac{h_3G_{a4}}{2}}}\right)\frac{\partial w^{(3)}}{\partial x}=\frac{\rho_3h_3^3}{12}\frac{\partial^2\psi_x^{(3)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(3)}}{\partial x}+\frac{\partial M_y^{(3)}}{\partial y}-Q_y^{(3)}-\underline{\frac{h_3G_{a1}}{4h_{a1}}\left(h_1\psi_y^{(1)}+h_3\psi_y^{(3)}\right)}-\underline{\frac{h_3G_{a4}}{4h_{a4}}\left(h_3\psi_y^{(3)}+h_4\psi_y^{(4)}\right)}\\
&-\left(\underline{\underline{\frac{h_3G_{a1}}{2}}}+\underline{\underline{\frac{h_3G_{a4}}{2}}}\right)\frac{\partial w^{(3)}}{\partial y}=\frac{\rho_3h_3^3}{12}\frac{\partial^2\psi_y^{(3)}}{\partial t^2}\\
&\frac{\partial Q_x^{(3)}}{\partial x}+\frac{\partial Q_y^{(3)}}{\partial y}-\underline{\frac{E_{a1}}{h_{a1}}\left(w^{(3)}-w^{(1)}\right)}+\underline{\frac{E_{a4}}{h_{a4}}\left(w^{(4)}-w^{(3)}\right)}=\rho_3h_3\frac{\partial^2w^{(3)}}{\partial t^2}
\end{aligned}\right. \qquad (6.6\ d)
$$

For Plate 4 in Part I;

$$
\left|\begin{aligned}
&\frac{\partial M_x^{(4)}}{\partial x}+\frac{\partial M_{yx}^{(4)}}{\partial y}-Q_x^{(4)}-\underline{\underline{\frac{h_4G_{a4}}{4h_{a4}}\left(h_2\psi_x^{(2)}+h_4\psi_x^{(4)}\right)}}-\frac{h_4G_{a4}}{\underline{\underline{2}}}\frac{\partial w^{(4)}}{\partial x}=\frac{\rho_4h_4^3}{12}\frac{\partial^2\psi_x^{(4)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(4)}}{\partial x}+\frac{\partial M_y^{(4)}}{\partial y}-Q_y^{(4)}-\underline{\frac{h_4G_{a4}}{4h_{a4}}\left(h_4\psi_y^{(2)}+h_4\psi_y^{(4)}\right)}-\frac{h_4G_{a4}}{2}\frac{\partial w^{(4)}}{\partial y}=\frac{\rho_4h_4^3}{12}\frac{\partial^2\psi_y^{(4)}}{\partial t^2}\\
&\frac{\partial Q_x^{(4)}}{\partial x}+\frac{\partial Q_y^{(4)}}{\partial y}-\underline{\frac{E_{a4}}{h_{a4}}\left(w^{(4)}-w^{(2)}\right)}=\rho_4h_4\frac{\partial^2w^{(4)}}{\partial t^2}
\end{aligned}\right. \qquad (6.6.e)
$$

For Plate 4 in Part I;

$$
\left|\begin{aligned}
&\frac{\partial M_x^{(4)}}{\partial x}+\frac{\partial M_{yx}^{(4)}}{\partial y}-Q_x^{(4)}-\underline{\underline{\frac{h_4G_{a4}}{4h_{a4}}\left(h_3\psi_x^{(3)}+h_4\psi_x^{(4)}\right)}}-\frac{h_4G_{a4}}{\underline{\underline{2}}}\frac{\partial w^{(4)}}{\partial x}=\frac{\rho_4h_4^3}{12}\frac{\partial^2\psi_x^{(4)}}{\partial t^2}\\
&\frac{\partial M_{yx}^{(4)}}{\partial x}+\frac{\partial M_y^{(4)}}{\partial y}-Q_y^{(4)}-\underline{\frac{h_4G_{a4}}{4h_{a4}}\left(h_3\psi_y^{(3)}+h_4\psi_y^{(4)}\right)}-\frac{h_4G_{a4}}{2}\frac{\partial w^{(4)}}{\partial y}=\frac{\rho_4h_4^3}{12}\frac{\partial^2\psi_y^{(4)}}{\partial t^2}\\
&\frac{\partial Q_x^{(4)}}{\partial x}+\frac{\partial Q_y^{(4)}}{\partial y}-\underline{\frac{E_{a4}}{h_{a4}}\left(w^{(4)}-w^{(3)}\right)}=\rho_4h_4\frac{\partial^2w^{(4)}}{\partial t^2}
\end{aligned}\right. \qquad (6.6.f)
$$

In the above equations, the singly and doubly "underlined terms" are the "coupling terms" between the middle plates and the upper and lower doublers.

At this stage, the "Classical Levy's Method" with the trigonometric series expansions is to be considered. The "Classical Lévy's Method" is restricted to rectangular plates with any two opposite edges simply supported. However, the other two edges may have arbitrary boundary conditions.

In the present study, the edges at x=0 and x=a are simply supported. "Boundary Conditions" which have to be satisfied along these edges are as follows,

$$\text{at } x=0,a \rightarrow \qquad w^{(j)}=0,\ M_x^{(j)}=0,\ \psi_y^{(j)}=0 \qquad (j=1,2,3,4) \tag{6.7}$$

By using the "Classical Lévy's Method" angles of rotations and displacements of the adherends can be expressed as,

Displacements and Angles of Rotation for Part I,

$$w^{(j)}(\eta,\xi_I,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}h_1\overline{W}_{mn}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$\psi_x^{(j)}(\eta,\xi_I,t)=\frac{a}{h_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Phi}_{mnx}^{(j)}(\xi_I)cos(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad (j=1,2,4) \tag{6.8}$$

$$\psi_y^{(j)}(\eta,\xi_I,t)=\frac{h_1}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Phi}_{mny}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

Displacements and Angles of Rotation, for Part II,

$$w^{(j)}(\eta,\xi_{II},t)=h_I\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{W}_{mn}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

$$\psi_x^{(j)}(\eta,\xi_{II},t)=\frac{a}{h_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Phi}_{mnx}^{(j)}(\xi_{II})cos(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad (j=1,3,4) \tag{6.9}$$

$$\psi_y^{(j)}(\eta,\xi_{II},t)=\frac{h_1}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\Phi}_{mny}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

where "m" is the number of half-waves in the x-direction, the superscript "j" denotes the middle adherend plates for j=2 and j=3, upper doubler for j=1 and the lower doubler for j=4. In the above equations, "i" is defined as $\sqrt{-1}$ and the "barred" (—) quantities are the dimensionless transverse displacements and angles of rotation. The nondimensional independent space variables η, ξ_I and ξ_{II} are defined as x/a, y_I/ℓ_I, and y_{II}/ℓ_{II} respectively. Here, "$\overline{\omega}_{mn}$" is the dimensionless circular frequency or the natural frequency of the flexural (or bending) vibrations of the entire composite bonded plate or panel system.

The "Stress Resultants" can also be expressed in trigonometric series in the x-direction as,

Stress Resultants for Part I;

$$M_x^{(j)}(\eta,\xi_I,t)=\frac{h_1^3 B_{11}^{(1)}}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnx}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$
$$M_{yx}^{(j)}(\eta,\xi_I,t)=h_1^2 B_{11}^{(1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnyx}^{(j)}(\xi_I)cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$
$$M_y^{(j)}(\eta,\xi_I,t)=h_1^2 B_{11}^{(1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mny}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad (j=1,2,4) \qquad (6.10)$$
$$Q_y^{(j)}(\eta,\xi_I,t)=h_1 B_{11}^{(1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mny}^{(j)}(\xi_I)sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$
$$Q_x^{(j)}(\eta,\xi_I,t)=\frac{h_1^2 B_{11}^{(1)}}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mnx}^{(j)}(\xi_I)cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

Stress Resultants for Part II;

$$M_x^{(j)}(\eta,\xi_{II},t)=\frac{h_1^3 B_{11}^{(1)}}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnx}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$
$$M_{yx}^{(j)}(\eta,\xi_{II},t)=h_1^2 B_{11}^{(1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mnyx}^{(j)}(\xi_{II})cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$
$$M_y^{(j)}(\eta,\xi_{II},t)=h_1^2 B_{11}^{(1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{M}_{mny}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t} \qquad (j=1,3,4) \qquad (6.11)$$
$$Q_y^{(j)}(\eta,\xi_{II},t)=h_1 B_{11}^{(1)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mny}^{(j)}(\xi_{II})sin(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$
$$Q_x^{(j)}(\eta,\xi_{II},t)=\frac{h_1^2 B_{11}^{(1)}}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{Q}_{mnx}^{(j)}(\xi_{II})cos(m\pi\eta)e^{i\overline{\omega}_{mn}t}$$

Where again the "barred" (—) $\overline{W}_{mn}^{(j)},\overline{\Psi}_{mnx}^{(j)},\overline{\Psi}_{mny}^{(j)},\overline{M}_{mnx}^{(j)},\overline{M}_{mnyx}^{(j)},\overline{M}_{mny}^{(j)},\overline{Q}_{mny}^{(j)},\overline{Q}_{mnx}^{(j)}$ are the "dimensionless fundamental dependent variables" of the problem under consideration.

For non dimensionalization of equations, the parameters in the governing differential equations shall be nondimensionalized with respect to the main or reference quantities which are chosen as "$B_{11}^{(1)}$", "h_1", "ρ_1" and "a".

The <u>dimensionless coordinates</u> or <u>independent space variables</u> are,

$$\eta = x/a,$$

$$\xi_I = y_I/l_I, \qquad \xi_{II} = y_{II}/l_{II}, \qquad \xi_{III} = y_{III}/l_{III} \qquad \xi_{IV} = y_{IV}/l_{IV} \tag{6.12}$$

The <u>dimensionless parameters</u> related to <u>orthotropic elastic constants</u> of adherends and the <u>adhesive layers</u> are,

$$\begin{aligned}
&\overline{B}_{ik}^{(j)} = B_{ik}^{(j)} \big/ B_{11}^{(1)} && (j = 1,2,3,4 \; and \; i,k = 1,2,3), \\
&\overline{B}_{\ell\ell}^{(j)} = B_{\ell\ell}^{(j)} \big/ B_{11}^{(1)} && (j = 1,2,3,4 \; and \; \ell = 4,5,6) \\
&\overline{G}_{a1} = G_{a1} \big/ B_{11}^{(1)}, && \overline{G}_{a4} = G_{a4} \big/ B_{11}^{(1)} \\
&\overline{E}_{a1} = E_{a1} \big/ B_{11}^{(1)}, && \overline{E}_{a4} = E_{a4} \big/ B_{11}^{(1)}
\end{aligned} \tag{6.13}$$

The <u>dimensionless parameters</u> related to the <u>densities</u> and the <u>geometry</u> of the plates and the adhesive layers are,

$$\begin{aligned}
&\overline{\rho}_2 = \rho_2 \big/ \rho_1, \quad \overline{\rho}_3 = \rho_3 \big/ \rho_1, \quad \overline{\rho}_4 = \rho_4 \big/ \rho_1 \\
&\overline{h}_{a1} = h_{a1} \big/ h_1, \quad \overline{h}_{a4} = h_{a4} \big/ h_1 \\
&\overline{L}_I = \ell_I \big/ a, \quad \overline{L}_{II} = \ell_{II} \big/ a, \quad \overline{L}_{III} = \ell_{III} \big/ a, \quad \overline{L}_{IV} = \ell_{IV} \big/ a \\
&\overline{h}_2 = h_2 \big/ h_1, \quad \overline{h}_3 = h_3 \big/ h_1, \quad \overline{h}_4 = h_4 \big/ h_1
\end{aligned} \tag{6.14}$$

The dimensionless frequency parameter $\overline{\omega}_{mn}$ of the entire, composite bonded joint plate or panel system is;

$$\left| \begin{array}{l} \overline{\omega}_{mn} = \rho_1 a^4 \omega_{mn}^2 / h_1^2 B_{11}^{(1)} \\ \overline{\Omega} = \overline{\omega}_{mn} \end{array} \right. \qquad (m,n=1,2,3...) \qquad (6.15)$$

where the dimensionless natural frequency parameter $\overline{\omega}_{mn}$ depending on the given m, n values, is ordered in terms of its magnitudes as $\overline{\Omega}_1<\overline{\Omega}_2<\overline{\Omega}_3<....$ with the subscript indicating the first, second, third, etc. dimensionless natural frequencies.

6.4.2.2 Reduction to "Governing Systems of First Order Ordinary Differential Equations" for "Main PROBLEM III",

As was done in previous "Main PROBLEMS" M_y, M_{xy}, Q_y, ψ_x, ψ_y and w are chosen as intrinsic variables and M_x, Q_x are chosen as auxiliary variables. Thus, from the Stress Resultants and Moment Resultant Equations given in (3.20), first order partial differential equations can be written with respect to the dimensionless independent variables ξ and η as,

For the "Bonded Region" or Part I;

$$\left| \begin{array}{l} \frac{1}{l_I}\frac{\partial \psi_y^{(j)}}{\partial \xi_I} = \frac{1}{B_{22}^{(j)}}\left(\frac{12}{h_j^3}M_y^{(j)} - B_{12}^{(j)}\frac{1}{a}\frac{\partial \psi_x^{(j)}}{\partial \eta}\right) \\ \frac{1}{l_I}\frac{\partial \psi_x^{(j)}}{\partial \xi_I} = \frac{12}{h_j^{(3)} B_{66}^{(j)}}M_{yx}^{(j)} - \frac{1}{a}\frac{\partial \psi_y^{(j)}}{\partial \eta} \\ \frac{1}{l_I}\frac{\partial w^{(j)}}{\partial \xi_I} = \frac{1}{\kappa_y^2 h_j B_{44}^{(j)}}Q_y^{(j)} - \psi_y^{(j)} \\ \frac{1}{l_I}\frac{\partial M_{yx}^{(j)}}{\partial \xi_I} = \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_x^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_x^{(j)}}{\partial \eta} + Q_x^{(j)} - \frac{h_j}{2}\left(q_{zx}^{(+)} + q_{zx}^{(-)}\right) \\ \frac{1}{l_\imath}\frac{\partial M_y^{(j)}}{\partial \xi_I} = \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_y^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_{yx}^{(j)}}{\partial \eta} + Q_y^{(j)} - \frac{h_j}{2}\left(q_{zy}^{(+)} + q_{zy}^{(-)}\right) \\ \frac{1}{l_I}\frac{\partial Q_y^{(j)}}{\partial \xi_I} = \rho_j h_j \frac{\partial^2 w^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial Q_x^{(j)}}{\partial \eta} - \left(q_z^{(+)} - q_z^{(-)}\right) \end{array} \right. \qquad (j=1,2,4) \quad (6.16)$$

For the <u>"Bonded Region" or Part II;</u>

$$\frac{1}{l_{II}}\frac{\partial \psi_y^{(j)}}{\partial \xi_{II}} = \frac{1}{B_{22}^{(j)}}\left(\frac{12}{h_j^3}M_y^{(j)} - B_{12}^{(j)}\frac{1}{a}\frac{\partial \psi_x^{(j)}}{\partial \eta}\right)$$

$$\frac{1}{l_{II}}\frac{\partial \psi_x^{(j)}}{\partial \xi_{II}} = \frac{12}{h_j^{(3)}B_{66}^{(j)}}M_{yx}^{(j)} - \frac{1}{a}\frac{\partial \psi_y^{(j)}}{\partial \eta}$$

$$\frac{1}{l_{II}}\frac{\partial w^{(j)}}{\partial \xi_{II}} = \frac{1}{\kappa_y^2 h_j B_{44}^{(j)}}Q_y^{(j)} - \psi_y^{(j)} \qquad \text{(j=1,3,4)} \quad (6.17)$$

$$\frac{1}{l_{II}}\frac{\partial M_{yx}^{(j)}}{\partial \xi_{II}} = \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_x^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_x^{(j)}}{\partial \eta} + Q_x^{(j)} - \frac{h_j}{2}\left(q_{zx}^{(+)} + q_{zx}^{(-)}\right)$$

$$\frac{1}{l_{II}}\frac{\partial M_y^{(j)}}{\partial \xi_{II}} = \frac{\rho_j h_j^3}{12}\frac{\partial^2 \psi_y^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial M_{yx}^{(j)}}{\partial \eta} + Q_y^{(j)} - \frac{h_j}{2}\left(q_{zy}^{(+)} + q_{zy}^{(-)}\right)$$

$$\frac{1}{l_{II}}\frac{\partial Q_y^{(j)}}{\partial \xi_{II}} = \rho_j h_j \frac{\partial^2 w^{(j)}}{\partial t^2} - \frac{1}{a}\frac{\partial Q_x^{(j)}}{\partial \eta} - \left(q_z^{(+)} - q_z^{(-)}\right)$$

where q's are surface loads and the stresses on the upper (-) and lower (+) surfaces of the plates. Also, note that the superscript and the subscript "j" denotes the middle adherends for j=2 and j=3, the upper doubler for j=1 and the lower doubler for j=4.

By substituting (6.10) and (6.11) in to (6.17) and making the necessary non-dimensionalizaitons with respect to (6.13), (6.14) and (6.15), "Governing System of First Order Ordinary Differential Equations" for the entire bonded joint plate or panel system are obtained as,

For Plate 1, in Part I (Bonded Region);

$$
\begin{aligned}
&\frac{d\overline{\Psi}^{(1)}_{mnx}}{d\xi_I} = \frac{12\overline{L}_I}{\overline{B}^{(1)}_{66}}\overline{M}^{(1)}_{mnyx} - \overline{L}_I\left(\frac{h_1}{a}\right)^2 m\pi\overline{\Psi}^{(1)}_{mny} \\
&\frac{d\overline{\Psi}^{(1)}_{mny}}{d\xi_I} = \frac{12\overline{L}_I}{\overline{B}^{(1)}_{22}}\left(\frac{a}{h_1}\right)^2\overline{M}^{(1)}_{mny} + \overline{L}_I\frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}}\left(\frac{a}{h_1}\right)^2 m\pi\overline{\Psi}^{(1)}_{mnx} \\
&\frac{d\overline{W}_{mn}^{(1)}}{d\xi_I} = \overline{L}_I\frac{a}{h_1}\frac{1}{\kappa_y^2\overline{B}^{(1)}_{44}}\overline{Q}^{(1)}_{mny} - \overline{L}_I\overline{\Psi}^{(1)}_{mny} \\
&\frac{d\overline{M}^{(1)}_{mnyx}}{d\xi_I} = \left\{\begin{aligned}&-\frac{\overline{L}_I}{12}\frac{a^4\rho_1\omega_{mn}^2}{h_1^2 B^{(1)}_{11}}\left(\frac{h_1}{a}\right)^2 + \frac{\overline{L}_I}{12}(m\pi)^2\left(\overline{B}^{(1)}_{11} - \frac{\left(\overline{B}^{(1)}_{12}\right)^2}{\overline{B}^{(1)}_{22}}\right)\\ &+\overline{L}_I\left(\frac{a}{h_1}\right)^2\kappa_x^2\overline{B}^{(1)}_{55} + \overline{L}_I\frac{\overline{G}_{a1}}{4\overline{h}_{a1}}\left(\frac{a}{h_1}\right)^2\end{aligned}\right\}\overline{\Psi}^{(1)}_{mnx} \\
&-\overline{L}_I\frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}}m\pi\overline{M}^{(1)}_{mny} + \left\{\overline{L}_I\kappa_x^2\overline{B}^{(1)}_{55}m\pi + \underline{\underline{\frac{\overline{L}_I}{2}\overline{G}_{a1}m\pi}}\right\}\overline{W}^{(1)}_{mn} \\
&+\overline{L}_I\frac{\overline{G}_{a1}}{4\overline{h}_{a1}}\left(\frac{a}{h_1}\right)^2\overline{h}_2\overline{\Psi}^{(2)}_{mnx} \\
&\frac{d\overline{M}^{(1)}_{mny}}{d\xi_I} = \left\{-\frac{\overline{L}_I}{12}\frac{\rho_1 a^4\omega_{mn}^2}{h_1^2 B^{(1)}_{11}}\left(\frac{h_1}{a}\right)^4 + \overline{L}_I\frac{\overline{G}_{a1}}{4\overline{h}_{a1}} - \underline{\underline{\frac{\overline{L}_I}{2}\overline{G}_{a1}}}\right\}\overline{\Psi}^{(1)}_{mny} + \overline{L}_I m\pi\overline{M}^{(1)}_{mnyx} \\
&+\left\{\overline{L}_I\frac{a}{h_1} + \underline{\underline{\frac{\overline{L}_I}{2\kappa_y^2\overline{B}^{(1)}_{44}}\overline{G}_{a1}\frac{a}{h_1}}}\right\}\overline{Q}_{mny}^{(1)} + \overline{L}_I\frac{\overline{G}_{a1}}{4\overline{h}_{a1}}\overline{h}_2\overline{\Psi}^{(2)}_{mny} \\
&\frac{d\overline{Q}^{(j)}_{mny}}{d\xi_I} = \left\{-\overline{L}_I\frac{\rho_1 a^4\omega_{mn}^2}{h_1^2 B^{(1)}_{11}}\left(\frac{h_1}{a}\right)^3 + \overline{L}_I\kappa_x^2\overline{B}^{(1)}_{55}\frac{h_1}{a}(m\pi)^2 + \overline{L}_I\frac{a}{h_1}\frac{\overline{E}_{a1}}{\overline{h}_{a1}}\right\}\overline{W}^{(1)}_{mn} \\
&+\overline{L}_I\kappa_x^2\overline{B}^{(1)}_{55}\frac{a}{h_1}m\pi\overline{\Psi}^{(1)}_{mnx} - \overline{L}_I\frac{a}{h_1}\frac{\overline{E}_{a1}}{\overline{h}_{a1}}\overline{W}^{(2)}_{mn}
\end{aligned}
\tag{6.18}
$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or “doubly underlined terms”).

For Plate 1, in Part II (Bonded Region);

$$\frac{d\overline{\Psi}_{mnx}^{(1)}}{d\xi_{II}} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(1)}}\overline{M}_{mnyx}^{(1)} - \overline{L}_{II}\left(\frac{h_1}{a}\right)^2 m\pi\overline{\Psi}_{mny}^{(1)}$$

$$\frac{d\overline{\Psi}_{mny}^{(1)}}{d\xi_{II}} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(1)}}\left(\frac{a}{h_1}\right)^2 \overline{M}_{mny}^{(1)} + \overline{L}_{II}\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}\left(\frac{a}{h_1}\right)^2 m\pi\overline{\Psi}_{mnx}^{(1)}$$

$$\frac{d\overline{W}_{mn}^{(1)}}{d\xi_{II}} = \overline{L}_{II}\frac{a}{h_1}\frac{1}{\kappa_y^2\overline{B}_{44}^{(1)}}\overline{Q}_{mny}^{(1)} - \overline{L}_{II}\overline{\Psi}_{mny}^{(1)}$$

$$\frac{d\overline{M}_{mnyx}^{(1)}}{d\xi_{II}} = \left\{\begin{matrix} -\dfrac{\overline{L}_{II}}{12}\dfrac{a^4\rho_1\omega_{mn}^2}{h_1^2 B_{11}^{(1)}}\left(\dfrac{h_1}{a}\right)^2 + \dfrac{\overline{L}_{II}}{12}(m\pi)^2\left(\overline{B}_{11}^{(1)} - \dfrac{\left(\overline{B}_{12}^{(1)}\right)^2}{\overline{B}_{22}^{(1)}}\right) \\ +\overline{L}_{II}\left(\dfrac{a}{h_1}\right)^2\kappa_x^2\overline{B}_{55}^{(1)} + \overline{L}_{II}\dfrac{\overline{G}_{a1}}{4\overline{h}_{a1}}\left(\dfrac{a}{h_1}\right)^2 \end{matrix}\right\}\overline{\Psi}_{mnx}^{(1)}$$

$$-\overline{L}_{II}\frac{\overline{B}_{12}^{(1)}}{\overline{B}_{22}^{(1)}}m\pi\overline{M}_{mny}^{(1)} + \left\{\overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(1)}m\pi + \underline{\underline{\frac{\overline{L}_{II}}{2}\overline{G}_{a1}m\pi}}\right\}\overline{W}_{mn}^{(1)}$$

$$+\overline{L}_{II}\frac{\overline{G}_{a1}}{4\overline{h}_{a1}}\left(\frac{a}{h_1}\right)^2\overline{h}_3\overline{\Psi}_{mnx}^{(3)}$$

$$\frac{d\overline{M}_{mny}^{(1)}}{d\xi_{II}} = \left\{-\frac{\overline{L}_{II}}{12}\frac{\rho_1 a^4\omega_{mn}^2}{h_1^2 B_{11}^{(1)}}\left(\frac{h_1}{a}\right)^4 \underline{\underline{-\frac{\overline{L}_{II}}{2}\overline{G}_{a1} + \overline{L}_{II}\frac{\overline{G}_{a1}}{4\overline{h}_{a1}}}}\right\}\overline{\Psi}_{mny}^{(1)} + \overline{L}_{II}m\pi\overline{M}_{mnyx}^{(1)}$$

$$+\left\{\overline{L}_{II}\frac{a}{h_1} + \underline{\underline{\frac{\overline{L}_{II}}{2\kappa_y^2\overline{B}_{44}^{(1)}}\overline{G}_{a1}\frac{a}{h_1}}}\right\}\overline{Q}_{mny}^{(1)} + \overline{L}_{II}\frac{\overline{G}_{a1}}{4\overline{h}_{a1}}\overline{h}_3\overline{\Psi}_{mny}^{(3)} \tag{6.19}$$

$$\frac{d\overline{Q}_{mny}^{(j)}}{d\xi_{II}} = \left\{-\overline{L}_{II}\frac{\rho_1 a^4\omega_{mn}^2}{h_1^2 B_{11}^{(1)}}\left(\frac{h_1}{a}\right)^3 + \overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(1)}\frac{h_1}{a}(m\pi)^2 + \overline{L}_{II}\frac{a}{h_1}\frac{\overline{E}_{a1}}{\overline{h}_{a1}}\right\}\overline{W}_{mn}^{(1)}$$

$$+\overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(1)}\frac{a}{h_1}m\pi\overline{\Psi}_{mnx}^{(1)} - \overline{L}_{II}\frac{a}{h_1}\frac{\overline{E}_{a1}}{\overline{h}_{a1}}\overline{W}_{mn}^{(3)}$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

For Plate 2, in Part I (Bonded Region);

$$
\begin{aligned}
\frac{d\overline{\Psi}^{(2)}_{mnx}}{d\xi_I} &= \overline{L}_I \frac{12}{\overline{h}_2^3 \overline{B}_{66}^{(2)}} \overline{M}^{(2)}_{mnyx} - \overline{L}_I \left(\frac{h_1}{a}\right)^2 m\pi \overline{\Psi}^{(2)}_{mny} \\
\frac{d\overline{\Phi}^{(2)}_{mny}}{d\xi_I} &= \overline{L}_I \left(\frac{a}{h_1}\right)^2 \frac{12}{\overline{B}_{22}^{(2)} \overline{h}_2^3} \overline{M}^{(2)}_{mny} + \overline{L}_I \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}} \left(\frac{a}{h_1}\right)^2 m\pi \overline{\Psi}^{(2)}_{mnx} \\
\frac{d\overline{W}_{mn}^{(2)}}{d\xi_I} &= \overline{L}_I \frac{a}{h_1} \frac{1}{\kappa_y^2 \overline{h}_2 \overline{B}_{44}^{(2)}} \overline{Q}^{(2)}_{mny} - \overline{L}_I \overline{\Psi}^{(2)}_{mny} \\
\frac{d\overline{M}^{(2)}_{mnyx}}{d\xi_I} &= \left\{
\begin{aligned}
&-\overline{L}_I \overline{\rho}_2 \overline{h}_2^3 \frac{\rho_1 a^4 \omega_{mn}^2}{12 h_1^2 B_{11}^{(1)}} \left(\frac{h_1}{a}\right)^2 + \overline{L}_I \kappa_x^2 \overline{h}_2 \overline{B}_{55}^{(2)} \left(\frac{a}{h_1}\right)^2 \\
&+ \overline{L}_I \frac{\overline{h}_2^3}{12} \left(\overline{B}_{11}^{(2)} - \frac{\left(\overline{B}_{12}^{(2)}\right)^2}{\overline{B}_{22}^{(2)}} \right) (m\pi)^2 + \overline{L}_I \left(\frac{a}{h_1}\right)^2 \frac{\overline{G}_{a1} \left(\overline{h}_2\right)^2}{4\overline{h}_{a1}} \\
&+ \overline{L}_I \left(\frac{a}{h_1}\right)^2 \frac{\left(\overline{h}_2\right)^2 \overline{G}_{a4}}{4\overline{h}_{a4}}
\end{aligned}
\right\} \overline{\Psi}^{(2)}_{mnx} \\
&- \overline{L}_I \frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}} m\pi \overline{M}^{(2)}_{mny} + \left\{ \overline{L}_I \kappa_x^2 \overline{h}_2 m\pi \overline{B}_{55}^{(2)} + \underline{\underline{\overline{L}_I \frac{\overline{h}_2 m\pi}{2} \left(\overline{G}_{a1} + \overline{G}_{a4}\right)}} \right\} \overline{W}^{(2)}_{mn} \\
&+ \overline{L}_I \left(\frac{a}{h_1}\right)^2 \frac{\overline{G}_{a1} \overline{h}_2}{4\overline{h}_{a1}} \overline{\Psi}^{(1)}_{mnx} + \overline{L}_I \left(\frac{a}{h_1}\right)^2 \frac{\overline{h}_2 \overline{G}_{a4}}{4\overline{h}_{a4}} \overline{h}_4 \overline{\Phi}^{(4)}_{mnx} \\
\frac{d\overline{M}^{(2)}_{mny}}{d\xi_I} &= \left\{
\begin{aligned}
&-\overline{L}_I \overline{\rho}_2 \overline{h}_2^3 \frac{a^4 \rho_1 \omega_{mn}^2}{12 h_1^2 B_{11}^{(1)}} \left(\frac{h_1}{a}\right)^4 + \overline{L}_I \frac{\left(\overline{h}_2\right)^2 \overline{G}_{a1}}{4\overline{h}_{a1}} + \overline{L}_I \frac{\left(\overline{h}_2\right)^2 \overline{G}_{a4}}{4\overline{h}_{a4}} \\
&\underline{\underline{-\overline{L}_I \frac{\overline{h}_2}{2} \left(\overline{G}_{a1} + \overline{G}_{a4}\right)}}
\end{aligned}
\right\} \overline{\Psi}^{(2)}_{mny} \\
&+ \overline{L}_I (m\pi) \overline{M}^{(2)}_{mnyx} + \overline{L}_I \frac{\overline{h}_2 \overline{G}_{a1}}{4\overline{h}_{a1}} \overline{\Psi}^{(1)}_{mny} + \overline{L}_I \frac{\overline{h}_2 \overline{G}_{a4}}{4\overline{h}_{a4}} \overline{h}_4 \overline{\Psi}^{(4)}_{mny} \\
&+ \left\{ \underline{\underline{+\overline{L}_I \frac{a}{h_1} \frac{1}{2\kappa_y^2 \overline{B}_{44}^{(2)}} \left(\overline{G}_{a1} + \overline{G}_{a4}\right)}} + \overline{L}_I \frac{a}{h_1} \right\} \overline{Q}^{(2)}_{mny} \\
\frac{d\overline{Q}^{(2)}_{mny}}{d\xi_I} &= \left\{
\begin{aligned}
&-\overline{L}_I \overline{\rho}_2 \overline{h}_2 \frac{\rho_1 a^4 \omega_{mn}^2}{h_1^2 B_{11}^{(1)}} \left(\frac{h_1}{a}\right)^3 + \overline{L}_I \kappa_x^2 \overline{h}_2 \overline{B}_{55}^{(2)} \frac{h_1}{a} (m\pi)^2 \\
&+ \overline{L}_I \frac{a}{h_1} \frac{\overline{E}_{a1}}{\overline{h}_{a1}} + \overline{L}_I \frac{a}{h_1} \frac{\overline{E}_{a4}}{\overline{h}_{a4}}
\end{aligned}
\right\} \overline{W}^{(2)}_{mn} + \\
&\overline{L}_I \kappa_x^2 \overline{h}_2 \overline{B}_{55}^{(2)} \frac{a}{h_1} m\pi \overline{\Psi}^{(2)}_{mnx} - \overline{L}_I \frac{a}{h_1} \frac{\overline{E}_{a4}}{\overline{h}_{a4}} \overline{W}^{(4)}_{mn} - \overline{L}_I \frac{a}{h_1} \frac{\overline{E}_{a1}}{\overline{h}_{a1}} \overline{W}^{(1)}_{mn}
\end{aligned}
\tag{6.20}
$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

For Plate 3, in Part II (Bonded Region);

$$
\begin{aligned}
&\frac{d\overline{\Psi}^{(3)}_{mnx}}{d\xi_{II}} = \overline{L}_{II}\,\frac{12}{\overline{h}_3^{\,3}\,\overline{B}^{(3)}_{66}}\,\overline{M}^{(3)}_{mnyx} - \overline{L}_{II}\left(\frac{h_1}{a}\right)^2 m\pi\,\overline{\Psi}^{(3)}_{mny} \\
&\frac{d\overline{\Phi}^{(3)}_{mny}}{d\xi_{I}} = \overline{L}_{II}\left(\frac{a}{h_1}\right)^2 \frac{12}{\overline{B}^{(3)}_{22}\,\overline{h}_3^{\,3}}\,\overline{M}^{(3)}_{mny} + \overline{L}_{II}\,\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}}\left(\frac{a}{h_1}\right)^2 m\pi\,\overline{\Psi}^{(3)}_{mnx} \\
&\frac{d\overline{W}^{(3)}_{mn}}{d\xi_{II}} = \overline{L}_{II}\,\frac{a}{h_1}\,\frac{1}{\kappa_y^2\,\overline{h}_3\,\overline{B}^{(3)}_{44}}\,\overline{Q}^{(3)}_{mny} - \overline{L}_{II}\,\overline{\Psi}^{(3)}_{mny} \\
&\frac{d\overline{M}^{(3)}_{mnyx}}{d\xi_{II}} = \left\{
\begin{aligned}
&-\overline{L}_{II}\,\overline{\rho}_3\,\overline{h}_3^{\,3}\,\frac{\rho_1 a^4\omega_{mn}^2}{12h_1^2 B^{(1)}_{11}}\left(\frac{h_1}{a}\right)^2 + \overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\,\overline{B}^{(3)}_{55}\left(\frac{a}{h_1}\right)^2 \\
&+\overline{L}_{II}\,\frac{\overline{h}_3^{\,3}}{12}(m\pi)^2\left(\overline{B}^{(3)}_{11} - \frac{\left(\overline{B}^{(3)}_{12}\right)^2}{\overline{B}^{(3)}_{22}}\right) + \overline{L}_{II}\left(\frac{a}{h_1}\right)^2 \frac{\overline{G}_{a1}\left(\overline{h}_3\right)^2}{4\overline{h}_{a1}} \\
&+\overline{L}_{II}\left(\frac{a}{h_1}\right)^2 \frac{\left(\overline{h}_3\right)^2\overline{G}_{a4}}{4\overline{h}_{a4}}
\end{aligned}
\right\}\overline{\Psi}^{(3)}_{mnx} \\
&-\overline{L}_{II}\,\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}}(m\pi)\overline{M}^{(3)}_{mny} + \left\{\overline{L}_{II}\,\kappa_x^2\,\overline{h}_3 m\pi\,\overline{B}^{(3)}_{55} + \overline{L}_{II}\,\frac{\overline{h}_3 m\pi}{2}\left(\overline{G}_{a1} + \overline{G}_{a4}\right)\right\}\overline{W}^{(3)}_{mn} \\
&+\overline{L}_{II}\left(\frac{a}{h_1}\right)^2 \frac{\overline{G}_{a1}\,\overline{h}_3}{4\overline{h}_{a1}}\,\overline{\Psi}^{(1)}_{mnx} + \overline{L}_{II}\left(\frac{a}{h_1}\right)^2 \frac{\overline{h}_3\,\overline{G}_{a4}}{4\overline{h}_{a4}}\,\overline{h}_4\,\overline{\Psi}^{(4)}_{mnx} \\
&\frac{d\overline{M}^{(3)}_{mny}}{d\xi_{II}} = \left\{
\begin{aligned}
&-\overline{L}_{II}\,\overline{\rho}_3\,\overline{h}_3^{\,3}\,\frac{a^4\rho_1\omega_{mn}^2}{12h_1^2 B^{(1)}_{11}}\left(\frac{h_1}{a}\right)^4 + \overline{L}_{II}\,\frac{\left(\overline{h}_3\right)^2\overline{G}_{a1}}{4\overline{h}_{a1}} + \overline{L}_{II}\,\frac{\left(\overline{h}_3\right)^2\overline{G}_{a4}}{4\overline{h}_{a4}} \\
&\underline{\underline{-\overline{L}_{II}\,\frac{\overline{h}_3}{2}\left(\overline{G}_{a1} + \overline{G}_{a4}\right)}}
\end{aligned}
\right\}\overline{\Psi}^{(3)}_{mny} \\
&+\overline{L}_{II}(m\pi)\overline{M}^{(3)}_{mnyx} + \overline{L}_{II}\,\frac{\overline{h}_3\,\overline{G}_{a1}}{4\overline{h}_{a1}}\,\overline{\Psi}^{(1)}_{mny} + \overline{L}_{II}\,\frac{\overline{h}_3\,\overline{G}_{a4}}{4\overline{h}_{a4}}\,\overline{h}_4\,\overline{\Psi}^{(4)}_{mny} \\
&+\left\{\underline{\underline{+\overline{L}_{II}\,\frac{a}{h_1}\,\frac{1}{2\kappa_y^2\,\overline{B}^{(3)}_{44}}\left(\overline{G}_{a1} + \overline{G}_{a4}\right)}} + \overline{L}_{II}\,\frac{a}{h_1}\right\}\overline{Q}^{(3)}_{mny} \\
&\frac{d\overline{Q}^{(3)}_{mny}}{d\xi_{II}} = \left\{
\begin{aligned}
&-\overline{L}_{II}\,\overline{\rho}_3\,\overline{h}_3\,\frac{\rho_1 a^4\omega_{mn}^2}{h_1^2 B^{(1)}_{11}}\left(\frac{h_1}{a}\right)^3 + \overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\,\overline{B}^{(3)}_{55}\,\frac{h_1}{a}(m\pi)^2 \\
&+\overline{L}_{II}\,\frac{a}{h_1}\,\frac{\overline{E}_{a1}}{\overline{h}_{a1}} + \overline{L}_{II}\,\frac{a}{h_1}\,\frac{\overline{E}_{a4}}{\overline{h}_{a4}}
\end{aligned}
\right\}\overline{W}^{(3)}_{mn} + \\
&\overline{L}_{II}\,\kappa_x^2\,\overline{h}_3\,\overline{B}^{(3)}_{55}\,\frac{a}{h_1}\,m\pi\,\overline{\Psi}^{(3)}_{mnx} - \overline{L}_{II}\,\frac{a}{h_1}\,\frac{\overline{E}_{a4}}{\overline{h}_{a4}}\,\overline{W}^{(4)}_{mn} - \overline{L}_{II}\,\frac{a}{h_1}\,\frac{\overline{E}_{a1}}{\overline{h}_{a1}}\,\overline{W}^{(1)}_{mn}
\end{aligned}
\qquad (6.21)
$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

For Plate 4, in Part I (Bonded Region);

$$
\begin{aligned}
\frac{d\overline{\Psi}^{(4)}_{mnx}}{d\xi_I} &= \overline{L}_I \frac{12}{\overline{h}_4^3 B_{66}^{(4)}} \overline{M}^{(4)}_{mnyx} - \overline{L}_I \left(\frac{h_1}{a}\right)^2 m\pi \overline{\Psi}^{(4)}_{mny} \\
\frac{d\overline{\Psi}^{(4)}_{mny}}{d\xi_I} &= \overline{L}_I \left(\frac{a}{h_1}\right)^2 \frac{12}{\overline{B}_{22}^{(4)} \overline{h}_4^3} \overline{M}^{(4)}_{mny} + \overline{L}_I \frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}} \left(\frac{a}{h_1}\right)^2 m\pi \overline{\Psi}^{(4)}_{mnx} \\
\frac{d\overline{W}_{mn}^{(4)}}{d\xi_I} &= \overline{L}_I \frac{a}{h_1} \frac{1}{\kappa_y^2 \overline{h}_4 \overline{B}_{44}^{(4)}} \overline{Q}^{(4)}_{mny} - \overline{L}_I \overline{\Psi}^{(4)}_{mny} \\
\frac{d\overline{M}^{(4)}_{mnyx}}{d\xi_I} &= \left\{ \begin{aligned} & -\overline{L}_I \overline{\rho}_4 \overline{h}_4^3 \frac{\rho_1 \omega_{mn}^2 a^4}{12 h_1^2 B_{11}^{(1)}} \left(\frac{h_1}{a}\right)^2 + \overline{L}_I \kappa_x^2 \overline{h}_4 \overline{B}_{55}^{(4)} \left(\frac{a}{h_1}\right)^2 \\ & + \overline{L}_I \frac{\overline{h}_4^3}{12} (m\pi)^2 \left(\overline{B}_{11}^{(4)} - \frac{\left(\overline{B}_{12}^{(4)}\right)^2}{\overline{B}_{22}^{(4)}} \right) + \overline{L}_I \frac{\left(\overline{h}_4\right)^2 \overline{G}_{a4}}{4h_{a4}} \left(\frac{a}{h_1}\right)^2 \end{aligned} \right\} \overline{\Psi}^{(4)}_{mnx} \\
& - \overline{L}_I \frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}} m\pi \overline{M}^{(4)}_{mny} + \left\{ \overline{L}_I \kappa_x^2 \overline{h}_4 \overline{B}_{55}^{(4)} m\pi + \overline{L}_I \underline{\underline{\frac{\overline{h}_4}{2} \overline{G}_{a4} m\pi}} \right\} \overline{W}^{(4)}_{mn} \\
& + \overline{L}_I \frac{\overline{h}_4 \overline{G}_{a4}}{4h_{a4}} \left(\frac{a}{h_1}\right)^2 \overline{h}_2 \overline{\Psi}^{(2)}_{mnx} \\
\frac{d\overline{M}^{(4)}_{mny}}{d\xi_I} &= -\left\{ \overline{L}_I \overline{\rho}_4 \overline{h}_4^3 \frac{\rho_1 \omega_{mn}^2 a^4}{12 h_1^2 B_{11}^{(1)}} \left(\frac{h_1}{a}\right)^4 - \overline{L}_I \underline{\underline{\frac{\overline{h}_4}{2} \overline{G}_{a4}}} + \overline{L}_I \frac{\left(\overline{h}_4\right)^2 \overline{G}_{a4}}{4\overline{h}_{a4}} \right\} \overline{\Psi}^{(4)}_{mny} \\
& + \overline{L}_I m\pi \overline{M}^{(4)}_{mnyx} + \left\{ \overline{L}_I \frac{a}{h_1} + \underline{\underline{\overline{L}_I \frac{a}{h_1} \frac{\overline{G}_{a4}}{2\kappa_y^2 \overline{B}_{44}^{(4)}}}} \right\} \overline{Q}^{(4)}_{mny} + \overline{L}_I \frac{\overline{h}_4 \overline{G}_{a4}}{4\overline{h}_{a4}} \overline{h}_2 \overline{\Psi}^{(2)}_{mny} \\
\frac{d\overline{Q}^{(4)}_{mny}}{d\xi_I} &= \left\{ -\overline{L}_I \overline{\rho}_4 \overline{h}_4 \frac{\rho_1 \omega_{mn}^2 a^4}{B_{11}^{(1)} h_1^2} \left(\frac{h_1}{a}\right)^3 + \overline{L}_I \frac{\overline{E}_{a4}}{\overline{h}_{a4}} \frac{a}{h_1} + \overline{L}_I \kappa_x^2 \overline{h}_4 \overline{B}_{55}^{(4)} \frac{h_1}{a} (m\pi)^2 \right\} \overline{W}^{(4)}_{mn} \\
& + \overline{L}_I \kappa_x^2 \overline{h}_4 \overline{B}_{55}^{(4)} \frac{a}{h_1} m\pi \overline{\Psi}^{(4)}_{mnx} - \overline{L}_I \frac{\overline{E}_{a4}}{\overline{h}_{a4}} \frac{a}{h_1} \overline{W}^{(2)}_{mn}
\end{aligned}
\tag{6.22}
$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or “doubly underlined terms”).

For Plate 4, in Part II (Bonded Region);

$$\frac{d\overline{\Psi}_{mnx}^{(4)}}{d\xi_{II}} = \overline{L}_{II}\frac{12}{\overline{h}_4^3 B_{66}^{(4)}}\overline{M}_{mnyx}^{(4)} - \overline{L}_{II}\left(\frac{h_1}{a}\right)^2 m\pi\overline{\Psi}_{mny}^{(4)}$$

$$\frac{d\overline{\Phi}_{mny}^{(4)}}{d\xi_{II}} = \overline{L}_{I}\left(\frac{a}{h_1}\right)^2 \frac{12}{\overline{B}_{22}^{(4)}\overline{h}_4^3}\overline{M}_{mny}^{(4)} + \overline{L}_{I}\frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}}\left(\frac{a}{h_1}\right)^2 m\pi\overline{\Psi}_{mnx}^{(4)}$$

$$\frac{d\overline{W}_{mn}^{(4)}}{d\xi_{I}} = \overline{L}_{II}\frac{a}{h_1}\frac{1}{\kappa_y^2\overline{h}_4\overline{B}_{44}^{(4)}}\overline{Q}_{mny}^{(4)} - \overline{L}_{II}\overline{\Psi}_{mny}^{(4)}$$

$$\frac{d\overline{M}_{mnyx}^{(4)}}{d\xi_{II}} = \left\{\begin{array}{l} -\overline{L}_{II}\overline{\rho}_4\overline{h}_4^3\dfrac{\rho_1\omega_{mn}^2 a^4}{12h_1^2 B_{11}^{(1)}}\left(\dfrac{h_1}{a}\right)^2 + \overline{L}_{II}\kappa_x^2\overline{h}_4\overline{B}_{55}^{(4)}\left(\dfrac{a}{h_1}\right)^2 \\ +\overline{L}_{II}\dfrac{\overline{h}_4^3}{12}(m\pi)^2\left(\overline{B}_{11}^{(4)} - \dfrac{\left(\overline{B}_{12}^{(4)}\right)^2}{\overline{B}_{22}^{(4)}}\right) + \overline{L}_{II}\dfrac{\left(\overline{h}_4\right)^2\overline{G}_{a4}}{4h_{a4}}\left(\dfrac{a}{h_1}\right)^2 \end{array}\right\}\overline{\Psi}_{mnx}^{(4)}$$

$$-\overline{L}_{II}\frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}}m\pi\overline{M}_{mny}^{(4)} + \left\{\overline{L}_{II}\kappa_x^2\overline{h}_4\overline{B}_{55}^{(4)}m\pi + \underline{\underline{\overline{L}_{II}\frac{\overline{h}_4}{2}\overline{G}_{a4}m\pi}}\right\}\overline{W}_{mn}^{(4)}$$

$$+\overline{L}_{II}\frac{\overline{h}_4\overline{G}_{a4}}{4h_{a4}}\left(\frac{a}{h_1}\right)^2\overline{h}_3\overline{\Psi}_{mnx}^{(3)}$$

$$\frac{d\overline{M}_{mny}^{(4)}}{d\xi_{II}} = -\left\{\begin{array}{l} \overline{L}_{II}\overline{\rho}_4\overline{h}_4^3\dfrac{\rho_1\omega_{mn}^2 a^4}{12h_1^2 B_{11}^{(1)}}\left(\dfrac{h_1}{a}\right)^4 - \underline{\underline{\overline{L}_{II}\dfrac{\overline{h}_4}{2}\overline{G}_{a4}}} \\ +\overline{L}_{II}\dfrac{\left(\overline{h}_4\right)^2\overline{G}_{a4}}{4\overline{h}_{a4}} \end{array}\right\}\overline{\Psi}_{mny}^{(4)}$$

$$+\overline{L}_{II}m\pi\overline{M}_{mnyx}^{(4)} + \left\{\overline{L}_{II}\frac{a}{h_1} + \underline{\underline{\overline{L}_{II}\frac{a}{h_1}\frac{\overline{G}_{a4}}{2\kappa_y^2\overline{B}_{44}^{(4)}}}}\right\}\overline{Q}_{mny}^{(4)} + \overline{L}_{II}\frac{\overline{h}_3\overline{G}_{a4}}{4\overline{h}_{a4}}\overline{h}_4\overline{\Psi}_{mny}^{(3)}$$

$$\frac{d\overline{Q}_{mny}^{(4)}}{d\xi_{II}} = \left\{-\overline{L}_{II}\overline{\rho}_4\overline{h}_4\frac{\rho_1\omega_{mn}^2 a^4}{B_{11}^{(1)}h_1^2}\left(\frac{h_1}{a}\right)^3 + \overline{L}_{II}\frac{\overline{E}_{a4}}{\overline{h}_{a4}}\frac{a}{h_1} + \overline{L}_{II}\kappa_x^2\overline{h}_4\overline{B}_{55}^{(4)}\frac{h_1}{a}(m\pi)^2\right\}\overline{W}_{mn}^{(4)}$$

$$+\overline{L}_{II}\kappa_x^2\overline{h}_4\overline{B}_{55}^{(4)}\frac{a}{h_1}m\pi\overline{\Psi}_{mnx}^{(4)} - \overline{L}_{II}\frac{\overline{E}_{a4}}{\overline{h}_{a4}}\frac{a}{h_1}\overline{W}_{mn}^{(3)} \qquad (6.23)$$

where double underlined terms come from $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms which can be neglected. (Later on, the natural frequencies and associated mode shapes will be obtained and compared with and without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms or "doubly underlined terms").

The quantities having the subscript “mn” are the “dimensionless fundamental dependent variables” of the problem in Part I and Part II Region (or “Bonded Region”). The ‘‘Two-Point Boundary Value Problem” is created in the ‘‘Bonded Regions’’ or (Part I and Part II regions) by reducing the system of partial differential equations to a ‘‘Governing System of First Order Ordinary Differential Equations’’ in ξ_I or y_I and ξ_{II} or y_{II} direction.

Thus, “Governing System of First Order Ordinary Differential Equations” in the compact matrix or the ‘‘state vector’’ form for the ‘‘Bonded Regions’’ (or Part I and Part II regions) can be written as,

$$\frac{d}{d\xi_I}\begin{Bmatrix} \overline{Y}_{mn}^{(1)} \\ \overline{Y}_{mn}^{(2)} \\ \overline{Y}_{mn}^{(4)} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{1,1} & \overline{C}_{1,2} & \overline{C}_{1,3} \\ \overline{C}_{2,1} & \overline{C}_{2,2} & \overline{C}_{2,3} \\ \overline{C}_{3,1} & \overline{C}_{3,2} & \overline{C}_{3,3} \end{bmatrix} \begin{Bmatrix} \overline{Y}_{mn}^{(1)} \\ \overline{Y}_{mn}^{(2)} \\ \overline{Y}_{mn}^{(4)} \end{Bmatrix}, \quad (0 < \xi_I < 1) \quad \text{(in Part I)} \quad (6.24)$$

with the ‘‘Arbitrary Boundary Conditions’’ and the ‘‘Continuity Conditions’’.

$$\frac{d}{d\xi_I}\begin{Bmatrix} \overline{Y}_{mn}^{(1)} \\ \overline{Y}_{mn}^{(3)} \\ \overline{Y}_{mn}^{(4)} \end{Bmatrix} = \begin{bmatrix} \overline{C}'_{1,1} & \overline{C}'_{1,2} & \overline{C}'_{1,3} \\ \overline{C}'_{2,1} & \overline{C}'_{2,2} & \overline{C}'_{2,3} \\ \overline{C}'_{3,1} & \overline{C}'_{3,2} & \overline{C}'_{3,3} \end{bmatrix} \begin{Bmatrix} \overline{Y}_{mn}^{(1)} \\ \overline{Y}_{mn}^{(3)} \\ \overline{Y}_{mn}^{(4)} \end{Bmatrix}, \quad (0 < \xi_{II} < 1) \quad \text{(in Part II)} \quad (6.25)$$

with the ‘‘Arbitrary Boundary Conditions’’ and the ‘‘Continuity Conditions’’.

where ξ_I and ξ_{II} are defined as y_I/ℓ_I, y_{II}/ℓ_I respectively. The superscripts show the related plate layer, the “Coefficient Sub-Matrices” $[\overline{C}_{i,j}]$ and $[\overline{C}'_{i,j}]$ are partitioned square matrices of dimension (6x6) which explicitly include the nondimensional geometric and material characteristics of the plate adherends, the doubler plates and of the adhesive layers and the dimensionless natural frequency parameter $\overline{\omega}_{mn}$ of the entire composite bonded joint system. $\overline{Y}_{mn}^{(j)}$ (j=1,2,3,4) are the ‘‘state vectors’ corresponding to “state variables” or the ‘‘dimensionless fundamental dependent variables’ of the problem under study,

$$\left\{\overline{Y}_{mn}^{(j)}\right\} = \left\{\overline{\Psi}_{mnx}^{(j)}, \overline{\Psi}_{mny}^{(j)}, \overline{W}_{mn}^{(j)}; \overline{M}_{mnyx}^{(j)}, \overline{M}_{mny}^{(j)}, \overline{Q}_{mny}^{(j)}\right\}^T, \quad (j=1,2,3,4) \qquad (6.26)$$

6.4.3 Analysis of Part III (or Single Layer) of Composite Plate System

The "Governing System of Ordinary Differential Equations" can be obtained for the plate adherend in Part III region by using the same procedure in the previous section. There is no adhesive layer in Part III. Therefore, coupling terms in (6.20) including the adhesive layer elastic constants are dropped. The "Governing System of First Order Ordinary Differential Equations" in the ''state vector'' form, for Part III region,

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}_{mn}^{(2)}\right\} = \left[\mathcal{D}\right]\left\{\overline{Y}_{mn}^{(2)}\right\} \qquad (0<\xi_{III}<1) \quad \text{(in Part III)} \qquad (6.27)$$

with the ''Arbitrary Boundary Conditions'' at $\xi_{III}=0$ and the ''Continuity Conditions'' at $\xi_{III}=1$ for the orthotropic plate adherend.

where ξ_{III} is defined as y_{III}/ℓ_{III} and [$\overline{\mathcal{D}}$] is the "Coefficient Matrix" of dimension (6x6) which explicitly includes dimensionless geometric and material characteristics of the plate adherend in the Part III region as well as the dimensionless natural frequencies $\overline{\omega}_{mn}$ of the entire composite plate system. The column matrix or the "state vector" $\left\{\overline{Y}_{mn}^{(2)}\right\}$ is defined as,

$$\left\{\overline{Y}_{mn}^{(2)}\right\} = \left\{\overline{\Psi}_{mnx}^{(2)}, \overline{\Psi}_{mny}^{(2)}, \overline{W}_{mn}^{(2)}; \overline{M}_{mnyx}^{(2)}, \overline{M}_{mny}^{(2)}, \overline{Q}_{mny}^{(2)}\right\}^T, \qquad (6.28)$$

6.4.4 Analysis of Part IV (or Single Layer) of Composite Plate System

The "Governing System of Ordinary Differential Equations" can be obtained for the plate adherend in Part IV region by using the same procedure in the previous section. There is no adhesive layer in Part IV. Therefore, coupling terms in (6.21) including the adhesive layer elastic constants are dropped. The "Governing System of First Order Ordinary Differential Equations" in the ''state vector'' form, for Part IV region,

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}_{mn}^{(3)}\right\} = \left[\overline{\mathcal{E}}\right]\left\{\overline{Y}_{mn}^{(3)}\right\}, \qquad (0<\xi_{IV}<1) \qquad \text{(in Part IV)} \qquad (6.29)$$

with the ''Arbitrary Boundary Conditions'' at ξ_{IV} =1 and the ''Continuity Conditions'' at ξ_{IV} =0 for the orthotropic plate adherend.

where ξ_{IV} is defined as y_{IV}/ℓ_{IV} and [$\overline{\mathcal{E}}$] is the "Coefficient Matrix" of dimension (6x6) which explicitly includes dimensionless geometric and material characteristics of the plate adherend in the Part IV region as well as the dimensionless natural frequencies $\overline{\omega}_{mn}$ of the entire composite bonded joint plate system. The column matrix or the "state vector" $\left\{\overline{Y}_{mn}^{(3)}\right\}$ is defined as,

$$\left\{\overline{Y}_{mn}^{(3)}\right\} = \left\{\overline{\Psi}_{mnx}^{(3)}, \overline{\Psi}_{mny}^{(3)}, \overline{W}_{mn}^{(3)}; \overline{M}_{mnyx}^{(3)}, \overline{M}_{mny}^{(3)}, \overline{Q}_{mny}^{(3)}\right\}^{T}, \qquad (6.30)$$

6.4.5 System of Governing First Order Ordinary Differential Equations for ("Main PROBLEM III")

In the previous sections, the "Governing System of First Order Ordinary Differential Equations" are obtained in the matrix or "state vector" form for the "Bonded Region" (or Part I and Part II regions), and for the "Single Layer Regions" (or Part III and Part IV regions). These equations can be written in "open matrix form" as, For Part I region or the "Bonded Region" (or Three-Layer Composite Plate Region),

$$
\frac{d}{d\xi_I}\begin{Bmatrix} \overline{\psi}^{(1)}_{mnx} \\ \overline{\psi}^{(1)}_{mny} \\ \overline{W}^{(1)}_{mn} \\ \overline{M}^{(1)}_{mnyx} \\ \overline{M}^{(1)}_{mny} \\ \overline{Q}^{(1)}_{mny} \\ \overline{\psi}^{(2)}_{mnx} \\ \overline{\psi}^{(2)}_{mny} \\ \overline{W}^{(2)}_{mn} \\ \overline{M}^{(2)}_{mnyx} \\ \overline{M}^{(2)}_{mny} \\ \overline{Q}^{(2)}_{mny} \\ \overline{\psi}^{(4)}_{mnx} \\ \overline{\psi}^{(4)}_{mny} \\ \overline{W}^{(4)}_{mn} \\ \overline{M}^{(4)}_{mnyx} \\ \overline{M}^{(4)}_{mny} \\ \overline{Q}^{(4)}_{mny} \end{Bmatrix} =
\left[\begin{array}{cccccc|cccccc|cccccc}
0 & c_{1,2} & 0 & c_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{2,1} & 0 & 0 & 0 & c_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{3,2} & 0 & 0 & 0 & c_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{4,1} & 0 & c_{4,3} & 0 & c_{4,5} & 0 & c_{4,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{5,2} & 0 & c_{5,4} & 0 & c_{5,6} & 0 & c_{5,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{6,1} & 0 & c_{6,3} & 0 & 0 & 0 & 0 & 0 & c_{6,9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{7,8} & 0 & c_{7,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{8,7} & 0 & 0 & 0 & c_{8,11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{9,8} & 0 & 0 & 0 & c_{9,12} & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{10,1} & 0 & 0 & 0 & 0 & 0 & c_{10,7} & 0 & c_{10,9} & 0 & c_{10,11} & 0 & c_{10,13} & 0 & 0 & 0 & 0 & 0 \\
0 & c_{11,2} & 0 & 0 & 0 & 0 & 0 & c_{11,8} & 0 & c_{11,10} & 0 & c_{11,12} & 0 & c_{11,14} & 0 & 0 & 0 & 0 \\
0 & 0 & c_{12,3} & 0 & 0 & 0 & c_{12,7} & 0 & c_{12,9} & 0 & 0 & 0 & 0 & 0 & c_{12,15} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{13,14} & 0 & c_{13,16} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14,13} & 0 & 0 & 0 & c_{14,17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{15,14} & 0 & 0 & 0 & c_{15,18} \\
0 & 0 & 0 & 0 & 0 & 0 & c_{16,7} & 0 & 0 & 0 & 0 & 0 & c_{16,13} & 0 & c_{16,15} & 0 & c_{16,17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{17,8} & 0 & 0 & 0 & 0 & 0 & c_{17,14} & 0 & c_{17,16} & 0 & c_{17,18} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{18,9} & 0 & 0 & 0 & c_{18,13} & 0 & c_{18,15} & 0 & 0 & 0
\end{array}\right]
\begin{Bmatrix} \overline{\psi}^{(1)}_{mnx} \\ \overline{\psi}^{(1)}_{mny} \\ \overline{W}^{(1)}_{mn} \\ \overline{M}^{(1)}_{mnyx} \\ \overline{M}^{(1)}_{mny} \\ \overline{Q}^{(1)}_{mny} \\ \overline{\psi}^{(2)}_{mnx} \\ \overline{\psi}^{(2)}_{mny} \\ \overline{W}^{(2)}_{mn} \\ \overline{M}^{(2)}_{mnyx} \\ \overline{M}^{(2)}_{mny} \\ \overline{Q}^{(2)}_{mny} \\ \overline{\psi}^{(4)}_{mnx} \\ \overline{\psi}^{(4)}_{mny} \\ \overline{W}^{(4)}_{mn} \\ \overline{M}^{(4)}_{mnyx} \\ \overline{M}^{(4)}_{mny} \\ \overline{Q}^{(4)}_{mny} \end{Bmatrix}
$$

$(0<\xi_I<1)$ (6.40 a)

The elements, in the "open matrix form", of the ''Coefficient Sub-Matrices" related to the plate layers are,

For plate 1 (Upper "Doubler" Plate),

$$c_{1,2} = -\overline{L}_1 m\pi\left(\frac{h_1}{a}\right)^2 \qquad c_{1,4} = \frac{12\overline{L}_1}{\overline{B}^{(1)}_{66}} \qquad c_{2,1} = \overline{L}_1 m\pi\frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}}\left(\frac{a}{h_1}\right)^2$$

$$c_{2,5} = \frac{12\overline{L}_1}{\overline{B}^{(1)}_{22}}\left(\frac{a}{h_1}\right)^2 \qquad c_{3,2} = -\overline{L}_1 \qquad c_{3,6} = \frac{\overline{L}_1}{\kappa_y^2\overline{B}^{(1)}_{44}}\left(\frac{a}{h_1}\right)$$

$$c_{4,1} = -\frac{\overline{L}_1}{12}\left(\frac{h_1}{a}\right)^2\overline{\Omega} + \overline{L}_1\kappa_x^2\overline{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^2 + \frac{\overline{L}_1}{12}(m\pi)^2\left(\overline{B}^{(1)}_{11} - \frac{\overline{B}^{(1)^2}_{12}}{\overline{B}^{(1)}_{22}}\right) + \frac{\overline{G}_{a1}\overline{L}_1}{4\overline{h}_{a1}}\left(\frac{a}{h_1}\right)^2$$

$$c_{4,3} = \overline{L}_1\kappa_x^2\overline{B}^{(1)}_{55}m\pi + \underline{\underline{\overline{L}_1\frac{m\pi}{2}\overline{G}_{a1}}} \qquad c_{4,5} = -\overline{L}_1 m\pi\frac{\overline{B}^{(1)}_{12}}{\overline{B}^{(1)}_{22}}$$

$$c_{5,2} = -\frac{\overline{L}_1}{12}\left(\frac{h_1}{a}\right)^4\overline{\Omega} + \frac{\overline{G}_{a1}\overline{L}_1}{4\overline{h}_{a1}} - \underline{\underline{\frac{\overline{L}_1\overline{G}_{a1}}{2}}} \qquad c_{5,4} = \overline{L}_1 m\pi$$

$$c_{5,6} = \overline{L}_1\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_1\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a1}}{2\kappa_y^2\overline{B}^{(1)}_{44}}}} \qquad c_{6,1} = \kappa_x^2\overline{L}_1\overline{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)m\pi$$

$$c_{6,3} = -\overline{L}_1\left(\frac{h_1}{a}\right)^3\overline{\Omega} + \overline{L}_1\kappa_x^2\left(\frac{h_1}{a}\right)(m\pi)^2\,\overline{B}_{55}^{(1)} + \frac{\overline{L}_1\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) \qquad c_{4,7} = \frac{\overline{L}_1\overline{G}_{a1}}{4\overline{h}_{a1}}\overline{h}_2\left(\frac{a}{h_1}\right)^2$$

$$c_{5,8} = \frac{\overline{L}_1\overline{G}_{a1}\overline{h}_2}{4\overline{h}_{a1}} \qquad c_{6,9} = -\frac{\overline{L}_1\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) \qquad (6.40\ \text{b})$$

<u>For Plate 2 (Adherend Plate),</u>

$$c_{7,8} = -\overline{L}_1 m\pi\left(\frac{h_1}{a}\right)^2 \qquad c_{7,10} = \frac{12\overline{L}_1}{\overline{B}_{66}^{(2)}\overline{h}_2^3} \qquad c_{8,7} = \overline{L}_1 m\pi\frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}\left(\frac{a}{h_1}\right)^2$$

$$c_{8,11} = \frac{12\overline{L}_1}{\overline{B}_{22}^{(2)}\overline{h}_2^3}\left(\frac{a}{h_1}\right)^2 \qquad c_{9,8} = -\overline{L}_1 \qquad c_{9,12} = \frac{\overline{L}_1}{\kappa_y^2\overline{B}_{44}^{(2)}\overline{h}_2}\left(\frac{a}{h_1}\right)$$

$$c_{10,7} = -\frac{\overline{L}_1}{12}\overline{\rho}_2\overline{h}_2^3\left(\frac{h_1}{a}\right)^2\overline{\Omega} + \overline{L}_1\kappa_x^2\overline{h}_2\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)^2 + \frac{\overline{L}_1\overline{h}_2^3}{12}(m\pi)^2\left(\overline{B}_{11}^{(2)} - \frac{\overline{B}_{12}^{(2)^2}}{\overline{B}_{22}^{(2)}}\right)$$
$$+\frac{\overline{G}_{a1}\overline{L}_1\overline{h}_2^2}{4\overline{h}_{a1}}\left(\frac{a}{h_1}\right)^2 + \frac{\overline{G}_{a4}\overline{L}_1\overline{h}_2^2}{4\overline{h}_{a4}}\left(\frac{a}{h_1}\right)^2$$

$$c_{10,9} = \overline{L}_1\kappa_x^2\overline{h}_2\overline{B}_{55}^{(2)}m\pi + \underline{\underline{\overline{L}_1\overline{h}_2\frac{m\pi}{2}\left(\overline{G}_{a1} + \overline{G}_{a4}\right)}} \qquad c_{10,11} = -\overline{L}_1 m\pi\frac{\overline{B}_{12}^{(2)}}{\overline{B}_{22}^{(2)}}$$

$$c_{11,8} = -\frac{\overline{L}_1}{12}\overline{\rho}_2\overline{h}_2^3\left(\frac{h_1}{a}\right)^4\overline{\Omega} + \frac{\overline{G}_{a1}\overline{h}_2^2\overline{L}_1}{4\overline{h}_{a1}} + \frac{\overline{G}_{a4}\overline{h}_2^2\overline{L}_1}{4\overline{h}_{a4}} - \underline{\underline{\frac{\overline{L}_1\overline{G}_{a1}\overline{h}_2}{2}}} - \underline{\underline{\frac{\overline{L}_1\overline{G}_{a4}\overline{h}_2}{2}}}$$

$$c_{11,10} = \overline{L}_1 m\pi \qquad c_{11,12} = \overline{L}_1\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_1\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a1}}{2\kappa_y^2\overline{B}_{44}^{(2)}}}} + \underline{\underline{\overline{L}_1\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a4}}{2\kappa_y^2\overline{B}_{44}^{(2)}}}}$$

$$c_{10,1} = \frac{\overline{L}_1\overline{G}_{a1}}{4\overline{h}_{a1}}\overline{h}_2\left(\frac{a}{h_1}\right)^2 \qquad c_{10,13} = \frac{\overline{L}_1\overline{G}_{a4}}{4\overline{h}_{a4}}\overline{h}_2\overline{h}_4\left(\frac{a}{h_1}\right)^2$$

$$c_{11,2} = \frac{\overline{L}_1\overline{G}_{a1}\overline{h}_2}{4\overline{h}_{a1}} \quad c_{12,7} = \overline{L}_1\overline{h}_2\kappa_x^2\overline{B}_{55}^{(2)}\left(\frac{a}{h_1}\right)m\pi \qquad c_{11,14} = \frac{\overline{L}_1\overline{G}_{a4}\overline{h}_2\overline{h}_4}{4\overline{h}_{a4}}$$

$$c_{12,9} = -\overline{L}_1\overline{\rho}_2\overline{h}_2\left(\frac{h_1}{a}\right)^3\overline{\Omega} + \overline{L}_1\kappa_x^2\left(\frac{h_1}{a}\right)\overline{h}_2(m\pi)^2\,\overline{B}_{55}^{(2)} + \frac{\overline{L}_1\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) + \frac{\overline{L}_1\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right)$$

$$c_{12,3} = -\frac{\overline{L}_1\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) \qquad c_{12,15} = -\frac{\overline{L}_1\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right) \qquad (6.40\ \text{c})$$

For plate 4 (Lower "Doubler" Plate),

$$c_{13,14} = -\overline{L}_1 m\pi \left(\frac{h_1}{a}\right)^2 \qquad c_{13,16} = \frac{12\overline{L}_1}{\overline{B}_{66}^{(4)}\overline{h}_4^3} \qquad c_{14,13} = \overline{L}_1 m\pi \frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}}\left(\frac{a}{h_1}\right)^2$$

$$c_{14,17} = \frac{12\overline{L}_1}{\overline{B}_{22}^{(4)}\overline{h}_4^3}\left(\frac{a}{h_1}\right)^2 \qquad c_{15,14} = -\overline{L}_1 \qquad c_{15,18} = \frac{\overline{L}_1}{\kappa_y^2 \overline{B}_{44}^{(4)}\overline{h}_4}\left(\frac{a}{h_1}\right)$$

$$c_{16,13} = -\frac{\overline{L}_1}{12}\overline{\rho}_4\overline{h}_4^3\left(\frac{h_1}{a}\right)^2\overline{\Omega} + \overline{L}_1\kappa_x^2\overline{B}_{55}^{(4)}\overline{h}_4\left(\frac{a}{h_1}\right)^2 + \frac{\overline{L}_1\overline{h}_4^3}{12}(m\pi)^2\left(\overline{B}_{11}^{(4)} - \frac{\overline{B}_{12}^{(4)^2}}{\overline{B}_{22}^{(4)}}\right)$$

$$+\frac{\overline{G}_{a4}\overline{L}_1\overline{h}_4^2}{4\overline{h}_{a4}}\left(\frac{a}{h_1}\right)^2 + \frac{\overline{G}_{a4}\overline{L}_1\overline{h}_2\overline{h}_4}{4\overline{h}_{a4}}\left(\frac{a}{h_1}\right)^2$$

$$c_{16,15} = \overline{L}_1\kappa_x^2\overline{h}_4\overline{B}_{55}^{(4)}m\pi + \underline{\underline{\overline{L}_1\frac{\overline{h}_4}{2}}}\overline{G}_{a4}m\pi \qquad c_{16,17} = -\overline{L}_1 m\pi\frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}}$$

$$c_{17,14} = -\frac{\overline{L}_1}{12}\overline{\rho}_4\overline{h}_4^3\left(\frac{h_1}{a}\right)^4\overline{\Omega} + \frac{\overline{L}_1\overline{G}_{a1}\overline{h}_4^2}{4\overline{h}_{a1}} - \underline{\underline{\frac{\overline{L}_1\overline{h}_4\overline{G}_{a4}}{2}}} \qquad c_{17,16} = \overline{L}_1 m\pi$$

$$c_{17,18} = \overline{L}_1\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_1\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a4}}{2\kappa_y^2\overline{B}_{44}^{(4)}}}} \qquad c_{18,13} = \kappa_x^2\overline{L}_1\overline{h}_4\overline{B}_{55}^{(4)}\left(\frac{a}{h_1}\right)m\pi$$

$$c_{18,15} = -\overline{L}_1\overline{\rho}_4\overline{h}_4\left(\frac{h_1}{a}\right)^3\overline{\Omega} + \overline{L}_1\kappa_x^2\overline{h}_4\left(\frac{h_1}{a}\right)(m\pi)^2\overline{B}_{55}^{(4)} + \frac{\overline{L}_1\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right)$$

$$c_{17,14} = \frac{\overline{L}_1\overline{G}_{a4}\overline{h}_2\overline{h}_4}{4\overline{h}_{a4}} \qquad c_{18,15} = -\frac{\overline{L}_1\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right) \qquad (6.40\text{ d})$$

For Part II region or the "Bonded Region" (or Three-Layer Composite Plate Region),

$$
\frac{d}{d\xi_{II}}\left\{\begin{array}{c}
\bar{\psi}^{(1)}_{mnx}\\ \bar{\psi}^{(1)}_{mny}\\ \bar{W}^{(1)}_{mn}\\ \bar{M}^{(1)}_{mnyx}\\ \bar{M}^{(1)}_{mny}\\ \bar{Q}^{(1)}_{mny}\\ \hline
\bar{\psi}^{(3)}_{mnx}\\ \bar{\psi}^{(3)}_{mny}\\ \bar{W}^{(3)}_{mn}\\ \bar{M}^{(3)}_{mnyx}\\ \bar{M}^{(3)}_{mny}\\ \bar{Q}^{(3)}_{mny}\\ \hline
\bar{\psi}^{(4)}_{mnx}\\ \bar{\psi}^{(4)}_{mny}\\ \bar{W}^{(4)}_{mn}\\ \bar{M}^{(4)}_{mnyx}\\ \bar{M}^{(4)}_{mny}\\ \bar{Q}^{(4)}_{mny}
\end{array}\right\}=
\left[\begin{array}{cccccc|cccccc|cccccc}
0 & c'_{1,2} & 0 & c'_{1,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c'_{2,1} & 0 & 0 & 0 & c'_{2,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & c'_{3,2} & 0 & 0 & 0 & c'_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c''_{4,1} & 0 & c'_{4,3} & 0 & c'_{4,5} & 0 & c''_{4,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & c'_{5,2} & 0 & c'_{5,4} & 0 & c'_{5,6} & 0 & c'_{5,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c'_{6,1} & 0 & c'_{6,3} & 0 & 0 & 0 & 0 & 0 & c'_{6,9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{7,8} & 0 & c'_{7,10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & c'_{8,7} & 0 & 0 & 0 & c'_{8,11} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{9,8} & 0 & 0 & 0 & c'_{9,12} & 0 & 0 & 0 & 0 & 0 & 0\\
c'_{10,1} & 0 & 0 & 0 & 0 & 0 & c'_{10,7} & 0 & c'_{10,9} & 0 & c'_{10,11} & 0 & c'_{10,13} & 0 & 0 & 0 & 0 & 0\\
0 & c'_{11,2} & 0 & 0 & 0 & 0 & 0 & c'_{11,8} & 0 & c'_{11,10} & 0 & c'_{11,12} & 0 & c'_{11,14} & 0 & 0 & 0 & 0\\
0 & 0 & c'_{12,3} & 0 & 0 & 0 & c'_{12,7} & 0 & c'_{12,9} & 0 & 0 & 0 & 0 & 0 & c'_{12,15} & 0 & 0 & 0\\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{13,14} & 0 & c'_{13,16} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{14,13} & 0 & 0 & 0 & c'_{14,17} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{15,14} & 0 & 0 & 0 & c'_{15,18}\\
0 & 0 & 0 & 0 & 0 & 0 & c'_{16,7} & 0 & 0 & 0 & 0 & 0 & c'_{16,13} & 0 & c'_{16,15} & 0 & c'_{16,17} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{17,8} & 0 & 0 & 0 & 0 & 0 & c'_{17,14} & 0 & c'_{17,16} & 0 & c'_{17,18}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c'_{18,9} & 0 & 0 & 0 & c'_{18,13} & 0 & c'_{18,15} & 0 & 0 & 0
\end{array}\right]
\left\{\begin{array}{c}
\bar{\psi}^{(1)}_{mnx}\\ \bar{\psi}^{(1)}_{mny}\\ \bar{W}^{(1)}_{mn}\\ \bar{M}^{(1)}_{mnyx}\\ \bar{M}^{(1)}_{mny}\\ \bar{Q}^{(1)}_{mny}\\ \hline
\bar{\psi}^{(3)}_{mnx}\\ \bar{\psi}^{(3)}_{mny}\\ \bar{W}^{(3)}_{mn}\\ \bar{M}^{(3)}_{mnyx}\\ \bar{M}^{(3)}_{mny}\\ \bar{Q}^{(3)}_{mny}\\ \hline
\bar{\psi}^{(4)}_{mnx}\\ \bar{\psi}^{(4)}_{mny}\\ \bar{W}^{(4)}_{mn}\\ \bar{M}^{(4)}_{mnyx}\\ \bar{M}^{(4)}_{mny}\\ \bar{Q}^{(4)}_{mny}
\end{array}\right\}
$$

$(0<\xi_{II}<1)$ (6.41 a)

The elements, in the open form, of the ''Coefficient Sub-Matrices" related to the plate layers are,

For plate 1 (Upper "Doubler" Plate),

$$c'_{1,2}=-\bar{L}_{II}m\pi\left(\frac{h_1}{a}\right)^2 \qquad c'_{1,4}=\frac{12\bar{L}_{II}}{\bar{B}^{(1)}_{66}} \qquad c'_{2,1}=\bar{L}_{II}m\pi\frac{\bar{B}^{(1)}_{12}}{\bar{B}^{(1)}_{22}}\left(\frac{a}{h_1}\right)^2$$

$$c'_{2,5}=\frac{12\bar{L}_{II}}{\bar{B}^{(1)}_{22}}\left(\frac{a}{h_1}\right)^2 \qquad c'_{3,2}=-\bar{L}_{II} \qquad c'_{3,6}=\frac{\bar{L}_{II}}{\kappa_y^2\bar{B}^{(1)}_{44}}\left(\frac{a}{h_1}\right)$$

$$c'_{4,1}=-\frac{\bar{L}_{II}}{12}\left(\frac{h_1}{a}\right)^2\bar{\Omega}+\bar{L}_{II}\kappa_x^2\bar{B}^{(1)}_{55}\left(\frac{a}{h_1}\right)^2+\frac{\bar{L}_{II}}{12}(m\pi)^2\left(\bar{B}^{(1)}_{11}-\frac{\bar{B}^{(1)^2}_{12}}{\bar{B}^{(1)}_{22}}\right)$$
$$+\frac{\bar{G}_{a1}\bar{L}_{II}}{4\bar{h}_{a1}}\left(\frac{a}{h_1}\right)^2$$

$$c'_{4,3}=\bar{L}_{II}\kappa_x^2\bar{B}^{(1)}_{55}m\pi+\underline{\underline{\bar{L}_{II}\frac{m\pi}{2}\bar{G}_{a1}}} \qquad c'_{4,5}=-\bar{L}_{II}m\pi\frac{\bar{B}^{(1)}_{12}}{\bar{B}^{(1)}_{22}}$$

$$c'_{5,2} = -\frac{\overline{L}_{II}}{12}\left(\frac{h_1}{a}\right)^4 \overline{\Omega} + \frac{\overline{G}_{a1}\overline{L}_{II}}{4\overline{h}_{a1}} - \underline{\underline{\frac{\overline{L}_{II}\,\overline{G}_{a1}}{2}}} \qquad c'_{5,4} = \overline{L}_{II} m\pi$$

$$c'_{5,6} = \overline{L}_{II}\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_{II}\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a1}}{2\kappa_y^2 \overline{B}_{44}^{(1)}}}} \qquad c'_{6,1} = \kappa_x^2 \overline{L}_{II}\overline{B}_{55}^{(1)}\left(\frac{a}{h_1}\right) m\pi$$

$$c'_{6,3} = -\overline{L}_{II}\left(\frac{h_1}{a}\right)^3 \overline{\Omega} + \overline{L}_{II}\kappa_x^2\left(\frac{h_1}{a}\right)(m\pi)^2 \overline{B}_{55}^{(1)} + \frac{\overline{L}_{II}\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) \qquad c'_{4,7} = \frac{\overline{L}_{II}\overline{G}_{a1}}{4\overline{h}_{a1}}\overline{h}_3\left(\frac{a}{h_1}\right)^2$$

$$c'_{5,8} = \frac{\overline{L}_{II}\overline{G}_{a1}\overline{h}_3}{4\overline{h}_{a1}} \qquad c'_{6,9} = -\frac{\overline{L}_{II}\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) \qquad (6.41\text{ b})$$

<u>For Plate 3 (Adherend Plate),</u>

$$c'_{7,8} = -\overline{L}_{II} m\pi\left(\frac{h_1}{a}\right)^2 \qquad c'_{7,10} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(3)}\overline{h}_3^3} \qquad c'_{8,7} = \overline{L}_{II} m\pi \frac{\overline{B}_{12}^{(3)}}{\overline{B}_{22}^{(3)}}\left(\frac{a}{h_1}\right)^2$$

$$c'_{8,11} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(3)}\overline{h}_3^3}\left(\frac{a}{h_1}\right)^2 \qquad c'_{9,8} = -\overline{L}_{II} \qquad c'_{9,12} = \frac{\overline{L}_{II}}{\kappa_y^2 \overline{B}_{44}^{(3)}\overline{h}_3}\left(\frac{a}{h_1}\right)$$

$$c'_{10,1} = \frac{\overline{L}_{II}\overline{G}_{a1}}{4\overline{h}_{a1}}\overline{h}_3\left(\frac{a}{h_1}\right)^2 \qquad c'_{10,13} = \frac{\overline{L}_{II}\overline{G}_{a4}}{4\overline{h}_{a4}}\overline{h}_3\overline{h}_4\left(\frac{a}{h_1}\right)^2$$

$$c'_{10,7} = -\frac{\overline{L}_{II}}{12}\overline{\rho}_3\overline{h}_3^3\left(\frac{h_1}{a}\right)^2 \overline{\Omega} + \overline{L}_{II}\kappa_x^2 \overline{h}_3 \overline{B}_{55}^{(3)}\left(\frac{a}{h_1}\right)^2 + \frac{\overline{L}_{II}\overline{h}_3^3}{12}(m\pi)^2\left(\overline{B}_{11}^{(3)} - \frac{\overline{B}_{12}^{(3)2}}{\overline{B}_{22}^{(3)}}\right)$$
$$+ \frac{\overline{G}_{a1}\overline{L}_{II}\overline{h}_3^2}{4\overline{h}_{a1}}\left(\frac{a}{h_1}\right)^2 + \frac{\overline{G}_{a4}\overline{L}_{II}\overline{h}_3^2}{4\overline{h}_{a4}}\left(\frac{a}{h_1}\right)^2$$

$$c'_{10,9} = \overline{L}_{II}\kappa_x^2 \overline{h}_3 \overline{B}_{55}^{(3)} m\pi + \underline{\underline{\overline{L}_{II}\overline{h}_3 \frac{m\pi}{2}\left(\overline{G}_{a1} + \overline{G}_{a4}\right)}} \qquad c'_{10,11} = -\overline{L}_{II} m\pi \frac{\overline{B}_{12}^{(3)}}{\overline{B}_{22}^{(3)}}$$

$$c'_{11,8} = -\frac{\overline{L}_{II}}{12}\overline{\rho}_2\overline{h}_3^3\left(\frac{h_1}{a}\right)^4 \overline{\Omega} + \frac{\overline{G}_{a1}\overline{h}_3^2\overline{L}_{II}}{4\overline{h}_{a1}} + \frac{\overline{G}_{a4}\overline{h}_3^2\overline{L}_{II}}{4\overline{h}_{a4}} - \underline{\underline{\frac{\overline{L}_{II}\overline{G}_{a1}\overline{h}_3}{2}}} - \underline{\underline{\frac{\overline{L}_{II}\overline{G}_{a4}\overline{h}_3}{2}}}$$

$$c'_{11,10} = \overline{L}_{II} m\pi \qquad c'_{11,12} = \overline{L}_{II}\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_{II}\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a1}}{2\kappa_y^2\overline{B}_{44}^{(3)}}}} + \underline{\underline{\overline{L}_{II}\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a4}}{2\kappa_y^2\overline{B}_{44}^{(3)}}}}$$

$$c'_{11,2} = \frac{\overline{L}_{II}\overline{G}_{a1}\overline{h}_3}{4\overline{h}_{a1}} \qquad c'_{11,14} = \frac{\overline{L}_{II}\overline{G}_{a4}\overline{h}_3\overline{h}_4}{4\overline{h}_{a4}}$$

$$c'_{12,3} = -\frac{\overline{L}_{II}\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) \qquad c'_{12,15} = -\frac{\overline{L}_{II}\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right) \qquad c'_{12,7} = \overline{L}_{II}\overline{h}_3\kappa_x^2\overline{B}_{55}^{(3)}\left(\frac{a}{h_1}\right)m\pi$$

$$c'_{12,9} = -\overline{L}_{II}\overline{\rho}_3\overline{h}_3\left(\frac{h_1}{a}\right)^3\overline{\Omega} + \overline{L}_{II}\kappa_x^2\left(\frac{h_1}{a}\right)\overline{h}_3(m\pi)^2\overline{B}_{55}^{(3)} + \frac{\overline{L}_{II}\overline{E}_{a1}}{\overline{h}_{a1}}\left(\frac{a}{h_1}\right) + \frac{\overline{L}_{II}\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right)$$

(6.41 c)

<u>For plate 4 (Lower "Doubler" Plate),</u>

$$c'_{13,14} = -\overline{L}_{II}m\pi\left(\frac{h_1}{a}\right)^2 \qquad c'_{13,16} = \frac{12\overline{L}_{II}}{\overline{B}_{66}^{(4)}\overline{h}_4^3} \qquad c'_{14,13} = \overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}}\left(\frac{a}{h_1}\right)^2$$

$$c'_{14,17} = \frac{12\overline{L}_{II}}{\overline{B}_{22}^{(4)}\overline{h}_4^3}\left(\frac{a}{h_1}\right)^2 \qquad c'_{15,14} = -\overline{L}_{II} \qquad c'_{15,18} = \frac{\overline{L}_{II}}{\kappa_y^2\overline{B}_{44}^{(4)}\overline{h}_4}\left(\frac{a}{h_1}\right)$$

$$c'_{16,13} = -\frac{\overline{L}_{II}}{12}\overline{\rho}_4\overline{h}_4^3\left(\frac{h_1}{a}\right)^2\overline{\Omega} + \overline{L}_{II}\kappa_x^2\overline{B}_{55}^{(4)}\overline{h}_4\left(\frac{a}{h_1}\right)^2 + \frac{\overline{L}_{II}\overline{h}_4^3}{12}(m\pi)^2\left(\overline{B}_{11}^{(4)} - \frac{\overline{B}_{12}^{(4)^2}}{\overline{B}_{22}^{(4)}}\right)$$

$$+\frac{\overline{G}_{a4}\overline{L}_{II}\overline{h}_4^2}{4\overline{h}_{a4}}\left(\frac{a}{h_1}\right)^2$$

$$c'_{16,13} = \frac{\overline{L}_{II}\overline{G}_{a4}}{4\overline{h}_{a4}}\overline{h}_3\overline{h}_4\left(\frac{a}{h_1}\right)^2 \qquad c'_{16,15} = \overline{L}_{II}\kappa_x^2\overline{h}_4\overline{B}_{55}^{(4)}m\pi + \underline{\underline{\overline{L}_{II}\frac{\overline{h}_4}{2}\overline{G}_{a4}m\pi}}$$

$$c'_{16,17} = -\overline{L}_{II}m\pi\frac{\overline{B}_{12}^{(4)}}{\overline{B}_{22}^{(4)}} \qquad c'_{17,14} = \frac{\overline{L}_{II}\overline{G}_{a4}\overline{h}_3\overline{h}_4}{4\overline{h}_{a4}}$$

$$c'_{17,14} = -\frac{\overline{L}_{II}}{12}\overline{\rho}_4\overline{h}_4^3\left(\frac{h_1}{a}\right)^4\overline{\Omega} + \frac{\overline{L}_{II}\overline{G}_{a1}\overline{h}_4^2}{4\overline{h}_{a1}} - \underline{\underline{\frac{\overline{L}_{II}\overline{h}_4\overline{G}_{a4}}{2}}} \qquad c'_{17,16} = \overline{L}_{II}m\pi$$

$$c'_{17,18} = \overline{L}_{II}\left(\frac{a}{h_1}\right) + \underline{\underline{\overline{L}_{II}\left(\frac{a}{h_1}\right)\frac{\overline{G}_{a4}}{2\kappa_y^2\overline{B}_{44}^{(4)}}}} \qquad c'_{18,13} = \kappa_x^2\overline{L}_{II}\overline{h}_4\overline{B}_{55}^{(4)}\left(\frac{a}{h_1}\right)m\pi$$

$$c'_{18,15} = -\overline{L}_{II}\overline{\rho}_4\overline{h}_4\left(\frac{h_1}{a}\right)^3\overline{\Omega} + \overline{L}_{II}\kappa_x^2\overline{h}_4\left(\frac{h_1}{a}\right)(m\pi)^2\overline{B}_{55}^{(4)} + \frac{\overline{L}_{II}\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right)$$

$$c'_{18,15} = -\frac{\overline{L}_{II}\overline{E}_{a4}}{\overline{h}_{a4}}\left(\frac{a}{h_1}\right)$$

(6.41 d)

For Part III region (or Single Layer Orthotropic Plate Adherend),

$$\frac{d}{d\xi_{III}}\begin{Bmatrix}\overline{\psi}^{(2)}_{mnx}\\ \overline{\psi}^{(2)}_{mny}\\ \overline{W}^{(2)}_{mn}\\ \overline{M}^{(2)}_{mnyx}\\ \overline{M}^{(2)}_{mny}\\ \overline{Q}^{(2)}_{mny}\end{Bmatrix}=\left[\begin{array}{ccc|ccc} 0 & d_{1,2} & 0 & d_{1,4} & 0 & 0\\ d_{2,1} & 0 & 0 & 0 & d_{2,5} & 0\\ 0 & d_{3,2} & 0 & 0 & 0 & d_{3,6}\\ \hline d_{4,1} & 0 & d_{4,3} & 0 & d_{4,5} & 0\\ 0 & d_{5,2} & 0 & d_{5,4} & 0 & d_{5,6}\\ d_{6,1} & 0 & d_{6,3} & 0 & 0 & 0\end{array}\right]\begin{Bmatrix}\overline{\psi}^{(2)}_{mnx}\\ \overline{\psi}^{(2)}_{mny}\\ \overline{W}^{(2)}_{mn}\\ \overline{M}^{(2)}_{mnyx}\\ \overline{M}^{(2)}_{mny}\\ \overline{Q}^{(2)}_{mny}\end{Bmatrix},\ (0<\xi_{III}<1)\quad (6.42\text{ a})$$

where the elements of the above ''Coefficient Matrix $\lfloor \overline{\mathcal{D}} \rfloor$'' are,

$$d_{1,2}=-\overline{L}_{III}m\pi\left(\frac{h_1}{a}\right)^2 \qquad d_{1,4}=\frac{12\overline{L}_{III}}{\overline{B}^{(2)}_{66}\overline{h}_2^3} \qquad d_{2,1}=\overline{L}_{III}m\pi\frac{\overline{B}^{(2)}_{12}}{\overline{B}^{(2)}_{22}}\left(\frac{a}{h_1}\right)^2$$

$$d_{2,5}=\frac{12\overline{L}_{III}}{\overline{B}^{(2)}_{22}\overline{h}_2^3}\left(\frac{a}{h_1}\right)^2 \qquad d_{3,2}=-\overline{L}_{III} \qquad d_{3,6}=\frac{\overline{L}_{III}}{\kappa_y^2\overline{B}^{(2)}_{44}\overline{h}_2}\left(\frac{a}{h_1}\right)$$

$$d_{4,1}=-\frac{\overline{L}_{III}}{12}\overline{\rho}\overline{h}_2^3\left(\frac{h_1}{a}\right)^2\overline{\Omega}+\overline{L}_{III}\kappa_x^2\overline{h}_2\overline{B}^{(2)}_{55}\left(\frac{a}{h_1}\right)^2+\frac{\overline{L}_{III}\overline{h}_2^3}{12}(m\pi)^2\left(\overline{B}^{(2)}_{11}-\frac{\overline{B}^{(2)^2}_{12}}{\overline{B}^{(2)}_{22}}\right)$$

$$d_{4,3}=\kappa_x^2\overline{L}_{III}\overline{B}^{(2)}_{55}\overline{h}_2 m\pi \qquad d_{4,5}=-\frac{\overline{B}^{(2)}_{12}}{\overline{B}^{(2)}_{22}}\overline{L}_{III}m\pi \qquad d_{5,2}=-\frac{\overline{L}_{III}}{12}\overline{\rho}\overline{h}_2^3\overline{\Omega}$$

$$d_{5,4}=\overline{L}_{III}m\pi \qquad d_{5,6}=\frac{\overline{L}_{III}a}{h_1} \qquad d_{6,1}=\overline{L}_{III}\kappa_x^2\overline{B}^{(2)}_{55}\overline{h}_2\left(\frac{a}{h_1}\right)m\pi$$

$$d_{6,3}=-\overline{L}_{III}\overline{\rho}\overline{h}_2\left(\frac{h_1}{a}\right)^3\overline{\Omega}+\overline{L}_{III}\kappa_x^2\overline{h}_2\overline{B}^{(2)}_{55}\left(\frac{h_1}{a}\right)(m\pi)^2 \qquad (6.42\text{ b})$$

For Part IV region (or Single Layer Plate Adherend),

$$\frac{d}{d\xi_{IV}}\begin{Bmatrix}\overline{\psi}^{(3)}_{mnx}\\ \overline{\psi}^{(3)}_{mny}\\ \overline{W}^{(3)}_{mn}\\ \overline{M}^{(3)}_{mnyx}\\ \overline{M}^{(3)}_{mny}\\ \overline{Q}^{(3)}_{mny}\end{Bmatrix}=\left[\begin{array}{ccc|ccc} 0 & e_{1,2} & 0 & e_{1,4} & 0 & 0\\ e_{2,1} & 0 & 0 & 0 & e_{2,5} & 0\\ 0 & e_{3,2} & 0 & 0 & 0 & e_{3,6}\\ \hline e_{4,1} & 0 & e_{4,3} & 0 & e_{4,5} & 0\\ 0 & e_{5,2} & 0 & e_{5,4} & 0 & e_{5,6}\\ e_{6,1} & 0 & e_{6,3} & 0 & 0 & 0\end{array}\right]\begin{Bmatrix}\overline{\psi}^{(3)}_{mnx}\\ \overline{\psi}^{(3)}_{mny}\\ \overline{W}^{(3)}_{mn}\\ \overline{M}^{(3)}_{mnyx}\\ \overline{M}^{(3)}_{mny}\\ \overline{Q}^{(3)}_{mny}\end{Bmatrix},\quad (0<\xi_{IV}<1)\quad (6.43.\text{a})$$

where the elements of the above ''Coefficient Matrix $\lfloor\overline{\mathcal{E}}\rfloor$'' are,

$$e_{1,2} = -\overline{L}_{IV}\, m\pi\left(\frac{h_1}{a}\right)^2 \qquad e_{1,4} = \frac{12\overline{L}_{IV}}{\overline{B}^{(3)}_{66}\overline{h}^3_3} \qquad e_{2,1} = \overline{L}_{IV}\, m\pi\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}}\left(\frac{a}{h_1}\right)^2$$

$$e_{2,5} = \frac{12\overline{L}_{IV}}{\overline{B}^{(3)}_{22}\overline{h}^3_3}\left(\frac{a}{h_1}\right)^2 \qquad e_{3,2} = -\overline{L}_{IV} \qquad e_{3,6} = \frac{\overline{L}_{IV}}{\kappa^2_y\overline{B}^{(3)}_{44}\overline{h}_3}\left(\frac{a}{h_1}\right)$$

$$e_{4,1} = -\frac{\overline{L}_{IV}}{12}\overline{\rho}_3\overline{h}^3_3\left(\frac{h_1}{a}\right)^2\overline{\Omega} + \overline{L}_{IV}\kappa^2_x\overline{B}^{(3)}_{55}\overline{h}_3\left(\frac{a}{h_1}\right)^2 + \frac{\overline{L}_{IV}\overline{h}^3_3}{12}(m\pi)^2\left(\overline{B}^{(3)}_{11} - \frac{\overline{B}^{(3)^2}_{12}}{\overline{B}^{(3)}_{22}}\right)$$

$$e_{4,3} = \kappa^2_x\overline{L}_{IV}\overline{B}^{(3)}_{55}\overline{h}_3 m\pi \qquad e_{4,5} = -\overline{L}_{IV}\, m\pi\frac{\overline{B}^{(3)}_{12}}{\overline{B}^{(3)}_{22}} \qquad e_{5,2} = -\frac{\overline{L}_{IV}}{12}\overline{\rho}_3\overline{h}^3_3\left(\frac{h_1}{a}\right)^4\overline{\Omega}$$

$$e_{5,4} = \overline{L}_{IV}\, m\pi \qquad e_{5,6} = \overline{L}_{IV}\left(\frac{a}{h_1}\right) \qquad e_{6,1} = \kappa^2_x\overline{L}_{IV}B^{(3)}_{55}\overline{h}_3\left(\frac{a}{h_1}\right)m\pi$$

$$e_{6,3} = -\overline{L}_{IV}\overline{\rho}_3\overline{h}_3\left(\frac{h_1}{a}\right)^3\overline{\Omega} + \overline{L}_{IV}\kappa^2_x\overline{B}^{(3)}_{55}\overline{h}_3\left(\frac{h_1}{a}\right)(m\pi)^2 \qquad (6.43.b)$$

The "Boundary Conditions" at x=0 and x=a are already satisfied by trigonometric expansion in ''Classical Lévy's Type Solution''. The ''Appropriate Boundary Conditions'' and the ''Continuity Conditions'' are needed to solve the ''Governing System of First Order Ordinary Differential Equations''. Then,

The ''Boundary Conditions'' along the edges in the y-direction,

F (Free): $\overline{M}_{yx}^{(j)} = \overline{M}_{y}^{(j)} = \overline{Q}_{y}^{(j)} = 0$

C (Clamped): $\overline{w}^{(j)} = \overline{\psi}_{x}^{(j)} = \overline{\psi}_{y}^{(j)} = 0$ (j=1,2,3,4) (6.44)

S (Simply Supported): $\overline{w}^{(j)} = \overline{\psi}_{x}^{(j)} = \overline{M}_{y}^{(j)} = 0$

The ''Continuity Conditions'' between Part I and Part II,

$$\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{I}=1}\right\}$$

$$\left\{\overline{\boldsymbol{Y}}^{(4)}_{\xi_{II}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(4)}_{\xi_{I}=1}\right\} \tag{6.45}$$

The ''Continuity Conditions'' between Part I and Part III,

$$\left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_{I}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1}\right\} \tag{6.46}$$

The ''Continuity Conditions'' between Part II and Part IV,

$$\left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{VI}=0}\right\} = \left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1}\right\} \tag{6.47}$$

Finally, as a summary, the entire set of the "Governing System of First Order Ordinary Differential Equations" for the "Main PROBLEM III" is given as;

$$\frac{d}{d\xi_I}\begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{mn} \\ \overline{\boldsymbol{Y}}^{(2)}_{mn} \\ \overline{\boldsymbol{Y}}^{(4)}_{mn} \end{Bmatrix} = \begin{bmatrix} \overline{\boldsymbol{C}}_{1,1} & \overline{\boldsymbol{C}}_{1,2} & \overline{\boldsymbol{C}}_{1,3} \\ \overline{\boldsymbol{C}}_{2,1} & \overline{\boldsymbol{C}}_{2,2} & \overline{\boldsymbol{C}}_{2,3} \\ \overline{\boldsymbol{C}}_{3,1} & \overline{\boldsymbol{C}}_{3,2} & \overline{\boldsymbol{C}}_{3,3} \end{bmatrix} \begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{mn} \\ \overline{\boldsymbol{Y}}^{(2)}_{mn} \\ \overline{\boldsymbol{Y}}^{(4)}_{mn} \end{Bmatrix}, \quad (0<\xi_I<1) \qquad \text{(in Part I)}$$

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(3)}\\ \overline{Y}_{mn}^{(4)}\end{Bmatrix}=\begin{bmatrix}\overline{\mathcal{C}'}_{1,1} & \overline{\mathcal{C}'}_{1,2} & \overline{\mathcal{C}'}_{1,3}\\ \overline{\mathcal{C}'}_{2,1} & \overline{\mathcal{C}'}_{2,2} & \overline{\mathcal{C}'}_{2,3}\\ \overline{\mathcal{C}'}_{3,1} & \overline{\mathcal{C}'}_{3,2} & \overline{\mathcal{C}'}_{3,3}\end{bmatrix}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(3)}\\ \overline{Y}_{mn}^{(4)}\end{Bmatrix},\ (0<\xi_{II}<1) \qquad \text{(in Part II)}$$

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}_{mn}^{(2)}\right\}=\left[\overline{\mathcal{D}}\right]\left\{\overline{Y}_{mn}^{(2)}\right\}, \qquad (0<\xi_{III}<1) \qquad \text{(in Part III)}$$

$$\frac{d}{d\xi_{IV}}\left\{\overline{Y}_{mn}^{(3)}\right\}=\left[\overline{\mathcal{E}}\right]\left\{\overline{Y}_{mn}^{(3)}\right\}, \qquad (0<\xi_{IV}<1) \qquad \text{(in Part IV)}$$

(6.48 a,b,c,d)

with the “Appropriate Boundary Conditions” and the “Continuity Conditions” in each Part I, Part II, Part III and Part III regions, respectively.

The above entire system of equations forms a “Two-Point Boundary Value Problem” for the “Main PROBLEM III” between the left and right supports in y-direction. It is obvious that, once the natural frequencies are obtained, then, the Equations (6.48.a,b,c,d) can be integrated numerically for a given particular geometry, materials and the support conditions by making the “Modified Transfer Matrix Method (with Interpolation Polynomials and/or Chebyshev Polynomials)”.

CHAPTER 7

METHOD OF SOLUTION

7.1 Introduction

Over the years, several different numerical methods and approximate methods have been developed for the solution of the “Initial and Boundary Value Problems of Plates”. Few analytical and/or closed form solutions are available in the open engineered and scientific literature and in some graduate level texts, therefore, the solution of complicated plate problems are, in general, attempted by means of the numerical and some approximate methods.

In the present “Thesis”, the “Modified Transfer Matrix Method (MTMM)” has been employed. This solution technique is a combination of the “Classical Levy’s Method”, the “Transfer Matrix Method” and the “Integrating Matrix Method”.

Yuceoglu and Özerciyes and also Özerciyes and Yuceoglu and Yuceoglu et al [VI.4-VI.13] have developed several versions of the present method of solution that is the “Modified Transfer Matrix Method (MTMM)”;

- “Modified Transfer Matrix Method (with Interpolation Polynomials)”
- “Modified Transfer Matrix Method (with Chebyshev Polynomials)”
- “Modified Transfer Matrix Method (with Eigenvalue Approach)”

Aforementioned methods are essentially semi-analytical and numerical techniques which can be easily applied to a certain class of plates, shell vibration problems.

In this “Thesis”, the first method mentioned above is employed. The present method of solution will be systematically applied to the “Main PROBLEM I, II, III”.

7.2 Method of Solution for "Main PROBLEM I a and "Main PROBLEM I b"

In this section, the application of the present solution technique will be explained in detail for the "Main PROBLEM I" without making any distinction between the "Main PROBLEM I.a" and the "Main PROBLEM I.b".

The initial step is to write the "Governing System of First Order Ordinary Differential Equations" in a "state vector" form as in (4.18), (4.20) and (4.22). In order to eliminate the first order ordinary differential operator and obtain fundamental dependent variables, only single step numerical integration is required. This is one of the advantages of the present method of solution.

The entire set of the "Governing System of Equations" may be rewritten in the compact "matrix form" with the unknown "state vectors" and the "Coefficient Matrices" as (recalling that the entire set, now, creates a "Two-point BVP"),

"Governing System of First Order Ordinary Differential Equations",

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{Y}^{(1)}_{mn}\\ \overline{Y}^{(2)}_{mn}\end{Bmatrix} = \left[\overline{\mathcal{C}}\right]\begin{Bmatrix}\overline{Y}^{(1)}_{mn}\\ \overline{Y}^{(2)}_{mn}\end{Bmatrix}, \qquad (0<\xi_I<1) \quad \text{(in Part I)}$$

$$\frac{d}{d\xi_{II}}\left\{\overline{Y}^{(1)}_{mn}\right\} = \left[\overline{\mathcal{D}}\right]\left\{\overline{Y}^{(1)}_{mn}\right\}, \qquad (0<\xi_{II}<1) \quad \text{(in Part II)} \qquad (7.1.a,b,c)$$

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}^{(2)}_{mn}\right\} = \left[\overline{\mathcal{E}}\right]\left\{\overline{Y}^{(2)}_{mn}\right\}, \qquad (0<\xi_{III}<1) \quad \text{(in Part III)}$$

with the "Appropriate Boundary Conditions" and the "Continuity Conditions" for the particular problem under consideration in Part I, Part II, Part III.

where ξ_I, ξ_{II}, ξ_{III} are defined as y_I/ℓ_I, y_{II}/ℓ_{II}, y_{III}/ℓ_{III}, respectively and the "Coefficient Matrix [$\overline{\mathcal{C}}$]" is of dimension (12x12) which explicitly includes the nondimensional geometric and material characteristics of the upper and lower plate adherends and the adhesive layer and the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$ of the composite bonded system. Similarly, the "Coefficient Matrix [$\overline{\mathcal{D}}$]" and the "Coefficient Matrix [$\overline{\mathcal{E}}$]" are both of dimensions (6x6) which explicitly include dimensionless geometric and material characteristics of the upper plate in Part II Region and lower plate in Part III Region, and the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$ .The "Column Matrix $\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(j)}\}$" (j=1,2) are the "state vectors" including state variables or "dimensionless fundamental dependent variables" of the problem,

$$\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{mn}}^{(j)}\right\} = \left\{\overline{\Psi}_{mnx}^{(j)}, \overline{\Psi}_{mny}^{(j)}, \overline{W}_{mn}^{(j)}; \overline{M}_{mnyx}^{(j)}, \overline{M}_{mny}^{(j)}, \overline{Q}_{mny}^{(j)}\right\}^T \qquad \text{(j=1,2)} \qquad (7.2)$$

The next step involves discretization of the "fundamental dependent variables" of the problem and the "Coefficient Matrices" in (7.1) with respect to the independent space variables ξ_I, ξ_{II}, ξ_{III}, (which are taken as dimensionless spatial coordinates) along Part I, Part II, and Part III, respectively.

The discretization procedure is performed by dividing Part I, Part II and Part III regions into sufficient number (n_1 for Part I, n_2 for Part II, and n_3 for Part III) of segments or stations along ξ_I, ξ_{II} and ξ_{III} directions, respectively and pre-multiplying discrete version of "Coefficient Matrices" by the appropriate "Global Integrating Matrix $[\mathcal{L}]$" which includes "Integrating Sub-Matrices [$\boldsymbol{L}$]". For convenience, "mn" subscript will be dropped from the equations. Then,

<u>For Part I Region (Two-Layer Composite Plate),</u>

$$\begin{Bmatrix} \dot{\overline{\boldsymbol{Y}}}^{(1)} \\ \dot{\overline{\boldsymbol{Y}}}^{(2)} \end{Bmatrix} - \begin{Bmatrix} \dot{\overline{\boldsymbol{Y}}}_{\boldsymbol{I}}^{(1)} \\ \dot{\overline{\boldsymbol{Y}}}_{\boldsymbol{I}}^{(2)} \end{Bmatrix} = [\mathcal{L}_{\mathrm{I}}]\left[\dot{\overline{\mathcal{C}}}\right]\begin{Bmatrix} \dot{\overline{\boldsymbol{Y}}}^{(1)} \\ \dot{\overline{\boldsymbol{Y}}}^{(2)} \end{Bmatrix}, \qquad \text{(in Part I)} \qquad (7.3)$$

<u>For Part II Region (Single Layer Orthotropic or Isotropic Upper Plate),</u>

$$\left\{\dot{\overline{\boldsymbol{Y}}}^{(1)}\right\} - \left\{\dot{\overline{\boldsymbol{Y}}}_{\boldsymbol{I}}^{(1)}\right\} = [\mathcal{L}_{\mathrm{II}}]\left[\dot{\overline{\mathcal{D}}}\right]\left\{\dot{\overline{\boldsymbol{Y}}}^{(1)}\right\}, \qquad \text{(in Part II)} \qquad (7.4)$$

<u>For Part III region (Single Layer Orthotropic or Isotropic Lower Plate),</u>

$$\left\{\dot{\bar{Y}}^{(2)}\right\} - \left\{\dot{\bar{Y}}_I^{(2)}\right\} = [\mathcal{L}_{\mathrm{III}}]\left[\dot{\bar{\mathcal{E}}}\right]\left\{\dot{\bar{Y}}^{(2)}\right\}, \qquad \text{(in Part III)} \qquad (7.5)$$

where subscripts I, II, III in $[\mathcal{L}]$ indicate integration performed for Part I, Part II, and Part III and $\left\{\dot{\bar{Y}}^{(j)}\right\}$, (j=1,2) is discrete versions of "state vector $\left\{\bar{Y}_{mn}^{(j)}\right\}$" and in matrices $\left[\dot{\bar{\mathcal{C}}}\right], \left[\dot{\bar{\mathcal{D}}}\right], \left[\dot{\bar{\mathcal{E}}}\right]$ ("dot" or "•" indicating the discretization along the ξ-direction) are the discrete versions of "Coefficient Matrices" $\left[\bar{\mathcal{C}}\right], \left[\bar{\mathcal{D}}\right], \left[\bar{\mathcal{E}}\right]$ in (7.1), respectively. The "state vector $\left\{\bar{Y}_I^{(j)}\right\}$" represents the "initial" end point, i.e. at $\xi_I=0$, $\xi_{II}=0$, and $\xi_{III}=0$.for Part I, Part II and Part III, respectively The superscripts "1" denotes the upper plate and the superscript "2" denotes lower plate.

The more detailed forms of the "state vectors" at "general station" and "state vectors" at "initial end points" is given below;

Discretization of "State Vectors" evaluated at the "general station" and "State Vectors" evaluated at the "initial end point"

$$\left\{\dot{\boldsymbol{Y}}^{(1)}\right\}=\begin{Bmatrix}\overline{\Psi}_{x1}^{(1)}\\ \overline{\Psi}_{x2}^{(1)}\\ .\\ .\\ \overline{\Psi}_{xn_l}^{(1)}\\ \overline{\Psi}_{y1}^{(1)}\\ \overline{\Psi}_{y2}^{(1)}\\ .\\ .\\ .\\ \overline{Q}_{y1}^{(1)}\\ .\\ .\\ \overline{Q}_{yn_l}^{(1)}\end{Bmatrix},\ \left\{\dot{\boldsymbol{Y}}^{(2)}\right\}=\begin{Bmatrix}\overline{\Psi}_{x1}^{(2)}\\ \overline{\Psi}_{x2}^{(2)}\\ .\\ .\\ \overline{\Psi}_{xn_l}^{(2)}\\ \overline{\Psi}_{y1}^{(2)}\\ \overline{\Psi}_{y2}^{(2)}\\ .\\ .\\ .\\ \overline{Q}_{y1}^{(2)}\\ .\\ .\\ \overline{Q}_{yn_l}^{(2)}\end{Bmatrix},\ \left\{\dot{\boldsymbol{Y}}_1^{(1)}\right\}=\begin{Bmatrix}\overline{\Psi}_{x1}^{(1)}\\ \overline{\Psi}_{x1}^{(1)}\\ .\\ .\\ \overline{\Psi}_{x1}^{(1)}\\ \overline{\Psi}_{y1}^{(1)}\\ \overline{\Psi}_{y1}^{(1)}\\ .\\ .\\ .\\ \overline{Q}_{y1}^{(1)}\\ .\\ .\\ \overline{Q}_{y1}^{(1)}\end{Bmatrix},\ \left\{\dot{\boldsymbol{Y}}_1^{(2)}\right\}=\begin{Bmatrix}\overline{\Psi}_{x1}^{(2)}\\ \overline{\Psi}_{x1}^{(2)}\\ .\\ .\\ \overline{\Psi}_{x1}^{(2)}\\ \overline{\Psi}_{y1}^{(2)}\\ \overline{\Psi}_{y1}^{(2)}\\ .\\ .\\ .\\ \overline{Q}_{y1}^{(2)}\\ .\\ .\\ \overline{Q}_{y1}^{(2)}\end{Bmatrix}, \tag{7.6}$$

Discretization of "Coefficient Matrix" for Part I,

$$\left[\dot{\overline{\boldsymbol{C}}}\right]=\begin{bmatrix}\left[\dot{\overline{\boldsymbol{C}}}_{1,1}\right]_{nxn} & \left[\dot{\overline{\boldsymbol{C}}}_{1,2}\right]_{nxn} & . & . & \left[\dot{\overline{\boldsymbol{C}}}_{1,12}\right]_{nxn}\\ . & . & . & . & \left[\dot{\overline{\boldsymbol{C}}}_{2,12}\right]_{nxn}\\ . & . & . & . & .\\ . & . & . & . & .\\ \left[\dot{\overline{\boldsymbol{C}}}_{12,1}\right]_{nxn} & . & . & . & \left[\dot{\overline{\boldsymbol{C}}}_{12,12}\right]_{nxn}\end{bmatrix} \tag{7.7}$$

Discretization of "Coefficient Matrix" for Part II,

$$\left[\dot{\bar{\mathcal{D}}}\right]=\begin{bmatrix} \left[\dot{\bar{\mathcal{D}}}_{1,1}\right]_{nxn} & \left[\dot{\bar{\mathcal{D}}}_{1,2}\right]_{nxn} & \cdot\ \cdot & \left[\dot{\bar{\mathcal{D}}}_{1,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \left[\dot{\bar{\mathcal{D}}}_{2,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \left[\dot{\bar{\mathcal{D}}}_{6,1}\right]_{nxn} & \cdot & \cdot\ \cdot & \left[\dot{\bar{\mathcal{D}}}_{6,6}\right]_{nxn} \end{bmatrix} \tag{7.8}$$

Discretization of "Coefficient Matrix" for Part III,

$$\left[\dot{\bar{\mathcal{E}}}\right]=\begin{bmatrix} \left[\dot{\bar{\mathcal{E}}}_{1,1}\right]_{nxn} & \left[\dot{\bar{\mathcal{E}}}_{1,2}\right]_{nxn} & \cdot\ \cdot & \left[\dot{\bar{\mathcal{E}}}_{1,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \left[\dot{\bar{\mathcal{E}}}_{2,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \left[\dot{\bar{\mathcal{E}}}_{6,1}\right]_{nxn} & \cdot & \cdot\ \cdot & \left[\dot{\bar{\mathcal{E}}}_{6,6}\right]_{nxn} \end{bmatrix} \tag{7.9}$$

In the above matrices, the second subscript in (7.6), (7.7), (7.8) and (7.9) indicate the discretization point or the "station" with which they are associated and $[\mathcal{C}_{i,j}]$, $[\mathcal{D}_{i,j}]$ and $[\mathcal{E}_{i,j}]$ are the diagonal "Sub-Matrices" composed of the elements of the related "Coefficient Matrix".

The relation between the "state vector" at a ''general station'' along ξ_I in Part I, ξ_{II} in Part II, and ξ_{III} in Part III, and the "state vector" at the "initial end points'' at ξ_I=0 in Part I, at ξ_{II}=0 in Part II, at ξ_{III}=0 in Part III, respectively, can be written by rearranging the above equations as, From Equation (7.3);

<u>For the "Overlap Region" or Part I region,</u>

$$\begin{Bmatrix} \dot{\bar{Y}}^{(1)} \\ \dot{\bar{Y}}^{(2)} \end{Bmatrix} = \left[\dot{\mathcal{U}} \right] \begin{Bmatrix} \dot{\bar{Y}}_1^{(1)} \\ \dot{\bar{Y}}_1^{(2)} \end{Bmatrix} \qquad \text{(in Part I)} \qquad (7.10)$$

where $\left[\dot{\mathcal{U}} \right]$ is the "Discretized Modified Transfer Matrix" for Part I region,

$$\left[\dot{\mathcal{U}} \right] = \left([I] - [\mathcal{L}_{\mathrm{I}}] \left[\dot{\bar{\mathcal{C}}} \right] \right)^{-1} \qquad (7.11)$$

Similarly, from Equation (7.4),

<u>For Part II region,</u>

$$\left\{ \dot{\bar{Y}}^{(1)} \right\} = \left[\dot{\mathcal{V}} \right] \left\{ \dot{\bar{Y}}_1^{(1)} \right\} \qquad \text{(in Part II)} \qquad (7.12)$$

where $\left[\dot{\mathcal{V}} \right]$ is the "Discretized Modified Transfer Matrix" for Part II region,

$$\left[\dot{\mathcal{V}} \right] = \left([I] - [\mathcal{L}_{\mathrm{II}}] \left[\dot{\bar{\mathcal{D}}} \right] \right)^{-1} \qquad (7.13)$$

Similarly, from Equation (7.5),

<u>For Part III region,</u>

$$\left\{ \dot{\bar{Y}}^{(2)} \right\} = \left[\dot{\mathcal{W}} \right] \left\{ \dot{\bar{Y}}_1^{(2)} \right\} \qquad \text{(in Part III)} \qquad (7.14)$$

where $\left[\dot{\mathcal{W}} \right]$ is the "Discretized Modified Transfer Matrix" for Part III region,

$$\left[\dot{\mathcal{W}} \right] = \left([I] - [\mathcal{L}_{\mathrm{III}}] \left[\dot{\bar{\mathcal{E}}} \right] \right)^{-1} \qquad (7.15)$$

In the above expressions, as defined before, $[\mathcal{L}_{I}]$, $[\mathcal{L}_{II}]$ and $[\mathcal{L}_{III}]$ are the "Global Integrating Matrices" for Part I, Part II and Part III, respectively.

The discretized versions of the "Modified Transfer Matrix" between a ''general station'' along ξ_I, ξ_{II}, ξ_{III} and the "initial end point $\xi_I=0$, $\xi_{II}=0$, $\xi_{III}=0$,respectively, can be expressed in open form as,

For Part I region (Two-Layer Composite Plate),

$$\begin{Bmatrix} \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x2}^{(1)} \\ \overline{\Psi}_{x3}^{(1)} \\ . \\ \overline{\Psi}_{xn_1}^{(2)} \\ . \\ . \\ \overline{Q}_{y1}^{(2)} \\ \overline{Q}_{y2}^{(2)} \\ . \\ . \\ \overline{Q}_{yn_1}^{(2)} \end{Bmatrix} = \begin{bmatrix} \dot{\mathcal{U}}_{1,1} & \dot{\mathcal{U}}_{1,2} & \dot{\mathcal{U}}_{1,3} & . & . & . & . & . & . & . & . & . & \dot{\mathcal{U}}_{1,12} \\ \dot{\mathcal{U}}_{2,1} & . & . & . & . & . & . & . & . & . & . & . & . \\ \dot{\mathcal{U}}_{1,3} & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ \dot{\mathcal{U}}_{12,1} & . & . & . & . & . & . & . & . & . & . & . & \dot{\mathcal{U}}_{12,12} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x1}^{(1)} \\ . \\ \overline{\Psi}_{x1}^{(1)} \\ . \\ . \\ \overline{Q}_{y1}^{(2)} \\ \overline{Q}_{y1}^{(2)} \\ . \\ . \\ \overline{Q}_{y1}^{(2)} \end{Bmatrix} \quad (7.16)$$

For Part II region (Single-Layer Orthotropic or Isotropic Plate),

$$\begin{Bmatrix} \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x2}^{(1)} \\ . \\ . \\ \overline{\Psi}_{xn_2}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \overline{\Psi}_{y2}^{(1)} \\ . \\ . \\ . \\ \overline{Q}_{y1}^{(1)} \\ . \\ . \\ \overline{Q}_{yn_2}^{(1)} \end{Bmatrix} = \begin{bmatrix} \dot{\mathcal{V}}_{1,1} & . & . & . & \dot{\mathcal{V}}_{1,6} \\ \dot{\mathcal{V}}_{2,1} & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \dot{\mathcal{V}}_{6,1} & . & . & . & \dot{\mathcal{V}}_{6,6} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x1}^{(1)} \\ . \\ . \\ \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ . \\ . \\ . \\ \overline{Q}_{y1}^{(1)} \\ . \\ . \\ \overline{Q}_{y1}^{(1)} \end{Bmatrix} \quad (7.17)$$

For Part III region (Single-Layer Orthotropic or Isotropic Plate),,

$$\begin{Bmatrix} \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{x2} \\ \cdot \\ \cdot \\ \overline{\Psi}^{(2)}_{xn_3} \\ \overline{\Psi}^{(2)}_{y1} \\ \overline{\Psi}^{(2)}_{y2} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_3} \end{Bmatrix} = \begin{bmatrix} \dot{w}_{1,1} & \cdot & \cdot & \cdot & \dot{w}_{1,6} \\ \dot{w}_{2,1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dot{w}_{6,1} & \cdot & \cdot & \cdot & \dot{w}_{6,6} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{x1} \\ \cdot \\ \cdot \\ \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{y1} \\ \overline{\Psi}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \end{Bmatrix} \quad (7.18)$$

The matrix $\left[\dot{u}\right]$ is composed of (12×12) square blocks of dimension n_1 and each individual block represents a relation between a “state variable” (an element of the appropriate “state vector”) at any “general station ξ_I” and the state variable at the “initial end point ξ_I=0”. The subscript n_1 is the number of discretization points in Part I region or in the “Overlap Region”.

The matrix $\left[\dot{v}\right]$ is composed of (6×6) blocks of dimension n_2 and each individual block represents a relation between a state variable at any “general station ξ_{II}” and the "state variable” (an element of the appropriate “state vector”) at the “initial end point ξ_{II}=0”. The subscript n_2 is the number of discretization points in Part II Region.

The matrix $\left[\dot{w}\right]$ is composed of (6×6) blocks of dimension n_3 and each individual block represents a relation between a “state variable” (an element of the appropriate “state vector”) at any “general station ξ_{III}” the state variable at the “initial end point ξ_{III}=0". The subscript n_3 is the number of discretization points in Part III Region.

Then, by summing the elements related with the each group of "state vectors" at "initial end point", on the integer multiples of n[th] row of $\left[\dot{\mathcal{U}}\right]$, $\left[\dot{\mathcal{V}}\right]$ and $\left[\dot{\mathcal{W}}\right]$ one can obtain the following,

For Part I region (Two-Layer Composite Plate),

$$\begin{Bmatrix} \overline{\Psi}^{(1)}_{xn_1} \\ \overline{\Psi}^{(1)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_1} \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\left(\dot{\mathcal{U}}_{1,1}\right)_{n,i} & \sum_{i=n+1}^{2n}\left(\dot{\mathcal{U}}_{1,2}\right)_{n,i} & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \sum_{i=11n+1}^{12n}\left(\dot{\mathcal{U}}_{12,12}\right)_{n,i} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \end{Bmatrix} \tag{7.19.a}$$

Then, in compact matrix form,

$$\begin{Bmatrix} \overline{\Psi}^{(1)}_{xn_1} \\ \overline{\Psi}^{(1)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_1} \end{Bmatrix} = \left[\tilde{\mathcal{U}}\right] \begin{Bmatrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \end{Bmatrix} \qquad \text{(in Part I)} \tag{7.19.b}$$

For Part II region (Single-Layer Orthotropic or Isotropic Plate),

$$\begin{Bmatrix} \overline{\Psi}^{(1)}_{xn_2} \\ \overline{\Psi}^{(1)}_{yn_2} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{yn_2} \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\left(\dot{\mathcal{V}}'_{1,1}\right)_{n,i} & \sum_{i=n+1}^{2n}\left(\dot{\mathcal{V}}_{1,2}\right)_{n,i} & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \sum_{i=5n+1}^{6n}\left(\dot{\mathcal{V}}_{1,6}\right)_{n,i} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{y1} \end{Bmatrix} \tag{7.20.a}$$

Then, in compact matrix form,

$$\begin{Bmatrix} \overline{\Psi}^{(1)}_{xn_2} \\ \overline{\Psi}^{(1)}_{yn_2} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{yn_2} \end{Bmatrix} = \left[\tilde{\mathcal{V}}\right] \begin{Bmatrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{y1} \end{Bmatrix} \qquad \text{(in Part II)} \qquad (7.20.b)$$

<u>For Part III region (Single-Layer Orthotropic or Isotropic Plate),</u>

$$\begin{Bmatrix} \overline{\Psi}^{(2)}_{xn_3} \\ \overline{\Psi}^{(2)}_{yn_3} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_3} \end{Bmatrix} = \begin{bmatrix} \sum\limits_{i=1}^{n}\left(\dot{\mathcal{W}}_{1,1}\right)_{n,i} & \sum\limits_{i=n+1}^{2n}\left(\dot{\mathcal{W}}_{1,2}\right)_{n,i} & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \sum\limits_{i=5n+1}^{6n}\left(\dot{\mathcal{W}}_{1,6}\right)_{n,i} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \end{Bmatrix} \qquad (7.21.a)$$

Then, in compact matrix form,

$$\begin{Bmatrix} \overline{\Psi}^{(2)}_{xn_3} \\ \overline{\Psi}^{(2)}_{yn_3} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_3} \end{Bmatrix} = \left[\tilde{\mathcal{W}}\right] \begin{Bmatrix} \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \end{Bmatrix} \qquad \text{(in Part III)} \qquad (7.21.b)$$

In the above equations, $\left[\tilde{\mathcal{U}}\right]$, $\left[\tilde{\mathcal{V}}\right]$ and $\left[\tilde{\mathcal{W}}\right]$ are the "final form" of the discretized "Modified Transfer Matrices" for Part I, Part II and Part III regions, respectively. They transfer the discretized quantities from the "initial end point ξ_t=0" (t=Parts I, II, III) to the general end point ξ_t=1" (t=Parts I, II, III), respectively.

At this stage, further combination of equations is needed. Thus, the equations (7.19), (7.20) and (7.21) can be rewritten in a compact matrix form, such that,

For the "Overlap Region" or Part I region,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=1} \\ \overline{Y}^{(2)}_{\xi_I=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} \\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} \end{bmatrix}_{01} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=0} \\ \overline{Y}^{(2)}_{\xi_I=0} \end{Bmatrix} \qquad (0\leq\xi_I\leq) \qquad (7.22)$$

For Part II region,

$$\left\{\overline{Y}^{(1)}_{\xi_{II}=1}\right\} = \left[\tilde{\mathcal{V}}_{1,1}\right]_{01} \left\{\overline{Y}^{(1)}_{\xi_{II}=0}\right\} \qquad (0\leq\xi_{II}\leq) \qquad (7.23)$$

For Part III region,

$$\left\{\overline{Y}^{(2)}_{\xi_{III}=1}\right\} = \left[\tilde{\mathcal{W}}_{1,1}\right]_{01} \left\{\overline{Y}^{(2)}_{\xi_{III}=0}\right\} \qquad (0\leq\xi_{III}\leq) \qquad (7.24)$$

where the subscript (01) in the above matrix expressions means that the "final form" of the appropriate "Modified Transfer Matrix" transfers the above quantities from the "initial end point, "0", to the "final end point, "1" along ξ_I, ξ_{II}, ξ_{III}, directions respectively.

The matrices $\left[\tilde{\mathcal{U}}_{i,j}\right]$,$\left[\tilde{\mathcal{V}}_{i,j}\right]$ and $\left[\tilde{\mathcal{W}}_{i,j}\right]$ are the partitioned square matrices of dimension (6x6) which implicitly includes the unknown dimensionless natural frequencies $\overline{\omega}_{mn}$ of the entire composite bonded plate system of the "Main PROBLEM I"

Now, natural frequencies of the entire system will be determined by using the "Boundary Conditions" at the supports and the "Continuity Conditions" between the regions or parts in the y-direction (see also the longitudinal cross section of the "Main PROBEM I"). Any combination of the "Boundary Conditions" at ξ_{II}=0, ξ_I=1 for the upper plate, and ξ_I=0, ξ_{III}=1 for the lower plate in the y-direction can be prescribed.

The "Continuity Conditions" between Part I and Part II can be written for upper plate adherend and the "Continuity Condition" between Part I and Part III can be written for lower plate adherend as follows,

The ''<u>Continuity Conditions</u>'' between the Part I and Part II,

$$\left\{\overline{Y}^{(1)}_{\xi_I=0}\right\}=\left\{\overline{Y}^{(1)}_{\xi_{II}=1}\right\} \tag{7.25}$$

The ''<u>Continuity Conditions</u>'' between the Part I and Part III,

$$\left\{\overline{Y}^{(2)}_{\xi_I=1}\right\}=\left\{\overline{Y}^{(2)}_{\xi_{III}=0}\right\} \tag{7.26}$$

By using Equations (7.23) and (7.25),

$$\left\{\overline{Y}^{(1)}_{\xi_I=0}\right\}=\left[\tilde{\mathcal{V}}_{1,1}\right]_{01}\left\{\overline{Y}^{(1)}_{\xi_{II}=0}\right\} \tag{7.27}$$

Then, one can write,

$$\begin{Bmatrix}\overline{Y}^{(1)}_{\xi_I=0}\\ \overline{Y}^{(2)}_{\xi_I=0}\end{Bmatrix}=\begin{bmatrix}\tilde{\mathcal{V}}_{1,1} & 0\\ 0 & I\end{bmatrix}\begin{Bmatrix}\overline{Y}^{(1)}_{\xi_{II}=0}\\ \overline{Y}^{(2)}_{\xi_I=0}\end{Bmatrix} \tag{7.28}$$

By using Equations (7.24) and (7.26),

$$\left\{\overline{Y}^{(2)}_{\xi_I=1}\right\}=\left[\tilde{\mathcal{W}}_{1,1}\right]^{-1}\left\{\overline{Y}^{(2)}_{\xi_{III}=1}\right\} \tag{7.29}$$

Then, one can write,

$$\begin{Bmatrix}\overline{Y}^{(1)}_{\xi_I=1}\\ \overline{Y}^{(2)}_{\xi_I=1}\end{Bmatrix}=\begin{bmatrix}I & 0\\ 0 & \tilde{\mathcal{W}}^{-1}_{1,1}\end{bmatrix}\begin{Bmatrix}\overline{Y}^{(1)}_{\xi_I=1}\\ \overline{Y}^{(2)}_{\xi_{III}=1}\end{Bmatrix} \tag{7.30}$$

After substituting (7.28) and (7.30) into (7.23) one can obtain,

$$\begin{bmatrix}\mathbf{I} & 0\\ 0 & \tilde{\mathcal{W}}^{-1}_{1,1}\end{bmatrix}\begin{Bmatrix}\overline{Y}^{(1)}_{\xi_I=1}\\ \overline{Y}^{(2)}_{\xi_{III}=1}\end{Bmatrix}=\begin{bmatrix}\tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2}\\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2}\end{bmatrix}\begin{bmatrix}\tilde{\mathcal{V}}_{1,1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\overline{Y}^{(1)}_{\xi_{II}=0}\\ \overline{Y}^{(2)}_{\xi_I=0}\end{Bmatrix} \tag{7.31}$$

where **[I]** is a unit matrix with dimension (6x6). And, by further rearrangement of (7.31) yields,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=1} \\ \overline{Y}^{(2)}_{\xi_{III}=1} \end{Bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \tilde{\mathcal{W}}_{1,1} \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} \\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{V}}_{1,1} & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=0} \\ \overline{Y}^{(2)}_{\xi_I=0} \end{Bmatrix} \qquad (7.32)$$

Then the final form of the "Discretized Modified Transfer Matrix $[Q]_{01}$" can be written as follows;,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=1} \\ \overline{Y}^{(2)}_{\xi_{III}=1} \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{1,1} & \mathcal{Q}_{1,2} \\ \mathcal{Q}_{2,1} & \mathcal{Q}_{2,2} \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=0} \\ \overline{Y}^{(2)}_{\xi_I=0} \end{Bmatrix} \qquad (1.33)$$

Thus, $[Q]_{01}$ transfers the "state variables" in the "state vectors" from left to support (initial end point) to the right support (final end point)along ξ or y-direction for the entire bonded plate system.

The "final form" of the "Modified Transfer Matrix" can be reduced to (6x6) by substitution of the "Boundary Conditions" in the y-direction at ξ_{II}=0, ξ_I=1 for the upper plate, and ξ_I=0, ξ_{III}=1 for the lower plate. After the boundary conditions are inserted, the following "matrix equation" (which implicitly includes the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$) can be obtained,

$[\mathbf{C_0}(\overline{\omega}_{mn})]_{\mathbf{(I)}}\{\mathbf{Y_0}\}_{\mathbf{(I)}} = \{\mathbf{0}\}$ $\Rightarrow$ Determinant of "Coefficient Matrix"

$= \left| [\mathbf{C_0}]_{\mathbf{(I)}} \right| = 0$ (7.34)

where subscript (I) in the above matrix expression indicate the "Main PROBLEM I"

The "Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(I)}}$" is obtained by eliminating the rows corresponding to nonzero "state variables" at the "final end point'' and the columns corresponding to zero "state variables" at the "initial end point'' in (7.33). Here, $\{\mathbf{Y_0}\}$ is a vector whose elements are the nonzero components of $\left\{ \boldsymbol{Y}^{(1)}_{\xi_{II}=0}, \boldsymbol{Y}^{(2)}_{\xi_{I}=0} \right\}^{T}$.

For a non-trivial solution of (7.34), determinant of the "Coefficient Matrix $\left[\mathbf{C_0}\right]_{\mathbf{(I)}}$" must be equal to zero. This procedure should be repeated for a specific set of "Boundary Conditions" and the given material and the Geometric Characteristics in order to find the roots of the determinant of the "Coefficient Matrix $\left[\mathbf{C_0}\right]_{\mathbf{(I)}}$" which includes the nondimensional natural frequencies, $\overline{\omega}_{mn}$ of the entire composite bonded plate system of the "Main PROBLEM I.". One can easily find the corresponding mode shapes, after the natural frequencies (and eigenvectors) are obtained for a particular set of "Boundary Conditions" in the y-directions.

7.3 Method of Solution for "Main PROBLEM II a" and "Main PROBLEM II b"

In this section, the application of the present solution technique will be explained in detail for the "Main PROBLEM II" without making any distinction between the "Main PROBLEM II.a" and the "Main PROBLEM II.b".

First step is to write the sets of the "Governing System of First Order Ordinary Differential Equations" in a "state vector" form in Part I, Part II, Part III and Part IV regions as in (5.23, 5.25, 5.27),

"<u>Governing System of First Order Ordinary Differential Equations</u>"

$$\frac{d}{d\xi_I}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(2)}\end{Bmatrix} = \left[\overline{\mathcal{C}}\right]\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(2)}\end{Bmatrix}, \qquad (0<\xi_I<1) \quad \text{(in Part I)}$$

$$\frac{d}{d\xi_{II}}\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(3)}\end{Bmatrix} = \left[\overline{\mathcal{C}'}\right]\begin{Bmatrix}\overline{Y}_{mn}^{(1)}\\ \overline{Y}_{mn}^{(3)}\end{Bmatrix}, \qquad (0<\xi_{II}<1) \quad \text{(in Part II)} \qquad (7.35.a,b,c,d)$$

$$\frac{d}{d\xi_{III}}\left\{\overline{Y}_{mn}^{(2)}\right\} = \left[\overline{\mathcal{D}}\right]\left\{\overline{Y}_{mn}^{(2)}\right\}, \qquad (0<\xi_{III}<1) \quad \text{(in Part III)}$$

$$\frac{d}{d\xi_{IV}}\left\{\overline{Y}_{mn}^{(3)}\right\} = \left[\overline{\mathcal{E}}\right]\left\{\overline{Y}_{mn}^{(3)}\right\}, \qquad (0<\xi_{IV}<1) \quad \text{(in Part IV)}$$

with the "Appropriate Boundary Conditions" and the "Continuity Conditions" for the particular problem under consideration.

where ξ_I, ξ_{II}, ξ_{III} and ξ_{IV} are defined as y_I/ℓ_I, y_{II}/ℓ_{II}, y_{III}/ℓ_{III} and y_{IV}/ℓ_{IV}, respectively. The "Coefficient Matrices $[\overline{\mathcal{C}}]$ and $[\overline{\mathcal{C}'}]$" are of dimension (12x12) and $[\overline{\mathcal{D}}]$, $[\overline{\mathcal{E}}]$ are of dimension (6x6). In the above, $[\overline{\mathcal{C}}]$ is the "Coefficient Matrix" for Part I of the "overlap region" and $[\overline{\mathcal{C}'}]$ is the "Coefficient Matrix" for Part II of the "Overlap Region". They include the dimensionless geometric and material characteristics of the adherends as well as the unknown dimensionless natural frequencies $\overline{\omega}_{mn}$. The "Column Matrix $\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{m}}^{(j)}\right\}$ (j=1,2,3) are the "state vectors" including the "dimensionless fundamental state variables" of the adherends,

$$\left\{\overline{\boldsymbol{Y}}_{\boldsymbol{m}}^{(j)}\right\}=\left\{\overline{\Psi}_{mx}^{(j)},\overline{\Psi}_{my}^{(j)},\overline{W}_{m}^{(j)};\overline{M}_{myx}^{(j)},\overline{M}_{my}^{(j)},\overline{Q}_{my}^{(j)}\right\}^{T} \qquad (j=1,2,3) \qquad (7.35\text{ e})$$

The Discretized "Modified Transfer Matrix" for Part I, Part II, Part III and Part IV can be obtained (similar to the "Main PROBLEM I") as for the "Main PROBLEM II",

For the "Overlap Region" Part I,

$$\begin{Bmatrix}\overline{\boldsymbol{Y}}_{\xi_I=1}^{(1)}\\ \overline{\boldsymbol{Y}}_{\xi_I=1}^{(2)}\end{Bmatrix}=\begin{bmatrix}\tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2}\\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2}\end{bmatrix}_{01}\begin{Bmatrix}\overline{\boldsymbol{Y}}_{\xi_I=0}^{(1)}\\ \overline{\boldsymbol{Y}}_{\xi_I=0}^{(2)}\end{Bmatrix}, \qquad \text{(in Part I)} \qquad (7.36)$$

For the "Overlap Region" Part II,

$$\begin{Bmatrix}\overline{\boldsymbol{Y}}_{\xi_{II}=1}^{(1)}\\ \overline{\boldsymbol{Y}}_{\xi_{II}=1}^{(3)}\end{Bmatrix}=\begin{bmatrix}\tilde{\mathcal{U}}'_{1,1} & \tilde{\mathcal{U}}'_{1,2}\\ \tilde{\mathcal{U}}'_{2,1} & \tilde{\mathcal{U}}'_{2,2}\end{bmatrix}_{01}\begin{Bmatrix}\overline{\boldsymbol{Y}}_{\xi_{II}=0}^{(1)}\\ \overline{\boldsymbol{Y}}_{\xi_{II}=0}^{(3)}\end{Bmatrix}, \qquad \text{(in Part II)} \qquad (7.37)$$

For Part III Region,

$$\left\{\overline{\boldsymbol{Y}}_{\xi_{III}=1}^{(2)}\right\}=\left[\tilde{\mathcal{V}}_{1,1}\right]_{01}\left\{\overline{\boldsymbol{Y}}_{\xi_{III}=0}^{(2)}\right\}, \qquad \text{(in Part III)} \qquad (7.38)$$

For Part IV Region,

$$\left\{\overline{\boldsymbol{Y}}_{\xi_{IV}=1}^{(3)}\right\}=\left[\dot{\mathcal{W}}_{1,1}\right]_{01}\left\{\overline{\boldsymbol{Y}}_{\xi_{IV}=0}^{(3)}\right\}, \qquad \text{(in Part IV)} \qquad (7.39)$$

where subscript (01) in the above expressions means that the "final form" of the "Modified Transfer Matrix" transferring the above quantities from the "initial end point 0", to the "final end point 1" in their respective parts and regions.

Next, the natural frequencies of the entire composite bonded plate system will be determined by using the "Boundary Conditions" and the "Continuity Conditions" in the y-direction. Any combination of the "Boundary Conditions" at ξ_I=0, ξ_{II}=1 for the upper plate, ξ_{III}=0, ξ_I=1 for the lower left plate, and ξ_{II}=0, ξ_{IV}=1 for the lower right plate in the y-direction can be prescribed.

The "Continuity Conditions" between Part I and Part II can be written for upper plate, the "Continuity Conditions" between Part I and Part III for lower left plate and the "Continuity Conditions" between Part II and Part IV for middle right plate can be written as in the following,

The ‘‘Continuity Conditions’’ between the Part I and Part II,

$$\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_I=1}\right\} = \left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0}\right\} \tag{7.40}$$

The ‘‘Continuity Conditions’’ between the Part I and Part III,

$$\left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1}\right\} = \left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_I=0}\right\} \tag{7.41}$$

The ‘‘Continuity Conditions’’ between the Part II and Part IV,

$$\left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1}\right\} = \left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=0}\right\} \tag{7.42}$$

From Equations (7.36), (7.38),(7.41) one can obtain,

$$\begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_I=1} \\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_I=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\mathcal{U}}}_{1,1} & \tilde{\boldsymbol{\mathcal{U}}}_{1,2} \\ \tilde{\boldsymbol{\mathcal{U}}}_{2,1} & \tilde{\boldsymbol{\mathcal{U}}}_{2,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{\mathcal{V}}}_{1,1} \end{bmatrix} \begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_I=0} \\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=0} \end{Bmatrix} \tag{7.43}$$

Similarly, by Equations (7.37), (7.39), (7.42),

$$\begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \tilde{\boldsymbol{\mathcal{W}}}_{1,1} \end{bmatrix}^{-1} \begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=1} \\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\mathcal{U}}}'_{1,1} & \tilde{\boldsymbol{\mathcal{U}}}'_{1,2} \\ \tilde{\boldsymbol{\mathcal{U}}}'_{2,1} & \tilde{\boldsymbol{\mathcal{U}}}'_{2,2} \end{bmatrix} \begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0} \\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=0} \end{Bmatrix} \tag{7.44}$$

Then, Equation (7.43) and (7.44) is manipulated by using Equation (7.40) and "final form" of the "Modified Transfer Matrix" which transfers the "state vectors" from the left support (initial end point) to the right support (final end point) is obtained as,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_{I}=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \widetilde{\mathcal{W}}_{1,1} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{U}}'_{1,1} & 0 & \widetilde{\mathcal{U}}'_{1,2} \\ 0 & I & 0 \\ \widetilde{\mathcal{U}}'_{2,1} & 0 & \widetilde{\mathcal{U}}'_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} \widetilde{\mathcal{U}}_{1,1} & \widetilde{\mathcal{U}}_{1,2} & 0 \\ \widetilde{\mathcal{U}}_{2,1} & \widetilde{\mathcal{U}}_{2,2} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \widetilde{\mathcal{V}}_{1,1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{I}=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \end{Bmatrix} \qquad (7.45)$$

or, finally, by simply multiplying and rearranging,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_{I}=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{1,1} & \mathcal{Q}_{1,2} & \mathcal{Q}_{1,3} \\ \mathcal{Q}_{2,1} & \mathcal{Q}_{2,2} & \mathcal{Q}_{2,3} \\ \mathcal{Q}_{3,1} & \mathcal{Q}_{3,2} & \mathcal{Q}_{3,3} \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{I}=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \end{Bmatrix} \qquad (7.46)$$

where $[Q_{01}]$ is the “final form” of the “Modified Transfer Matrix”.

The “final form” of the “Modified Transfer Matrix” can be reduced to (9x9) by the substitution of the “Boundary Conditions” in the y-direction at ξ_I=0, ξ_{II}=1 for the upper plate, ξ_{III}=0, ξ_I=1 for the lower left plate and. ξ_{II}=0, ξ_{IV}=1 for the lower right plate. After inserting the “Boundary Condition” the following matrix equation which implicitly includes the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$ can be obtained,

$[\mathbf{C_0}(\omega_{mn})]_{\mathbf{(II)}}\{\mathbf{Y_0}\}_{\mathbf{(II)}} = \{\mathbf{0}\}$ $\Rightarrow$ Determinant of “Coefficient Matrix”=

$$\left| [\mathbf{C_0}]_{\mathbf{(II)}} \right| = 0 \qquad (7.47)$$

where subscript “II” indicates the “Main PROBLEM II”, and the “Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(II)}}$” is obtained by eliminating the rows corresponding to nonzero “state variables” at the “final end point’’ and the columns corresponding to zero state variables at the "initial end point'' in (7.46). Here, $\{\mathbf{Y_0}\}$ is a vector whose elements are the nonzero components of $\left\{ \boldsymbol{Y}^{(1)}_{\xi_I=0}, \boldsymbol{Y}^{(2)}_{\xi_{III}=0}, \boldsymbol{Y}^{(3)}_{\xi_{II}=0} \right\}^T$.

For a non-trivial solution, the determinant of the "Coefficient Matrix $\left[\mathbf{C_0}\right]_{(\mathbf{II})}$" must be equal to zero. Again this procedure should be repeated for a given specific set of "Boundary Conditions" with given the material and geometric characteristics for the problem under consideration. The roots of the determinant of the "Coefficient Matrix $\left[\mathbf{C_0}\right]_{(\mathbf{II})}$" correspond to the nondimensionalized natural frequencies, $\overline{\omega}_{mn}$ of the entire composite bonded plate system of the "Main PROBLEM II". The roots of the above determinant should be numerically obtained. One can easily find the corresponding mode shapes, after the natural frequencies (and eigenvectors) are obtained for a particular set of "Boundary Conditions" in the y-directions.

7.4 Method of Solution for Special case of "Main PROBLEM II a" and "Main PROBLEM II b"

In this section, the application of the present solution technique will be explained in detail for the Special Case of the "Main PROBLEM IIa" and the "Main PROBLEM IIb".

"Modified Transfer Matrix" for Part I, Part II, Part III, Part IV and Part V can be obtained similarly for "Main PROBLEM II" as;

For the "Overlap Region" Part I,

$$\begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_I=1} \\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_I=1} \end{Bmatrix} = \begin{bmatrix} \widetilde{\mathcal{U}}_{1,1} & \widetilde{\mathcal{U}}_{1,2} \\ \widetilde{\mathcal{U}}_{2,1} & \widetilde{\mathcal{U}}_{2,2} \end{bmatrix}_{01} \begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_I=0} \\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_I=0} \end{Bmatrix} \qquad \text{(in Part I)} \qquad (7.48)$$

For the "Overlap Region" Part II,

$$\begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=1} \\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} \widetilde{\mathcal{U}}'_{1,1} & \widetilde{\mathcal{U}}'_{1,2} \\ \widetilde{\mathcal{U}}'_{2,1} & \widetilde{\mathcal{U}}'_{2,2} \end{bmatrix}_{01} \begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0} \\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=0} \end{Bmatrix} \qquad \text{(in Part II)} \qquad (7.49)$$

For Part III Region,

$$\left\{ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1} \right\} = \left[\tilde{\boldsymbol{\mathcal{V}}}_{\mathbf{1,1}} \right]_{01} \left\{ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=0} \right\} \quad \text{(in Part III)} \qquad (7.50)$$

For Part IV Region,

$$\left\{ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=1} \right\} = \left[\tilde{\boldsymbol{\mathcal{W}}}_{\mathbf{1,1}} \right]_{01} \left\{ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=0} \right\} \quad \text{(in Part IV)} \qquad (7.51)$$

For Part V Region,

$$\left\{ \overline{\boldsymbol{Y}}^{(1)}_{\xi_{V}=1} \right\} = \left[\tilde{\boldsymbol{\mathcal{Z}}}_{\mathbf{1,1}} \right]_{01} \left\{ \overline{\boldsymbol{Y}}^{(1)}_{\xi_{V}=0} \right\} \quad \text{(in Part V)} \qquad (7.52)$$

where subscript (01) in the above expressions means that the "final form" of the "Modified Transfer Matrix" transferring the above quantities from the "initial end point 0", to the "final end point 1".

Now natural frequencies of the entire system will be determined by using the "Boundary Conditions" and the "Continuity Conditions" in the y-direction.

The ''Continuity Conditions'' between the Part I and Part V,

$$\left\{ \overline{\boldsymbol{Y}}^{(1)}_{\xi_{I}=1} \right\} = \left\{ \overline{\boldsymbol{Y}}^{(1)}_{\xi_{V}=0} \right\} \qquad (7.53)$$

The ''Continuity Conditions'' between the Part I and Part III,

$$\left\{ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1} \right\} = \left\{ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{I}=0} \right\} \qquad (7.54)$$

The ''Continuity Conditions'' between the Part II and Part IV,

$$\left\{ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1} \right\} = \left\{ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=0} \right\} \qquad (7.55)$$

The ''Continuity Conditions'' between the Part II and Part V,

$$\left\{ \overline{\boldsymbol{Y}}^{(1)}_{\xi_{V}=1} \right\} = \left\{ \overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0} \right\} \qquad (7.56)$$

Then, from Equations (7.48) and (7.56), final form of the “Modified Transfer Matrix” is obtained as,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_{I}=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \widetilde{\mathcal{W}}_{1,1} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{U}}'_{1,1} & 0 & \widetilde{\mathcal{U}}'_{1,2} \\ 0 & I & 0 \\ \widetilde{\mathcal{U}}'_{2,1} & 0 & \widetilde{\mathcal{U}}'_{2,2} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{Z}}_{1,1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$\begin{bmatrix} \widetilde{\mathcal{U}}_{1,1} & \widetilde{\mathcal{U}}_{1,2} & 0 \\ \widetilde{\mathcal{U}}_{2,1} & \widetilde{\mathcal{U}}_{2,2} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \widetilde{\mathcal{V}}_{1,1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{I}=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \end{Bmatrix} \tag{7.57}$$

or, simply rearranging,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_{I}=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{1,1} & \mathcal{Q}_{1,2} & \mathcal{Q}_{1,3} \\ \mathcal{Q}_{2,1} & \mathcal{Q}_{2,2} & \mathcal{Q}_{2,3} \\ \mathcal{Q}_{3,1} & \mathcal{Q}_{3,2} & \mathcal{Q}_{3,3} \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{I}=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \end{Bmatrix} \tag{7.58}$$

where $[\boldsymbol{Q_{01}}]$ is the “final form” of the “Modified Transfer Matrix”.

When the boundary condition applied the following matrix equation which implicitly includes the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$ can be obtained,

$[\mathbf{C_0}(\omega_{mn})]_{\mathbf{(II)}}\{\mathbf{Y_0}\}_{\mathbf{(II)}} = \{\mathbf{0}\}$ $\Rightarrow$ Determinant of the “Coefficient Matrix”=

$$\left| [\mathbf{C_0}]_{\mathbf{(II)}} \right| = 0 \tag{7.59}$$

where the subscript (II) indicates the “Special Case of Main PROBLEM II”.

The “Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(II)}}$” is obtained by eliminating the rows corresponding to nonzero state variable at the “final end point’’ and the columns corresponding to zero state variables at the “initial end point’’. Here, $\{\mathbf{Y_0}\}$ is a vector whose elements are the nonzero components of $\left\{ \boldsymbol{Y}^{(1)}_{\xi_I=0}, \boldsymbol{Y}^{(2)}_{\xi_{III}=0}, \boldsymbol{Y}^{(3)}_{\xi_{II}=0} \right\}^{T}$ similarly as "Main PROBLEM II".

For non-trivial solution, determinant of the "Coefficient Matrix $[\mathbf{C_0}]_{(\mathbf{II})}$" shall be equal to zero which correspond to the nondimensionalized natural frequencies, $\overline{\omega}_{mn}$ of the entire composite plate system of the Special Case of "Main PROBLEM II".

7.5 Method of Solution for "Main PROBLEM III a" and "Main PROBLEM III b"

In this section, the application of the present solution technique will be explained in detail for the "Main PROBLEM III" without making any distinction between the "Main PROBLEM III.a" and the "Main PROBLEM III.b".

The first step in the solution procedure is to write, the sets of the "Governing System of First Order Ordinary Differential Equations" in a "state vector" form in Part I, Part II, Part III and Part IV regions" as was done in (6.24, 6.25, 6.27, 6.29),

"Governing System of First Order Ordinary Differential Equations"

$$\frac{d}{d\xi_I}\begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{mn} \\ \overline{\boldsymbol{Y}}^{(2)}_{mn} \\ \overline{\boldsymbol{Y}}^{(4)}_{mn} \end{Bmatrix} = \left[\overline{\mathcal{C}}\right]\begin{Bmatrix} \overline{\boldsymbol{Y}}^{(1)}_{mn} \\ \overline{\boldsymbol{Y}}^{(2)}_{mn} \\ \overline{\boldsymbol{Y}}^{(4)}_{mn} \end{Bmatrix}, \qquad (0<\xi_I<1) \quad \text{(in Part I)}$$

$$\frac{d}{d\xi_{II}}\begin{Bmatrix} \overline{Y}^{(1)}_{mn} \\ \overline{Y}^{(3)}_{mn} \\ \overline{Y}^{(4)}_{mn} \end{Bmatrix} = \left[\overline{\mathcal{C}'}\right]\begin{Bmatrix} \overline{Y}^{(1)}_{mn} \\ \overline{Y}^{(3)}_{mn} \\ \overline{Y}^{(4)}_{mn} \end{Bmatrix}, \qquad (0<\xi_{II}<1) \quad \text{(in Part II)} \quad (7.60\ \text{a,b,c,d})$$

$$\frac{d}{d\xi_{III}}\left\{\overline{\boldsymbol{Y}}^{(2)}_{mn}\right\} = \left[\overline{\mathcal{D}}\right]\left\{\overline{\boldsymbol{Y}}^{(2)}_{mn}\right\}, \qquad (0<\xi_{III}<1) \quad \text{(in Part III)}$$

$$\frac{d}{d\xi_{IV}}\left\{\overline{\boldsymbol{Y}}^{(3)}_{mn}\right\} = \left[\overline{\mathcal{E}}\right]\left\{\overline{\boldsymbol{Y}}^{(3)}_{mn}\right\}, \qquad (0<\xi_{IV}<1) \quad \text{(in Part IV)}$$

with the "Appropriate Boundary Conditions" and the "Continuity Conditions" for the particular problem under consideration.

where ξ_I, ξ_{II}, ξ_{III} and ξ_{IV} are defined as y_I/ℓ_I, y_{II}/ℓ_{II}, y_{III}/ℓ_{III} and y_{IV}/ℓ_{IV}, respectively. The "Coefficient Matrices [$\overline{\mathcal{C}}$] and [$\overline{\mathcal{C}}'$]" are of dimension (18x18) and [$\overline{\mathcal{D}}$], $[\mathcal{E}]$ are of dimension (6x6). [$\overline{\mathcal{C}}$] is the "Coefficient Matrix" for Part I of the "Bonded Region" and [$\overline{\mathcal{C}}'$] is the "Coefficient Matrix" for Part II of the "Bonded Region". They include the dimensionless geometric and material characteristics of the adherends as well as the unknown dimensionless natural frequencies $\overline{\omega}_{mn}$. The "Column Matrix $\{\overline{\boldsymbol{Y}}_{\boldsymbol{m}}^{(j)}\}$ (j=1,2,3,4) are the "state vectors" including the "dimensionless fundamental state variables" of the adherends (or rather of the problems under study),

$$\{\overline{\boldsymbol{Y}}_{\boldsymbol{m}}^{(j)}\} = \{\overline{\Psi}_{mx}^{(j)}, \overline{\Psi}_{my}^{(j)}, \overline{W}_{m}^{(j)}; \overline{M}_{myx}^{(j)}, \overline{M}_{my}^{(j)}, \overline{Q}_{my}^{(j)}\}^{T} \qquad \text{(j=1,2,3,,4)} \qquad (7.48.e)$$

The next step involves the discretization of the "fundamental dependent variables" of the problem under investigation and the "Coefficient Matrices" with respect to the independent space variables ξ_I, ξ_{II}, ξ_{III}, ξ_{IV} respectively.

The discretization is performed by dividing the Part I, Part II, Part III and Part IV regions into sufficient number (n_1 for Part I, n_2 for Part II, n_3 for Part III and n_4 for Part IV) of segments or stations along ξ_I, ξ_{II}, ξ_{III} and ξ_{IV} directions respectively and by pre-multiplying the discrete version of "Coefficient Matrices" by the appropriate "Global Integrating Matrix $[\mathcal{L}]$" which includes integrating sub-matrices [$\boldsymbol{L}$]. For convenience, "mn" subscript will be dropped from the equations. Then,

<u>For Part I region (Three-Layer Composite Plates),</u>

$$\begin{Bmatrix} \dot{\overline{\boldsymbol{Y}}}^{(1)} \\ \dot{\overline{\boldsymbol{Y}}}^{(2)} \\ \dot{\overline{\boldsymbol{Y}}}^{(4)} \end{Bmatrix} - \begin{Bmatrix} \dot{\overline{\boldsymbol{Y}}}_{1}^{(1)} \\ \dot{\overline{\boldsymbol{Y}}}_{1}^{(2)} \\ \dot{\overline{\boldsymbol{Y}}}_{1}^{(4)} \end{Bmatrix} = [\mathcal{L}_{I}] \left[\dot{\overline{\mathcal{C}}} \right] \begin{Bmatrix} \dot{\overline{\boldsymbol{Y}}}^{(1)} \\ \dot{\overline{\boldsymbol{Y}}}^{(2)} \\ \dot{\overline{\boldsymbol{Y}}}^{(4)} \end{Bmatrix}, \qquad \text{(in Part I)} \qquad (7.61)$$

For Part II region (Three-Layer Composite Plates),

$$\begin{Bmatrix} \dot{\overline{Y}}^{(1)} \\ \dot{\overline{Y}}^{(3)} \\ \dot{\overline{Y}}^{(4)} \end{Bmatrix} - \begin{Bmatrix} \dot{\overline{Y}}_I^{(1)} \\ \dot{\overline{Y}}_I^{(3)} \\ \dot{\overline{Y}}_I^{(4)} \end{Bmatrix} = [\mathcal{L}_{\mathrm{II}}] \left[\dot{\mathcal{C}}'\right] \begin{Bmatrix} \dot{\overline{Y}}^{(1)} \\ \dot{\overline{Y}}^{(3)} \\ \dot{\overline{Y}}^{4} \end{Bmatrix}, \qquad \text{(in Part II)} \qquad (7.62)$$

For Part III region (Single Layer Orthotropic or Isotropic Base Plate),

$$\left\{\dot{\overline{Y}}^{(2)}\right\} - \left\{\dot{\overline{Y}}_I^{(2)}\right\} = [\mathcal{L}_{\mathrm{III}}] \left[\dot{\mathcal{D}}\right] \left\{\dot{\overline{Y}}^{(2)}\right\}, \qquad \text{(in Part III)} \qquad (7.63)$$

For Part IV region (Single Layer Orthotropic or Isotropic Base Plate),

$$\left\{\dot{\overline{Y}}^{(3)}\right\} - \left\{\dot{\overline{Y}}_I^{(3)}\right\} = [\mathcal{L}_{\mathrm{IV}}] \left[\dot{\mathcal{E}}\right] \left\{\dot{\overline{Y}}^{(3)}\right\}, \qquad \text{(in Part IV)} \qquad (7.64)$$

where the subscript in $[\mathcal{L}]$ indicates the corresponding part (Part I, Part II, Part III and Part IV) $\left\{\dot{\overline{Y}}^{(j)}\right\}$ is discrete version of "state vector" $\left\{\overline{Y}_{mn}^{(j)}\right\}$ and $\left[\dot{\mathcal{C}}\right], \left[\dot{\mathcal{C}}'\right], \left[\dot{\mathcal{D}}\right], \left[\dot{\mathcal{E}}\right]$ ("dot" or "·" indicating the discretization along the ξ-direction) are the discrete versions of the "Coefficient Matrices" $[\overline{\mathcal{C}}], [\overline{\mathcal{C}}'], [\overline{\mathcal{D}}], [\overline{\mathcal{E}}]$. The "state vector" $\left\{\overline{Y}_I^{(j)}\right\}$ is evaluated at the "initial end point", (i.e. at ξ_I=0, ξ_{II}=0, ξ_{III}=0 and ξ_{IV}=0.for Part I, Part II, Part III and Part IV, respectively) The superscripts (1,2,3,4) denote the adherends.

One can write the "state vector" at a "general station" as $\left\{\dot{\overline{Y}}^{(j)}\right\}$, with respect to the "state vector" at the "initial end point" with the subscript "1" as $\left\{\overline{Y}_I^{(j)}\right\}$ and discretized version of the "Coefficient Matrix" as, “State Vectors” evaluated at the “general station” and “State Vectors” evaluated at the “initial end point”

$$\left\{\dot{\bar{\boldsymbol{Y}}}^{(1)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(1)}\\ \bar{\Psi}_{x2}^{(1)}\\ .\\ .\\ \bar{\Psi}_{xn_1}^{(1)}\\ \bar{\Psi}_{y1}^{(1)}\\ \bar{\Psi}_{y2}^{(1)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(1)}\\ .\\ .\\ \bar{Q}_{yn_1}^{(1)}\end{Bmatrix},\left\{\dot{\bar{\boldsymbol{Y}}}^{(2)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(2)}\\ \bar{\Psi}_{x2}^{(2)}\\ .\\ .\\ \bar{\Psi}_{xn_2}^{(2)}\\ \bar{\Psi}_{y1}^{(2)}\\ \bar{\Psi}_{y2}^{(2)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(2)}\\ .\\ .\\ \bar{Q}_{yn_2}^{(2)}\end{Bmatrix},\left\{\dot{\bar{\boldsymbol{Y}}}^{(3)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(3)}\\ \bar{\Psi}_{x2}^{(3)}\\ .\\ .\\ \bar{\Psi}_{xn_3}^{(3)}\\ \bar{\Psi}_{y1}^{(3)}\\ \bar{\Psi}_{y2}^{(3)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(3)}\\ .\\ .\\ \bar{Q}_{yn_3}^{(3)}\end{Bmatrix},\left\{\dot{\bar{\boldsymbol{Y}}}^{(4)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(4)}\\ \bar{\Psi}_{x2}^{(4)}\\ .\\ .\\ \bar{\Psi}_{xn_4}^{(4)}\\ \bar{\Psi}_{y1}^{(4)}\\ \bar{\Psi}_{y2}^{(4)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(4)}\\ .\\ .\\ \bar{Q}_{yn_4}^{(4)}\end{Bmatrix},$$

$$\left\{\dot{\bar{\boldsymbol{Y}}}_1^{(1)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(1)}\\ \bar{\Psi}_{x1}^{(1)}\\ .\\ .\\ \bar{\Psi}_{x1}^{(1)}\\ \bar{\Psi}_{y1}^{(1)}\\ \bar{\Psi}_{y1}^{(1)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(1)}\\ .\\ .\\ \bar{Q}_{y1}^{(1)}\end{Bmatrix},\left\{\dot{\bar{\boldsymbol{Y}}}_1^{(2)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(2)}\\ \bar{\Psi}_{x1}^{(2)}\\ .\\ .\\ \bar{\Psi}_{x1}^{(2)}\\ \bar{\Psi}_{y1}^{(2)}\\ \bar{\Psi}_{y1}^{(2)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(2)}\\ .\\ .\\ \bar{Q}_{y1}^{(2)}\end{Bmatrix},\left\{\dot{\bar{\boldsymbol{Y}}}_1^{(3)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(3)}\\ \bar{\Psi}_{x1}^{(3)}\\ .\\ .\\ \bar{\Psi}_{x1}^{(3)}\\ \bar{\Psi}_{y1}^{(3)}\\ \bar{\Psi}_{y1}^{(3)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(3)}\\ .\\ .\\ \bar{Q}_{y1}^{(3)}\end{Bmatrix},\left\{\dot{\bar{\boldsymbol{Y}}}_1^{(4)}\right\}=\begin{Bmatrix}\bar{\Psi}_{x1}^{(4)}\\ \bar{\Psi}_{x1}^{(4)}\\ .\\ .\\ \bar{\Psi}_{x1_4}^{(4)}\\ \bar{\Psi}_{y1}^{(4)}\\ \bar{\Psi}_{y1}^{(4)}\\ .\\ .\\ .\\ \bar{Q}_{y1}^{(4)}\\ .\\ .\\ \bar{Q}_{y1}^{(4)}\end{Bmatrix}, \qquad (7.65)$$

and the “Coefficient Matrices” $\left[\dot{\mathcal{C}}\right], \left[\dot{\mathcal{C}}'\right], \left[\dot{\mathcal{D}}\right], \left[\dot{\mathcal{E}}\right]$ are,

$$\left[\dot{\mathcal{C}}\right] = \begin{bmatrix} \left[\dot{\mathcal{C}}_{1,1}\right]_{nxn} & \left[\dot{\mathcal{C}}_{1,2}\right]_{nxn} & \cdot\ \cdot & \left[\dot{\mathcal{C}}_{1,18}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{C}}_{2,18}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \left[\dot{\mathcal{C}}_{18,1}\right]_{nxn} & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{C}}_{18,18}\right]_{nxn} \end{bmatrix} \tag{7.66}$$

$$\left[\dot{\mathcal{C}}'\right] = \begin{bmatrix} \left[\dot{\mathcal{C}}'_{1,1}\right]_{nxn} & \left[\dot{\mathcal{C}}'_{1,2}\right]_{nxn} & \cdot\ \cdot & \left[\dot{\mathcal{C}}'_{1,18}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{C}}'_{2,18}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \left[\dot{\mathcal{C}}'_{18,1}\right]_{nxn} & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{C}}'_{18,18}\right]_{nxn} \end{bmatrix} \tag{7.67}$$

$$\left[\dot{\mathcal{D}}\right] = \begin{bmatrix} \left[\dot{\mathcal{D}}_{1,1}\right]_{nxn} & \left[\dot{\mathcal{D}}_{1,2}\right]_{nxn} & \cdot\ \cdot & \left[\dot{\mathcal{D}}_{1,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{D}}_{2,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \left[\dot{\mathcal{D}}_{6,1}\right]_{nxn} & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{D}}_{6,6}\right]_{nxn} \end{bmatrix} \tag{7.68}$$

$$\left[\dot{\mathcal{E}}\right] = \begin{bmatrix} \left[\dot{\mathcal{E}}_{1,1}\right]_{nxn} & \left[\dot{\mathcal{E}}_{1,2}\right]_{nxn} & \cdot\ \cdot & \left[\dot{\mathcal{E}}_{1,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{E}}_{2,6}\right]_{nxn} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \left[\dot{\mathcal{E}}_{6,1}\right]_{nxn} & \cdot & \cdot\ \cdot & \left[\dot{\mathcal{E}}_{6,6}\right]_{nxn} \end{bmatrix} \tag{7.69}$$

The second subscripts in (7.66), (7.67), (7.68) and (7.69) indicate the discretization point or the "station" with which they are associated and $[\mathcal{C}_{i,j}]$, $[\mathcal{C}'_{i,j}]$,

[$\mathcal{D}_{i,j}$] and [$\mathcal{E}_{i,j}$] are the diagonal "Sub-Matrices" composed of the elements of the related "Coefficient Matrix".

The equations in more compact form are,

For the "Bonded Region" or Part I region,

$$\begin{Bmatrix} \dot{\bar{Y}}^{(1)} \\ \dot{\bar{Y}}^{(2)} \\ \dot{\bar{Y}}^{(4)} \end{Bmatrix} = \left[\dot{\mathcal{U}} \right] \begin{Bmatrix} \dot{\bar{Y}}_1^{(1)} \\ \dot{\bar{Y}}_1^{(2)} \\ \dot{\bar{Y}}_1^{(4)} \end{Bmatrix} \quad \text{(in Part I)} \qquad (7.70)$$

where $[\dot{\mathcal{U}}]$ is the discretized form of "Modified Transfer Matrix" for Part I region,

$$\left[\dot{\mathcal{U}} \right] = \left([I] - [\mathcal{L}_{I}] \left[\dot{\mathcal{C}} \right] \right)^{-1} \qquad (7.71)$$

For the "Bonded Region" or Part II region,

$$\begin{Bmatrix} \dot{\bar{Y}}^{(1)} \\ \dot{\bar{Y}}^{(3)} \\ \dot{\bar{Y}}^{(4)} \end{Bmatrix} = \left[\dot{\mathcal{U}}' \right] \begin{Bmatrix} \dot{\bar{Y}}_1^{(1)} \\ \dot{\bar{Y}}_1^{(3)} \\ \dot{\bar{Y}}_1^{(4)} \end{Bmatrix} \quad \text{(in Part II)} \qquad (7.72)$$

where $\left[\dot{\mathcal{U}}' \right]$ is the discretized form of "Modified Transfer Matrix" for Part II region,

$$\left[\dot{\mathcal{U}}' \right] = \left([I] - [\mathcal{L}_{II}] \left[\dot{\mathcal{C}}' \right] \right)^{-1} \qquad (7.73)$$

For Part III region,

$$\left\{ \dot{\bar{Y}}^{(2)} \right\} = \left[\dot{\mathcal{V}} \right] \left\{ \dot{\bar{Y}}_1^{(2)} \right\} \quad \text{(in Part III)} \qquad (7.74)$$

where $\left[\dot{\mathcal{V}}\right]$ is the discretized version of "Modified Transfer Matrix" for Part III region,

$$\left[\dot{\mathcal{V}}\right] = \left([I] - [\mathcal{L}_{\mathbf{III}}]\left[\dot{\mathcal{D}}\right]\right)^{-1} \tag{7.75}$$

<u>For Part IV region,</u>

$$\left\{\dot{\bar{Y}}^{(3)}\right\} = [\dot{\mathcal{W}}]\left\{\dot{\bar{Y}}_1^{(3)}\right\} \quad \text{(in Part IV)} \tag{7.76}$$

where $\left[\dot{\mathcal{W}}\right]$ is the discretized version of "Modified Transfer Matrix" for Part IV region,

$$[\dot{\mathcal{W}}] = \left([I] - [\mathcal{L}_{\mathbf{IV}}]\left[\dot{\mathcal{E}}\right]\right)^{-1} \tag{7.77}$$

The matrices, $[\mathcal{L}_{\mathbf{I}}]$, $[\mathcal{L}_{\mathbf{II}}]$, $[\mathcal{L}_{\mathbf{III}}]$ and $[\mathcal{L}_{\mathbf{IV}}]$ are the "Global Integrating Matrices" for Part I, Part II, Part III and Part IV, respectively. They include the "Integrating Sub-Matrices $[\boldsymbol{L}]$". Rewriting the equations (7.58-7.65) in "open form", For the “Bonded Region” or Part I region (Three-Layer Composite Plate),

$$
\begin{Bmatrix}
\overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x2}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{xn_1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \overline{\Psi}_{y2}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{x1}^{(2)} \\ \cdot \\ \cdot \\ \overline{Q}_{y1}^{(4)} \\ \overline{Q}_{y2}^{(4)} \\ \cdot \\ \cdot \\ \overline{Q}_{yn_1}^{(4)}
\end{Bmatrix}
=
\begin{bmatrix}
\dot{\mathcal{U}}_{1,1} & \dot{\mathcal{U}}_{1,2} & \dot{\mathcal{U}}_{1,3} & \cdots & \dot{\mathcal{U}}_{1,18} \\
\dot{\mathcal{U}}_{2,1} & \dot{\mathcal{U}}_{2,2} & \cdot & \cdots & \cdot \\
\dot{\mathcal{U}}_{3,1} & \cdot & \cdot & \cdots & \cdot \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\dot{\mathcal{U}}_{1,18} & \cdot & \cdot & \cdots & \dot{\mathcal{U}}_{18,18}
\end{bmatrix}
\begin{Bmatrix}
\overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x1}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{x1}^{(2)} \\ \cdot \\ \cdot \\ \overline{Q}_{y1}^{(4)} \\ \overline{Q}_{y1}^{(4)} \\ \cdot \\ \cdot \\ \overline{Q}_{y1}^{(4)}
\end{Bmatrix}
\quad (7.78)
$$

For the “Bonded Region” or Part II region (Three-Layer Composite Plate),

$$
\begin{Bmatrix}
\overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x2}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{xn_1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \overline{\Psi}_{y2}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{x1}^{(3)} \\ \cdot \\ \cdot \\ \overline{Q}_{y1}^{(4)} \\ \overline{Q}_{y2}^{(4)} \\ \cdot \\ \cdot \\ \overline{Q}_{yn_1}^{(4)}
\end{Bmatrix}
=
\begin{bmatrix}
\dot{\mathcal{U}}'_{1,1} & \dot{\mathcal{U}}'_{1,2} & \dot{\mathcal{U}}'_{1,3} & \cdots & \dot{\mathcal{U}}''_{1,18} \\
\dot{\mathcal{U}}'_{2,1} & \dot{\mathcal{U}}'_{2,2} & \cdot & \cdots & \cdot \\
\dot{\mathcal{U}}'_{3,1} & \cdot & \cdot & \cdots & \cdot \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\dot{\mathcal{U}}'_{1,18} & \cdot & \cdot & \cdots & \dot{\mathcal{U}}'_{18,18}
\end{bmatrix}
\begin{Bmatrix}
\overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{x1}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{x1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \overline{\Psi}_{y1}^{(1)} \\ \cdot \\ \cdot \\ \overline{\Psi}_{x1}^{(3)} \\ \cdot \\ \cdot \\ \overline{Q}_{y1}^{(4)} \\ \overline{Q}_{y1}^{(4)} \\ \cdot \\ \cdot \\ \overline{Q}_{y1}^{(4)}
\end{Bmatrix}
\quad (7.79)
$$

For Part III region (Single-Layer Base Plate),

$$\begin{Bmatrix} \overline{\Psi}_{x1}^{(2)} \\ \overline{\Psi}_{x2}^{(2)} \\ . \\ . \\ \overline{\Psi}_{xn_3}^{(2)} \\ \overline{\Psi}_{y1}^{(2)} \\ \overline{\Psi}_{y2}^{(2)} \\ . \\ . \\ . \\ \overline{Q}_{y1}^{(2)} \\ . \\ . \\ \overline{Q}_{yn_3}^{(2)} \end{Bmatrix} = \begin{bmatrix} \dot{\mathcal{V}}_{1,1} & . & . & . & \dot{\mathcal{V}}_{1,6} \\ \dot{\mathcal{V}}_{2,1} & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \dot{\mathcal{V}}_{6,1} & . & . & . & \dot{\mathcal{V}}_{6,6} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}_{x1}^{(2)} \\ \overline{\Psi}_{x1}^{(2)} \\ . \\ . \\ \overline{\Psi}_{x1}^{(2)} \\ \overline{\Psi}_{y1}^{(2)} \\ \overline{\Psi}_{y1}^{(2)} \\ . \\ . \\ . \\ \overline{Q}_{y1}^{(2)} \\ . \\ . \\ \overline{Q}_{y1}^{(2)} \end{Bmatrix} = \qquad (7.80)$$

For Part IV region (Single-Layer Base Plate),

$$\begin{Bmatrix} \overline{\Psi}_{x1}^{(3)} \\ \overline{\Psi}_{x2}^{(3)} \\ . \\ . \\ \overline{\Psi}_{xn_4}^{(3)} \\ \overline{\Psi}_{y1}^{(3)} \\ \overline{\Psi}_{y2}^{(3)} \\ . \\ . \\ . \\ \overline{Q}_{y1}^{(3)} \\ . \\ . \\ \overline{Q}_{yn_4}^{(3)} \end{Bmatrix} = \begin{bmatrix} \dot{\mathcal{W}}_{1,1} & . & . & . & \dot{\mathcal{W}}_{1,6} \\ \dot{\mathcal{W}}_{2,1} & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \dot{\mathcal{W}}_{6,1} & . & . & . & \dot{\mathcal{W}}_{6,6} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}_{x1}^{(3)} \\ \overline{\Psi}_{x1}^{(2)} \\ . \\ . \\ \overline{\Psi}_{x1}^{(3)} \\ \overline{\Psi}_{y1}^{(3)} \\ \overline{\Psi}_{y1}^{(3)} \\ . \\ . \\ . \\ \overline{Q}_{y1}^{(3)} \\ . \\ . \\ \overline{Q}_{y1}^{(3)} \end{Bmatrix} \qquad (7.81)$$

The matrices $\left[\dot{\mathcal{U}}\right]$ and $\left[\dot{\mathcal{U}}'\right]$ are composed of (18×18) square blocks of dimension n_1 and dimension n_2 respectively, and each individual block represents a relation between a “state variable” at any “general station” and the “state variable” at the “initial end point”. The subscript n_1 and n_2 are the number of discretization points in Part I and Part II.

The matrix $\left[\dot{v}\right]$ is composed of (6×6) blocks of dimension n_3 and each individual block represents a relation between a “state variable” at any “general station and “state variable” at the “initial end point. The subscript n_3 is the number of discretization points in Part III.

The matrix $\left[\dot{w}\right]$ is composed of (6×6) blocks of dimension n_4 and each individual block represents a relation between a “state variable” at any “general station and “state variable” at the “initial end point. The subscript n_4 is the number of discretization points in Part IV.

Then, from (7.78), (7.79), (7.80) and (7.81), one can obtain the relation between the “state vector” at the “initial end points ξ_I=0, ξ_{II}=0, ξ_{III}=0 and ξ_{IV}=0” and the “final end points ξ_I=1, ξ_{II}=1, ξ_{III}=1 and ξ_{IV}=1” and along Part I, Part II, Part III and Part IV regions, respectively.

Then, by summing up the elements related with the each group of “state vectors” at “initial end point”, on the integer multiples of n^{th} row of the discretized version of “Modified Transfer Matrix one can obtain the following, For the “Bonded Region” or Part I region (Three-Layer Composite Plate),

$$\begin{Bmatrix} \overline{\Psi}^{(1)}_{xn_1} \\ \overline{\Psi}^{(1)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{yn_1} \\ \overline{\Psi}^{(2)}_{xn_1} \\ \overline{\Psi}^{(2)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_1} \\ \overline{\Psi}^{(4)}_{xn_1} \\ \overline{\Psi}^{(4)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{yn_1} \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \left(\dot{\mathcal{Z}}_{1,1}\right)_{n,i} & \cdots & \sum_{i=17n+1}^{18n} \left(\dot{\mathcal{Z}}_{1,18}\right)_{n,i} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} \left(\dot{\mathcal{Z}}_{18,1}\right)_{n,i} & \cdots & \sum_{i=17n+1}^{18n} \left(\dot{\mathcal{Z}}_{18,18}\right)_{n,i} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{y1} \\ \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \\ \overline{\Psi}^{(4)}_{x1} \\ \overline{\Psi}^{(4)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{y1} \end{Bmatrix}$$

(7.82)

or in more compact from,

$$\begin{Bmatrix} \overline{\Psi}^{(1)}_{xn_1} \\ \overline{\Psi}^{(1)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{yn_1} \end{Bmatrix} = \left[\tilde{\mathcal{Z}}\right] \begin{Bmatrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{y1} \end{Bmatrix} \qquad (7.83)$$

For the “Bonded Region” or Part II region (Three-Layer Composite Plate),

$$\left\{\begin{matrix} \overline{\Psi}^{(1)}_{xn_1} \\ \overline{\Psi}^{(1)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{yn_1} \\ \overline{\Psi}^{(2)}_{xn_1} \\ \overline{\Psi}^{(2)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_1} \\ \overline{\Psi}^{(4)}_{xn_1} \\ \overline{\Psi}^{(4)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{yn_1} \end{matrix}\right\} = \begin{bmatrix} \sum_{i=1}^{n}\left(\dot{\mathcal{U}}'_{1,1}\right)_{n,i} & \cdots & \sum_{i=17n+1}^{18n}\left(\dot{\mathcal{U}}'_{1,18}\right)_{n,i} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}\left(\dot{\mathcal{U}}'_{18,1}\right)_{n,i} & \cdots & \sum_{i=17n+1}^{18n}\left(\dot{\mathcal{U}}'_{18,18}\right)_{n,i} \end{bmatrix} \left\{\begin{matrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(1)}_{y1} \\ \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \\ \overline{\Psi}^{(4)}_{x1} \\ \overline{\Psi}^{(4)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{y1} \end{matrix}\right\} \tag{7.84}$$

or in more compact from,

$$\left\{\begin{matrix} \overline{\Psi}^{(1)}_{xn_1} \\ \overline{\Psi}^{(1)}_{yn_1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{yn_1} \end{matrix}\right\} = \left[\tilde{\mathcal{U}}'\right] \left\{\begin{matrix} \overline{\Psi}^{(1)}_{x1} \\ \overline{\Psi}^{(1)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(4)}_{y1} \end{matrix}\right\} \tag{7.85}$$

For Part III region (Single-Layer Base Plate),

$$\begin{Bmatrix} \overline{\Psi}^{(3)}_{xn_3} \\ \overline{\Psi}^{(3)}_{yn_3} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(3)}_{yn_3} \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\left(\dot{\mathcal{V}}_{1,1}\right)_{n,i} & \sum_{i=n+1}^{2n}\left(\dot{\mathcal{V}}_{1,2}\right)_{n,i} & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \sum_{i=5n+1}^{6n}\left(\dot{\mathcal{V}}_{1,6}\right)_{n,i} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(3)}_{x1} \\ \overline{\Psi}^{(3)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(3)}_{y1} \end{Bmatrix} \tag{7.86}$$

or in more compact from,

$$\begin{Bmatrix} \overline{\Psi}^{(2)}_{xn_3} \\ \overline{\Psi}^{(2)}_{yn_3} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{yn_3} \end{Bmatrix} = \left[\tilde{\mathcal{V}}\right] \begin{Bmatrix} \overline{\Psi}^{(2)}_{x1} \\ \overline{\Psi}^{(2)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(2)}_{y1} \end{Bmatrix} \tag{7.87}$$

For Part IV region (Single-Layer Base Plate),

$$\begin{Bmatrix} \overline{\Psi}^{(3)}_{xn_4} \\ \overline{\Psi}^{(3)}_{yn_4} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(3)}_{yn_4} \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\left(\dot{\mathcal{W}}_{1,1}\right)_{n,i} & \sum_{i=n+1}^{2n}\left(\dot{\mathcal{W}}_{1,2}\right)_{n,i} & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \sum_{i=5n+1}^{6n}\left(\dot{\mathcal{W}}_{1,6}\right)_{n,i} \end{bmatrix} \begin{Bmatrix} \overline{\Psi}^{(3)}_{x1} \\ \overline{\Psi}^{(3)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(3)}_{y1} \end{Bmatrix} \tag{7.88}$$

or in more compact from,

$$\begin{Bmatrix} \overline{\Psi}^{(3)}_{xn_4} \\ \overline{\Psi}^{(3)}_{yn_4} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(3)}_{yn_4} \end{Bmatrix} = \left[\tilde{\mathcal{W}}\right] \begin{Bmatrix} \overline{\Psi}^{(3)}_{x1} \\ \overline{\Psi}^{(3)}_{y1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Q}^{(3)}_{y1} \end{Bmatrix} \tag{7.89}$$

where in the above expressions $[\tilde{\mathcal{U}}]$, $[\tilde{\mathcal{U}}']$, $[\tilde{\mathcal{V}}]$ and $[\tilde{\mathcal{W}}]$ are the "final form" of the "Modified Transfer Matrices" for Part I, Part II, Part III and Part IV regions, respectively. They transfer the discretized quantities from the "initial end point ξ_t=0" (t=Parts I, II, III,IV) to the "final end point ξ_t=1" (t=Parts I, II, III,IV), respectively.

For the "Bonded Region" Part I,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=1} \\ \overline{Y}^{(2)}_{\xi_I=1} \\ \overline{Y}^{(4)}_{\xi_I=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} & \tilde{\mathcal{U}}_{1,3} \\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} & \tilde{\mathcal{U}}_{2,3} \\ \tilde{\mathcal{U}}_{3,1} & \tilde{\mathcal{U}}_{3,2} & \tilde{\mathcal{U}}_{3,3} \end{bmatrix}_{01} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=0} \\ \overline{Y}^{(2)}_{\xi_I=0} \\ \overline{Y}^{(4)}_{\xi_I=0} \end{Bmatrix} \quad \text{(in Part I)} \qquad (7.90)$$

For the "Bonded Region" Part II,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(3)}_{\xi_{II}=1} \\ \overline{Y}^{(4)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathcal{U}}'_{1,1} & \tilde{\mathcal{U}}'_{1,2} & \tilde{\mathcal{U}}'_{1,3} \\ \tilde{\mathcal{U}}'_{2,1} & \tilde{\mathcal{U}}'_{2,2} & \tilde{\mathcal{U}}'_{2,3} \\ \tilde{\mathcal{U}}'_{3,1} & \tilde{\mathcal{U}}'_{3,2} & \tilde{\mathcal{U}}'_{3,3} \end{bmatrix}_{01} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \\ \overline{Y}^{(4)}_{\xi_{II}=0} \end{Bmatrix} \quad \text{(in Part II)} \qquad (7.91)$$

For Part III Region,

$$\left\{\overline{Y}^{(2)}_{\xi_{III}=1}\right\} = \left[\tilde{\mathcal{V}}_{1,1}\right]_{01} \left\{\overline{Y}^{(2)}_{\xi_{III}=0}\right\} \quad \text{(in Part III)} \qquad (7.92)$$

For Part IV Region,

$$\left\{\overline{Y}^{(3)}_{\xi_{IV}=1}\right\} = \left[\dot{\mathcal{W}}_{1,1}\right]_{01} \left\{\overline{Y}^{(3)}_{\xi_{IV}=0}\right\} \quad \text{(in Part IV)} \qquad (7.93)$$

where the subscript (01) in the above expressions means that the "final form" of the "Modified Transfer Matrix" transferring the above quantities from the "initial end point 0", to the "final end point 1".

Now the natural frequencies of the entire system will be determined by using the "Boundary Conditions" and the "Continuity Conditions" in the y-direction. Any combination of the "Boundary Conditions" at ξ_I=0, ξ_{II}=1 for the upper plate, ξ_{III}=0, ξ_I=1 for the middle left plate, ξ_{II}=0, ξ_{IV}=1 for the middle right plate and ξ_I=0, ξ_{II}=1 for the lower plate in the y-direction can be prescribed.

"Continuity Conditions" between Part I and Part II can be written for upper plate and lower plate, the "Continuity Condition" between Part I and Part III for middle left plate, the "Continuity Condition" between Part II and Part IV for middle right plate can be written as follows,

The ''<u>Continuity Conditions</u>'' between the Part I and Part II,

$$\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_I=1}\right\}=\left\{\overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0}\right\} \tag{7.94}$$

$$\left\{\overline{\boldsymbol{Y}}^{(4)}_{\xi_I=1}\right\}=\left\{\overline{\boldsymbol{Y}}^{(4)}_{\xi_{II}=0}\right\} \tag{7.95}$$

The ''<u>Continuity Conditions</u>'' between the Part I and Part III,

$$\left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1}\right\}=\left\{\overline{\boldsymbol{Y}}^{(2)}_{\xi_I=0}\right\} \tag{7.96}$$

The ''<u>Continuity Conditions</u>'' between the Part II and Part IV,

$$\left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1}\right\}=\left\{\overline{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=0}\right\} \tag{7.97}$$

From the Equations (7.90), (7.92), (7.94), (7.96),one can obtain,

$$\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{\xi_I=1}\\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_I=1}\\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=0}\\ \overline{\boldsymbol{Y}}^{(4)}_{\xi_I=1}\end{Bmatrix}=\begin{bmatrix}\tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} & 0 & \tilde{\mathcal{U}}_{1,3}\\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} & 0 & \tilde{\mathcal{U}}_{2,3}\\ 0 & 0 & \boldsymbol{I} & 0\\ \tilde{\mathcal{U}}_{3,1} & \tilde{\mathcal{U}}_{3,2} & 0 & \tilde{\mathcal{U}}_{3,3}\end{bmatrix}\begin{bmatrix}\boldsymbol{I} & 0 & 0 & 0\\ 0 & \tilde{\mathcal{V}}_{1,1} & 0 & 0\\ 0 & 0 & \boldsymbol{I} & 0\\ 0 & 0 & 0 & \boldsymbol{I}\end{bmatrix}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{\xi_I=0}\\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=0}\\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=0}\\ \overline{\boldsymbol{Y}}^{(4)}_{\xi_I=0}\end{Bmatrix}$$

(7.98)

Similarly by Equations (7.91), (7.93), (7.95), (7.97),

$$\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0}\\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_I=1}\\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=0}\\ \overline{\boldsymbol{Y}}^{(4)}_{\xi_{II}=0}\end{Bmatrix}=\begin{bmatrix}\tilde{\mathcal{U}}'_{1,1} & 0 & \tilde{\mathcal{U}}'_{1,2} & \tilde{\mathcal{U}}'_{1,3}\\ 0 & \boldsymbol{I} & 0 & 0\\ \tilde{\mathcal{U}}'_{2,1} & 0 & \tilde{\mathcal{U}}'_{2,2} & \tilde{\mathcal{U}}'_{2,3}\\ \tilde{\mathcal{U}}'_{3,1} & 0 & \tilde{\mathcal{U}}'_{3,2} & \tilde{\mathcal{U}}'_{3,3}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{I} & 0 & 0 & 0\\ 0 & \boldsymbol{I} & 0 & 0\\ 0 & 0 & \tilde{\mathcal{W}}_{1,1} & 0\\ 0 & 0 & 0 & \boldsymbol{I}\end{bmatrix}^{-1}\begin{Bmatrix}\overline{\boldsymbol{Y}}^{(1)}_{\xi_I=0}\\ \overline{\boldsymbol{Y}}^{(2)}_{\xi_{III}=0}\\ \overline{\boldsymbol{Y}}^{(3)}_{\xi_{II}=0}\\ \overline{\boldsymbol{Y}}^{(4)}_{\xi_I=0}\end{Bmatrix}$$

(7.99)

After some manipulation the "final form" of the "Modified Transfer Matrix" is obtained as,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_I=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \\ \overline{Y}^{(4)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \tilde{\mathcal{W}}_{1,1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{U}}'_{1,1} & 0 & \tilde{\mathcal{U}}'_{1,2} & \tilde{\mathcal{U}}'_{1,3} \\ 0 & I & 0 & 0 \\ \tilde{\mathcal{U}}'_{2,1} & 0 & \tilde{\mathcal{U}}'_{2,2} & \tilde{\mathcal{U}}'_{2,3} \\ \tilde{\mathcal{U}}'_{3,1} & 0 & \tilde{\mathcal{U}}'_{3,2} & \tilde{\mathcal{U}}'_{3,3} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} & 0 & \tilde{\mathcal{U}}_{1,3} \\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} & 0 & \tilde{\mathcal{U}}_{2,3} \\ 0 & 0 & I & 0 \\ \tilde{\mathcal{U}}_{3,1} & \tilde{\mathcal{U}}_{3,2} & 0 & \tilde{\mathcal{U}}_{3,3} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{V}}_{1,1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \\ \overline{Y}^{(4)}_{\xi_I=0} \end{Bmatrix} \qquad (7.100)$$

Or finally by multiplying and, simply rearranging,

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_I=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \\ \overline{Y}^{(4)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{1,1} & \mathcal{Q}_{1,2} & \mathcal{Q}_{1,3} & \mathcal{Q}_{1,4} \\ \mathcal{Q}_{2,1} & \mathcal{Q}_{2,2} & \mathcal{Q}_{2,3} & \mathcal{Q}_{2,4} \\ \mathcal{Q}_{3,1} & \mathcal{Q}_{3,2} & \mathcal{Q}_{3,3} & \mathcal{Q}_{3,4} \\ \mathcal{Q}_{4,1} & \mathcal{Q}_{4,2} & \mathcal{Q}_{4,3} & \mathcal{Q}_{4,4} \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \\ \overline{Y}^{(4)}_{\xi_I=0} \end{Bmatrix} \qquad (7.101)$$

where $[\boldsymbol{Q}]_{01}$ is the final form of the “Modified Transfer Matrix”.

The final form of the “Modified Transfer Matrix” can be reduced to (12x12) by substitution of the “Boundary Conditions” in the y-direction at ξ_I=0, ξ_{II}=1 for the upper plate and lower plate, ξ_{III}=0, ξ_I=1 for the middle left plate and. ξ_{II}=0, ξ_{IV}=1 for the middle right plate. After inserting the “Boundary Conditions” the following matrix equation which implicitly includes the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$ can be obtained,

$[\mathbf{C_0}(\omega_{mn})]_{\mathbf{(III)}}\{\mathbf{Y_0}\}_{\mathbf{(III)}} = \{\mathbf{0}\}$ $\Rightarrow$ Determinant of the “Coefficient Matrix”=

$$\left| [\mathbf{C_0}]_{\mathbf{(III)}} \right| = 0 \qquad (7.102)$$

where the subscript (III) indicates the “Main PROBLEM III”.

The “Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(III)}}$” is obtained by eliminating the rows corresponding to nonzero “state variables” at the “final end point’’ and the columns corresponding to zero “state variables” at the “initial end point’’ in (1.89). Here,

$\{\mathbf{Y_0}\}$ is a vector whose elements are the nonzero components of $\left\{ \boldsymbol{Y}^{(1)}_{\xi_I=0}, \boldsymbol{Y}^{(2)}_{\xi_{III}=0}, \boldsymbol{Y}^{(3)}_{\xi_{II}=0}, \boldsymbol{Y}^{(4)}_{\xi_I=0} \right\}^T$

For a non-trivial solution, the determinant of the "Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(III)}}$" mustl be equal to zero. This procedure should be repeated for a specific set of "Boundary Conditions". The roots of the determinant of the "Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(III)}}$" correspond to the nondimensionalized natural frequencies, $\overline{\boldsymbol{\omega}}_{mn}$ of the entire composite bonded plate system of the "Main PROBLEM III". The roots of the determinant must carefully be obtained numerically. One can easily find the corresponding mode shapes, after the natural frequencies (and eigenvectors) are obtained for a particular set of "Boundary Conditions" in the y-directions.

7.6 Method of Solution for Special case of “Main PROBLEM III a” and “Main PROBLEM III b”

In this section, the application of the present solution technique will be explained in detail for the Special Case of the “Main PROBLEM IIIa” and the “Main PROBLEM IIIb”.

“Modified Transfer Matrix” for Part I, Part II, Part III, Part IV and Part V can be obtained similarly for “Main PROBLEM III” as;

<u>For the “Bonded Region” Part I,</u>

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=1} \\ \overline{Y}^{(2)}_{\xi_I=1} \\ \overline{Y}^{(4)}_{\xi_I=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} & \tilde{\mathcal{U}}_{1,3} \\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} & \tilde{\mathcal{U}}_{2,3} \\ \tilde{\mathcal{U}}_{3,1} & \tilde{\mathcal{U}}_{3,2} & \tilde{\mathcal{U}}_{3,3} \end{bmatrix}_{01} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_I=0} \\ \overline{Y}^{(2)}_{\xi_I=0} \\ \overline{Y}^{(4)}_{\xi_I=0} \end{Bmatrix} \quad \text{(in Part I)} \qquad (7.103)$$

<u>For the “Bonded Region” Part II,</u>

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(3)}_{\xi_{II}=1} \\ \overline{Y}^{(4)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathcal{U}}'_{1,1} & \tilde{\mathcal{U}}'_{1,2} & \tilde{\mathcal{U}}'_{1,3} \\ \tilde{\mathcal{U}}'_{2,1} & \tilde{\mathcal{U}}'_{2,2} & \tilde{\mathcal{U}}'_{2,3} \\ \tilde{\mathcal{U}}'_{3,1} & \tilde{\mathcal{U}}'_{3,2} & \tilde{\mathcal{U}}'_{3,3} \end{bmatrix}_{01} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \\ \overline{Y}^{(4)}_{\xi_{II}=0} \end{Bmatrix} \quad \text{(in Part II)} \qquad (7.104)$$

<u>For Part III Region,</u>

$$\left\{\overline{Y}^{(2)}_{\xi_{III}=1}\right\} = \left[\tilde{\mathcal{V}}_{1,1}\right]_{01} \left\{\overline{Y}^{(2)}_{\xi_{III}=0}\right\} \quad \text{(in Part III)} \qquad (7.105)$$

<u>For Part IV Region,</u>

$$\left\{\overline{Y}^{(3)}_{\xi_{IV}=1}\right\} = \left[\dot{\mathcal{W}}_{1,1}\right]_{01} \left\{\overline{Y}^{(3)}_{\xi_{IV}=0}\right\} \quad \text{(in Part IV)} \qquad (7.106)$$

<u>For Part V Region,</u>

$$\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_V=1} \\ \overline{Y}^{(4)}_{\xi_V=1} \end{Bmatrix} = \begin{bmatrix} \tilde{\mathcal{Z}}_{1,1} & 0 \\ 0 & \tilde{\mathcal{Z}}_{4,4} \end{bmatrix}_{01} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_V=0} \\ \overline{Y}^{(4)}_{\xi_V=0} \end{Bmatrix} \quad \text{(in Part V)} \qquad (7.107)$$

where subscript (01) in the above expressions means that the “final form” of the “Modified Transfer Matrix” transferring the above quantities from the “initial end point 0”, to the “final end point 1”.

Now natural frequencies of the entire system will be determined by using the “Boundary Conditions” and the “Continuity Conditions” in the y-direction.

The ‘‘Continuity Conditions’’ between the Part I and Part V,

$$\left\{\overleftarrow{\boldsymbol{Y}}^{(1)}_{\xi_I=1}\right\}=\left\{\overleftarrow{\boldsymbol{Y}}^{(1)}_{\xi_V=0}\right\} \qquad (7.108)$$

$$\left\{\overleftarrow{\boldsymbol{Y}}^{(4)}_{\xi_I=1}\right\}=\left\{\overleftarrow{\boldsymbol{Y}}^{(4)}_{\xi_V=0}\right\}$$

The ‘‘Continuity Conditions’’ between the Part I and Part III,

$$\left\{\overleftarrow{\boldsymbol{Y}}^{(2)}_{\xi_{III}=1}\right\}=\left\{\overleftarrow{\boldsymbol{Y}}^{(2)}_{\xi_I=0}\right\} \qquad (7.109)$$

The ‘‘Continuity Conditions’’ between the Part II and Part IV,

$$\left\{\overleftarrow{\boldsymbol{Y}}^{(3)}_{\xi_{II}=1}\right\}=\left\{\overleftarrow{\boldsymbol{Y}}^{(3)}_{\xi_{IV}=0}\right\} \qquad (7.110)$$

The ‘‘Continuity Conditions’’ between the Part II and Part V,

$$\left\{\overleftarrow{\boldsymbol{Y}}^{(1)}_{\xi_V=1}\right\}=\left\{\overleftarrow{\boldsymbol{Y}}^{(1)}_{\xi_{II}=0}\right\} \qquad (7.111)$$

$$\left\{\overleftarrow{\boldsymbol{Y}}^{(4)}_{\xi_V=1}\right\}=\left\{\overleftarrow{\boldsymbol{Y}}^{(4)}_{\xi_{II}=0}\right\}$$

Then, from Equations (7.103) and (7.111), final form of the “Modified Transfer Matrix” is obtained as,

$$
\begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{II}=1} \\ \overline{Y}^{(2)}_{\xi_{I}=1} \\ \overline{Y}^{(3)}_{\xi_{IV}=1} \\ \overline{Y}^{(4)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \tilde{\mathcal{W}}_{1,1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{U}}'_{1,1} & 0 & \tilde{\mathcal{U}}'_{1,2} & \tilde{\mathcal{U}}'_{1,3} \\ 0 & I & 0 & 0 \\ \tilde{\mathcal{U}}'_{2,1} & 0 & \tilde{\mathcal{U}}'_{2,2} & \tilde{\mathcal{U}}'_{2,3} \\ \tilde{\mathcal{U}}'_{3,1} & 0 & \tilde{\mathcal{U}}'_{3,2} & \tilde{\mathcal{U}}'_{3,3} \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{Z}}_{1,1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{Z}}_{4,4} \end{bmatrix}
$$

$$
\begin{bmatrix} \tilde{\mathcal{U}}_{1,1} & \tilde{\mathcal{U}}_{1,2} & 0 & \tilde{\mathcal{U}}_{1,3} \\ \tilde{\mathcal{U}}_{2,1} & \tilde{\mathcal{U}}_{2,2} & 0 & \tilde{\mathcal{U}}_{2,3} \\ 0 & 0 & I & 0 \\ \tilde{\mathcal{U}}_{3,1} & \tilde{\mathcal{U}}_{3,2} & 0 & \tilde{\mathcal{U}}_{3,3} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{V}}_{1,1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \overline{Y}^{(1)}_{\xi_{I}=0} \\ \overline{Y}^{(2)}_{\xi_{III}=0} \\ \overline{Y}^{(3)}_{\xi_{II}=0} \\ \overline{Y}^{(4)}_{\xi_{I}=0} \end{Bmatrix}
$$

(7.112)

or, simply rearranging,

$$
\begin{Bmatrix} \overline{\mathbf{Y}}^{(1)}_{\xi_{II}=1} \\ \overline{\mathbf{Y}}^{(2)}_{\xi_{I}=1} \\ \overline{\mathbf{Y}}^{(3)}_{\xi_{IV}=1} \\ \overline{\mathbf{Y}}^{(4)}_{\xi_{II}=1} \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{1,1} & \mathcal{Q}_{1,2} & \mathcal{Q}_{1,3} & \mathcal{Q}_{1,4} \\ \mathcal{Q}_{2,1} & \mathcal{Q}_{2,2} & \mathcal{Q}_{2,3} & \mathcal{Q}_{2,4} \\ \mathcal{Q}_{3,1} & \mathcal{Q}_{3,2} & \mathcal{Q}_{3,3} & \mathcal{Q}_{3,4} \\ \mathcal{Q}_{4,1} & \mathcal{Q}_{4,2} & \mathcal{Q}_{4,3} & \mathcal{Q}_{4,4} \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{Y}}^{(1)}_{\xi_{I}=0} \\ \overline{\mathbf{Y}}^{(2)}_{\xi_{III}=0} \\ \overline{\mathbf{Y}}^{(3)}_{\xi_{II}=0} \\ \overline{\mathbf{Y}}^{(4)}_{\xi_{I}=0} \end{Bmatrix} \qquad (7.113)
$$

where $[\boldsymbol{Q}]_{01}$ is the final form of the “Modified Transfer Matrix”.

When the boundary condition applied the following matrix equation which implicitly includes the unknown dimensionless natural frequency parameter $\overline{\omega}_{mn}$ can be obtained,

$[\mathbf{C_0}]_{\mathbf{(III)}}\{\mathbf{Y_0}\}_{\mathbf{(III)}} = \{\mathbf{0}\}$ $\Rightarrow$ Determinant of the “Coefficient Matrix”=

$$
\left| [\mathbf{C_0}]_{\mathbf{(III)}} \right| = 0 \qquad (7.115)
$$

where the subscript (III) indicates the “Special Case of Main PROBLEM III”.

The “Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(III)}}$” is obtained by eliminating the rows corresponding to nonzero state variable at the “final end point'' and the columns corresponding to zero state variables at the “initial end point''. Here, $\{\mathbf{Y_0}\}$ is a vector whose elements are the nonzero components of $\left\{ \boldsymbol{Y}^{(1)}_{\xi_I=0}, \boldsymbol{Y}^{(2)}_{\xi_{III}=0}, \boldsymbol{Y}^{(3)}_{\xi_{II}=0}, \boldsymbol{Y}^{(4)}_{\xi_I=0} \right\}^T$ similarly as "Main PROBLEM III".

For non-trivial solution, determinant of the "Coefficient Matrix $[\mathbf{C_0}]_{\mathbf{(III)}}$" shall be equal to zero which correspond to the nondimensionalized natural frequencies, $\overline{\omega}_{mn}$ of the entire composite plate system of the Special Case of "Main PROBLEM III".

7.7 Integrating Matrix Method

Here, in this section, the "Integrating Matrix Method" will be explained in general terms for all the "Main PROBLEMS" considered in the present "Thesis".

The "Integrating Matrices" can be obtained from Hunter's [VII.1] detailed discussion. In this part, the algorithm of the "Integrating Matrix" which is used to integrate the discretized "Coefficient Matrices" will be explained. The "Integrating Matrix" is a means by which a continuous function may be integrated with the use of a finite-difference approach. This numerical method is based upon the assumption that the function f(x) may be represented by a polynomial of degree "r" as follows,

Interpolation Polynomial (of degree r)

$$f(x)=a_0+a_1x+a_2x^2+..........+a_rx^r \tag{7.116}$$

The "Integrating Matrix" is developed by expressing (7.116) in the form of "Newton's Forward Difference Interpolation" formula. The integrand may be represented conveniently by polynomials of any degree. When the "Integrating Matrix" is employed, any number of stations "n" may be chosen (so long as n≥r) (see Hunter [VII.1]).

The "Global Integrating Matrix [$\mathcal{L}$]" works as an operator, in a similar way to the conventional integral symbol such that,

$$\mathscr{L} \leftrightarrow \int_0^s (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)ds \tag{7.117}$$

$$[\mathscr{L}] = \begin{bmatrix} L & 0 & . & . & . & 0 \\ 0 & L & 0 & . & . & .0 \\ 0 & 0 & L & 0 & . & .0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & L \end{bmatrix} = \text{(Diagonal Matrix)} \tag{7.118}$$

In the above, each basic block "Integrating Sub-Matrix [L]″ is responsible for the integration of one single "state variable" (or "fundamental dependent variable") of the problem.

Here, $[\mathscr{L}_{\mathrm{I}}]$ for "Main PROBLEM I", $[\mathscr{L}_{\mathrm{I}}]$ and $[\mathscr{L}_{\mathrm{II}}]$ for "Main PROBLEM II" are used as a "Global Integrating Matrix" for the "Two-Layer Region", which are square block diagonal matrices composed of (since the "state vector" is composed of 12 "dimensionless fundamental dependent variables") (12×12) ''basic'' square blocks of dimension n_s, (i.e. n_s=12). And, "n" is the number of discretization points along the "Overlap Region". Therefore, the dimension of the "Global Integrating Matrix $[\mathscr{L}]$" is (12n×12n) for the "Two-Layer Region(s)" for the "Main PROBLEM I" and for the "Main PROBLEM II" respectively.

Similarly, $[\mathscr{L}_{\mathrm{I}}]$ for "Main PROBLEM III" is used as a "Global Integrating Matrix" for "Three-Layer Region", which is a square block diagonal matrix composed of (since the "state vector" is composed of 18 "dimensionless fundamental dependent variables" for "Overlap Region" or Part I region) (18×18) ''basic'' square blocks of dimension n_s, (i.e. n_s=18). And "n" is the number of discretization points along the "Overlap Region". Therefore, the dimension of the "Global Integrating Matrix $[\mathscr{L}]$" is (18n×18n) for "Overlap Region" for the "Main PROBLEM III"

Similarly, $[\mathscr{L}_{\mathrm{II}}]$ and $[\mathscr{L}_{\mathrm{III}}]$ (subscripts here corresponds to Part I, Part II, and Part III respectively) for "Main PROBLEM I", $[\mathscr{L}_{\mathrm{III}}]$ and $[\mathscr{L}_{\mathrm{IV}}]$ for "Main

PROBLEM II" and "Main PROBLEM III" are used as the "Global Integrating Matrices" for "Single Layer" region. They are square block diagonal matrices composed of (since the "state vector"" is composed of six "dimensionless fundamental dependent variables" for the "Single Layer" region) (6x6) ''basic'' square blocks of dimension n_s, (i.e. n_s=6). And "n" is the number of "discretization points" along the "Single Layer" region. Therefore, the dimensions of the "Global Integrating Matrices $[\mathcal{L}]$ is (6n×6n) for "Single Layer" region for all "Main PROBLEM(s)"

It is important to note here that, the accuracy and convergence of the present method are affected by the "degree" of the "Approximating Polynomial (or Interpolation Polynomial)" and the "number" of "discretization points". A proper degree of the "Assumed Interpolation Polynomial" may result in convergence with fewer discretization points. However, it is not expected to obtain more accurate results as the degree of the "Assumed Polynomial" increases arbitrarily since the polynomial shows more oscillating behavior as the degree of it increases. This does not necessarily resemble the actual behavior of the plate system. Therefore, one has to be careful in that regard. Another important point is that, one has to check zero of the support conditions at the far end right side boundary.

The "Integrating Sub-Matrix [L]" which is a component of the "Global Integrating Matrix [$\mathcal{L}$]" is normalized for integrations in the unitary interval [0, 1] in the following way,

$$[L]=[\boldsymbol{S}][\boldsymbol{W}_n] \qquad (7.119)$$

where [S] is an (n×n) "Lower Triangular Matrix". Here, n is the number of discretization point in the related part. The "Lower Triangular Matrix [$\boldsymbol{S}$]" is given as,

$$[S] = \begin{bmatrix} 1 & 0 & . & . & 0 \\ 1 & 1 & . & . & 0 \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ 1 & 1 & . & . & 1 \end{bmatrix} \tag{7.120}$$

where $[W_n]$ is the "Weighting Matrix":

$$[W_n] = \frac{\Delta n}{1440} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & ... & ... & 0 \\ 475 & 1427 & -798 & 482 & -173 & 27 & 0 & ... & ... & 0 \\ -27 & 637 & 1022 & -258 & 77 & -11 & 0 & ... & ... & 0 \\ 11 & -93 & 802 & 802 & -93 & 11 & 0 & ... & ... & 0 \\ 0 & 11 & -93 & 802 & 802 & -93 & 11 & 0 & ... & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & ... & ... & 11 & -93 & .802 & .802 & .-93 & 11 & 0 \\ 0 & ... & ... & 0 & 11 & -93 & 802 & 802 & -93 & 11 \\ 0 & ... & ... & 0 & -11 & 77 & -258 & 1022 & 637 & -27 \\ 0 & ... & ... & 0 & 27 & -173 & 482 & -798 & 1427 & 475 \end{bmatrix}$$

(7.121)

where Δn is the "step size", given by,

$$\Delta n = \frac{1}{(n-1)} \tag{7.97}$$

The above expression (7.121), is based on a "5th Order Interpolation Polynomial". Moreover, the "Weighting Matrix $[W_n]$" for the "6th Order Interpolation Polynomial" that is given in Hunter [VII.1] which is also used in this present "Thesis" is given.

“FREE FLEXURAL (or BENDING) VIBRATIONS ANALYSIS OF COMPOSITE, ORTHOTROPIC PLATE AND/OR PANELS WITH VARIOUS BONDED JOINTS”
(- - -IN AERO-STRUCTURAL SYSTEMS - - -)

VOLUME II

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULLFILMENT OF THE REQUIREMENTS
FOR
THE DEGREE OFMASTER OF SCIENCE
IN
AEROSPACE ENGINEERING

SEPTEMBER 2004

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8 NUMERICAL SOLUTIONS

8.1 Some Remarks on Material and Geometric Characteristics and Boundary Conditions

In this section, the mode shapes corresponding to the several support conditions, in the “hard” and the “soft” adhesive cases and some parametric studies will be presented and discussed in some detail.

The theoretical formulation of the “Composite Plate and/or Panel System with Bonded Lap Joints” were given as “Main PROBLEM I”, “Main PROBLEM II and Main PROBLEM III in the previous chapters.

For “Main PROBLEM I”, the “Material and Geometric Characteristics” are given in Table 8.1 for the “Composite, Orthotropic Plate and/or Panel System (Upper Plate Adherend is Graphite-Epoxy and Lower Plate Adherend is Kevlar-Epoxy)”.

For “Main PROBLEM II”, the “Material and Geometric Characteristics” are presented in Table 8.2 for “Composite, Orthotropic Plate and/or Panel System (Doubler is Graphite-Epoxy and Plate Adherends are Kevlar Epoxy)”.

For “Main PROBLEM III the “Material and Geometric Characteristics” are shown in Table 8.3 for “Composite, Orthotropic Plate and/or Panel System (Doublers are Graphite-Epoxy and Plate Adherends are Kevlar Epoxy)”.

Two different adhesive cases, “hard” and “soft” adhesive cases are considered to show the significant effects of the adhesive layer elastic constants on the mode shapes and natural frequencies.

The “Classical Levy’s Solution” is used for the theoretical formulation of all cases of “Main PROBLEMS”. The “Boundary Conditions” are assumed to be simply supported are at x=0 and x=a. Therefore, in the following numerical results only “Boundary Conditions” at y=0 and y=L will be given (since these are arbitrarily prescribed support conditions).

In the figures and parametric studies, the "Boundary Conditions" are indicated in terms of the following notations:

(S) "Simply Support Condition"

(C) "Clamped Support Condition"

(F) "Free Edge Condition"

The "Boundary Conditions" read from left to right for "Main PROBLEMs". For the "Main PROBLEM Ia", the first two letters are for the "Upper Plate Adherend", third and fourth letters are for the "Lower Plate Adherend". For example, (CFFC) means the "Upper Plate Adherend" has "Clamped Support Condition" at the left (y=0), "Free Edge Condition" at the right ($y=b_1$), and "Lower Plate Adherend" has "Free Edge Condition" at the left ($y=\ell_I$), "Clamped Support Condition" at the right (y=L).

For the "Main PROBLEM II", the first two letters are for the "Doubler", third and fourth letters are for the "Left Plate Adherend" and fifth and sixth letters are for the "Right Plate Adherend" For example, (FFCFFC) means the "Doubler" has "Free Support Condition" at the left and the right edges, "Left Plate Adherend" has "Clamped Support Condition" at the left (y=0), "Free Edge Condition" at the right ($y=b_2$) and "Right Plate Adherend" has "Free Edge Condition" at the left ($y= \ell_I+ \ell_{II}$), "Clamped Support Condition" at the right (y=L).

For the "Main PROBLEM III", the first two letters are for the "Upper Doubler", third and fourth letters are for the "Left Plate Adherend" and fifth and sixth letters are for the "Right Plate Adherend" and last two letters are for the "Lower Doubler" For example, (FFCFFCFF) means the "Upper and Lower Doublers" have "Free Support Condition" at the left and the right edges, "Left Plate Adherend" has "Clamped Support Condition" at the left (y=0), "Free Edge Condition" at the right ($y=b_2$) and "Right Plate Adherend" has "Free Edge Condition" at the left ($y= \ell_I+ \ell_{II}$), "Clamped Support Condition" at the right (y=L).

Table 8.1 Material and Geometric Characteristics of "Composite Plate or Panel System" for "Main PROBLEM I"

Kevlar-Epoxy (Upper Plate Adherend) j=1	*Graphite-Epoxy (Lower Plate Adherend) j=2*	*Adhesive layer (Hard)*	*Adhesive layer (Soft)*
E_{xj}=5.5. GPa **E_{yj}=76.0 GPa** **G_{xyj}=2.10 GPa** **G_{xzj}=1.5 GPa** **G_{yzj}=2.0 Gpa** **ν_{xyj}=0.024** **ν_{yxj}=0.34** **ρ_j=1.3 gr/cm^3** **h_j=0.01 m** **a=0.5 m**	**E_{xj}=11.71. GPa** **E_{yj}=137.8 GPa** **G_{xyj}=5.51 GPa** **G_{xzj}=2.5 GPa** **G_{yzj}=3.0 Gpa** **ν_{xyj}=0.0213** **ν_{yxj}=0.25** **ρ_j=1.6 gr/cm^3** **h_j=0.01 m** **a=0.5 m**	**E_a=4.0 Gpa** **G_a= 1.4 GPa** **υ_a=0.43** **h_a=0.15x10^{-3} m** **ρ_a=neglected**	**$E_a/B_{11}^{(1)}=10^{-4}$** **$G_a/B_{11}^{(1)}=\frac{10^{-4}}{2(1+\upsilon_a)}$** **$\upsilon_a$=0.3** **$h_a$=0.15x10^{-3} m** **$\rho_a$=neglected**

Table 8.2 Material and Geometric Characteristics of "Composite Plate or Panel System" for "Main PROBLEM II"

Kevlar-Epoxy Doubler Plate j=1	*Graphite-Epoxy (Plate Adherends) j=2.3*	*Adhesive layer (Hard)*	*Adhesive layer (Soft)*
E_{xj}=5.5. GPa **E_{yj}=76.0 GPa** **G_{xyj}=2.10 GPa** **G_{xzj}=1.5 GPa** **G_{yzj}=2.0 Gpa** **ν_{xyj}=0.024** **ν_{yxj}=0.34** **ρ_j=1.3 gr/cm^3** **h_j=0.01 m** **a=0.5 m**	**E_{xj}=11.71. GPa** **E_{yj}=137.8 GPa** **G_{xyj}=5.51 GPa** **G_{xzj}=2.5 GPa** **G_{yzj}=3.0 Gpa** **ν_{xyj}=0.0213** **ν_{yxj}=0.25** **ρ_j=1.6 gr/cm^3** **h_j=0.01 m** **a=0.5 m**	**E_a=4.0 Gpa** **G_a= 1.4 GPa** **υ_a=0.43** **h_a=0.15x10^{-3} m** **ρ_a=neglected**	**$E_a/B_{11}^{(1)}=10^{-4}$** **$G_a/B_{11}^{(1)}=\dfrac{10^{-4}}{2(1+\upsilon_a)}$** **$\upsilon_a$=0.3** **$h_a$=0.15x10^{-3} m** **$\rho_a$=neglected**

Table 8.3 Material and Geometric Characteristics of "Composite Plate or Panel System" for "Main PROBLEM III"

Kevlar-Epoxy Doubler Plates j=1,4	*Graphite-Epoxy (Plate Adherends) j=2,3*	*Adhesive layer (Hard)*	*Adhesive layer (Soft)*
E_{xj}=5.5. GPa **E_{yj}=76.0 GPa** **G_{xyj}=2.10 GPa** **G_{xzj}=1.5 GPa** **G_{yzj}=2.0 Gpa** **ν_{xyj}=0.024** **ν_{yxj}=0.34** **ρ_j=1.3 gr/cm^3** **h_j=0.01 m** **a=0.5 m**	**E_{xj}=11.71. GPa** **E_{yj}=137.8 GPa** **G_{xyj}=5.51 GPa** **G_{xzj}=2.5 GPa** **G_{yzj}=3.0 Gpa** **ν_{xyj}=0.0213** **ν_{yxj}=0.25** **ρ_j=1.6 gr/cm^3** **h_j=0.01 m** **a=0.5 m**	**E_{a1}=4.0 Gpa** **G_{a1}= 1.4 GPa** **υ_{a1}=0.43** **E_{a4}=4.0 Gpa** **G_{a4}= 1.4 GPa** **υ_{a4}=0.43** **h_{a1}=0.15x10^{-3} m** **h_{a4}=0.15x10^{-3} m** **ρ_a=neglected**	**$E_{a1}/B_{11}^{(1)}=10^{-4}$** **$G_{a1}/B_{11}^{(1)}=\dfrac{10^{-4}}{2(1+\upsilon_a)}$** **$\upsilon_{a1}$=0.3** **$E_{a4}/B_{11}^{(1)}=10^{-4}$** **$G_{a4}/B_{11}^{(1)}=\dfrac{10^{-4}}{2(1+\upsilon_a)}$** **$\upsilon_{a4}$=0.3** **$h_{a1}$=0.15x10^{-3} m** **$h_{a4}$=0.15x10^{-3} m** **$\rho_a$=neglected**

8.2 Numerical Results and Discussion for "Main PROBLEM I.a"

In the "Main PROBLEM Ia.", the "Composite Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint" is analyzed. The upper adherend is made of Graphite-Epoxy and the lower plate adherend is Kevlar-Epoxy. For the in-between adhesive layer, the "hard" and the "soft" adhesive cases are taken into account. The "Geometric and the Material Characteristics" of the single lap joint system are given in Table 8.1.

In Figures 8.1 – 8.10, the mode shapes and the corresponding natural frequencies (from the first to fifth), in the "hard" and the subsequent "soft" adhesive cases with various boundary conditions are presented.

From aforementioned Figures, in the "hard" adhesive case it is easy to observe that regardless of the boundary conditions, there exists an almost "stationary region" in the mode shapes. And this region moves from left to the right part (or vice versa) in the composite single lap joint system. In the "soft" adhesive case, however, an almost "stationary region" does not exist in mode shapes. The general trend in the mode shapes, for the "soft" adhesive case is that, the "Overlap Region" moves or bends with the rest of the lap joint system. And the mode shapes are completely different in comparison with those of the "hard" adhesive cases with the same support conditions.

Next, for the "Main PROBLEM Ia", in Figures 8.11 through 8.28, several important parametric studies are presented. In Figures 8.11-8.16, the "Dimensionless Natural Frequency $\overline{\Omega}$" versus "Joint Length Ratio ℓ_I/L" from the first up to the fifth mode, are plotted, for both the "hard" and the "soft" adhesive cases, corresponding to the various support conditions.

From Figures 8.11, 8.13, 8.15, in the "hard" adhesive case, it is obvious that, as the wet area or the "Overlap Region" spreads (in the y-direction), the natural frequencies, at first gradually, and then, relatively sharply increases. These results of course, are the consequences of the increasing overall stiffness of the lap joint system due to the spreading of the “Overlap Region”.

In the “soft” adhesive case, in Figures 8.12, 8.14, 8.16, the increases in the natural frequencies are relatively gradual. And no sharp increases can be observed as the “Overlap Region” spreads along the y-direction. This also can be expected. It is because, due to the “soft” adhesive, the “Overlap Region” connects both adherends rather loosely and thus, a relatively loose lap joint system is created.

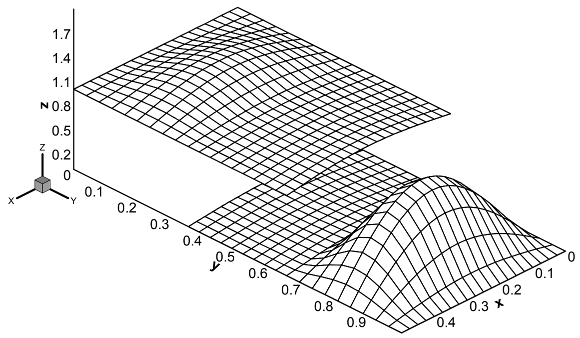
In Figures 8.17 through 8.22, the effect of the “Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$” on the natural frequencies (from the first up to the fifth) in the “hard” and “soft” adhesive cases, are investigated for various boundary conditions. In the “hard” adhesive case, in Figures 8.17, 8.19, 8.21, the first natural frequency, in spite of the increasing “Bending Rigidity Ratio”, does remain practically constant. In the third and higher modes, the natural frequencies increase sharply at first and after the “Bending Rigidity Ratio=1.8” they become almost flat or constant regardless of the increase in “Bending Rigidity Ratio”.

In the “soft” adhesive cases, in the Figures 8.18, 8.20, 8.22, the first and the second frequencies remain more or less constant as the “Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$” increases. In the third and higher modes, the natural frequencies increase significantly. In some cases, though, after “Bending Rigidity Ratio=1.0”, the fifth frequency reaches a constant value.

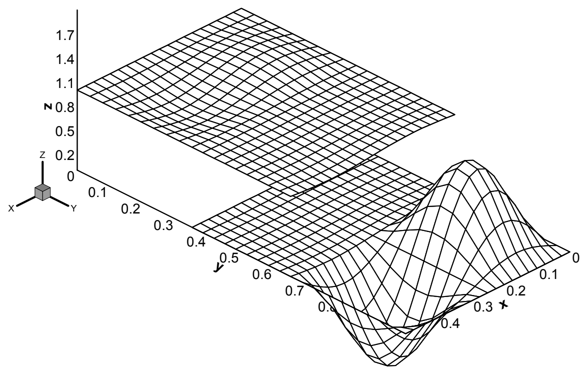
Lastly, the direct effects of the adhesive layer elastic constants E_a, and also G_a on the dimensionless natural frequencies are investigated for the “Main PROBLEM I.a”. In order to show these effects, the “Dimensionless Natural Frequencies” versus the “Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$” are plotted (while the other elastic constant kept constant) in Figures 8.23 through 8.25 for various boundary condition. Similarly, the “Dimensionless Natural Frequencies” versus the “Adhesive Shear Modulus Ratio $G_a/B_{11}^{(1)}$” are presented in Figures 8.26 through 8.28 for various support condition.

It can be seen from the Figures 8.23, 8.24, 8.25, the influence of the "Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" on the natural frequencies, is not significant. In Figures 8.26, 8.27, 8.28, we can see that the "Shear Modulus Ratio $G_a/B_{11}^{(1)}$", after the first and second modes, in higher modes, significantly affects the natural frequencies. Also, in those Figures, one can observe a "transition region" which takes the frequencies to considerably higher levels. After then, the frequencies don't increase and they remain practically constant.

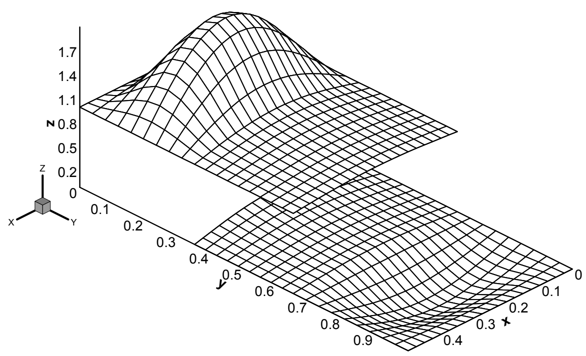
8.2.1. Natural Frequencies and Corresponding Mode Shapes for "Main PROBLEM Ia"



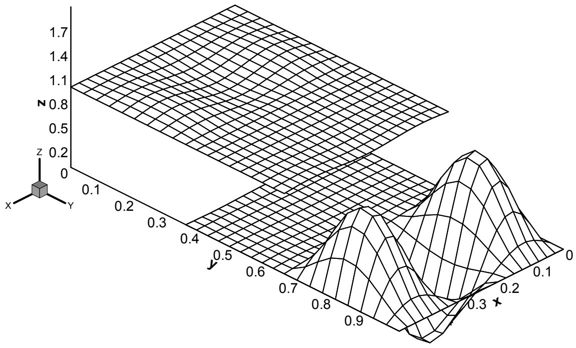
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=1027.406



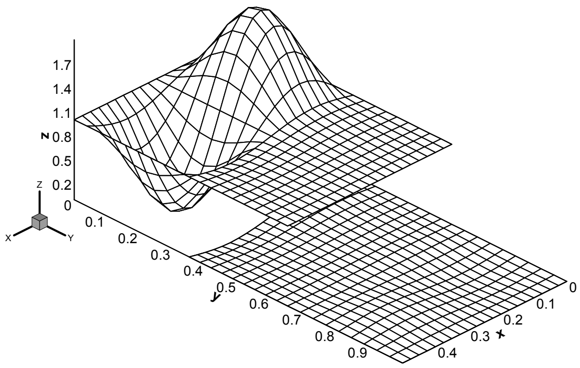
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=1213.235



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{12}$=1355.670



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{31}$=1622.912



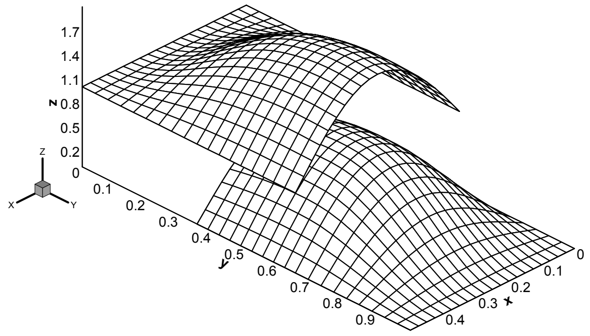
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=1672.817

("Hard" Adhesive Case)

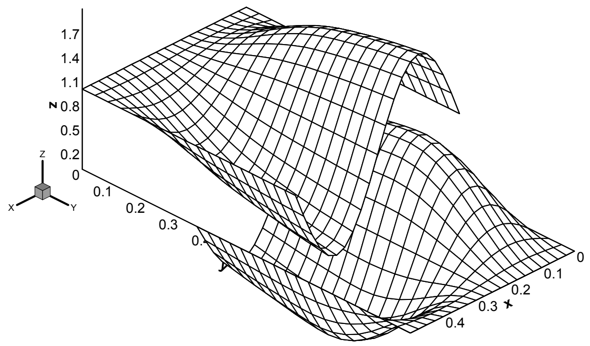
Fig 8.1 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

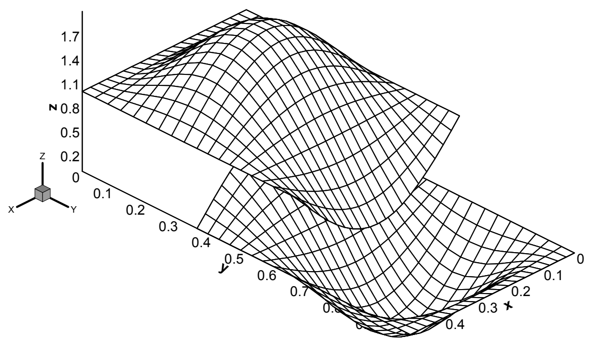
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction CFFC)



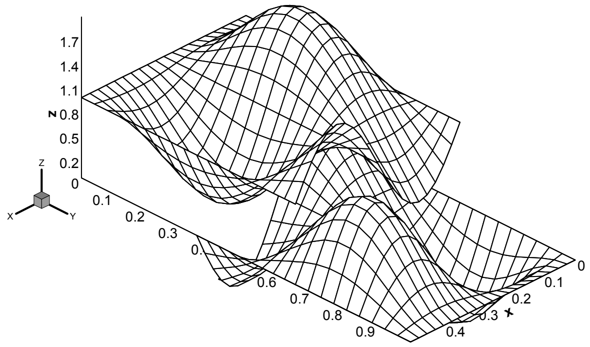
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=54.007



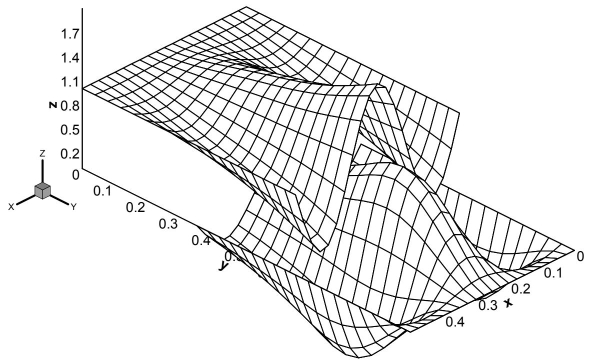
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=2331.807



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=233.171



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{22}$=417.335



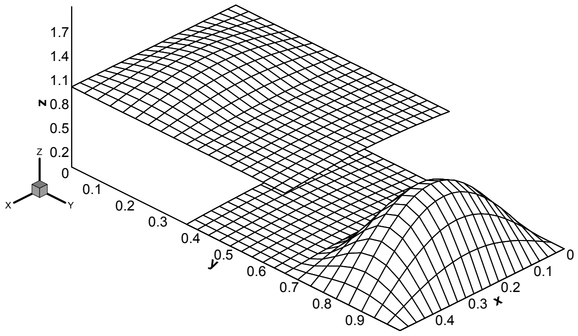
e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{31}$=739.349

("<u>Soft</u>" Adhesive Case)

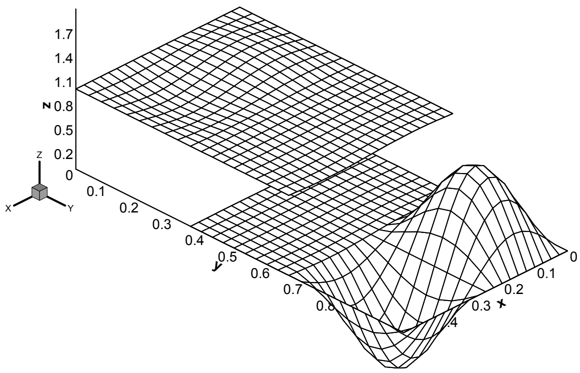
Fig.8.2 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

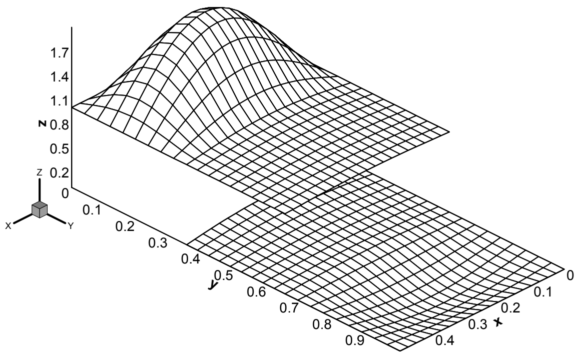
(Joint Length ℓ_I=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction CFFC)



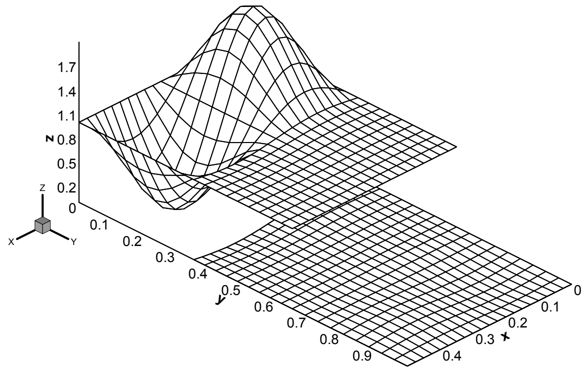
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=544.104



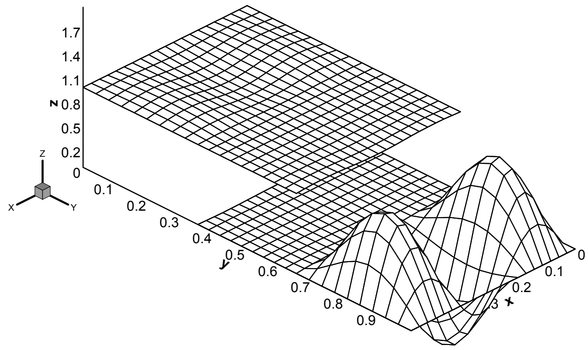
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=692.859



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{12}$=733.913



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=993.558



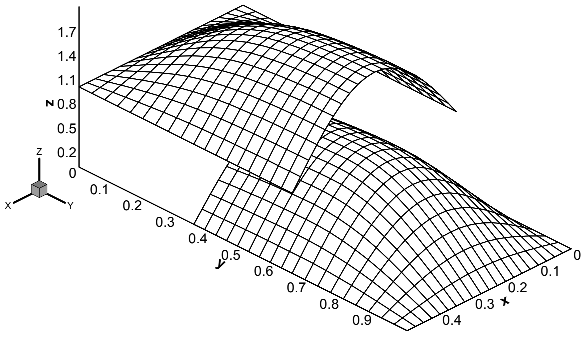
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=1098.416

("<u>Hard</u>" Adhesive Case)

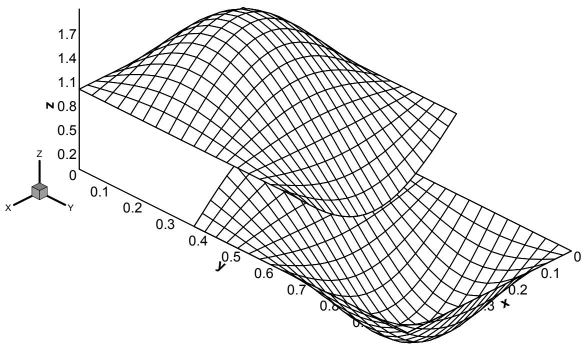
Fig.8.3 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

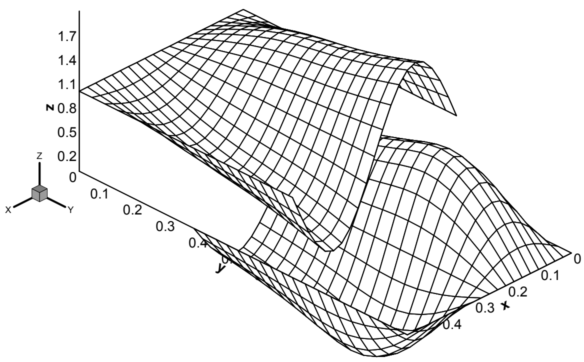
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction SFFS)



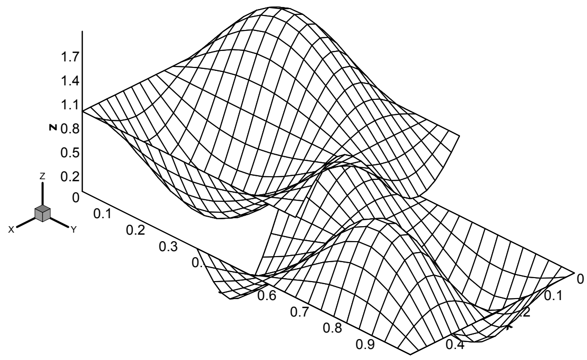
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=36.374



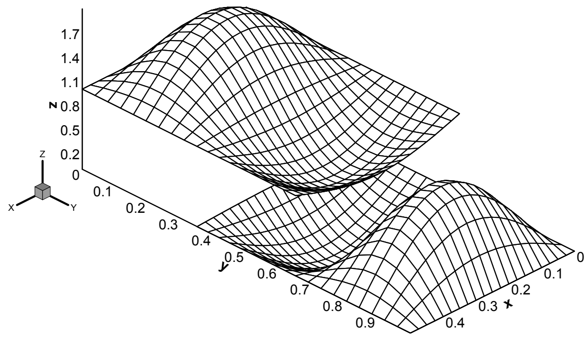
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}$=127.987



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{21}$=200.521



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=296.208

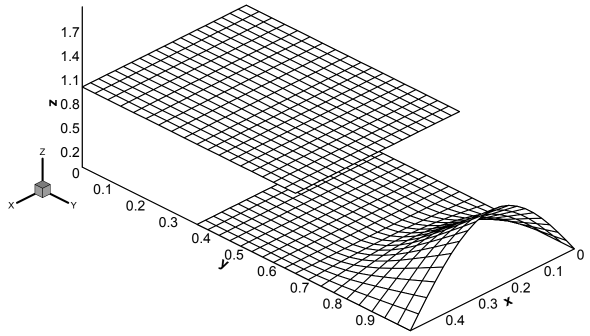


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=448.798

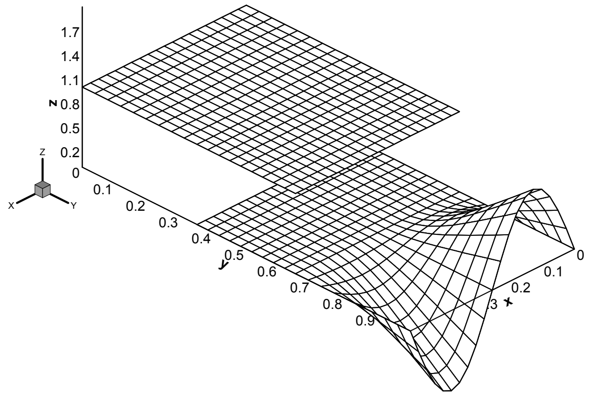
("<u>Soft</u>" Adhesive Case)

Fig.8.4 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint"

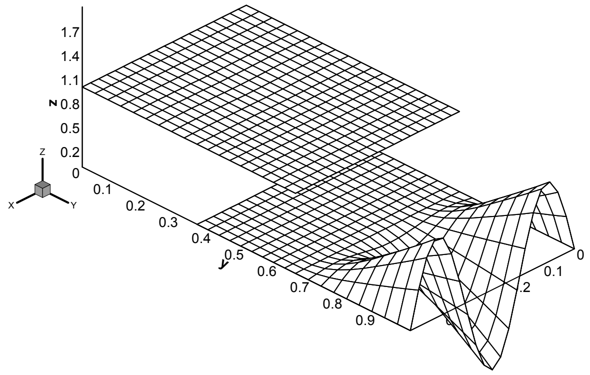
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction SFFS)



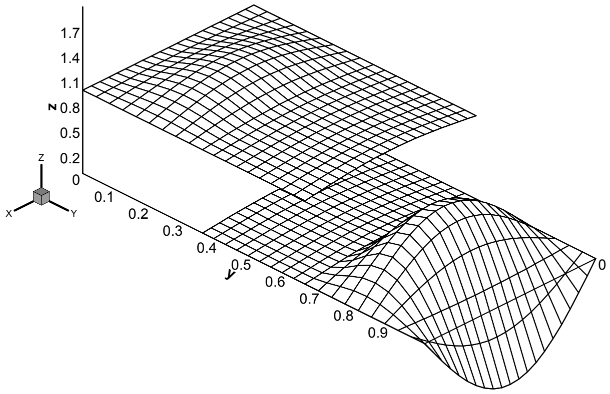
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=40.084



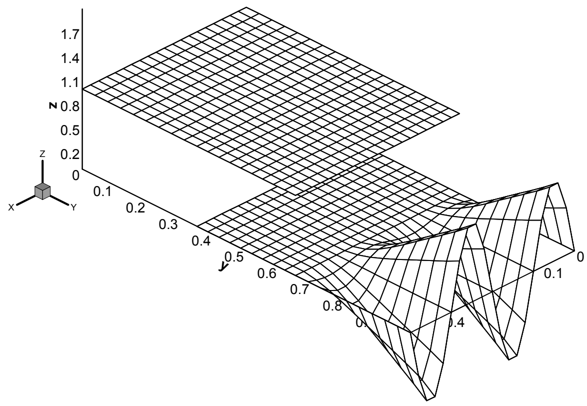
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=126.280



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=451.374



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=1054.832

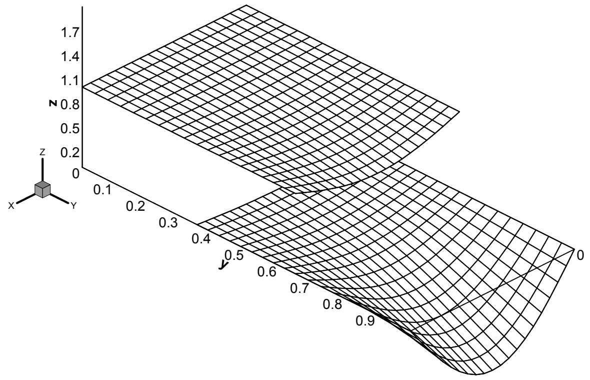


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{41}$=1278.577

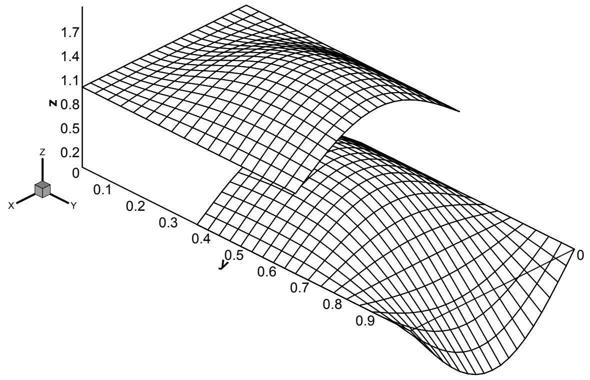
("Hard" Adhesive Case)

Fig.8.5 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint"

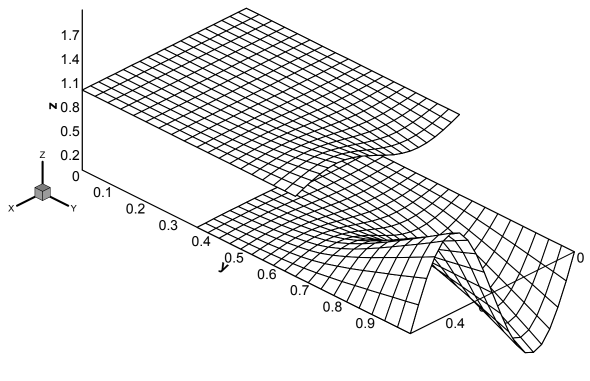
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction CFFF)



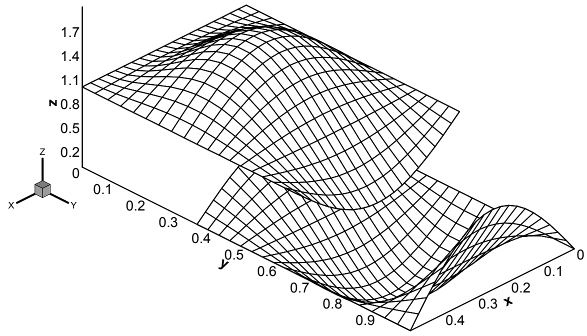
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=15.423



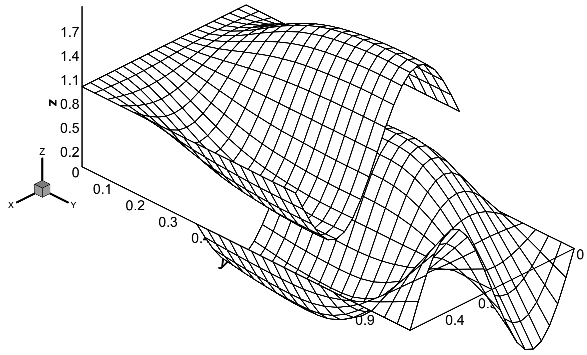
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=56.689



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=98.994



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=238.560



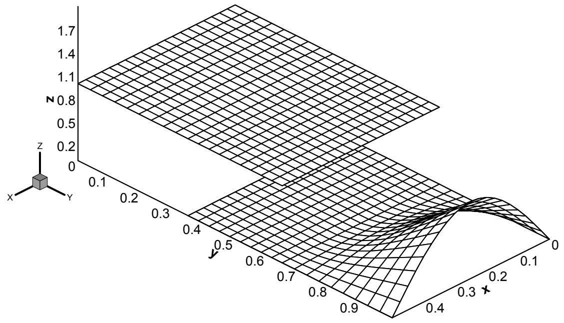
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=238.649

("<u>Soft</u>" Adhesive Case)

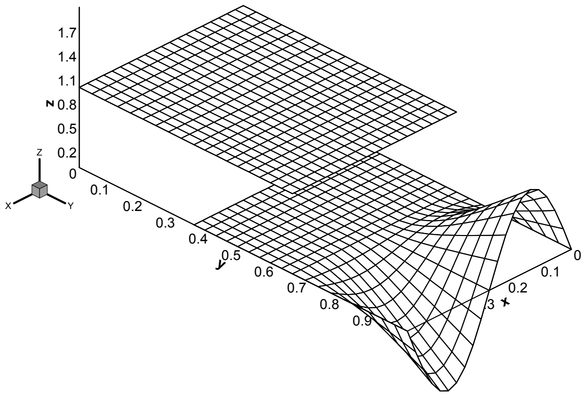
Fig.8.6 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

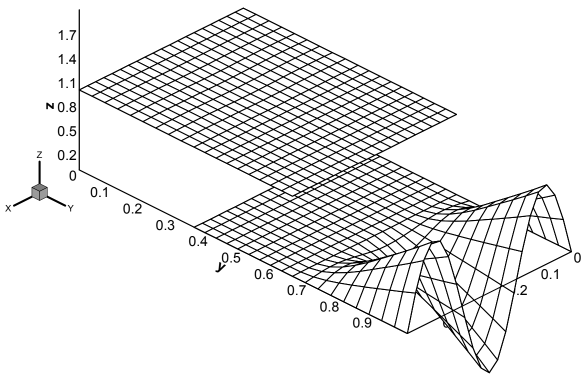
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction CFFF)



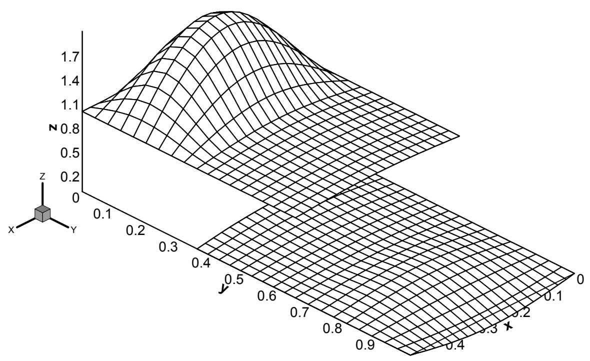
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=40.077



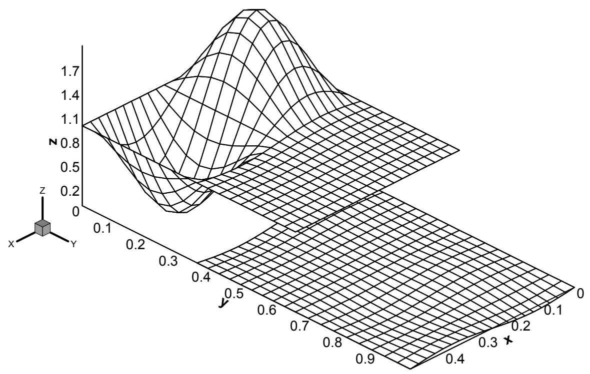
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{21}$=126.277



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{31}$=451.372



d) Fourth Mode with $\overline{\Omega}_4$= $\overline{\omega}_{12}$=723.534

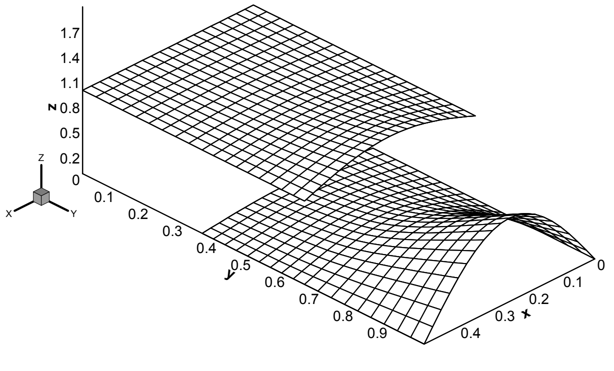


e) Fifth Mode with $\overline{\Omega}_5$= $\overline{\omega}_{22}$=991.487

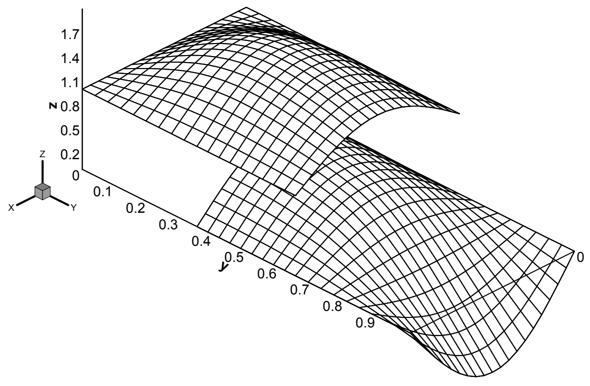
("Hard" Adhesive Case)

Fig.8.7 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint"

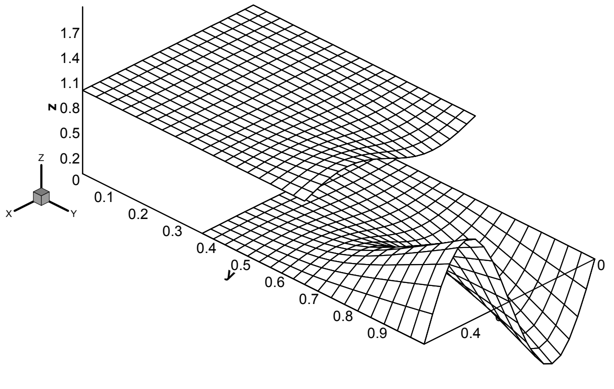
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction SFFF)



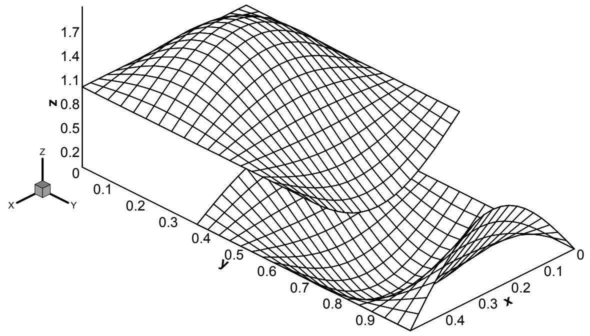
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=15.350



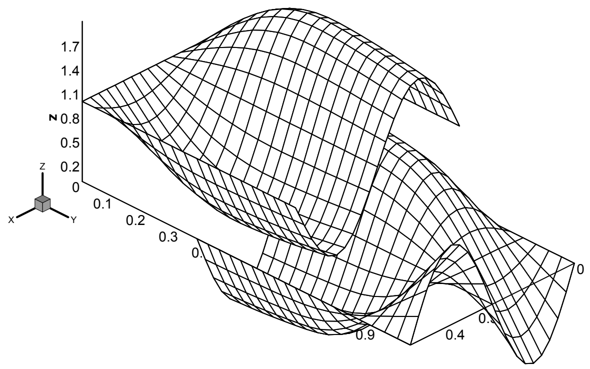
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=45.005



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=98.984



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=175.062



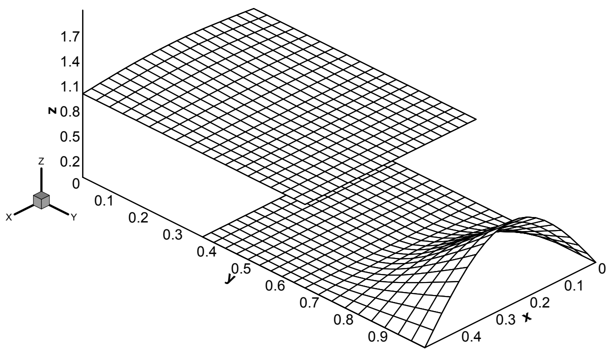
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=223.906

("Soft" Adhesive Case)

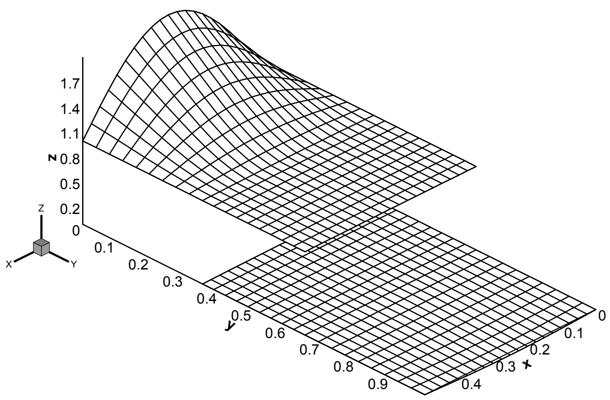
Fig.8.8 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

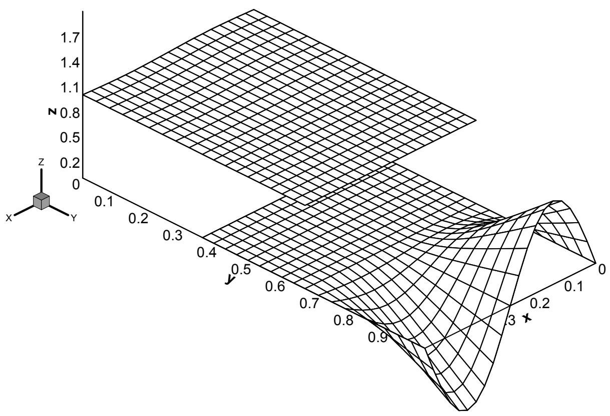
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction SFFF)



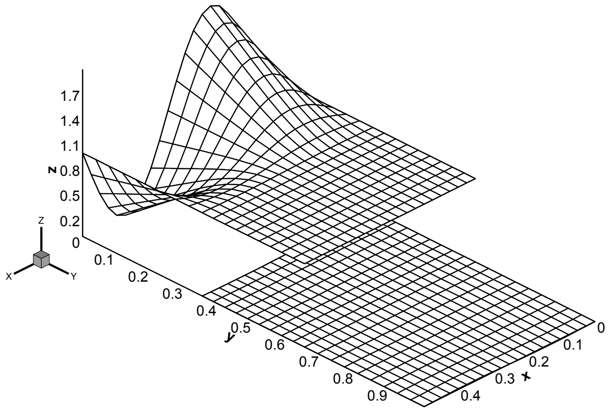
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=40.035



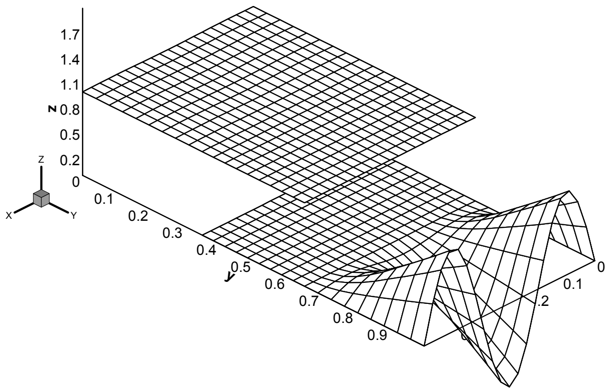
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=60.490



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=126.268



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{22}$=214.437

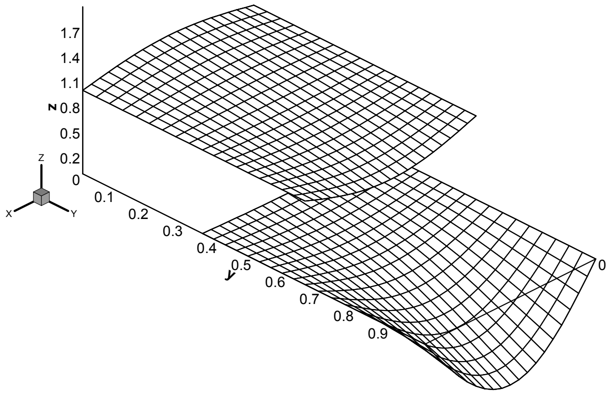


e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{31}$=451.370

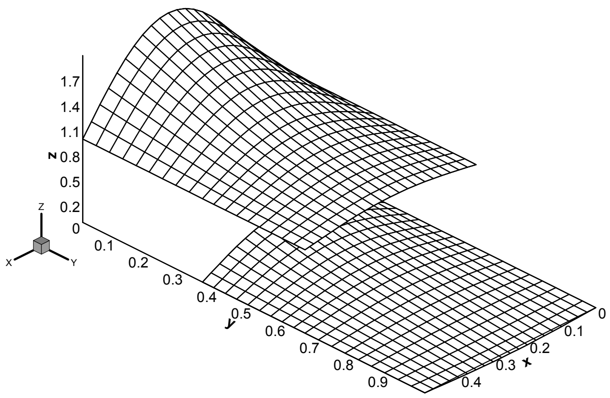
("<u>Hard</u>" Adhesive Case)

Fig.8.9 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint"

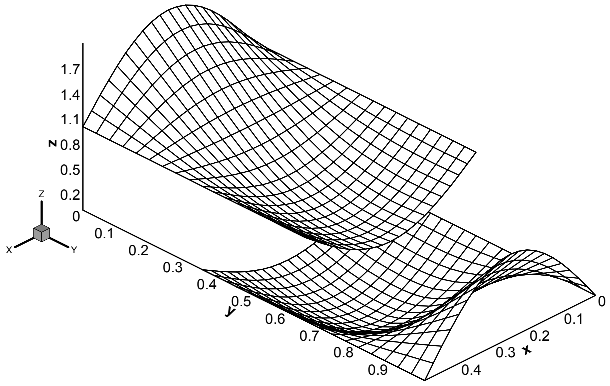
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFF)



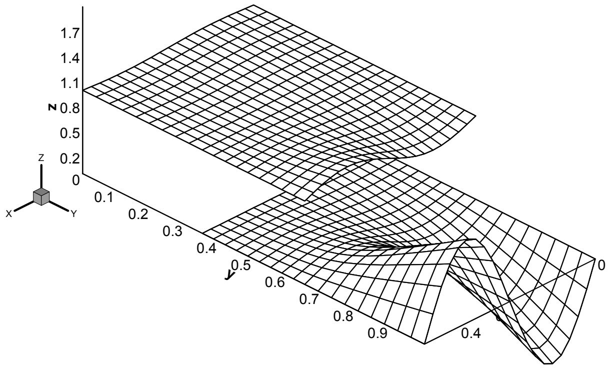
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 15.240$



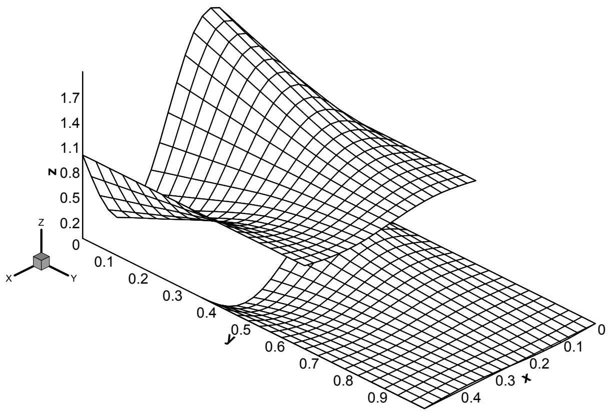
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 18.801$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{13} = 61.620$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{21}$=98.932



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=157.029

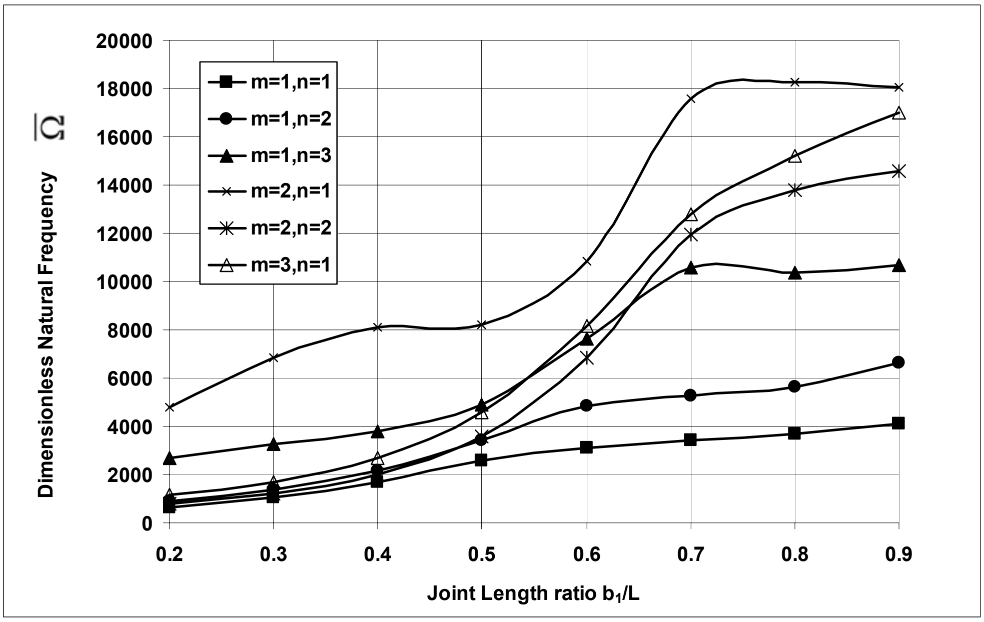
("Soft" Adhesive Case)

Fig.8.10 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

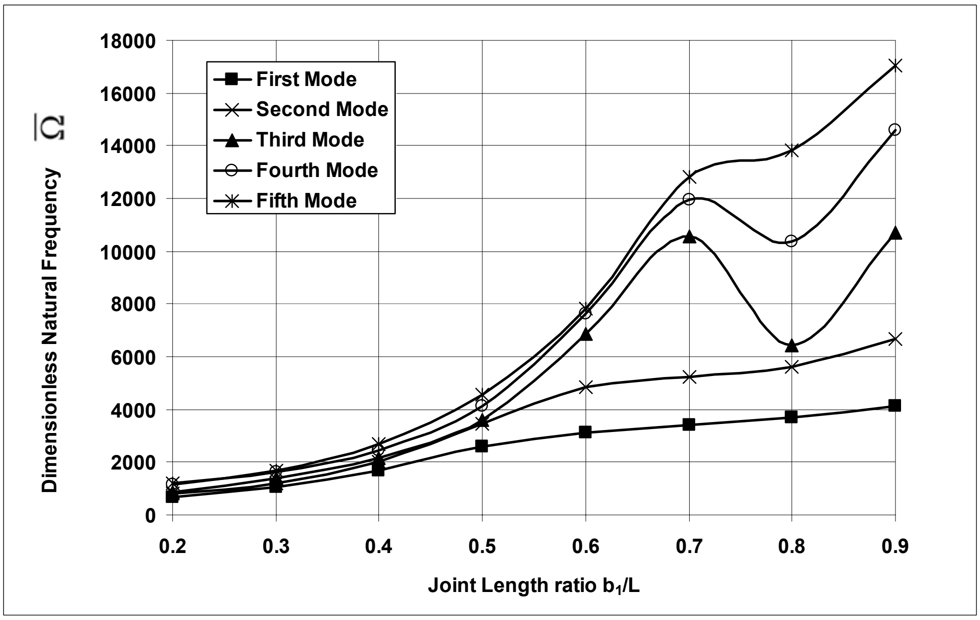
(Joint Length ℓ_l=0.3 m, b_1=0.65 m, b_2=0.65m, $\tilde{b}$ =0.5m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFF)

8.2.2. Some Parametric Studies for "Main PROBLEM Ia"



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, "Hard" Adhesive

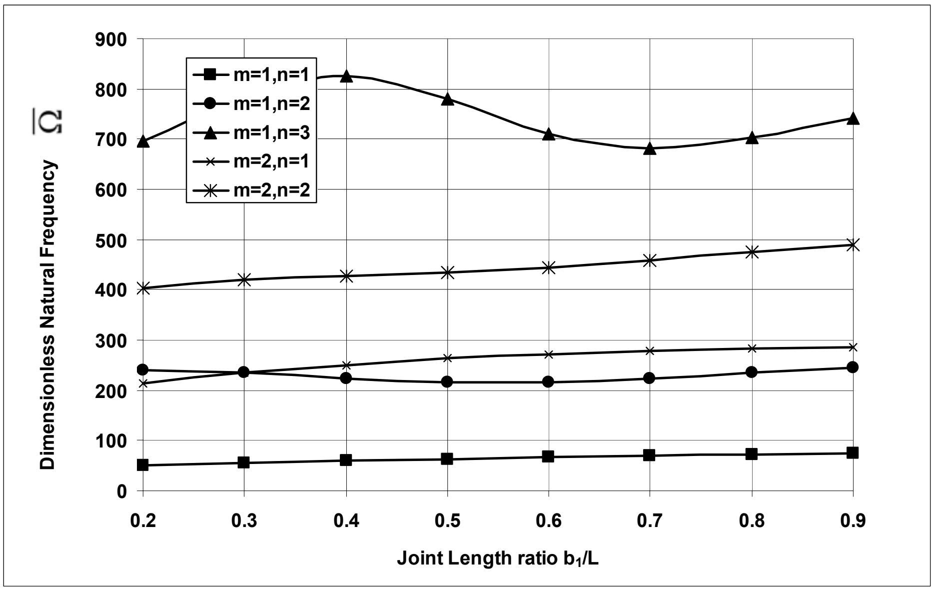
b)



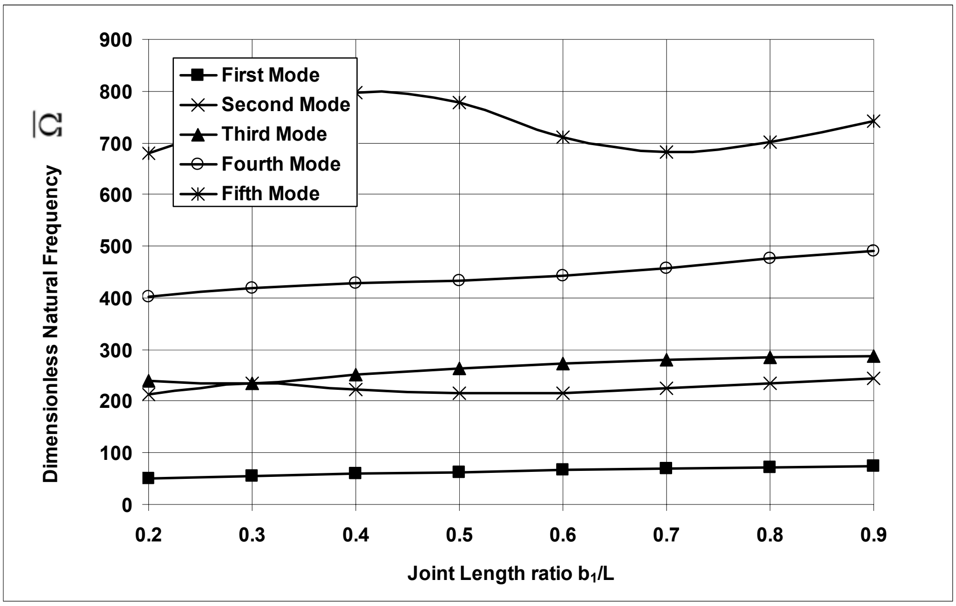
b) "Various Modes with (CFFC) B.C.'s, "Hard" Adhesive

Fig 8.11 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Joint Length ℓ_I/L" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =varies, $\tilde{b} = 0.5$, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



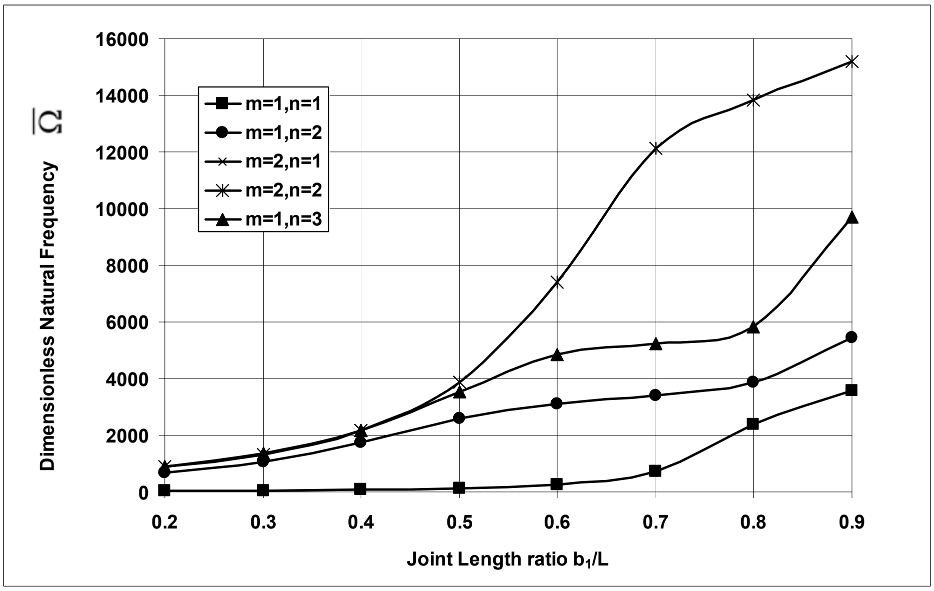
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, ''Soft'' Adhesive



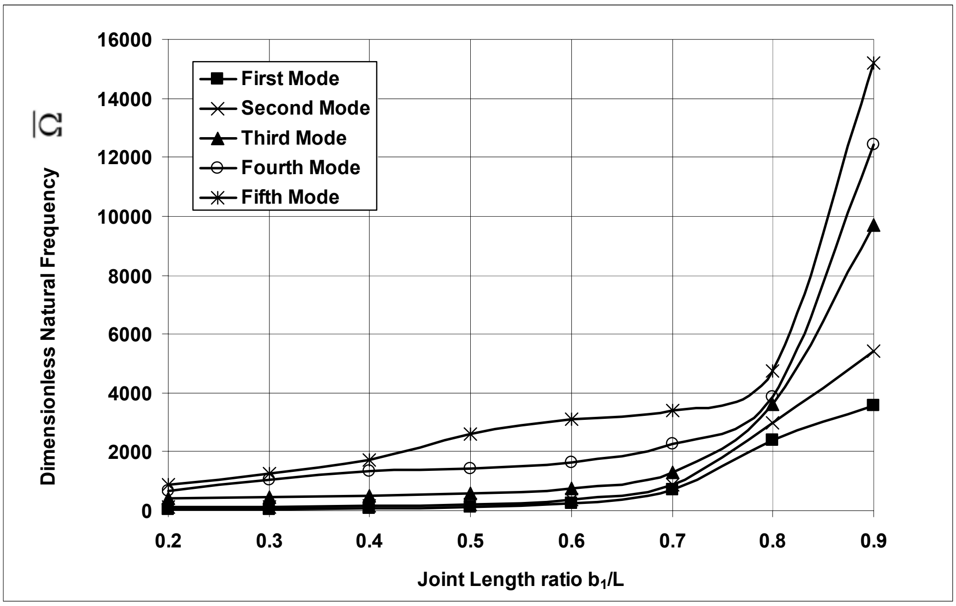
b) "Various Modes with (CFFC) B.C.'s, ''Soft'' Adhesive

Fig 8.12 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Joint Length ℓ_I /L" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =varies, $\tilde{b} = 0.5$, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



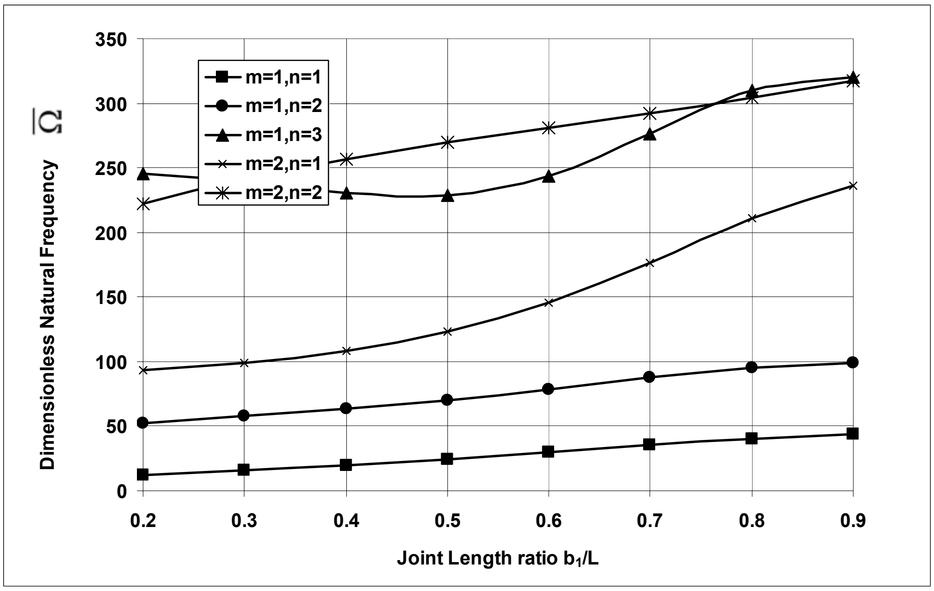
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, ''Hard'' Adhesive



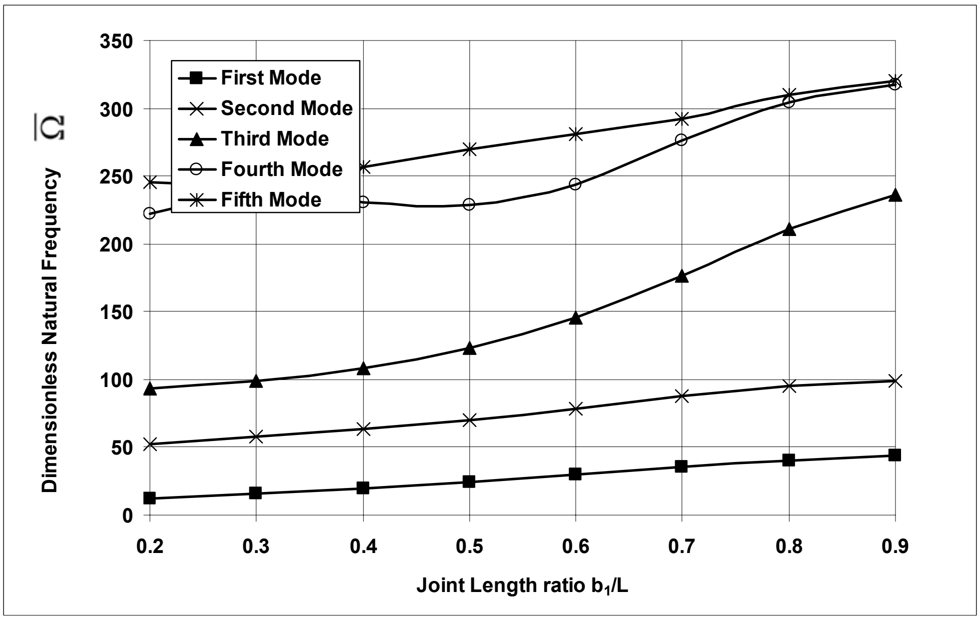
b) "Various Modes with (CFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.13 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Joint Length ℓ_I/L" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =varies, $\tilde{b} = 0.5$, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



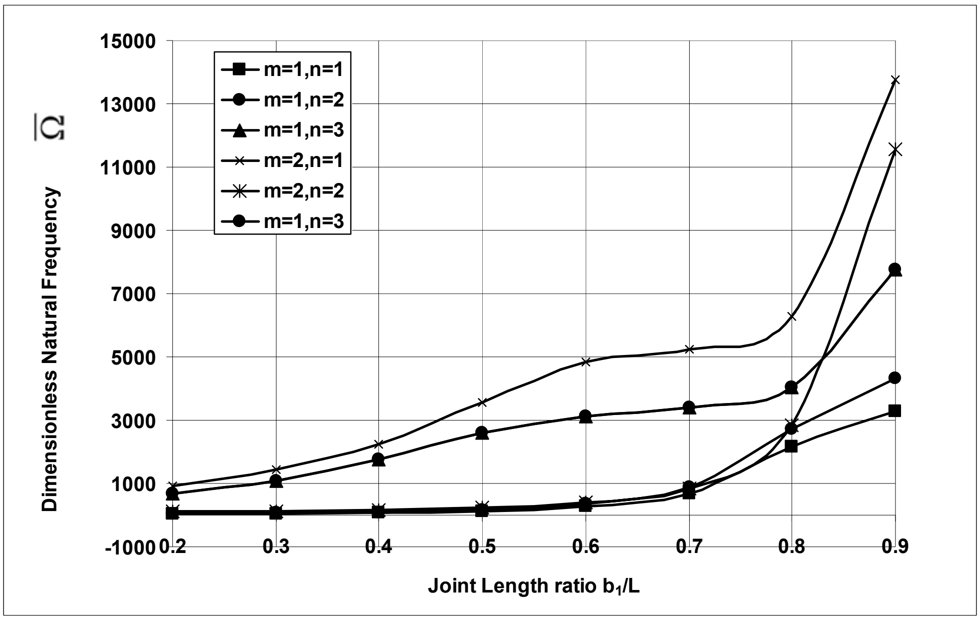
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, ''Soft'' Adhesive



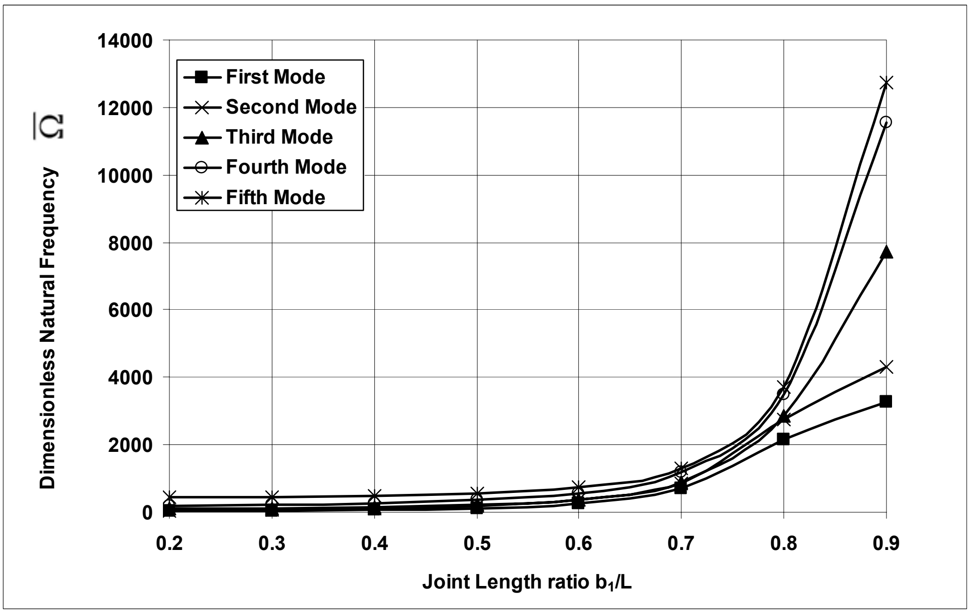
b) "Various Modes with (CFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.14 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Joint Length ℓ_I /L" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =varies, $\tilde{b} = 0.5$, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



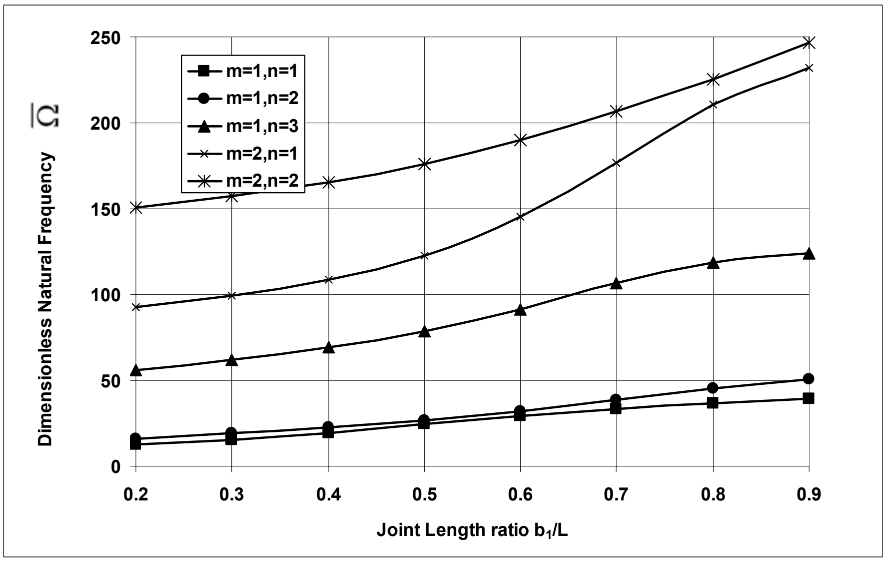
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, ''Hard' Adhesive



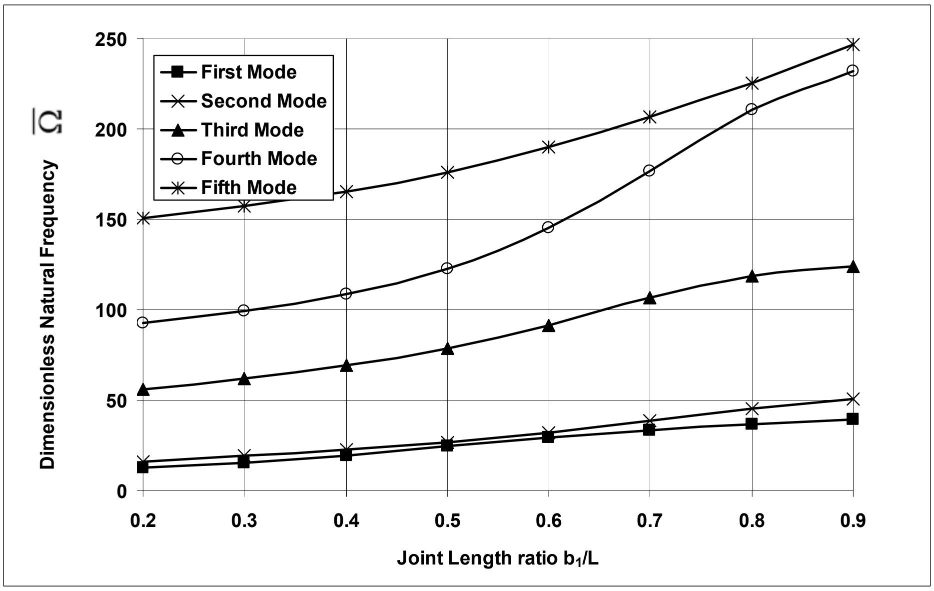
b) "Various Modes with (FFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.15 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Joint Length ℓ_I/L" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =varies, $\tilde{b} = 0.5$, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



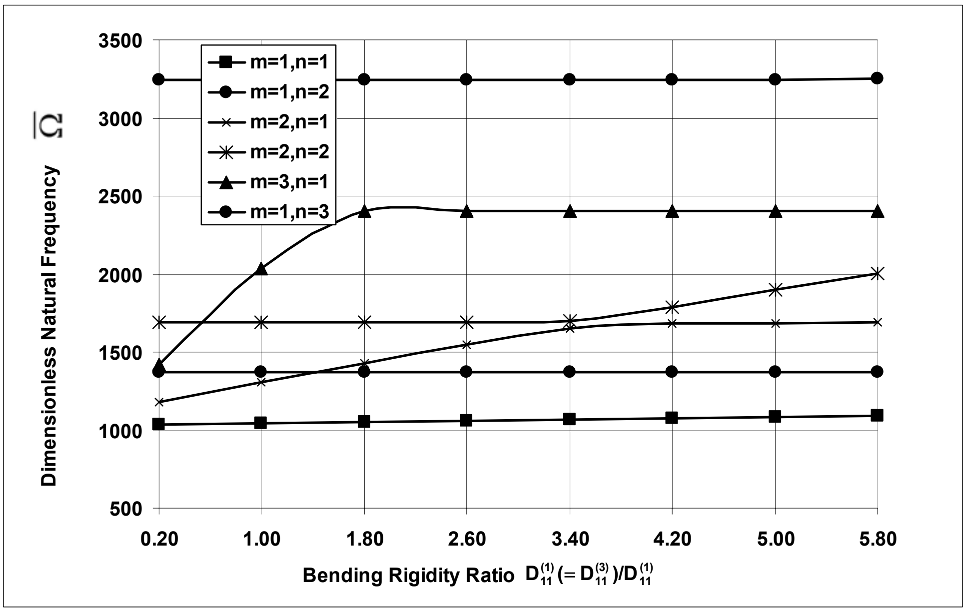
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, "Soft" Adhesive



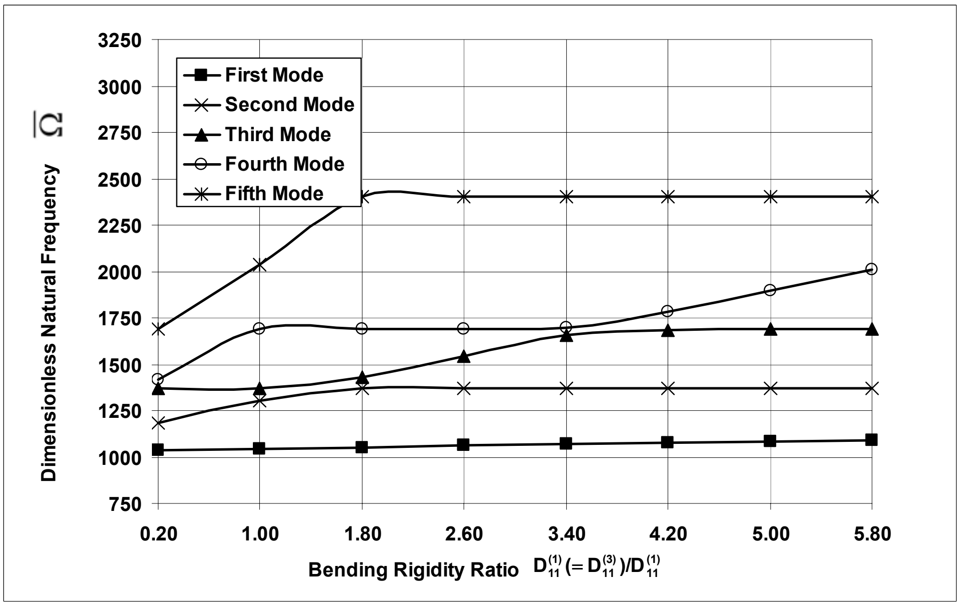
b) "Various Modes with (FFFF) B.C.'s, "Soft" Adhesive

Fig 8.16 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Joint Length ℓ_I/L" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =varies, $\tilde{b} = 0.5$, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



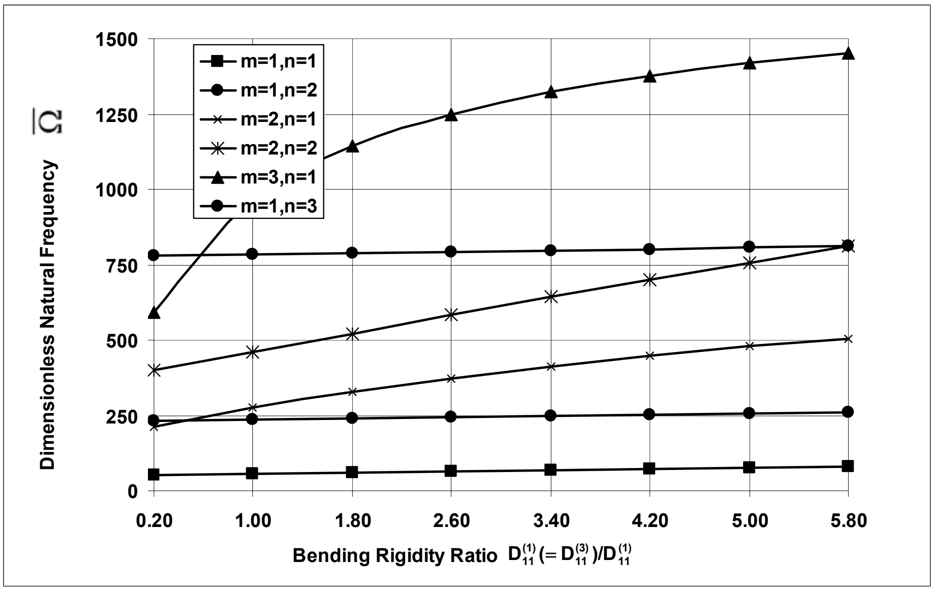
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, ''Hard'' Adhesive



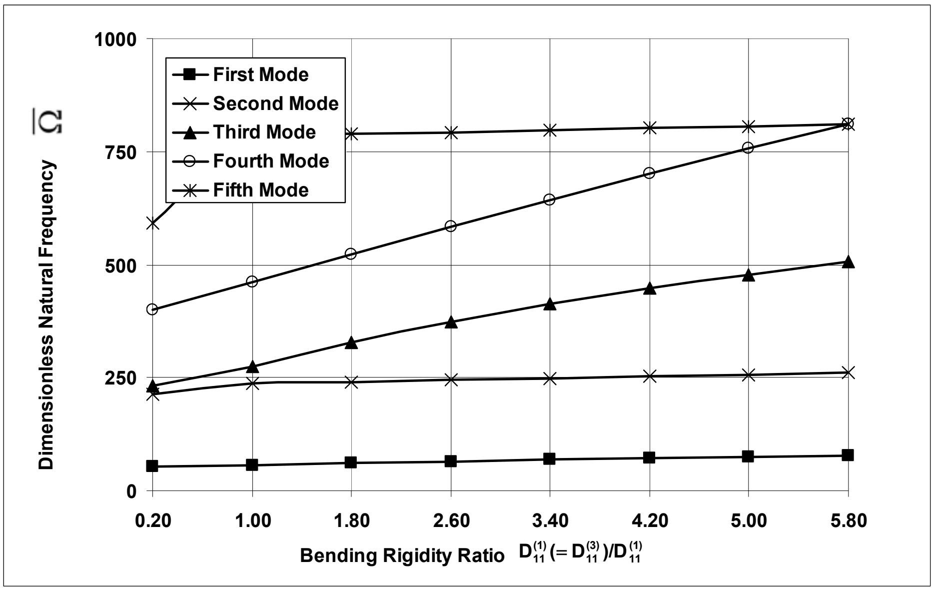
b) "Various Modes with (CFFC) B.C.'s, ''Hard'' Adhesive

Fig 8.17 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



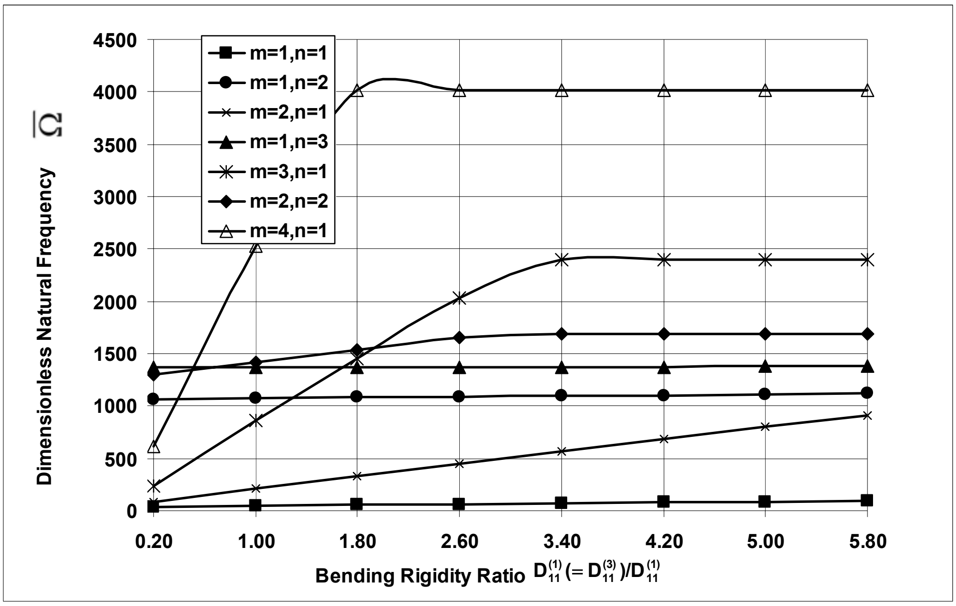
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, "Soft" Adhesive



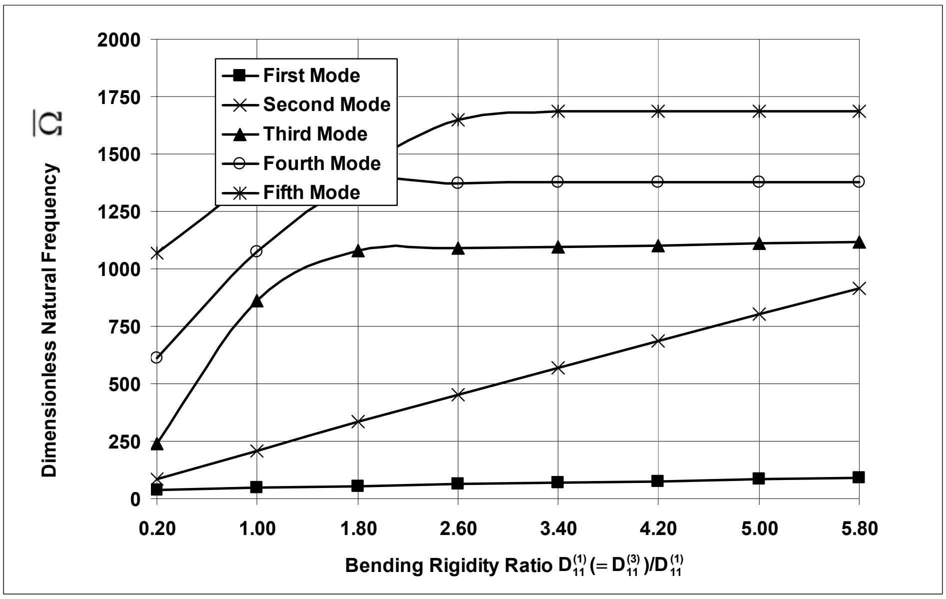
b) "Various Modes with (CFFC) B.C.'s, "Soft" Adhesive

Fig 8.18 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



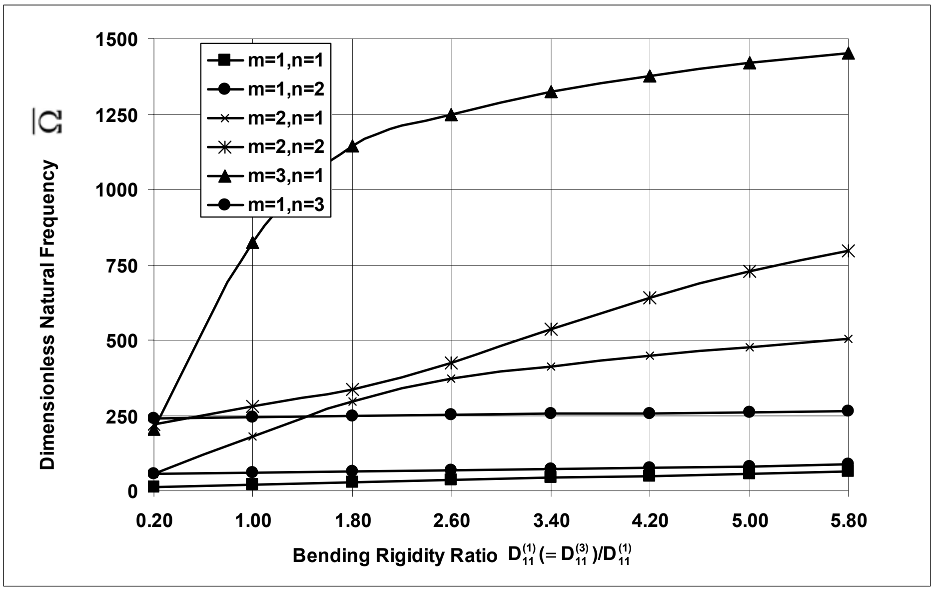
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, "Hard" Adhesive



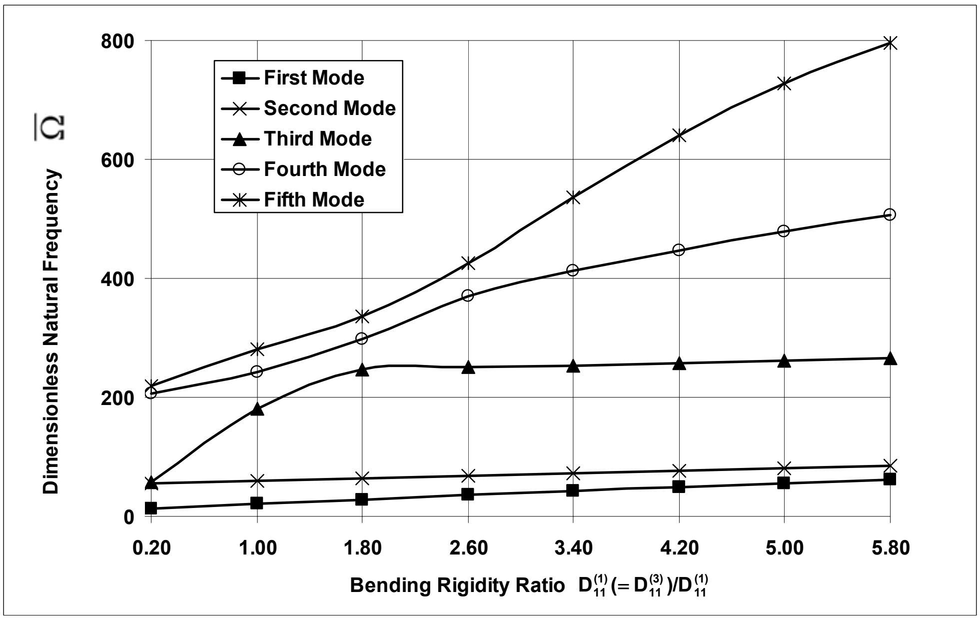
b) "Various Modes with (CFFF) B.C.'s, "Hard" Adhesive

Fig 8.19 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness characteristics are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, ''Soft'' Adhesive



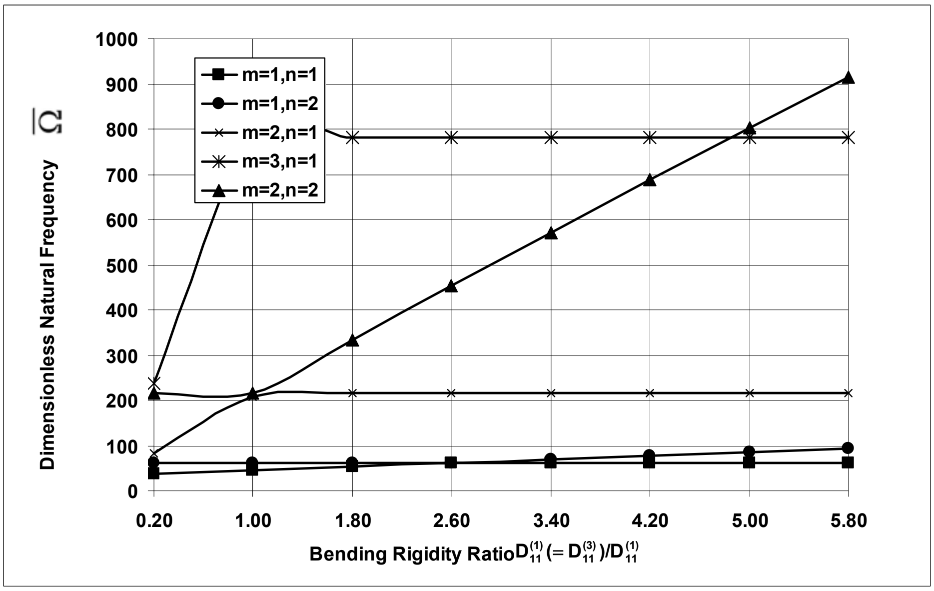
b) "Various Modes with (CFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.20 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

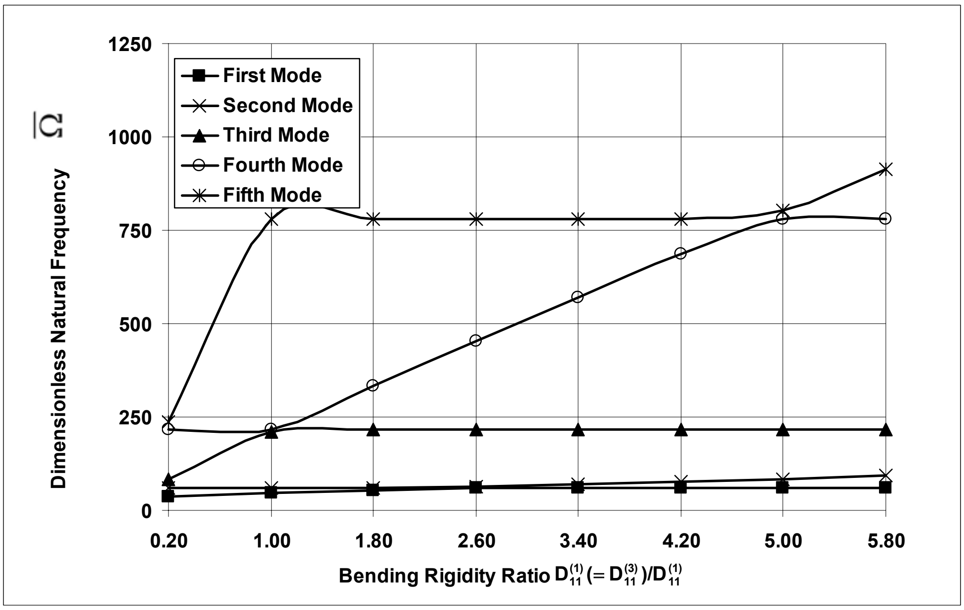
($D_{11}^{(2)}$ increases while other stiffness characteristics are kept constant)

(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)

(Boundary Conditions in y-direction CFFF)



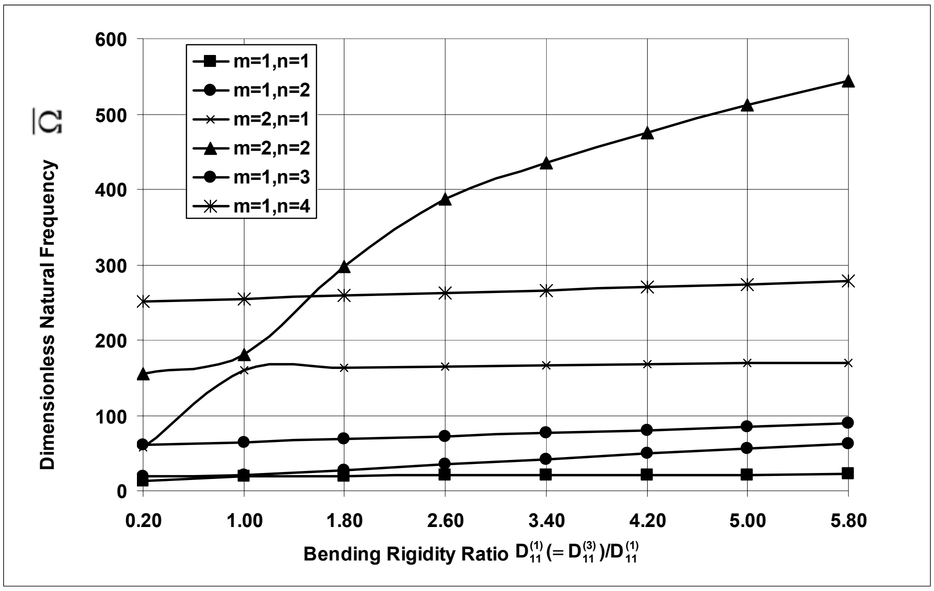
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



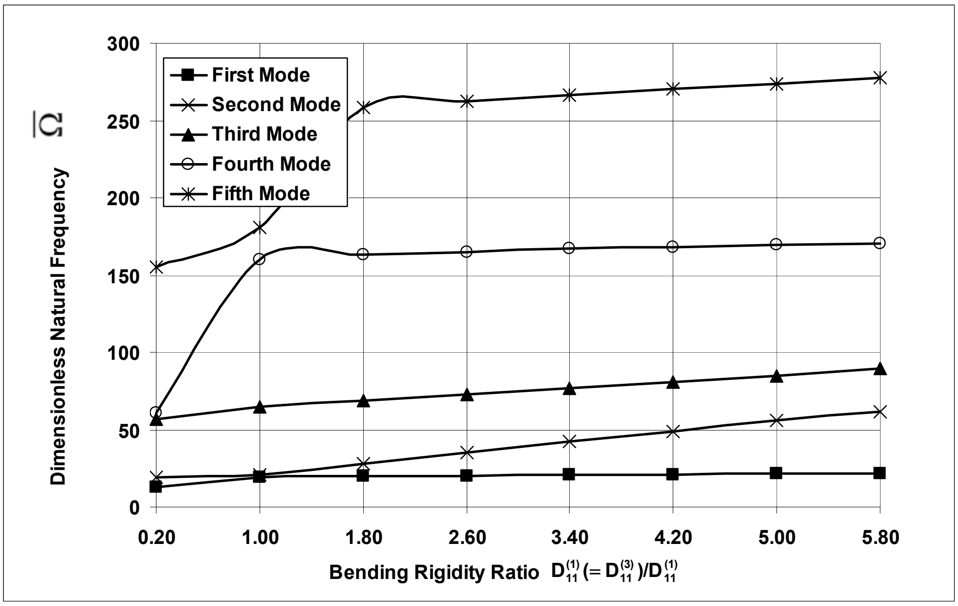
b) "Various Modes with (FFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.21 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Centrally</u> Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness characteristics are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



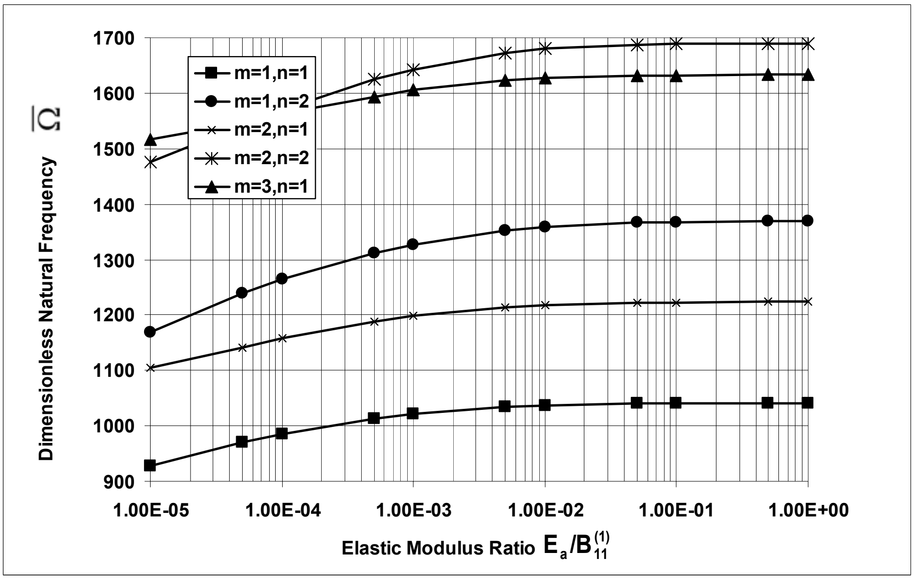
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, ''Soft'' Adhesive



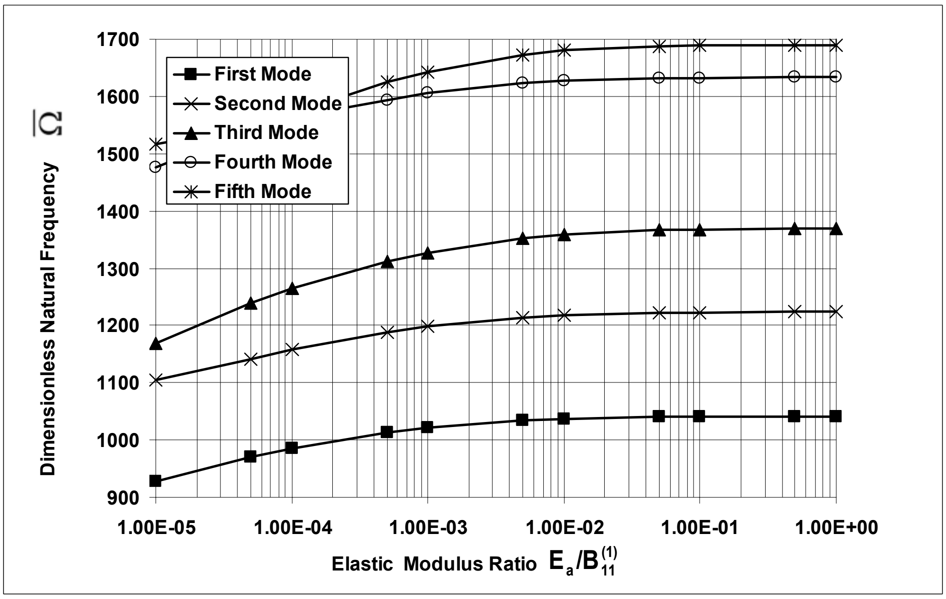
b) "Various Modes with (FFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.22 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness characteristics are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



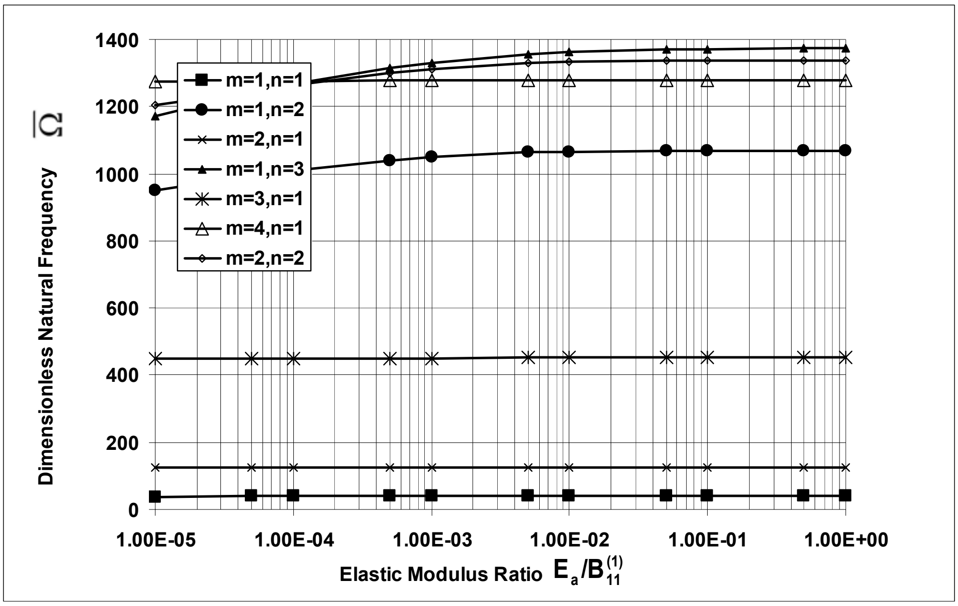
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s



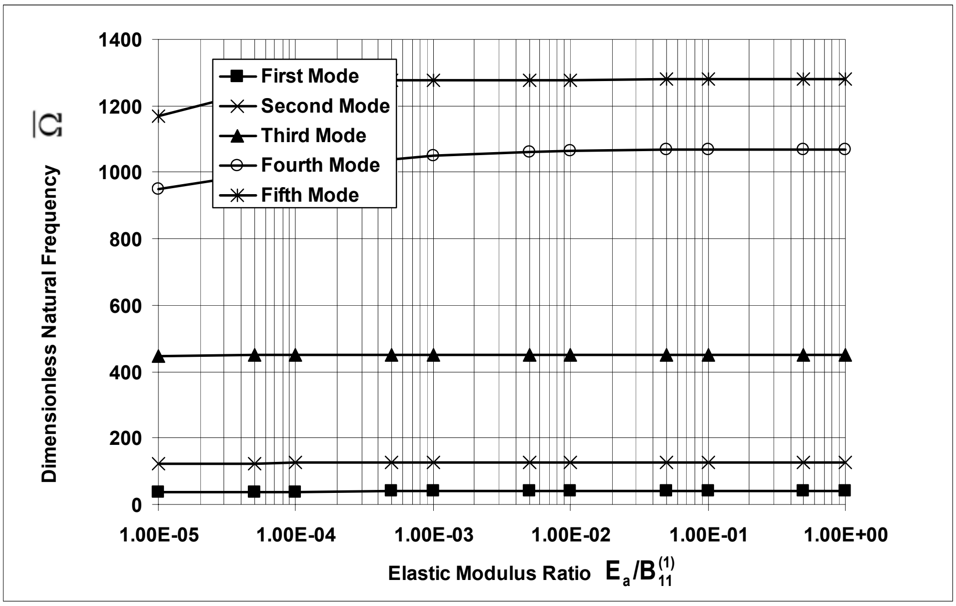
b) "Various Modes with (CFFC) B.C.'s

Fig 8.23 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)
Elastic Modulus Ratio axis is plotted in Log Scale



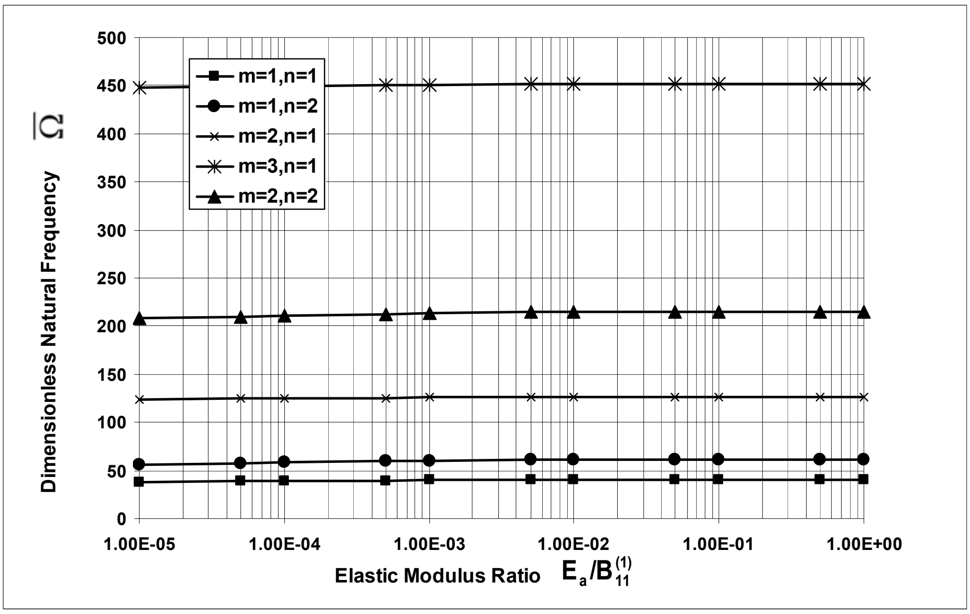
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s



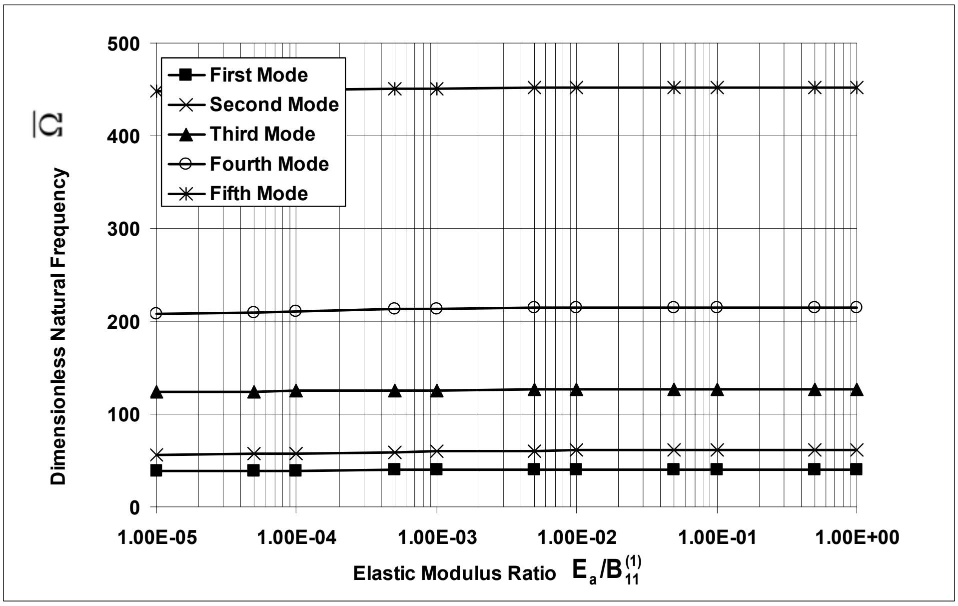
b) "Various Modes with (CFFF) B.C.'s

Fig 8.24 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



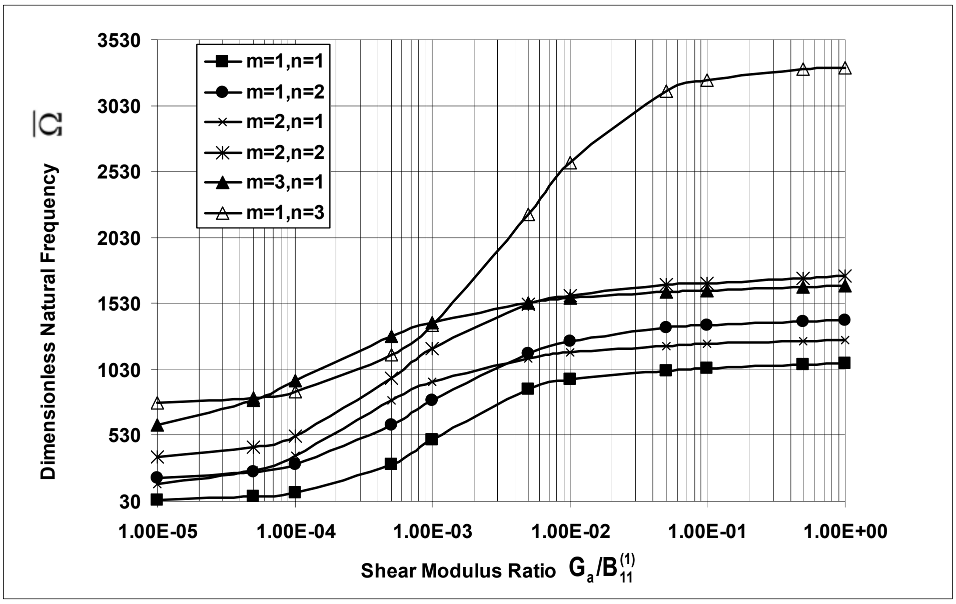
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s



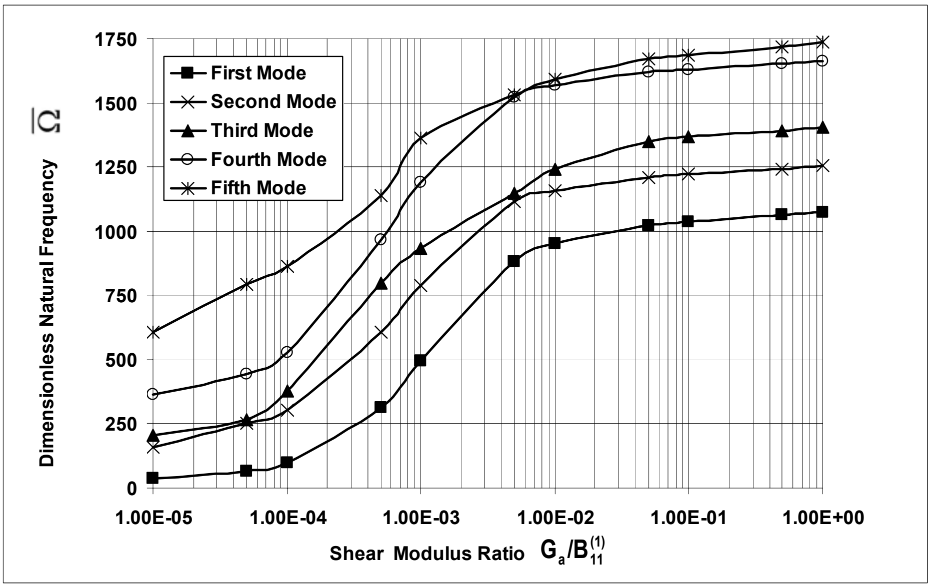
b) "Various Modes with (FFFF) B.C.'s

Fig 8.25 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



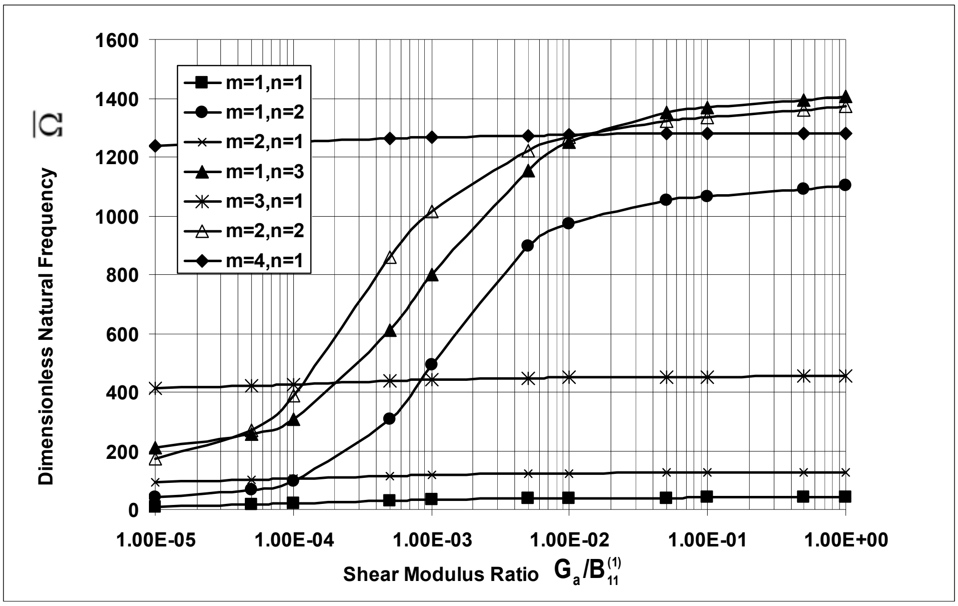
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s



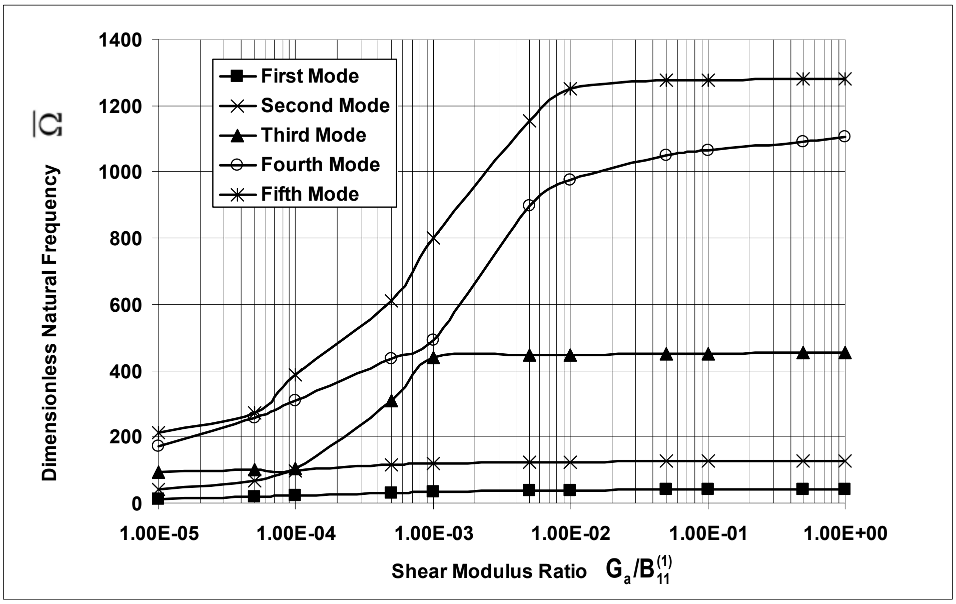
b) "Various Modes with (CFFC) B.C.'s

Fig 8.26 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$ " of "Composite, Orthotropic Plates and/or Panels with <u>Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)
Shear Modulus Ratio axis is plotted in Log Scale



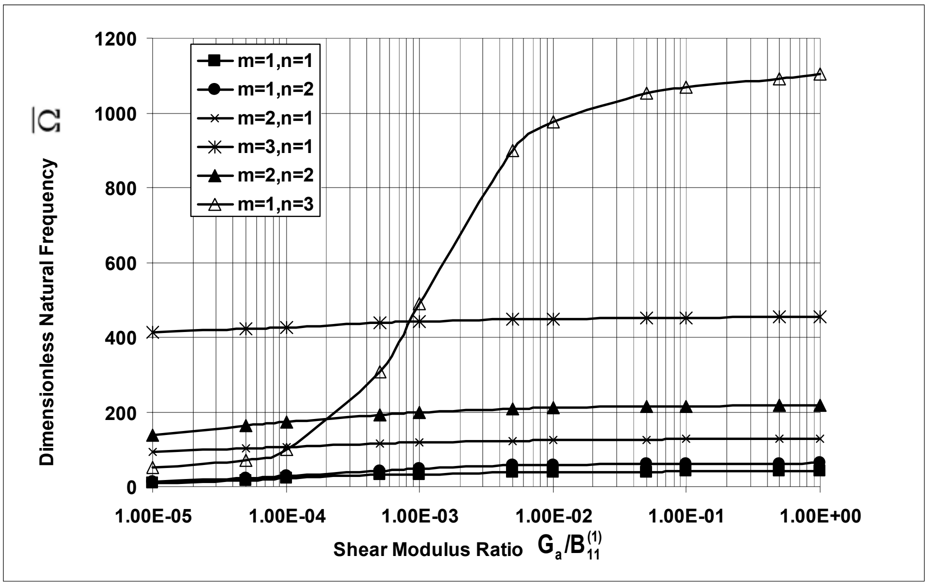
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s



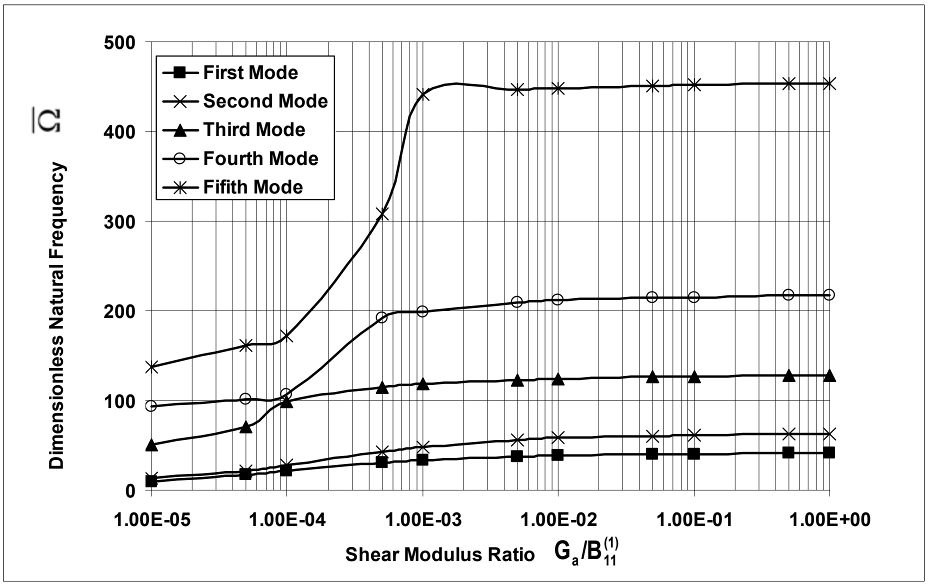
b) "Various Modes with (CFFF) B.C.'s

Fig 8.27 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" of "Composite, Orthotropic Plates and/or Panels with <u>Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)
Shear Modulus Ratio axis is plotted in Log Scale



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s



b) "Various Modes with (FFFF) B.C.'s

Fig 8.28 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$ " of "Composite, Orthotropic Plates and/or Panels with Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.5 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF
Shear Modulus Ratio axis is plotted in Log Scale

8.2.3. Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequencies"

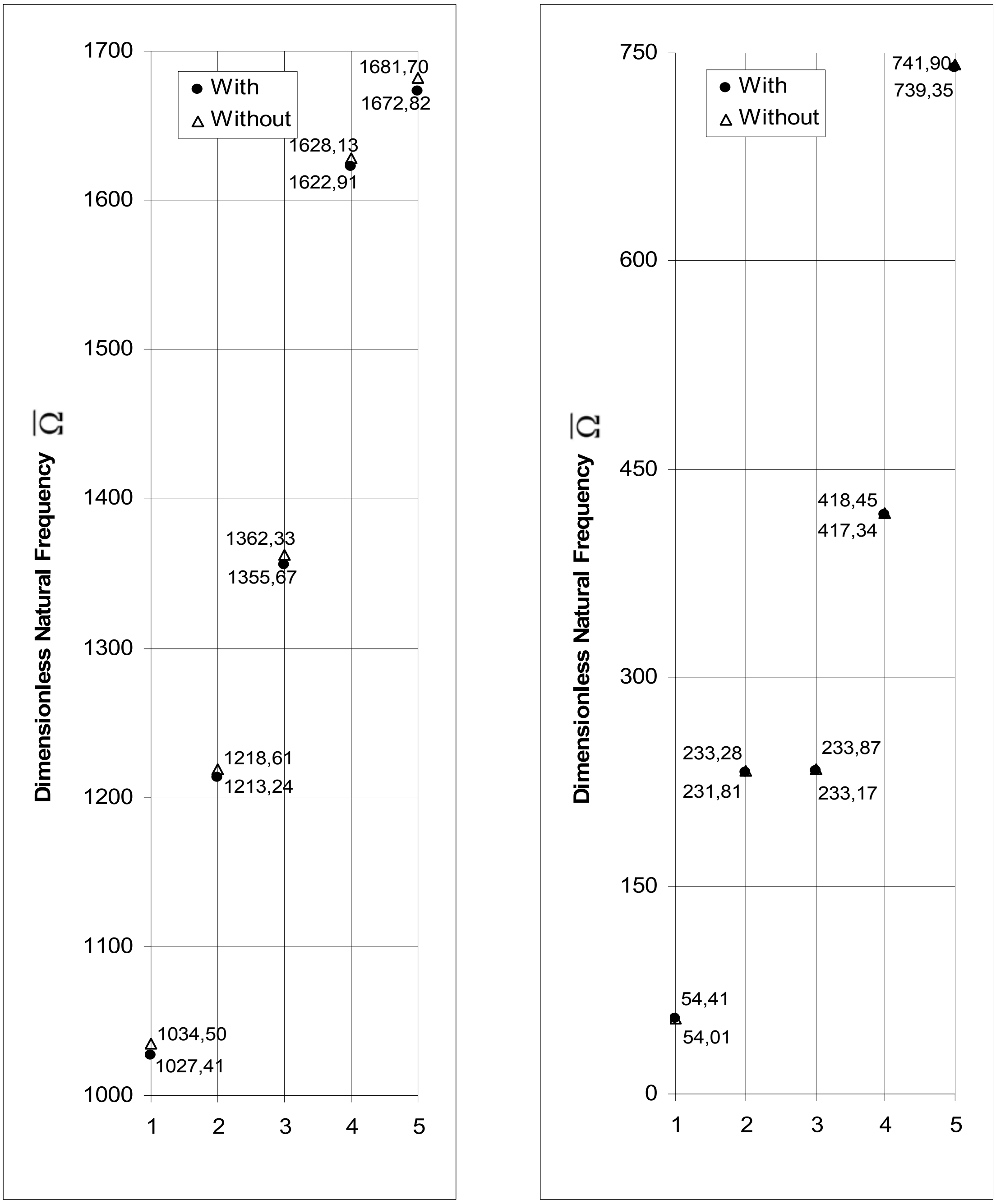
Table 8.4 Comparison of "Dimensionless Natural Frequencies" obtained by adding $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms to adhesive layer equations for "Main PROBLEM Ia"

a) "Hard" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	1034.499	1027.406	7.093
	2	1218.612	1213.235	5.377
	3	1362.327	1355.670	6.657
	4	1628.133	1622.912	5.221
	5	1681.695	1672.817	8.878
SFFS	1	547.414	544.104	3.310
	2	695.704	692.859	2.845
	3	737.971	733.913	4.058
	4	998.561	993.558	5.003
	5	1101.310	1098.416	2.894
CFFF	1	40.232	40.084	0.148
	2	126.465	126.280	0.185
	3	451.613	451.374	0.239
	4	1062.629	1054.832	7.797
	5	1278.883	1278.577	0.306
FFFF	1	40.201	40.035	0.166
	2	60.808	60.490	0.318
	3	126.457	126.268	0.189
	4	214.881	214.437	0.444
	5	451.611	451.370	0.241

b) "Soft" Adhesive Case

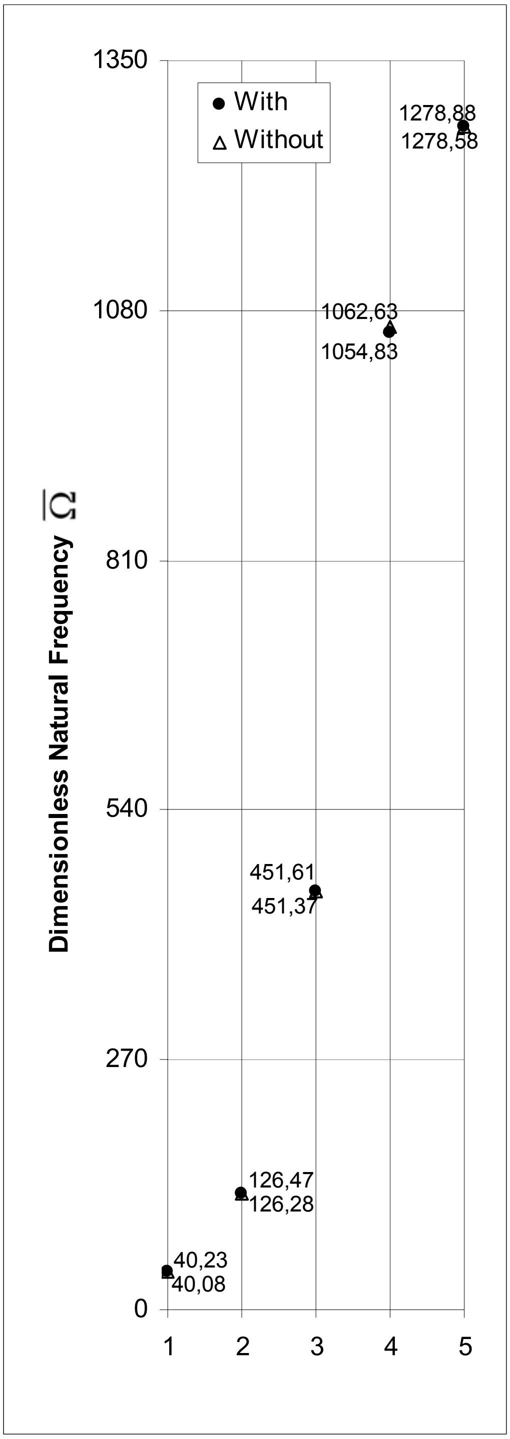
Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	54.410	54.007	0.403
	2	233.281	231.807	1.474
	3	233.871	233.171	0.700
	4	418.450	417.335	1.115
	5	741.900	739.349	2.551
SFFS	1	36.724	36.374	0.350
	2	128.471	127.987	0.484
	3	201.702	200.521	1.181
	4	297.005	296.208	0.797
	5	449.189	448.798	0.391
CFFF	1	15.502	15.423	0.079
	2	57.047	56.689	0.358
	3	99.076	98.994	0.082
	4	239.235	238.560	0.675
	5	240.109	238.649	1.460
FFFF	1	15.317	15.24	0.077
	2	18.911	18.801	0.11
	3	61.929	61.62	0.309
	4	99.014	98.932	0.082
	5	157.258	157.029	0.229



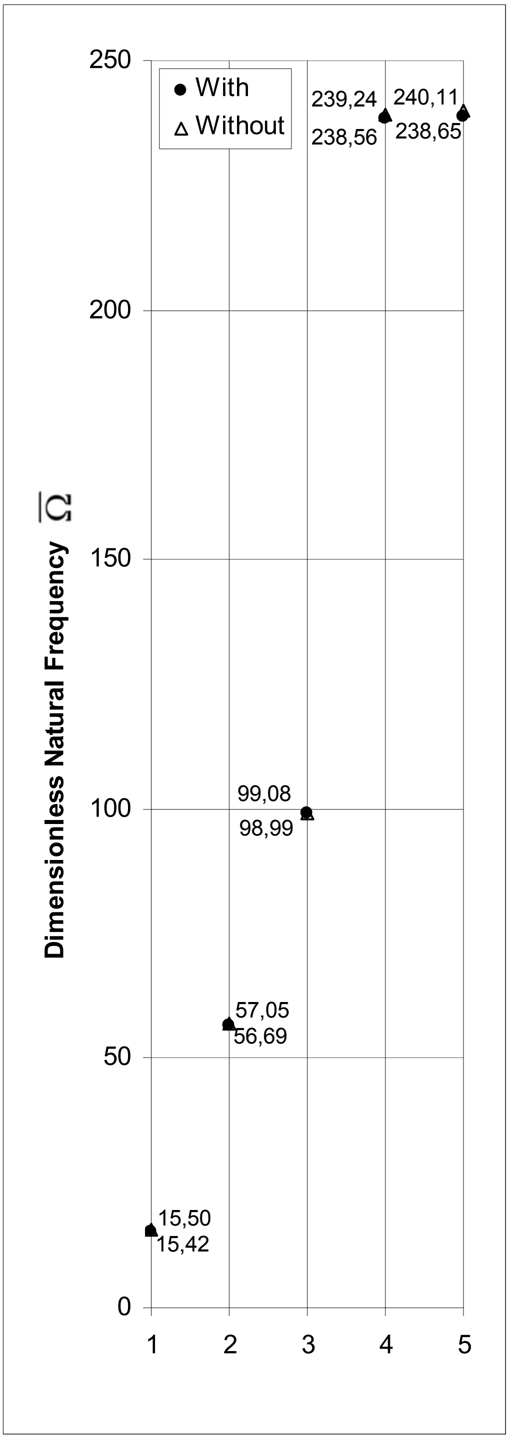
a) "Hard" Adhesive Case **b) "Soft" Adhesive Case**

Figure 8.29 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Single Lap Joint"

(Boundary Conditions in y-direction CFFC)



a) “Hard” Adhesive Case



b) “Soft” Adhesive Case

Figure 8.30 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on “Dimensionless Natural Frequency $\overline{\Omega}$ ” in “Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Single Lap Joint”

(Boundary Conditions in y-direction CFFF)

8.3 Numerical Results and Discussion for "Main PROBLEM I.b"

In the "Main PROBLEM Ib.", the "Composite Orthotropic Plates and/or Panels with a Non-Centrally Bonded Single Lap Joint" is analyzed. The upper adherend is made of Graphite-Epoxy and the lower plate adherend is Kevlar-Epoxy. For the in-between adhesive layer, the "hard" and the "soft" adhesive cases are taken into account. The "Geometric and the Material Characteristics" of the single lap joint system are given in Table 8.1.

In Figures 8.29 – 8.38, the mode shapes and the corresponding natural frequencies (from the first to fifth), in the "hard" and the subsequent "soft" adhesive cases with various boundary conditions are presented.

From aforementioned Figures, in the "hard" adhesive case it is easy to observe that, regardless of the boundary conditions, there exists an almost "stationary region" in the mode shapes. And this region moves from left to the right part (or vice versa) in the composite single lap joint system. In the "soft" adhesive case, however, an almost "stationary region" does not exist in mode shapes. The general trend in the mode shapes, for the "soft" adhesive case is that, the "Overlap Region" moves or bends with the rest of the lap joint system. And the mode shapes are completely different in comparison with those of the "hard" adhesive cases with the same support conditions.

Next, for the "Main PROBLEM Ib", in Figures 8.39 through 8.56, several important parametric studies are presented. In Figures 8.39-8.44, the "Dimensionless Natural Frequency $\overline{\Omega}$" versus "Position Ratio $\tilde{b}/L$" from the first up to the fifth mode are plotted, for both the "hard" and "soft" adhesive cases, corresponding to the various support conditions.

From Figures 8.39, 8.41 8.43, in the "hard" adhesive case, it is obvious that as position of the "Overlap Region" changes (in the y-direction), the natural frequencies gradually increase up to a certain position and then decreases.

These results are consequences of the movement of the half waves from left to right because of the change in the position of the “Overlap Region”.

In the “soft” adhesive case, in Figures 8.40, 8.42, 8.44, the natural frequencies increase with the position of the “Overlap Region”. This also can be expected due to the “soft” adhesive which makes the system loose and which shows a similar behavior in mode shapes up to $\tilde{b}$ =0.5m.

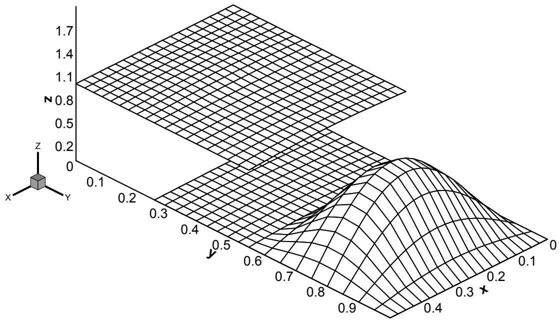
In Figures 8.45 through 8.50, the effect of the “Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$” on the natural frequencies (from the first up to the fifth) in the “hard” and “soft” adhesive cases, are investigated for various boundary conditions. In the “hard” adhesive case, in Figures 8.45, 8.47, 8.49, the first natural frequency, in spite of the increasing “Bending Rigidity Ratio”, does remain practically constant. In the third and higher modes, the natural frequencies increase sharply at first and after the “Bending Rigidity Ratio=2.6” they become almost flat or constant regardless of the increase in “Bending Rigidity Ratio”.

In the “soft” adhesive cases, in the Figures 8.46, 8.48, 8.50, the first and the second frequencies remain more or less constant as the “Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$” increases. In the third and higher modes, the natural frequencies increase significantly. In some cases, though, in the fifth frequency, after “Bending Rigidity Ratio=1.8”, the fifth frequency reaches a constant value.

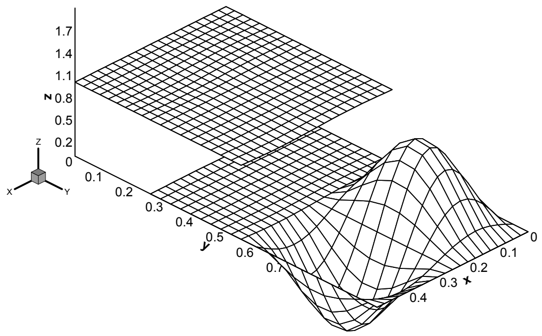
Lastly, the direct effects of the adhesive layer elastic constants E_a, and also G_a on the dimensionless natural frequencies are investigated for the “Main PROBLEM I.b”. In order to show these effects, the “Dimensionless Natural Frequencies” versus the “Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$” are plotted (while the other elastic <u>constant kept constant</u>) in Figures 8.51 through 8.53 for various boundary condition. Similarly, the “Dimensionless Natural Frequencies” versus the “Adhesive Shear Modulus Ratio $G_a/B_{11}^{(1)}$” are presented in Figures 8.54 through 8.56 for various support condition.

It can be seen from the Figures 8.51, 8.52, 8.53, the influence of the "Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" on the natural frequencies, is not significant. In Figures 8.54, 8.55, 8.56, we can see that the "Shear Modulus Ratio $G_a/B_{11}^{(1)}$", after the first and second modes, in higher modes, significantly affects the natural frequencies. Also, in those Figures, one can observe a "transition region" which takes the frequencies to considerable higher levels. After then, the frequencies don't increase and they remain practically constant.

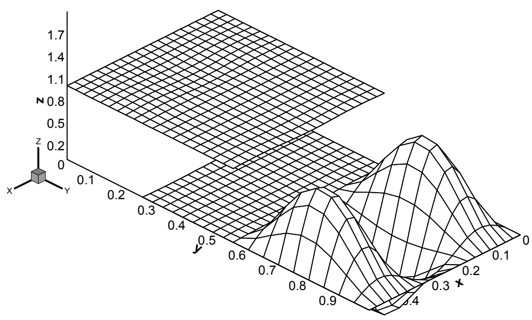
8.3.1 Natural Frequencies and Corresponding Mode Shapes for "Main PROBLEM Ib"



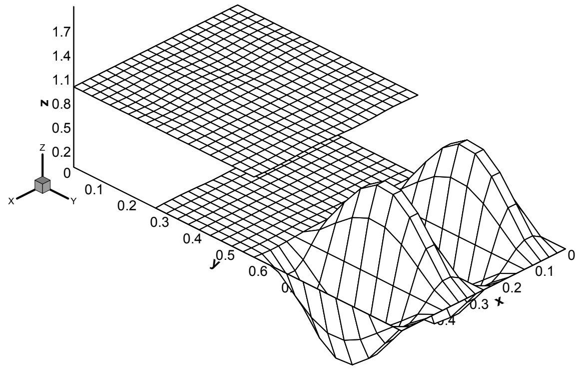
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=433.304



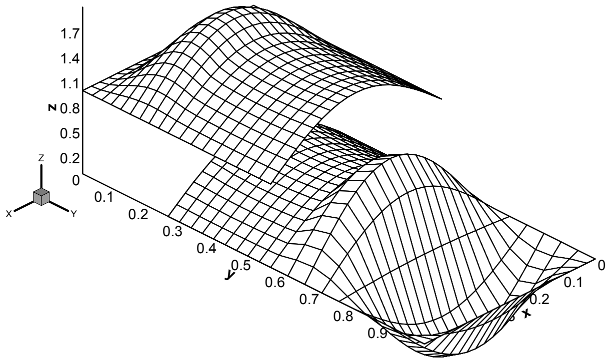
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=549.786



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=919.501



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41} = 1800.220$



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12} = 2314.154$

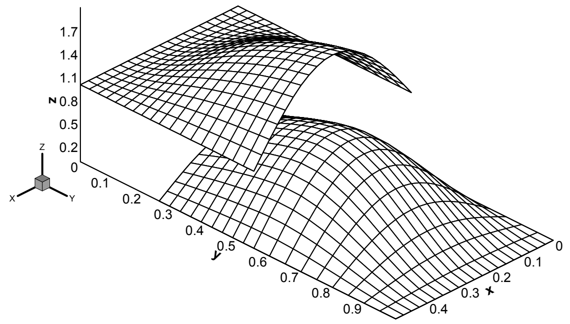
("Hard" Adhesive Case)

Fig.8.31 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

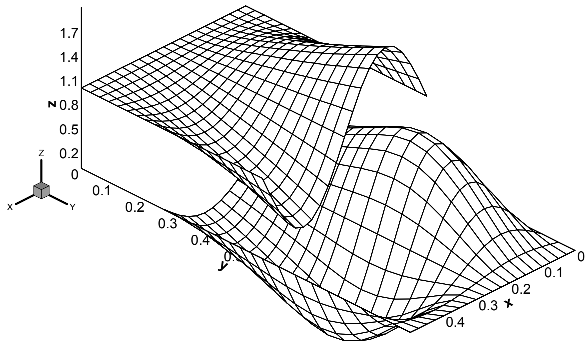
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_j=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b} = 0.4$, a=0.5 m. L=1 m)

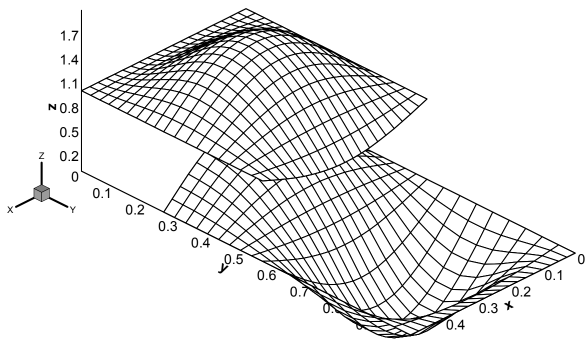
(Boundary Conditions in y-direction CFFC)



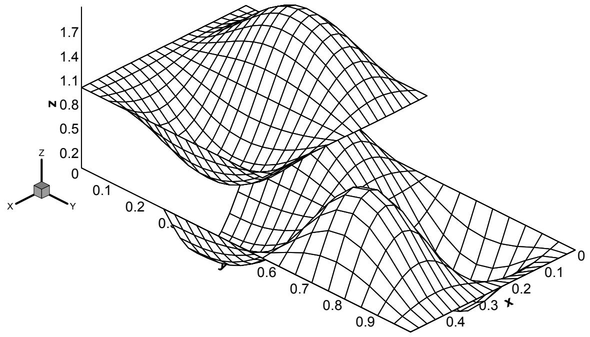
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=52.953



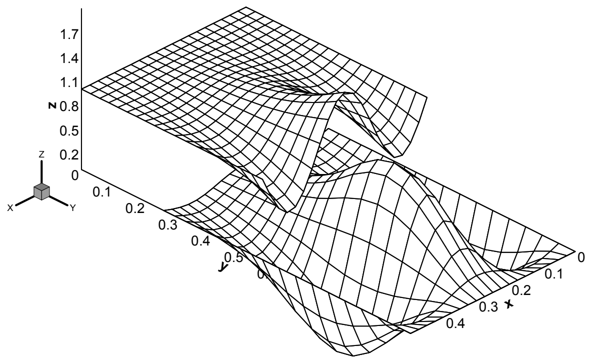
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=208.986



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{12}$=222.164



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=392.751



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=627.854

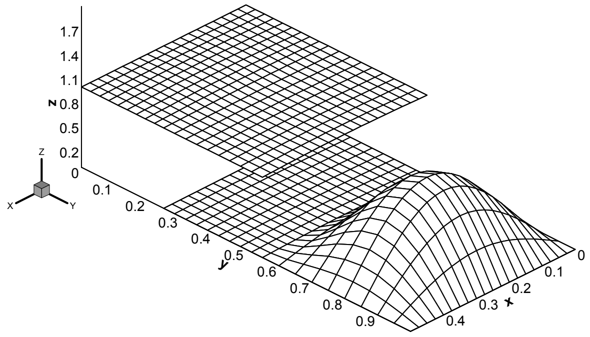
("Soft" Adhesive Case)

Fig.8.32 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

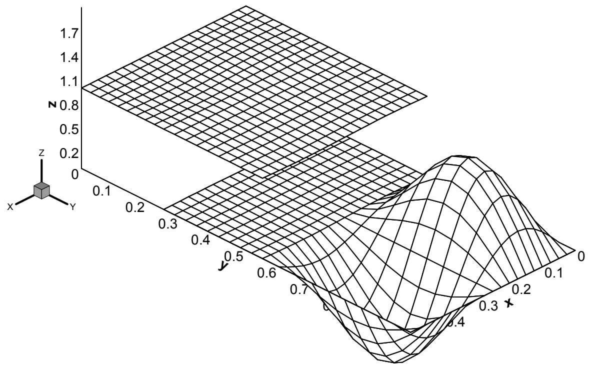
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b} = 0.4$m, a=0.5 m. L=1 m)

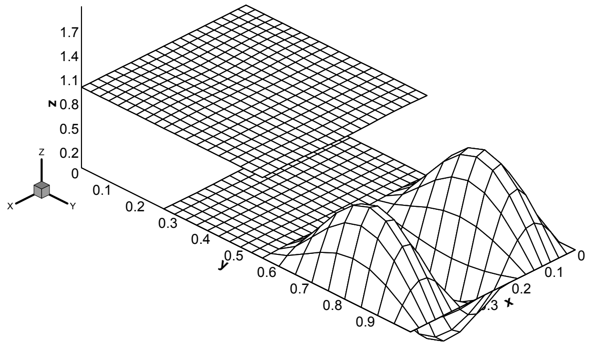
(Boundary Conditions in y-direction CFFC)



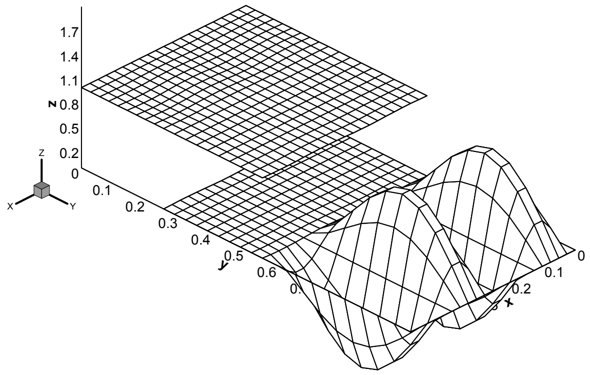
a) **First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=224.047**



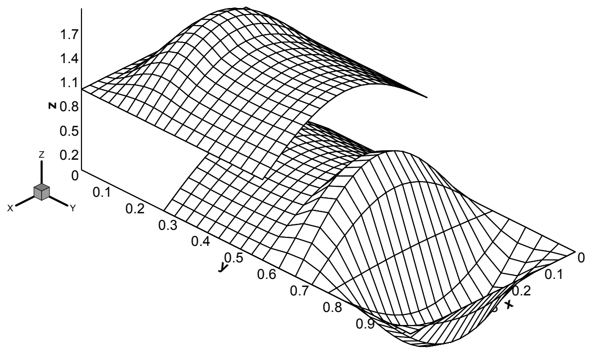
b) **Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=335.817**



c) **Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=700.731**



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41}$=1582.946



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12}$=1743.746

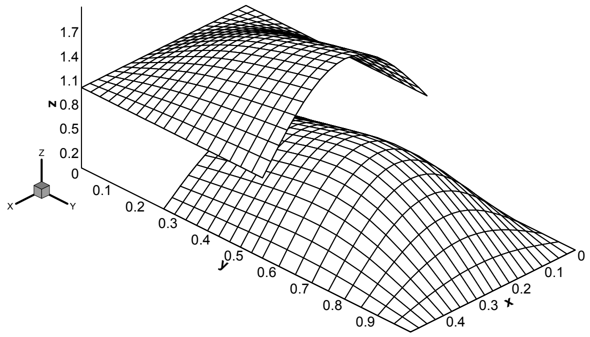
("Hard" Adhesive Case)

Fig.8.33 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

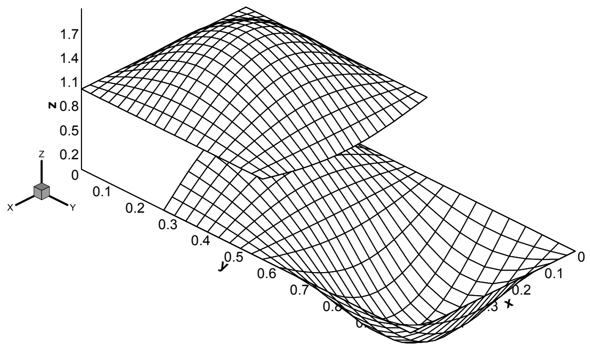
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b} = 0.4$m, a=0.5 m. L=1 m)

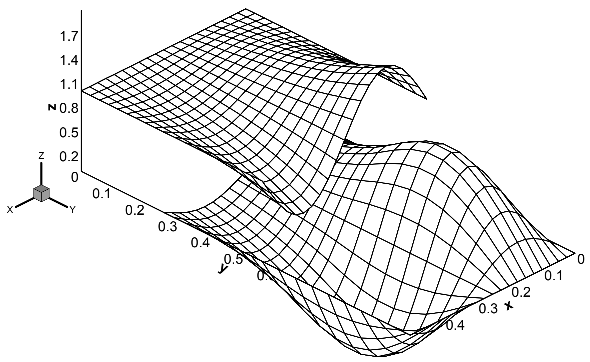
(Boundary Conditions in y-direction SFFS)



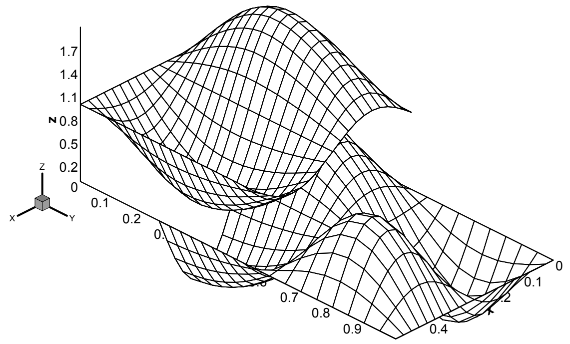
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=34.595



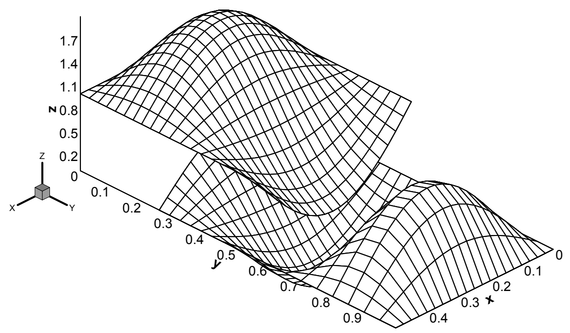
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}$=118.174



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{21}$=170.651



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=290.486



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=400.059

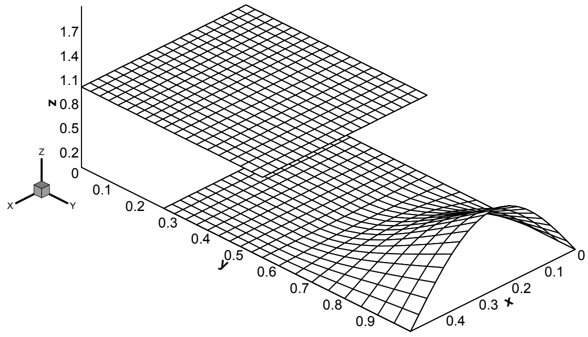
("<u>Soft</u>" Adhesive Case)

Fig.8.34 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Non-Central</u> Single Lap Joint"

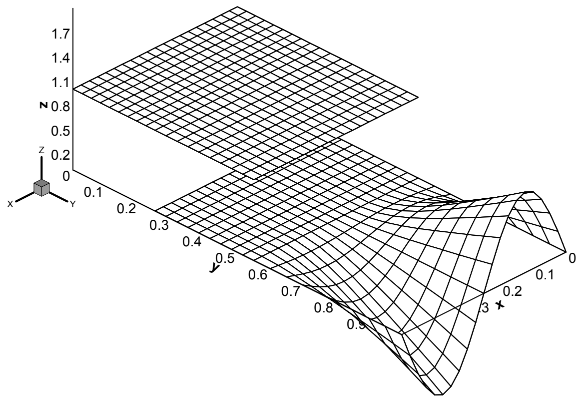
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b}=$0.4m, a=0.5 m. L=1 m)

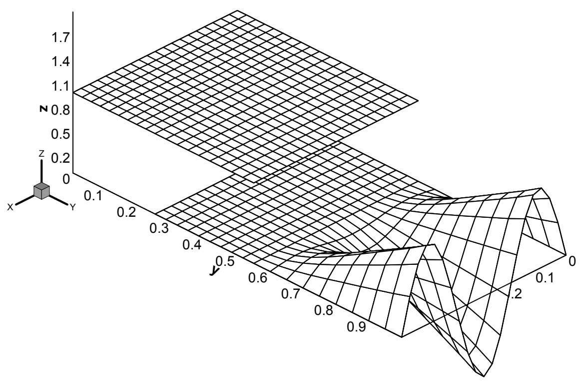
(Boundary Conditions in y-direction SFFS)



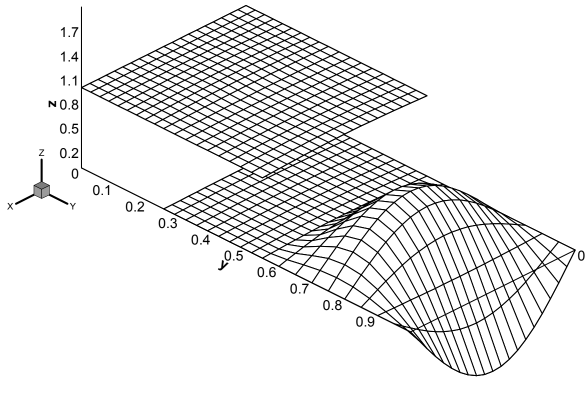
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 19.441$



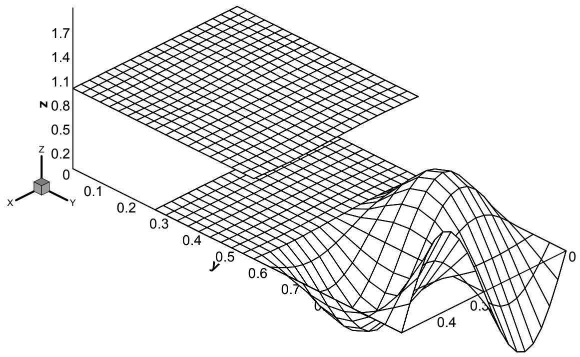
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 99.265$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 414.284$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12} = 447.394$



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22} = 612.756$

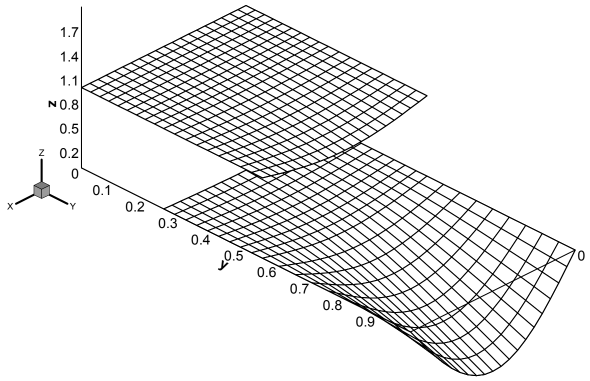
("Hard" Adhesive Case)

Fig.8.35 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

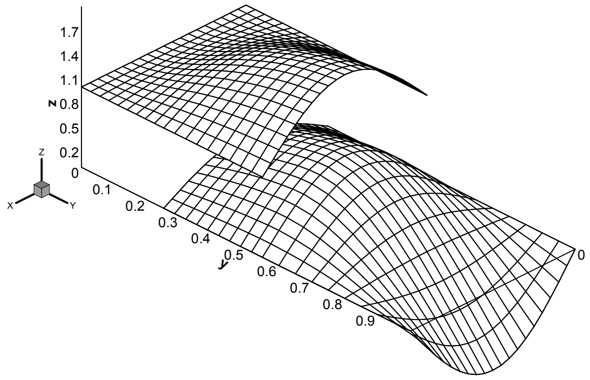
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b} = 0.4$m, a=0.5 m. L=1 m)

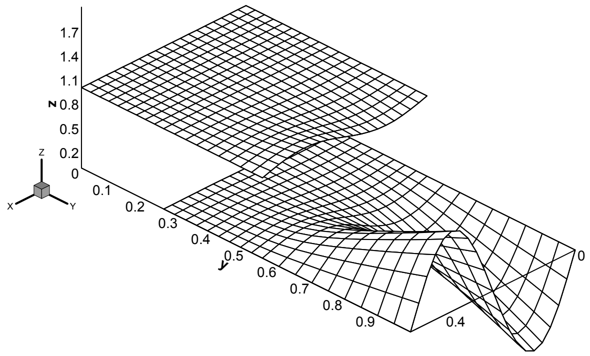
(Boundary Conditions in y-direction CFFF)



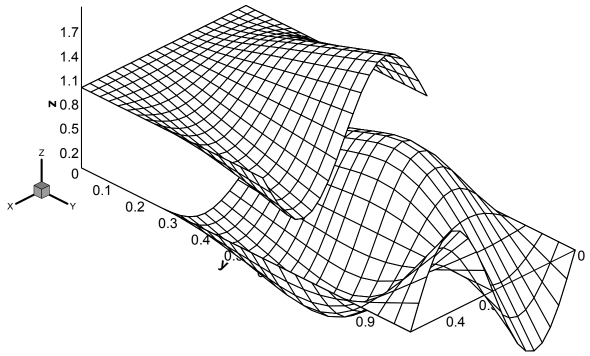
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 10.958$



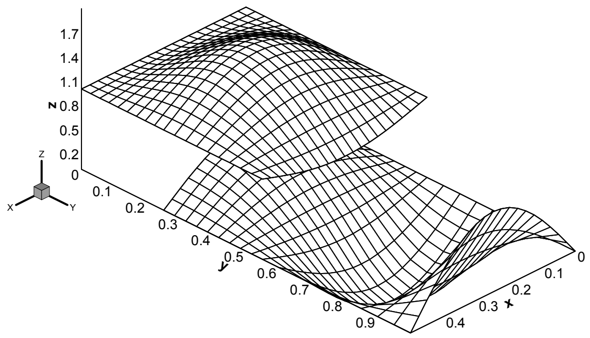
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 54.475$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 88.852$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=221.455



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=228.393

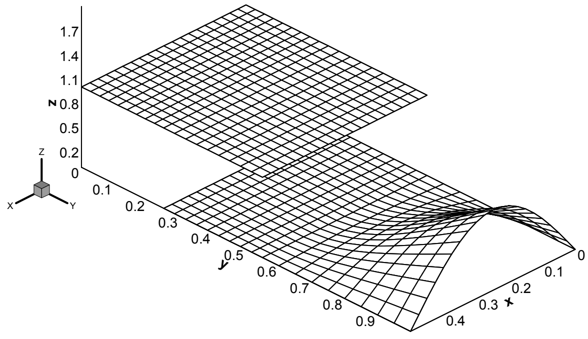
("Soft" Adhesive Case)

Fig.8.36 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

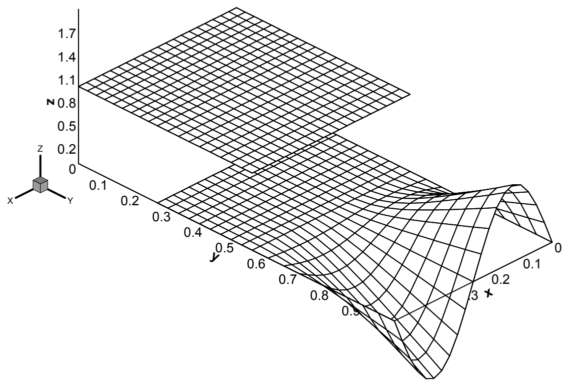
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b} = 0.4$m, a=0.5 m. L=1 m)

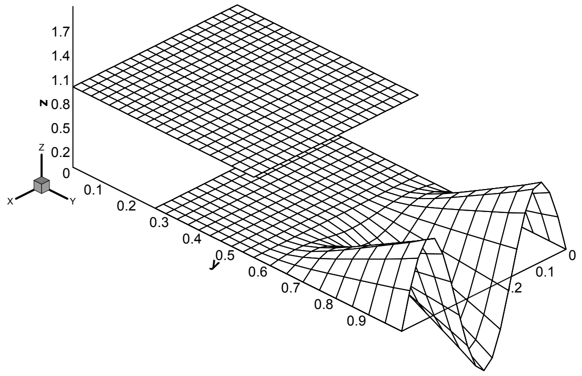
(Boundary Conditions in y-direction CFFF)



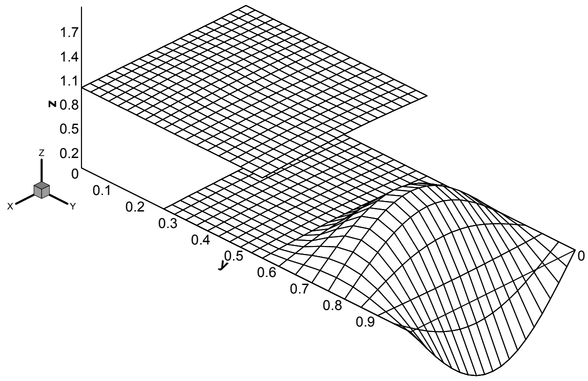
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=19.439



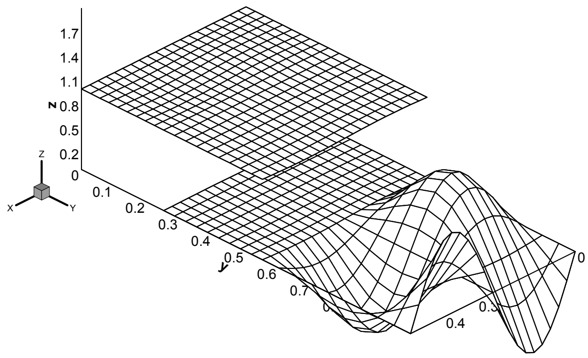
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=99.264



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=414.282



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=447.012



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=612.672

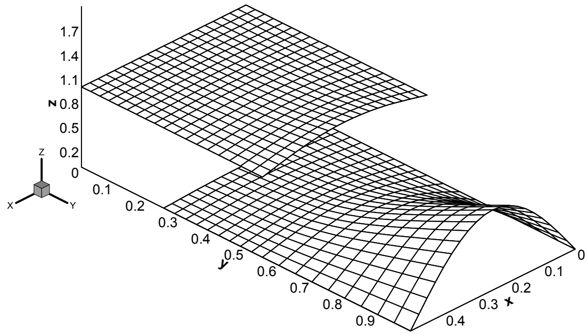
("Hard" Adhesive Case)

Fig.8.37 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

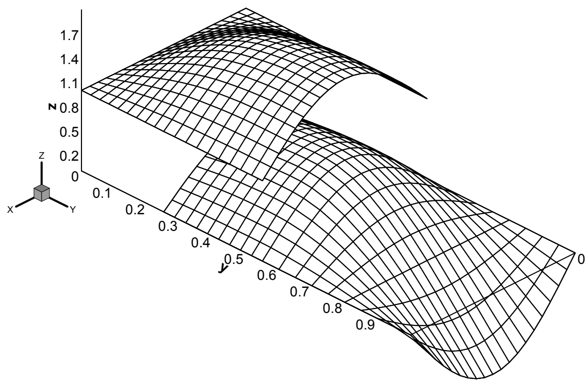
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b}=0.4$m, a=0.5 m. L=1 m)

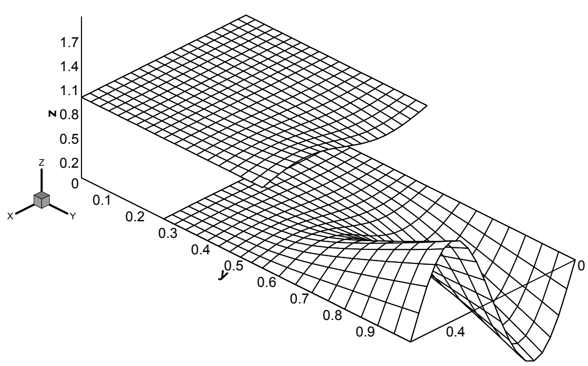
(Boundary Conditions in y-direction SFFF)



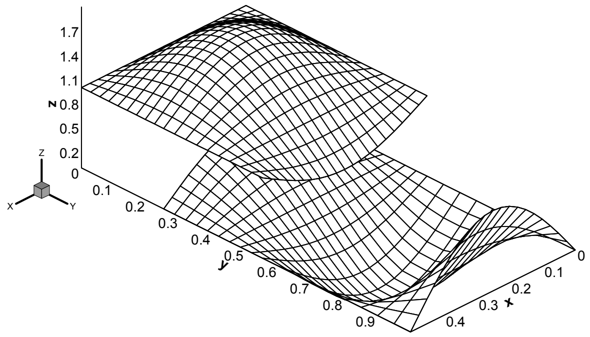
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 10.914$



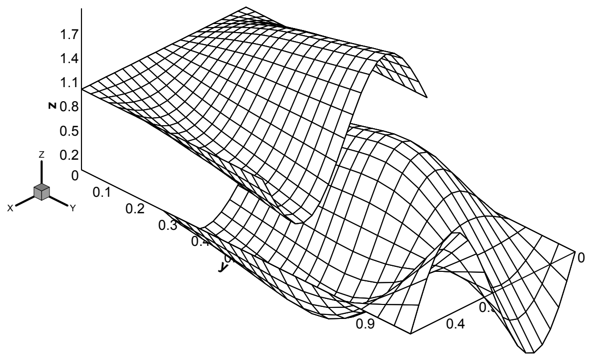
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 43.875$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 88.848$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=174.532



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=214.527

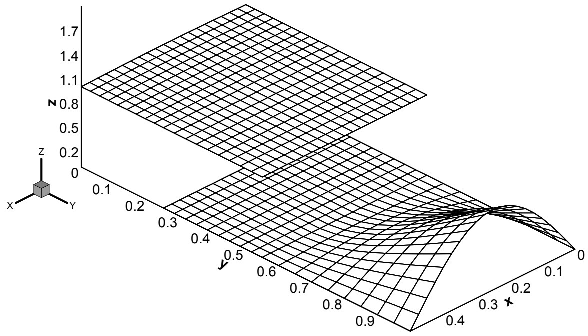
("Soft" Adhesive Case)

Fig.8.38 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

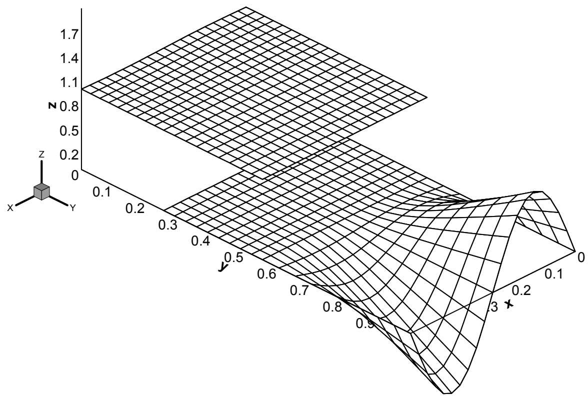
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.75 m, b_2=0.75m, $\tilde{b} = 0.4$m, a=0.5 m. L=1 m)

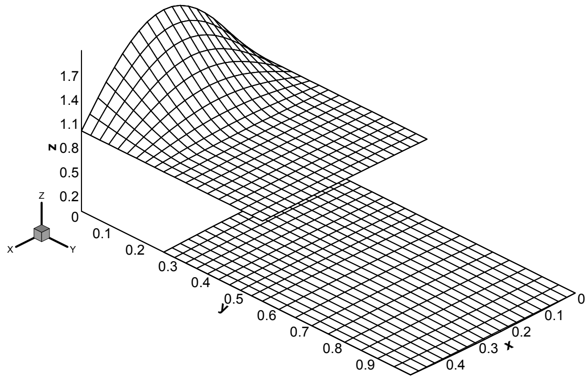
(Boundary Conditions in y-direction SFFF)



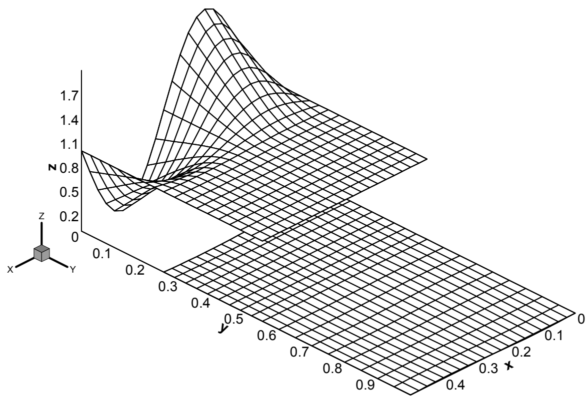
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=19.432



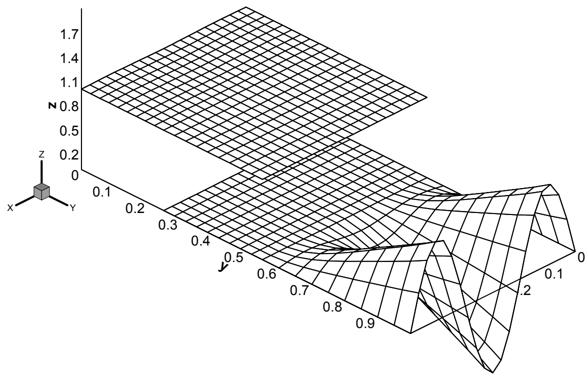
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{21}$=99.259



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{12}$=170.514



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=355.359



e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=414.280

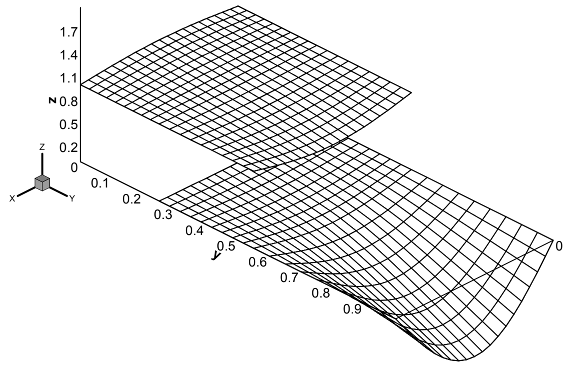
("<u>Hard</u>" Adhesive Case)

Fig.8.39 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a <u>Non-Central</u> Single Lap Joint"

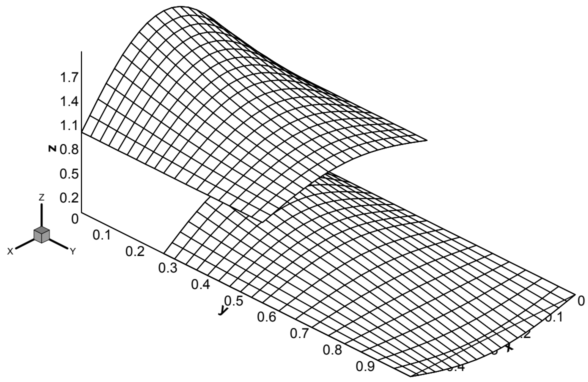
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b} = 0.4$m, a=0.5 m. L=1 m)

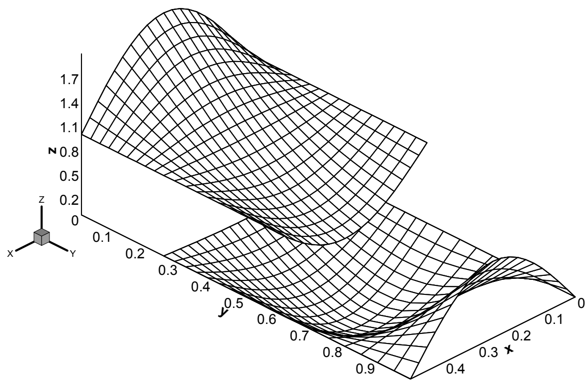
(Boundary Conditions in y-direction FFFF)



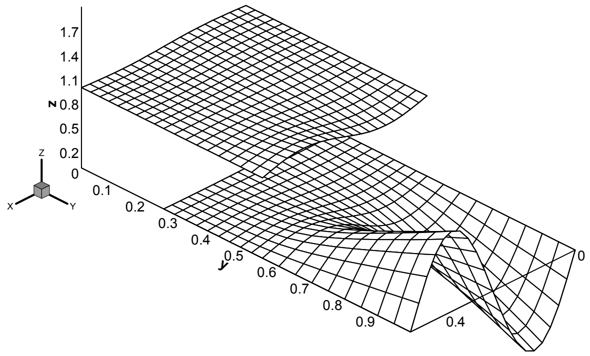
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 10.825$



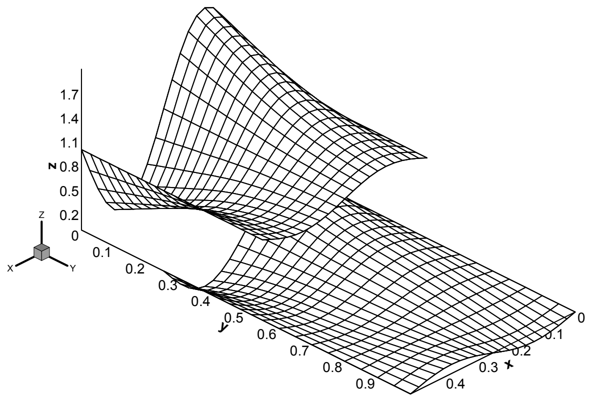
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 24.204$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{13} = 62.211$



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{21}$=88.803



e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{22}$=173.182

("Soft" Adhesive Case)

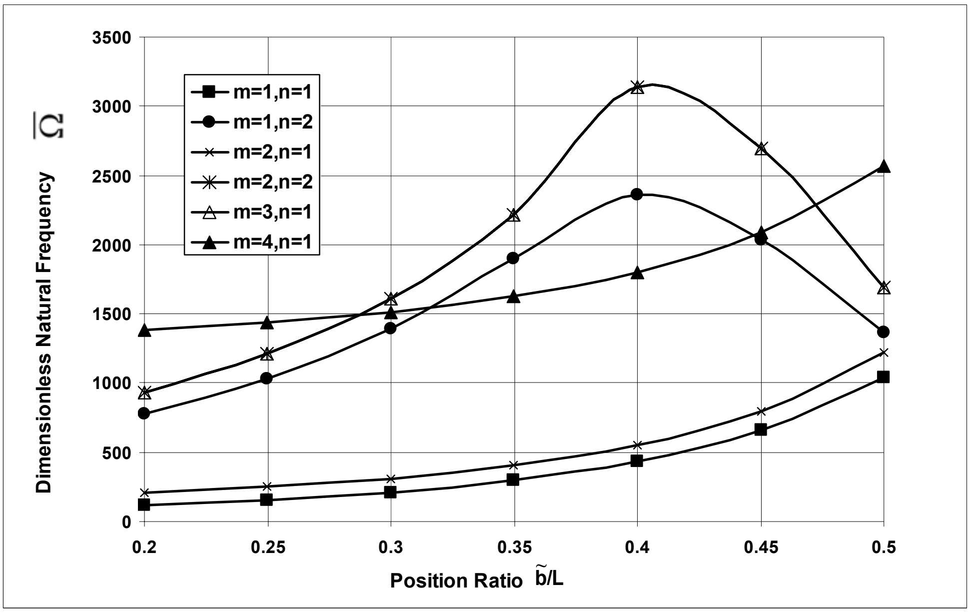
Fig.8.40 Mode Shapes and Dimensionless Natural Frequencies of " Composite, Orthotropic Plates and/or Panels with a Non-Central Single Lap Joint"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy)

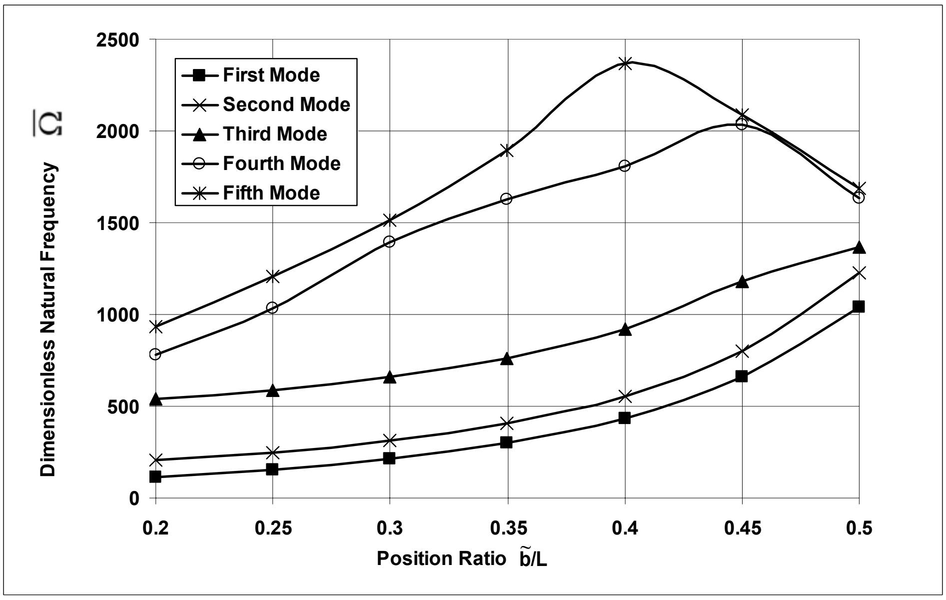
(Joint Length ℓ_y=0.3m., b_1=0.55 m, b_2=0.75m, $\tilde{b}=$0.4m, a=0.5 m. L=1 m)

(Boundary Conditions in y-direction FFFF)

8.3.2 Some Parametric Studies for "Main PROBLEM Ib"



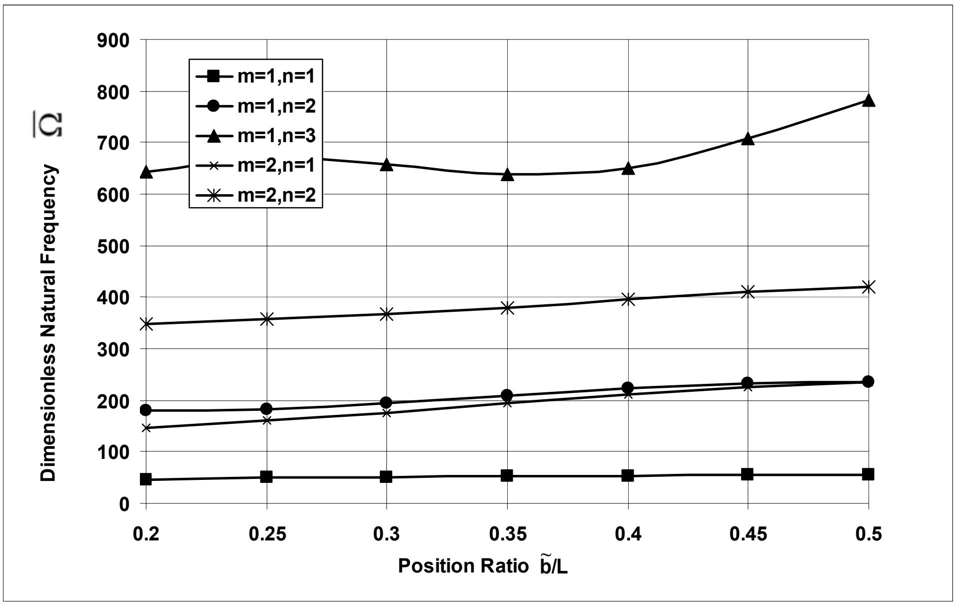
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, "<u>Hard</u>" Adhesive



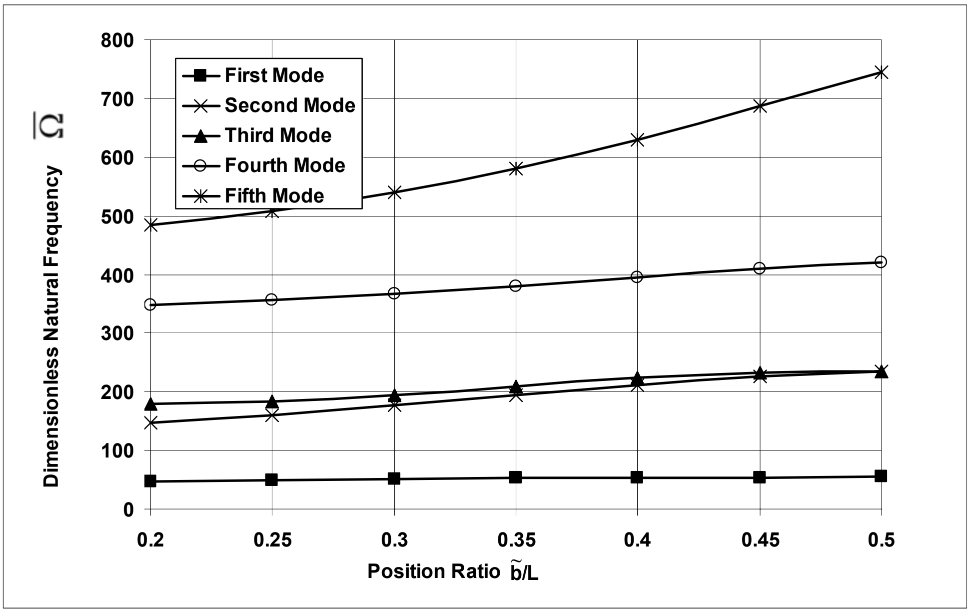
b) "Various Modes with (CFFC) B.C.'s, "<u>Hard</u>" Adhesive

Fig 8.41 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$ / L" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =varies, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



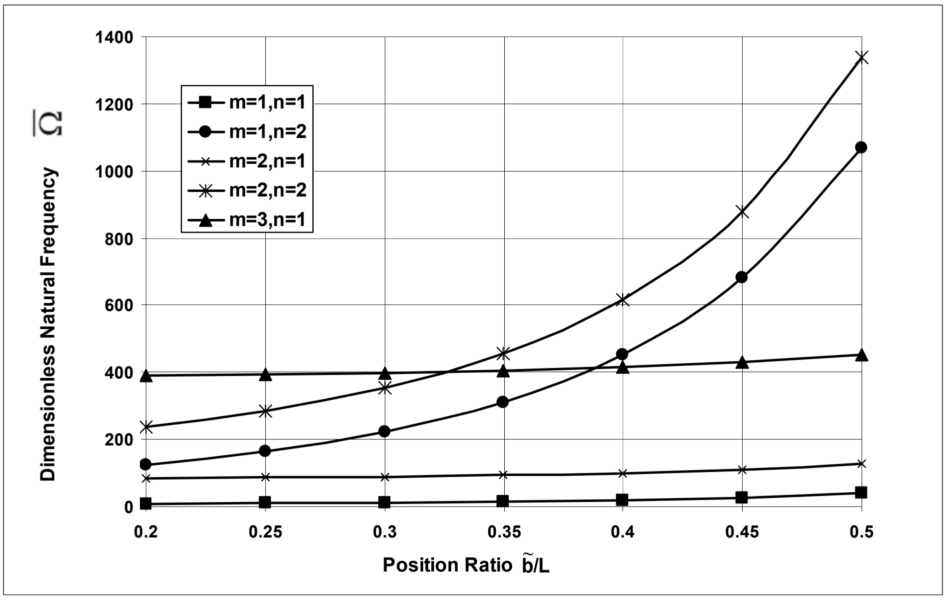
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, "Soft" Adhesive



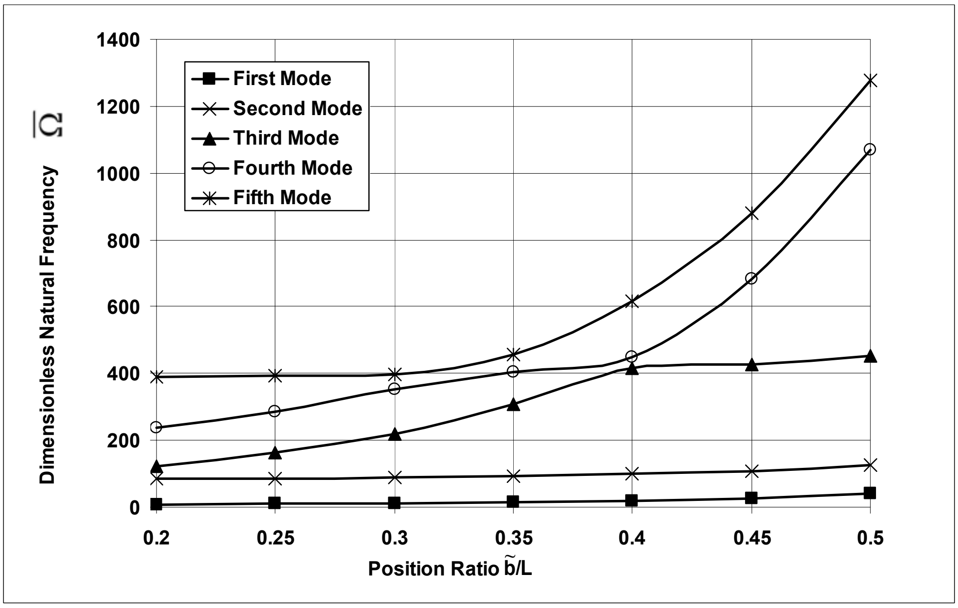
b) "Various Modes with (CFFC) B.C.'s, "Soft" Adhesive

Fig 8.42 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}/L$" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =varies, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



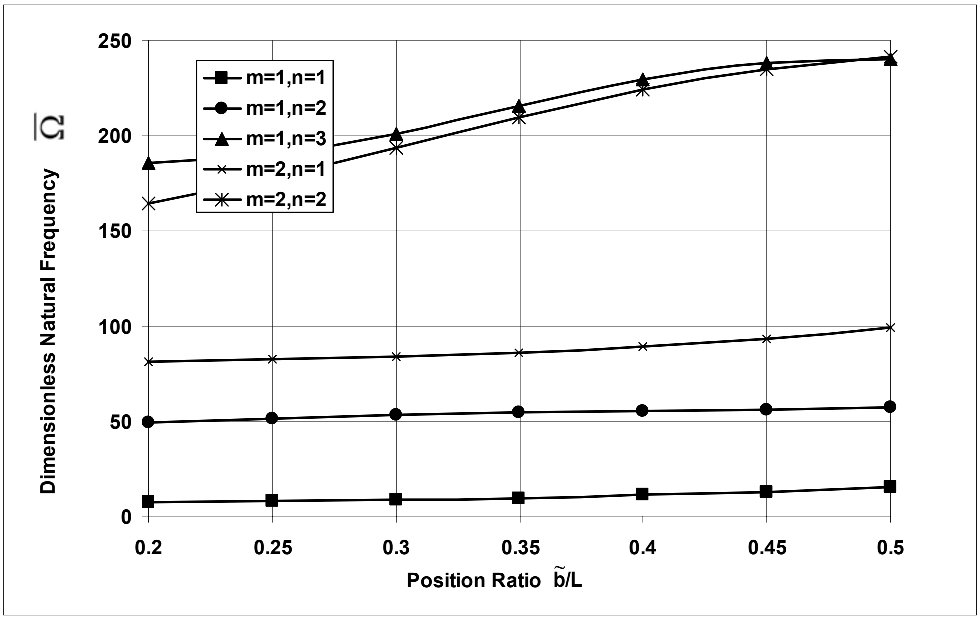
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, ''Hard'' Adhesive



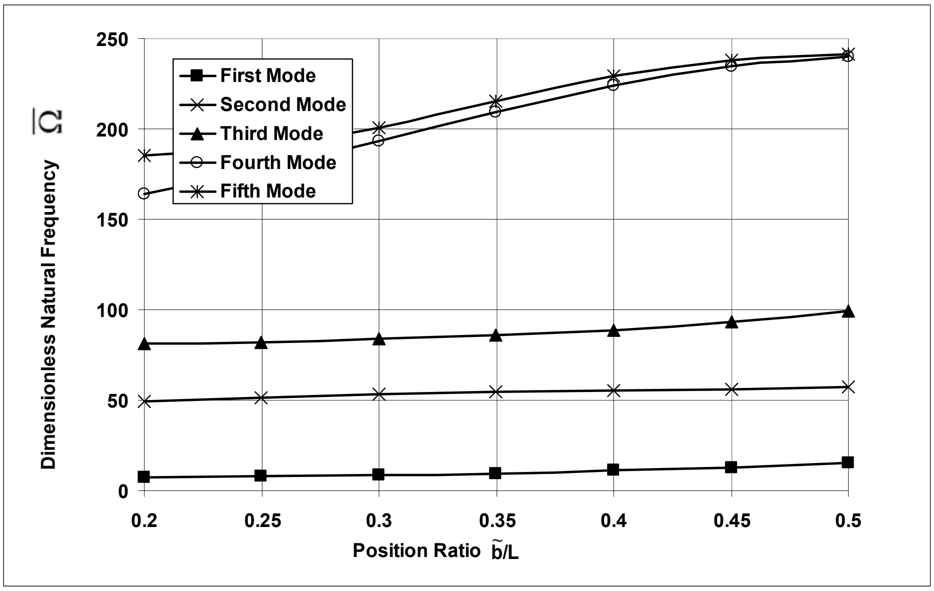
b) "Various Modes with (CFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.43 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$ / L" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =varies, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



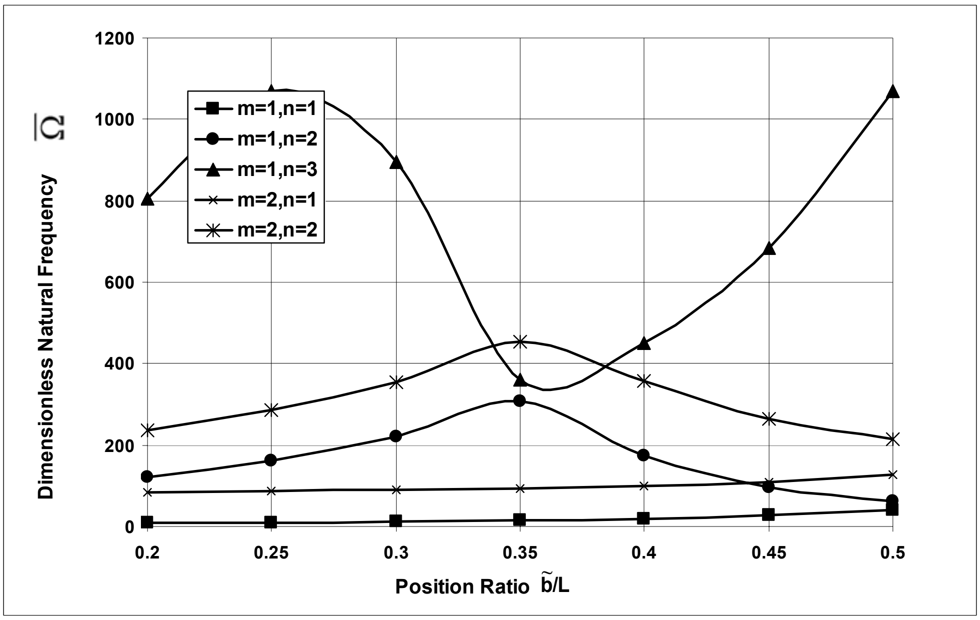
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, "<u>Soft</u>" Adhesive



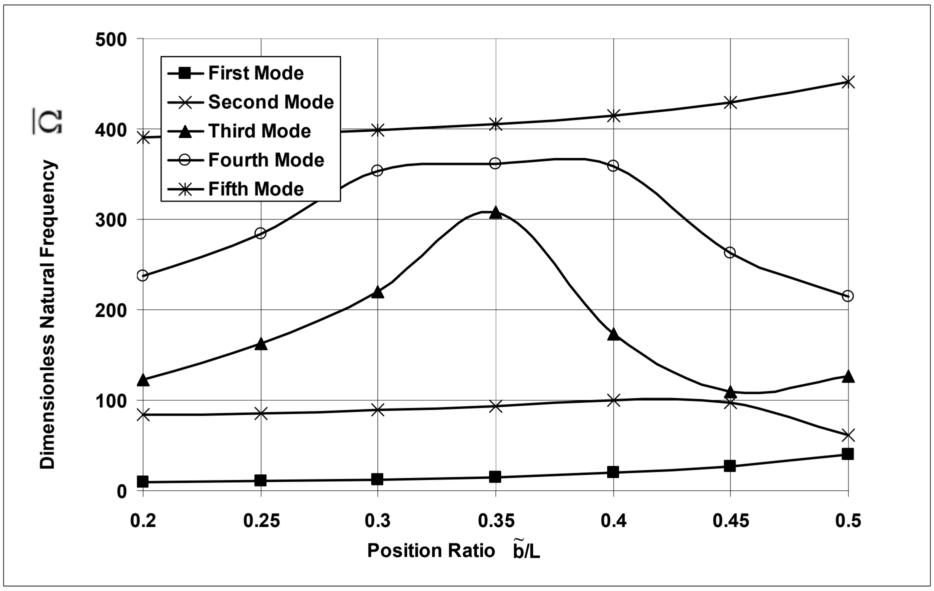
b) "Various Modes with (CFFF) B.C.'s, "<u>Soft</u>" Adhesive

Fig 8.44 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$ / L" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =varies, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



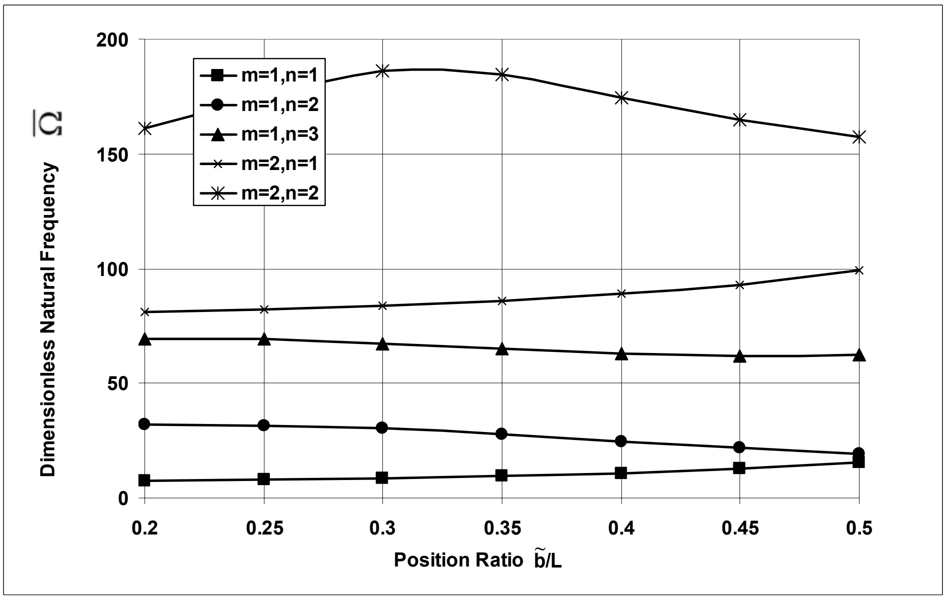
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, "<u>Hard</u>" Adhesive



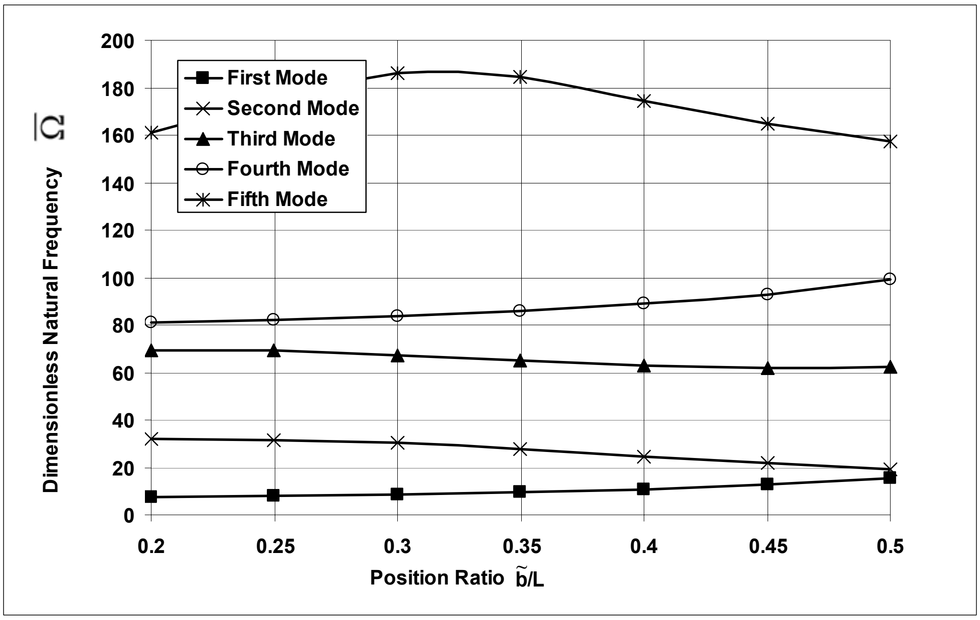
b) "Various Modes with (FFFF) B.C.'s, "<u>Hard</u>" Adhesive

Fig 8.45 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$ / L" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =varies, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



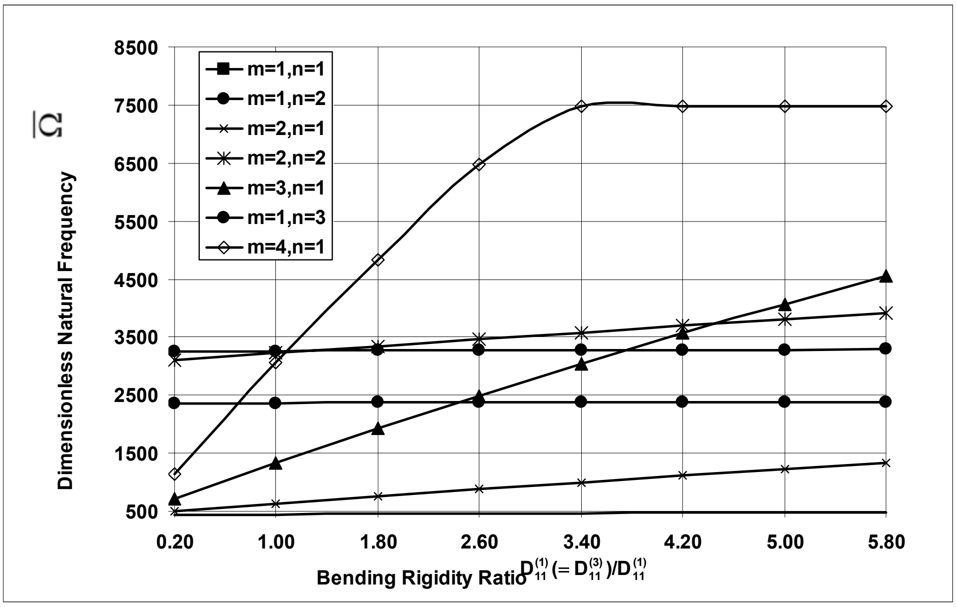
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



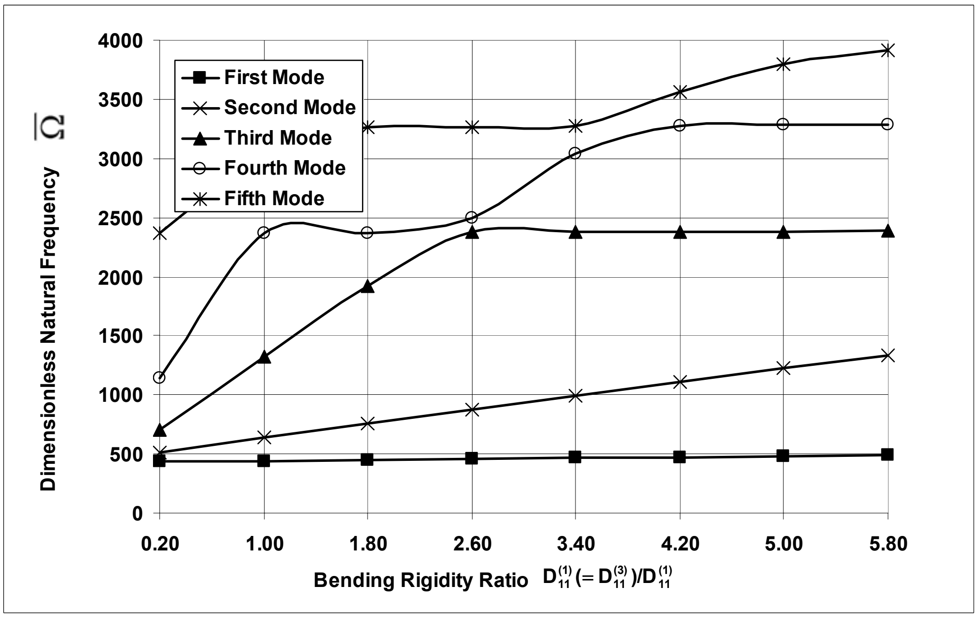
b) "Various Modes with (FFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.46 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$ / L" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =varies, a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



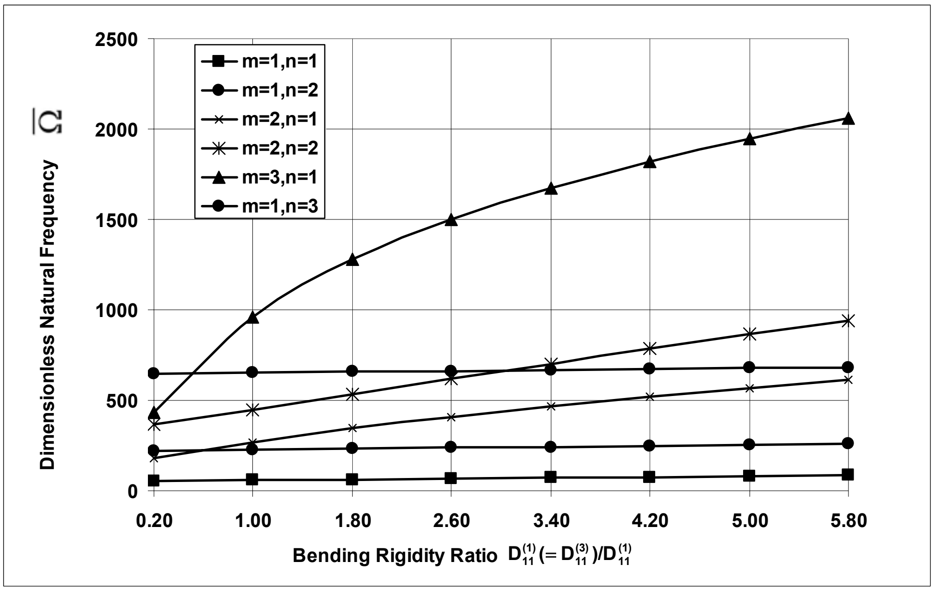
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, ''Hard'' Adhesive



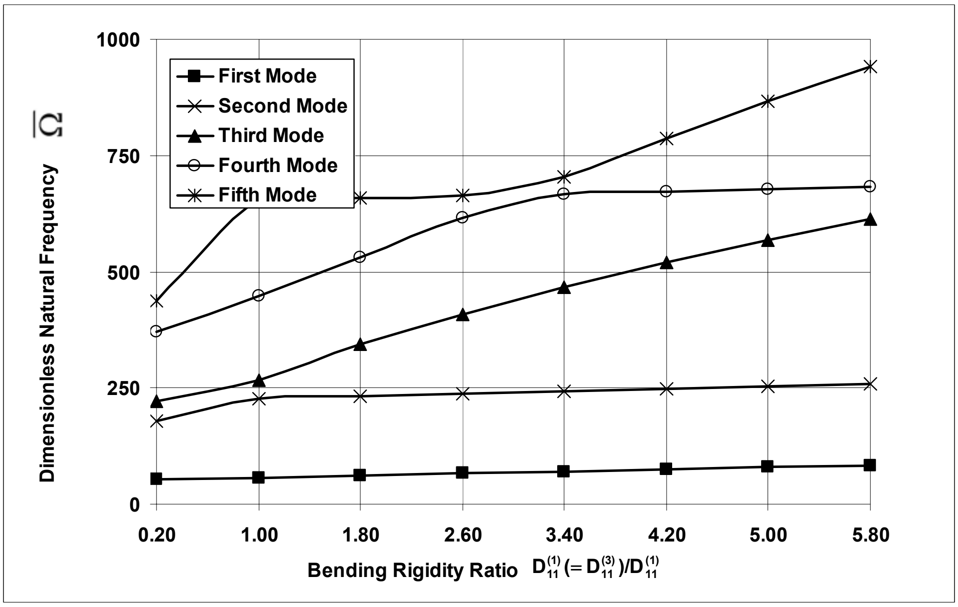
b) "Various Modes with (CFFC) B.C.'s, ''Hard'' Adhesive

Fig 8.47 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



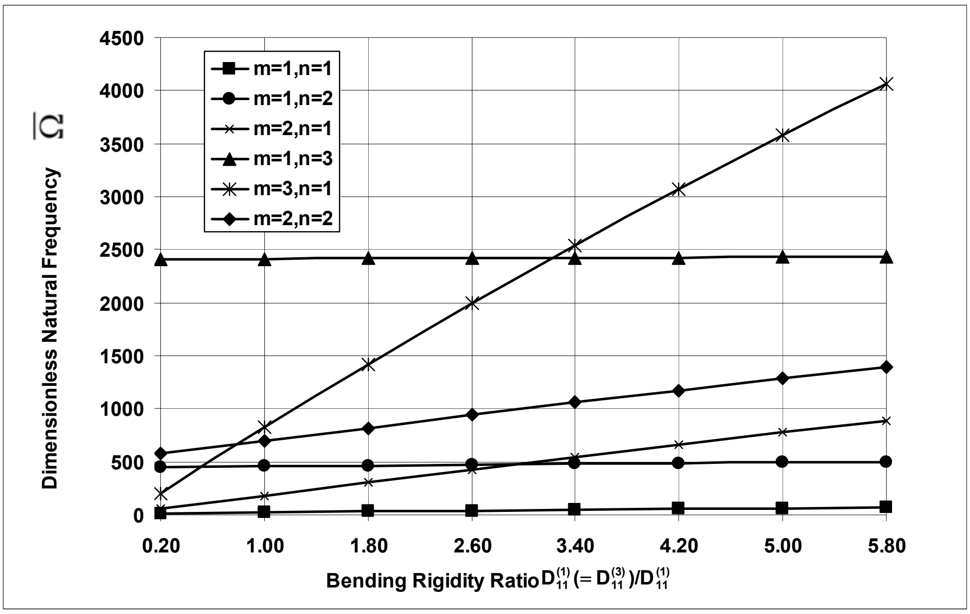
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s, "Soft" Adhesive



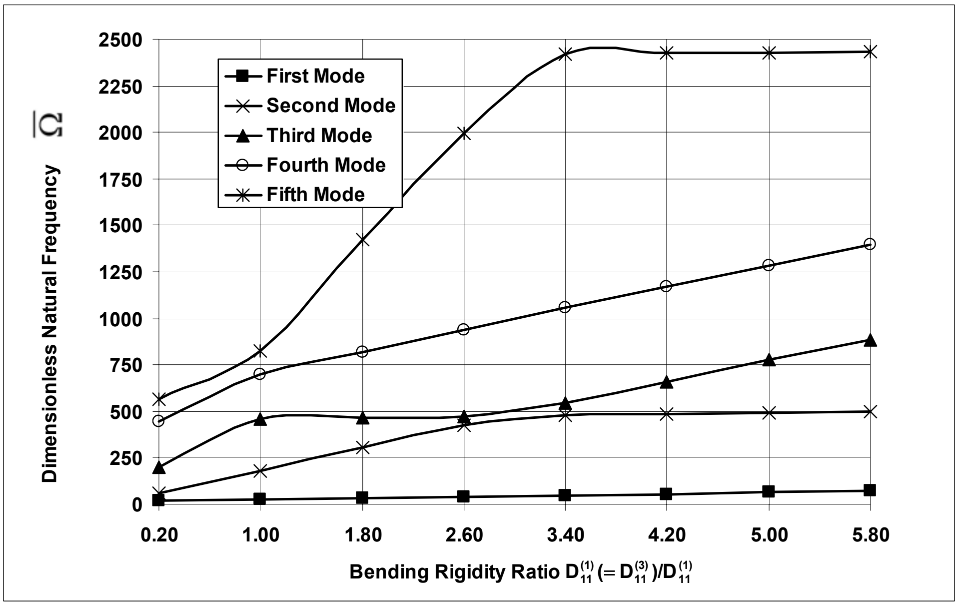
b) "Various Modes with (CFFC) B.C.'s, "Soft" Adhesive

Fig 8.48 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)



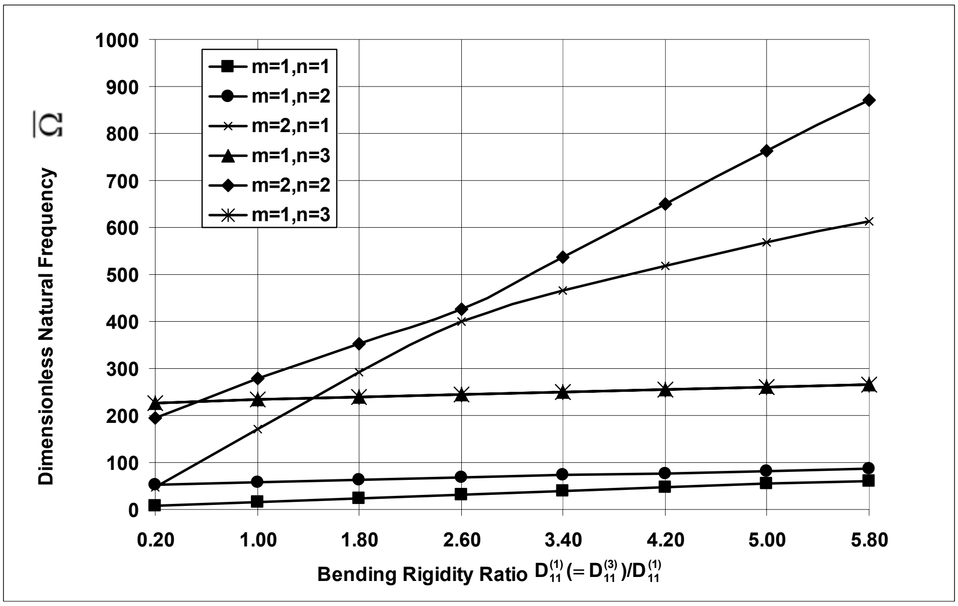
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, "<u>Hard</u>" Adhesive



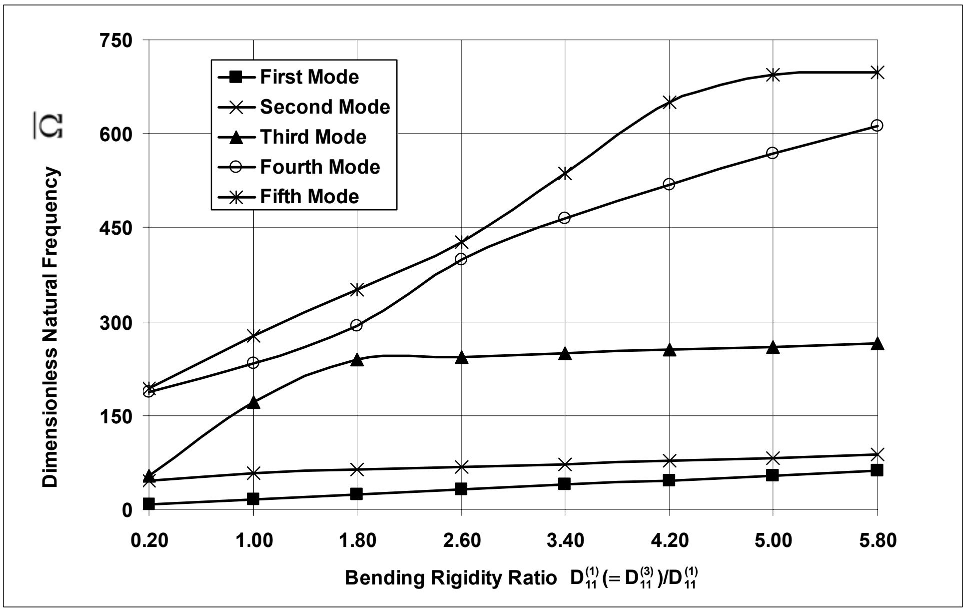
b) "Various Modes with (CFFF) B.C.'s, "<u>Hard</u>" Adhesive

Fig 8.49 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



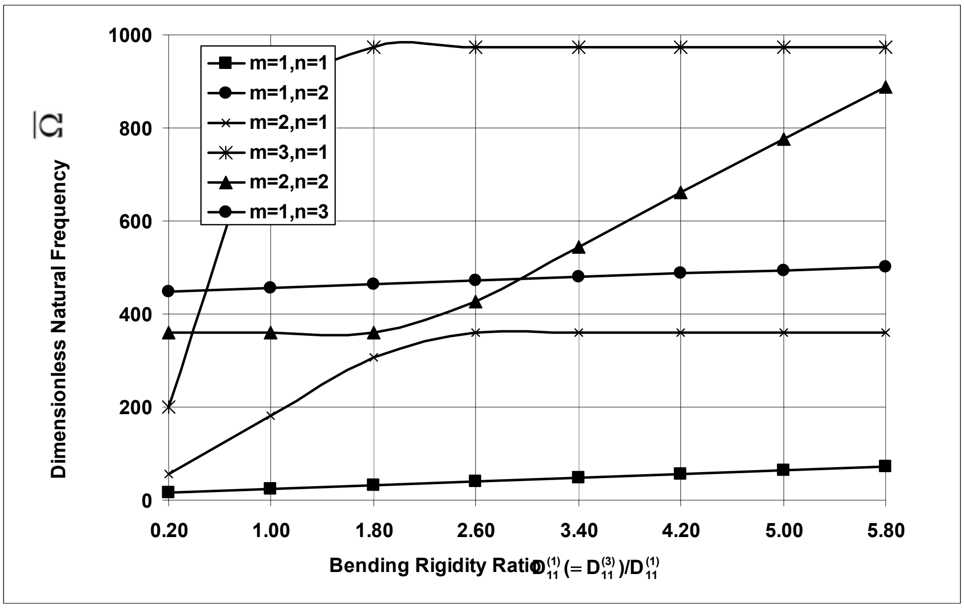
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



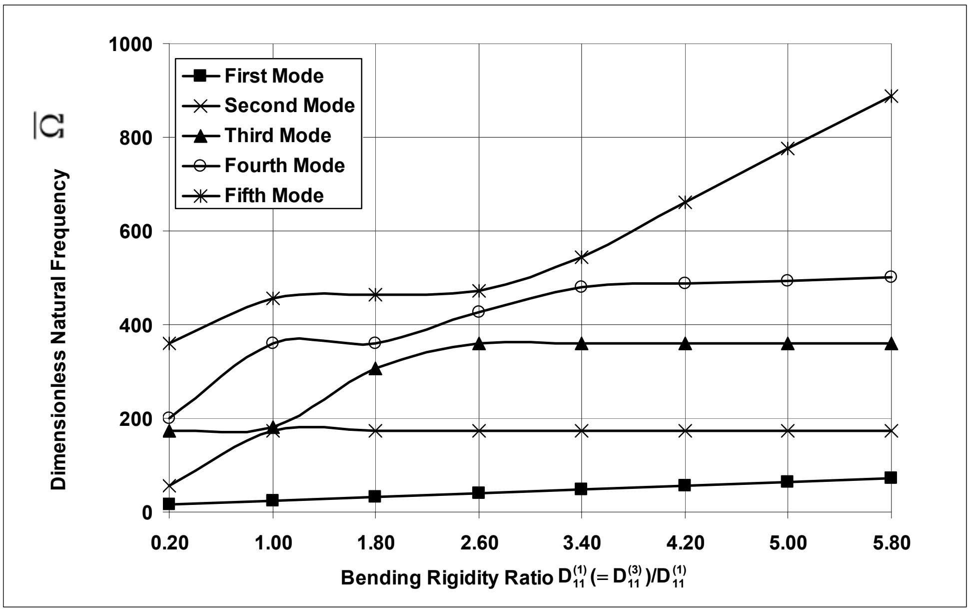
b) "Various Modes with (CFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.50 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)



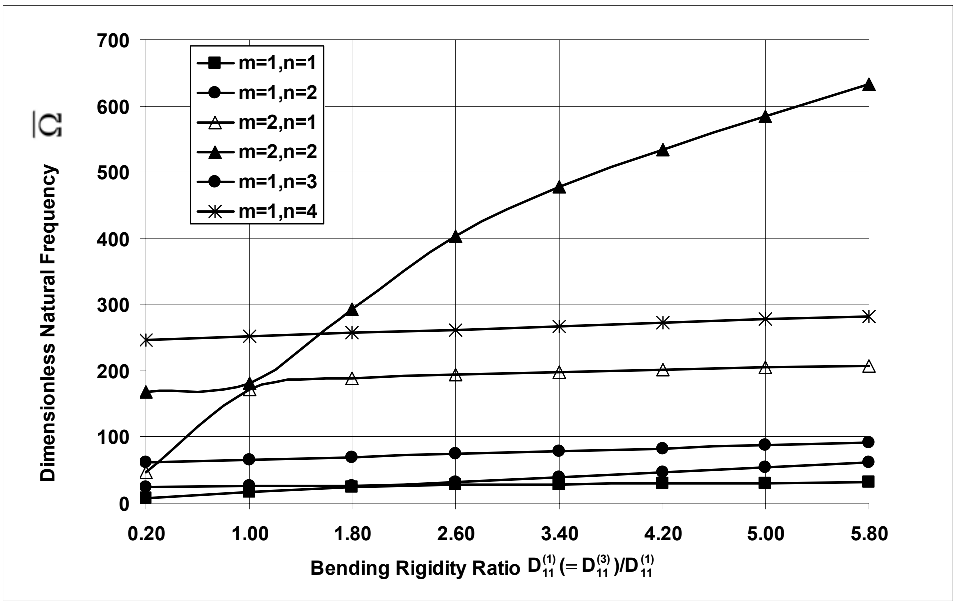
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



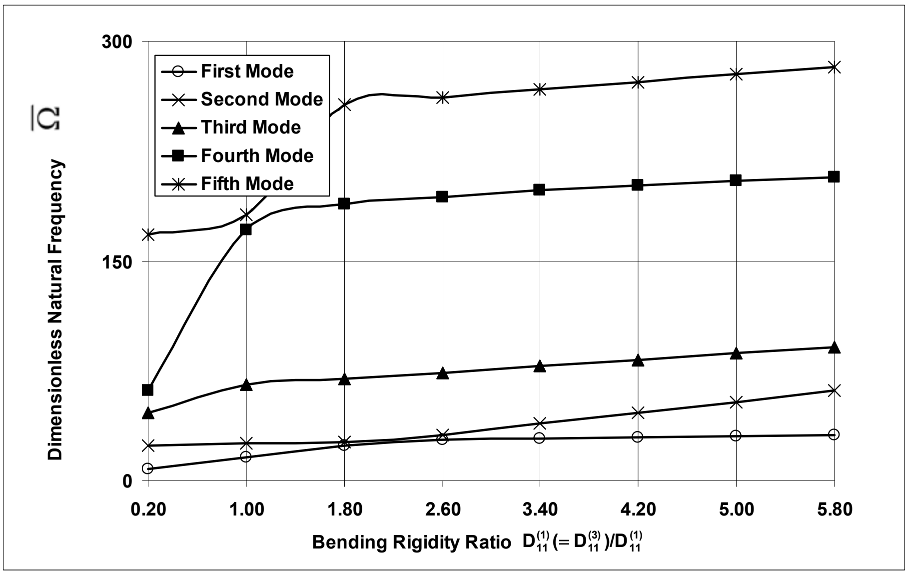
b) "Various Modes with (FFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.51 ""Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



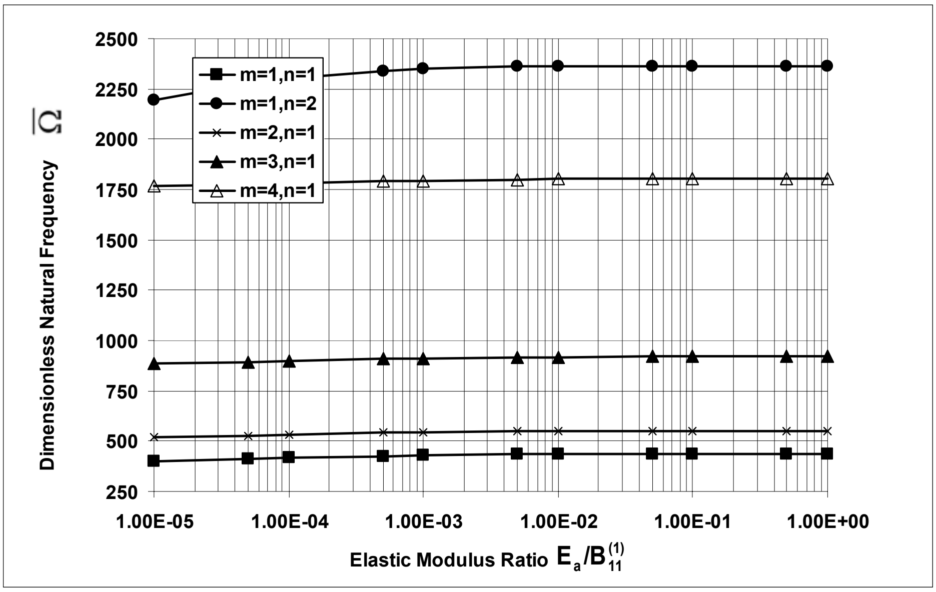
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s, ''Soft'' Adhesive



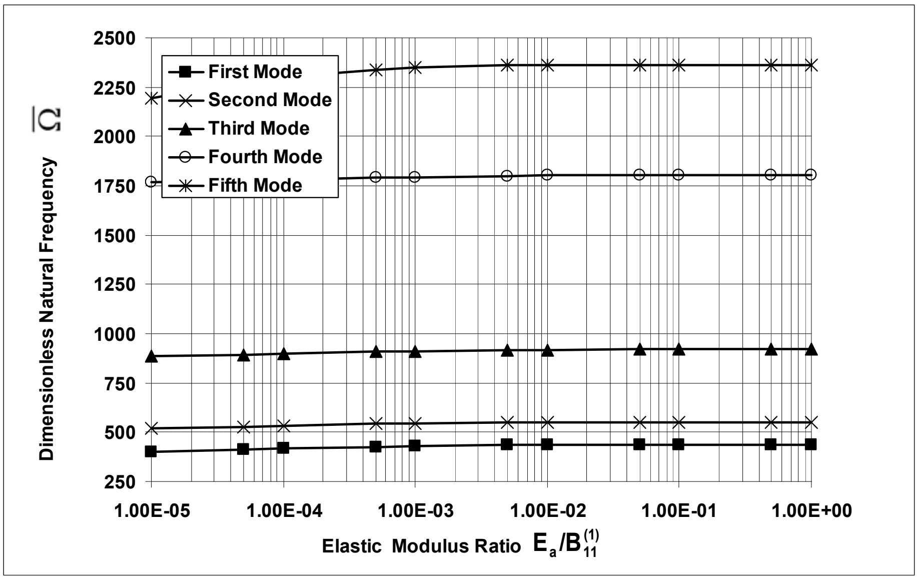
b) "Various Modes with (FFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.52 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Bonded Single Lap Joint"

($D_{11}^{(2)}$ increases while other stiffness constants are kept constant)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)



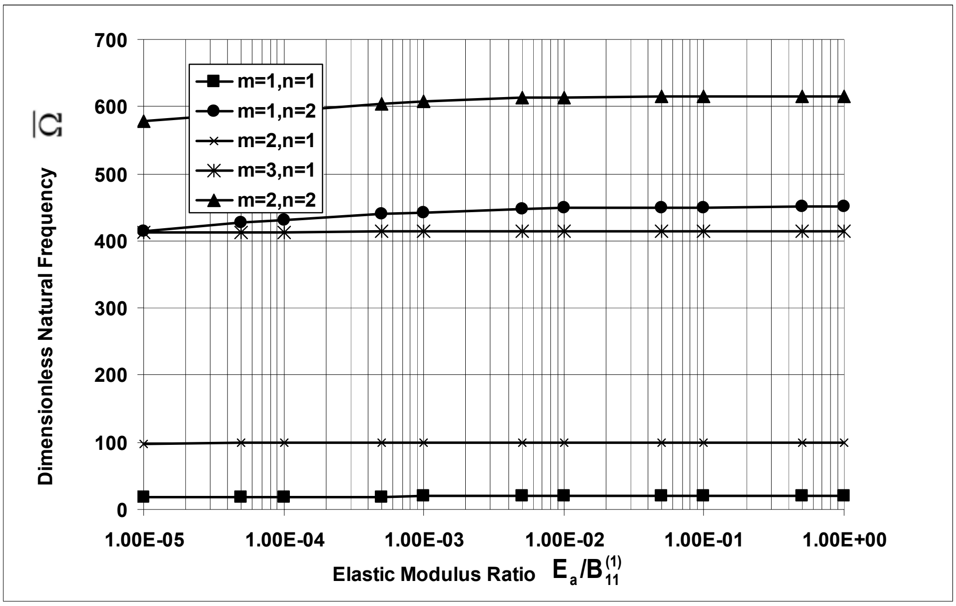
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s



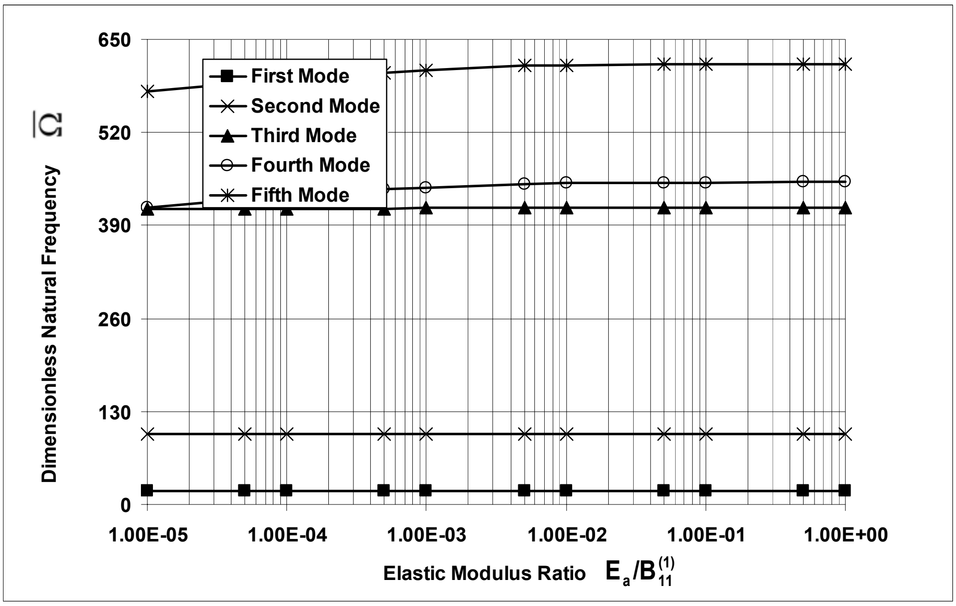
b) "Various Modes with (CFFC) B.C.'s

Fig 8.53 "Dimensionless Nat. Freq's ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)
Elastic Modulus Ratio axis is plotted in Log Scale



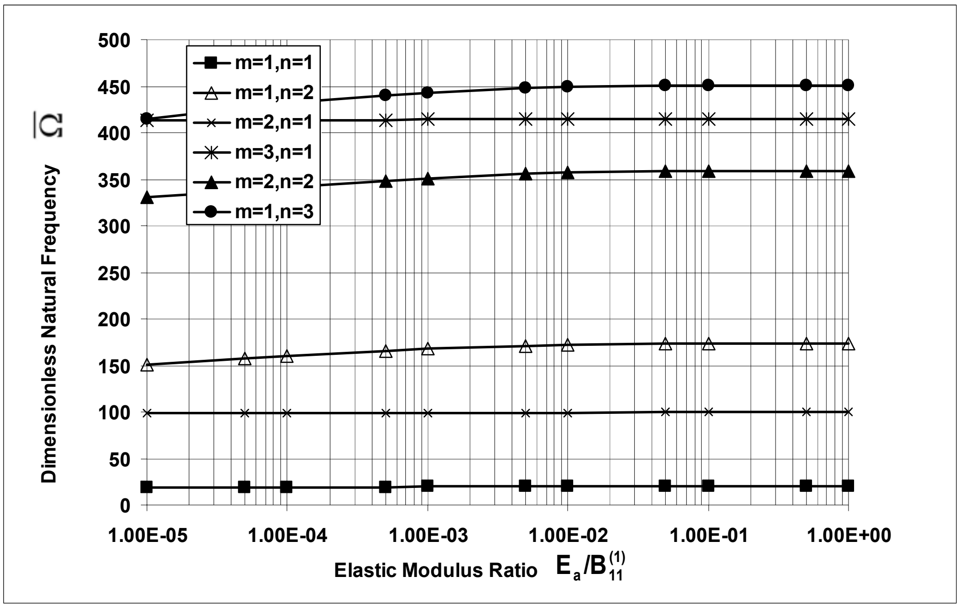
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFF) B.C.'s



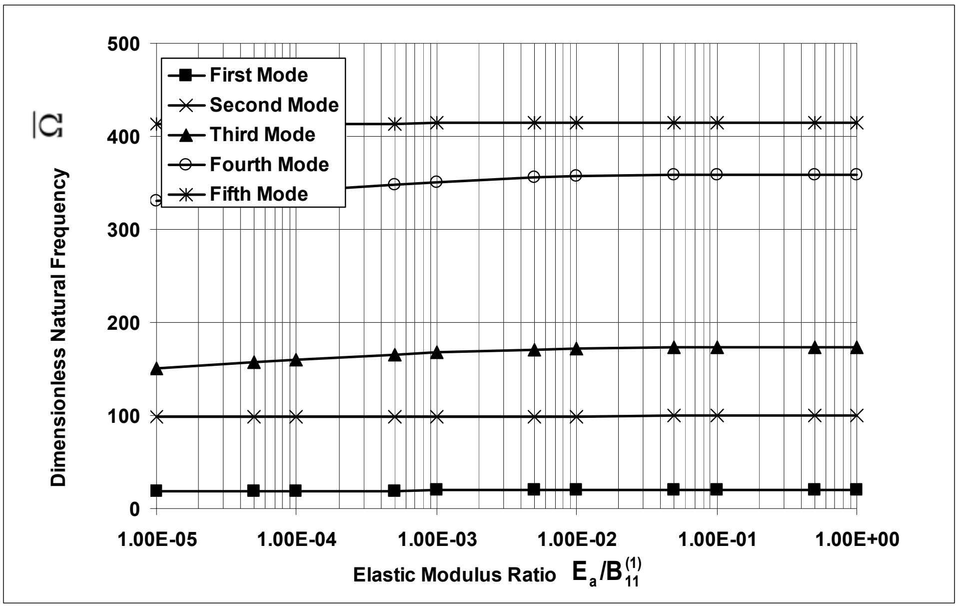
b) "Various Modes with (CFFF) B.C.'s

Fig 8.54 "Dimensionless Nat. Freq's ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



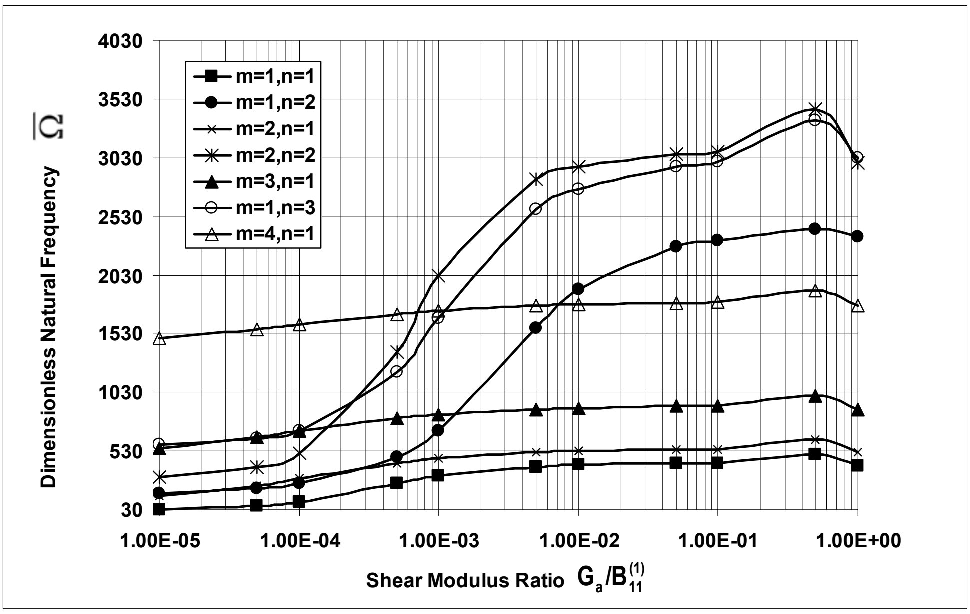
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s



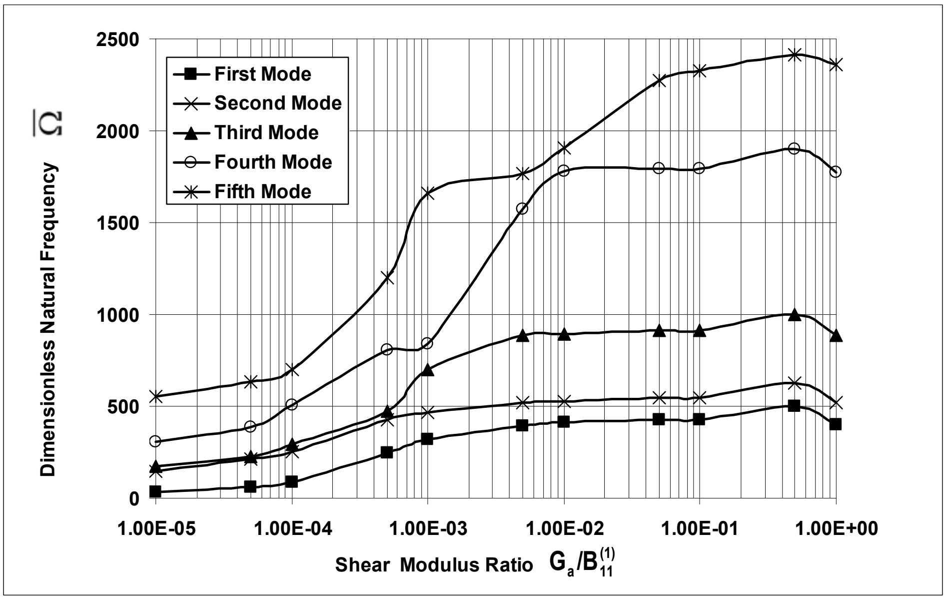
b) "Various Modes with (FFFF) B.C.'s

Fig 8.55 "Dimensionless Nat. Freq's ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Bonded Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



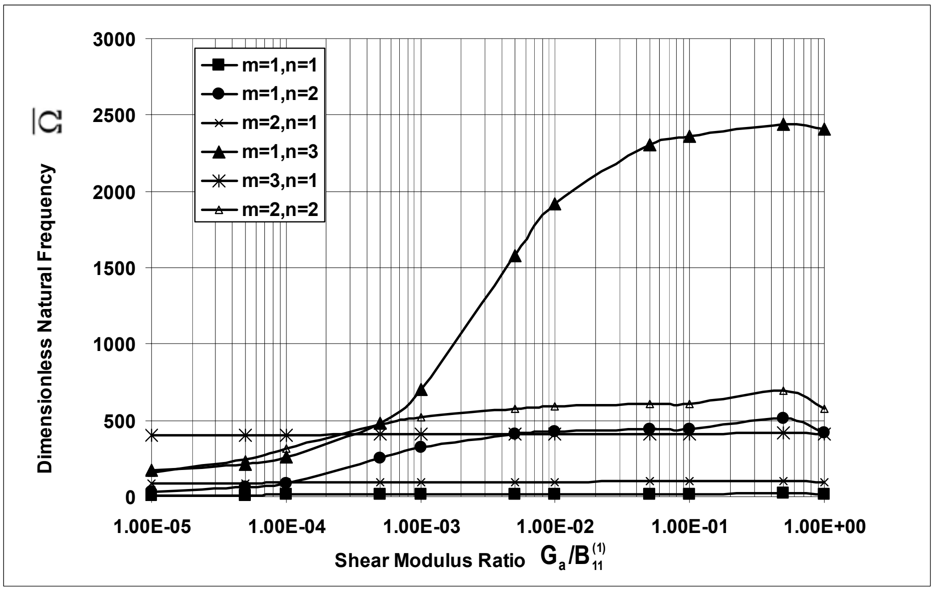
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (CFFC) B.C.'s



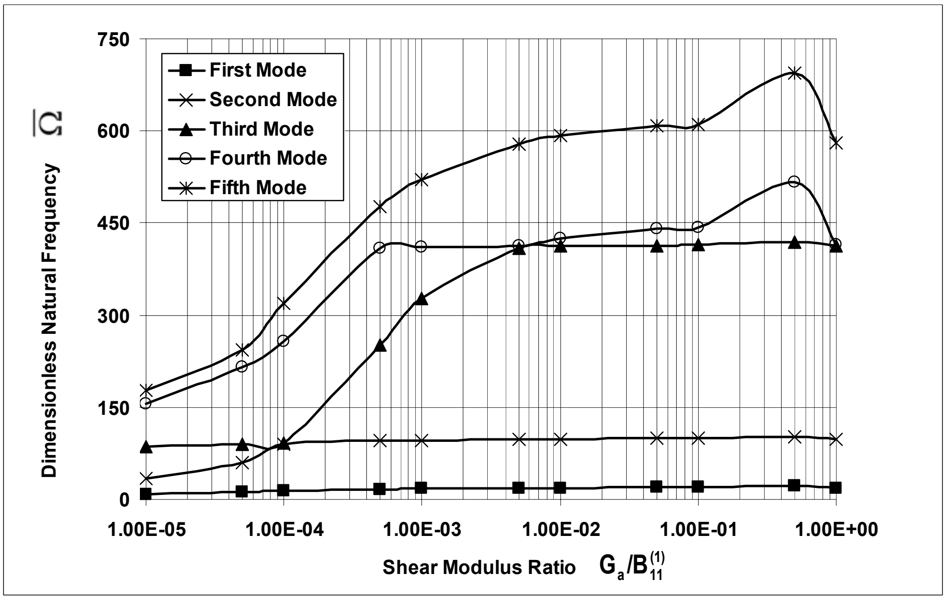
b) "Various Modes with (CFFC) B.C.'s

Fig 8.56 "Dimensionless Natural Freq's ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with Non-Centrally Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFC)
Shear Modulus Ratio axis is plotted in Log Scale



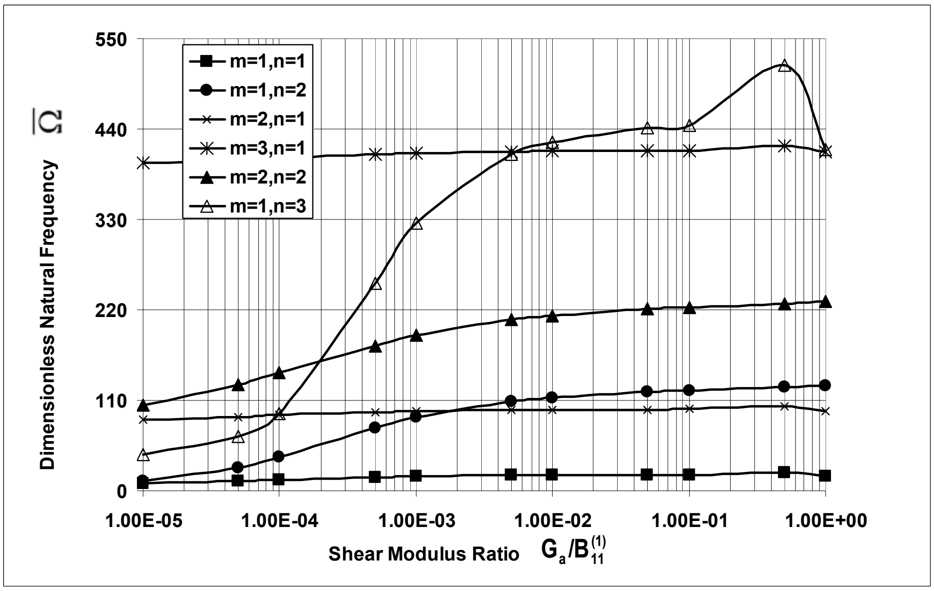
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (SFFS) B.C.'s



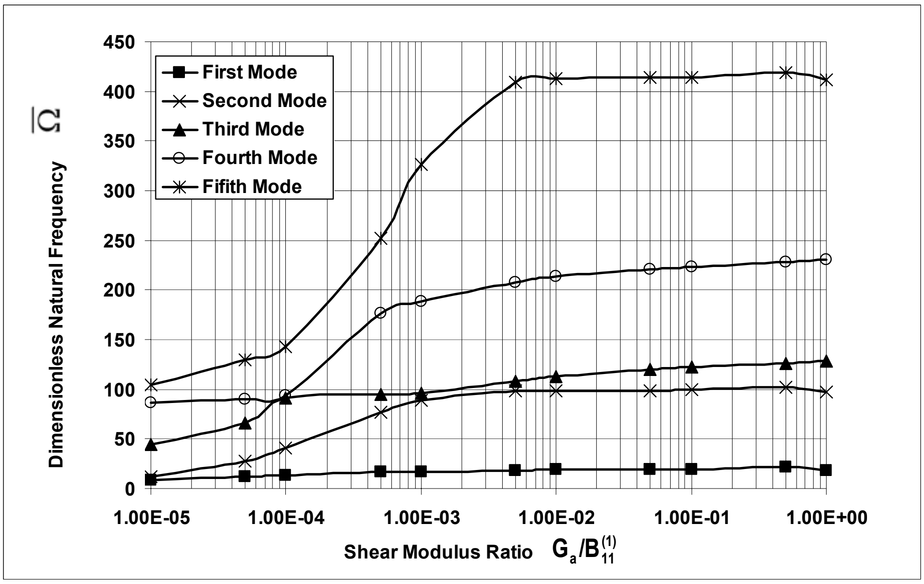
b) "Various Modes with (SFFS) B.C.'s

Fig 8.57 "Dimensionless Natural Freq's ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$ =0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction CFFF)
Shear Modulus Ratio axis is plotted in Log Scale



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFF) B.C.'s



b) "Various Modes with (FFFF) B.C.'s

Fig 8.58 "Dimensionless Natural Freq's ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with <u>Non-Centrally</u> Single Lap Joint"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy)
(Joint Length ℓ_I =0.3m, $\tilde{b}$=0.4 m., a=0.5 m, L=1.0m)
(Boundary Conditions in y-direction FFFF)
Shear Modulus Ratio axis is plotted in Log Scale

8.3.3 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequencies"

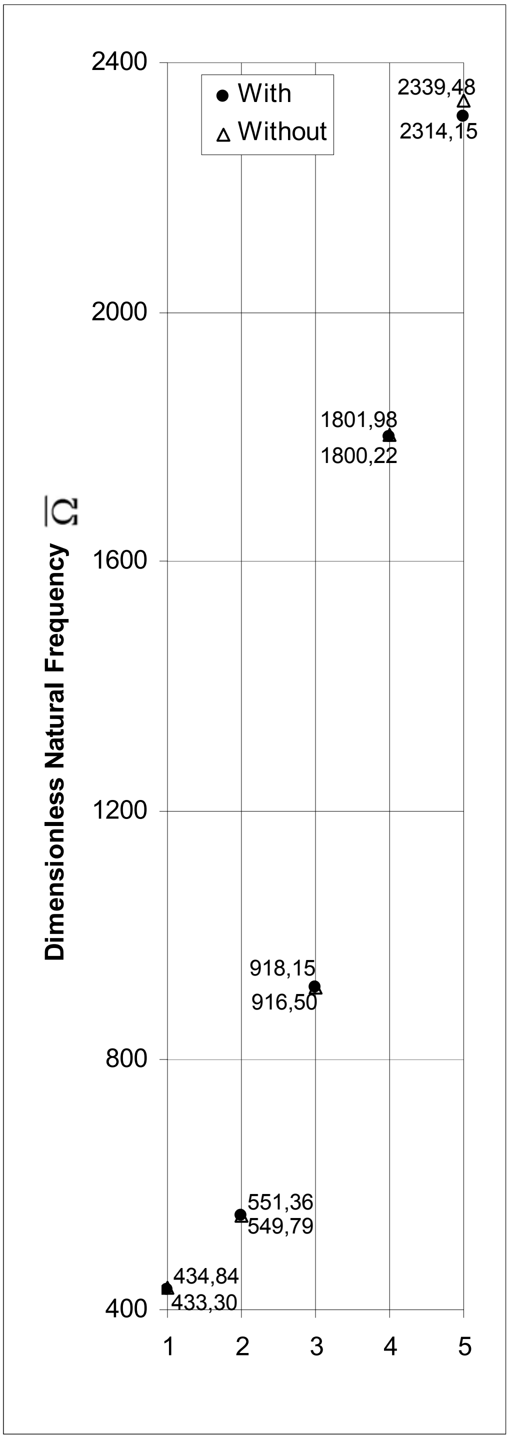
Table 8.5 Comparison of "Dimensionless Natural Frequencies" obtained by adding $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms to adhesive layer equations for "Main PROBLEM Ib"

a) "<u>Hard</u>" Adhesive Case

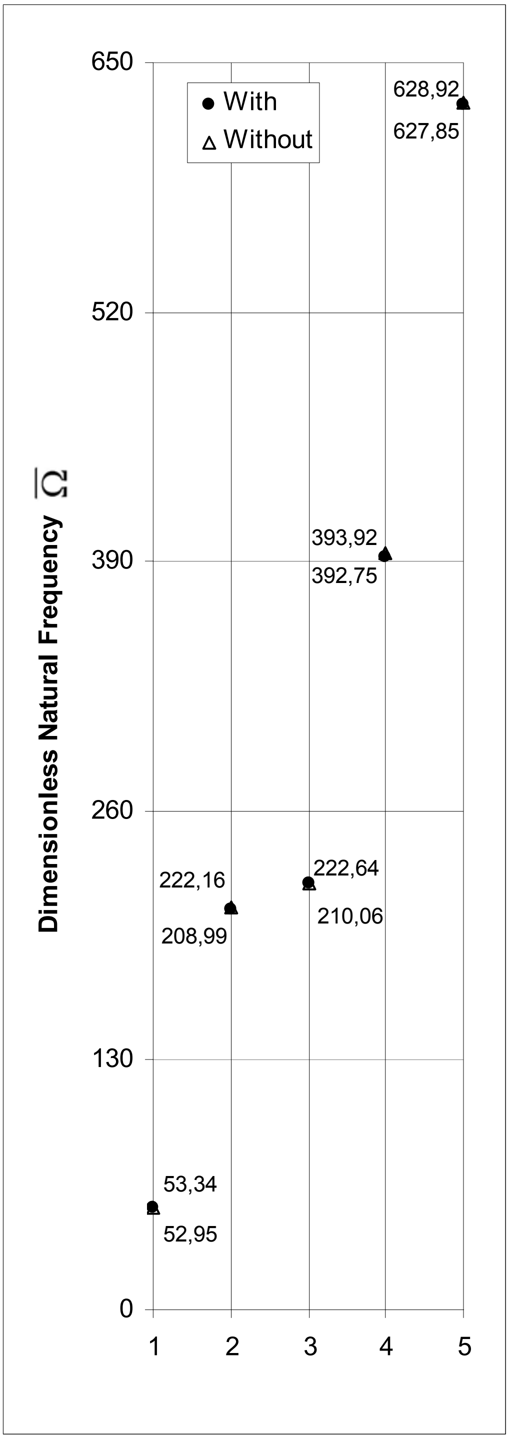
Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	434.843	433.304	1.539
	2	551.360	549.786	1.574
	3	918.154	916.501	1.653
	4	1801.978	1800.220	1.758
	5	2339.476	2314.154	25.322
SFFS	1	224.837	224.047	0.790
	2	336.660	335.817	0.843
	3	701.671	700.731	0.940
	4	1584.013	1582.946	1.067
	5	1763.952	1743.746	20.206
CFFF	1	19.487	19.441	0.046
	2	99.327	99.265	0.062
	3	414.373	414.284	0.089
	4	448.998	447.394	1.604
	5	614.441	612.756	1.685
FFFF	1	19.481	19.432	0.049
	2	99.325	99.259	0.066
	3	172.054	170.514	1.540
	4	357.180	355.359	1.821
	5	414.370	414.280	0.090

b) "<u>Soft</u>" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	53.342	52.953	0.389
	2	210.056	208.986	1.070
	3	222.640	222.164	0.476
	4	393.920	392.751	1.169
	5	628.916	627.854	1.062
SFFS	1	34.915	34.595	0.320
	2	118.530	118.174	0.356
	3	171.289	170.651	0.638
	4	291.735	290.486	1.249
	5	400.745	400.059	0.686
CFFF	1	10.990	10.958	0.032
	2	54.841	54.475	0.366
	3	88.887	88.852	0.035
	4	222.622	221.455	1.167
	5	228.876	228.393	0.483
FFFF	1	10.859	10.825	0.034
	2	24.411	24.204	0.207
	3	62.535	62.211	0.324
	4	88.838	88.803	0.035
	5	173.712	173.182	0.53



a) "Hard" Adhesive Case

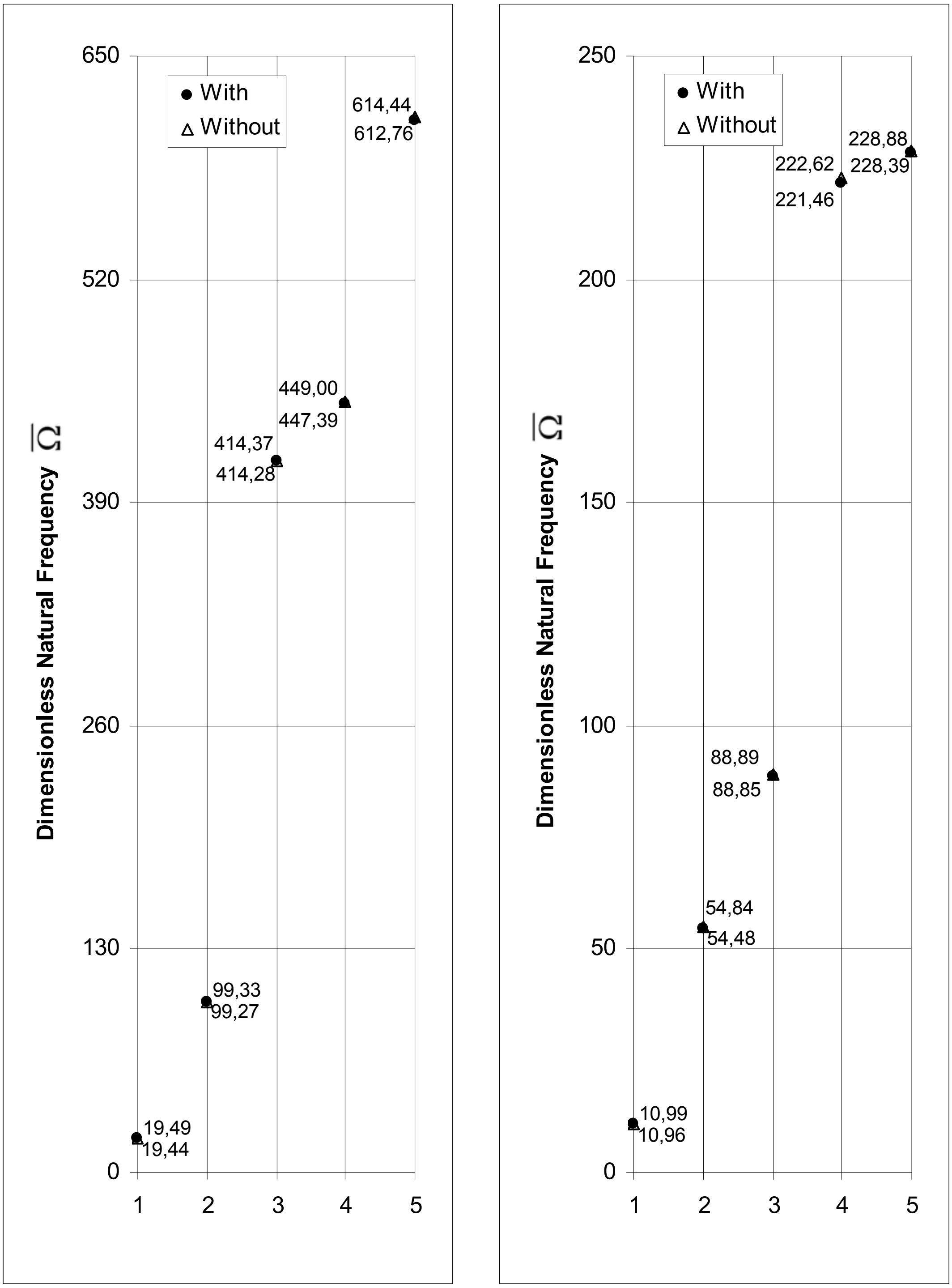


b) "Soft" Adhesive Case

Figure 59 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$

on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Single Lap Joint"

(Boundary Conditions in y-direction CFFC)



a) "Hard" Adhesive Case

b) "Soft" Adhesive Case

Figure 60 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Single Lap Joint" (Boundary Conditions in y-direction CFFF)

8.4 Numerical Results and Discussion for "Main PROBLEM II.a"

In the "Main PROBLEM IIa.", the "Composite Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint" is analyzed. The doubler is made of Graphite-Epoxy and the lower plate adherends are Kevlar-Epoxy. For the in-between adhesive layer, the "hard" and the "soft" adhesive cases are taken into account. The "Geometric and the Material Characteristics" of the symmetric single lap joint system are given in Table 8.2.

In Figures 8.57 – 8.66, the mode shapes and the corresponding natural frequencies (from the first to fifth), in the "hard" and the subsequent "soft" adhesive cases with various boundary conditions are presented.

From aforementioned Figures, in the "hard" adhesive case it is easy to observe that that there exists an almost "stationary region" in the mode shapes with respect to the symmetry of the "Boundary Conditions". And symmetric and skew symmetric modes flow each other in the composite symmetric single lap joint system. If the boundary conditions are not symmetric the "almost stationary area" changes the position from left to right. In the "soft" adhesive case, however, an almost "stationary region" does not exist in mode shapes. The general trend in the mode shapes, for the "soft" adhesive case is that, the "Bonded Region" moves or bends with the rest of the lap joint system. And the mode shapes are completely different in comparison with those of the "hard" adhesive cases with the same support conditions.

Next, for the "Main PROBLEM IIa", in Figures 8.67 through 8.84, several important parametric studies are presented. In Figures 8.67-8.72, the "Dimensionless Natural Frequency $\overline{\Omega}$" versus "Joint Length Ratio $(\ell_I + \ell_I)/L$" from the first up to the fifth mode are plotted, for both the "hard" and the "soft" adhesive cases, corresponding to the various support conditions.

From Figures 8.67, 8.69 8.71, in the "hard" adhesive case, it is obvious that as the wet area or the "Bonded Region" spreads (in the y-direction), the natural frequencies, at first gradually, and then, relatively sharply increases. These results of course, are the consequences of the increasing overall stiffness of the lap joint system due to the spreading of the "Bonded Region".

In the "soft" adhesive case, in Figures 8.67-8.72, the natural frequencies does not significantly change. And no sharp increases can be observed as the "Bonded Region" spreads along the y-direction. This also can be expected. It is because, due to the "soft" adhesive, the "Bonded Region" connects both adherends rather loosely and a thus relatively loose doubler joint system is created.

In Figures 8.74, 8.76 8.78, the effect of the "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" on the natural frequencies (from the first up to the fifth) in the "hard" and "soft" adhesive cases, are investigated for various boundary conditions. In the "hard" adhesive case, in Figures 8.73, 8.75, 8.77, the first two natural frequencies, in spite of the increasing "Bending Rigidity Ratio", does remain practically constant. In the higher modes, the natural frequencies increase sharply at first and after the "Bending Rigidity Ratio=2.8" they become almost flat or constant regardless of the increase in "Bending Rigidity Ratio".

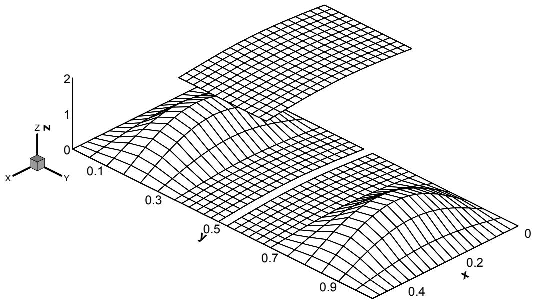
In the "soft" adhesive cases, in the Figures 8.74, 8.76, 8.78, the first and the second frequencies remain more or less constant as the "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" increases. In the third and higher modes, the natural frequencies increase.

Lastly, the direct effects of the adhesive layer elastic constants E_a, and also G_a on the dimensionless natural frequencies are investigated for the "Main PROBLEM II.a". In order to show these effects, the "Dimensionless Natural Frequencies" versus the "Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" are plotted (while the other elastic constant kept constant) in Figures 8.79 through 8.81 for various boundary condition. Similarly, the "Dimensionless Natural Frequencies" versus the "Adhesive Shear

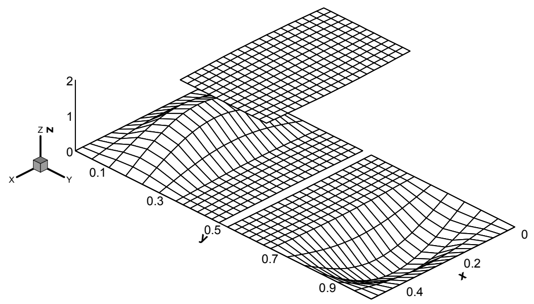
Modulus Ratio $G_a/B_{11}^{(1)}$" are presented in Figures 8.82 through 8.84 for various support condition.

It can be seen from the Figures 8.79, 8.80, 8.81, the influence of the "Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" on the natural frequencies, is not significant. In Figures 8.82, 8.83, 8.84, we can see that the "Shear Modulus Ratio $G_a/B_{11}^{(1)}$", significantly affects the natural frequencies. Also, in those Figures, one can observe a "transition region" which takes the frequencies to considerable higher levels. After then, no change is observed in the frequencies.

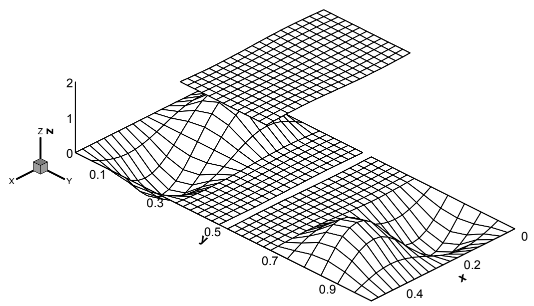
8.4.1 Natural Frequencies and Corresponding Mode Shapes for "Main PROBLEM IIa"



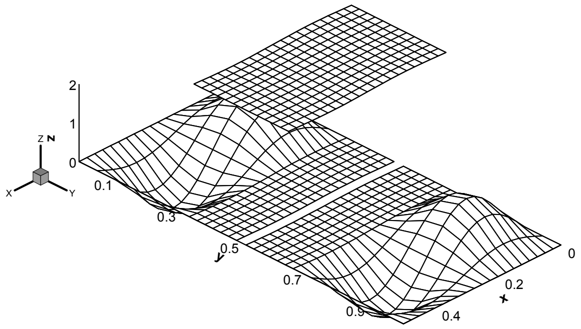
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 1005.579$



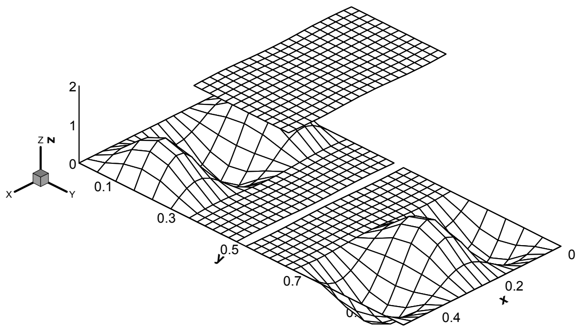
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 1057.842$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 1204.653$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22} = 1209.066$

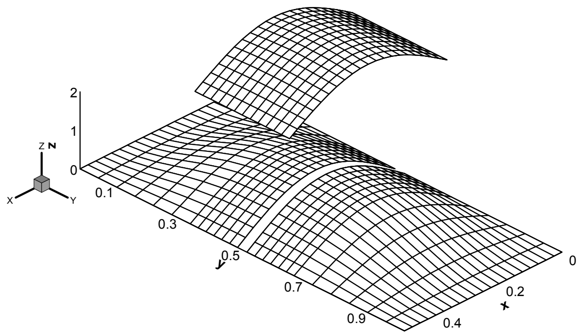


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31} = 1613.390$

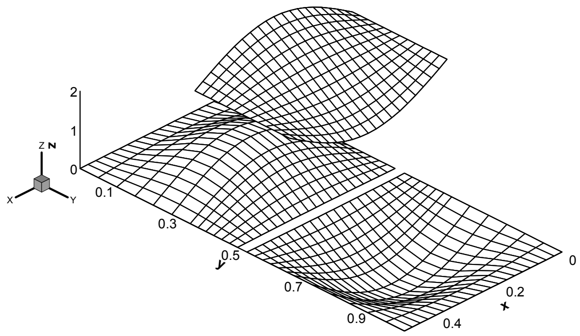
("Hard" Adhesive Case)

Fig 8.61 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

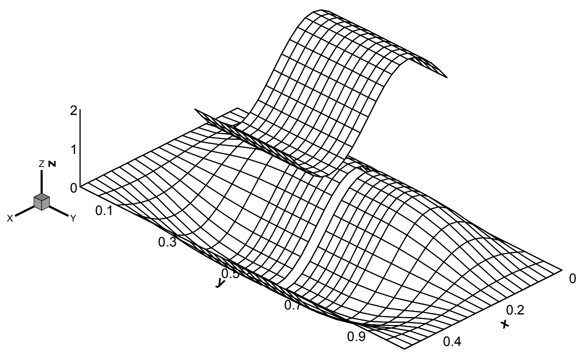
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



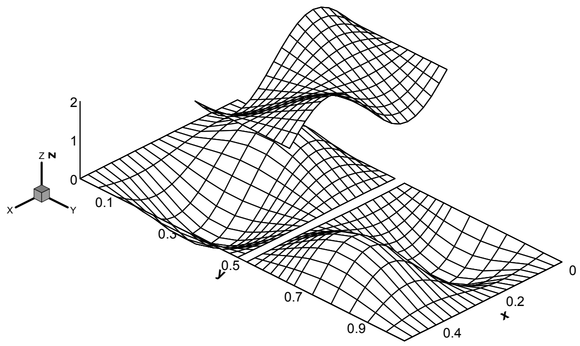
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=49.307



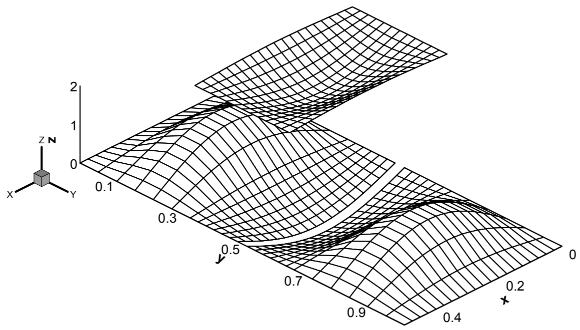
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}$=200.524



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{21}$=220.317



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=363.804

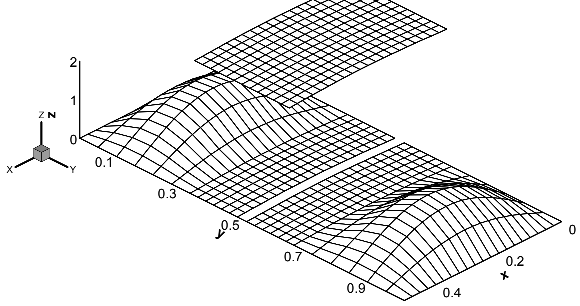


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=659.942

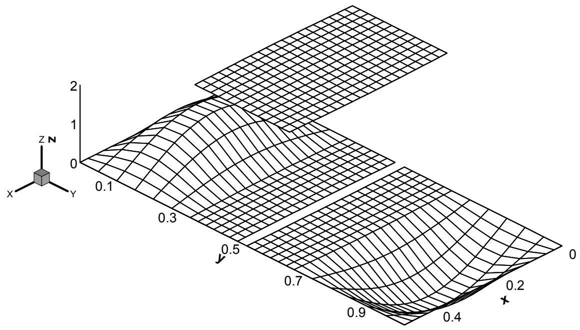
("Soft" Adhesive Case)

Fig.8.62 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

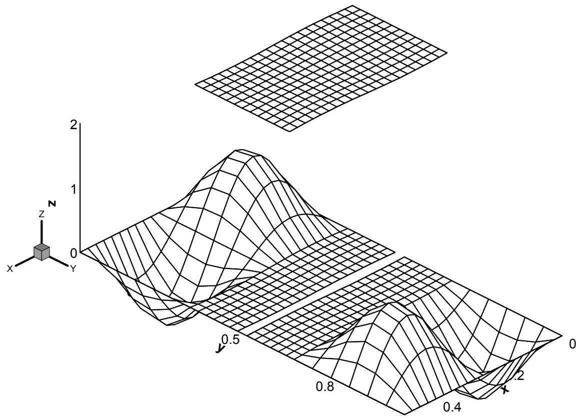
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



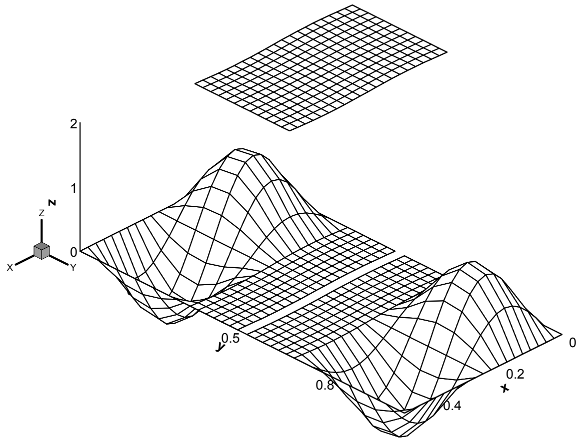
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=535.732



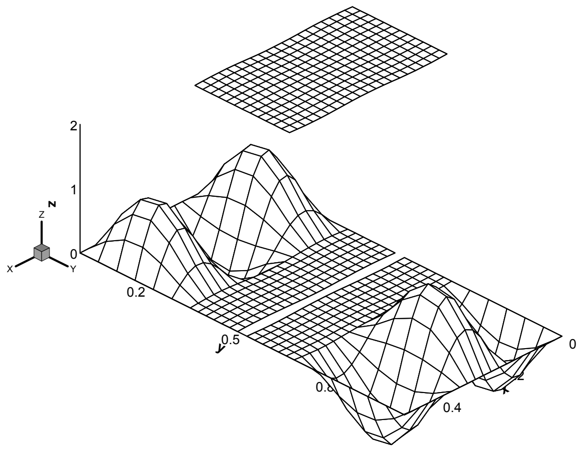
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=549.712



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=687.711



d) Fourth Mode with $\overline{\Omega}_4$= $\overline{\omega}_{22}$=690.588

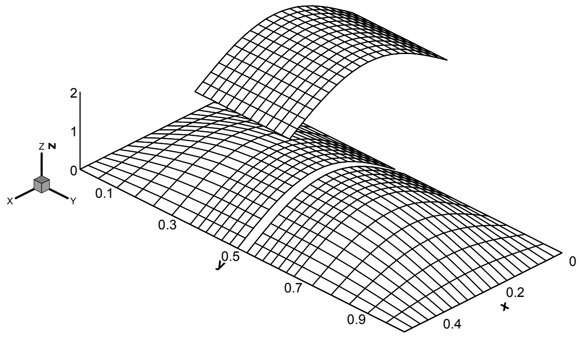


e) Fifth Mode with $\overline{\Omega}_5$= $\overline{\omega}_{31}$=1092.787

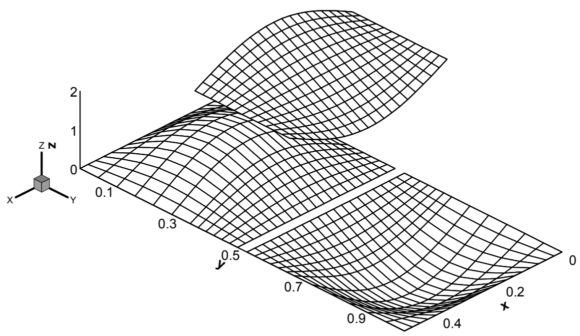
("<u>Hard</u>" Adhesive Case)

Fig.8.63 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

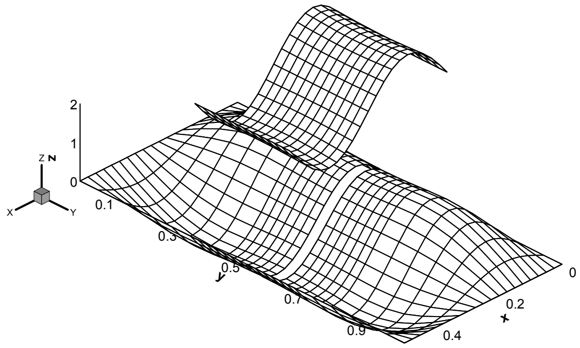
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFS)



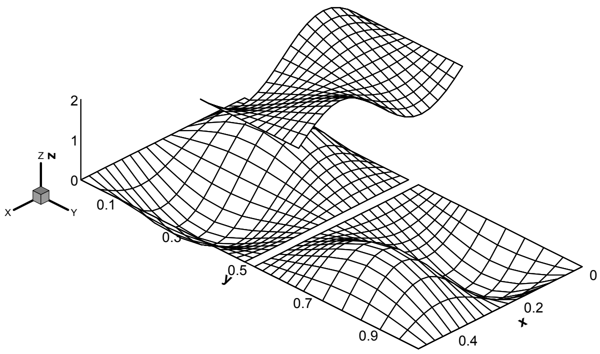
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 35.776$



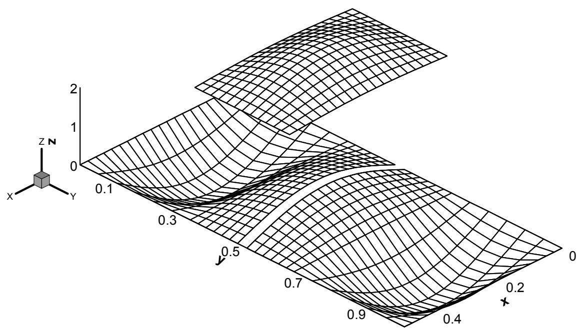
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 112.564$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 191.054$



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{22}$=253.241

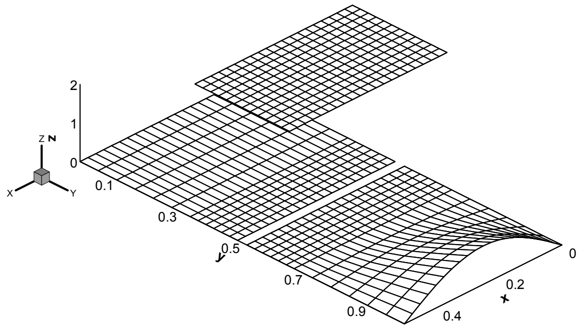


e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{13}$=378.531

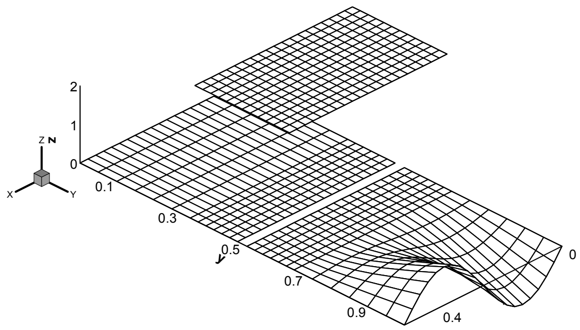
("Soft" Adhesive Case)

Fig.8.64 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

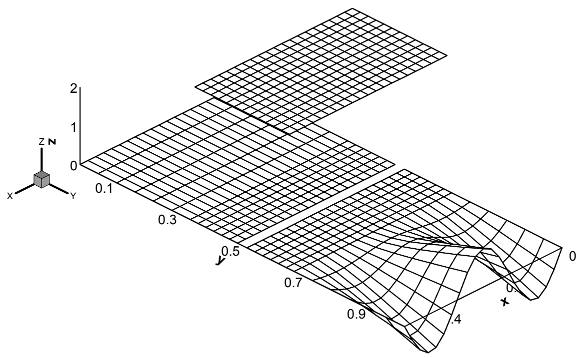
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFS)



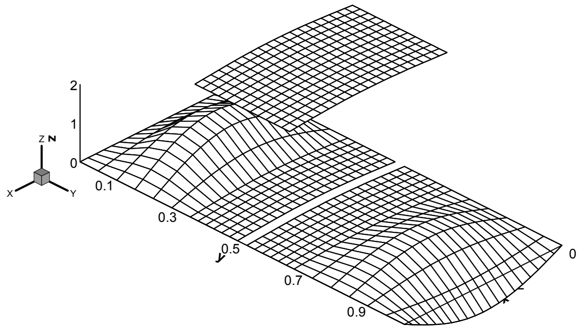
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=39.800



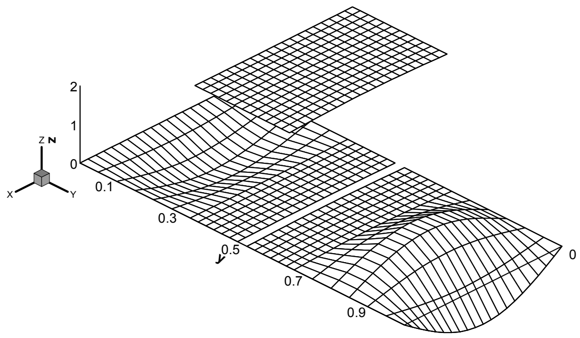
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=125.909



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=450.831



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=1015.306

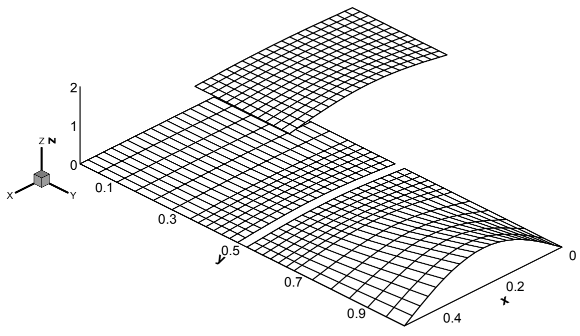


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=1080.462

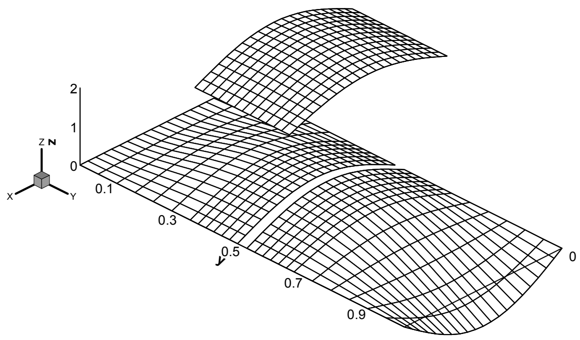
("Hard" Adhesive Case)

Fig.8.65 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

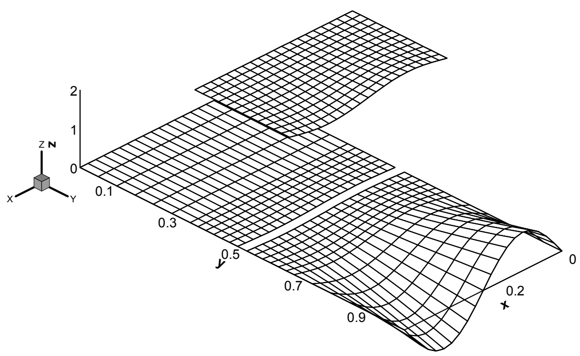
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



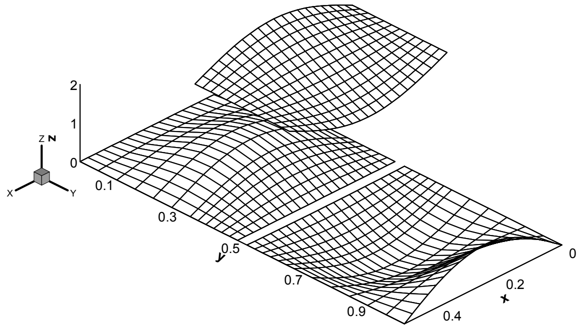
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 15.161$



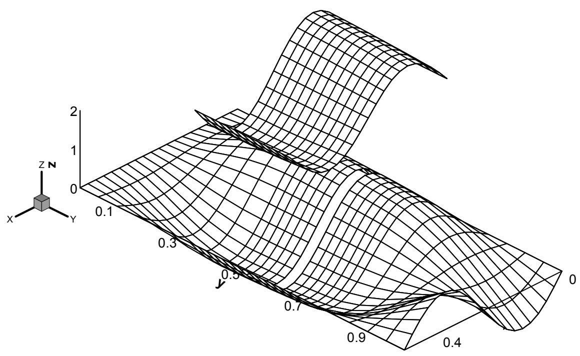
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 51.276$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 98.636$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=205.961

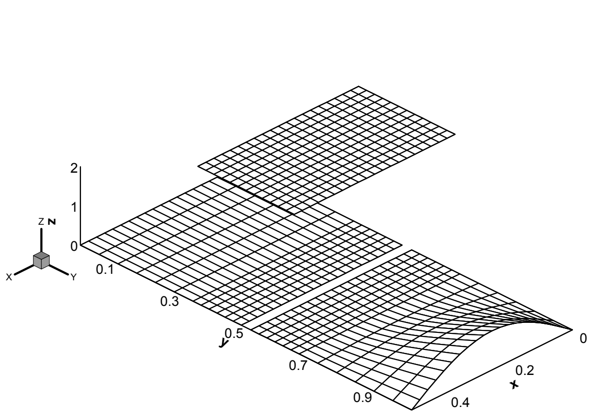


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=205.961

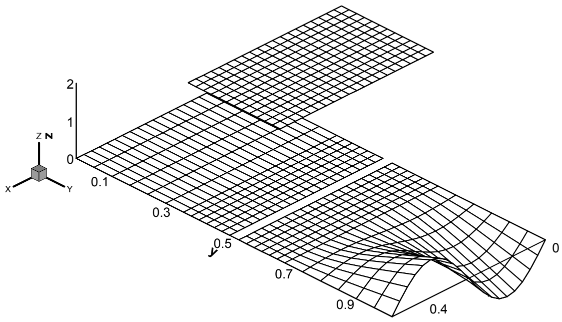
("Soft" Adhesive Case)

Fig.8.66 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

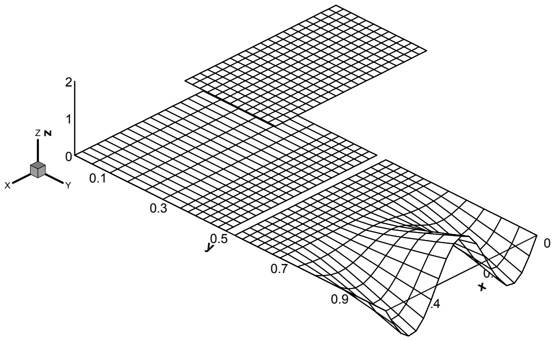
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



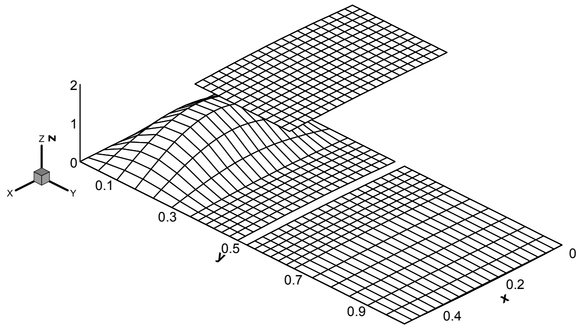
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 39.800$



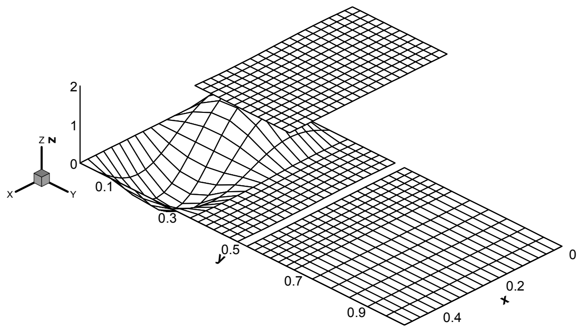
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 125.909$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 450.831$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=542.453

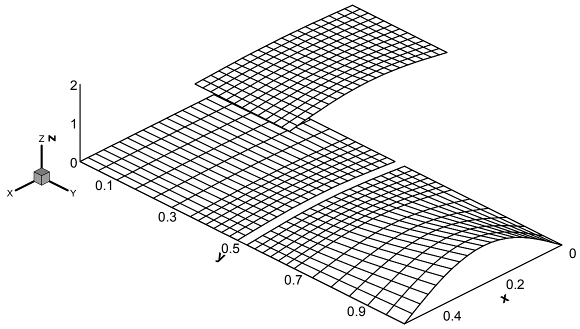


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=689.145

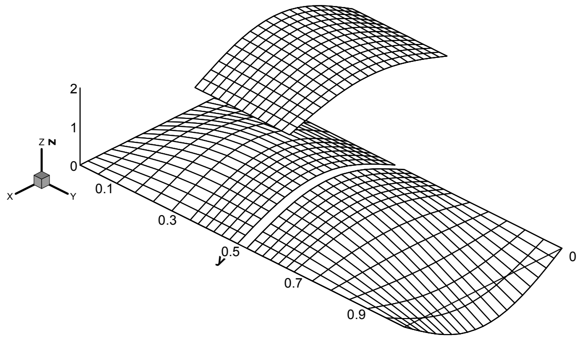
("Hard" Adhesive Case)

Fig.8.67 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

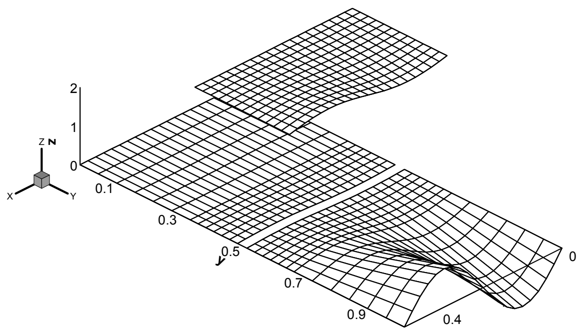
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFF)



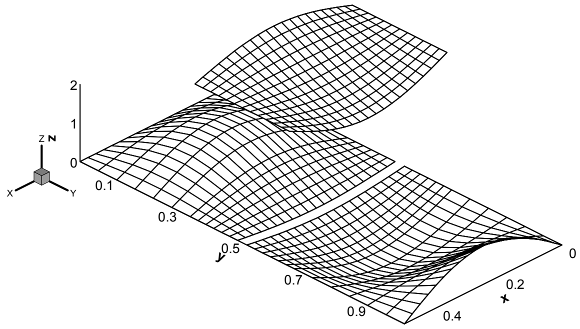
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 15.132$



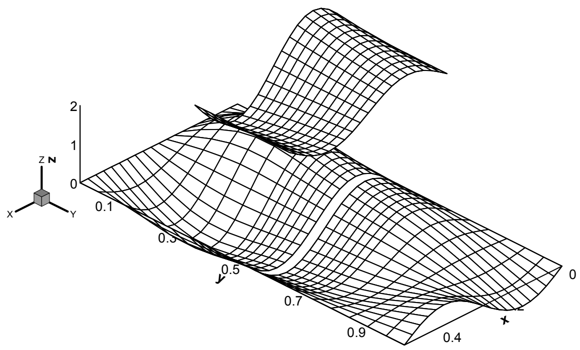
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 41.579$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 98.618$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=153.876

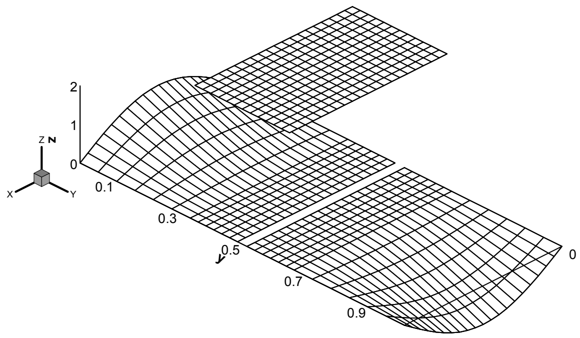


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=201.673

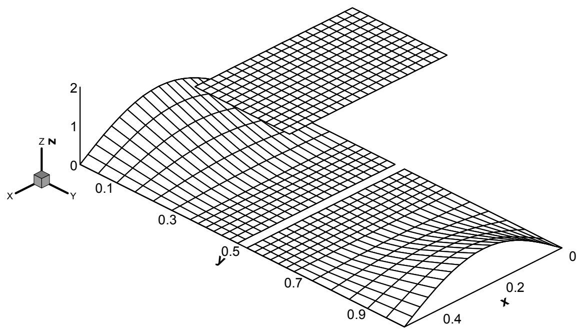
("Soft" Adhesive Case)

Fig.8.68 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

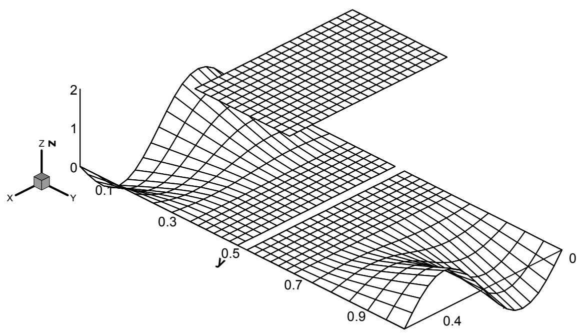
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFF)



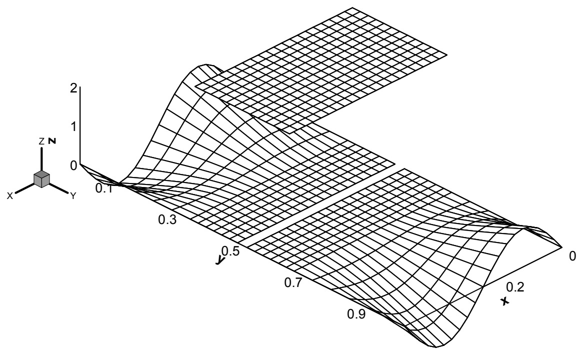
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 39.755$



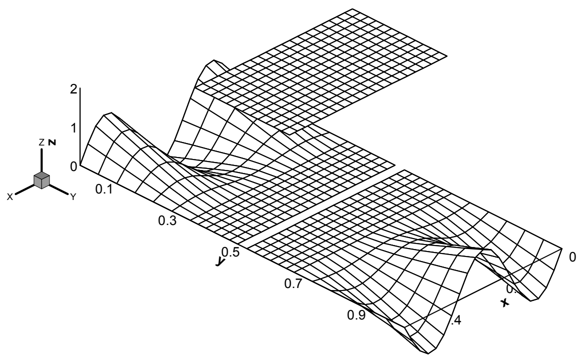
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 39.845$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 125.780$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=126.038

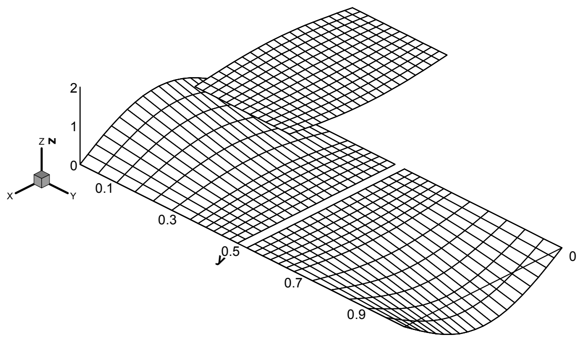


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=450.666

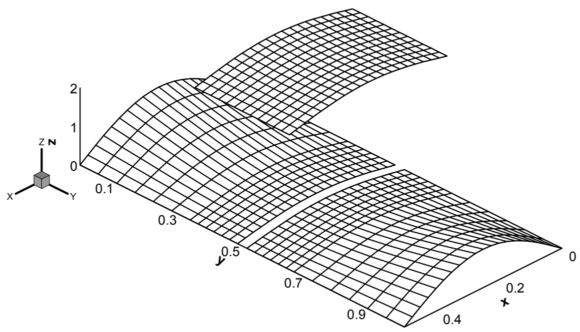
("Hard" Adhesive Case)

Fig.8.69 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

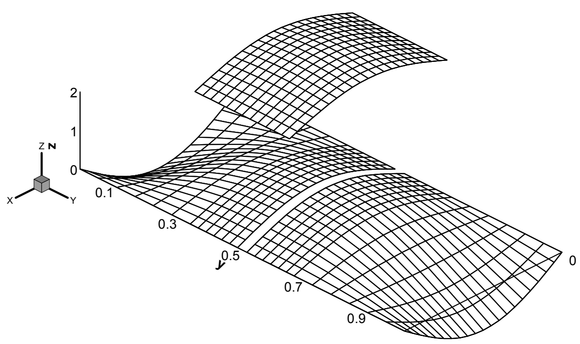
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



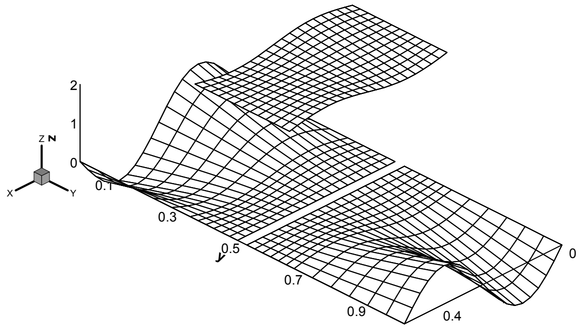
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=14.435



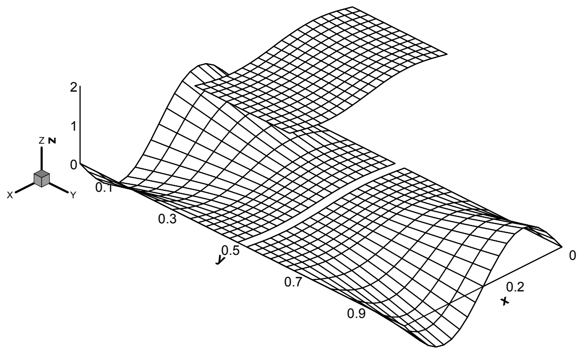
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}$=15.654



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{13}$=53.687



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{21}$=97.015



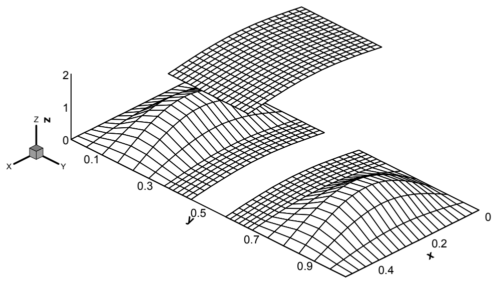
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=100.244

("Soft" Adhesive Case)

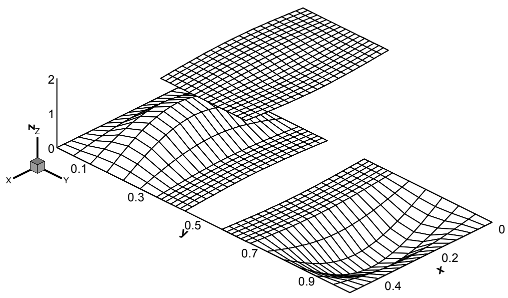
Fig.8.70 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., b_1=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)

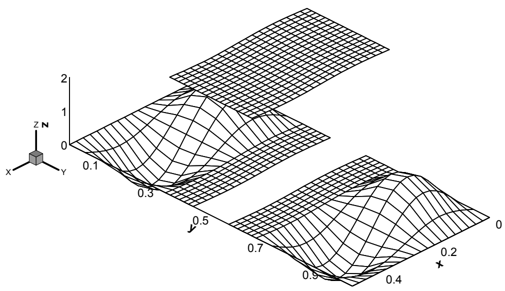
8.4.2 Natural Frequencies and Corresponding Mode Shapes for "Special Case of Main PROBLEM IIa"



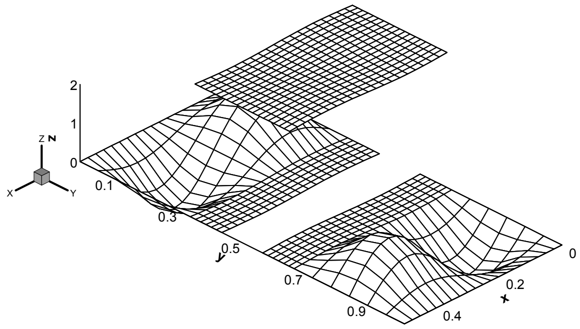
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=925.059



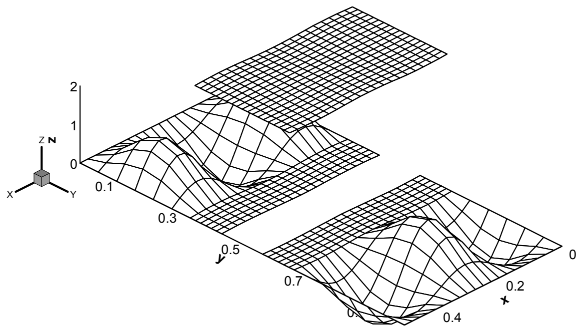
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}$=1023.842



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{21}$=1208.468



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=1209.277

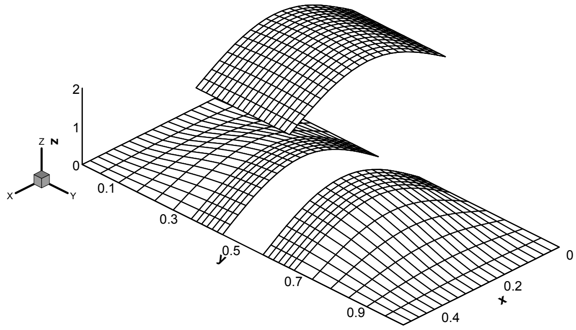


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=1623.867

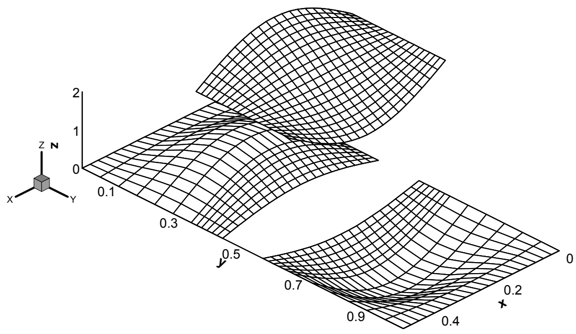
("Hard" Adhesive Case)

Fig 8.71 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

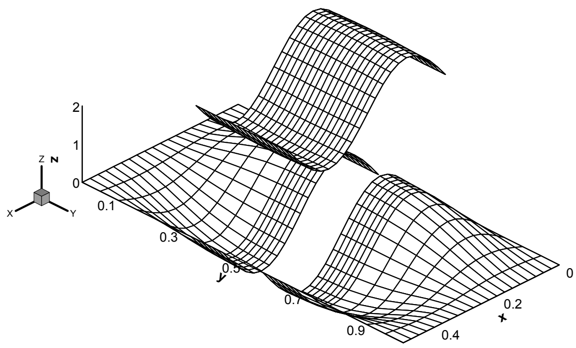
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



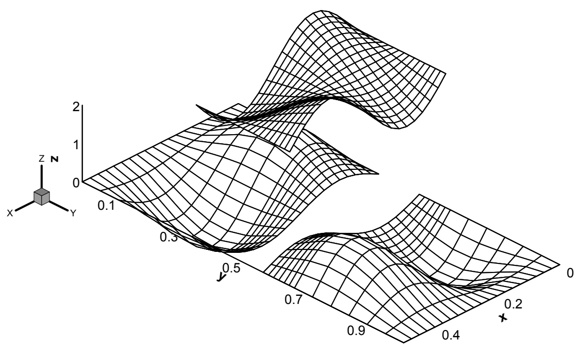
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=44.380



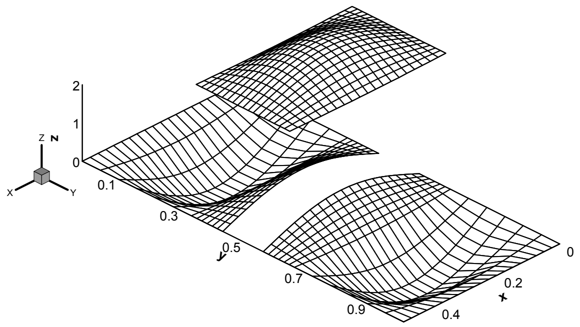
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=181.109



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=200.948



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=343.393

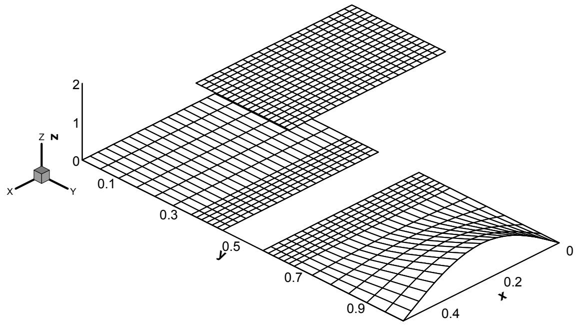


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=673.881

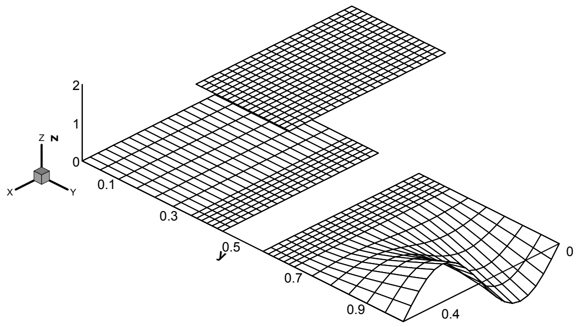
(“Soft” Adhesive Case)

Fig 8.72 Mode Shapes and Dimensionless Natural Frequencies of “Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap”

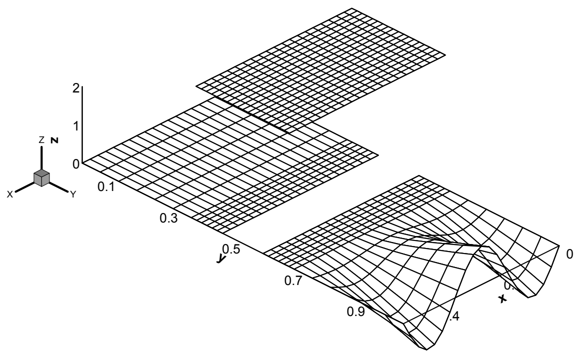
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



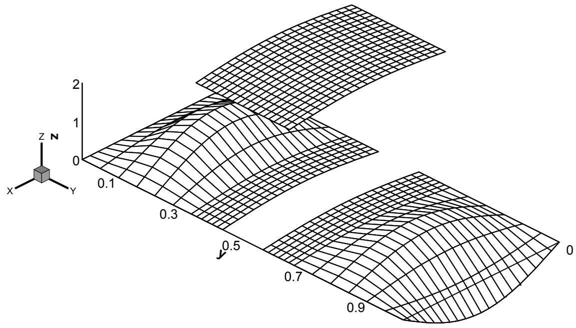
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=39.930



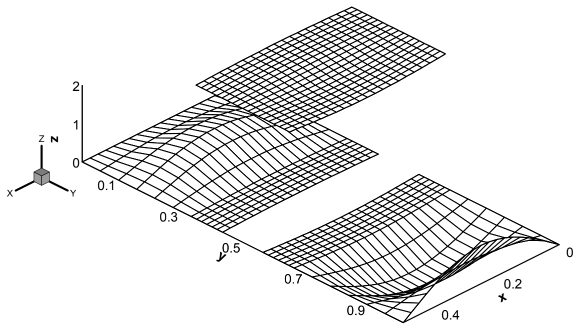
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=126.206



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=451.278



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=970.103

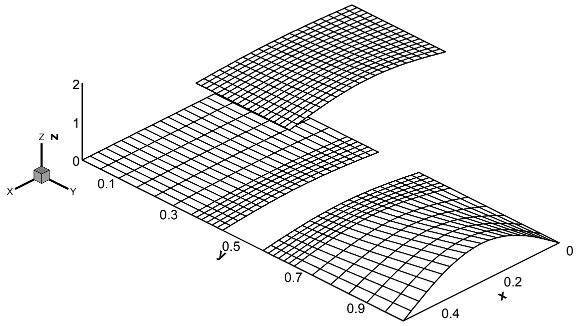


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=1041.537

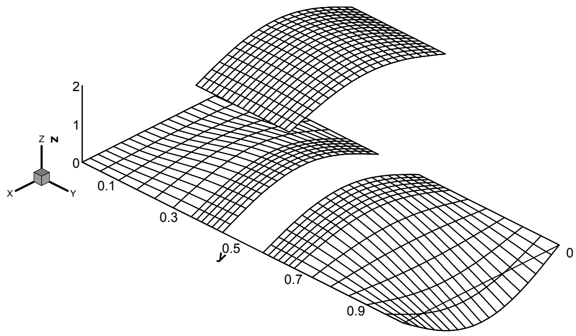
("Hard" Adhesive Case)

Fig 8.73 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

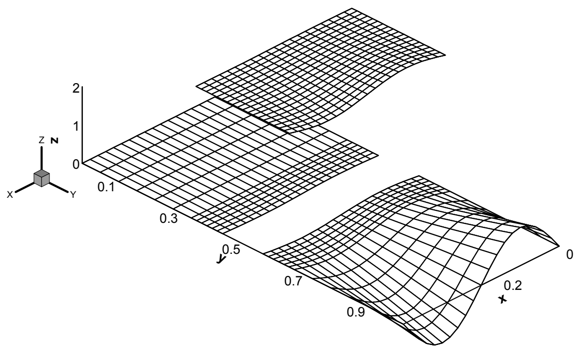
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



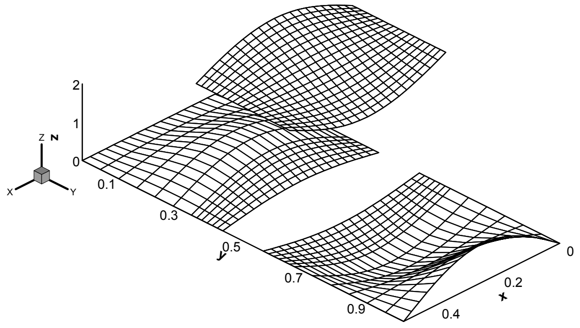
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 13.513$



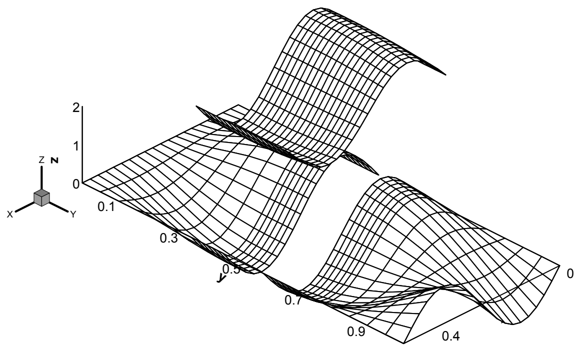
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 46.648$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 97.136$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=186.796



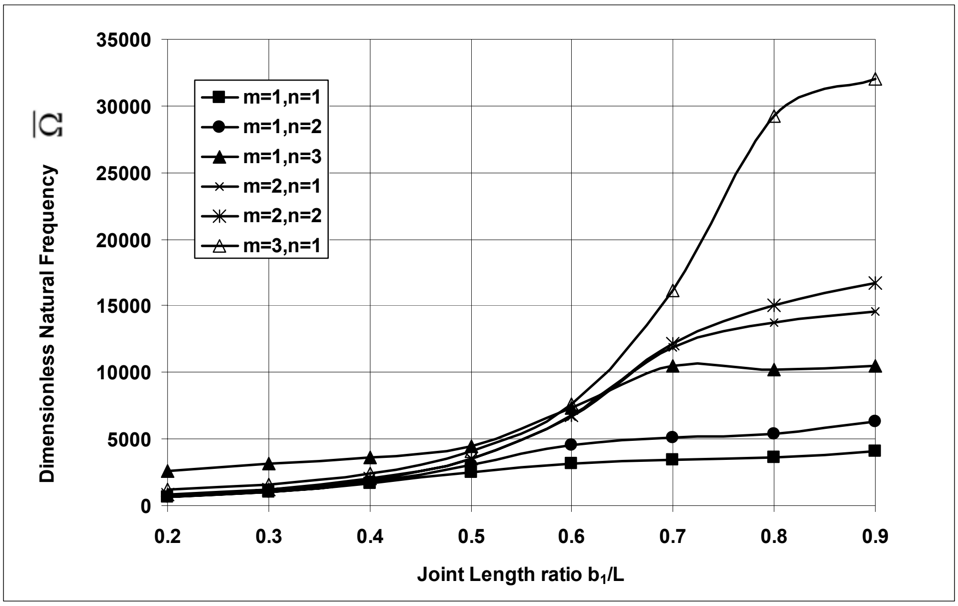
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=241.403

("Soft" Adhesive Case)

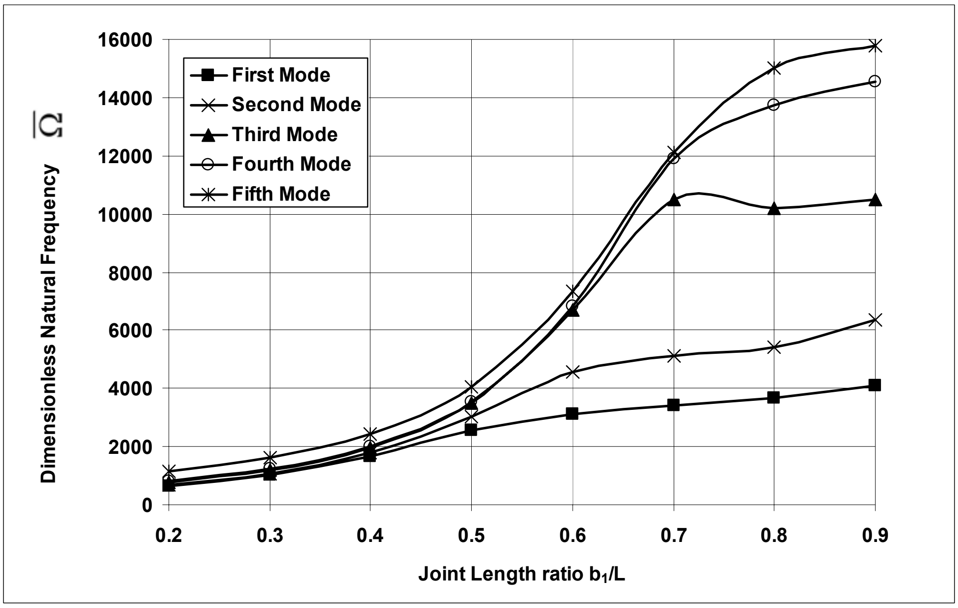
Fig 8.74 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)

8.4.3 Some Parametric Studies for “Main PROBLEM Ia”



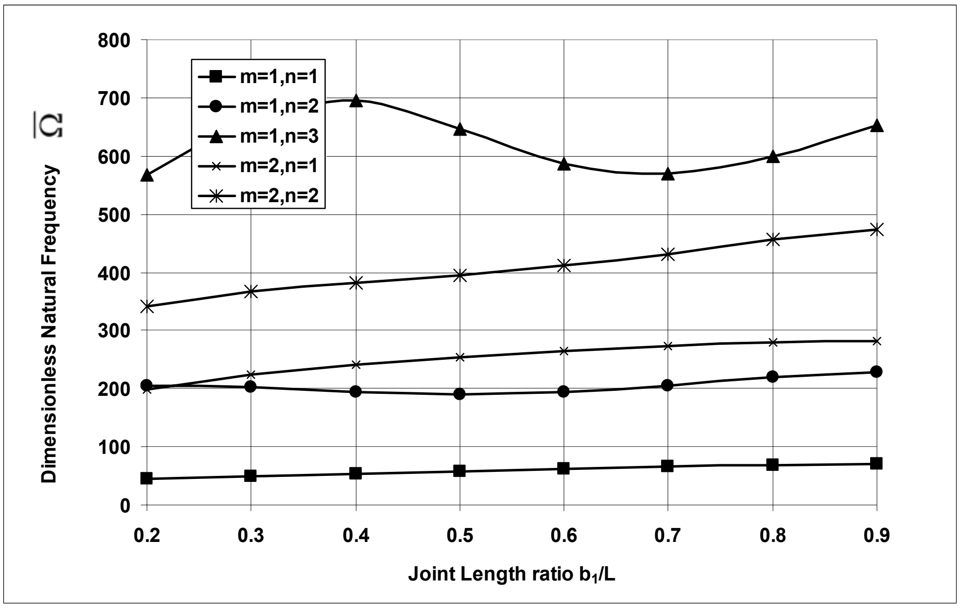
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFC) B.C.’s, ‘‘<u>Hard</u>’’ Adhesive



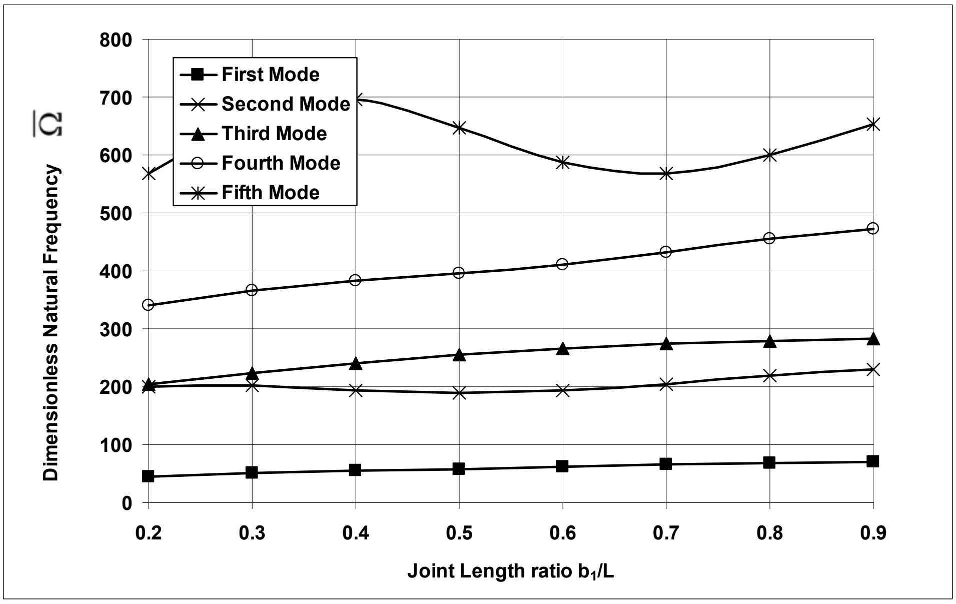
b) “Various Modes with (FFCFFC) B.C.’s, ‘‘<u>Hard</u>’’ Adhesive

Fig 8.75 “Dimensionless Natural Freq. ($\overline{\Omega}$)” versus “Joint Length ($\ell_I+\ell_{II}$)/ L” in “Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)”

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=varies, $\widetilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



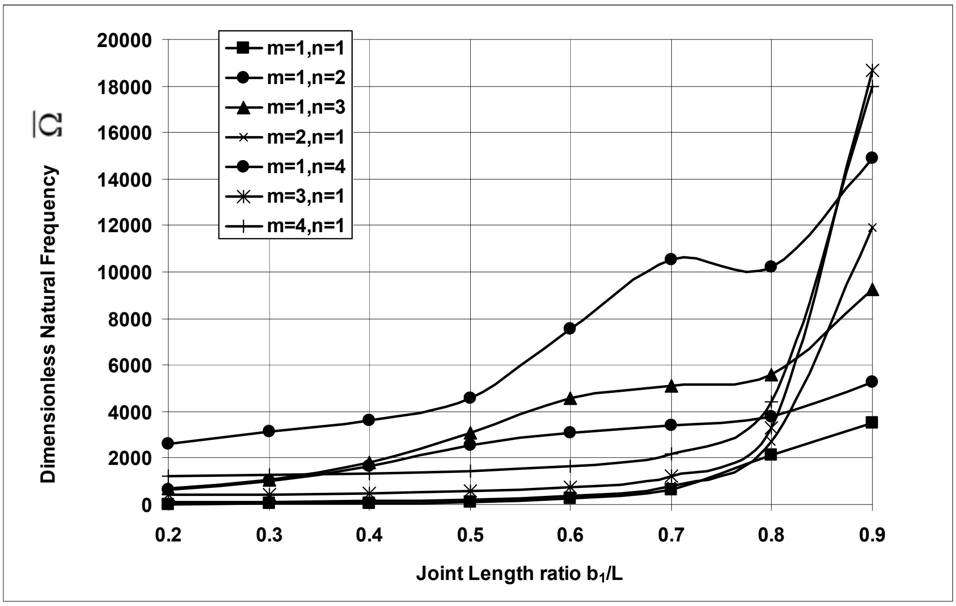
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFC) B.C.'s, ''<u>Soft</u>'' Adhesive



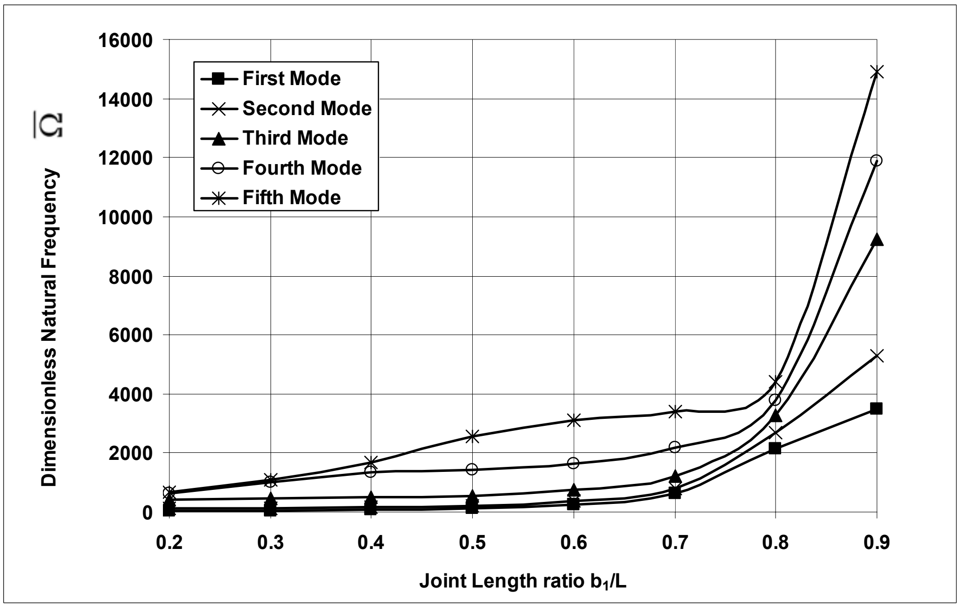
b) "Various Modes with (FFCFFC) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.76 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length (ℓ_I+ℓ_{II})/ L" in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



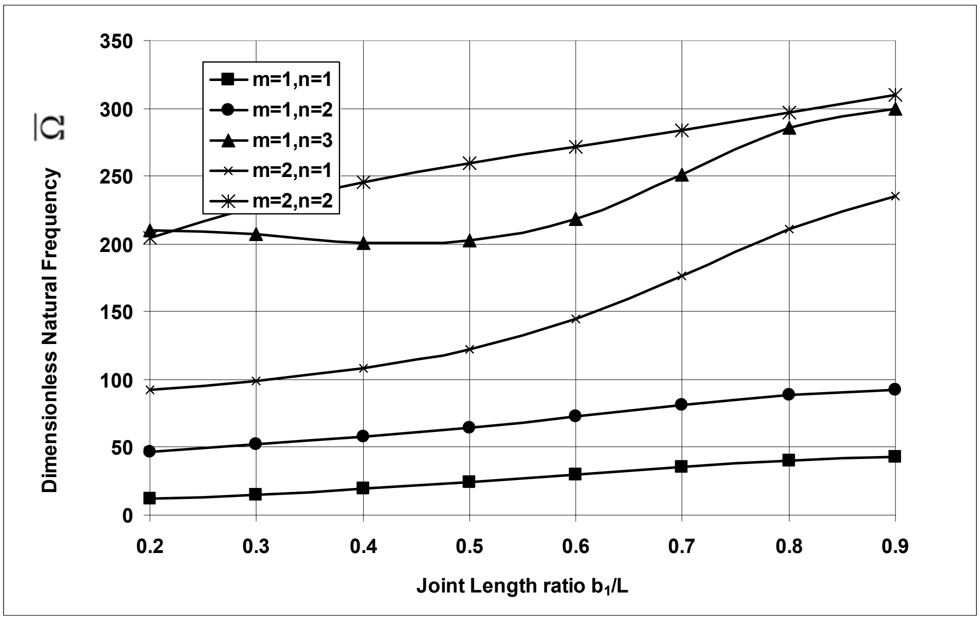
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFF) B.C.'s, ''Hard'' Adhesive



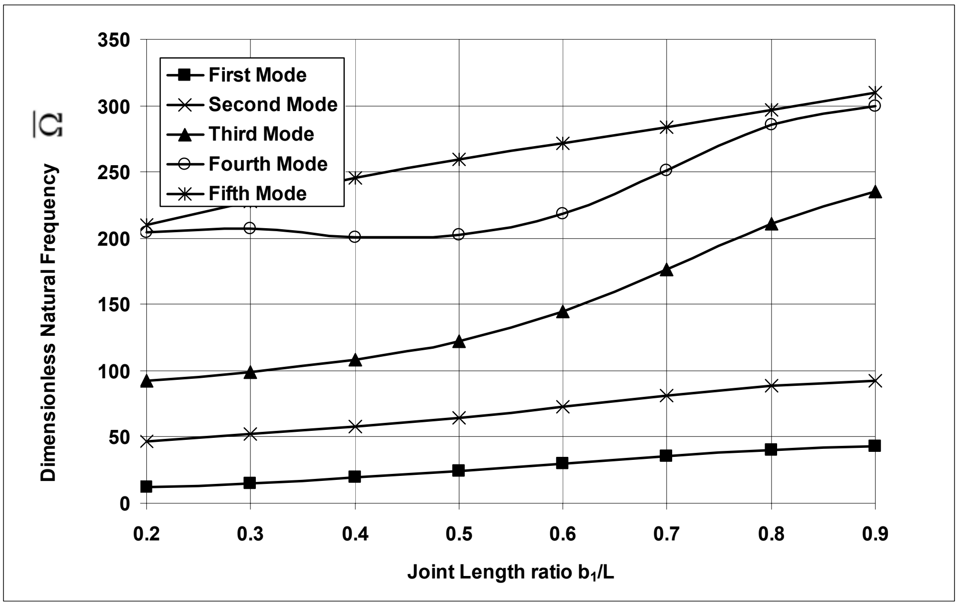
b) "Various Modes with (FFCFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.77 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length ($\ell_I+\ell_{II}$)/ L" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



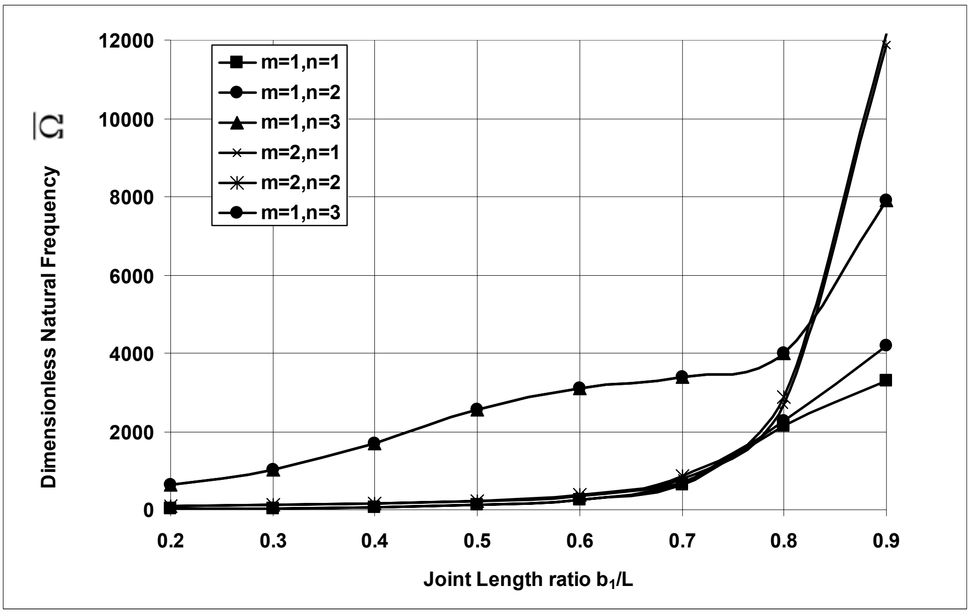
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFF) B.C.'s, ''Soft'' Adhesive



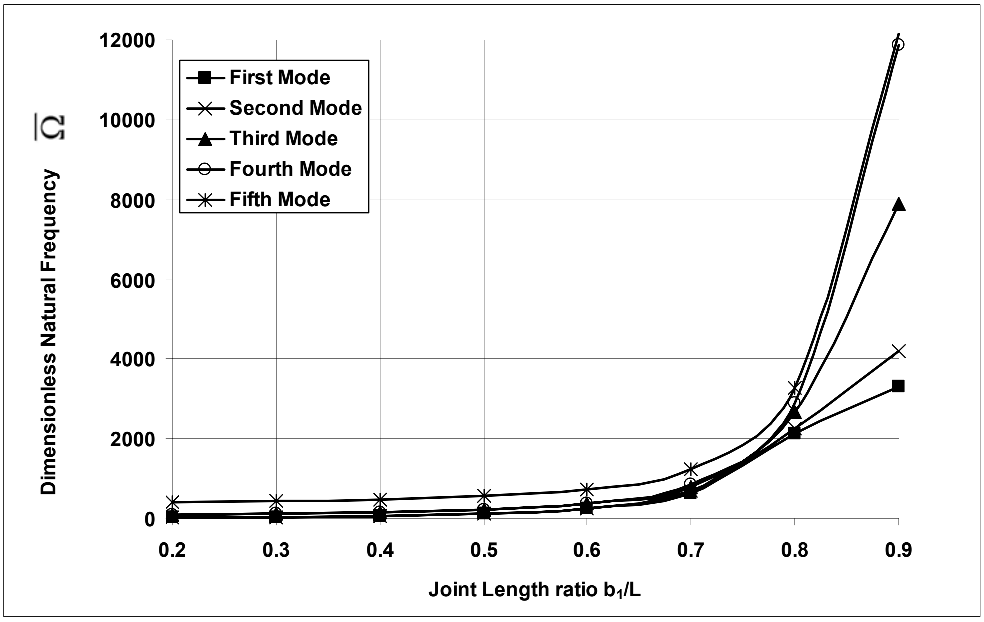
b) "Various Modes with (FFCFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.78 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length (ℓ_I+ℓ_{II})/ L" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=varies, $\widetilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



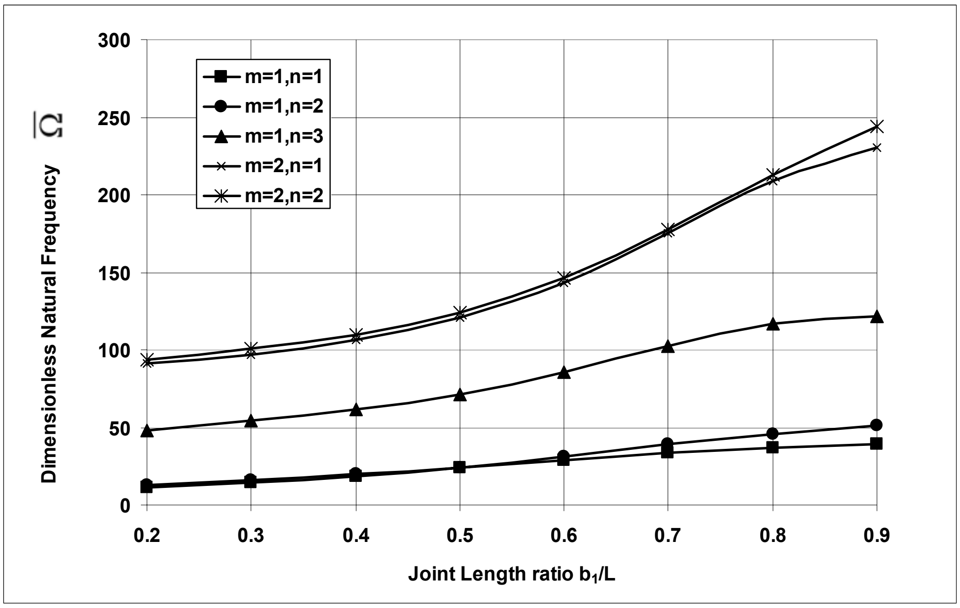
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFFFFF) B.C.'s, "Hard" Adhesive



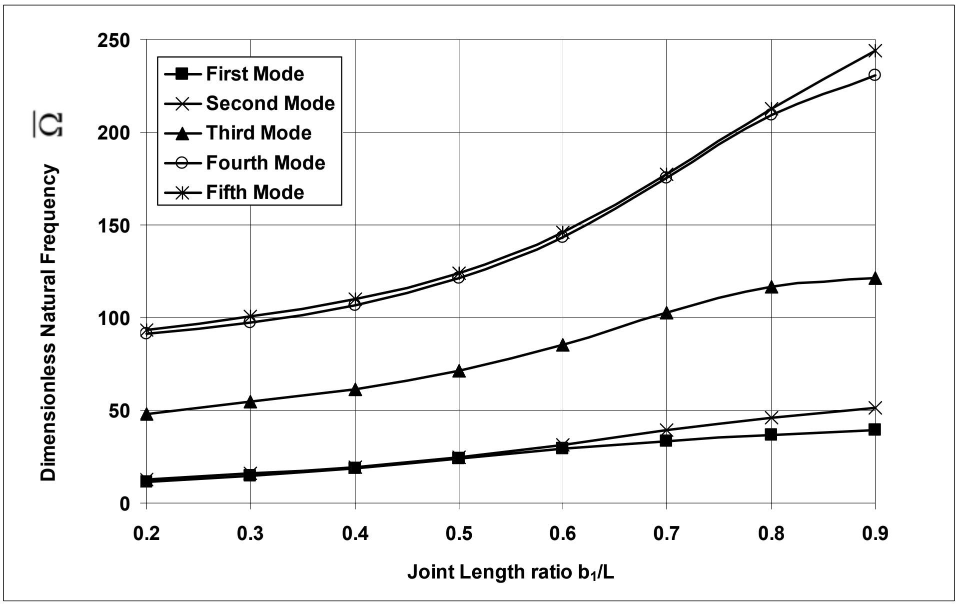
b) "Various Modes with (FFFFFF) B.C.'s, "Hard" Adhesive

Fig 8.79 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length ($\ell_I+\ell_{II}$)/ L" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



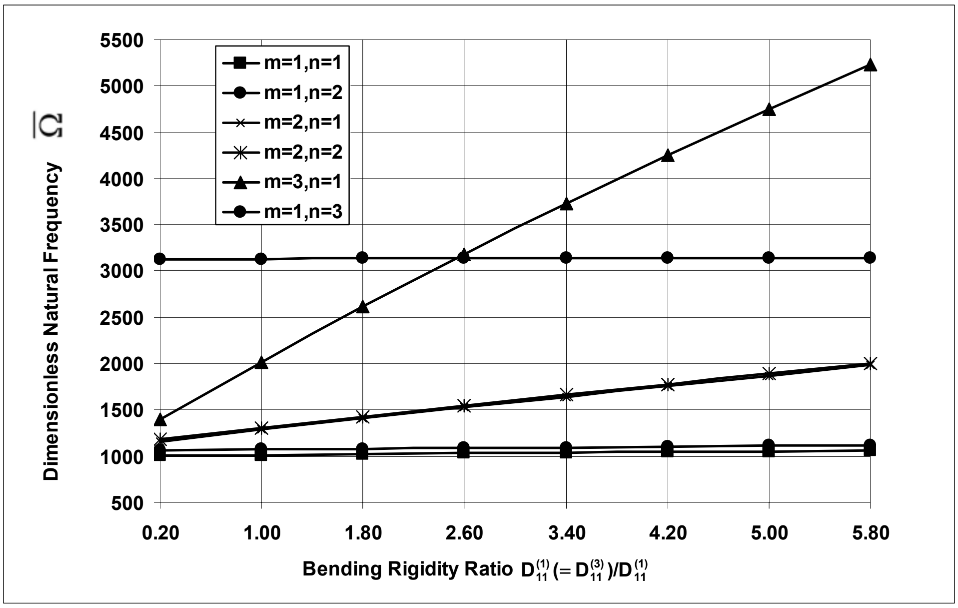
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFFFFF) B.C.'s, ''Soft'' Adhesive



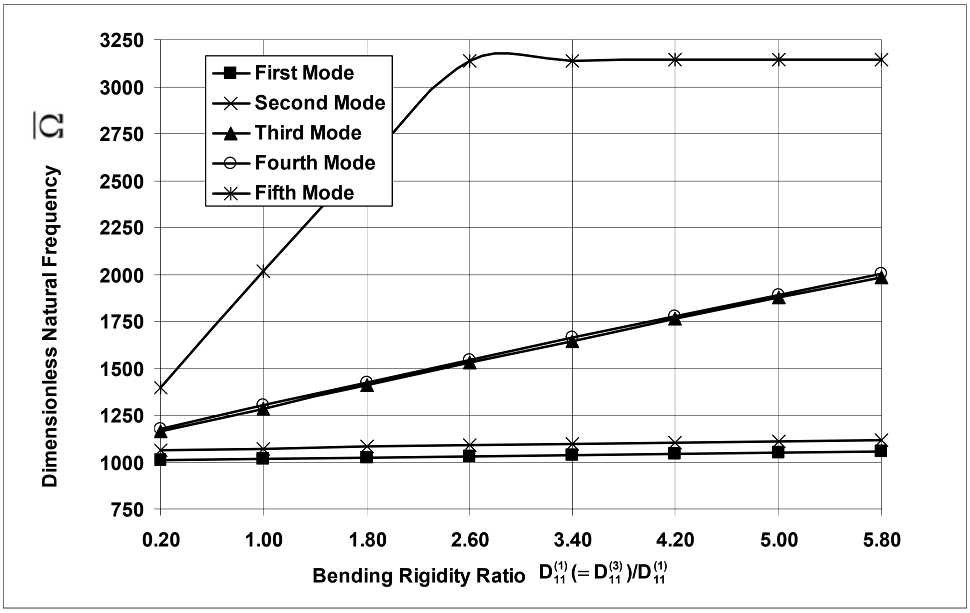
b) "Various Modes with (FFFFFF) B.C.'s, ''Soft' Adhesive

Fig 8.80 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length ($\ell_I+\ell_{II}$)/ L" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



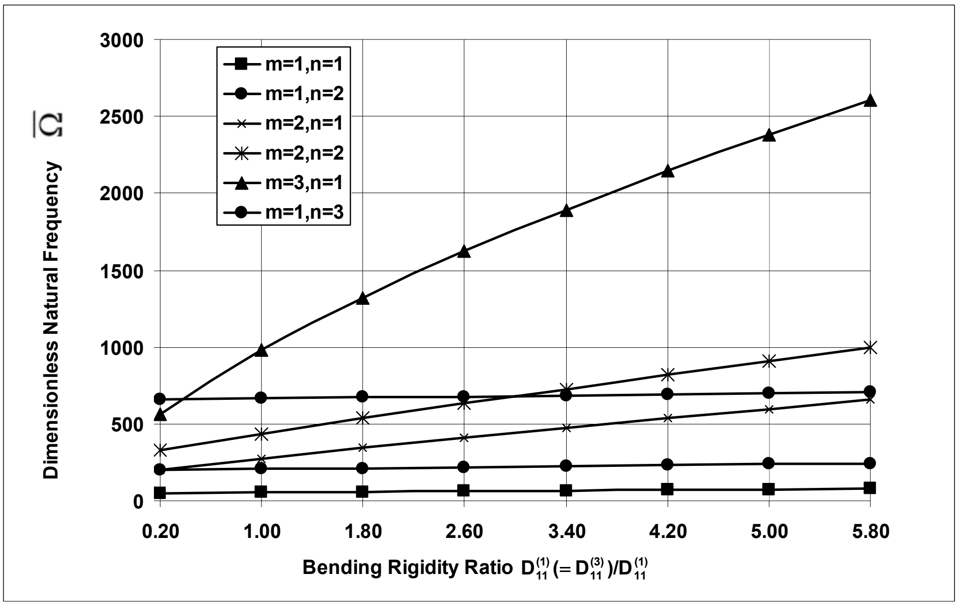
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFC) B.C.'s, ''Hard'' Adhesive



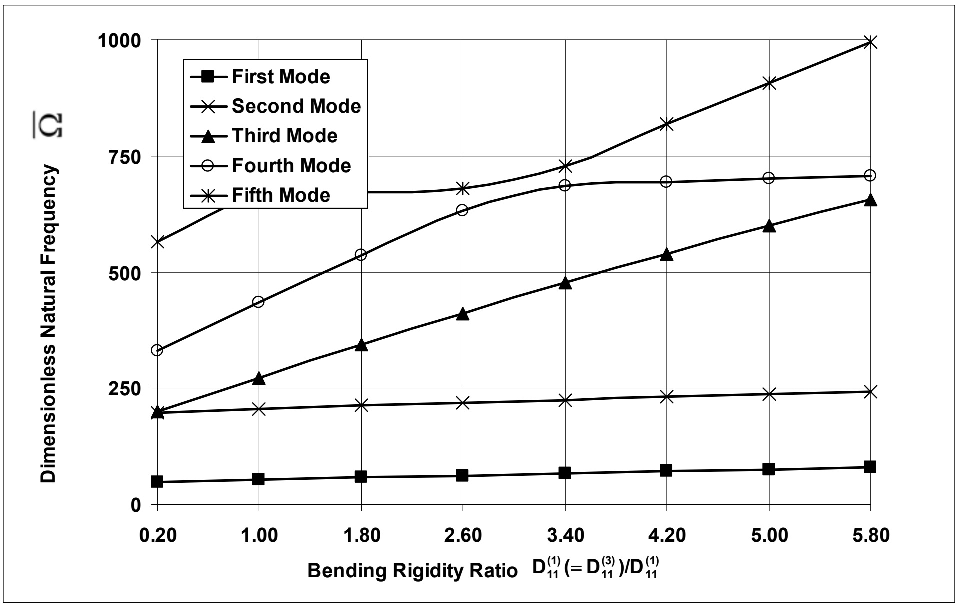
b) "Various Modes with (FFCFFC) B.C.'s, ''Hard'' Adhesive

Fig 8.81 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



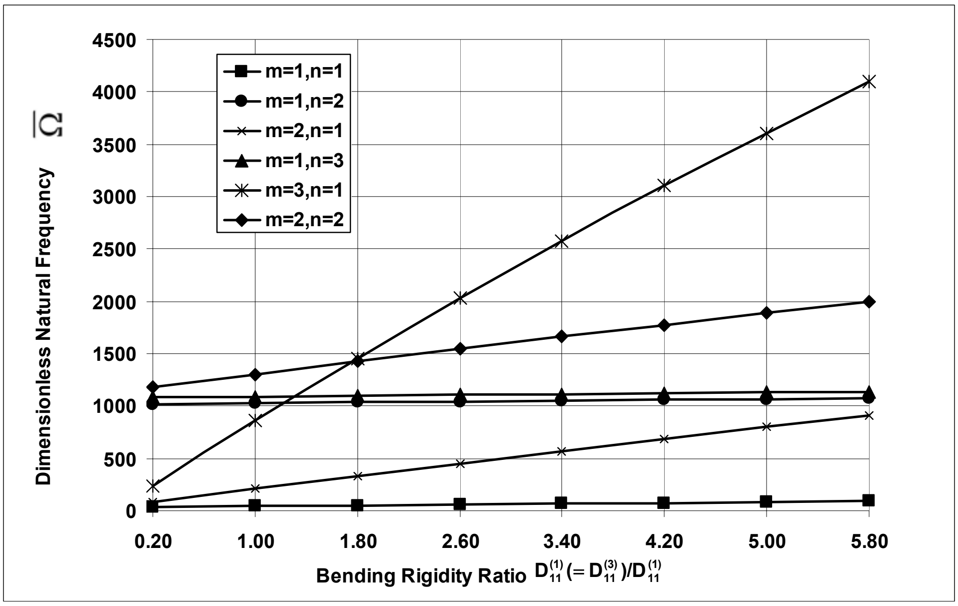
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFC) B.C.'s, ''Soft'' Adhesive



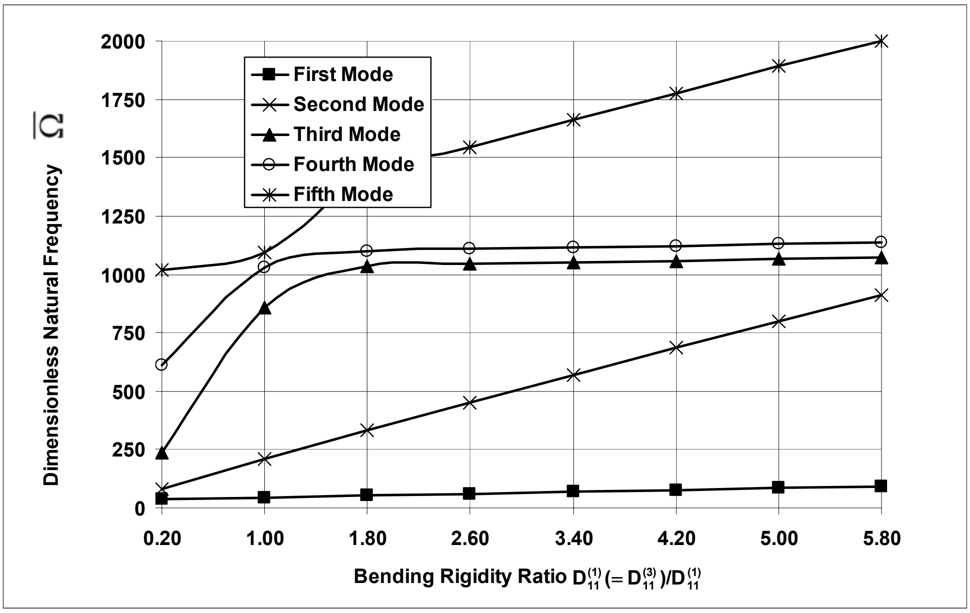
b) "Various Modes with (FFCFFC) B.C.'s, ''Soft'' Adhesive

Fig 8.82 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



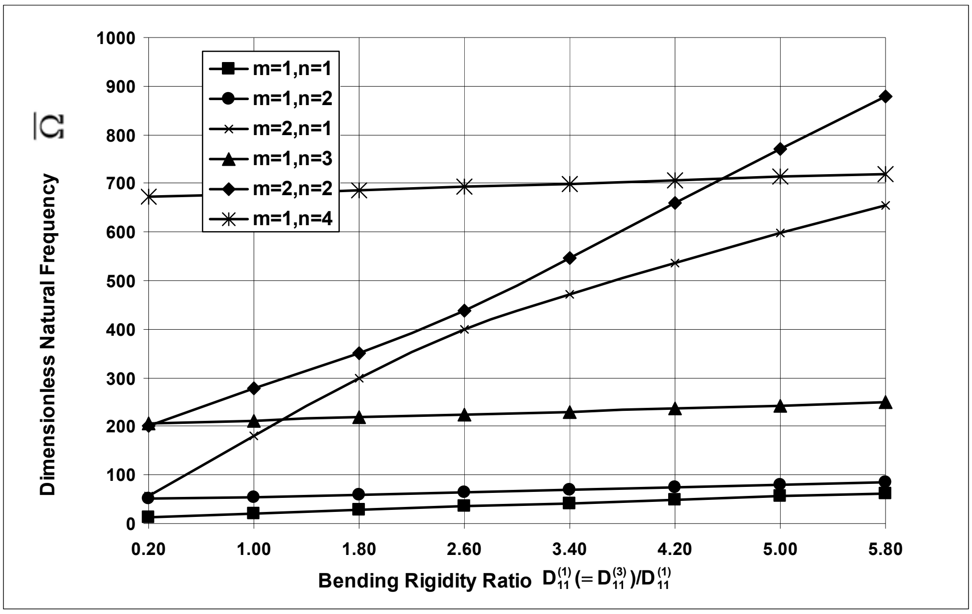
b) "Various Modes with (FFCFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.83 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D^{(2)}_{11} \left(=D^{(3)}_{11}\right)/ D^{(1)}_{11}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

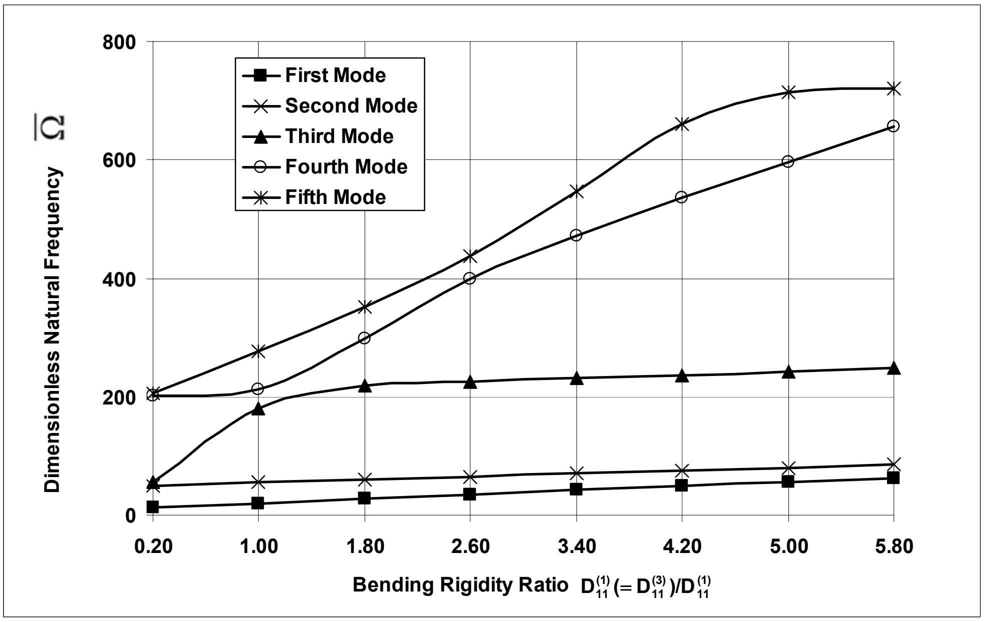
($D^{(2)}_{11}$ and $D^{(3)}_{11}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\widetilde{b}$ =0.5 m, a=0.5 m. L=1 m)

(Boundary Conditions in y-direction FFCFFF)



a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFF) B.C.'s, ''Soft'' Adhesive

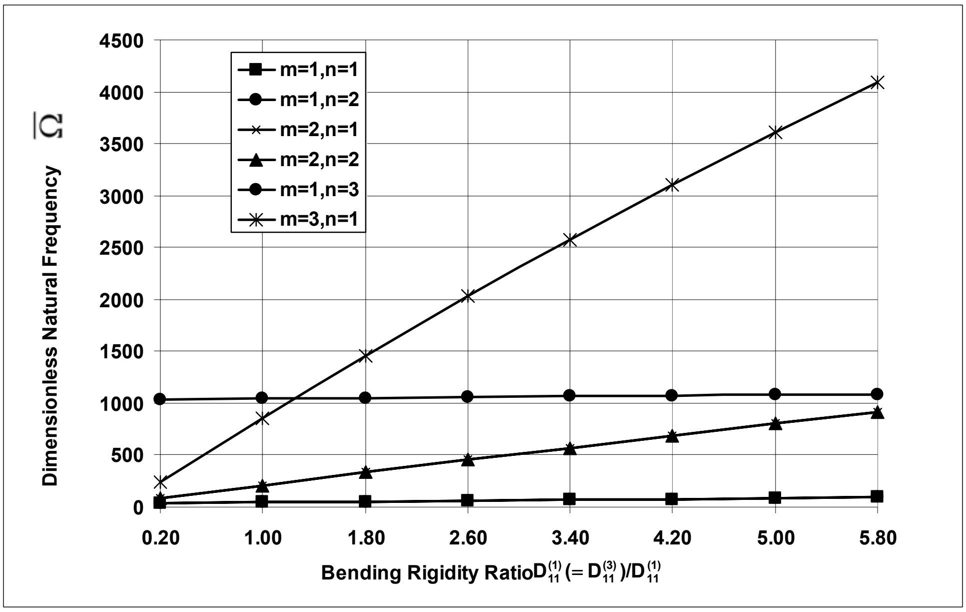


b) "Various Modes with (FFCFFF) B.C.'s, ''Soft'' Adhesive

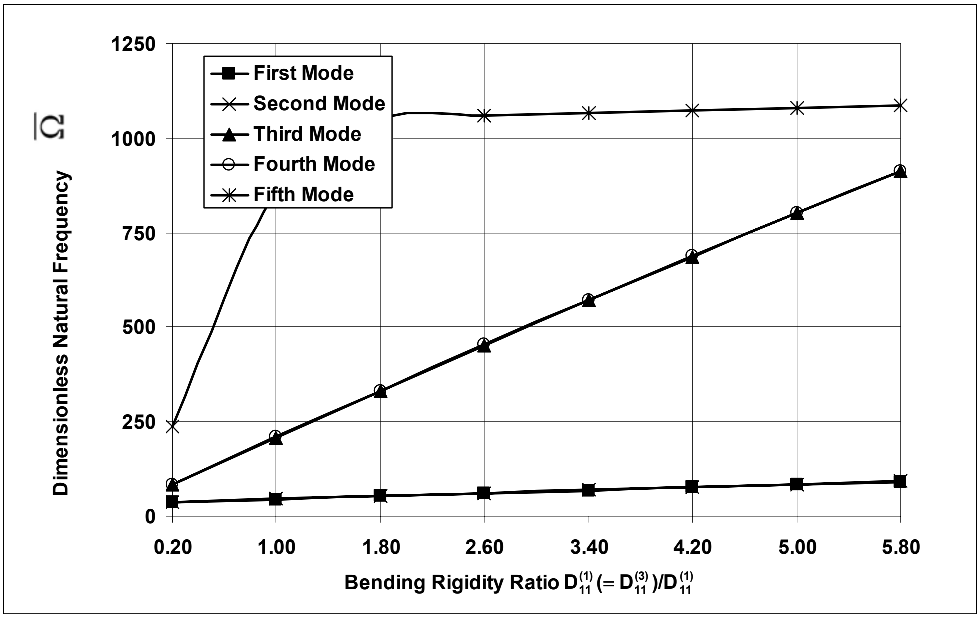
Fig 8.84 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



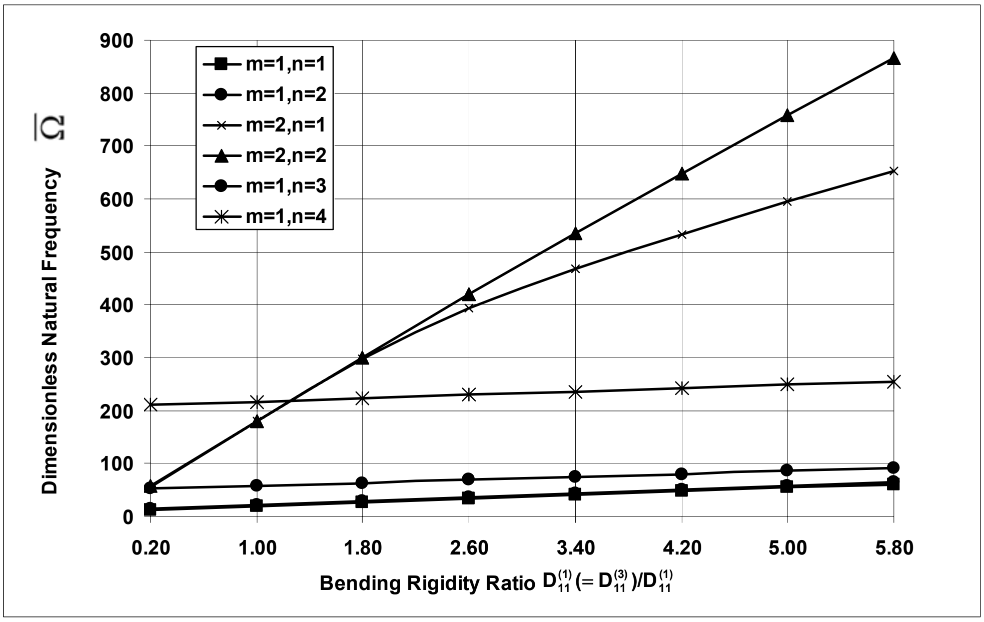
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFFFFF) B.C.'s, ''Hard'' Adhesive



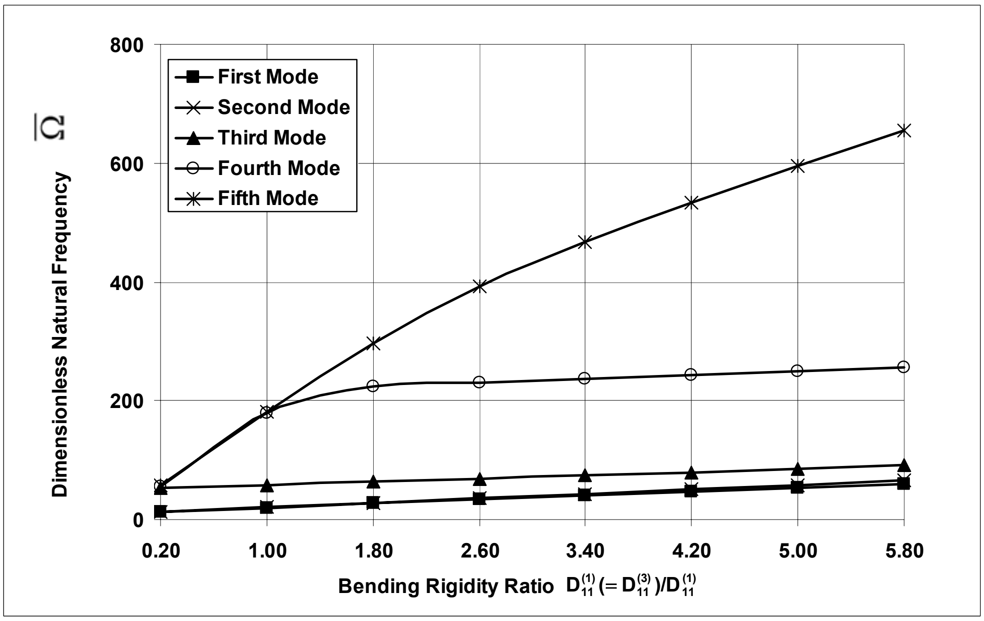
b) "Various Modes with (FFFFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.85 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



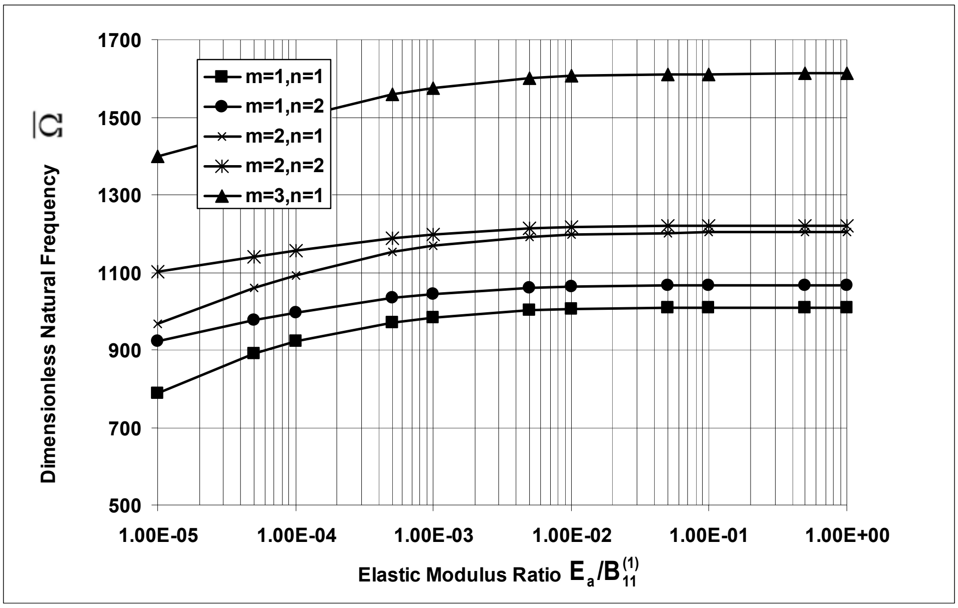
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



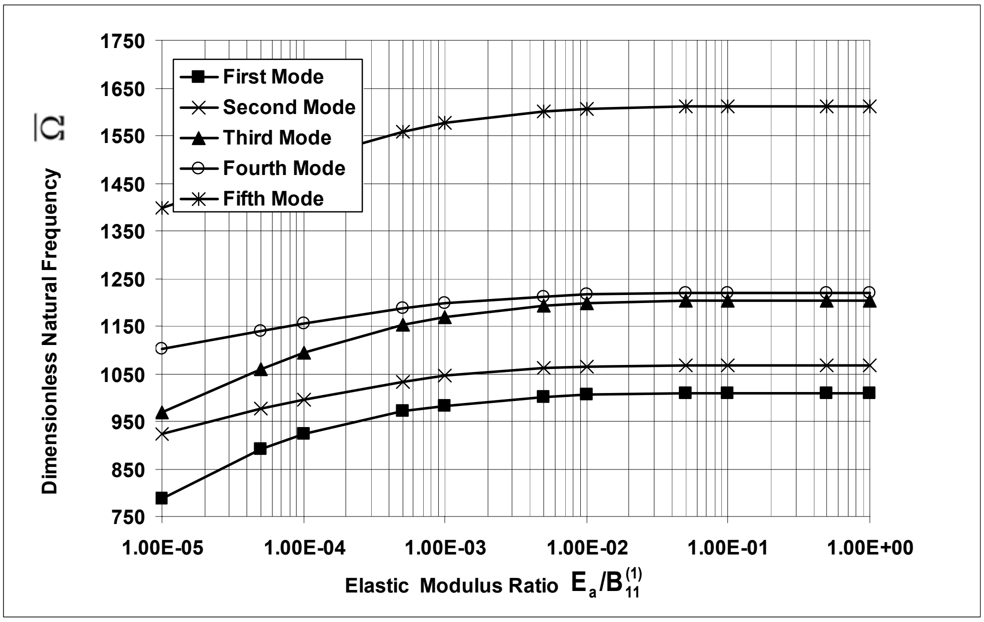
b) "Various Modes with (FFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.86 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\widetilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



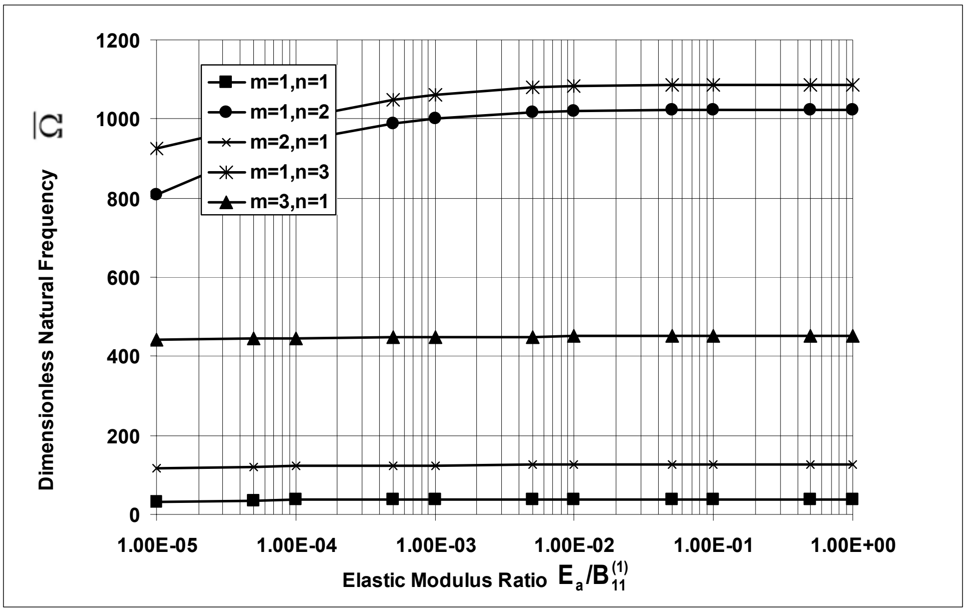
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFC) B.C.'s



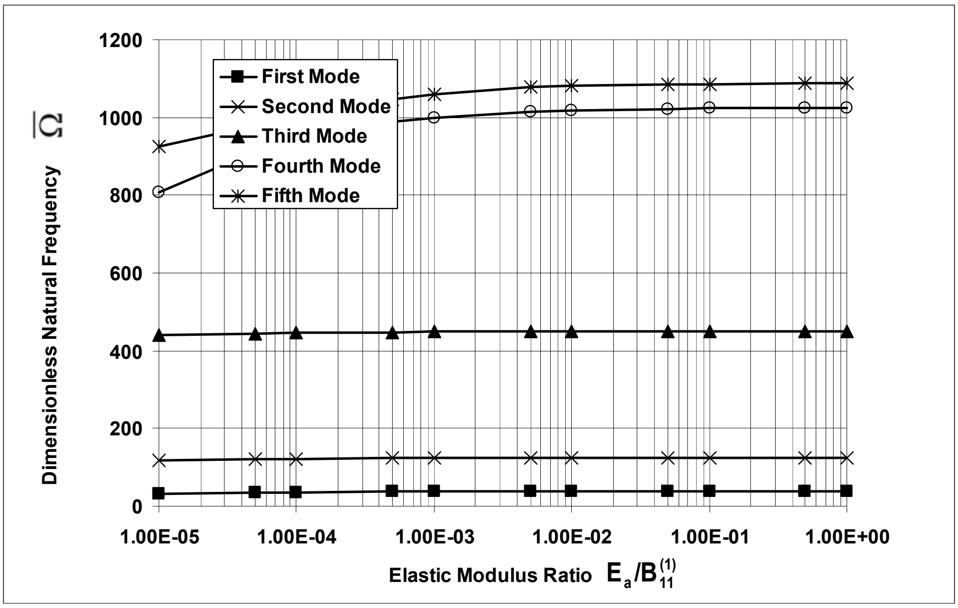
b) "Various Modes with (FFCFFC) B.C.'s

Fig 8.87 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)
Elastic Modulus Ratio axis is plotted in Log Scale



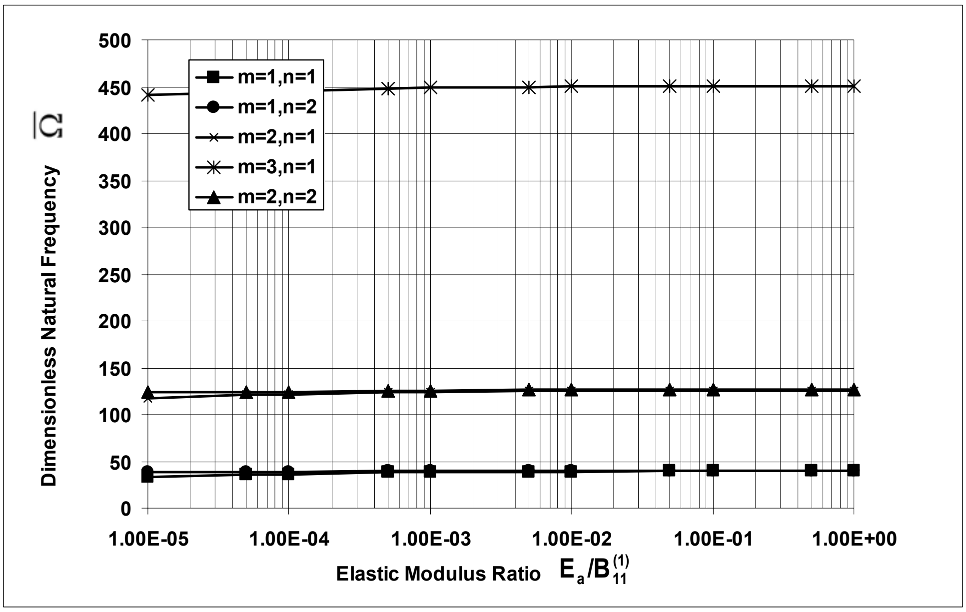
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFF) B.C.'s



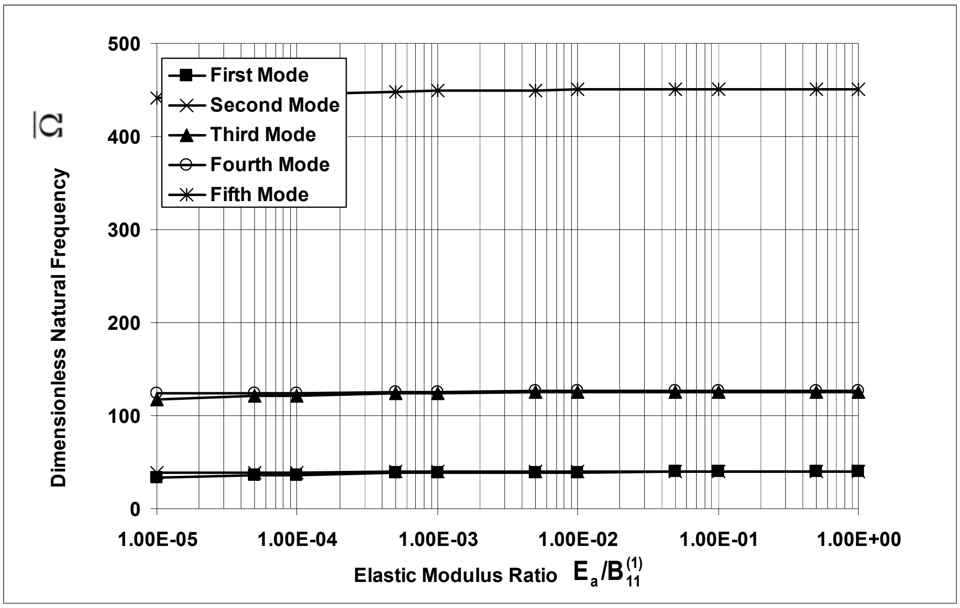
b) "Various Modes with (FFCFFF) B.C.'s

Fig 8.88 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



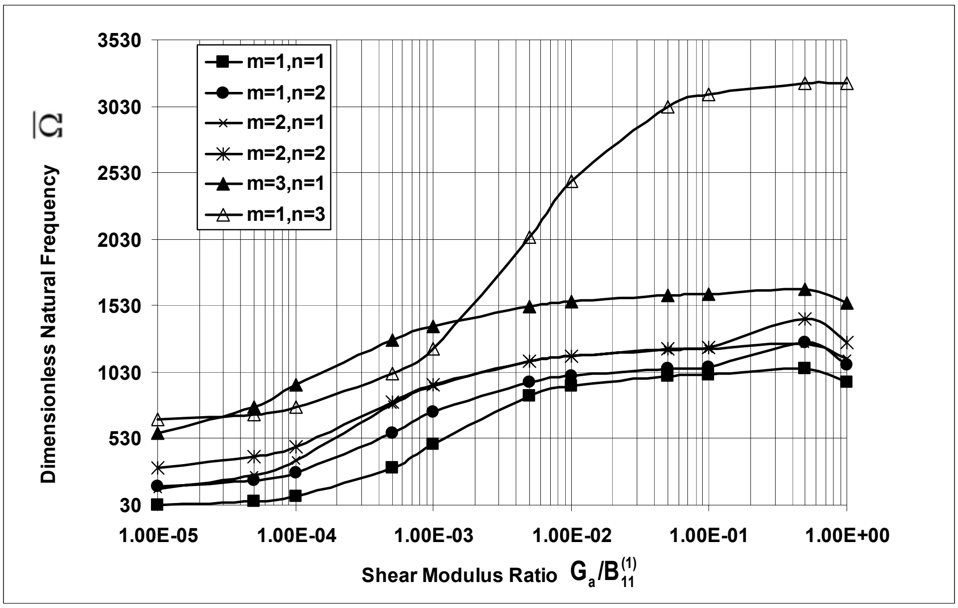
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFFFFF) B.C.'s



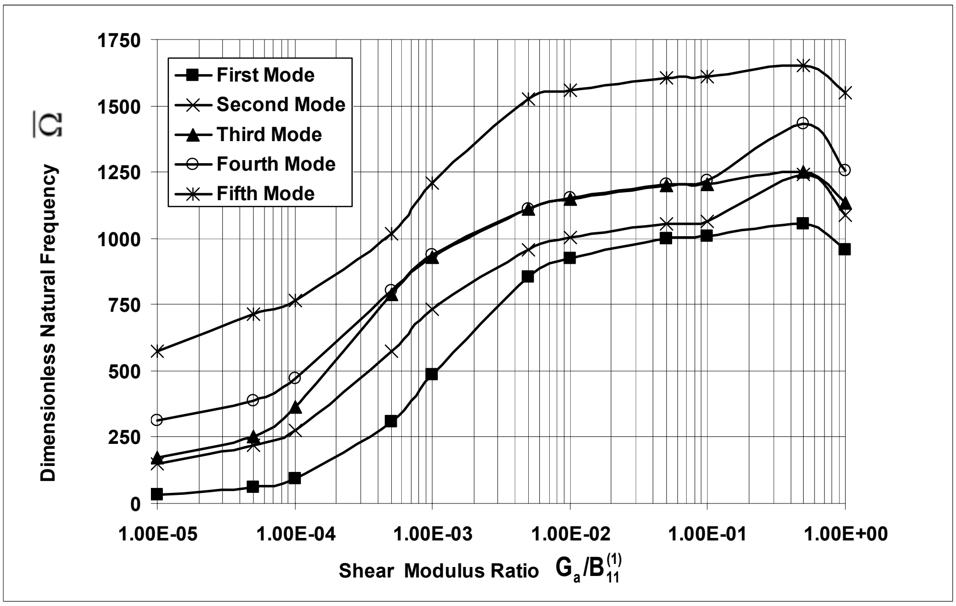
b) "Various Modes with (FFFFFF) B.C.'s

Fig 8.89 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



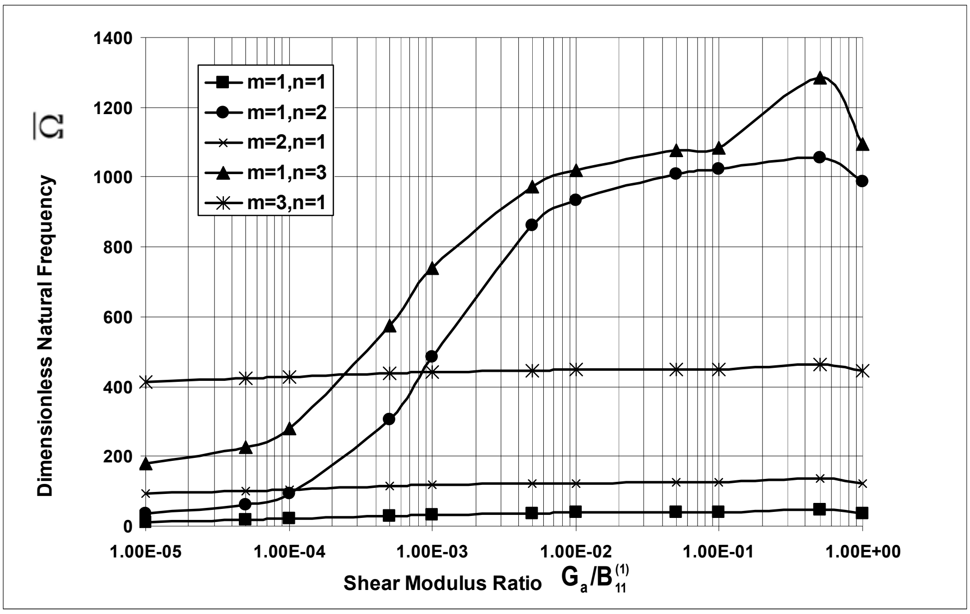
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFC) B.C.'s



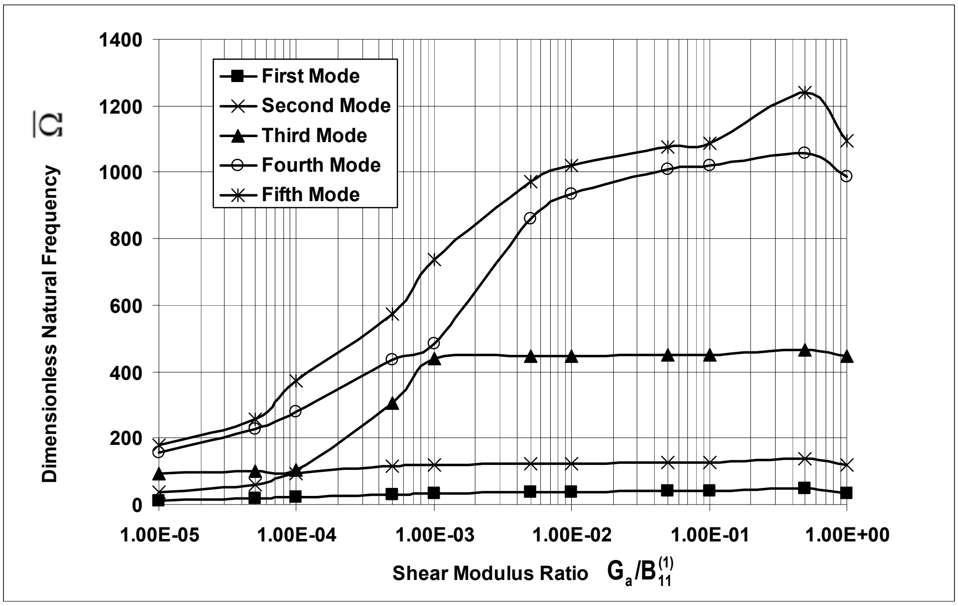
b) "Various Modes with (FFCFFC) B.C.'s

Fig 8.90 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)
Shear Modulus Ratio axis is plotted in Log Scale



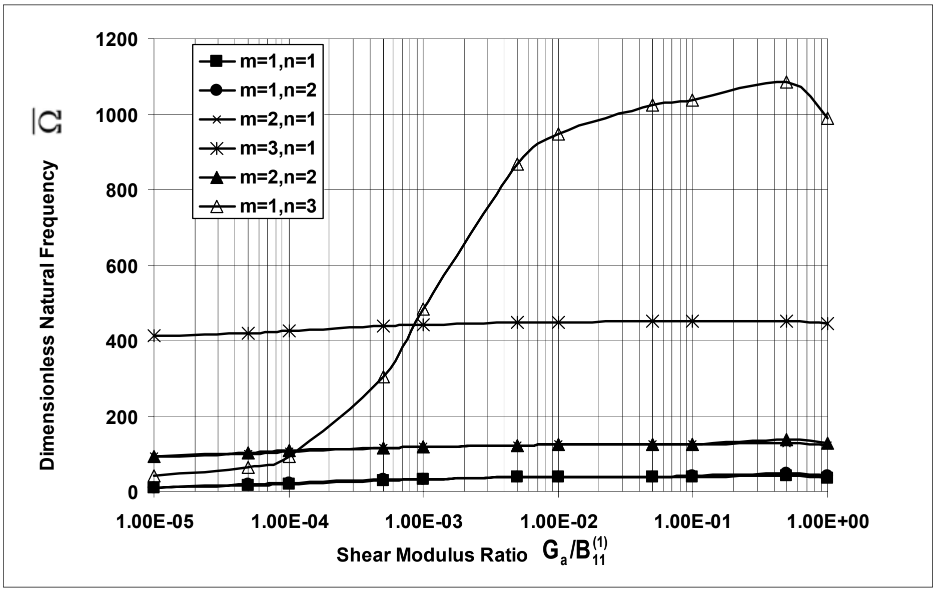
a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFCFFF) B.C.'s



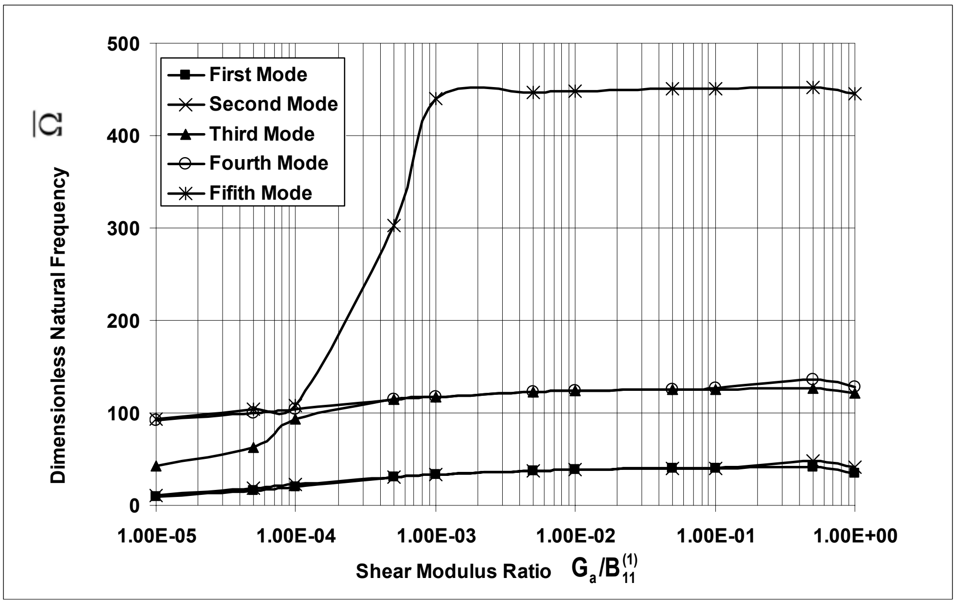
b) "Various Modes with (FFCFFF) B.C.'s

Fig 8.91 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)
Shear Modulus Ratio axis is plotted in Log Scale



a) Dependency in natural frequency on the number in half waves in y- and x-direction with (FFFFFF) B.C.'s



b) "Various Modes with (FFFFFF) B.C.'s

Fig 8.92 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)
Shear Modulus Ratio axis is plotted in Log Scale

8.4.4 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequencies"

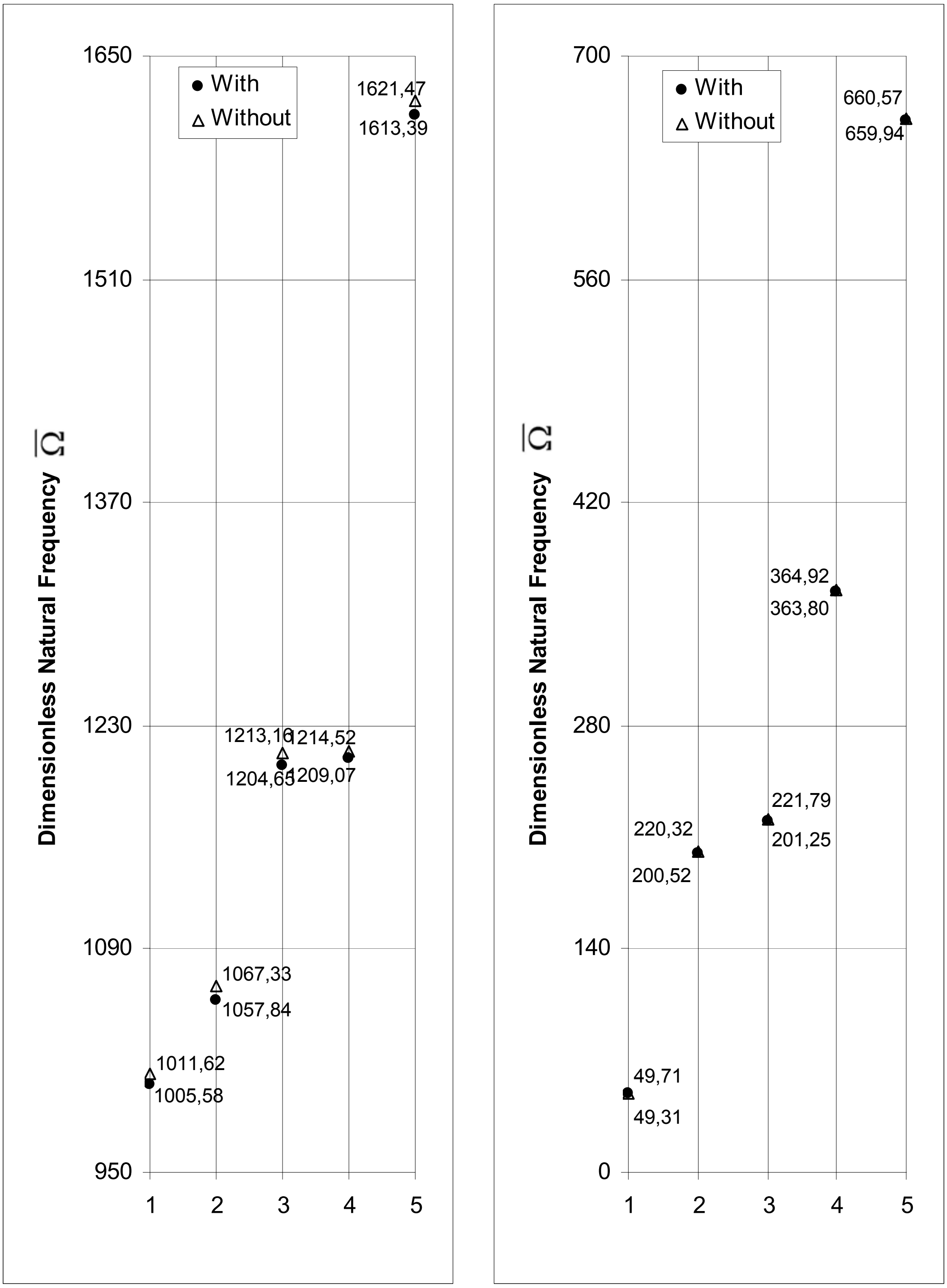
Table 8.6 Comparison of "Dimensionless Natural Frequencies" obtained by adding $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms to adhesive layer equations for "Main PROBLEM IIa"

a) "Hard" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	1011.615	1005.579	6.036
	2	1067.327	1057.842	9.485
	3	1213.155	1204.653	8.502
	4	1214.521	1209.066	5.455
	5	1621.474	1613.390	8.084
SFFS	1	538.675	535.732	2.943
	2	554.514	549.712	4.801
	3	692.221	687.711	4.510
	4	693.497	690.588	2.910
	5	1097.295	1092.787	4.508
CFFF	1	40.016	39.800	0.216
	2	126.173	125.909	0.264
	3	451.174	450.831	0.342
	4	1021.641	1015.306	6.335
	5	1090.067	1080.462	9.605
FFFF	1	40.011	39.755	0.256
	2	40.020	39.845	0.175
	3	126.099	125.780	0.319
	4	126.247	126.038	0.209
	5	451.072	450.666	0.406

b) "Soft" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	49.714	49.307	0.407
	2	201.252	200.524	0.727
	3	221.791	220.317	1.475
	4	364.919	363.804	1.115
	5	660.568	659.942	0.626
SFFS	1	35.130	34.776	0.354
	2	113.069	112.564	0.504
	3	192.234	191.054	1.180
	4	253.967	253.241	0.726
	5	378.945	378.531	0.414
CFFF	1	15.240	15.161	0.079
	2	51.647	51.276	0.371
	3	98.718	98.636	0.083
	4	206.656	205.961	0.695
	5	226.211	224.750	1.461
FFFF	1	14.505	14.435	0.070
	2	15.746	15.654	0.092
	3	54.015	53.687	0.328
	4	97.096	97.015	0.081
	5	100.330	100.244	0.086



a) "Hard" Adhesive Case **b) "Soft" Adhesive Case**

Figure 93 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint" (Boundary Conditions in y-direction FFCFFC)

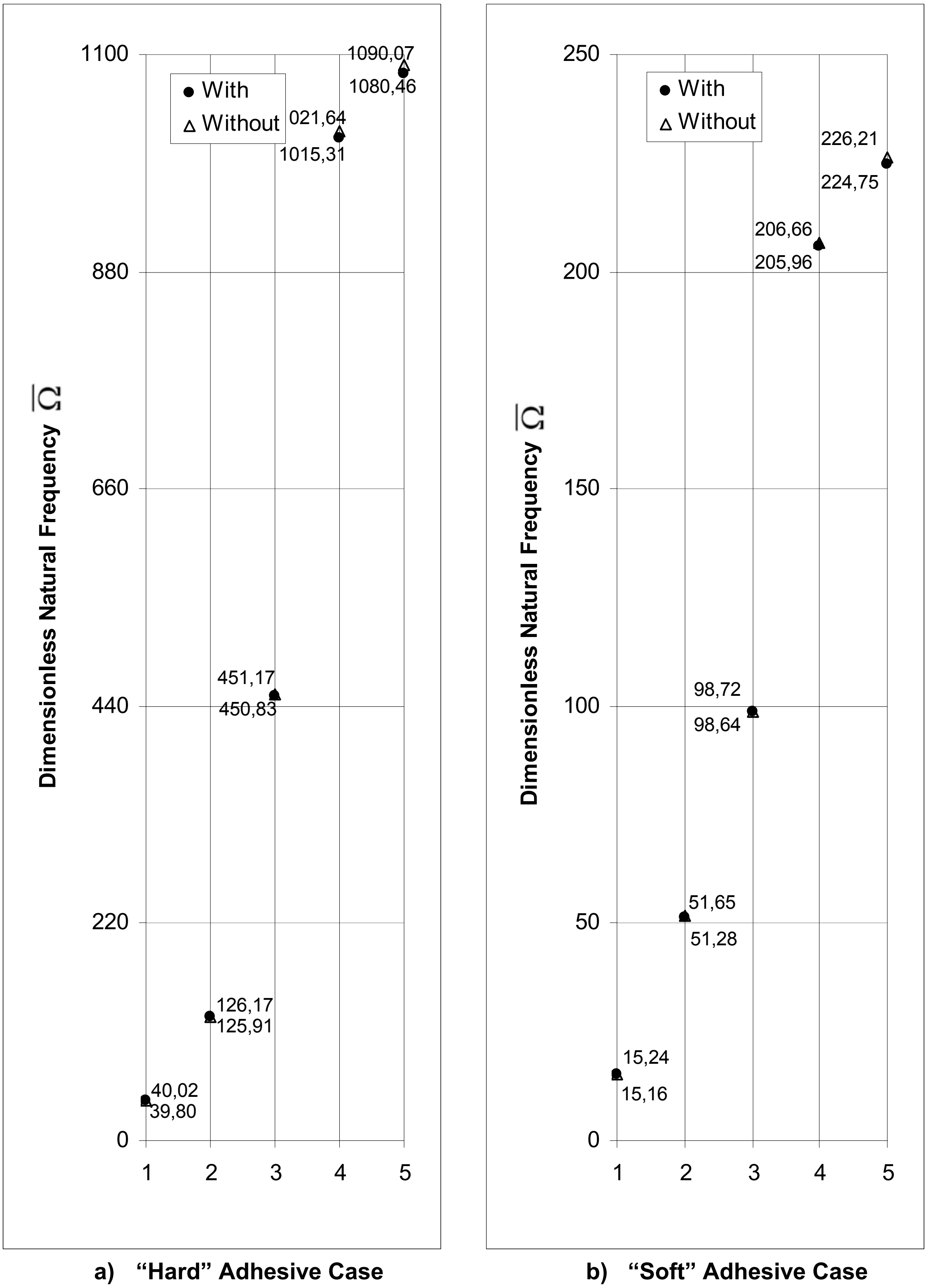


Figure 94 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Single Lap Joint" (Boundary Conditions in y-direction FFCFFF)

8.5 Numerical Results and Discussion for "Main PROBLEM II.b"

In the "Main PROBLEM IIb.", the "Composite Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint" is analyzed. The doubler is made of Graphite-Epoxy and the lower plate adherends are Kevlar-Epoxy. For the in-between adhesive layer, the "hard" and the "soft" adhesive cases are taken into account. The "Geometric and the Material Characteristics" of the single lap joint system are given in Table 8.2.

In Figures 8.85 – 8.94, the mode shapes and the corresponding natural frequencies (from the first to fifth), in the "hard" and the subsequent "soft" adhesive cases with various boundary conditions are presented.

From aforementioned Figures, in the "hard" adhesive case it is easy to observe that with respect to the position of the "Bonded Region", there exists an almost "stationary region" in the mode shapes. And this region moves from left to the right part (or vice versa) in the composite symmetric single lap joint system. In the "soft" adhesive case, however, an almost "stationary region" does not exist in mode shapes. The general trend in the mode shapes, for the "soft" adhesive case is that, the "Bonded Region" moves or bends with the rest of the lap joint system. And the mode shapes are completely different in comparison with those of the "hard" adhesive cases with the same support conditions.

Next, for the "Main PROBLEM IIb", in Figures 8.95 through 8.112, the several important parametric studies are presented. In Figures 8.95-8.100, the "Dimensionless Natural Frequency $\overline{\Omega}$" versus "Position Ratio $\tilde{b}/L$" from the first up to the fifth mode are plotted, for both the "hard" and the "soft" adhesive cases, corresponding to the various support conditions.

From Figures 8.95, 8.97 8.99, in the "hard" adhesive case, it is obvious that, as position of the "Bonded Region" changes (in the y-direction), the natural frequencies gradually increase up to a certain position and then decreases.

These results are the consequences of the movement of the half waves from left to right because of the change in the position of the “Bonded Region”.

In the “soft” adhesive case, in Figures 8.96, 8.98, 8.100, the natural frequencies increase with the position of the “Bonded Region”. This also can be expected due to the “soft” adhesive which makes the system loose and which shows a similar behavior in mode shapes up to $\widetilde{b}$ =0.5m.

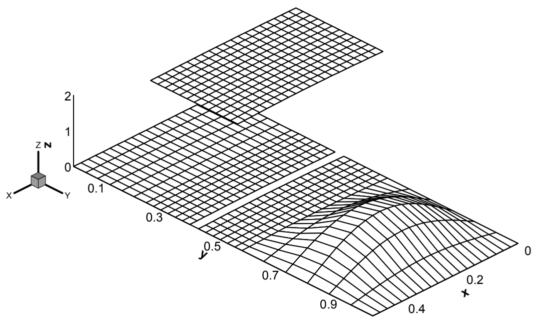
In Figures 8.101 through 8.106, the effect of the “Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$” on the natural frequencies (from the first up to the fifth) in the “hard” and “soft” adhesive cases, are investigated for various boundary conditions. In the “hard” adhesive case, in Figures 8.101, 8.103, 8.105, the first two natural frequencies, in spite of the increasing “Bending Rigidity Ratio”, do remain practically constant. In the higher modes, the natural frequencies increase sharply at first and after the “Bending Rigidity Ratio=2.6” they become almost flat or constant regardless of the increase in “Bending Rigidity Ratio”.

In the “soft” adhesive cases, in the Figures 8.102, 8.104, 8.106, the first and the second frequencies remain more or less constant as the “Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$” increases. In the third and higher modes, the natural frequencies increase.

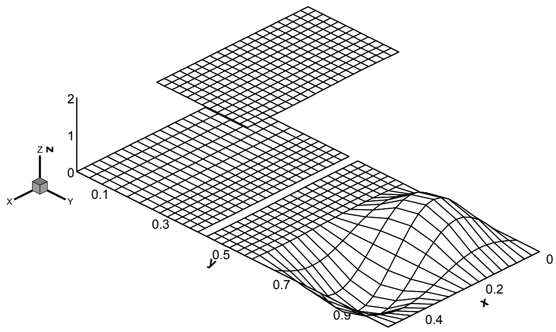
Lastly, the direct effects of the adhesive layer elastic constants E_a, and also G_a on the dimensionless natural frequencies are investigated for the “Main PROBLEM II.b”. In order to show these effects, the “Dimensionless Natural Frequencies” versus the “Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$” are plotted (while the other elastic constant kept constant) in Figures 8.107 through 8.109 for various boundary condition. Similarly, the “Dimensionless Natural Frequencies” versus the “Adhesive Shear Modulus Ratio $G_a/B_{11}^{(1)}$” are presented in Figures 8.110 through 8.112 for various support condition.

It can be seen from the Figures 8.107-8.109, the influence of the "Adhesive Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" on the natural frequencies, is not significant. In Figures 8.110-8.112, we can see that the "Shear Modulus Ratio $G_a/B_{11}^{(1)}$", significantly affects the natural frequencies. Also, in those Figures, one can observe a "transition region" which takes the frequencies to considerable higher levels. After then, no change is observed in the frequencies.

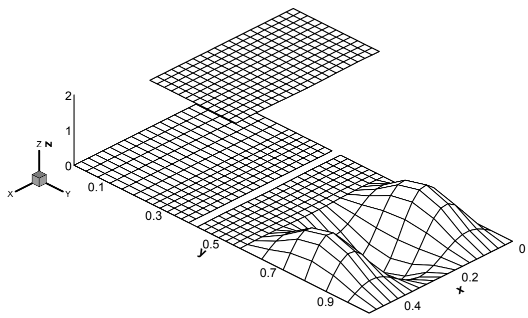
8.5.1 Natural Frequencies and Corresponding Mode Shapes for "Main PROBLEM IIb"



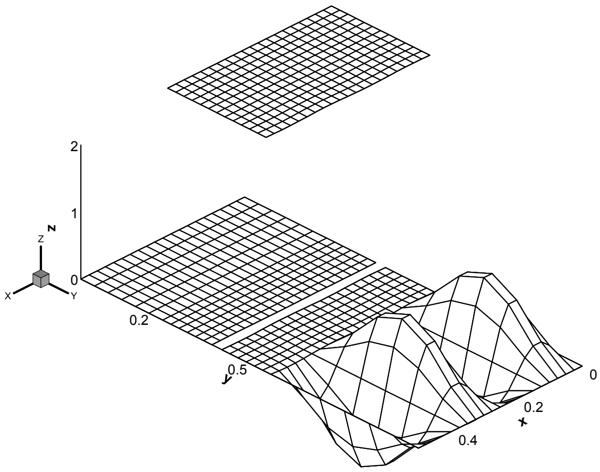
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 430.476$



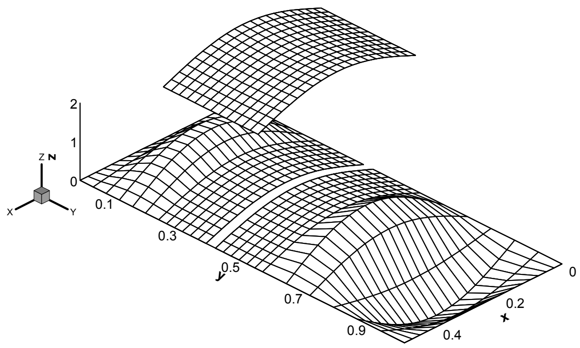
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 547.545$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 914.429$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41}$=1798.165

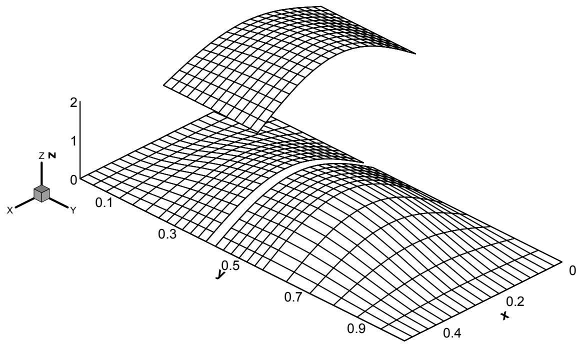


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12}$=2295.530

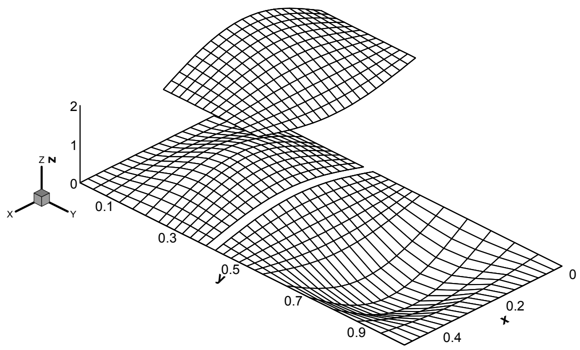
("Hard" Adhesive Case)

Fig 8.95 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

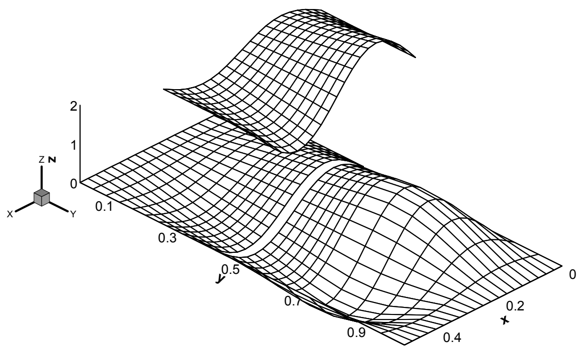
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



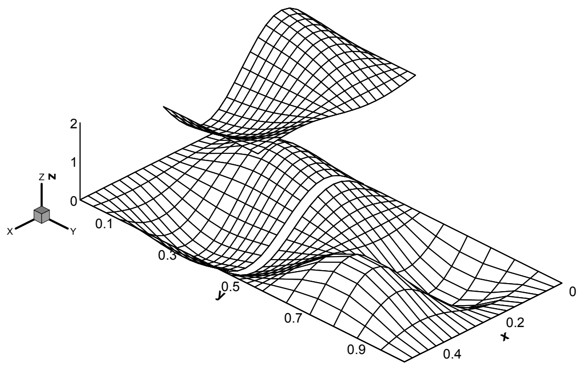
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=49.116



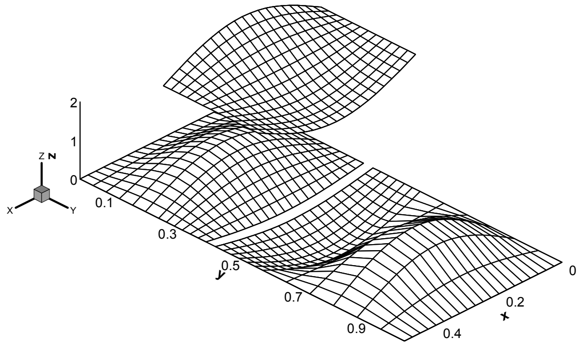
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=191.144



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=205.773



d) Fourth Mode with $\overline{\Omega}_4$= $\overline{\omega}_{22}$=353.509

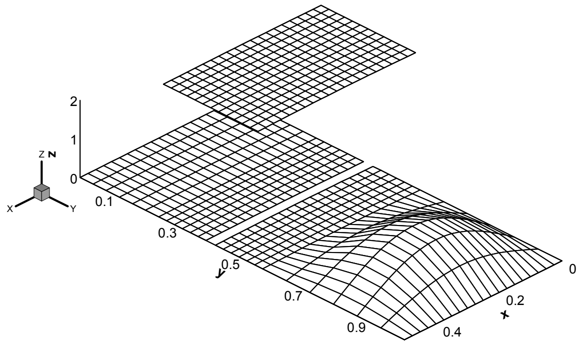


e) Fifth Mode with $\overline{\Omega}_5$= $\overline{\omega}_{13}$=578.172

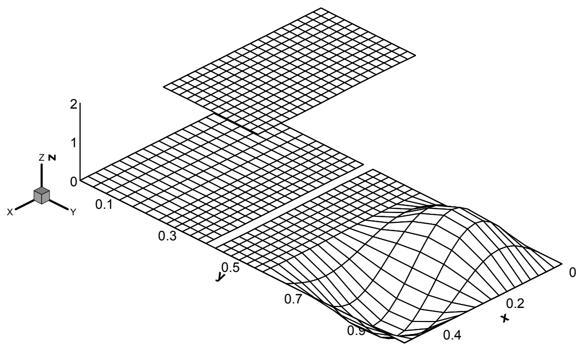
("<u>Soft</u>" Adhesive Case)

Fig.8.96 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

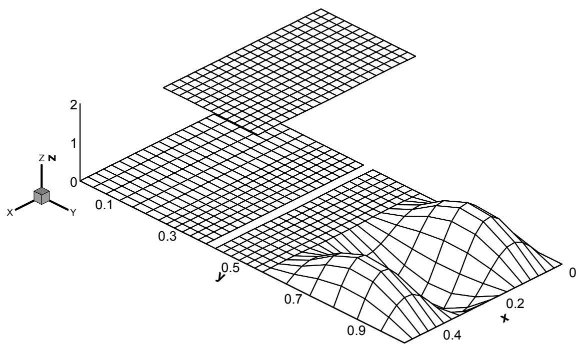
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



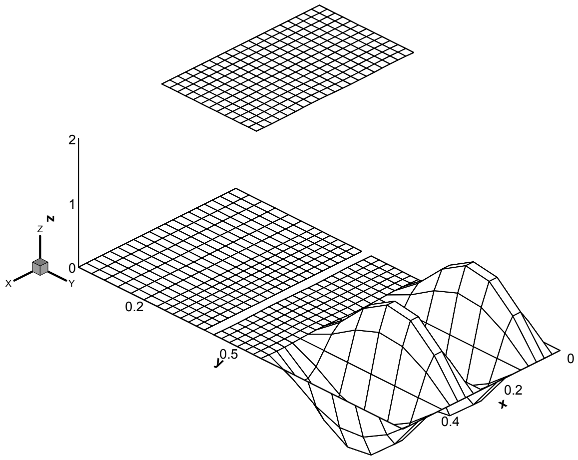
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 222.617$



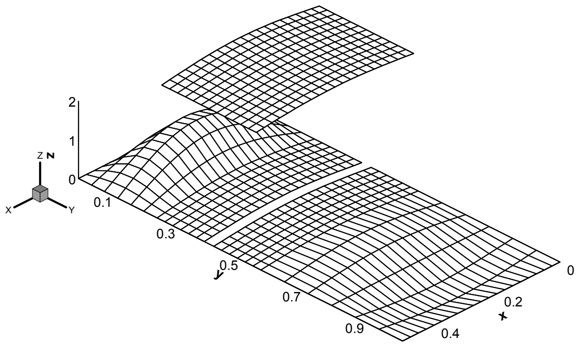
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 334.522$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 699.417$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41}$=1581.499

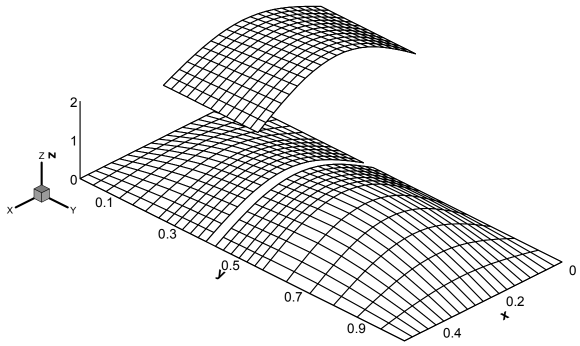


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12}$=1606.664

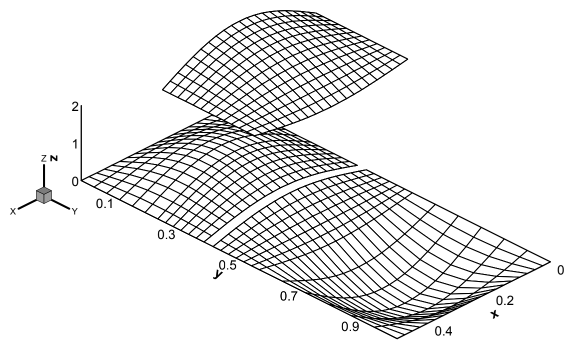
("<u>Hard</u>" Adhesive Case)

Fig.8.97 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

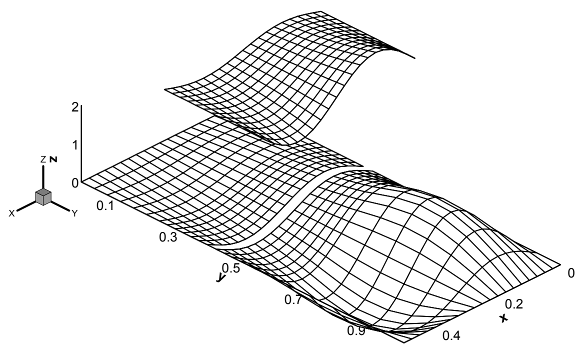
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFS)



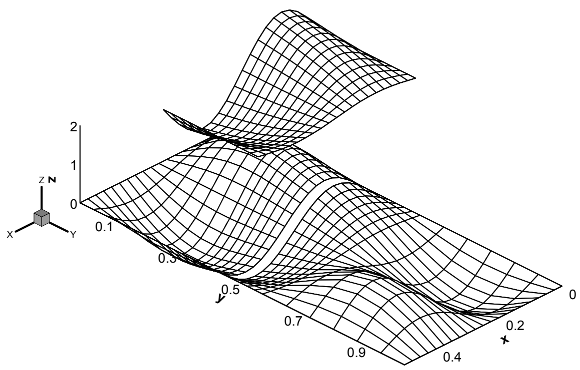
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=33.635



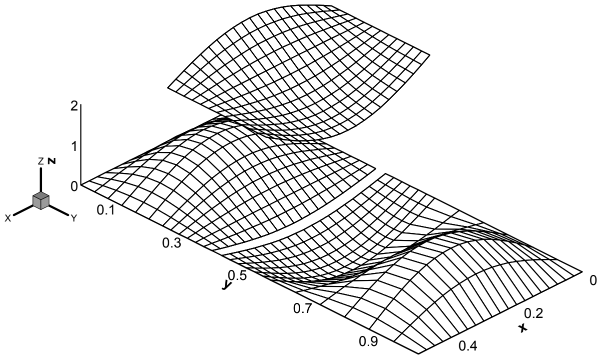
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=104.878



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=169.516



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{22}$=262.938

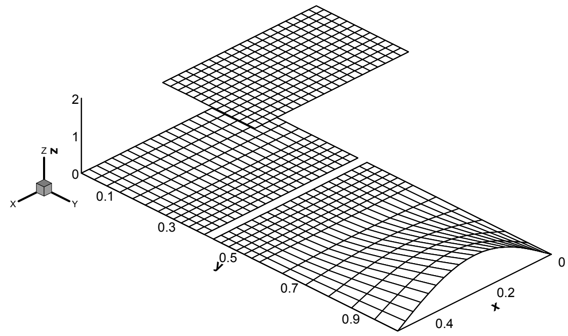


e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{13}$=358.832

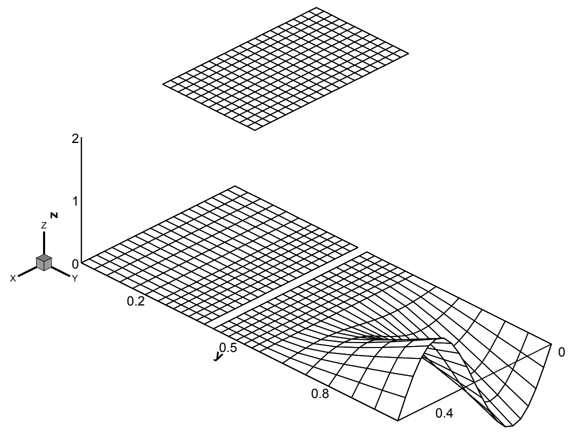
("Soft" Adhesive Case)

Fig.8.98 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

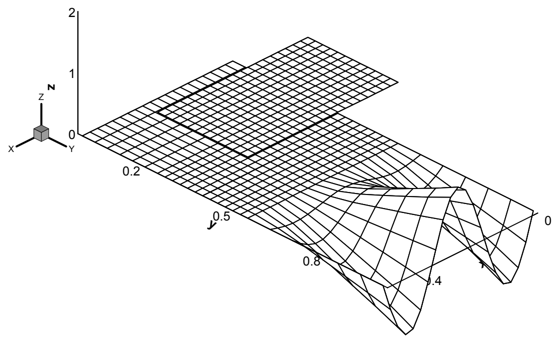
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFS)



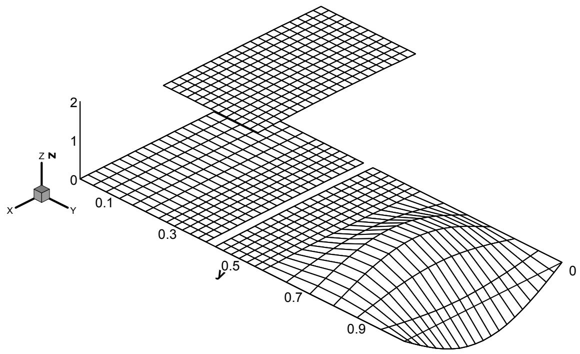
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=19.349



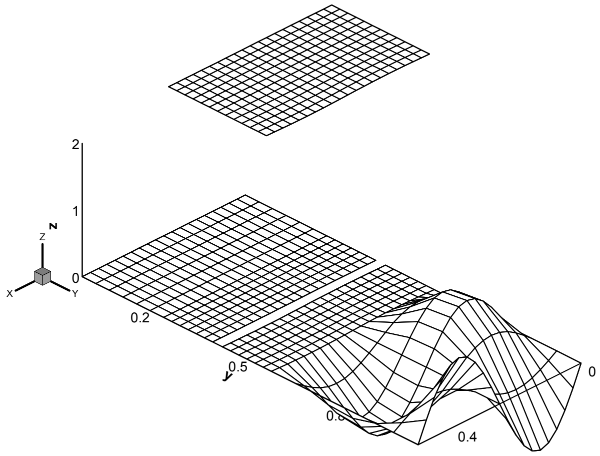
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=99.125



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=414.051



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=445.060

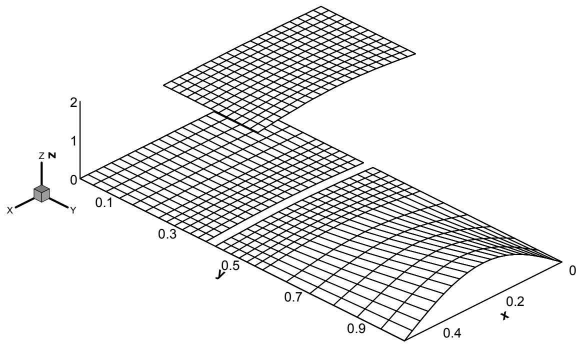


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=611.701

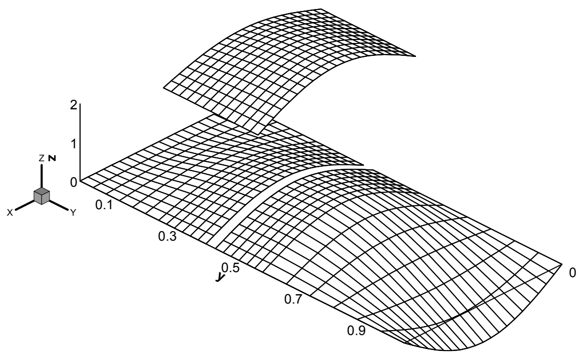
("Hard" Adhesive Case)

Fig.8.99 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

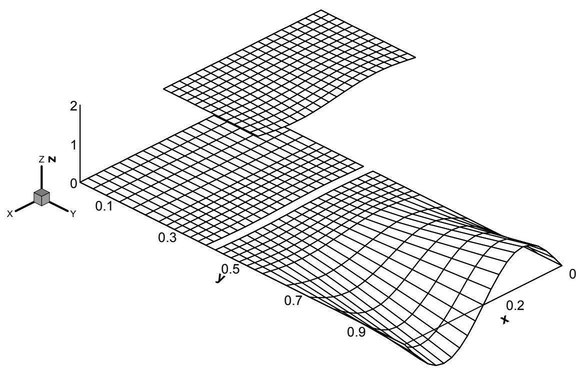
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$=0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



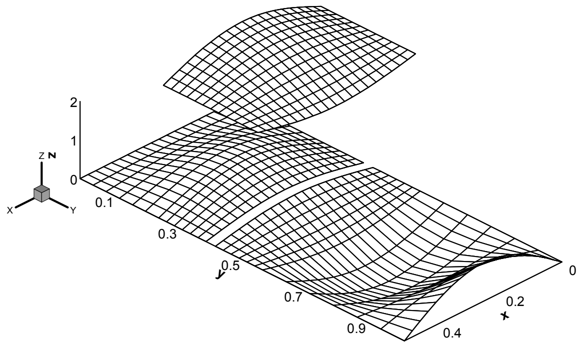
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=10.807



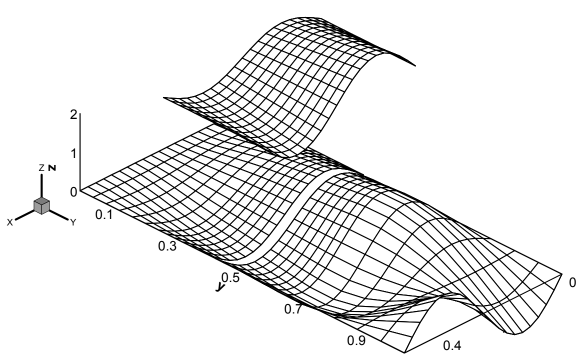
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=50.353



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=88.680



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=196.562

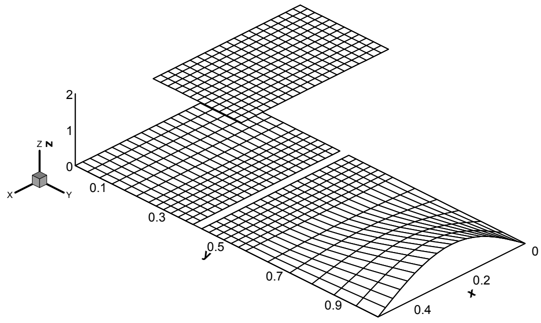


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=216.953

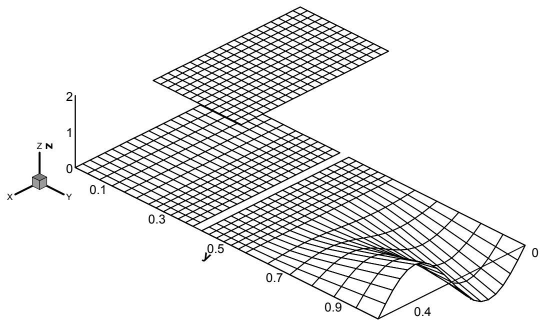
("Soft" Adhesive Case)

Fig.8.100 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

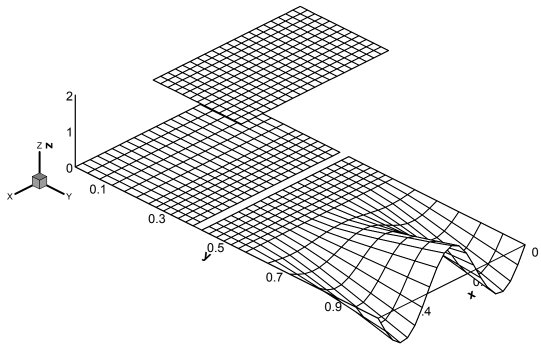
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



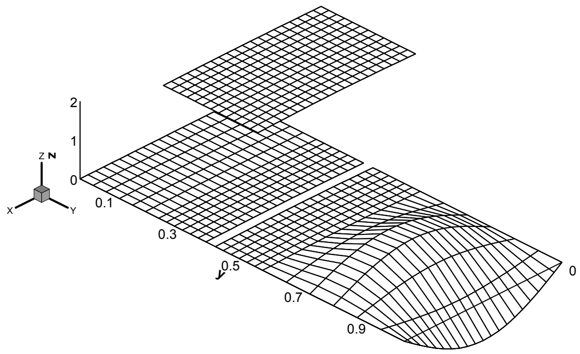
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=19.349



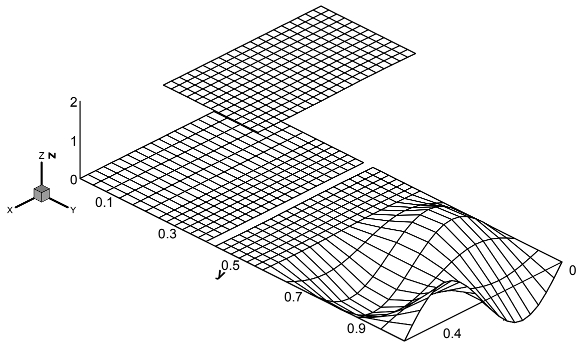
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=99.125



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=414.051



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=445.022

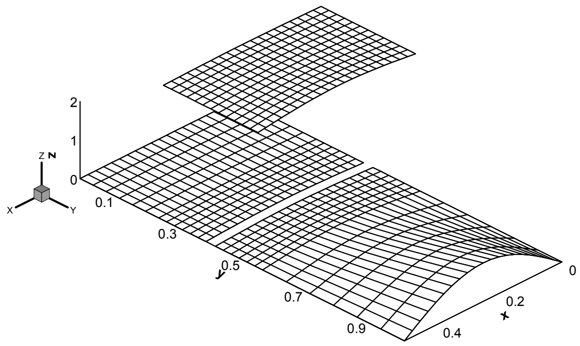


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=611.701

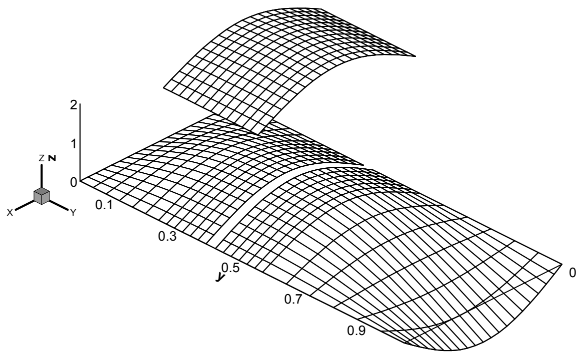
("Hard" Adhesive Case)

Fig.8.101 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

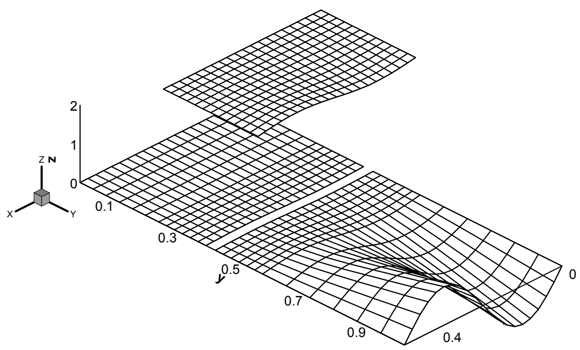
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFF)



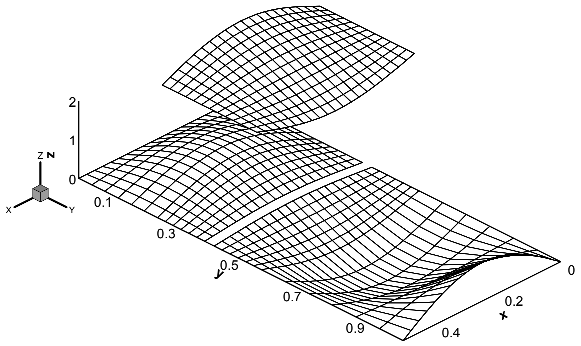
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 10.793$



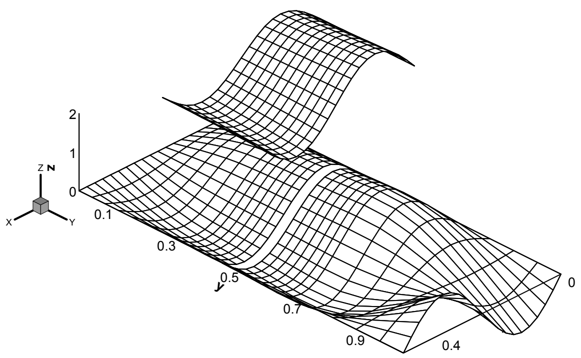
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 41.547$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 88.674$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=154.011

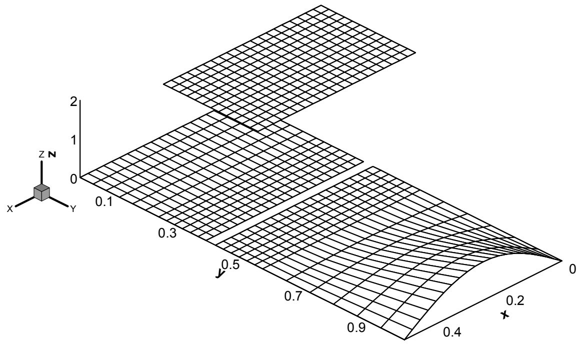


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=208.673

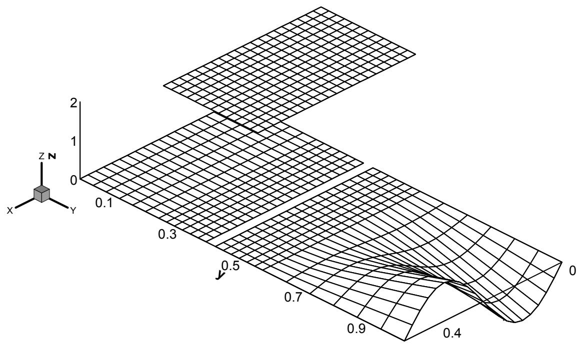
("<u>Soft</u>" Adhesive Case)

Fig.8.102 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

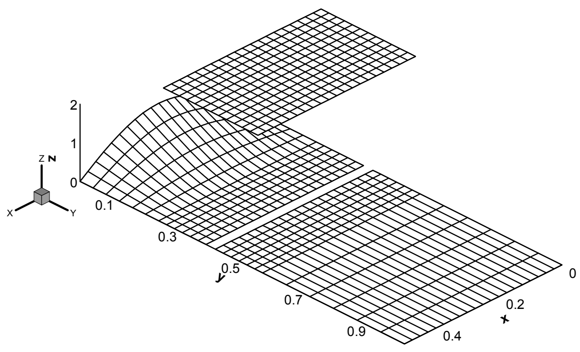
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFF)



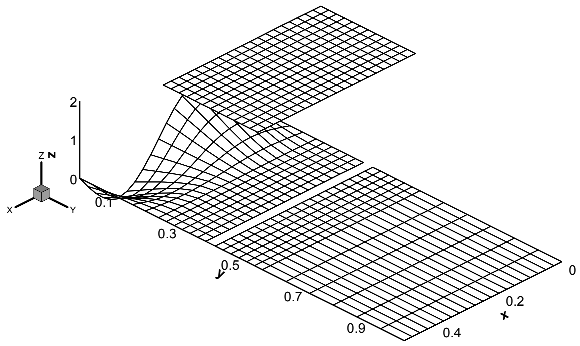
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 19.349$



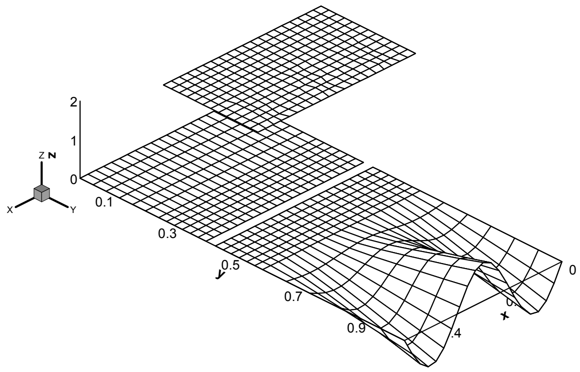
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 99.125$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{12} = 120.837$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=221.938

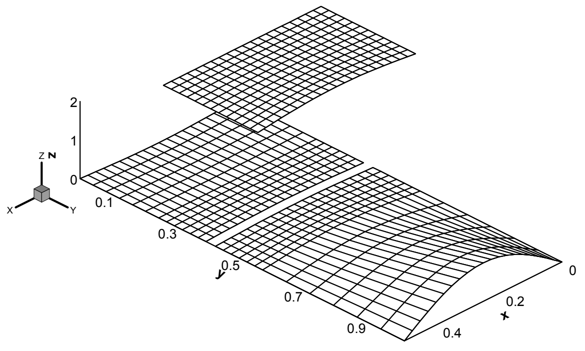


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=414.051

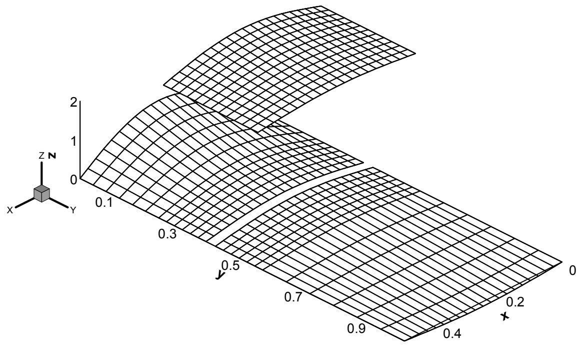
("Hard" Adhesive Case)

Fig.8.103 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

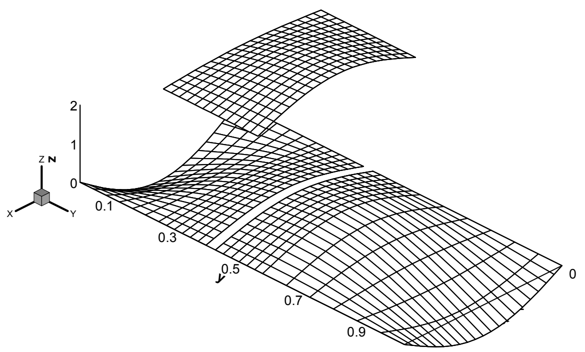
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



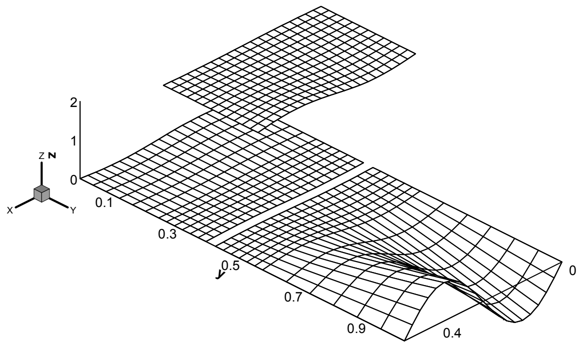
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=10.701



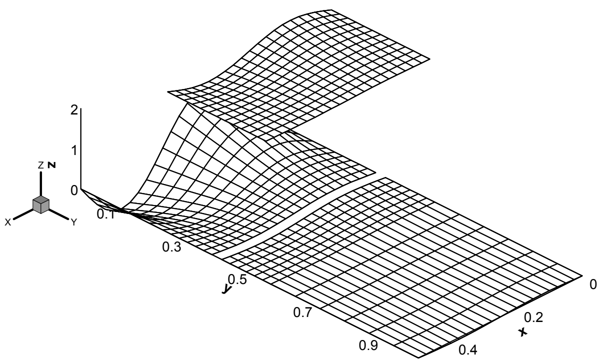
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=22.477



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{13}$=56.748



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{21}$=88.584



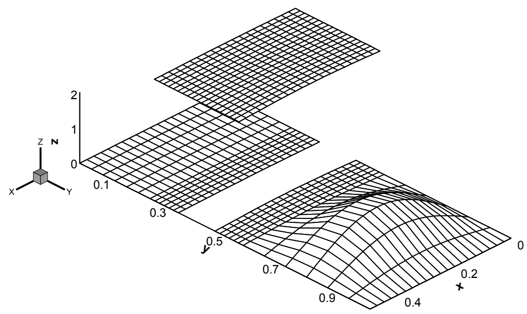
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=121.365

("<u>Soft</u>" Adhesive Case)

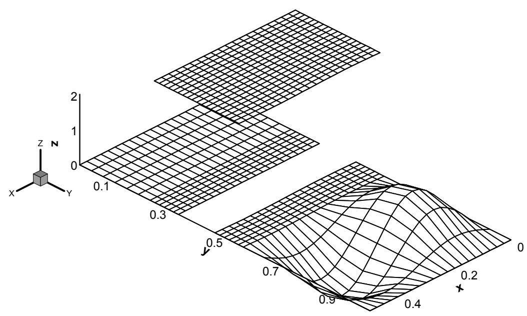
Fig.8.104 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m, b_1=0.3 m, b_2=0.4m, b_3=0.6m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)

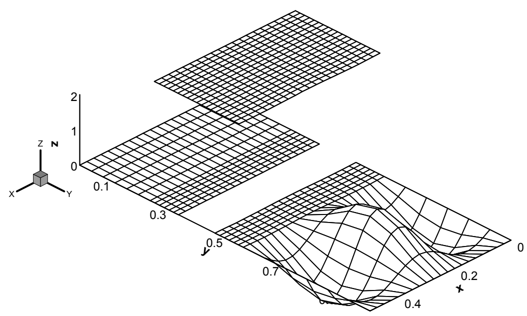
8.5.2 Natural Frequencies and Corresponding Mode Shapes for "Special Case of Main PROBLEM IIb"



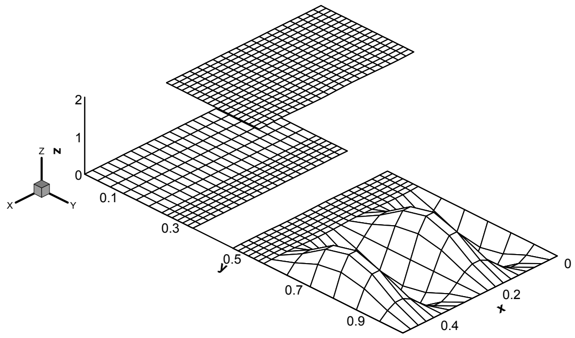
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=425.639



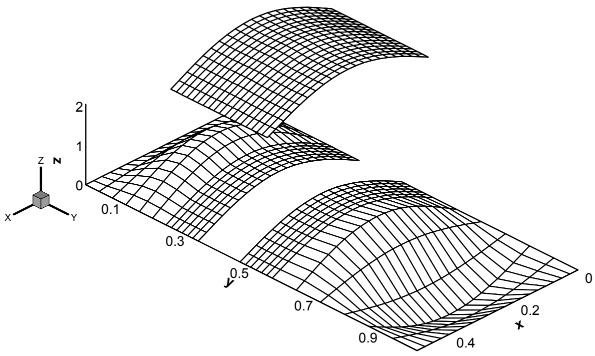
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=549.082



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=917.091



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41}$=1801.284

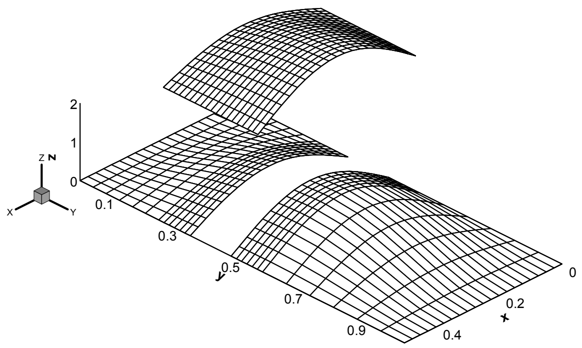


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12}$=1925.877

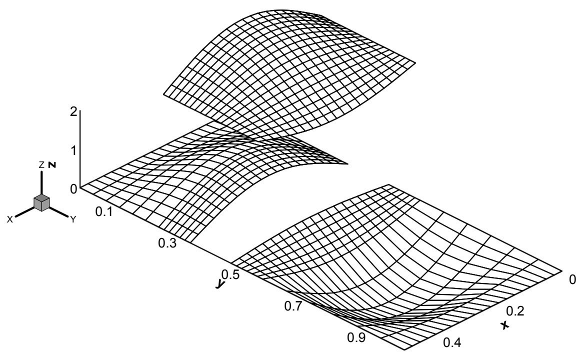
("Hard" Adhesive Case)

Fig 8.105 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

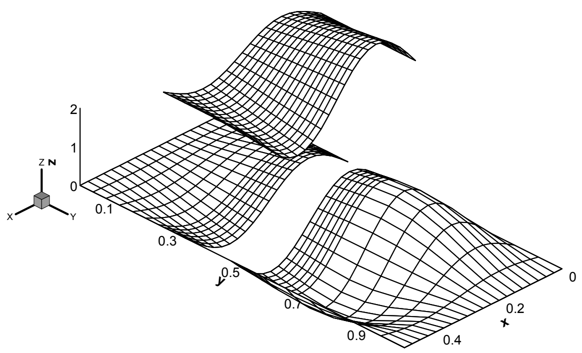
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ℓ_I=0.1m, ℓ_{II}=0.1m, b_1=0.3m, b_2=0.35m, b_3=0.55m, $\tilde{b}$ =0.5m, a=0.5m, L=1m)
(Boundary Conditions in y-direction FFCFFC)



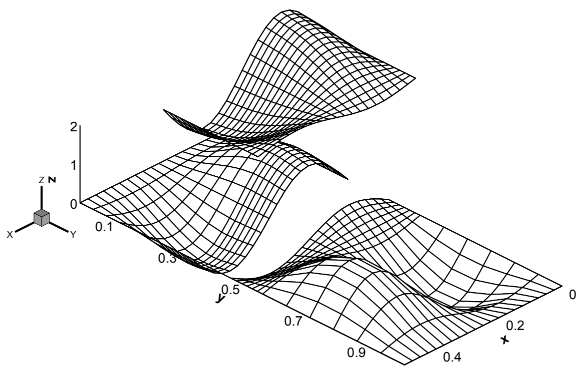
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=44.145



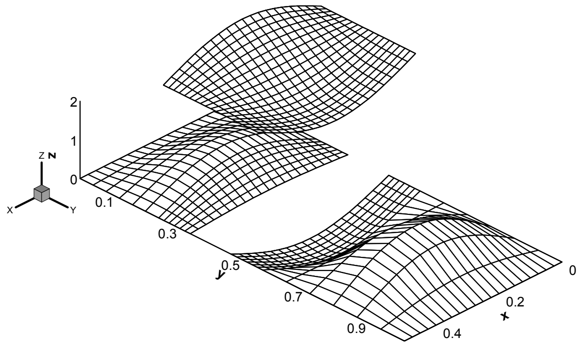
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=180.932



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=190.066



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=339.208

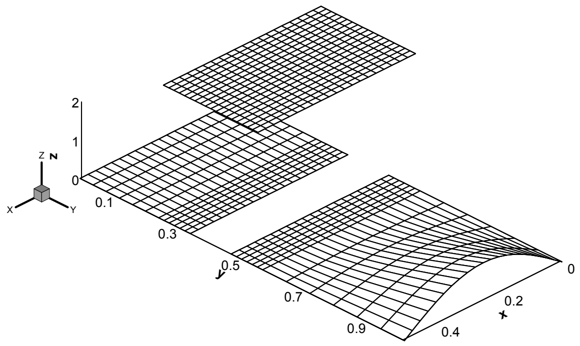


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=547.773

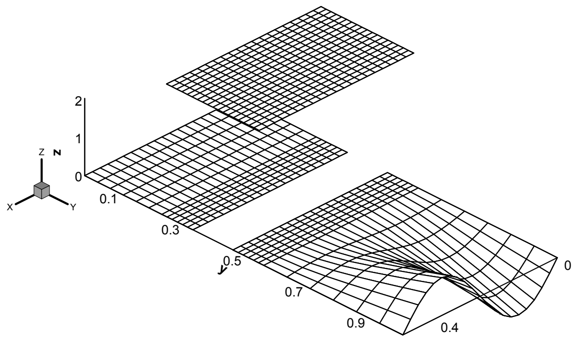
("Soft" Adhesive Case)

Fig 8.106 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

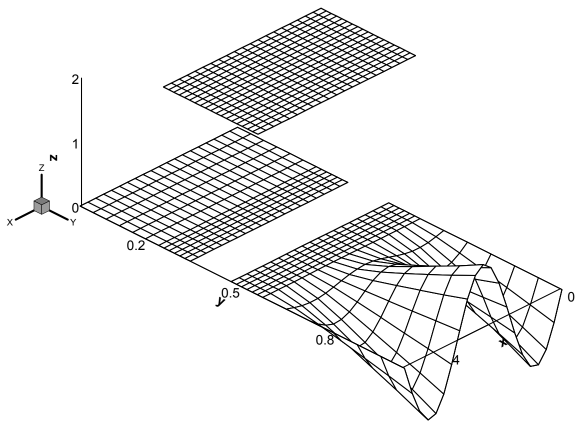
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ℓ_I=0.1m, ℓ_{II}=0.1m, b_1=0.3m, b_2=0.35m, b_3=0.55m, $\tilde{b}$ =0.5m, a=0.5m, L=1m)
(Boundary Conditions in y-direction FFCFFC)



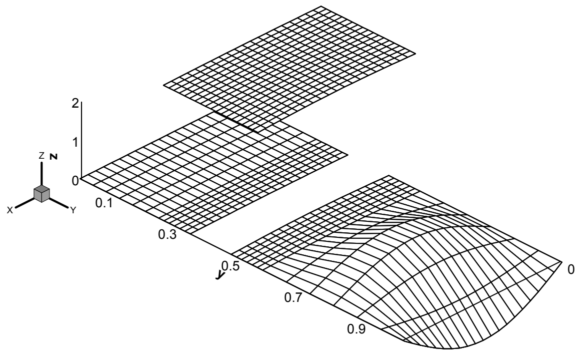
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=19.402



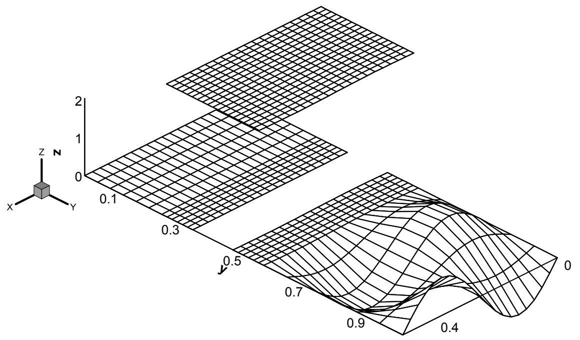
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=99.235



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=414.233



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=439.504

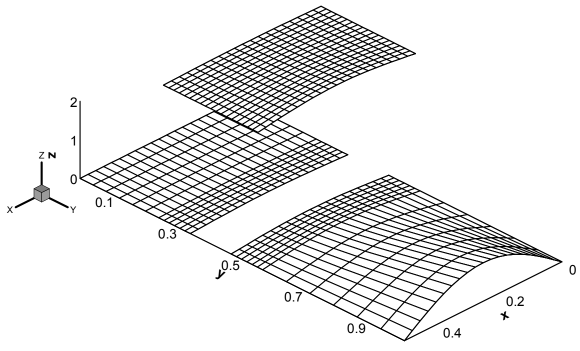


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=613.532

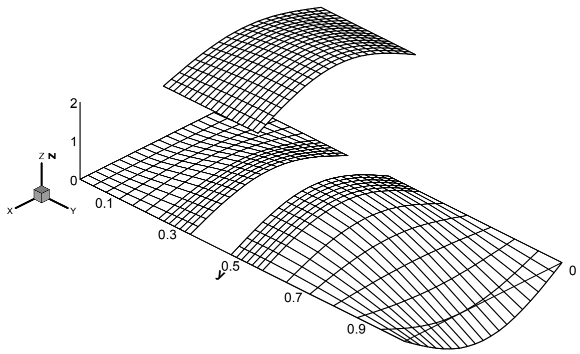
("Hard" Adhesive Case)

Fig 8.107 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

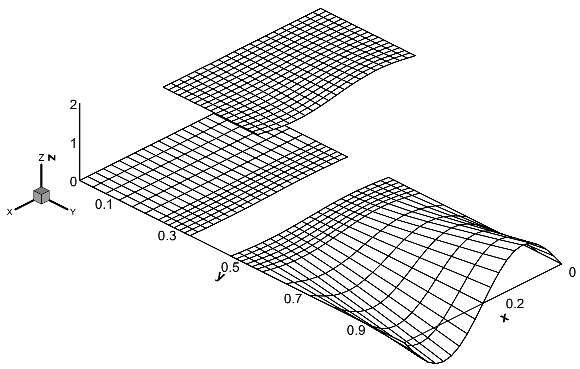
(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ℓ_I=0.1m, ℓ_{II}=0.1m, b_1=0.3m, b_2=0.35m, b_3=0.55m, $\tilde{b}$ =0.5m, a=0.5m, L=1m)
(Boundary Conditions in y-direction FFCFFF)



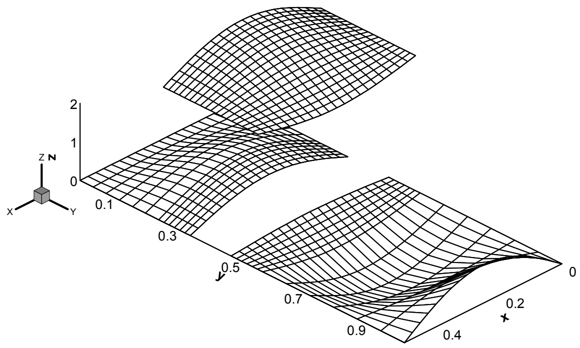
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 10.069$



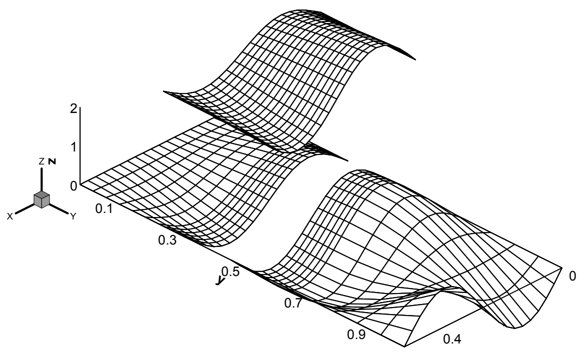
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 45.679$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 88.017$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=186.090



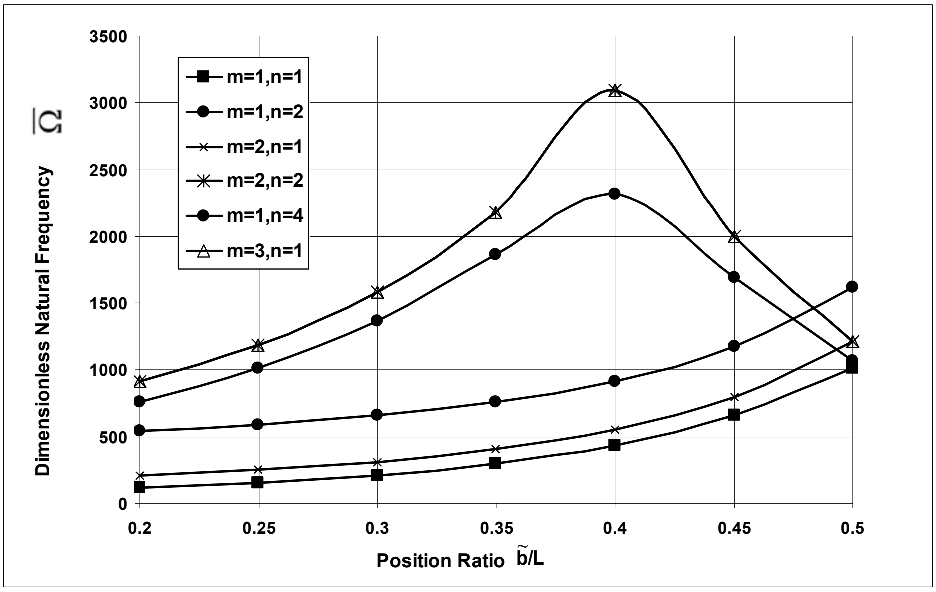
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=200.332

("Soft" Adhesive Case)

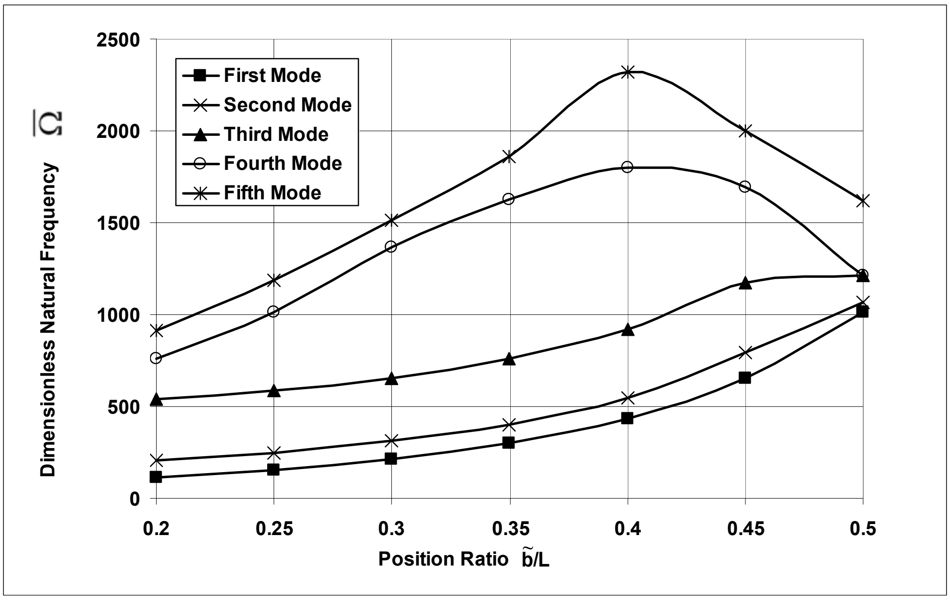
Fig 8.108 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint) with a Gap"

(Plate 1= Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ℓ_I=0.1m, ℓ_{II}=0.1m, b_1=0.3m, b_2=0.35m, b_3=0.55m,
$\tilde{b}$ =0.5m, a=0.5m, L=1m)
(Boundary Conditions in y-direction FFCFFF)

8.5.3 Some Parametric Studies for "Main PROBLEM IIb"



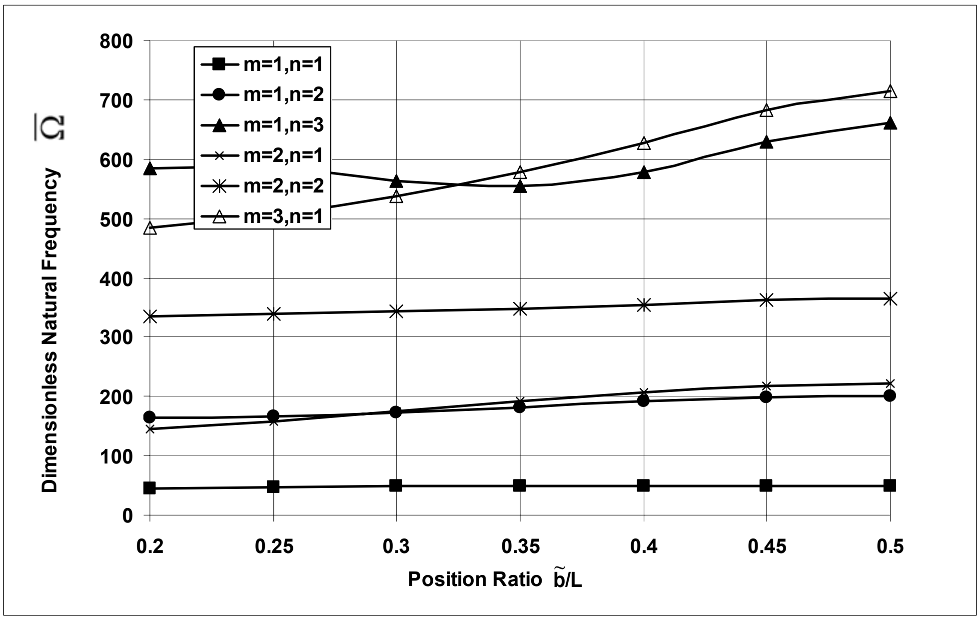
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFC) B.C.'s, ''<u>Hard</u>'' Adhesive



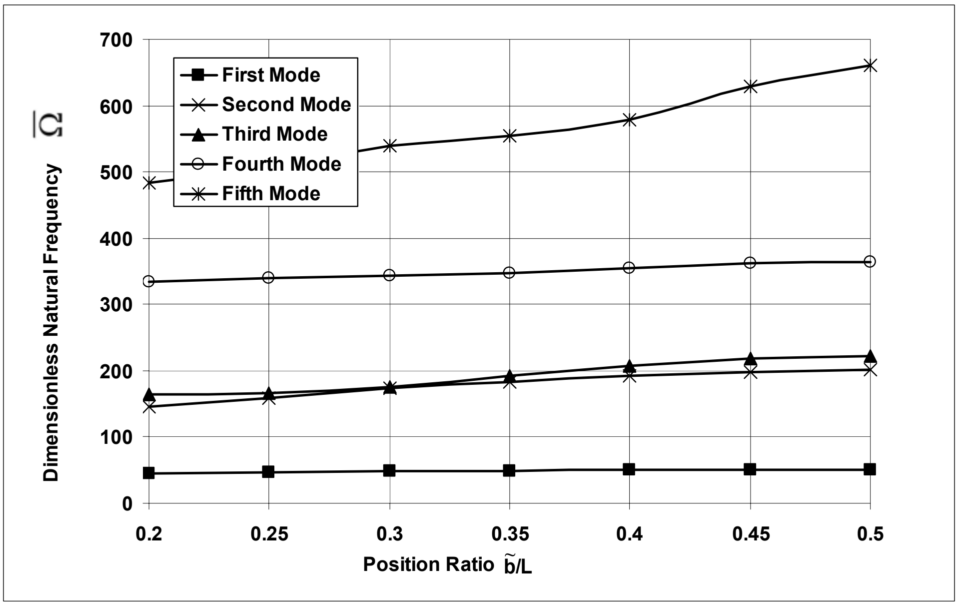
b) "Various Modes with (FFCFFC) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.109 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}/L$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



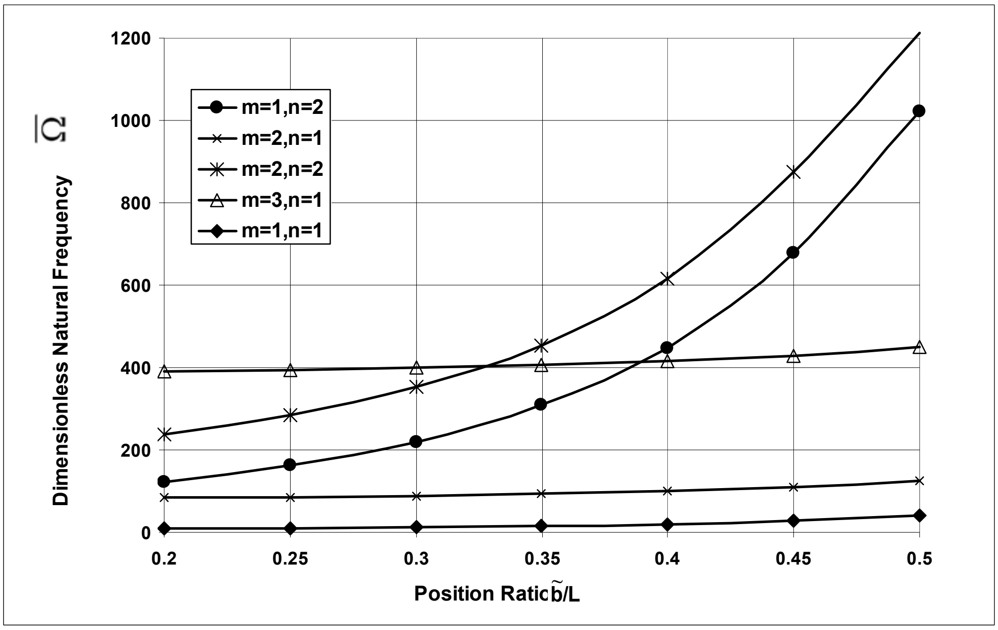
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFC) B.C.'s, "Soft" Adhesive



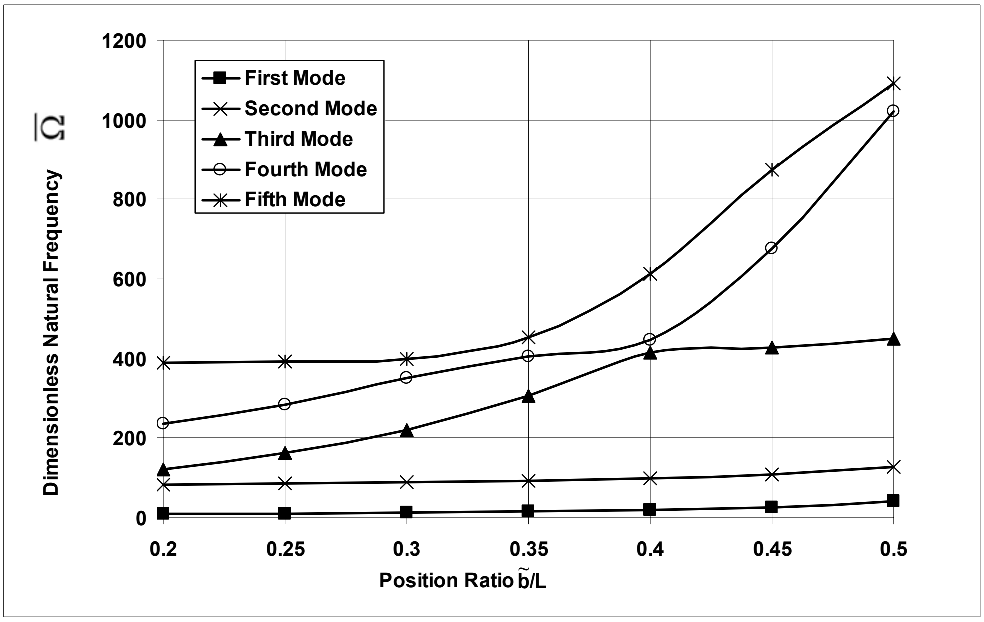
b) "Various Modes with (FFCFFC) B.C.'s, "Soft" Adhesive

Fig 8.110 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



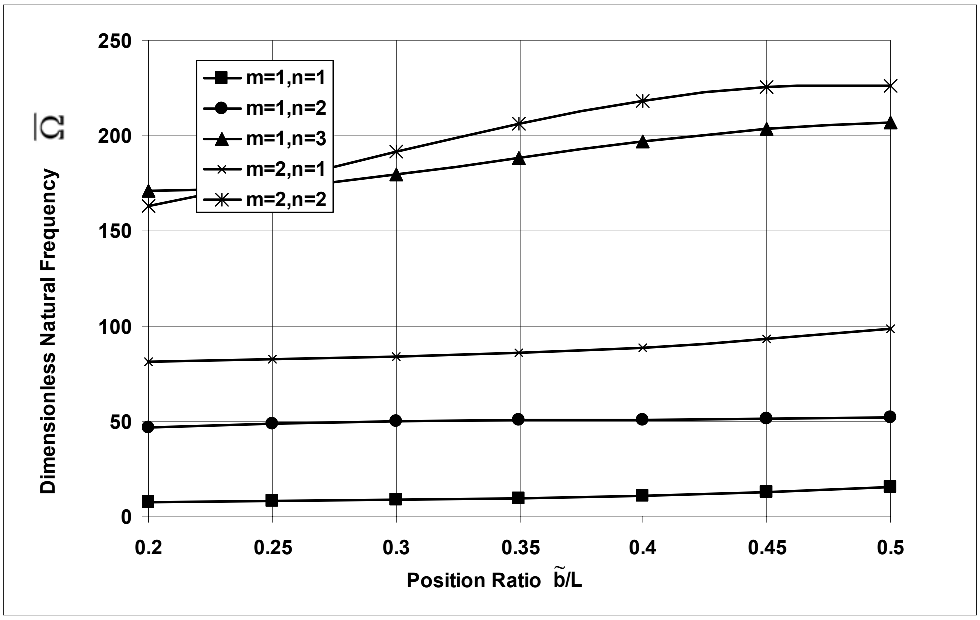
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFF) B.C.'s, "Hard" Adhesive



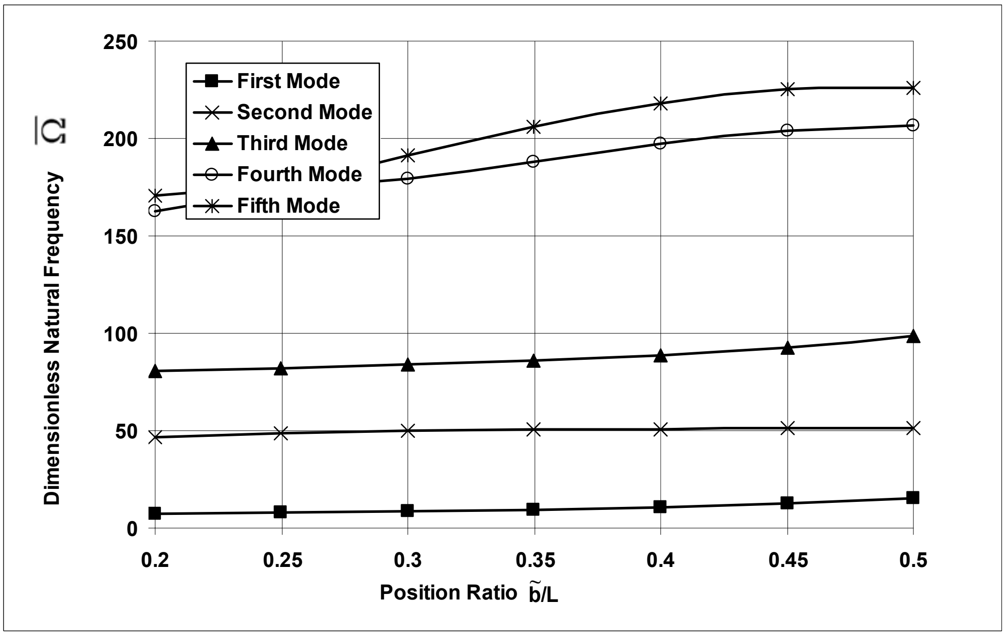
b) "Various Modes with (FFCFFF) B.C.'s, "Hard" Adhesive

Fig 8.111 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



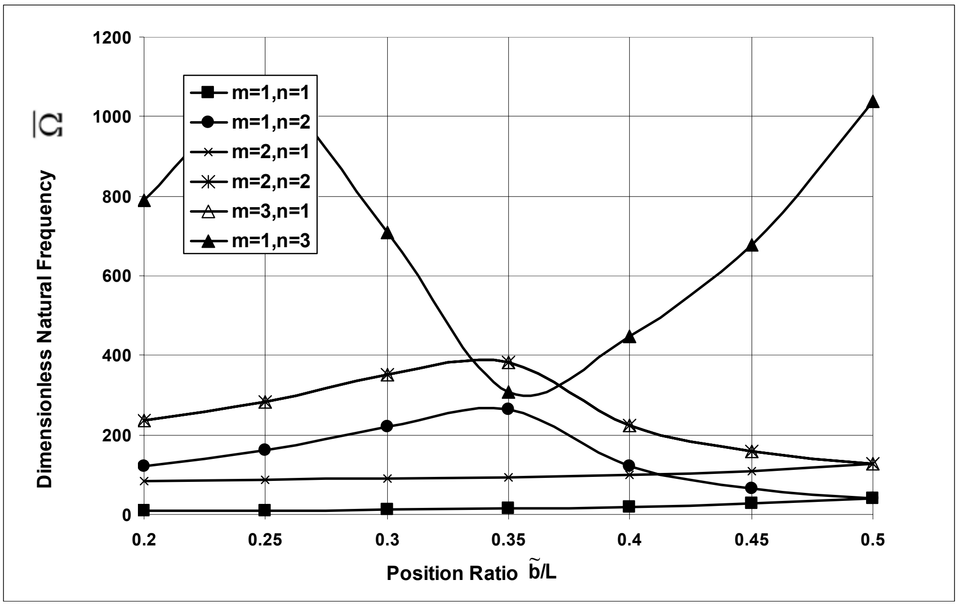
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



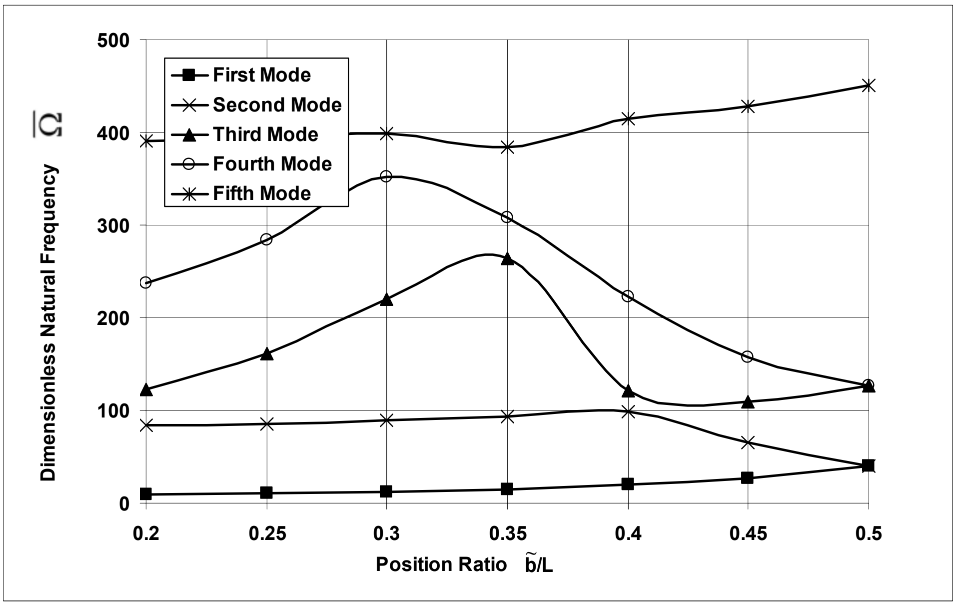
b) "Various Modes with (FFCFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.112 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}/L$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



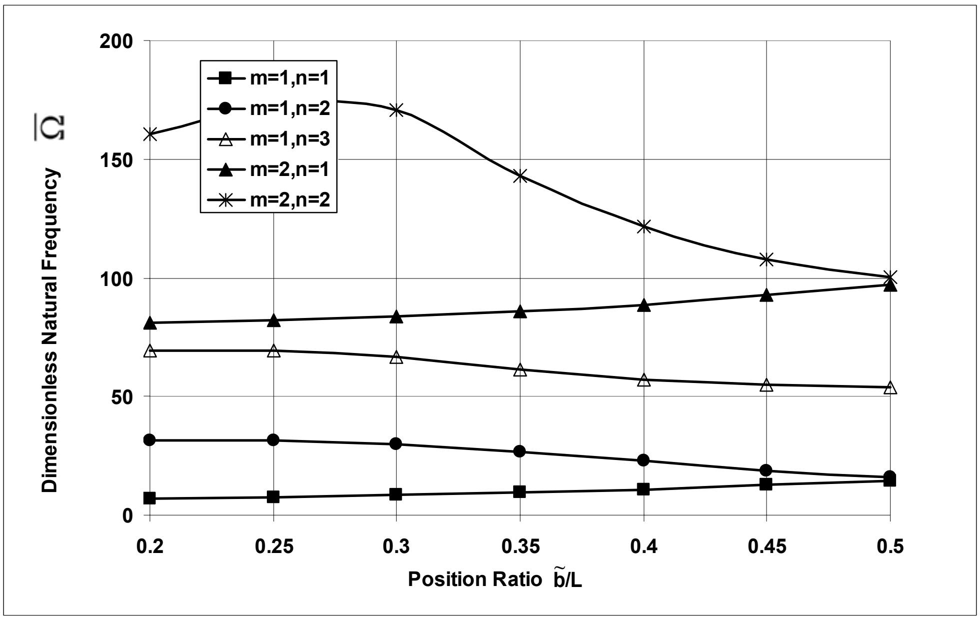
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFF) B.C.'s, ''Hard'' Adhesive



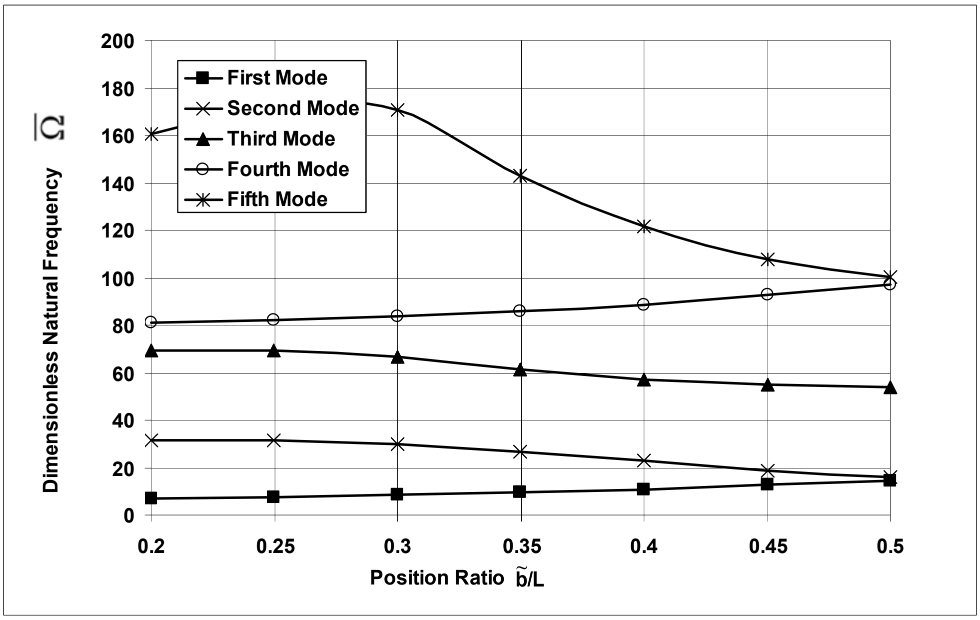
b) "Various Modes with (FFFFFF) B.C.'s, ''Hard' Adhesive

Fig 8.113 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



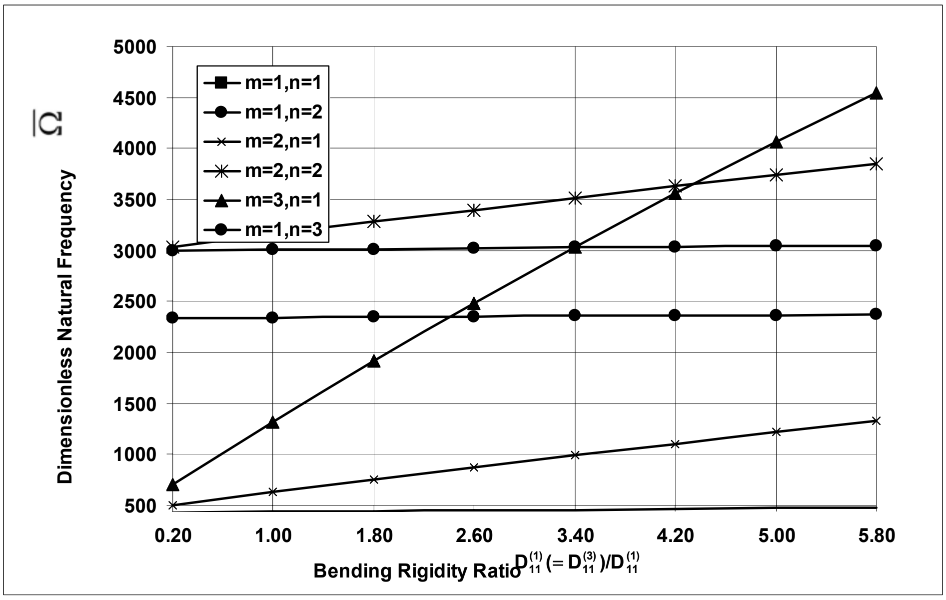
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



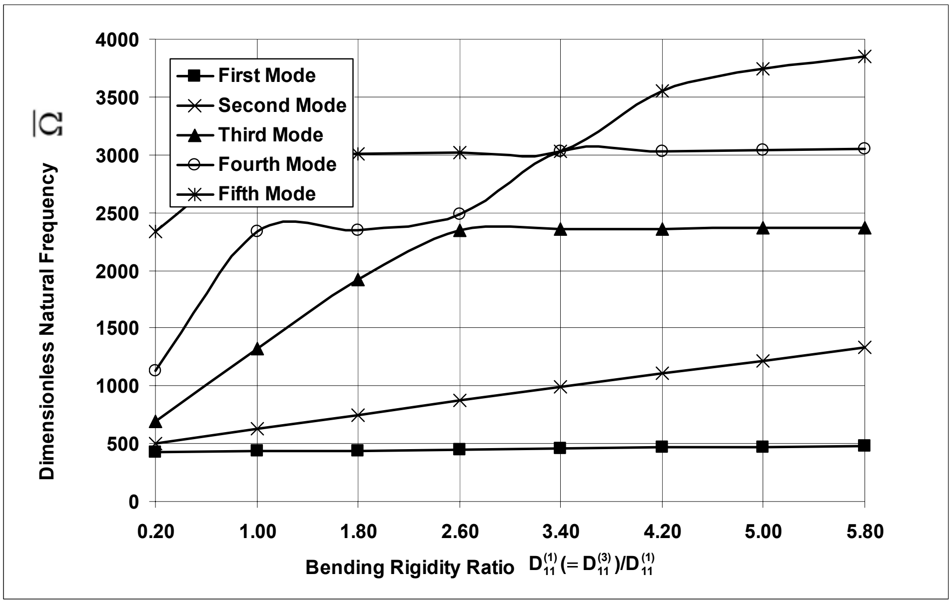
b) "Various Modes with (FFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.114 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



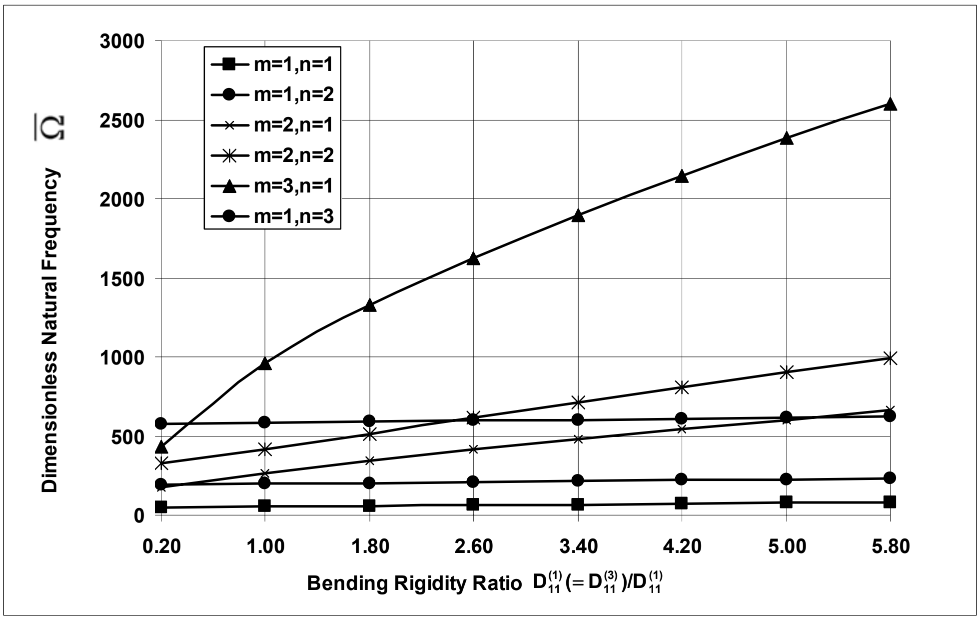
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFC) B.C.'s, ''Hard'' Adhesive



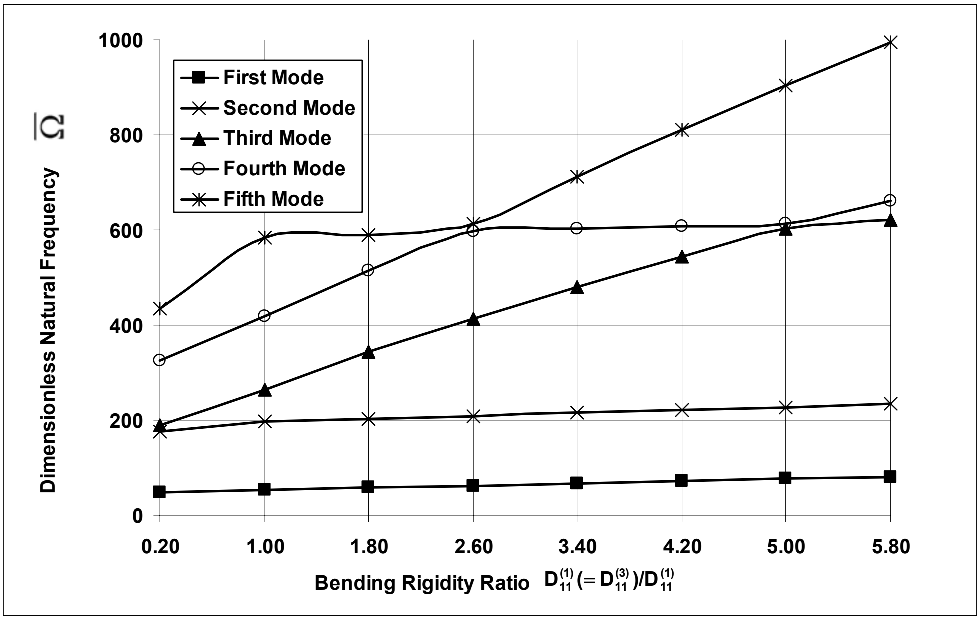
b) "Various Modes with (FFCFFC) B.C.'s, ''Hard'' Adhesive

Fig 8.115 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non--Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



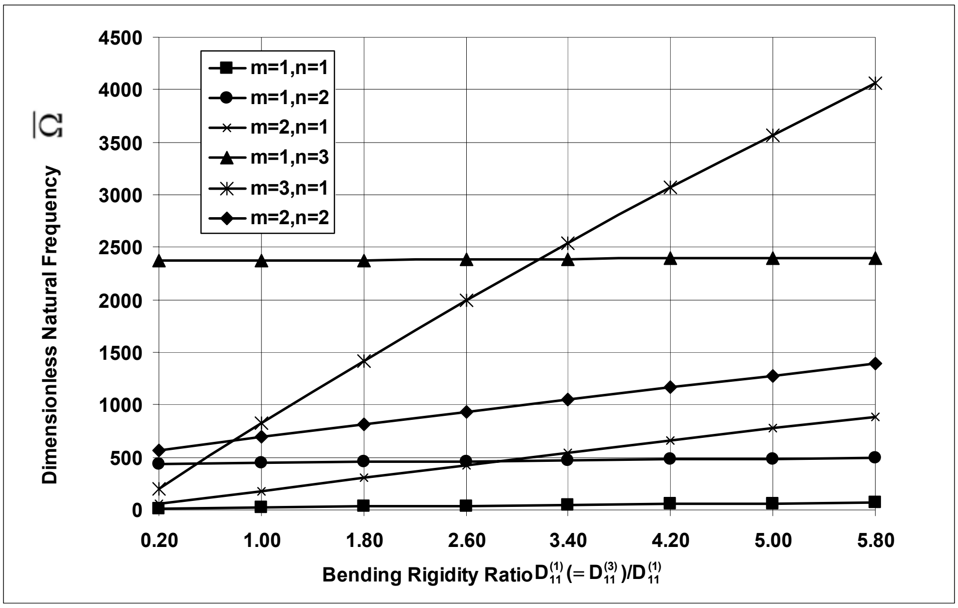
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFC) B.C.'s, ''<u>Soft</u>'' Adhesive



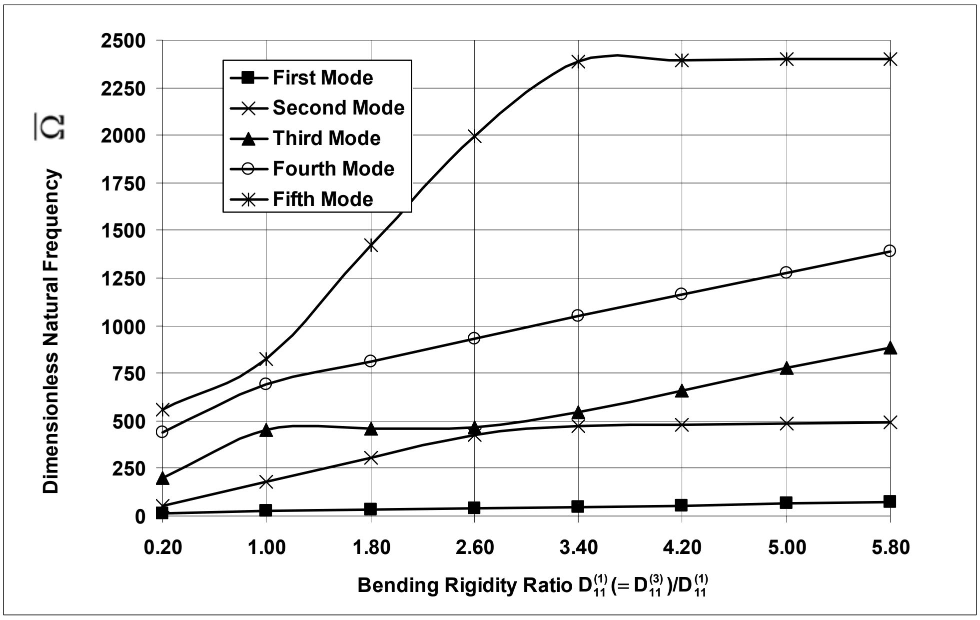
b) "Various Modes with (FFCFFC) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.116 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

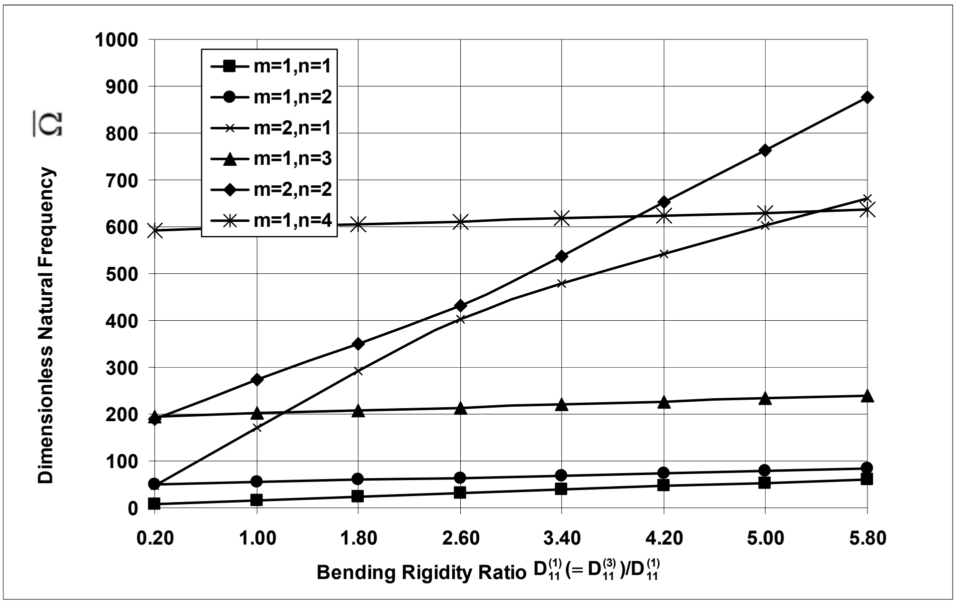


b) "Various Modes with (FFCFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

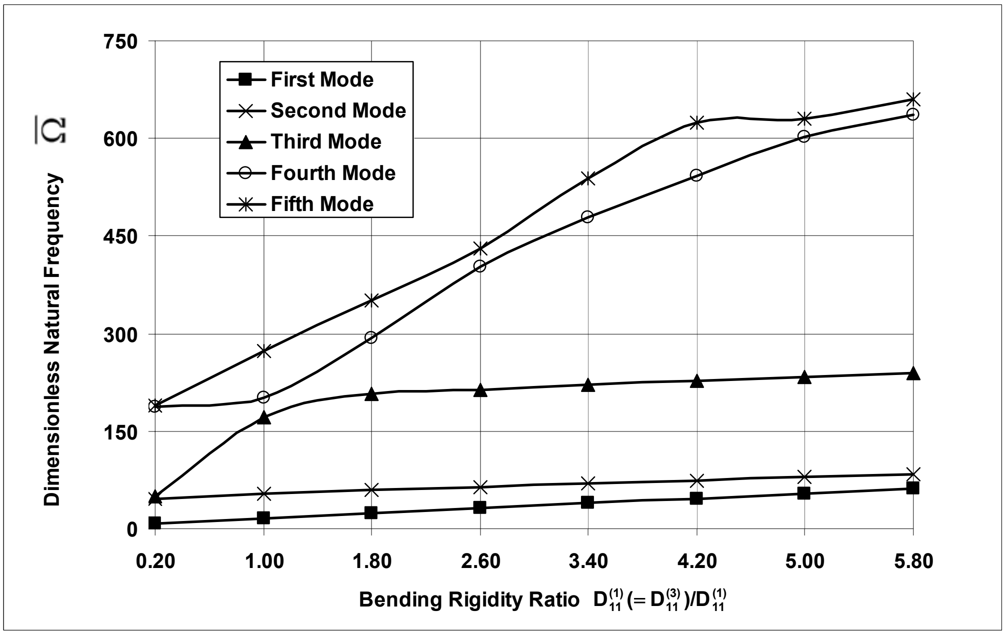
Fig 8.117 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I + \ell_{II}$)=0.3m, $\widetilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



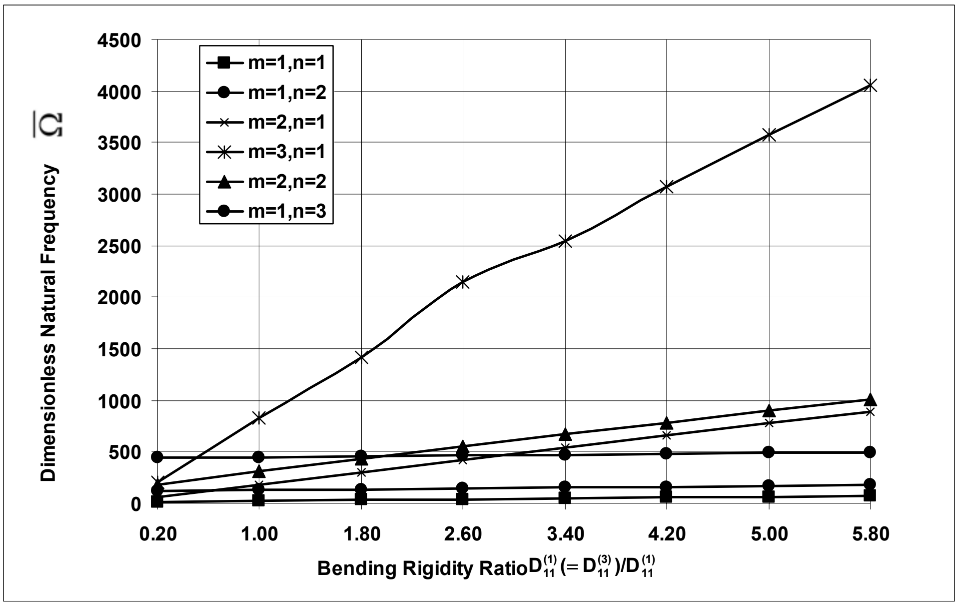
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFF) B.C.'s, "<u>Soft</u>" Adhesive



b) "Various Modes with (FFCFFF) B.C.'s, "<u>Soft</u>" Adhesive

Fig 8.118 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFF) B.C.'s, "Hard" Adhesive

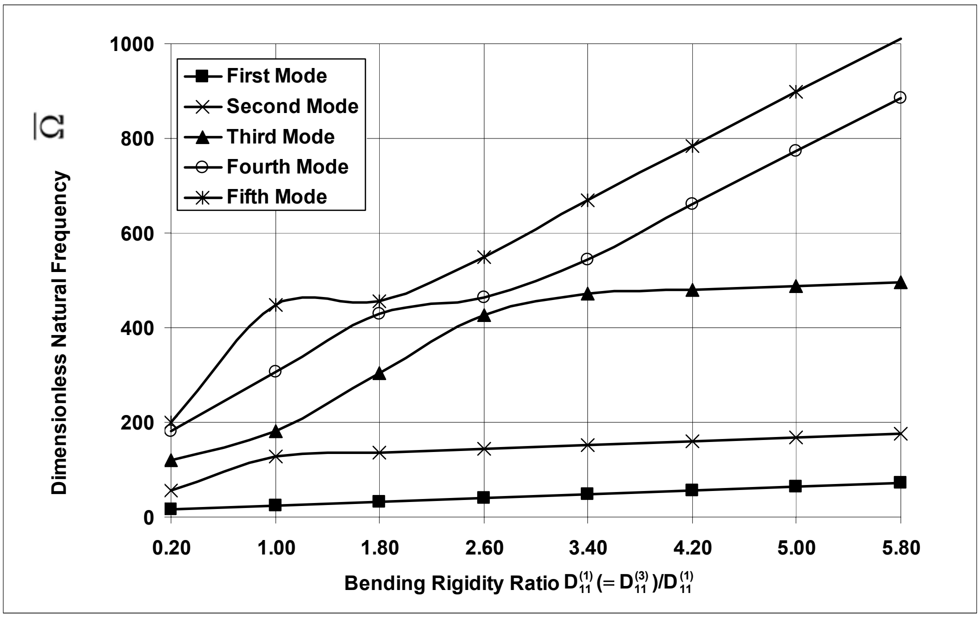
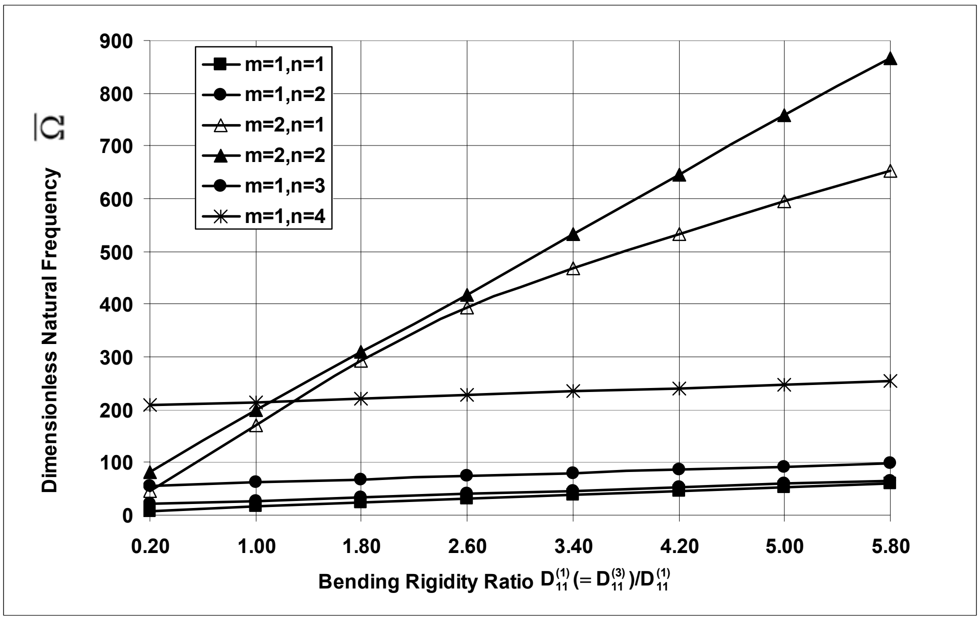
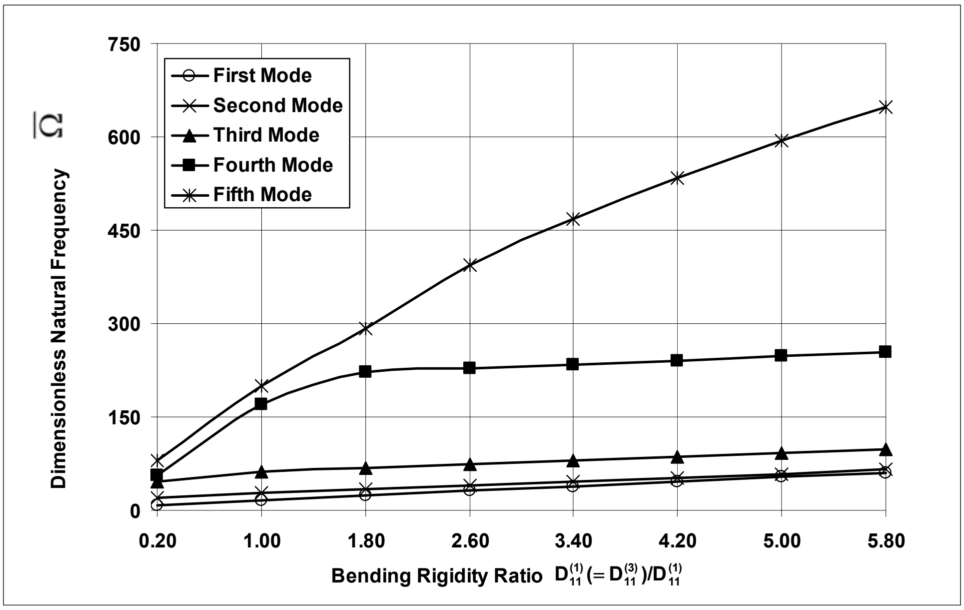


Fig 8.119 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFF) B.C.'s, ''Soft'' Adhesive

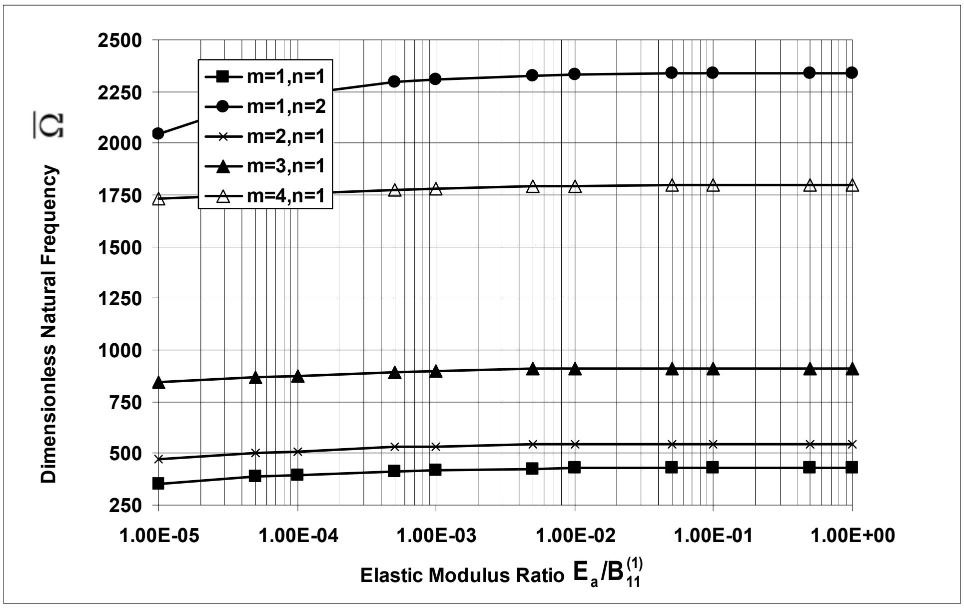


b) "Various Modes with (FFFFFF) B.C.'s, ''Soft'' Adhesive

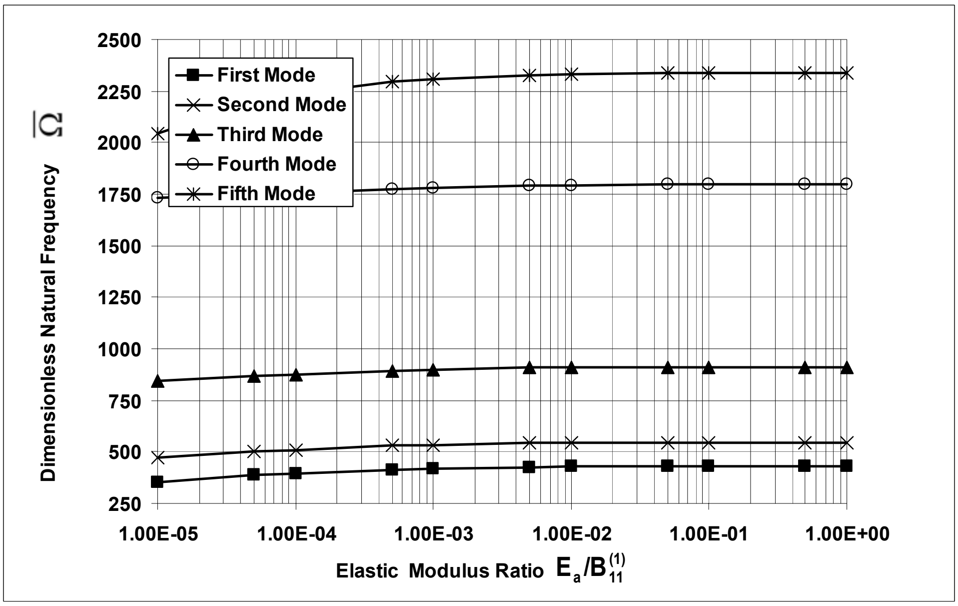
Fig 8.120 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)



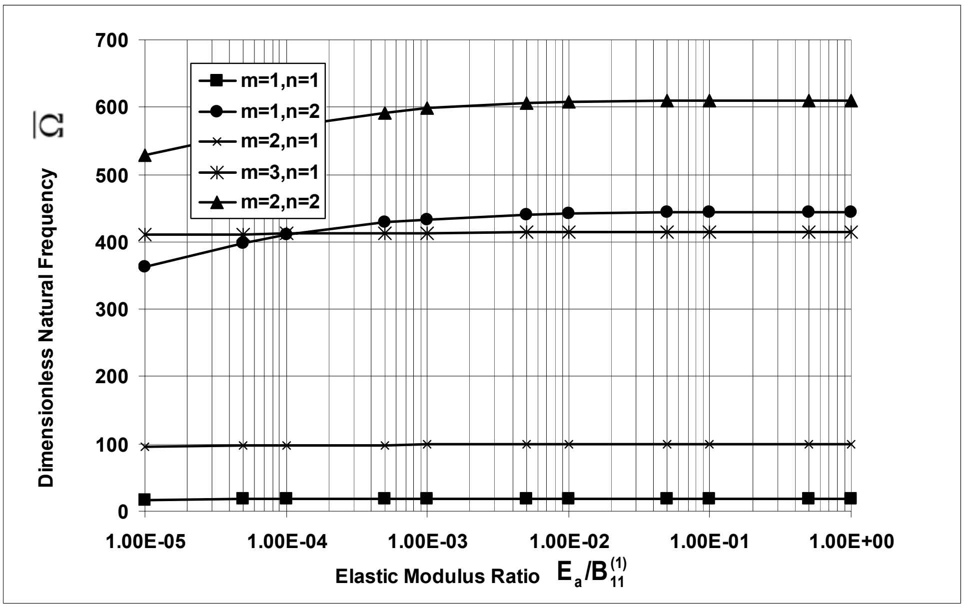
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFC) B.C.'s



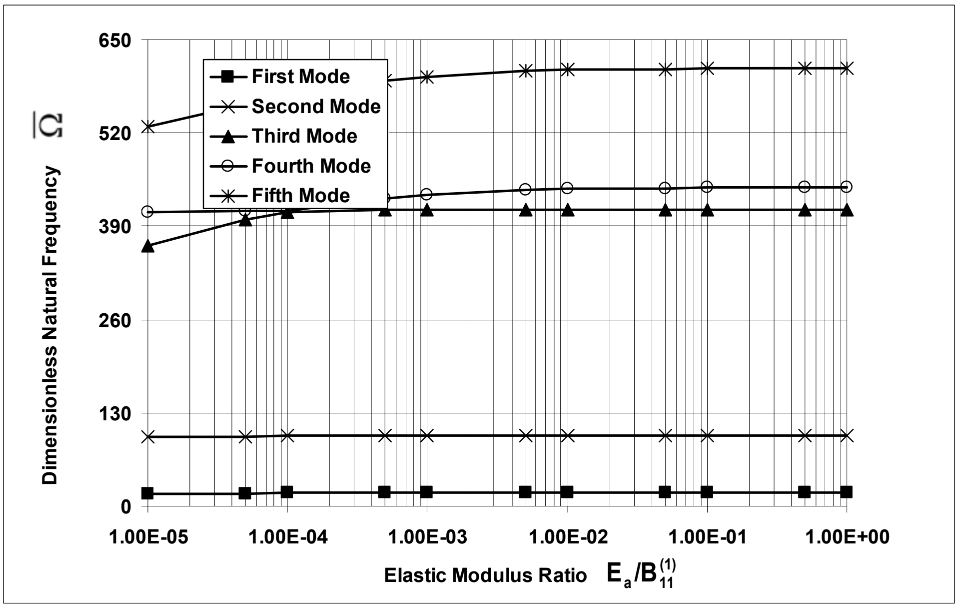
b) "Various Modes with (FFCFFC) B.C.'s

Fig 8.121 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)
Elastic Modulus Ratio axis is plotted in Log Scale



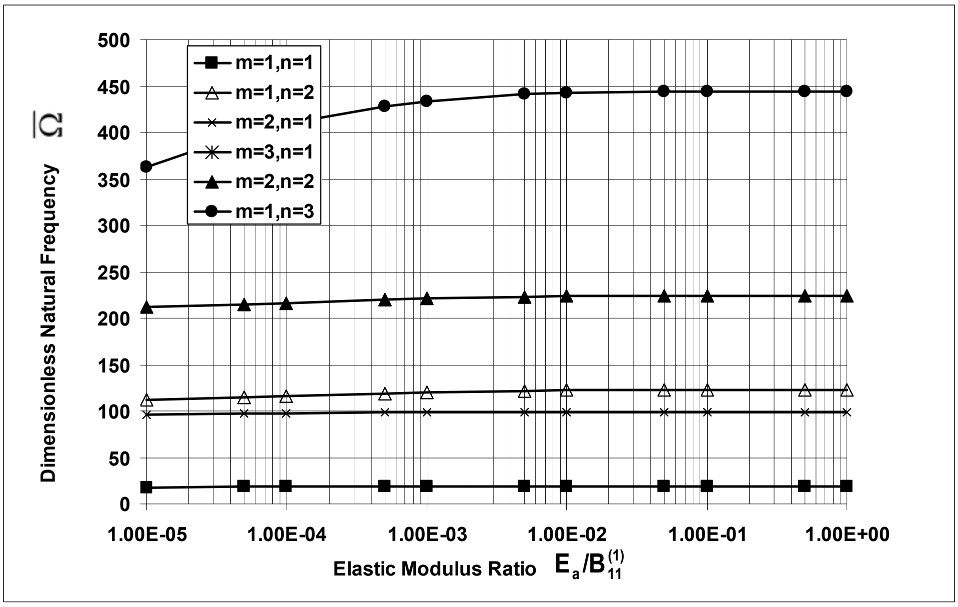
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFF) B.C.'s



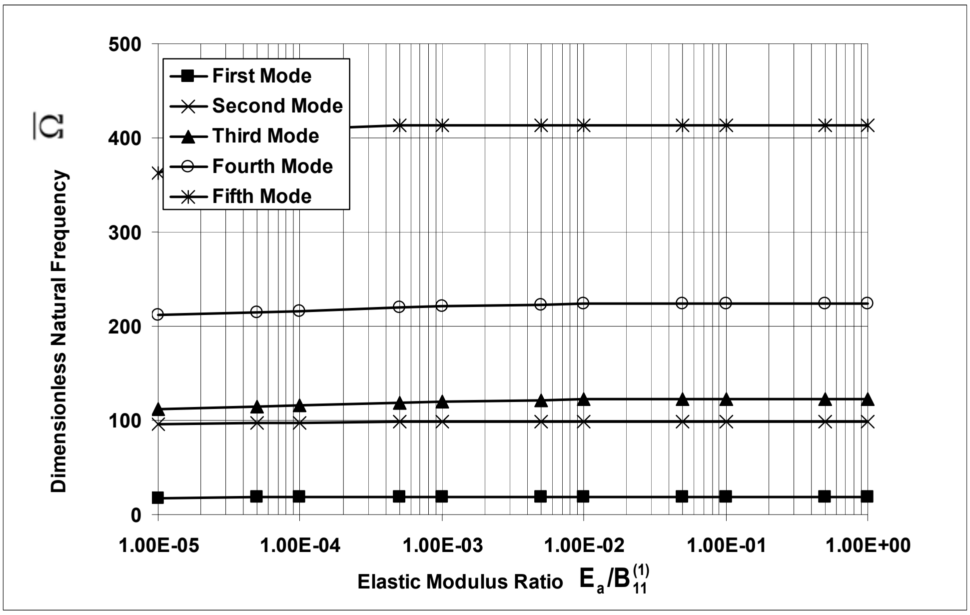
b) "Various Modes with (FFCFFF) B.C.'s

Fig 8.122 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



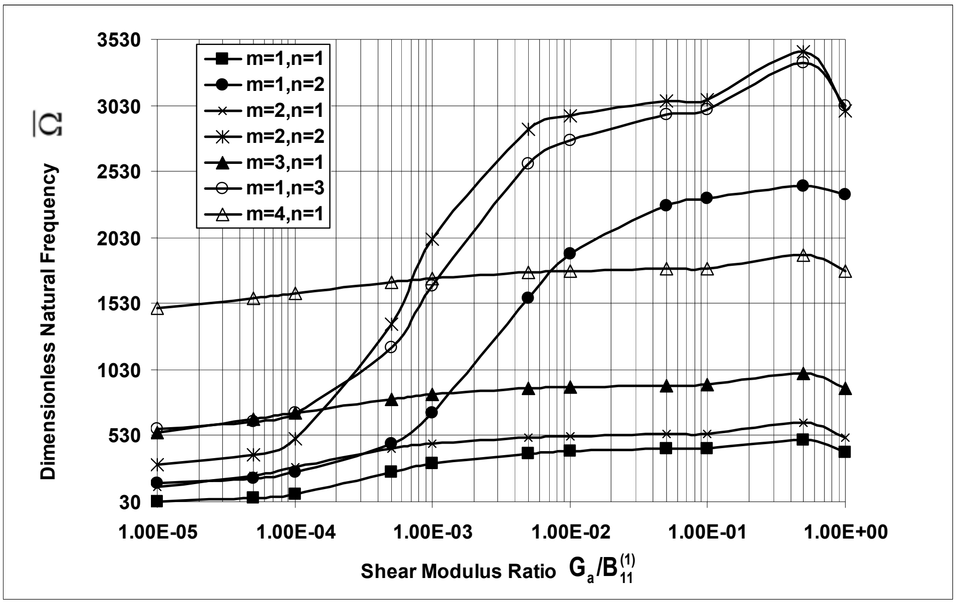
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFF) B.C.'s



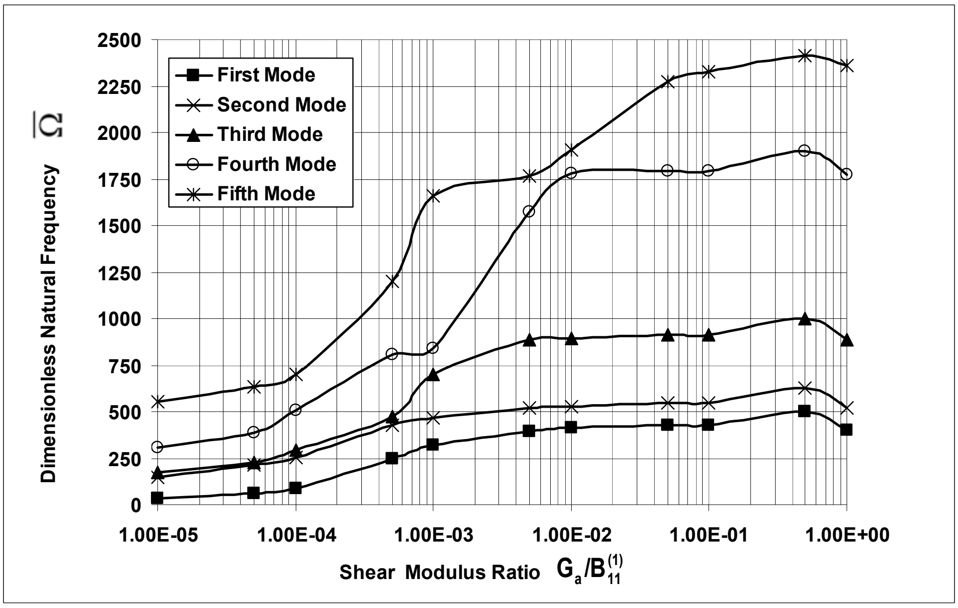
b) "Various Modes with (FFFFFF) B.C.'s

Fig 8.123 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



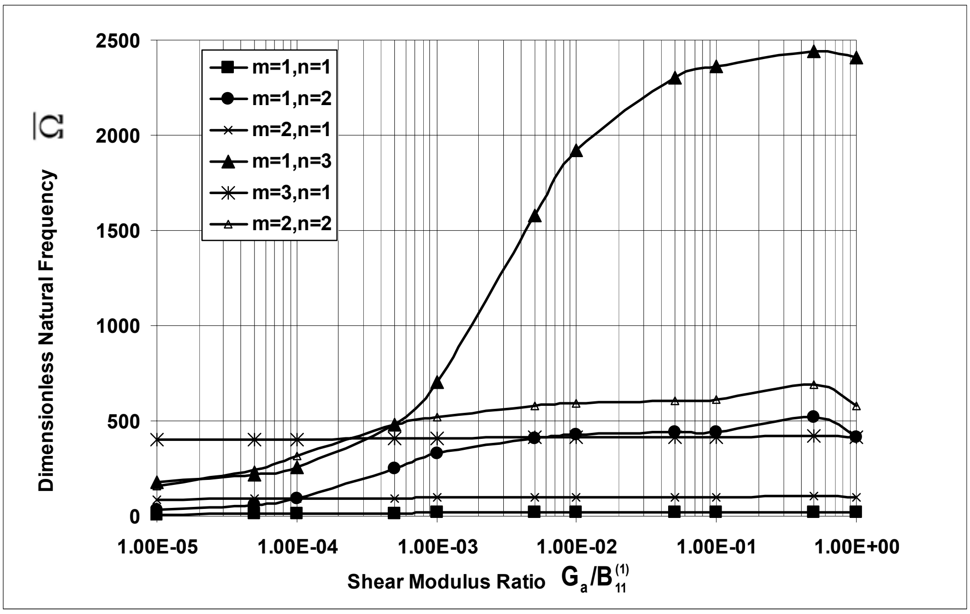
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFC) B.C.'s



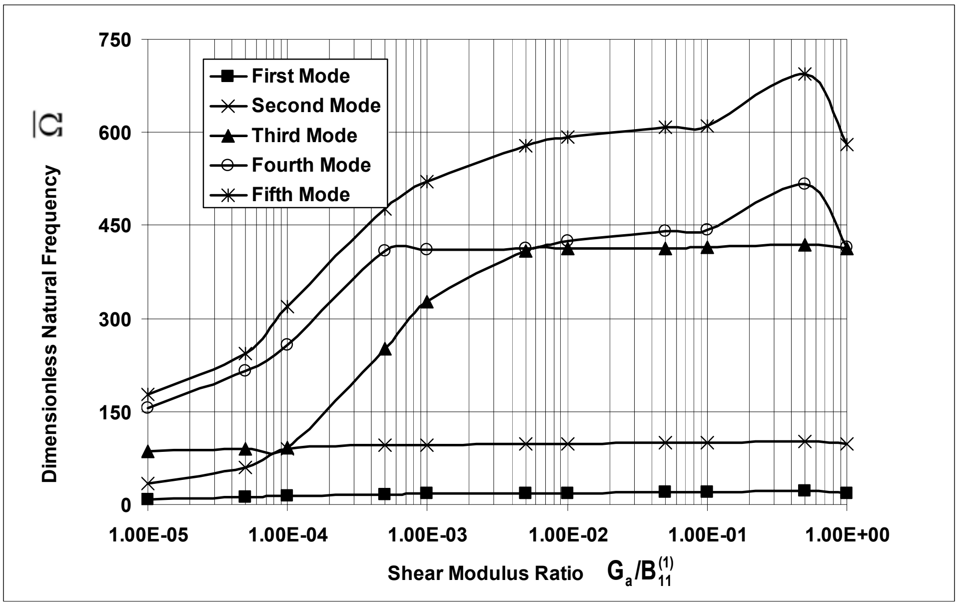
b) "Various Modes with (FFCFFC) B.C.'s

Fig 8.124 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFC)
Shear Modulus Ratio axis is plotted in Log Scale



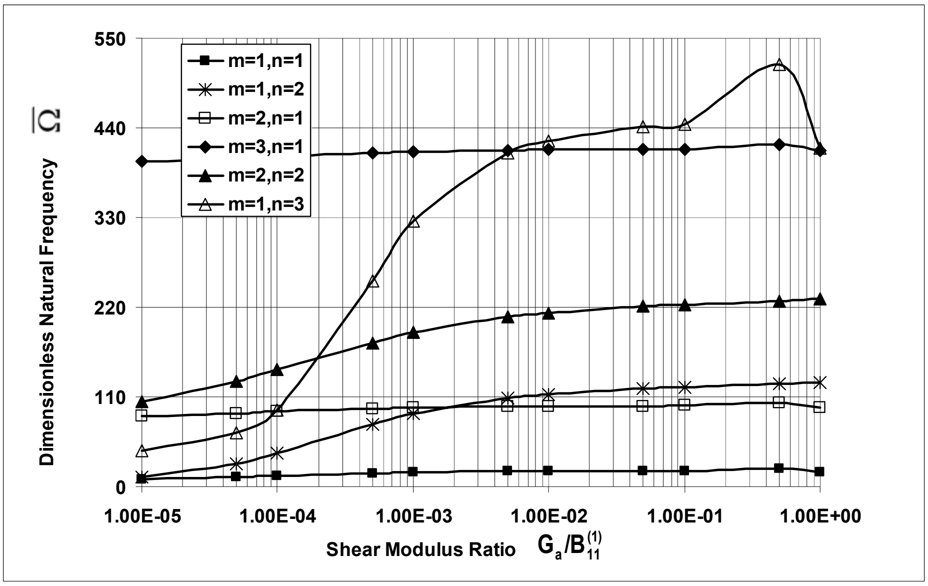
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFF) B.C.'s



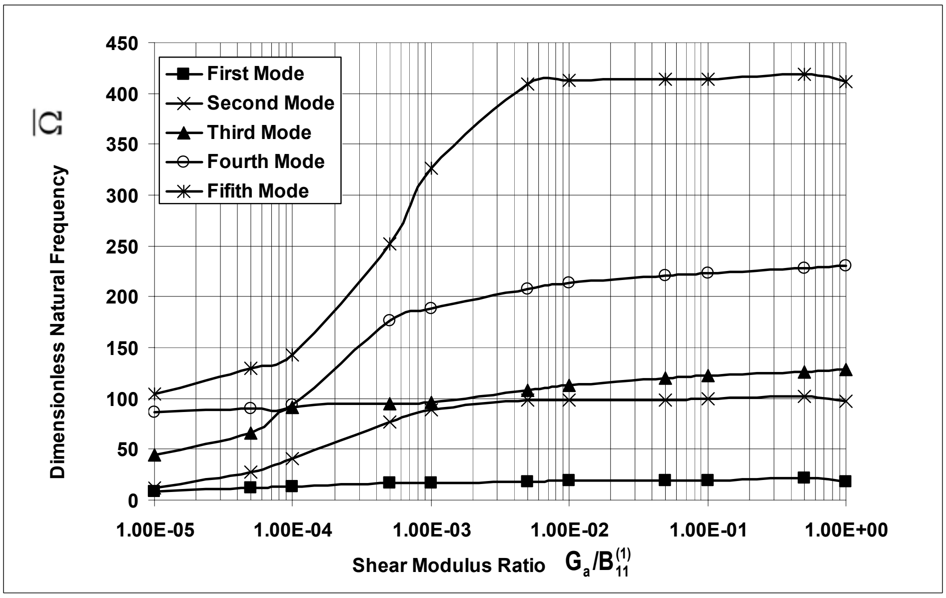
b) "Various Modes with (FFCFFF) B.C.'s

Fig 8.125 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFF)
Shear Modulus Ratio axis is plotted in Log Scale



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFF) B.C.'s



b) "Various Modes with (FFFFFF) B.C.'s

Fig 8.126 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_a/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint (or Symmetric Doubler Joint)"

(Plate 1=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFF)
Shear Modulus Ratio axis is plotted in Log Scale

8.5.4 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequencies"

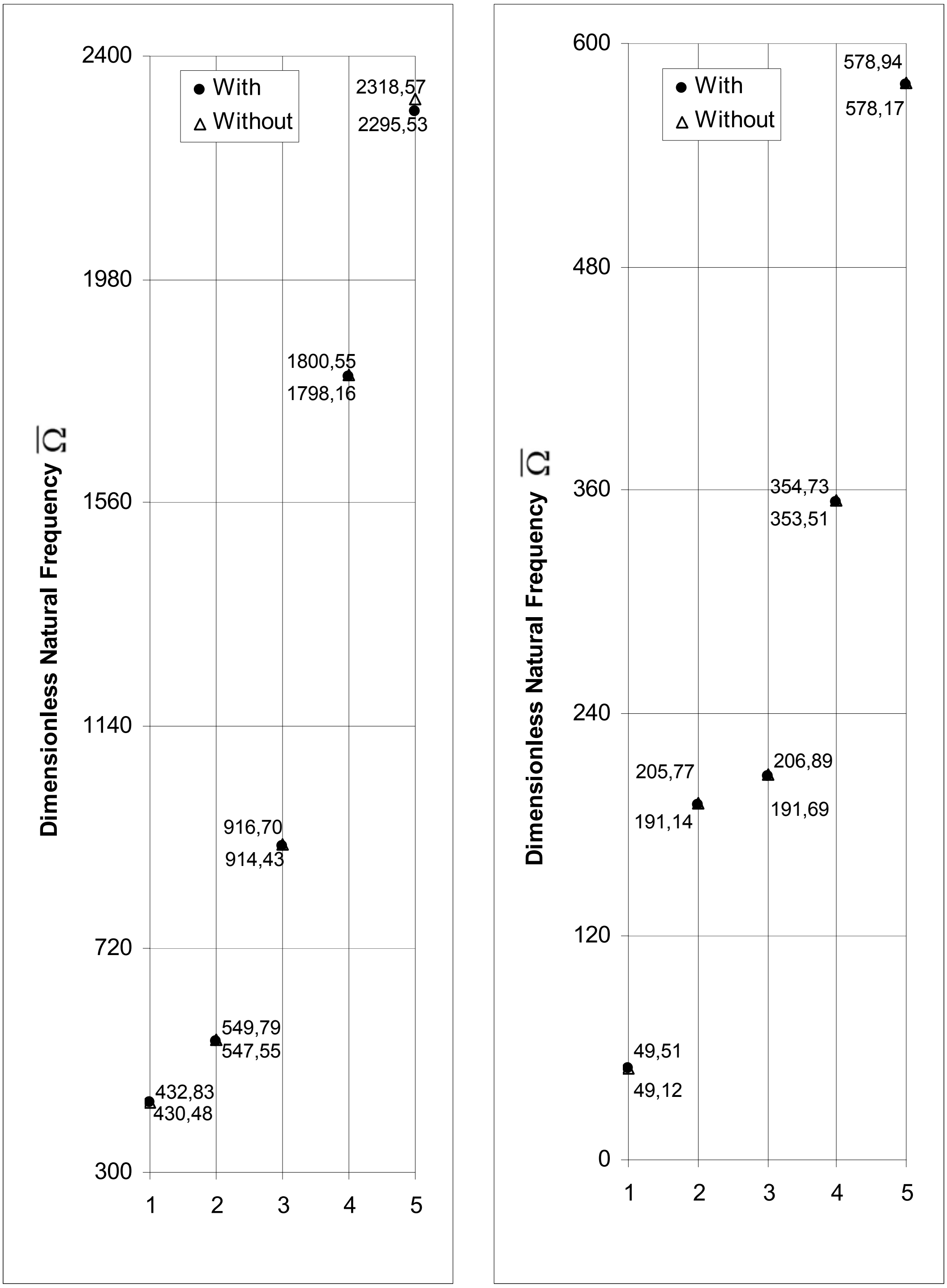
Table 8.7 Comparison of "Dimensionless Natural Frequencies" obtained by adding $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms to adhesive layer equations for "Main PROBLEM IIb"

a) "Hard" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	432.834	430.476	2.358
	2	549.791	547.545	2.245
	3	916.705	914.429	2.275
	4	1800.549	1798.165	2.384
	5	2318.570	2295.530	23.040
SFFS	1	223.785	222.617	1.168
	2	335.704	334.522	1.183
	3	700.695	699.417	1.278
	4	1582.930	1581.499	1.431
	5	1622.422	1606.664	15.758
CFFF	1	19.417	19.349	0.067
	2	99.217	99.125	0.092
	3	414.183	414.051	0.131
	4	447.498	445.060	2.438
	5	614.117	611.702	2.415
FFFF	1	19.417	19.349	0.067
	2	99.217	99.125	0.092
	3	121.874	120.837	1.037
	4	223.120	221.983	1.137
	5	414.183	414.051	0.131

b) "Soft" Adhesive Case

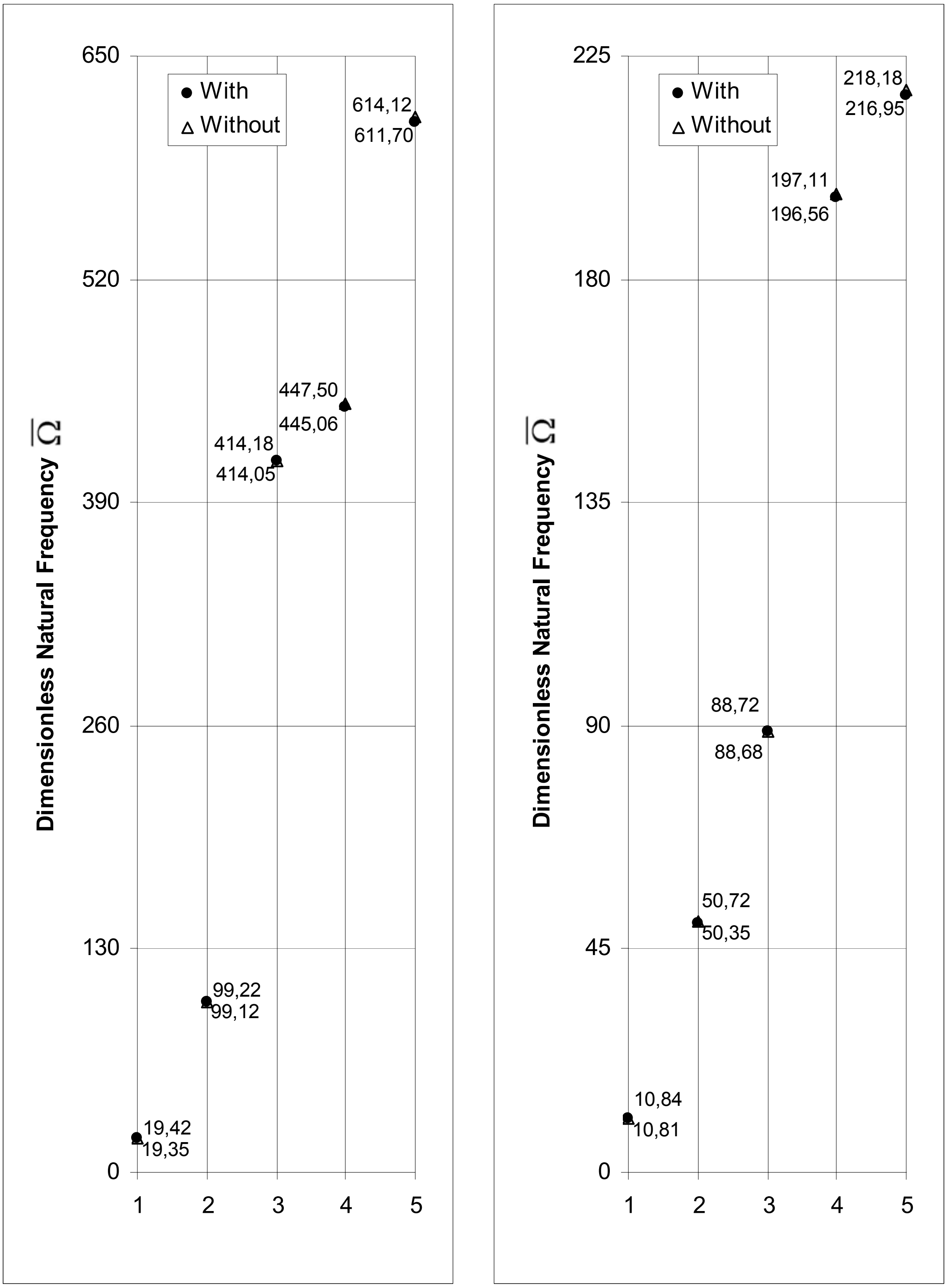
Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	49.507	49.116	0.391
	2	191.691	191.144	0.547
	3	206.889	205.773	1.116
	4	354.729	353.509	1.221
	5	578.942	578.172	0.770
SFFS	1	33.960	33.635	0.325
	2	105.281	104.878	0.403
	3	170.173	169.516	0.658
	4	264.247	262.983	1.264
	5	359.521	358.832	0.689
CFFF	1	10.840	10.807	0.033
	2	50.724	50.353	0.371
	3	88.716	88.680	0.035
	4	197.113	196.562	0.550
	5	218.179	216.953	1.225
FFFF	1	10.735	10.701	0.034
	2	22.674	22.477	0.197
	3	57.067	56.748	0.319
	4	88.620	88.584	0.036
	5	121.613	121.365	0.248



a) "Hard" Adhesive Case **b) "Soft" Adhesive Case**

Figure 127 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint"

(Boundary Conditions in y-direction FFCFFC)



a) “Hard” Adhesive Case

b) “Soft” Adhesive Case

Figure 128 **Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on “Dimensionless Natural Frequency $\overline{\Omega}$” in “Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Single Lap Joint” (Boundary Conditions in y-direction FFCFFF)**

8.6 Numerical Results and Discussion for "Main PROBLEM III.a"

In the "Main PROBLEM IIIa.", the "Composite Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint" is analyzed. The doublers are made of Graphite-Epoxy and the plate adherends are Kevlar-Epoxy. For the in-between adhesive layer, the "hard" and the "soft" adhesive cases are taken into account. The "Geometric and the Material Characteristics" of the symmetric double lap joint system are given in Table 8.3.

In Figures 8.113 – 8.122, the mode shapes and the corresponding natural frequencies (from the first to fifth), in the "hard" and the subsequent "soft" adhesive cases with various boundary conditions are presented.

From aforementioned Figures, in the "hard" adhesive case it is easy to observe that, that there exists an almost "stationary region" in the mode shapes with respect to the symmetry of the "Boundary Conditions". And symmetric and skew symmetric modes flow each other in the composite symmetric double lap joint system. If the boundary conditions are not symmetric the "almost stationary area" changes the position from left to right. In the "soft" adhesive case, however, an almost "stationary region" does not exist in mode shapes. The general trend in the mode shapes, for the "soft" adhesive case, the "Bonded Region" moves or bends with the rest of the lap joint system. And the mode shapes are completely different in comparison with those of the "hard" adhesive cases with the same support conditions.

Next, for the "Main PROBLEM IIIa", in Figures 8.123 through 8.140, several important parametric studies are presented. In Figures 8.123-8.128, the "Dimensionless Natural Frequency $\overline{\Omega}$" versus "Joint Length Ratio $(\ell_I + \ell_I)/L$" from the first up to the fifth mode are plotted, for both the "hard" and the "soft" adhesive cases, corresponding to the various support conditions.

From Figures 8.123, 8.125, 8.127, in the "hard" adhesive case, it is obvious that as the wet area or the "Bonded Region" spreads (in the y-direction), the natural frequencies, at first gradually, and then, relatively sharply increases. These results of course, are consequences of the increasing stiffness of the lap joint system due to the spreading of the "Bonded Region".

In the "soft" adhesive case, in Figures 8.124, 8.126, 8.128, the natural frequencies does not significantly change. And no sharp increases can be observed as the "Bonded Region" spreads along the y-direction. This also can be expected. It is because, due to the "soft" adhesive, the "Bonded Region" connects both adherends rather loosely and thus a relatively loose doubler joint system is created.

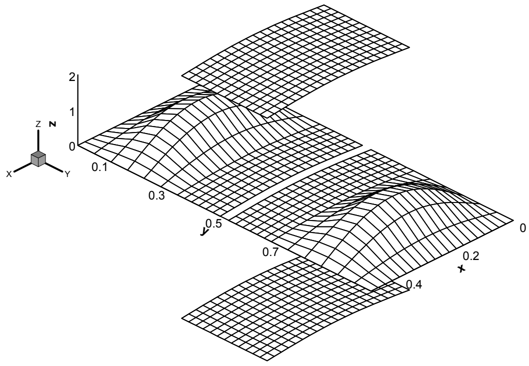
In Figures 8.129 through 8.134, the effect of the "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" on the natural frequencies (from the first up to the fifth) in the "hard" and "soft" adhesive cases, are investigated for various boundary conditions. In the "hard" adhesive case, in Figures 8.129, 8.131, 8.133, the first two natural frequencies, in spite of the increasing "Bending Rigidity Ratio", does remain practically constant. In the higher modes, the natural frequencies increase sharply at first and after the "Bending Rigidity Ratio=1.8" they become almost flat or constant regardless of the increase in "Bending Rigidity Ratio".

In the "soft" adhesive cases, in the Figures 8.130, 8.132, 8.134, the first and the second frequencies remain more or less constant as the "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" increases. In the third and higher modes, the natural frequencies increase.

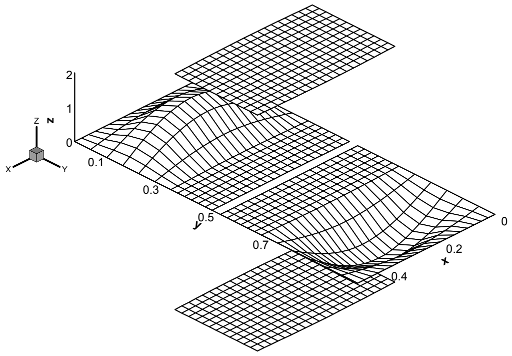
Lastly, the direct effects of the adhesive layer elastic constants E_{a1}, E_{a4}, and also $G_{a1,}$ G_{a4} on the dimensionless natural frequencies are investigated for the "Main PROBLEM III.a". In order to show these effects, the "Dimensionless Natural Frequencies" versus the "Adhesive Elastic Modulus Ratio $(E_{a1}=E_{a4})/B_{11}^{(1)}$" are plotted (while the other elastic constant kept constant) in Figures 8.135 through 8.137 for various boundary condition. Similarly, the "Dimensionless Natural Frequencies" versus the “Adhesive Shear Modulus Ratio (G_{a1}=G_{a4})/ $B_{11}^{(1)}$” are presented in Figures 8.138 through 8.140 for various support condition.

It can be seen from the Figures 8.135-8.137, the influence of the “Adhesive Elastic Modulus Ratio (E_{a1}=E_{a4})/ $B_{11}^{(1)}$” on the natural frequencies, is not significant. In Figures 8.138- 8.140, we can see that the “Shear Modulus Ratio (G_{a1}=G_{a4})/ $B_{11}^{(1)}$”, significantly affects the natural frequencies. Also, in those Figures, one can observe a “transition region” which takes the frequencies to significant higher levels. After then, no change is observed in the frequencies. In Figure 8.140, the “Dimensionless Natural Frequency” versus “Shear Modulus Ratio (G_{a1}=G_{a4})/ $B_{11}^{(1)}$” is given for FFFFFF Boundary Condition. The natural frequencies are almost constant for lower frequencies but changes significantly for the fifth mode. Since this is not the expected trend for the effect of the shear modulus ratio, this parametric study will be observed in detail in the future studies.

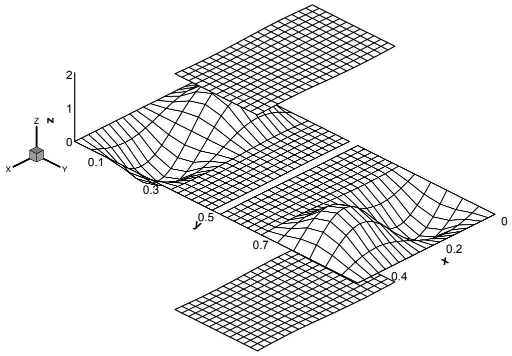
8.6.1 Natural Frequencies and Corresponding Mode Shapes for "Main PROBLEM IIIa"



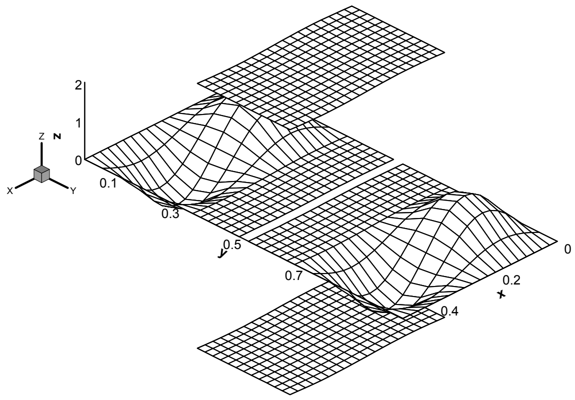
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=1113.502



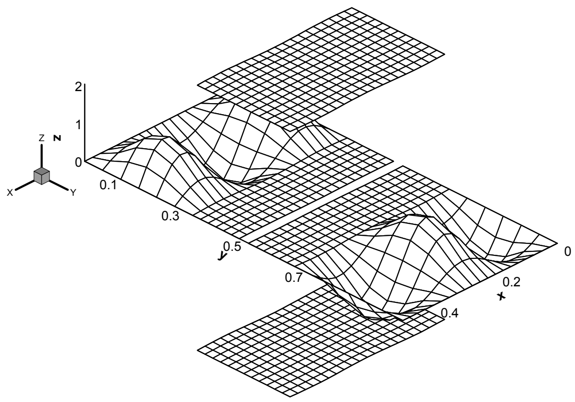
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=1140.849



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=1280.051



d) Fourth Mode with $\overline{\Omega}_4$= $\overline{\omega}_{22}$=1334.096

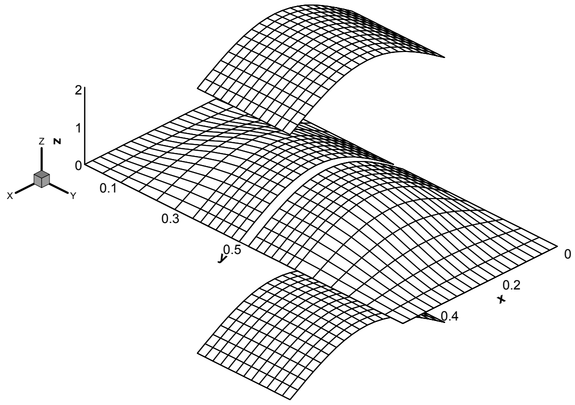


e) Fifth Mode with $\overline{\Omega}_5$= $\overline{\omega}_{31}$=1686.086

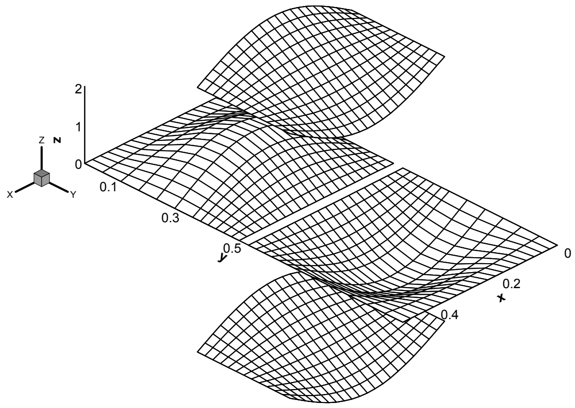
("Hard" Adhesive Case)

Fig 8.129 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

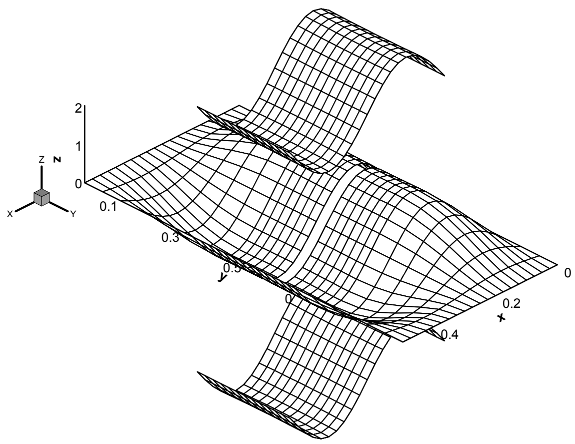
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3 m., b_1=b_4=0.3 m, b_2=b_3=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



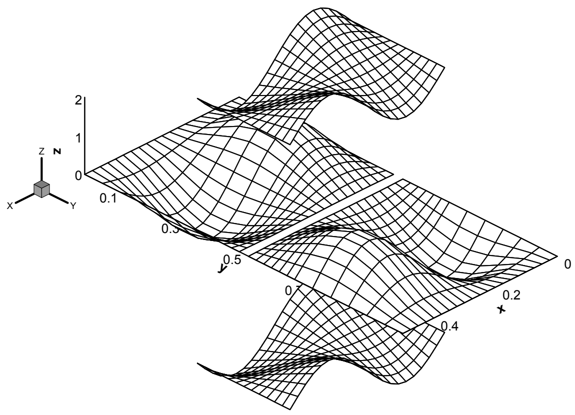
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 55.997$



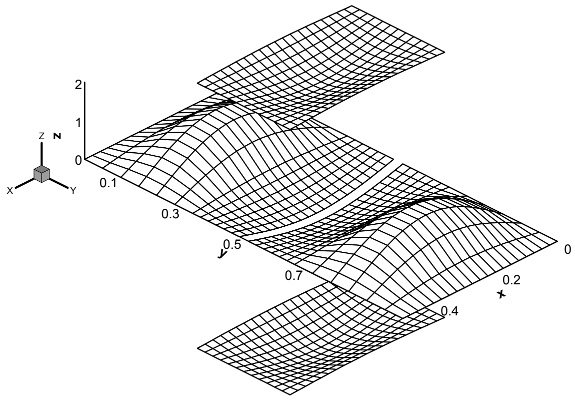
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 211.316$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 261.748$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=411.243

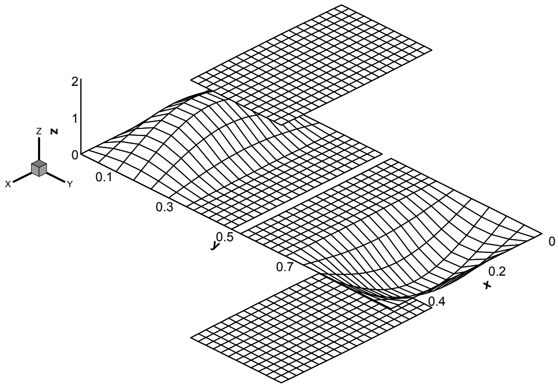


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=732.919

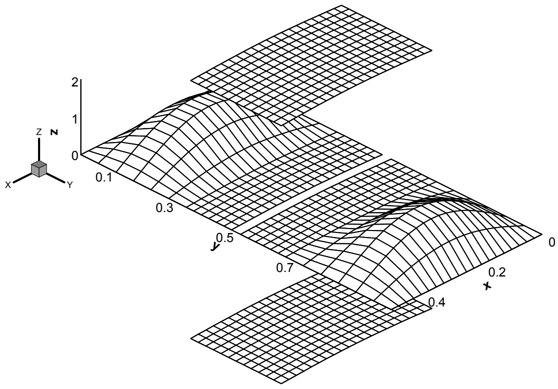
("Soft" Adhesive Case)

Fig.8.130 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

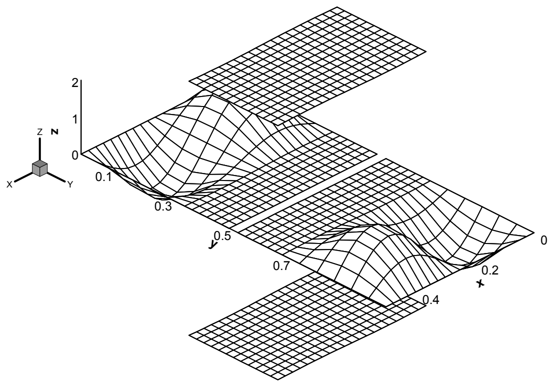
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



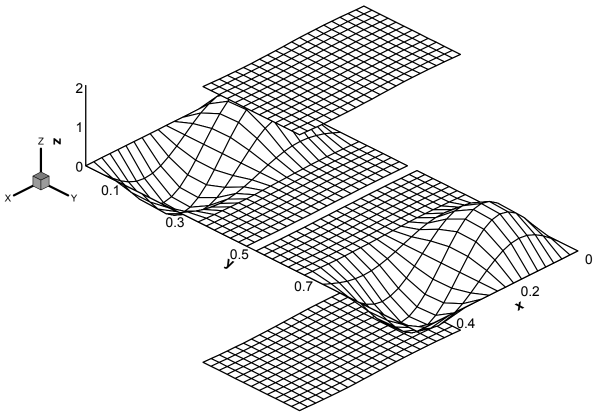
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=592.791



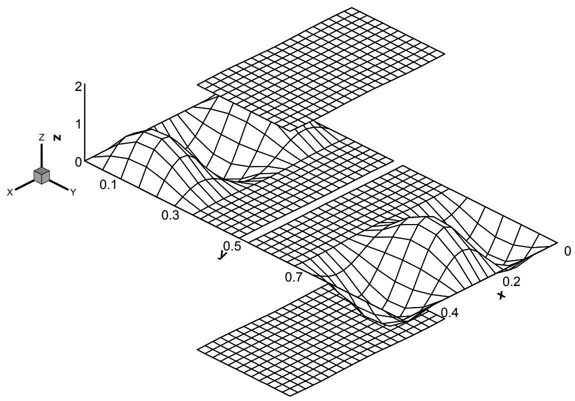
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=600.165



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{12}$=729.047



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=759.748

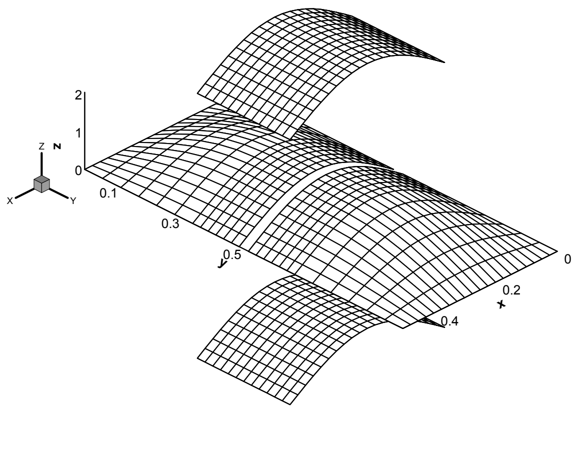


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=1134.688

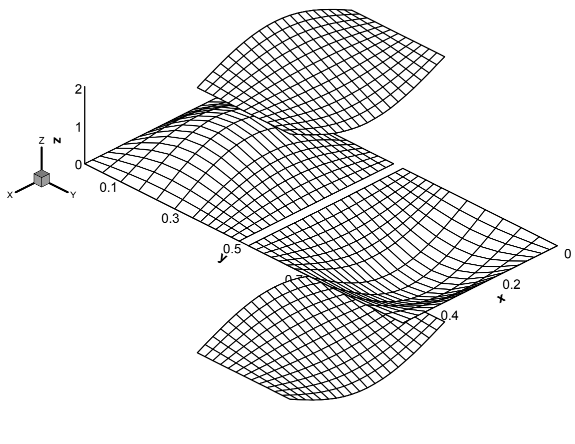
("Hard" Adhesive Case)

Fig.8.131 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

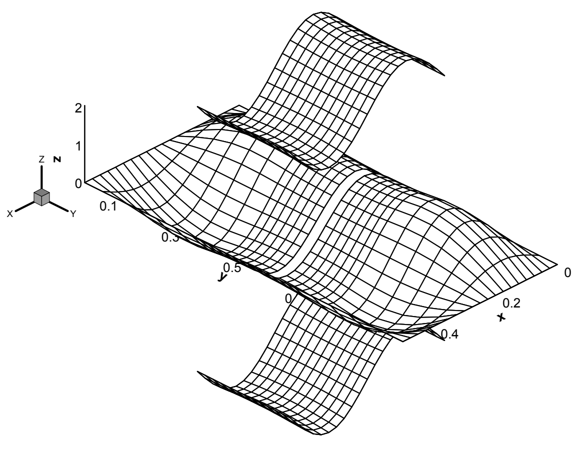
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3 m., $b_1 = b_4$=0.3 m, $b_2 = b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFSFF)



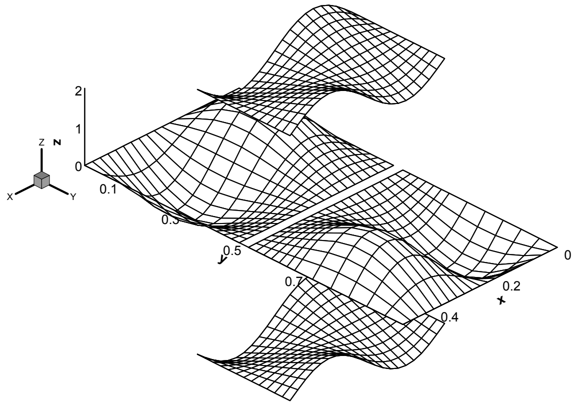
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 44.097$



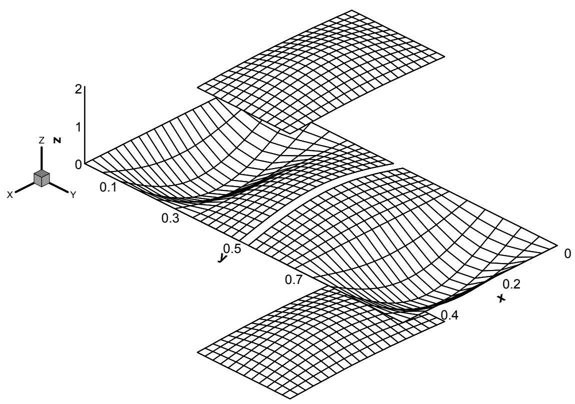
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 129.450$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 232.746$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=294.211

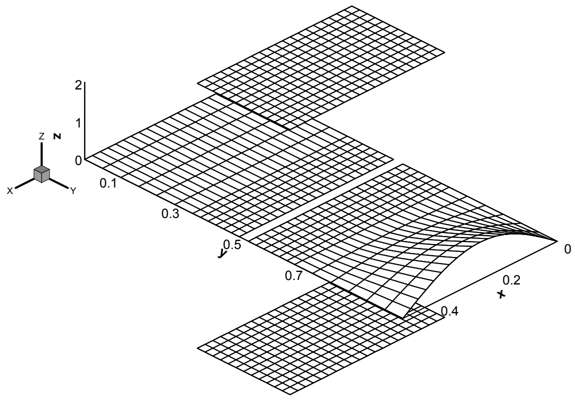


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=410.652

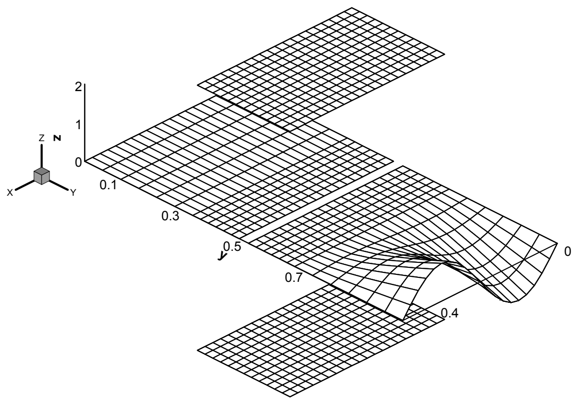
("Soft" Adhesive Case)

Fig.8.132 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

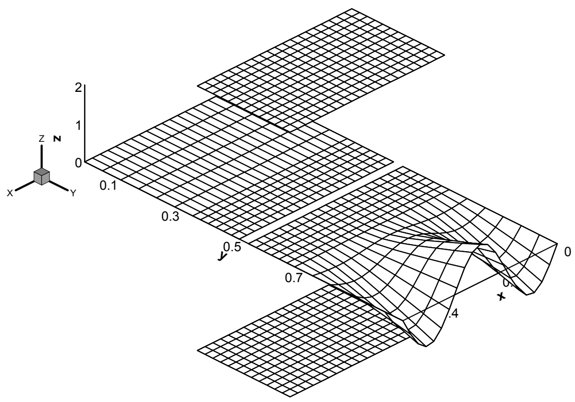
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFSFF)



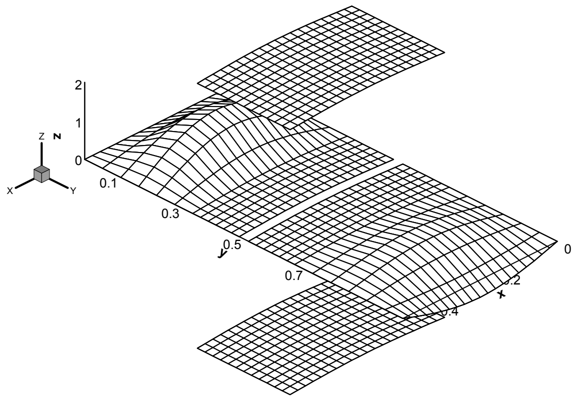
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=43.395



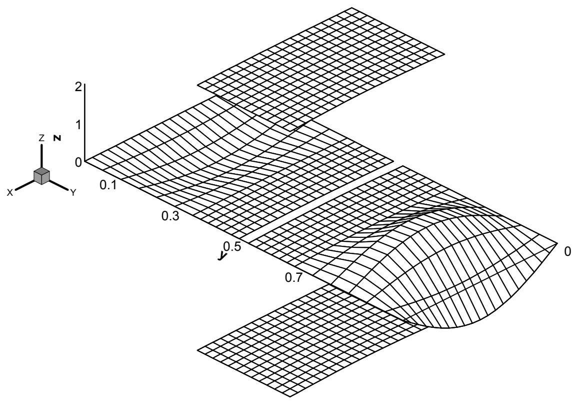
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{21}$=130.238



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{31}$=456.283



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{12}$=1120.601

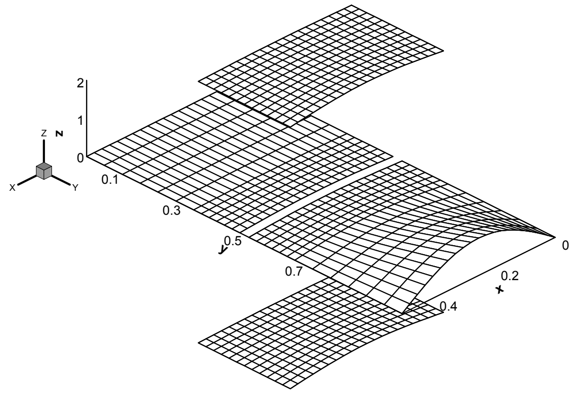


e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{13}$=1168.753

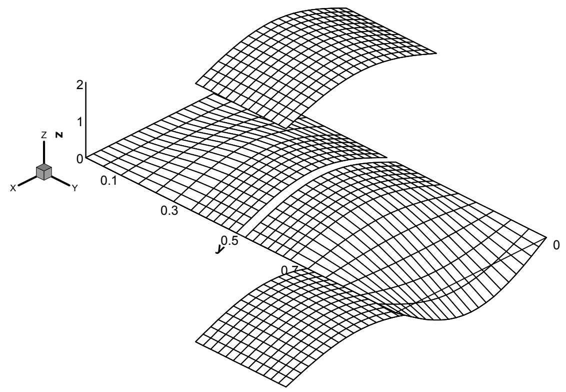
("Hard" Adhesive Case)

Fig.8.133 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

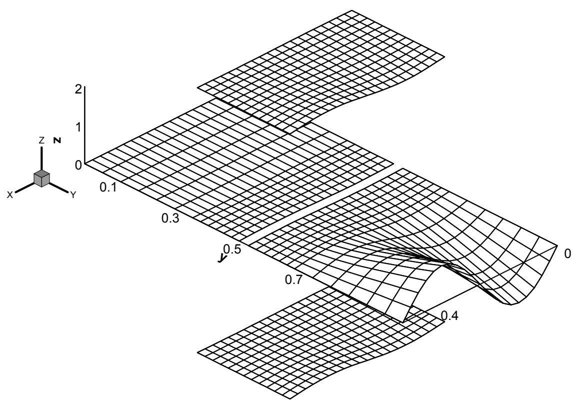
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



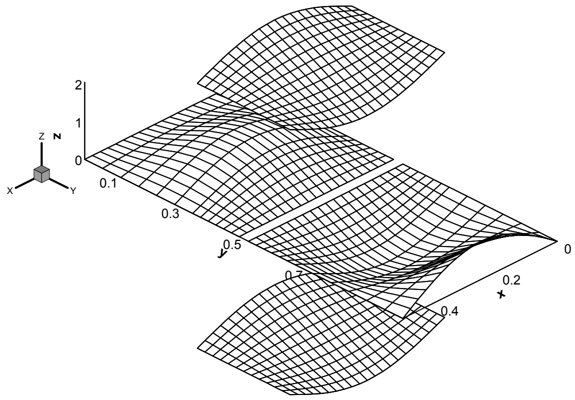
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=19.342



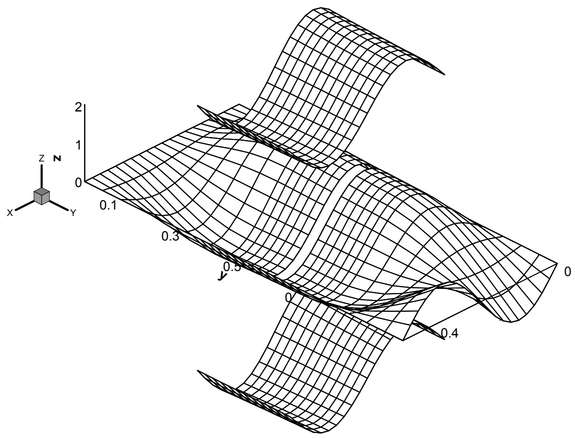
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=57.769



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=104.508



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=215.871

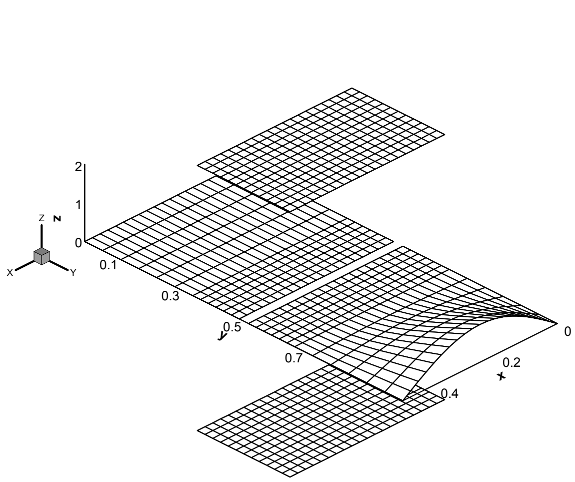


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=265.094

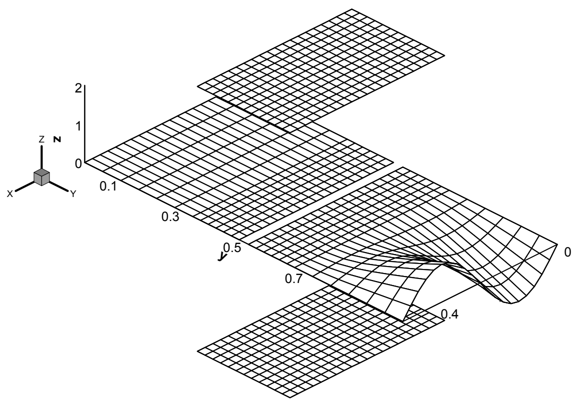
("<u>Soft</u>" Adhesive Case)

Fig.8.134 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

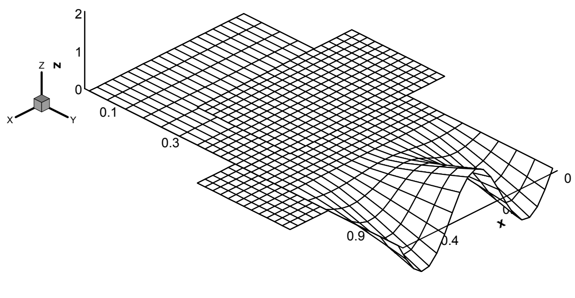
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



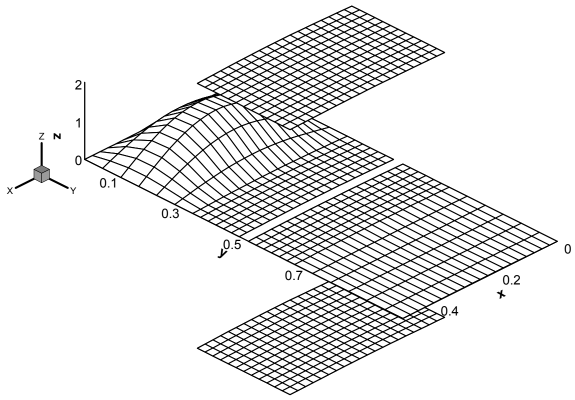
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 43.391$



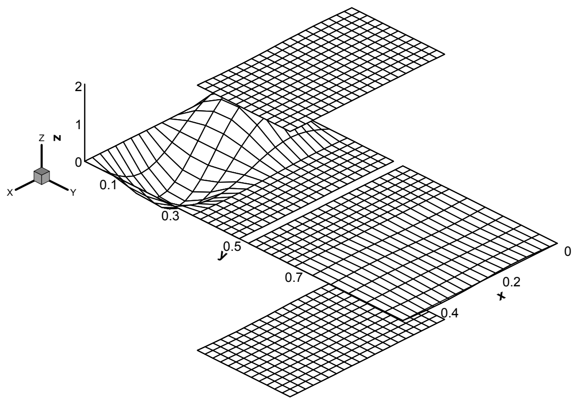
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 130.231$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 456.274$



d) Fourth Mode with $\overline{\Omega}_4=\overline{\omega}_{12}$=596.477

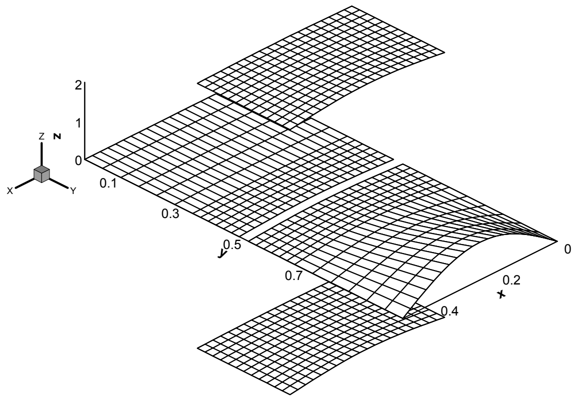


e) Fifth Mode with $\overline{\Omega}_5=\overline{\omega}_{22}$=743.880

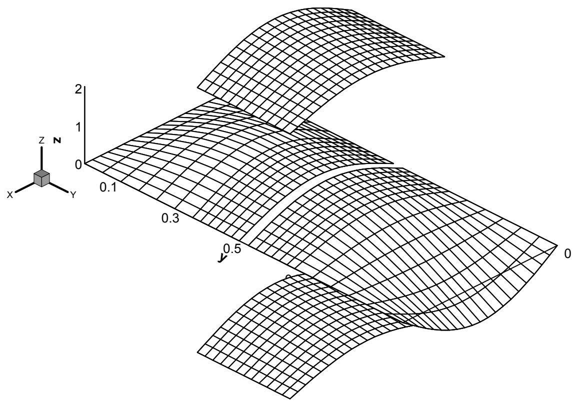
("<u>Hard</u>" Adhesive Case)

Fig.8.135 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

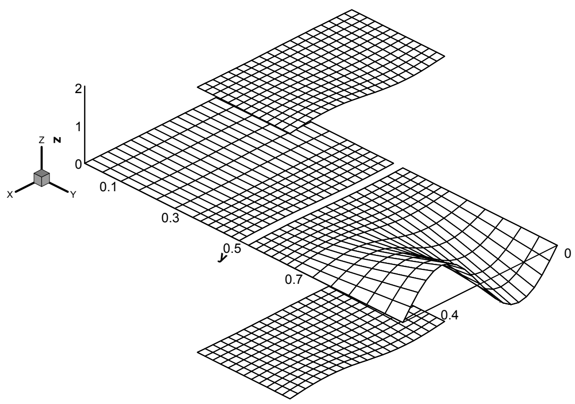
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFFFF)



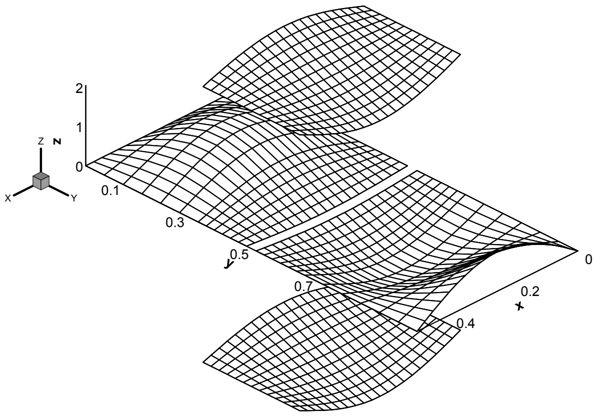
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 19.341$



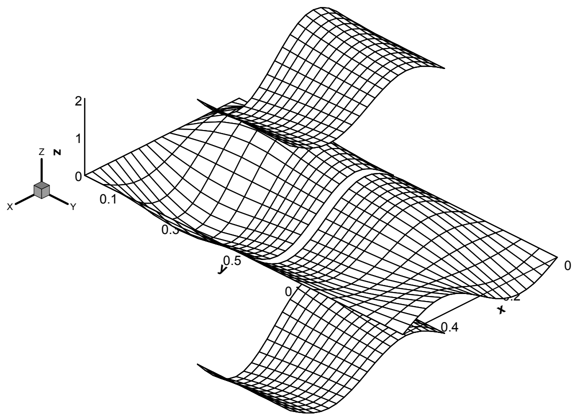
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 49.979$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 104.481$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=167.815

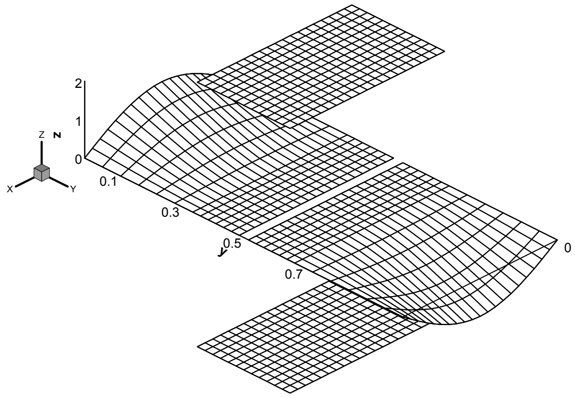


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=242.699

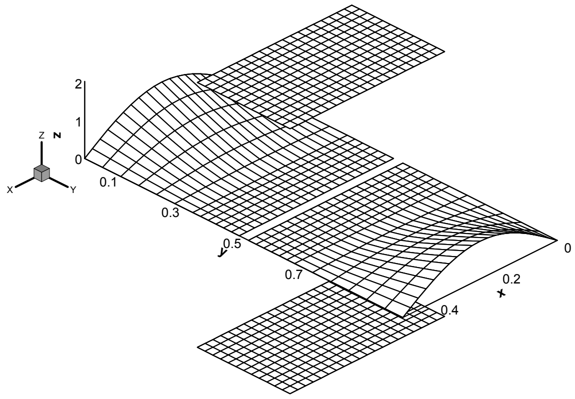
("Soft" Adhesive Case)

Fig.8.136 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

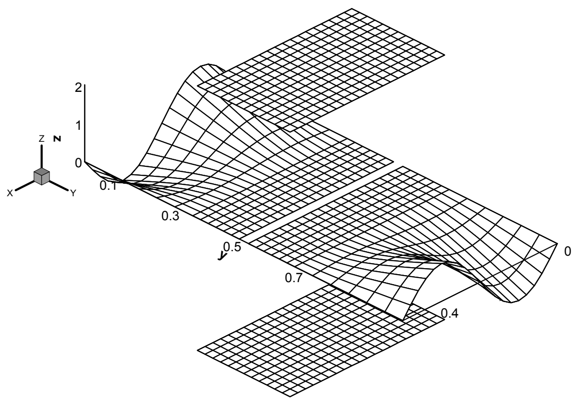
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFSFFFFF)



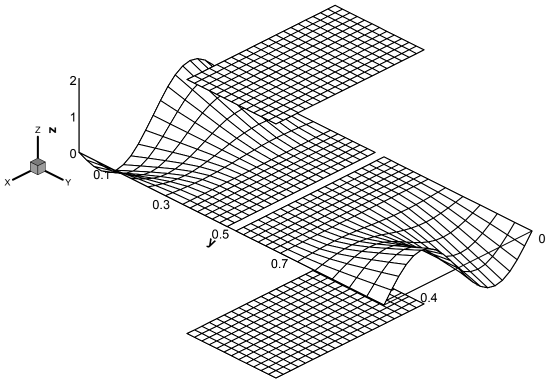
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=42.512



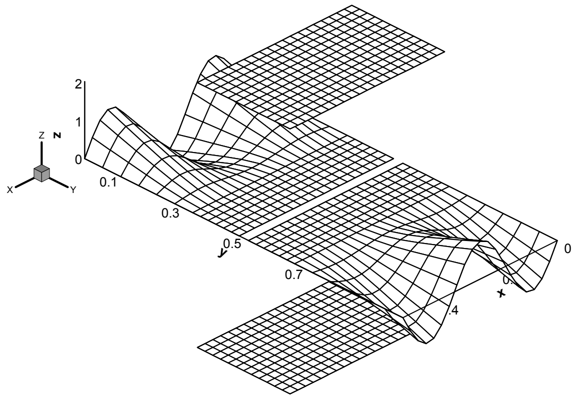
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=44.280



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=129.043



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=131.438

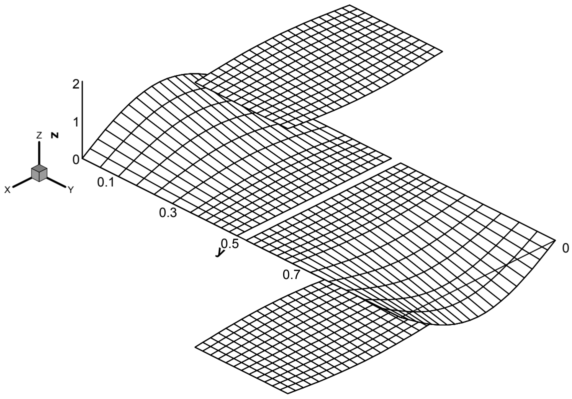


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=454.695

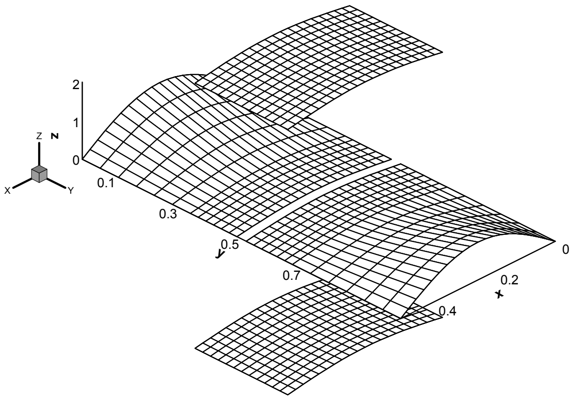
("Hard" Adhesive Case)

Fig.8.137 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

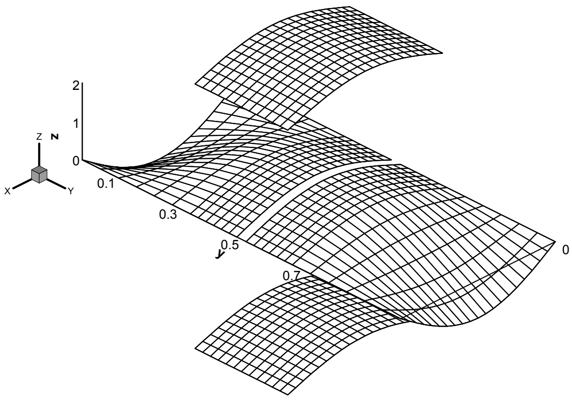
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)



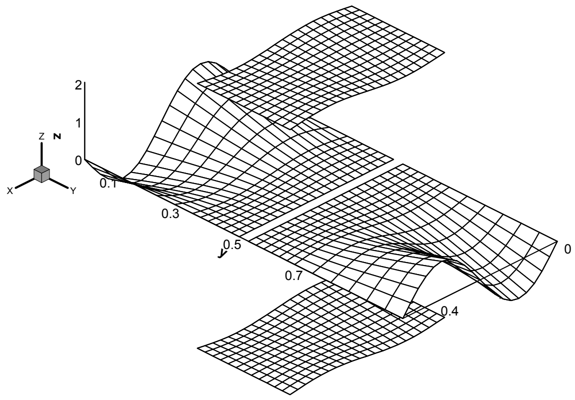
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}=18.190$



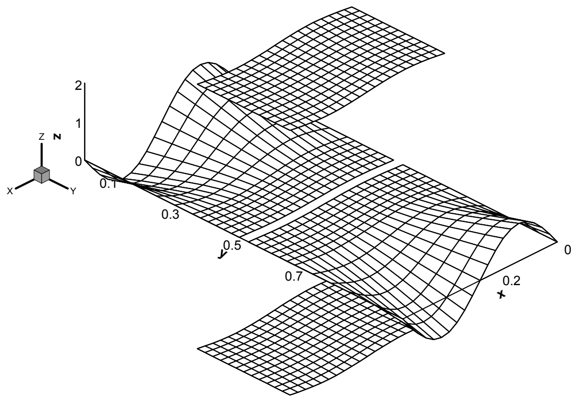
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}=20.308$



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{13}=59.902$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{21}$=102.747



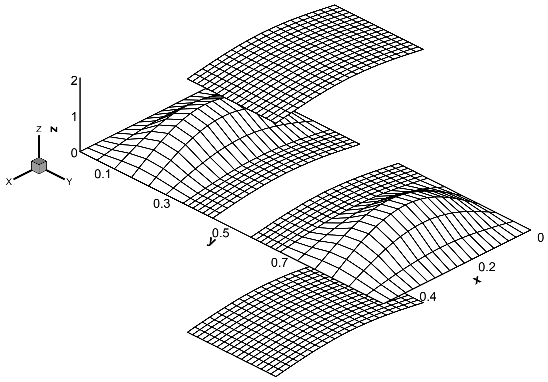
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=106.283

("Soft" Adhesive Case)

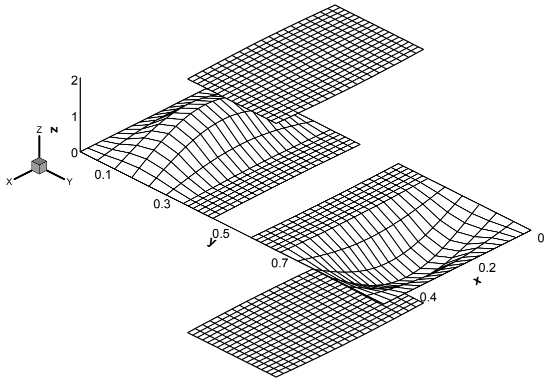
Fig.8.138 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3 m., $b_1=b_4$=0.3 m, $b_2=b_3$=0.5m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)

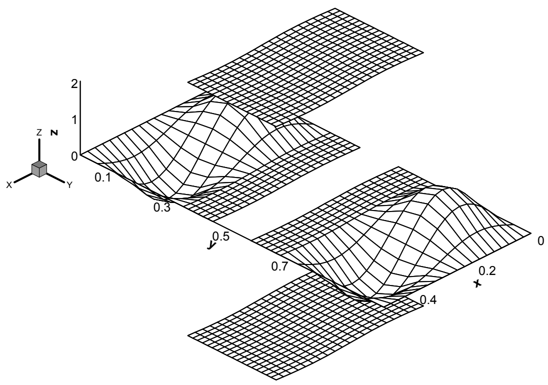
8.6.2 Natural Frequencies and Corresponding Mode Shapes for "Special Case of Main PROBLEM IIIa"



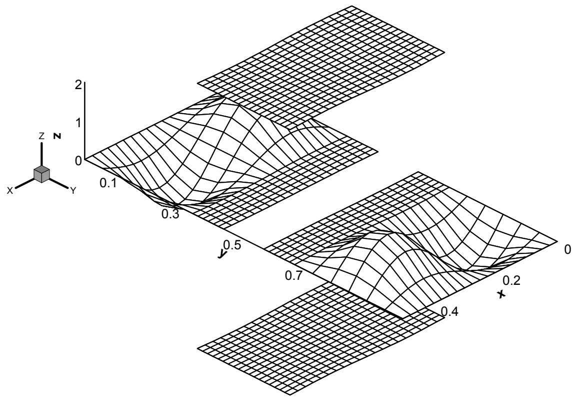
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=1016.715



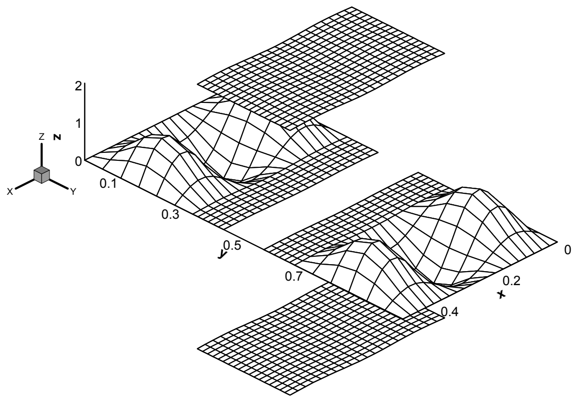
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=1169.859



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=1293.002



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=1312.249

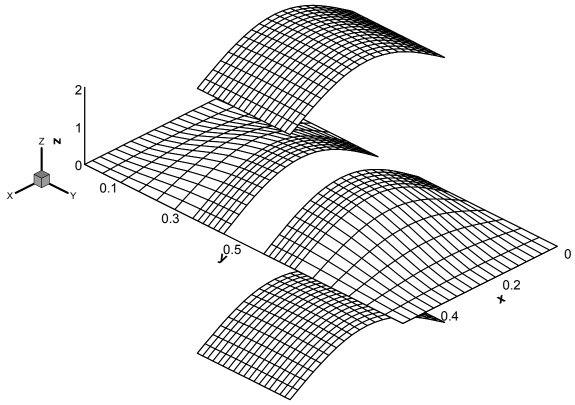


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=1710.846

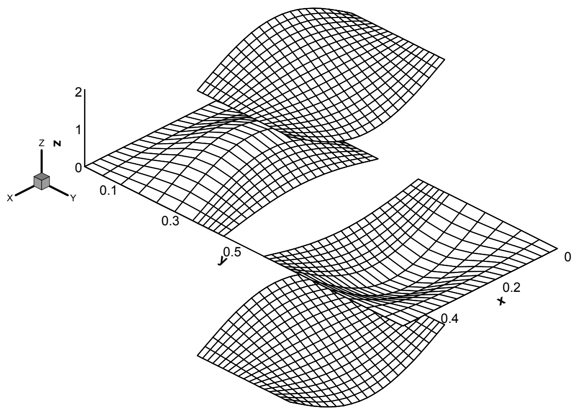
("Hard" Adhesive Case)

Fig 8.139 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

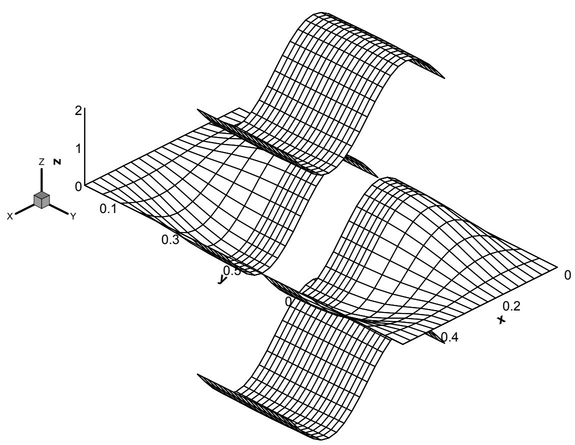
(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



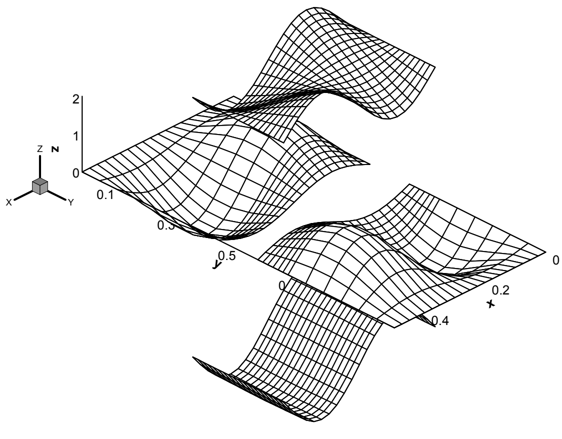
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=46.845



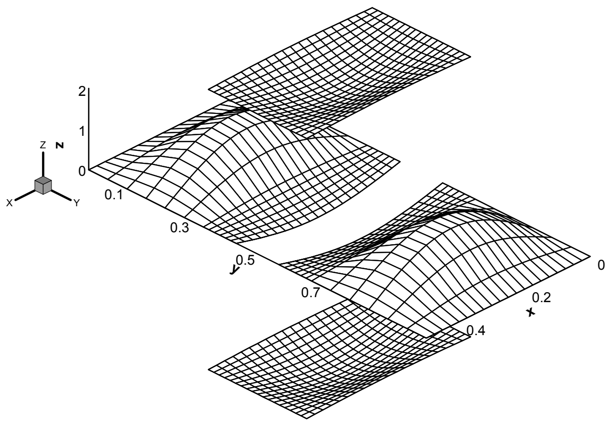
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=188.090



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=228.259



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=388.537

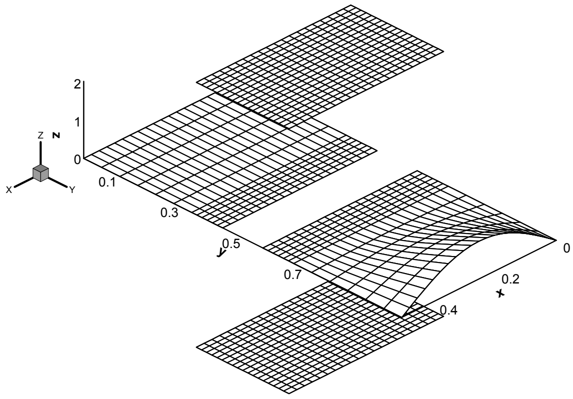


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=735.333

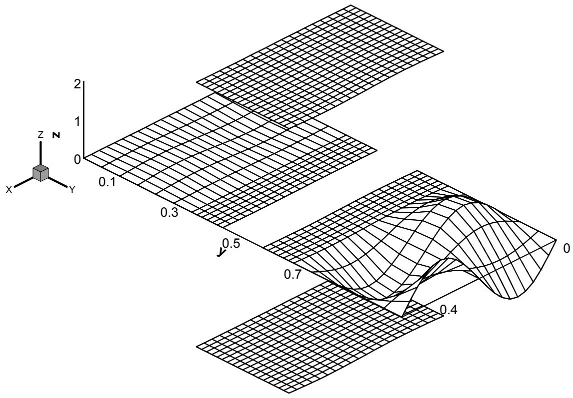
("Soft" Adhesive Case)

Fig 8.140 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

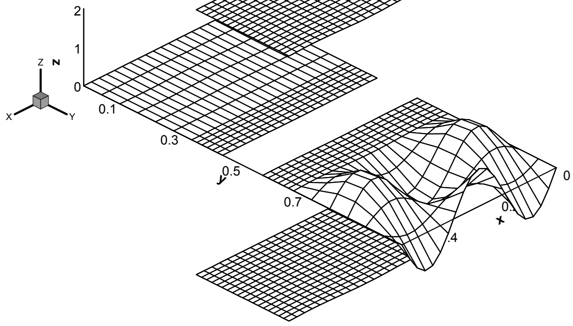
(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



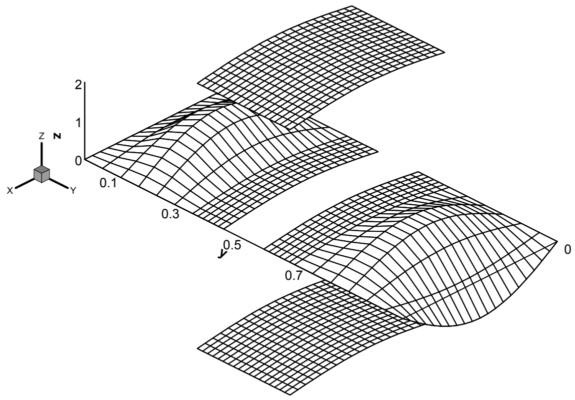
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=43.224



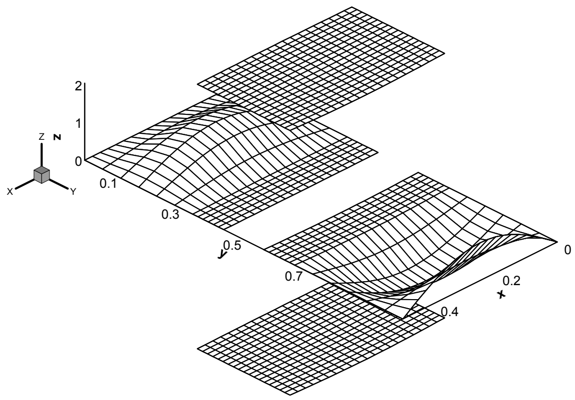
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=130.101



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=456.166



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=1024.385

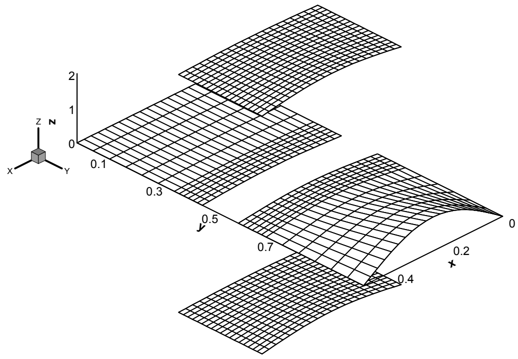


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=1191.559

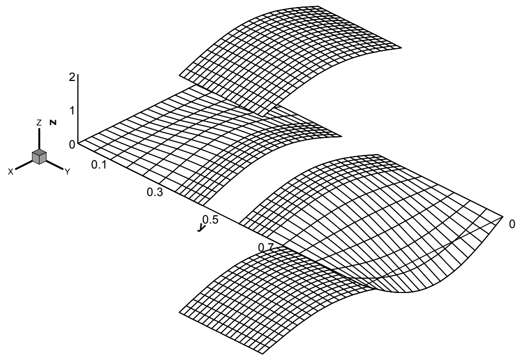
("Hard" Adhesive Case)

Fig 8.141 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

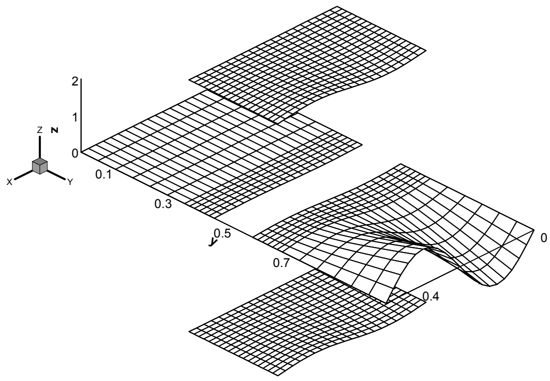
(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



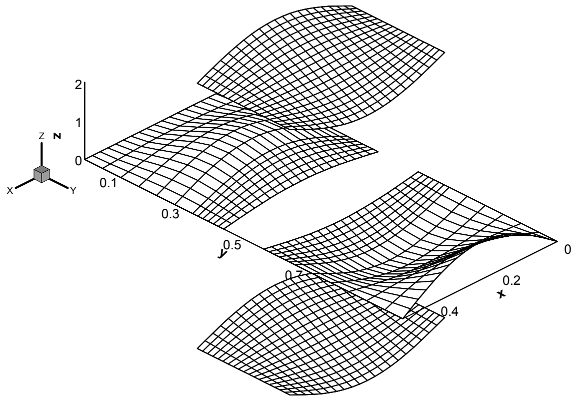
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 17.445$



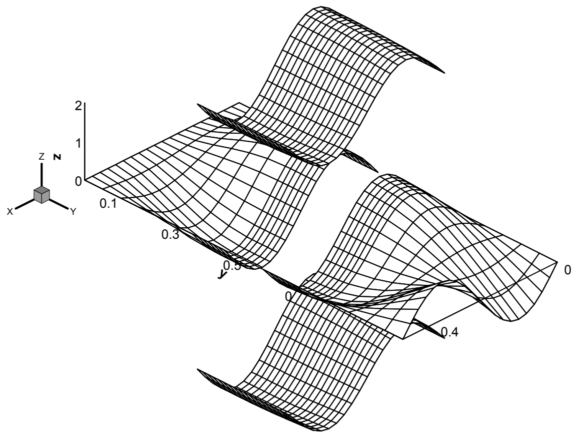
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12} = 49.432$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21} = 103.417$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=193.253



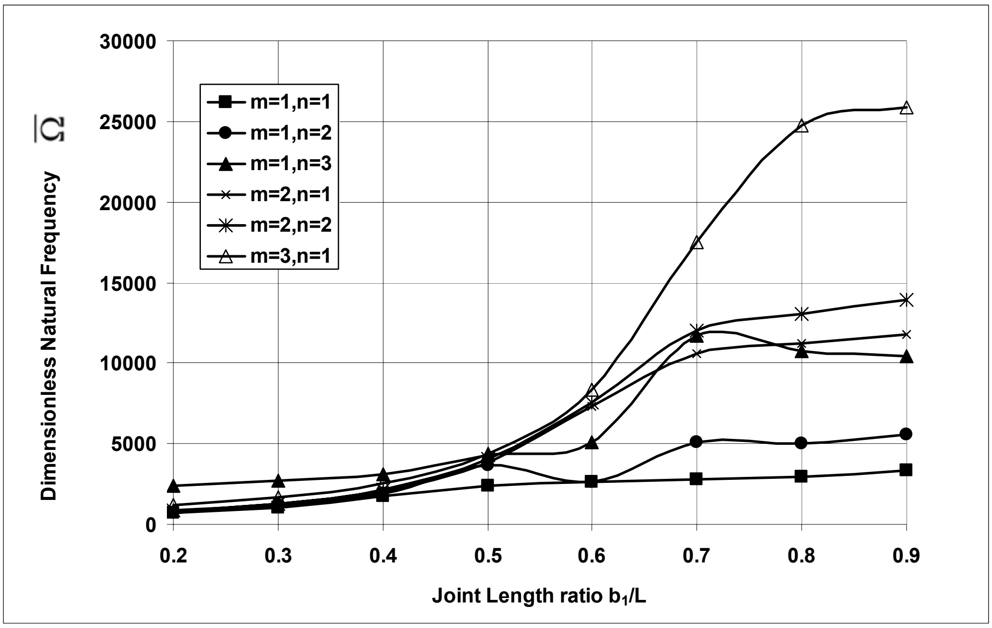
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=231.705

(“Soft” Adhesive Case)

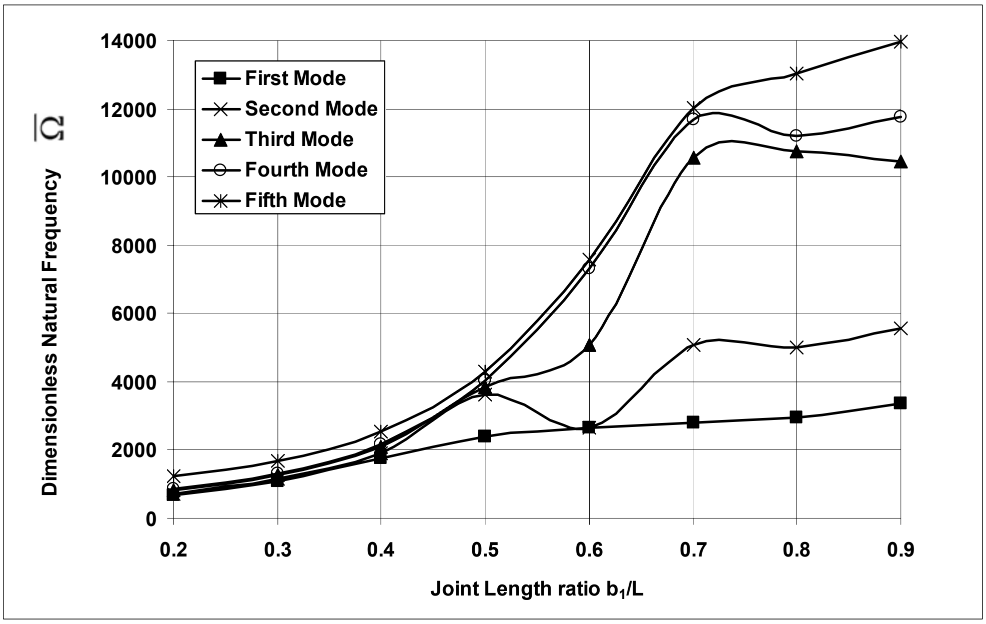
Fig 8.142 Mode Shapes and Dimensionless Natural Frequencies of “Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap”

(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1 m., b_1=0.3 m, b_2=b_3=0.4m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)

8.6.3 Some Parametric Studies for "Main PROBLEM IIIa"



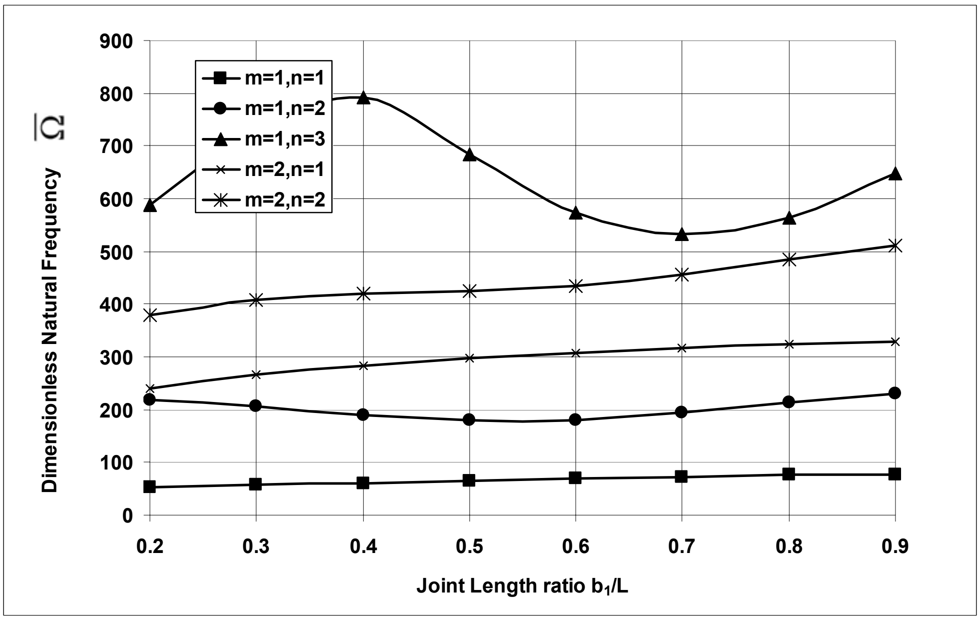
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''Hard'' Adhesive



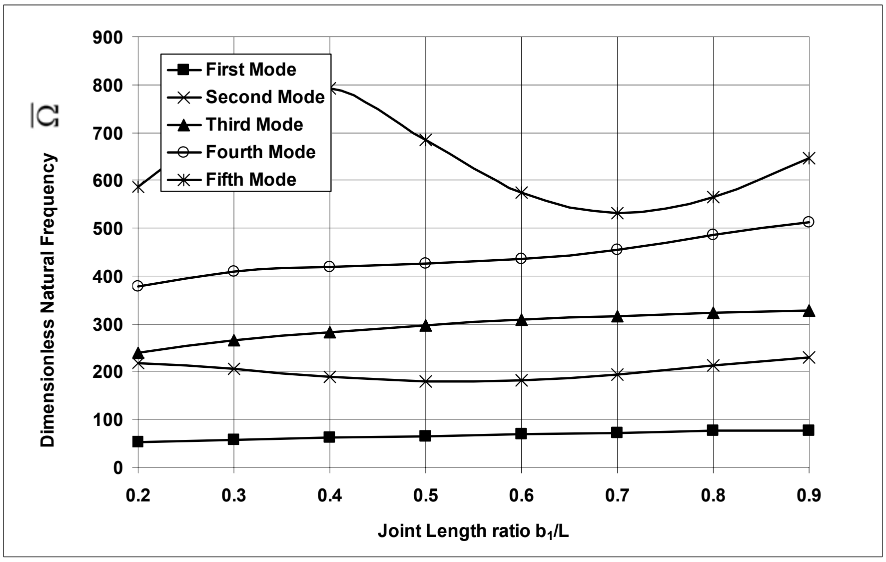
b) "Various Modes with (FFCFFCFF) B.C.'s, ''Hard'' Adhesive

Fig 8.143 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length (ℓ_I+ℓ_{II})/ L" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



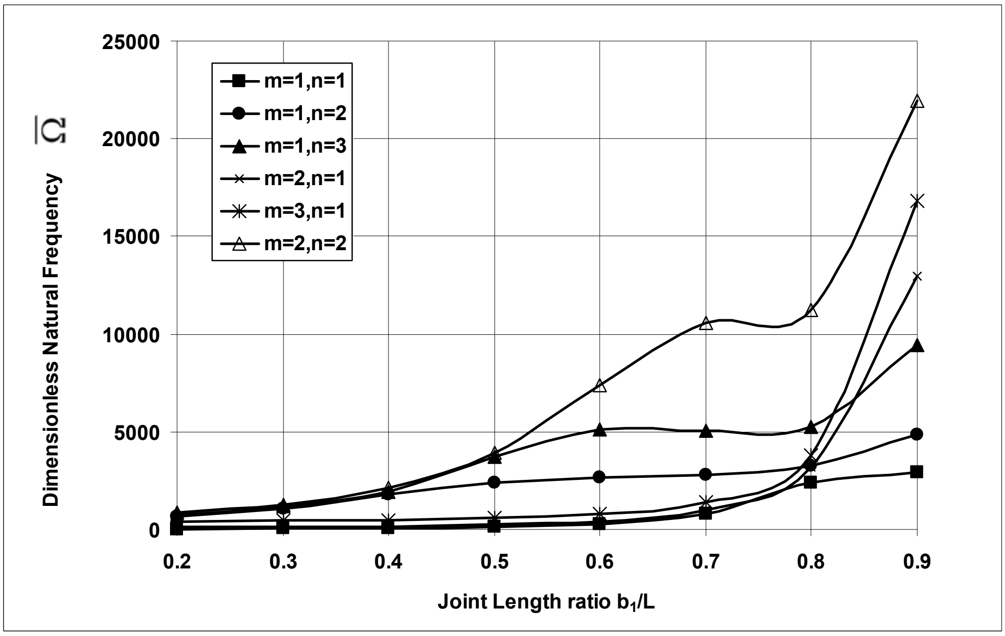
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''Soft'' Adhesive



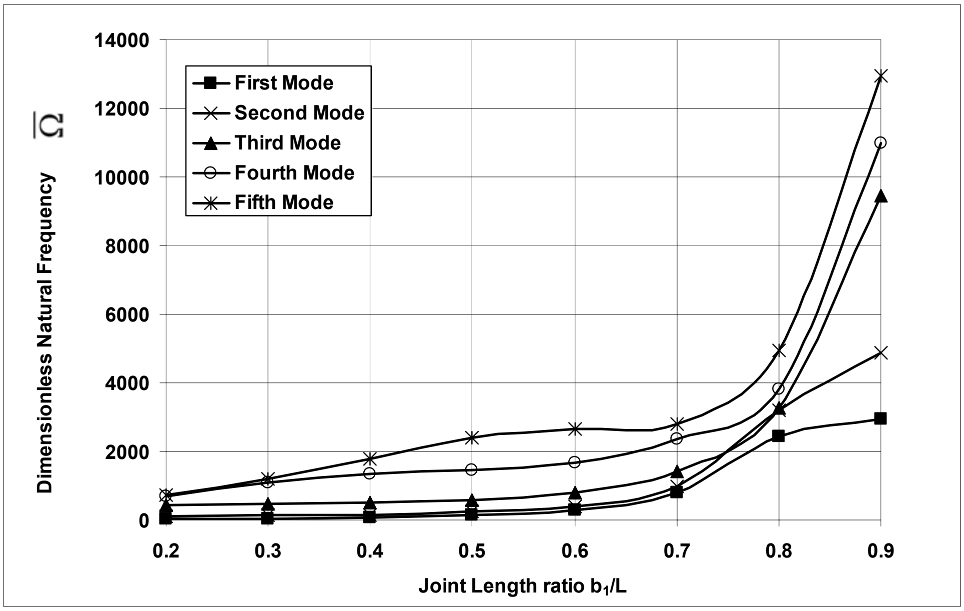
b) "Various Modes with (FFCFFCFF) B.C.'s, ''Soft'' Adhesive

Fig 8.144 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length (ℓ_I+ℓ_{II})/ L" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



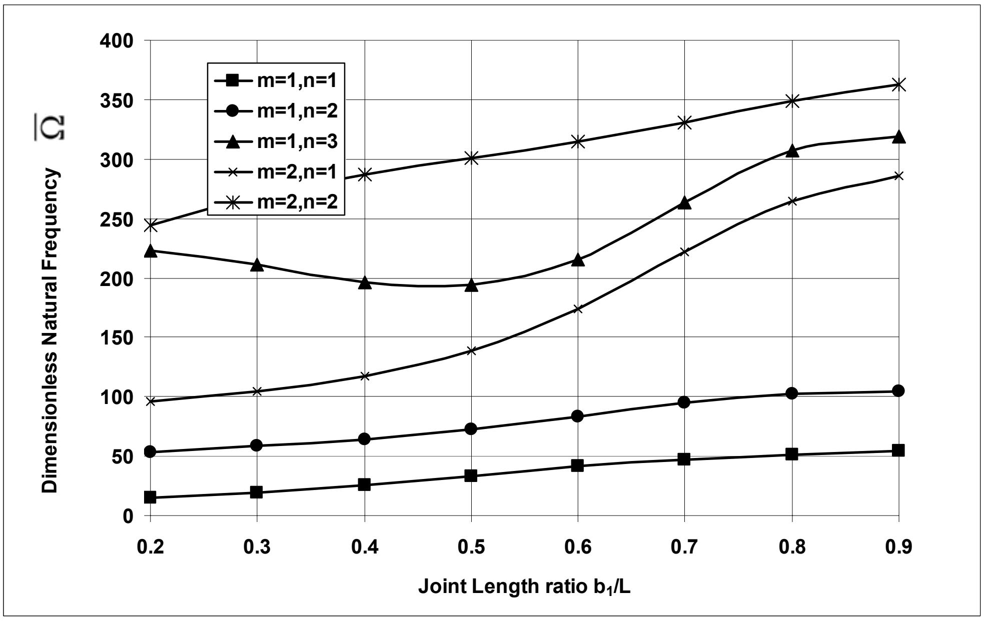
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



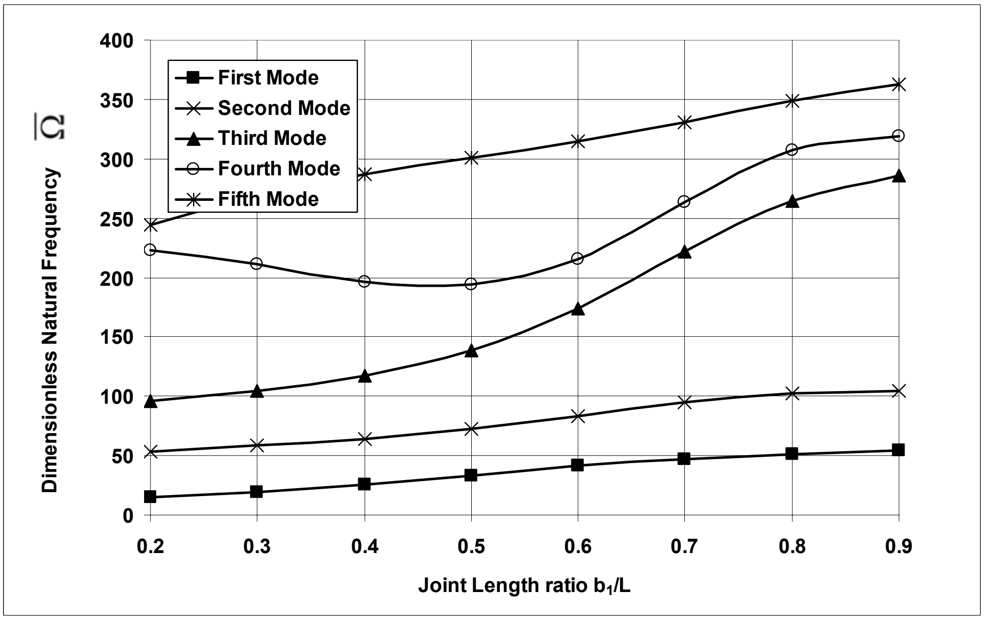
b) "Various Modes with (FFCFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.145 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length ($\ell_I+\ell_{II}$)/ L" in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



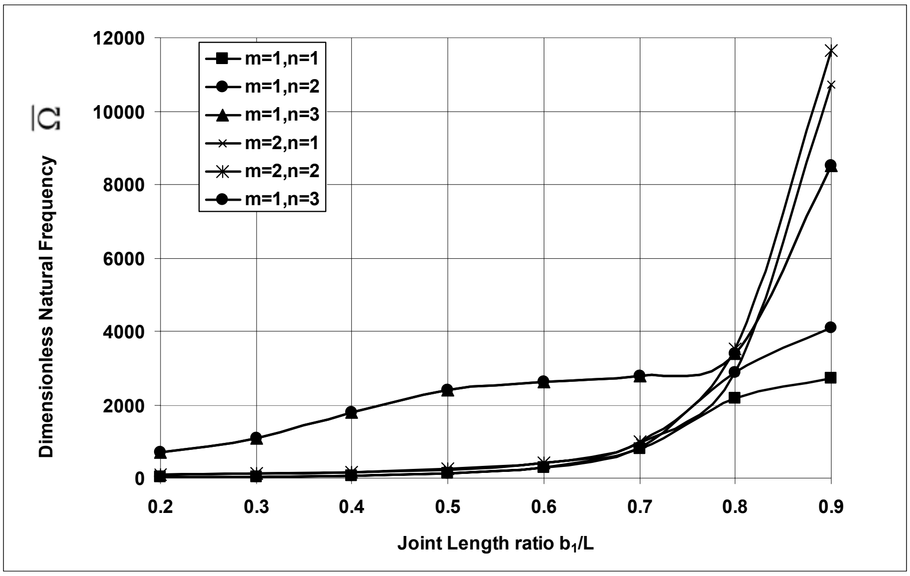
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



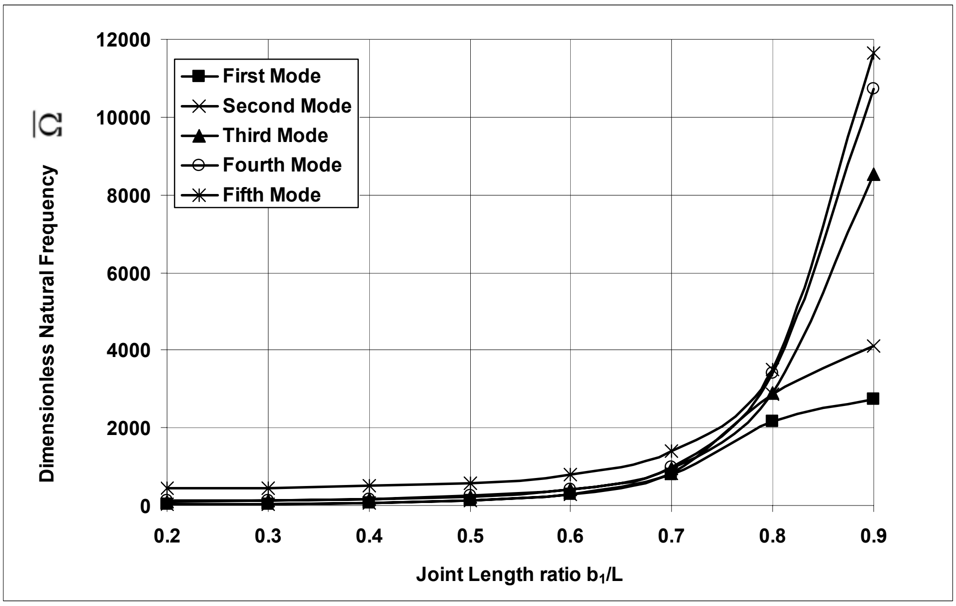
b) "Various Modes with (FFCFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.146 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length (ℓ_I+ℓ_{II})/ L" in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



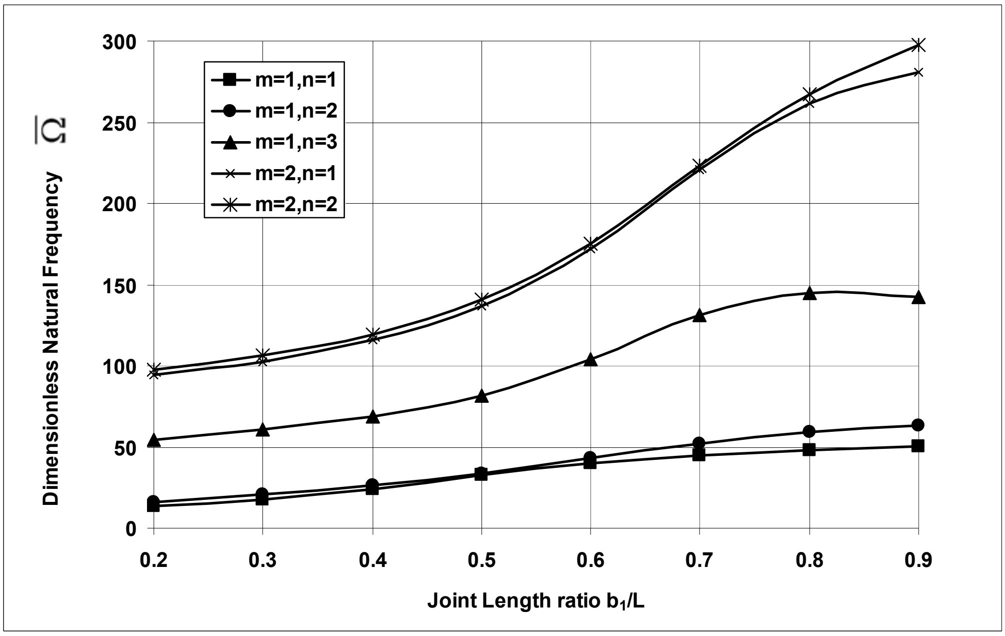
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



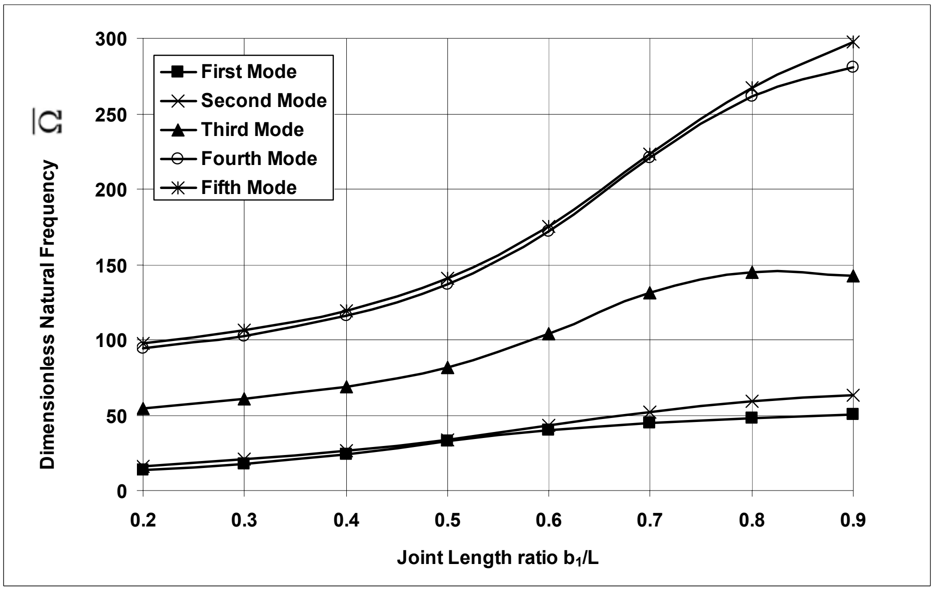
b) "Various Modes with (FFFFFFFF) B.C.'s, ''<u>Hard</u>' Adhesive

Fig 8.147 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length ($\ell_I+\ell_{II}$)/ L" in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)



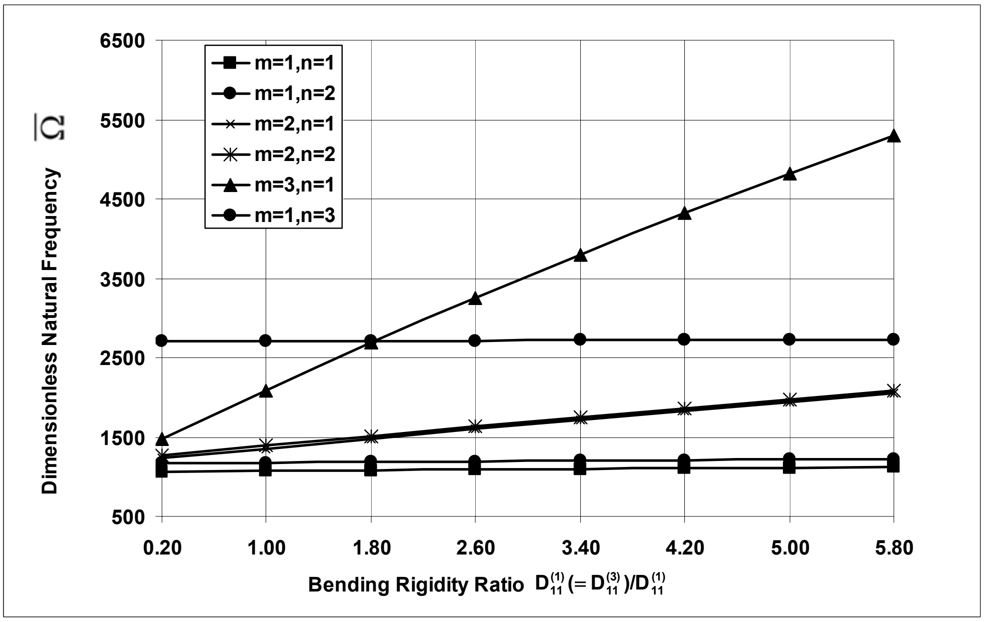
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



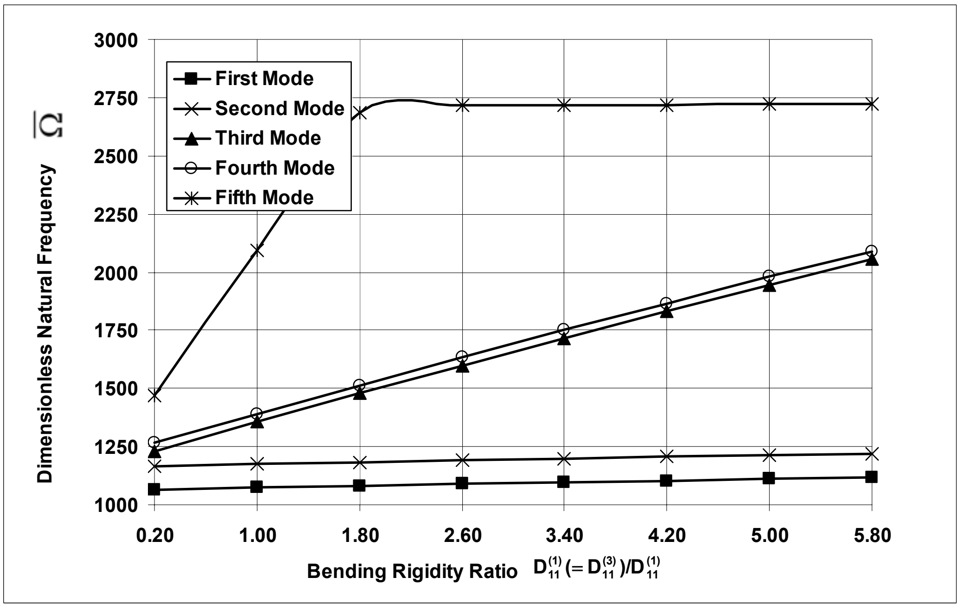
b) "Various Modes with (FFFFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.148 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Joint Length (ℓ_I+ℓ_{II})/ L" in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=varies, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''<u>Hard</u>'' Adhesive

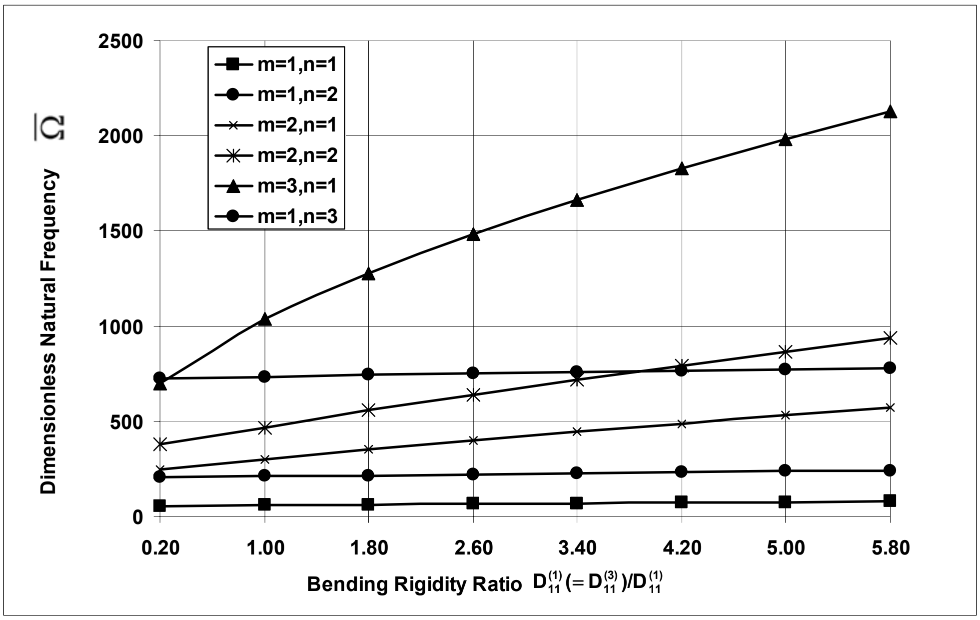


b) "Various Modes with (FFCFFCFF) B.C.'s, ''<u>Hard</u>'' Adhesive

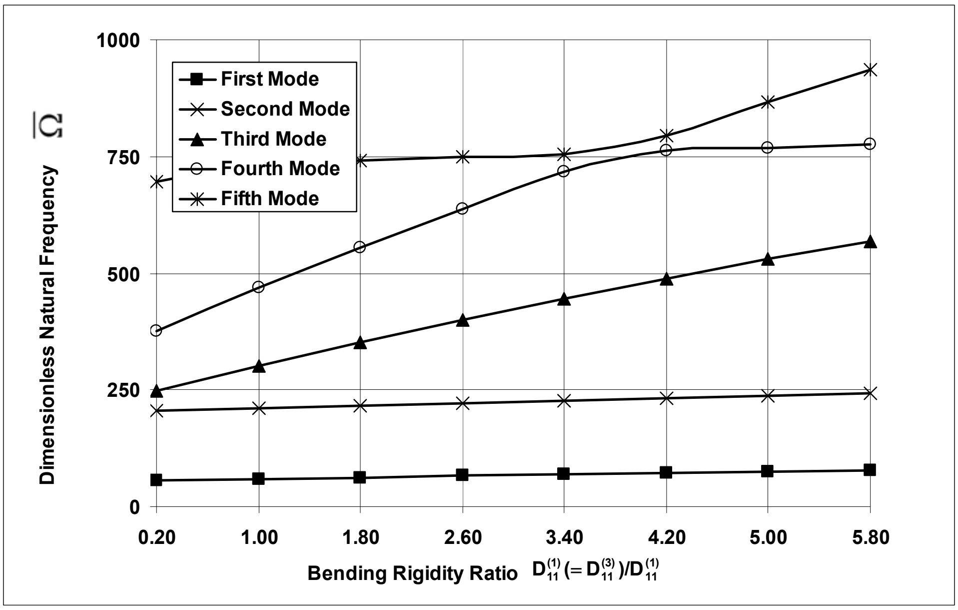
Fig 8.149 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length (ℓ_I+ℓ_{II})=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''Soft'' Adhesive

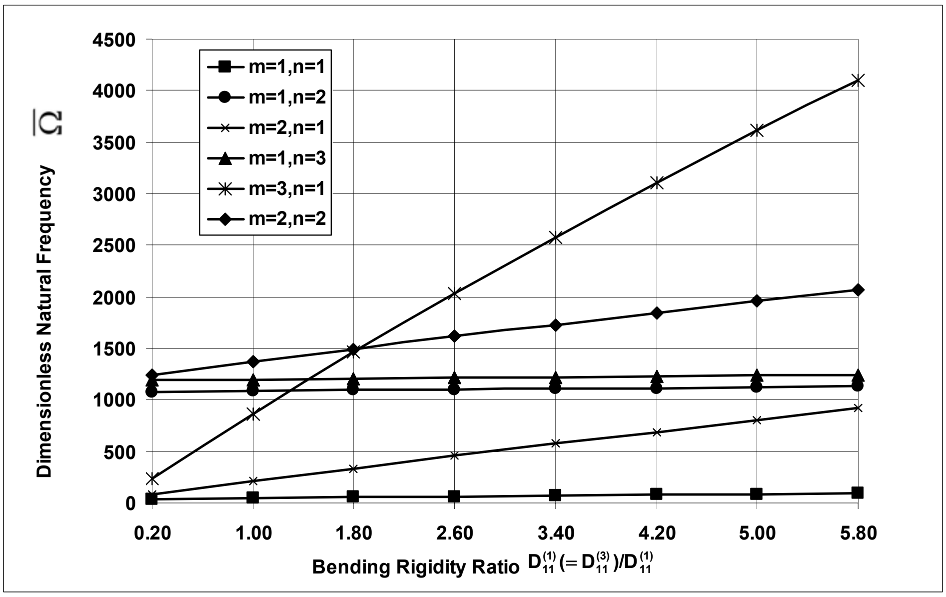


b) "Various Modes with (FFCFFCFF) B.C.'s, ''Soft'' Adhesive

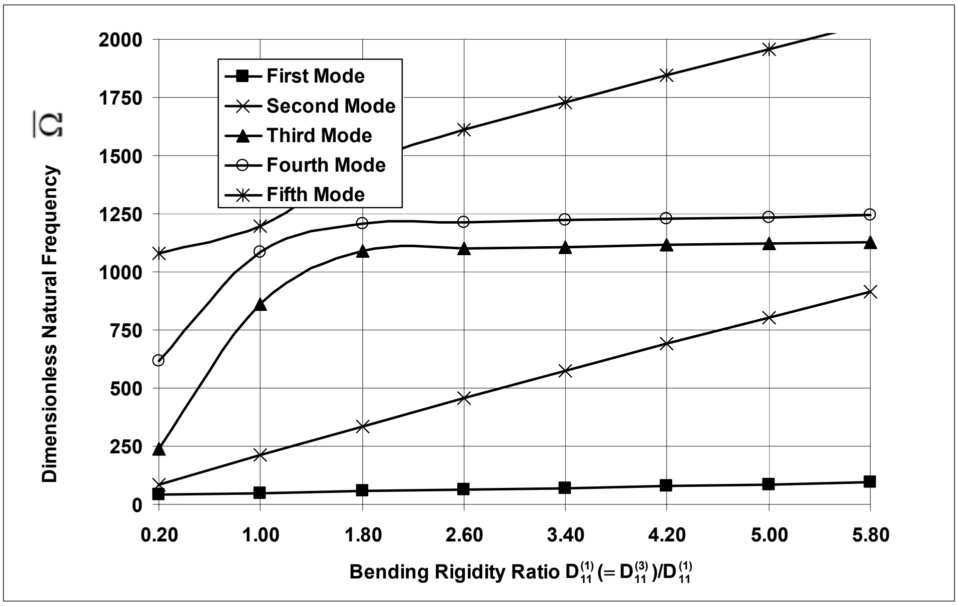
Fig 8.150 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(= D_{11}^{(3)}\right) / D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_{I}+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



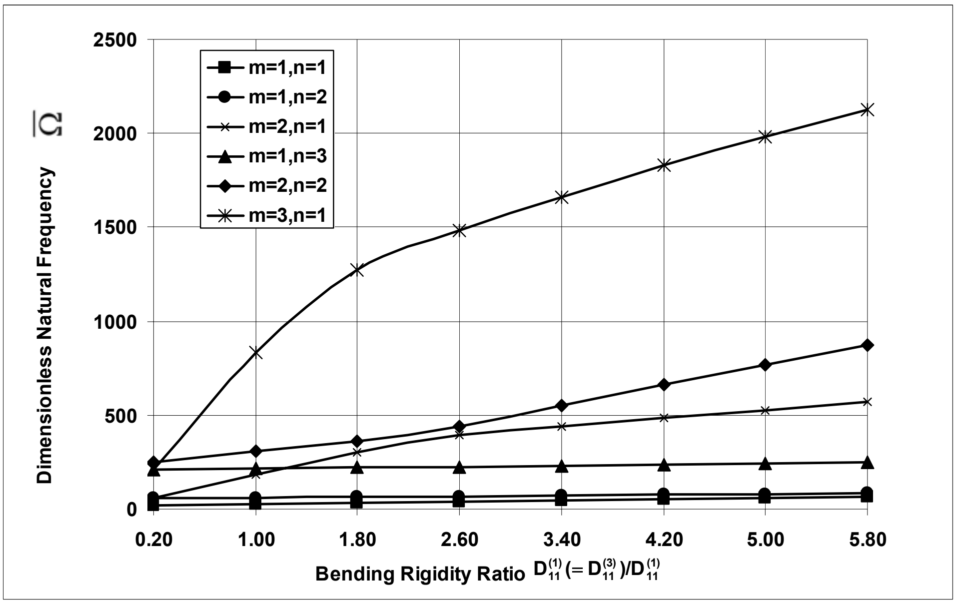
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



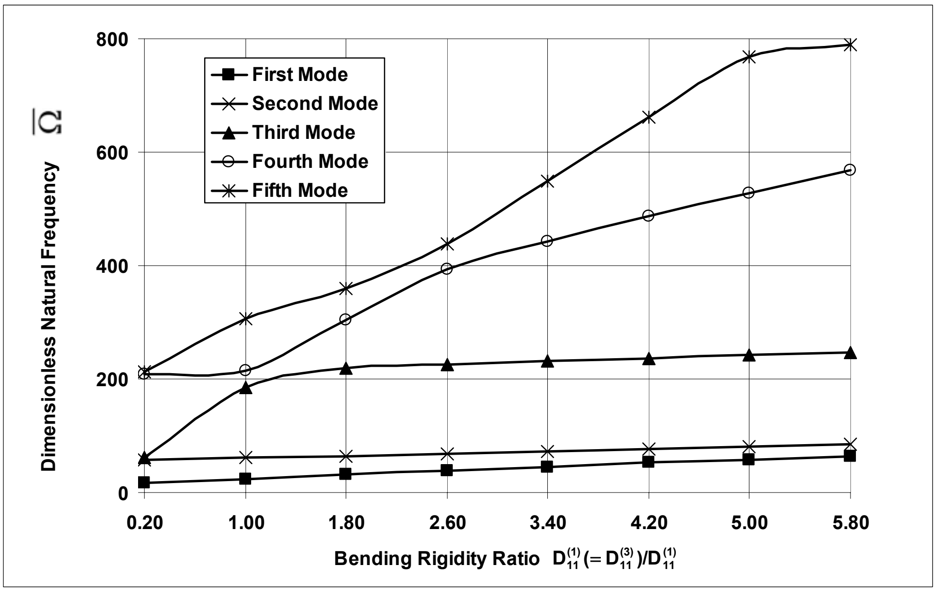
b) "Various Modes with (FFCFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.151 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(= D_{11}^{(3)}\right) / D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I + \ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



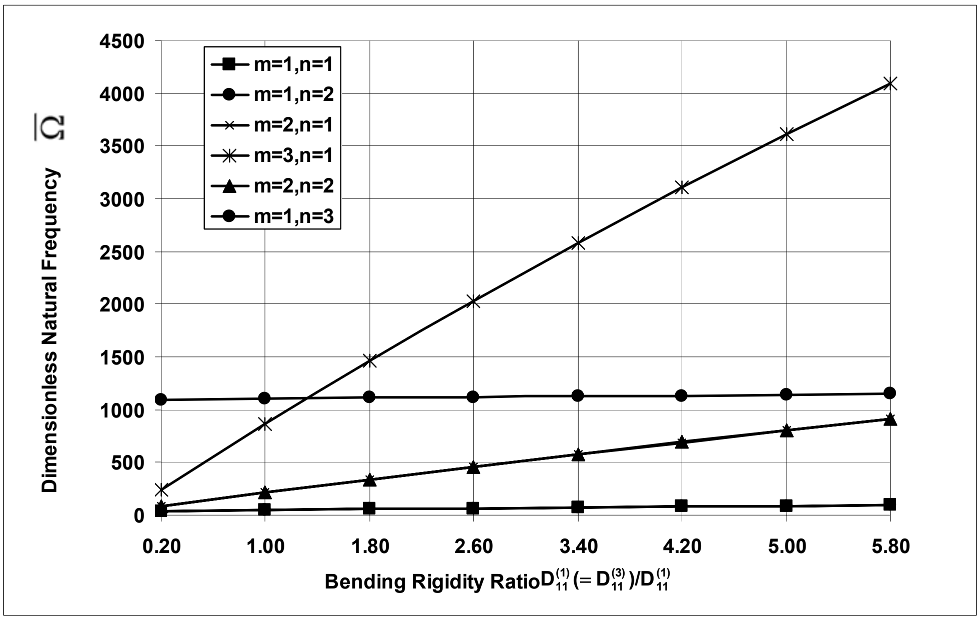
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive



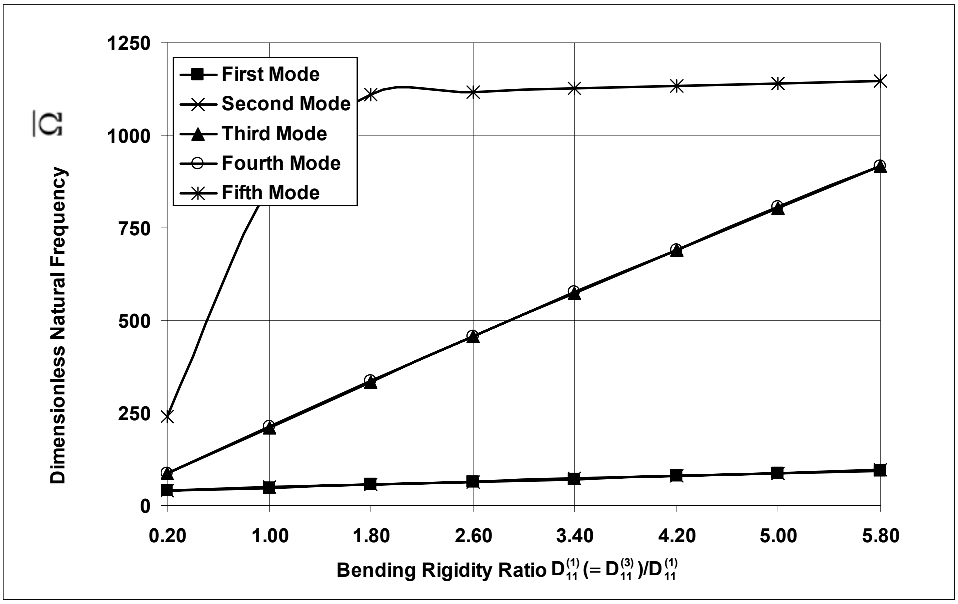
b) "Various Modes with (FFCFFFFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.152 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\widetilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''Hard'' Adhesive

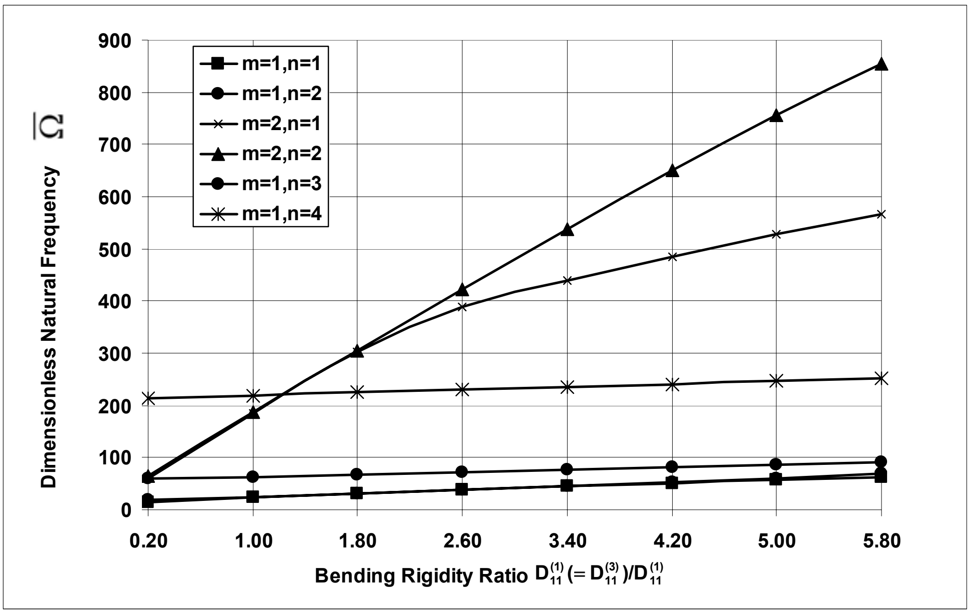


b) "Various Modes with (FFFFFFFF) B.C.'s, ''Hard'' Adhesive

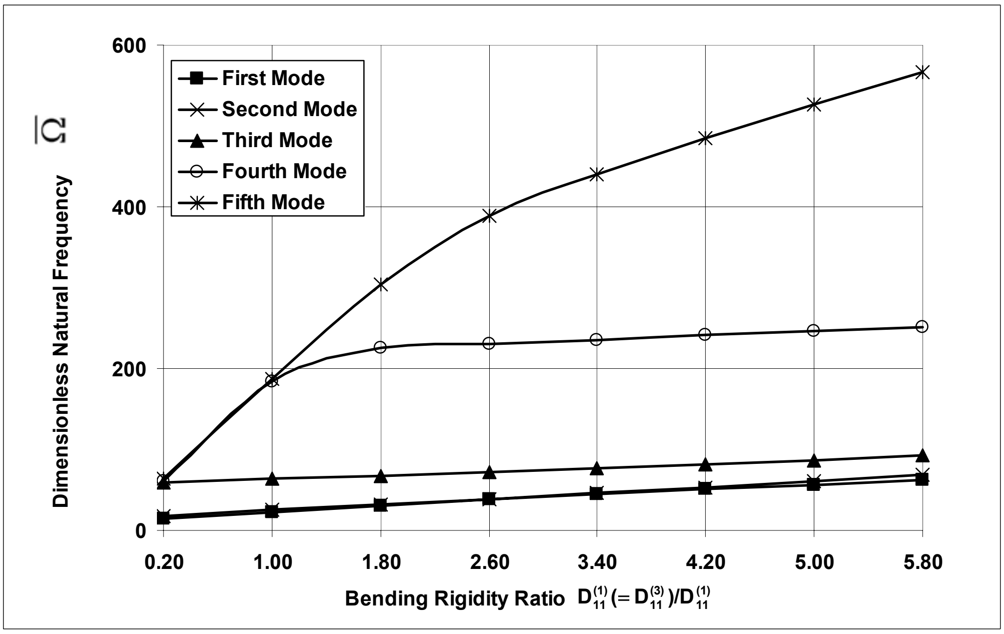
Fig 8.153 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$=0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)



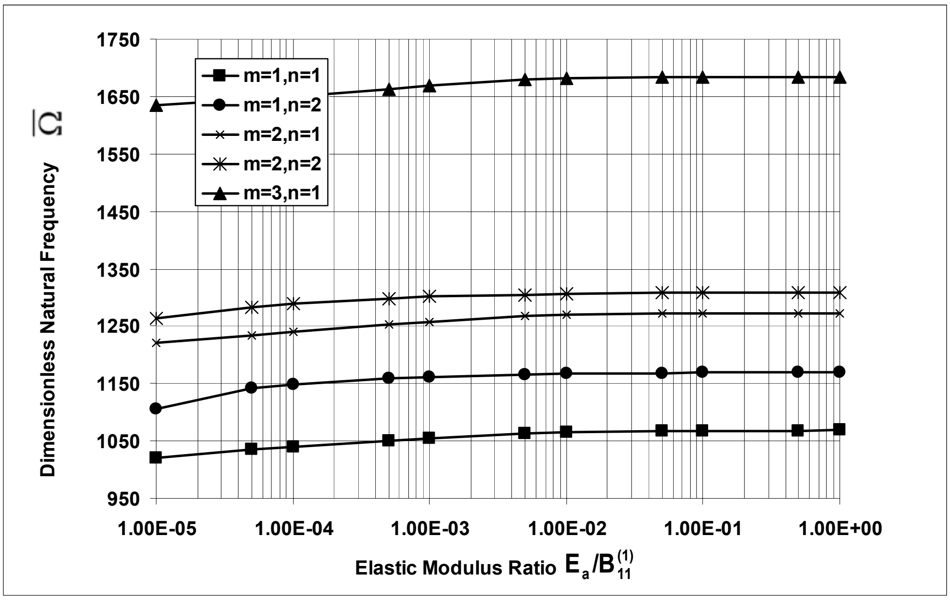
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''Soft'' Adhesive



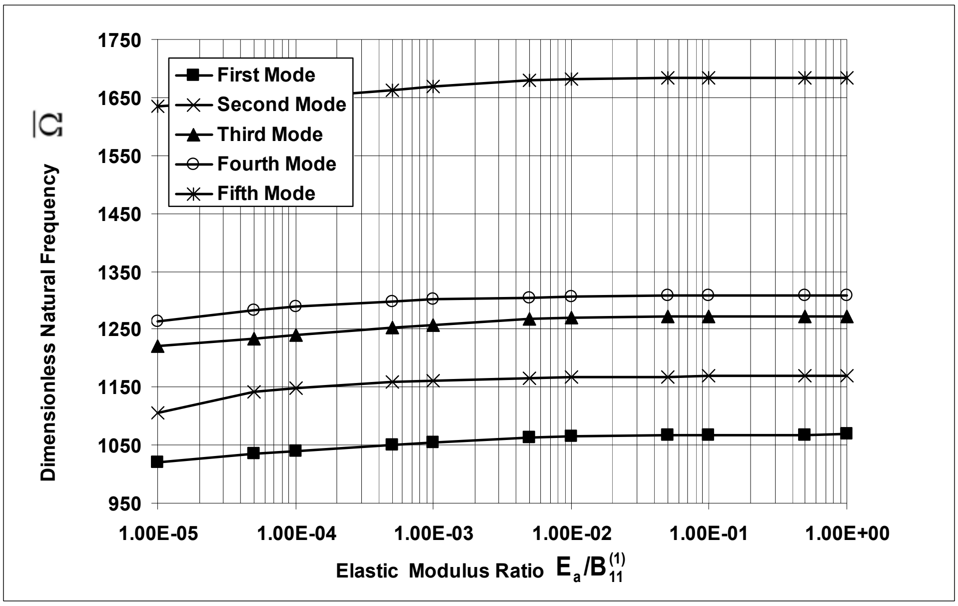
b) "Various Modes with (FFFFFFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.154 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)



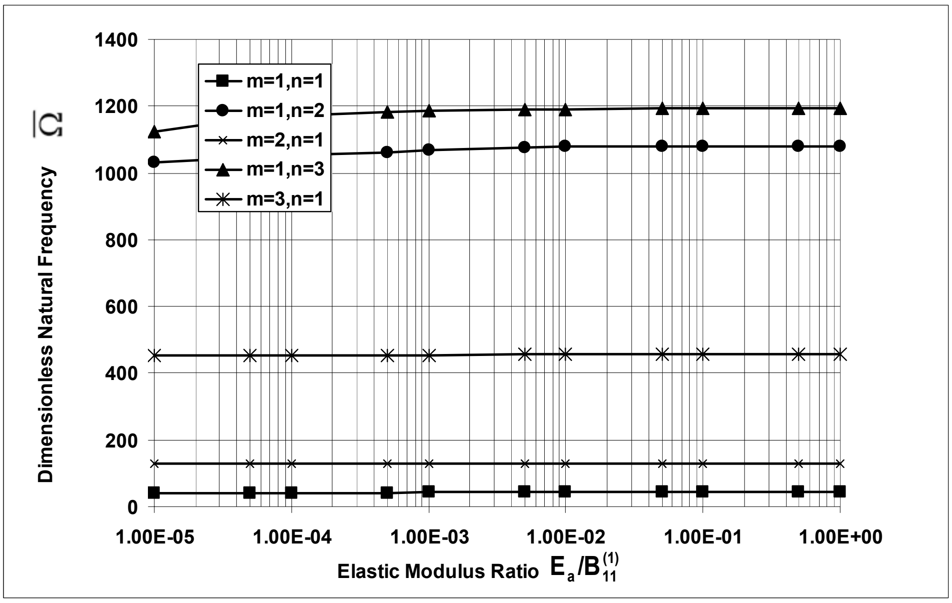
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s



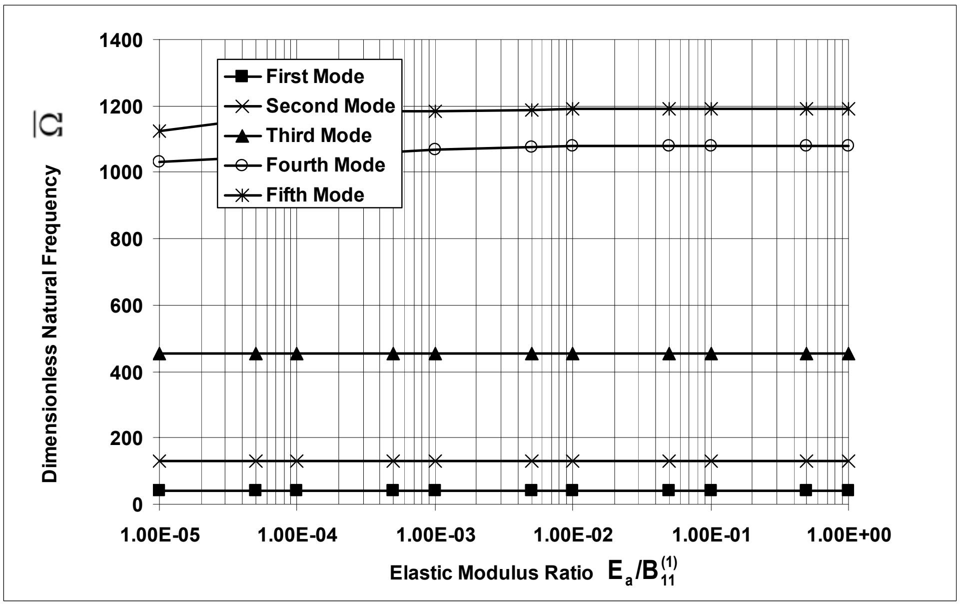
b) "Various Modes with (FFCFFCFF) B.C.'s

Fig 8.155 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_{a1}(=E_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)
Elastic Modulus Ratio axis is plotted in Log Scale



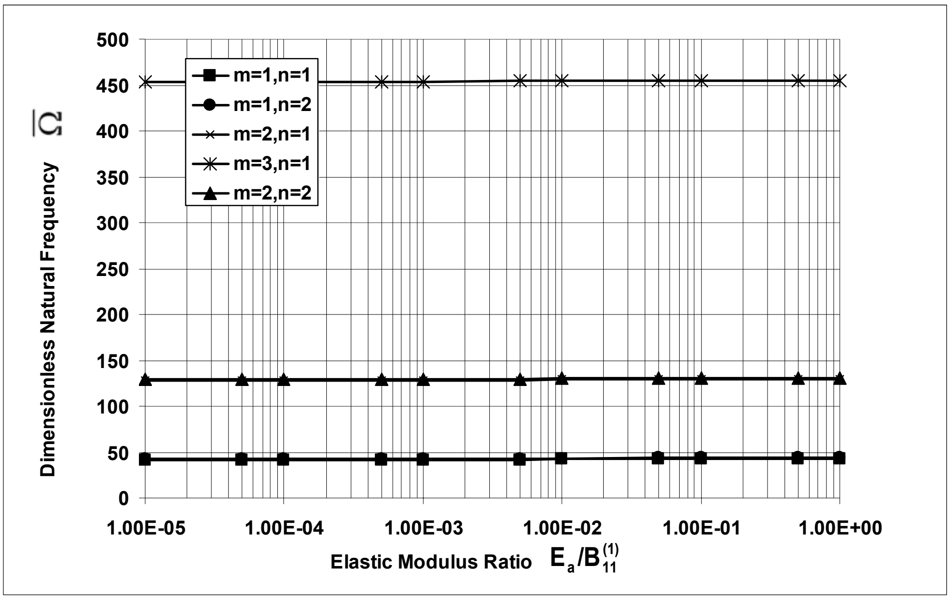
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s



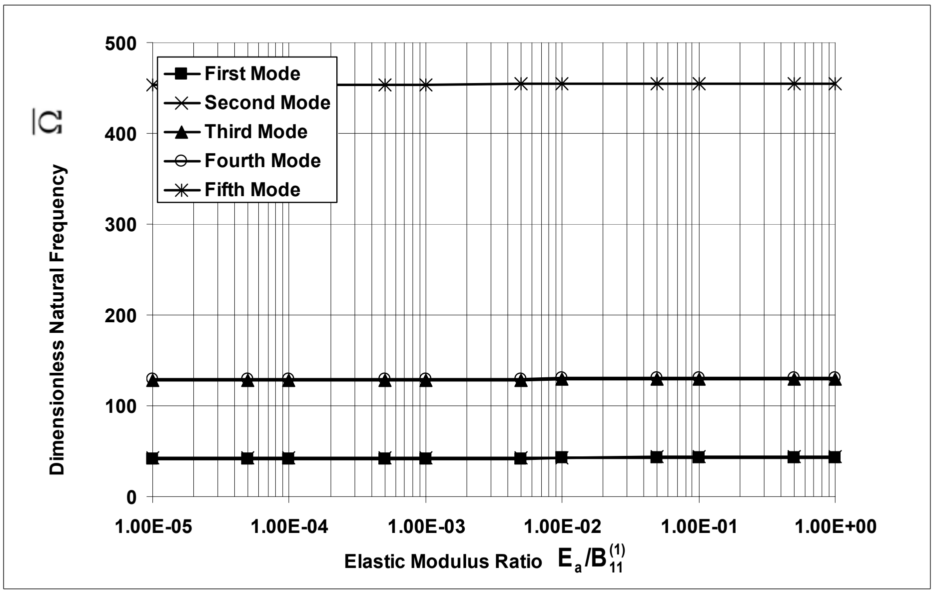
b) "Various Modes with (FFCFFFFF) B.C.'s

Fig 8.156 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_{a1}(=E_{a4})/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\widetilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



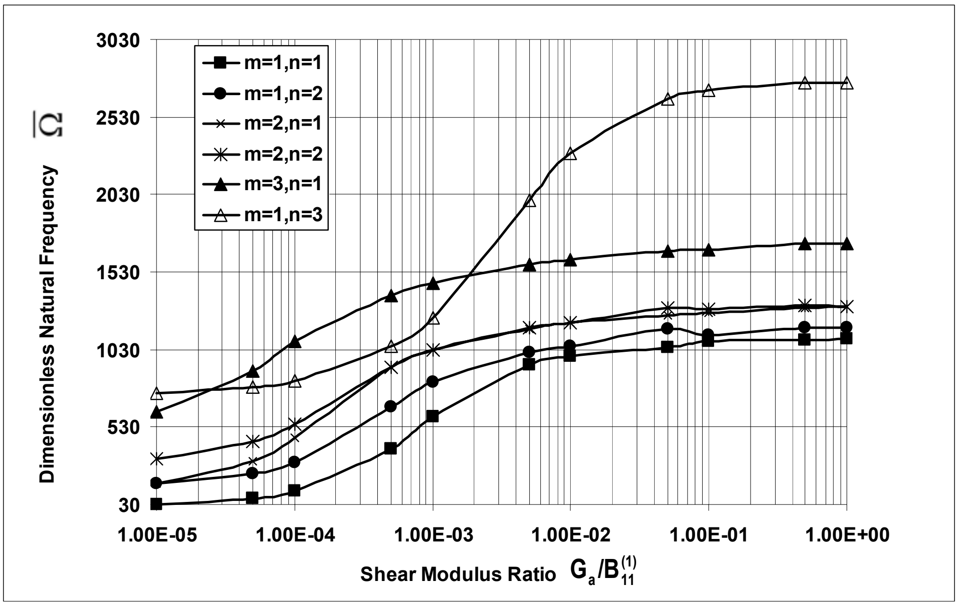
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s



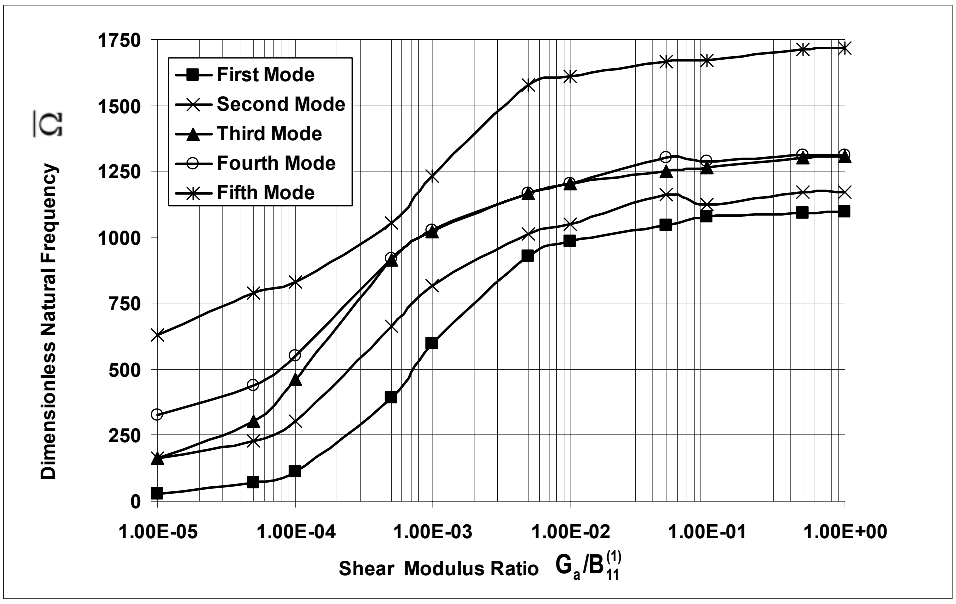
b) "Various Modes with (FFFFFFFF) B.C.'s

Fig 8.157 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_{a1}(=E_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



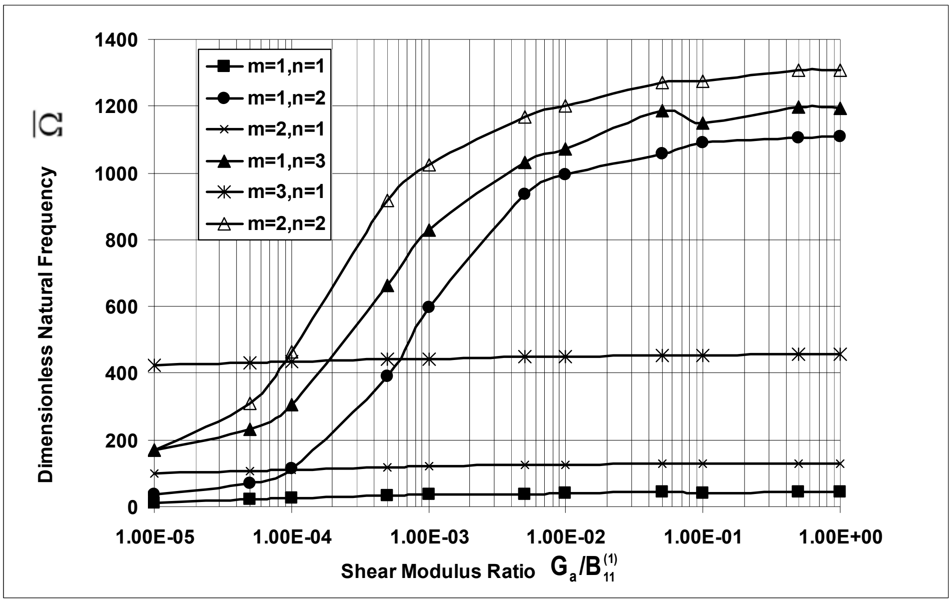
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFCFF) B.C.'s



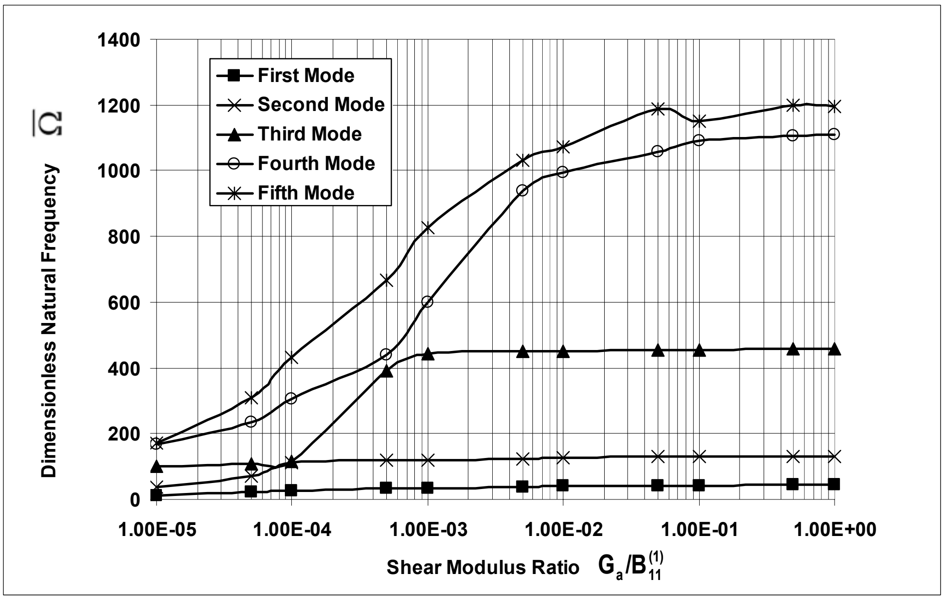
b) "Various Modes with (FFCFFCFF) B.C.'s

Fig 8.158 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_{a1}(=G_{a4})/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)
Shear Modulus Ratio axis is plotted in Log Scale



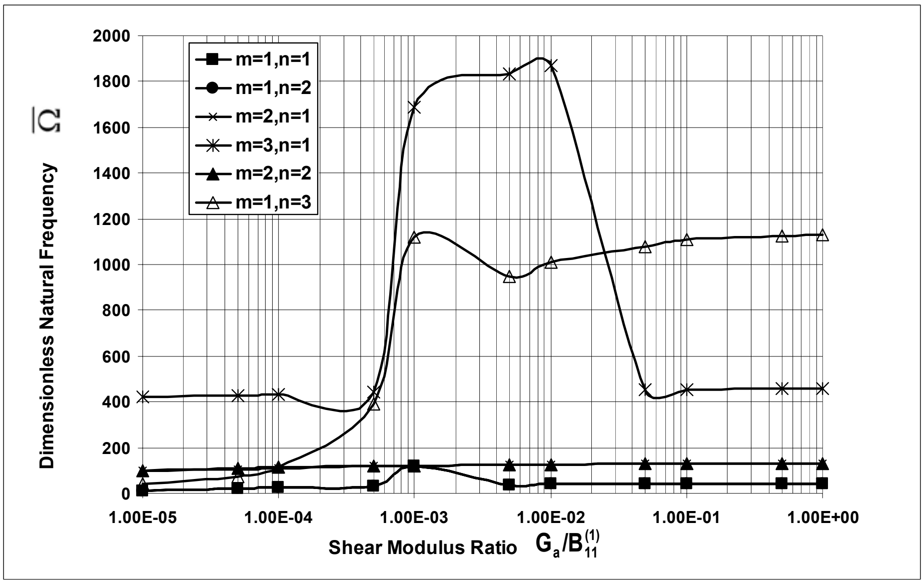
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s



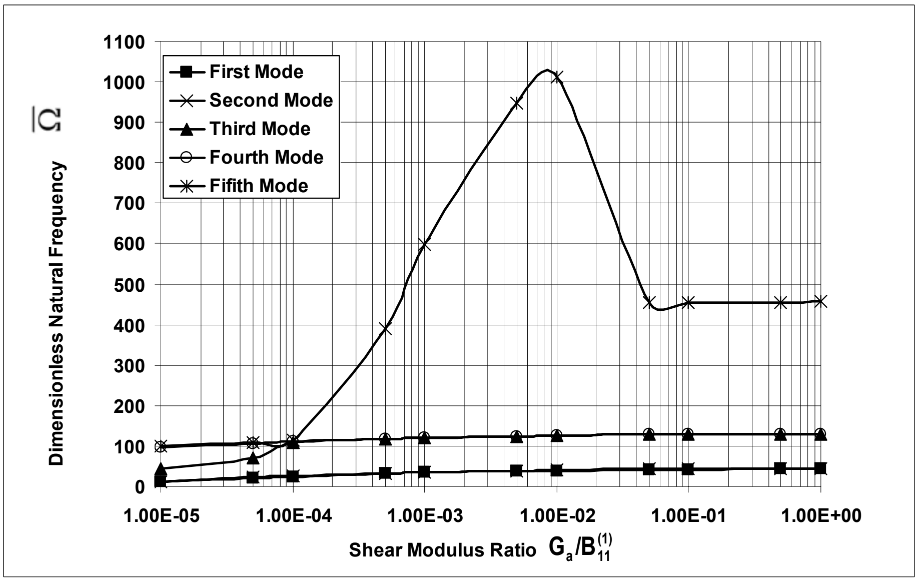
b) "Various Modes with (FFCFFFFF) B.C.'s

Fig 8.159 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_{a1}(=G_{a4})/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)
Shear Modulus Ratio axis is plotted in Log Scale



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s



b) "Various Modes with (FFFFFFFF) B.C.'s

Fig 8.160 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_{a1}(= G_{a4})/B_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.5 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)
Shear Modulus Ratio axis is plotted in Log Scale

8.6.4 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequencies"

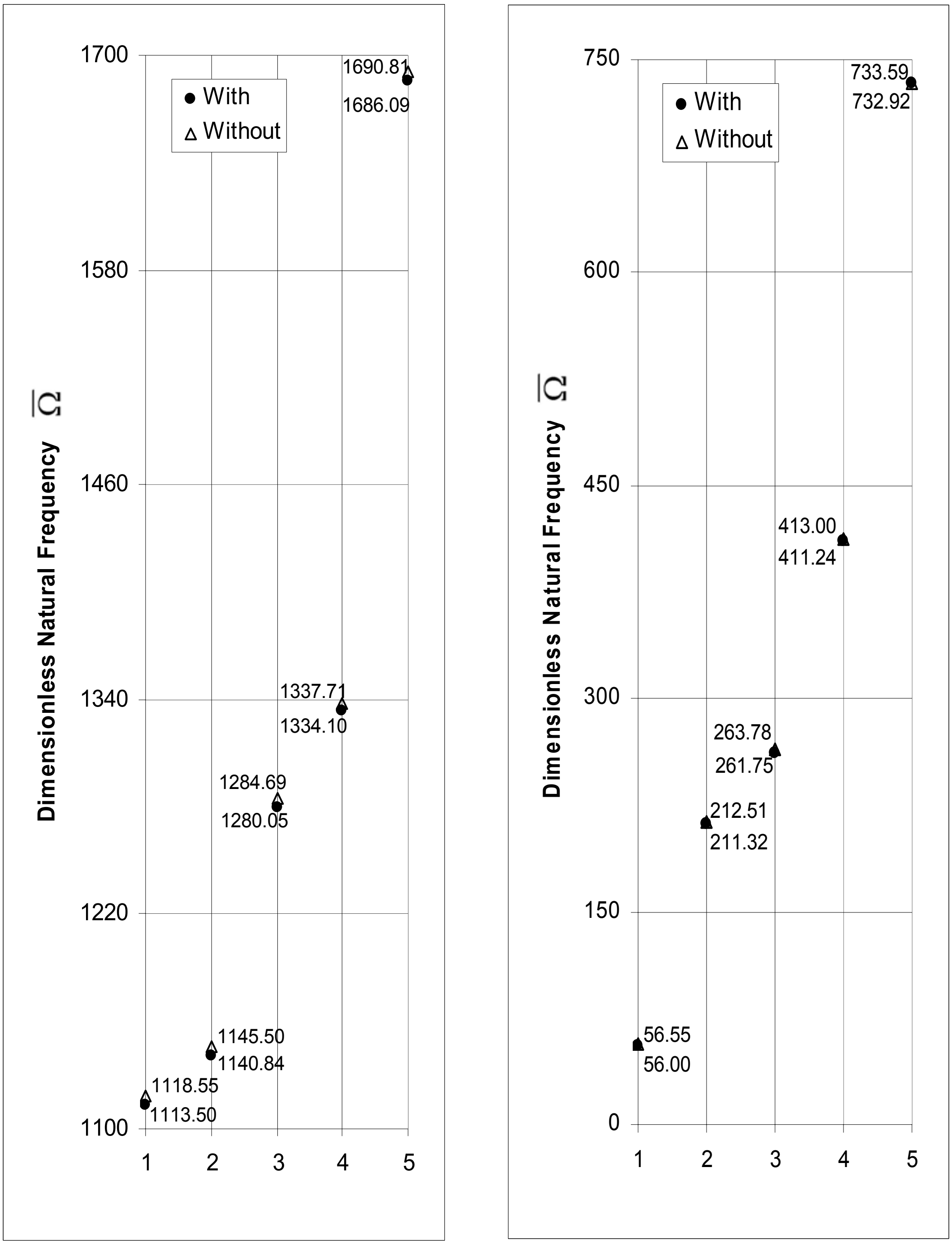
Table 8.8 Comparison of "Dimensionless Natural Frequencies" obtained by adding $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms to adhesive layer equations for "Main PROBLEM IIIa"

a) "Hard" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	1118.553	1113.502	5.050
	2	1145.500	1140.840	4.661
	3	1284.693	1280.051	4.643
	4	1337.712	1334.096	3.616
	5	1690.808	1686.086	4.722
SFFS	1	595.064	592.791	2.273
	2	602.327	600.165	2.162
	3	731.395	729.047	2.348
	4	761.677	759.748	1.929
	5	1137.201	1134.688	2.513
CFFF	1	43.503	43.395	0.108
	2	130.376	130.238	0.138
	3	456.471	456.283	0.188
	4	1125.654	1120.601	5.053
	5	1173.786	1168.753	5.033
FFFF	1	42.614	42.512	0.102
	2	44.393	44.280	0.114
	3	129.180	129.043	0.137
	4	131.576	131.438	0.138
	5	454.891	454.695	0.196

b) "Soft" Adhesive Case

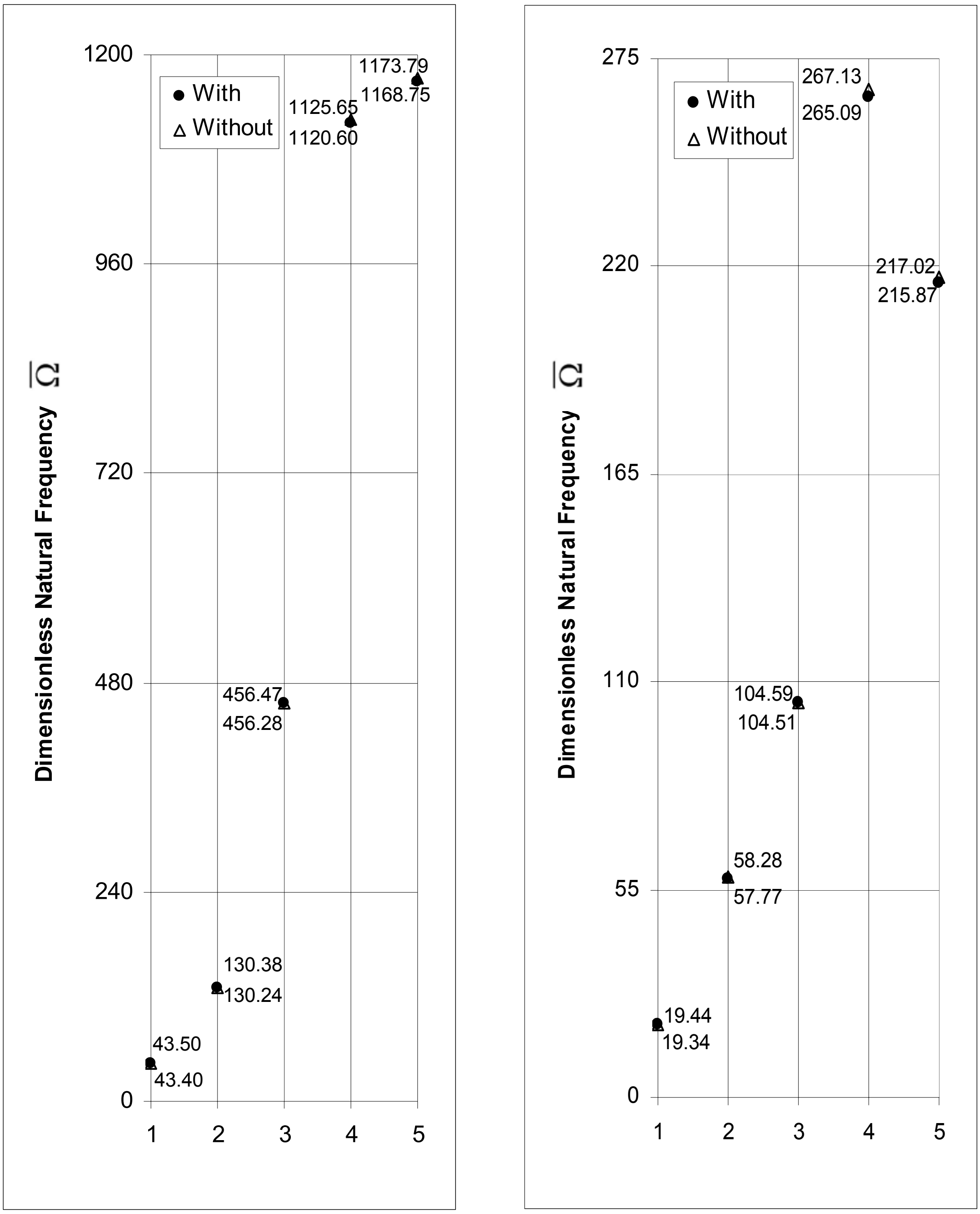
Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	56.551	55.997	0.553
	2	212.514	211.316	1.197
	3	263.784	261.748	2.036
	4	412.997	411.243	1.755
	5	733.591	732.919	0.672
SFFS	1	44.598	44.097	0.501
	2	130.291	129.450	0.842
	3	234.398	232.746	1.652
	4	295.312	294.211	1.101
	5	411.070	410.652	0.419
CFFF	1	19.437	19.342	0.096
	2	58.283	57.769	0.513
	3	104.586	104.508	0.079
	4	267.126	265.094	2.031
	5	217.024	215.871	1.153
FFFF	1	18.284	18.190	0.094
	2	20.411	20.308	0.103
	3	60.365	59.902	0.463
	4	102.835	102.747	0.088
	5	106.352	106.283	0.069



a) "Hard" Adhesive Case **b) "Soft" Adhesive Case**

Figure 161 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$ " in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint"

(Boundary Conditions in y-direction FFCFFCFF)



a) "Hard" Adhesive Case　　　**b) "Soft" Adhesive Case**

Figure 162　Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a Centrally Bonded Symmetric Double Lap Joint"

(Boundary Conditions in y-direction FFCFFFFF)

8.7 Numerical Results and Discussion for "Main PROBLEM III.b"

In the "Main PROBLEM IIIb.", the "Composite Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint" is analyzed. The doublers are made of Graphite-Epoxy and the plate adherends are Kevlar-Epoxy. For the in-between adhesive layer, the "hard" and the "soft" adhesive cases are taken into account. The "Geometric and the Material Characteristics" of the single lap joint system are given in Table 8.2.

In Figures 8.366 – 8.386, the mode shapes and the corresponding natural frequencies (from the first to fifth), in the "hard" and the subsequent "soft" adhesive cases with various boundary conditions are presented.

From aforementioned Figures, in the "hard" adhesive case it is easy to observe that with respect to the position of the "Bonded Region", there exists an almost "stationary region" in the mode shapes. And this region moves from left to the right part (or vice versa) in the composite symmetric double lap joint system. In the "soft" adhesive case, however, an almost "stationary region" does not exist in mode shapes. The general trend in the mode shapes, for the "soft" adhesive case is that, the "Bonded Region" moves or bends with the rest of the lap joint system. And the mode shapes are completely different in comparison with those of the "hard" adhesive cases with the same support conditions.

Next, for the "Main PROBLEM IIIb", in Figures 8.151 through 8.168, several important parametric studies are presented. In Figures 8.151-8.156, the "Dimensionless Natural Frequency $\overline{\Omega}$" versus "Position Ratio $\tilde{b}/L$" from the first up to the fifth mode are plotted, for both the "hard" and the "soft" adhesive cases, corresponding to the various support conditions.

From Figures 8.151, 8.153, 8.155, in the "hard" adhesive case, it is obvious that as position of the "Bonded Region" changes (in the y-direction), the natural frequencies gradually increase up to a certain position and then decreases.

These results are consequences of the movement of the half waves from left to right because of the change in the position of the "Bonded Region".

In the "soft" adhesive case, in Figures 8.152, 8.154, 8.156, the natural frequencies increase with the position of the "Bonded Region". This also can be expected due to the "soft" adhesive which makes the system loose and which shows a similar behavior in mode shapes up to $\tilde{b}$ =0.5m.

In Figures 8.157 through 8.162, the effect of the "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" on the natural frequencies (from the first up to the fifth) in the "hard" and "soft" adhesive cases, are investigated for various boundary conditions. In the "hard" adhesive case, in Figures 8.157, 8.159, 8.161, the first natural frequency, in spite of the increasing "Bending Rigidity Ratio", does remain practically constant. In the higher modes, the natural frequencies increase sharply at first and after the "Bending Rigidity Ratio=2.6" they become almost flat or constant regardless of the increase in "Bending Rigidity Ratio".

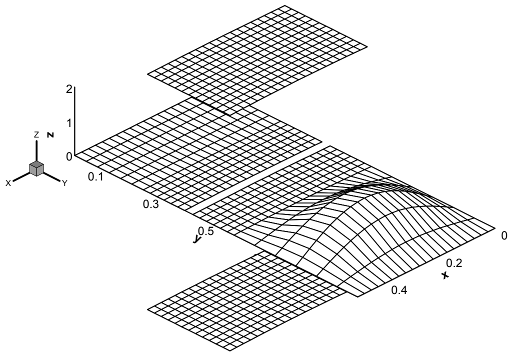
In the "soft" adhesive cases, in the Figures 8.158, 8.160, 8.162, the first three frequencies remain more or less constant as the "Bending Rigidity Ratio $D_{11}^{(2)}/D_{11}^{(1)}$" increases. In the higher modes, the natural frequencies increase.

Lastly, the direct effects of the adhesive layer elastic constants Ea1, Ea4, and also Ga1, Ga4 on the dimensionless natural frequencies are investigated for the "Main PROBLEM III.b". In order to show these effects, the "Dimensionless Natural Frequencies" versus the "Adhesive Elastic Modulus Ratio $(E_{a1}=E_{a4})/B_{11}^{(1)}$" are plotted (while the other elastic <u>constant kept constant</u>) in Figures 8.163 through 8.165 for various boundary condition. Similarly, the "Dimensionless Natural Frequencies" versus the "Adhesive Shear Modulus Ratio $(G_{a1}=G_{a4})/B_{11}^{(1)}$" are presented in Figures 8.166 through 8.168 for various support condition.

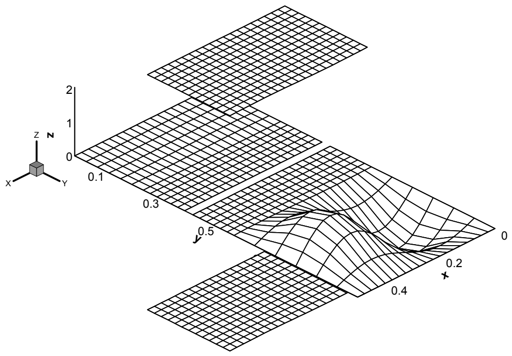
It can be seen from the Figures 8.163-8.165, the influence of the "Adhesive Elastic Modulus Ratio $(E_{a1}=E_{a4})/B_{11}^{(1)}$" on the natural frequencies, is not significant.

In Figures 8.166-8.168, we can see that the "Shear Modulus Ratio(G_{a1}=G_{a4}) / $B_{11}^{(1)}$", significantly affects the natural frequencies. Also, in those Figures, one can observe a "transition region" which takes the frequencies to considerable higher levels. After then, no change is observed in the frequencies

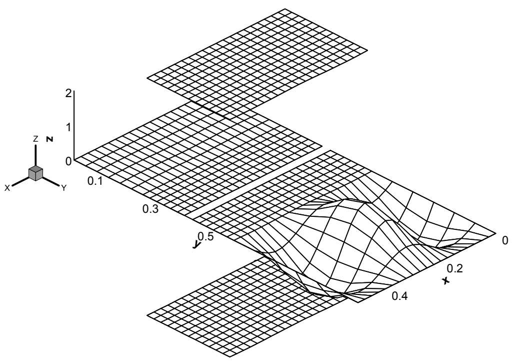
8.7.1 Natural Frequencies and Corresponding Mode Shapes for "Main PROBLEM IIIb"



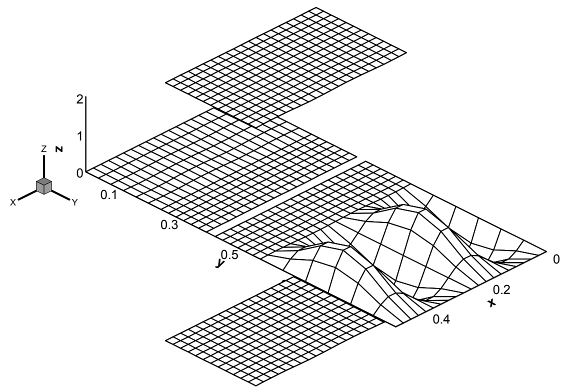
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=463.952



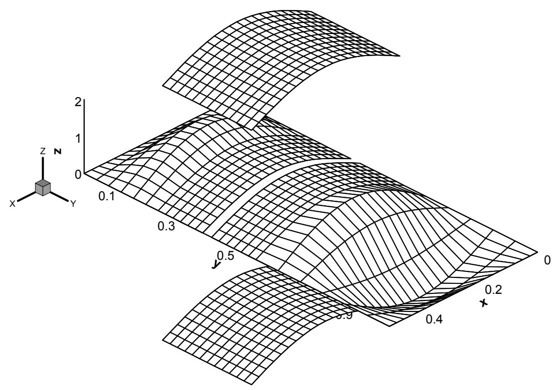
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=581.094



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=948.958



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41}$=1834.358

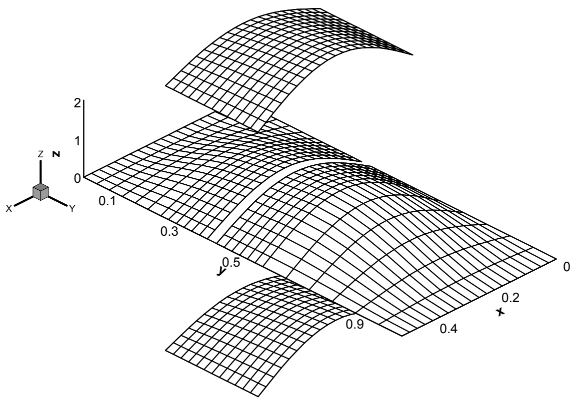


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12}$=2195.680

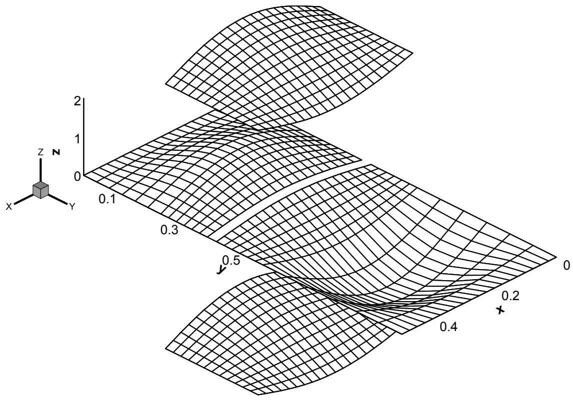
("Hard" Adhesive Case)

Fig 8.163 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

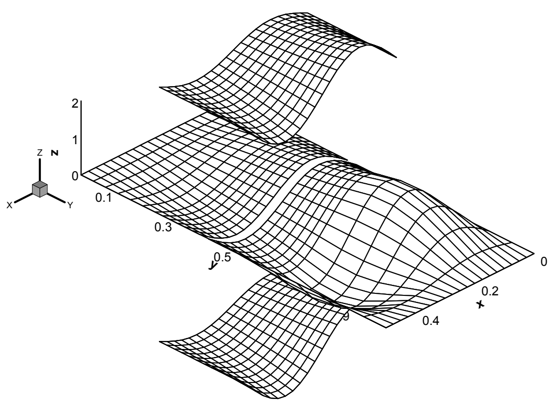
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFCFFCFF)



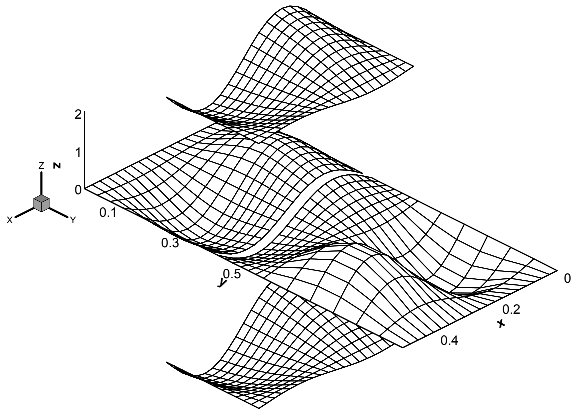
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}=56.262$



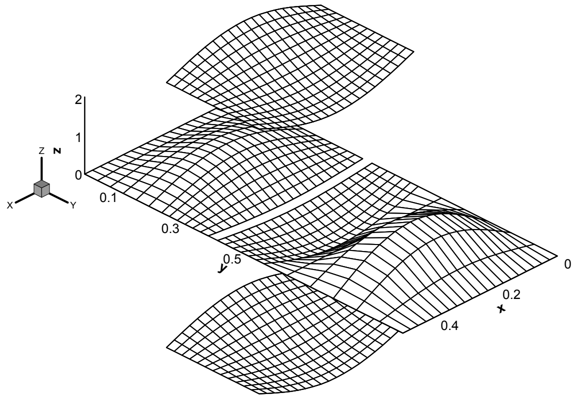
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}=200.966$



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{21}=244.320$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=385.357

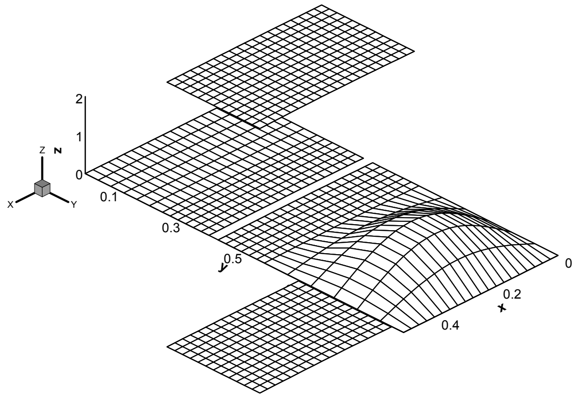


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=557.076

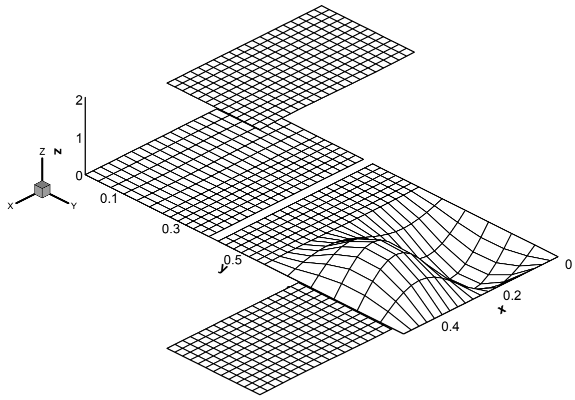
("Soft" Adhesive Case)

Fig.8.164 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

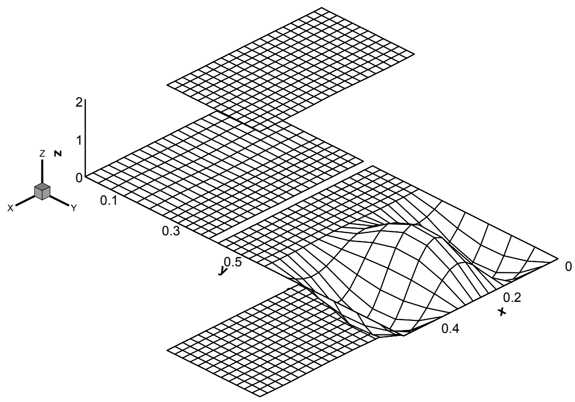
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3m, $b_1 = b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFCFFCFF)



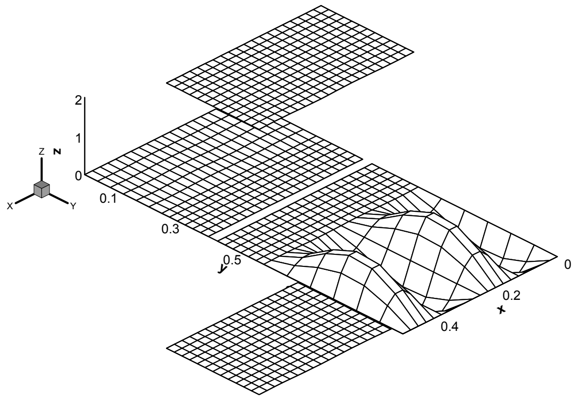
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=239.941



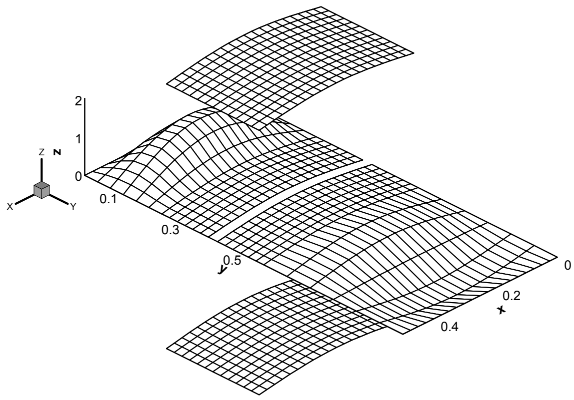
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=352.640



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=719.203



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{41}$=1603.578

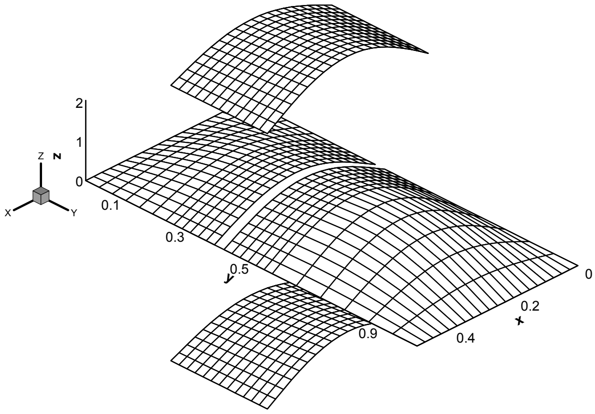


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{12}$=1733.026

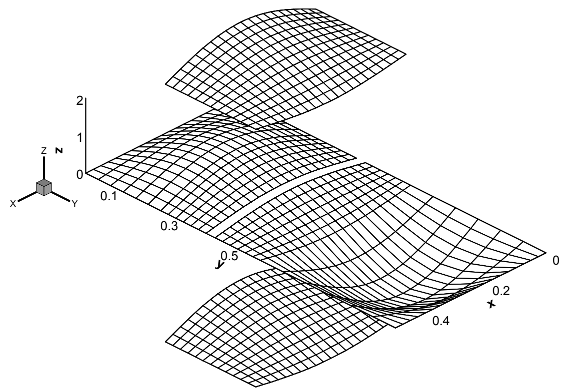
("Hard" Adhesive Case)

Fig.8.165 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

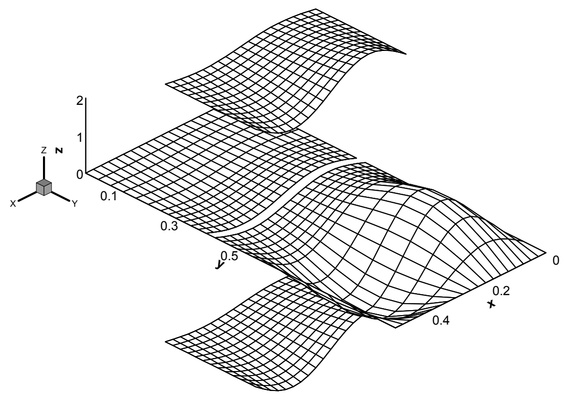
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFSFFSFF)



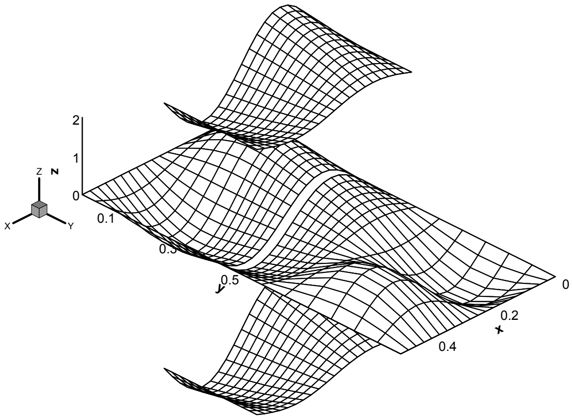
a) First Mode with $\overline{\Omega}_1=\overline{\omega}_{11}$=42.914



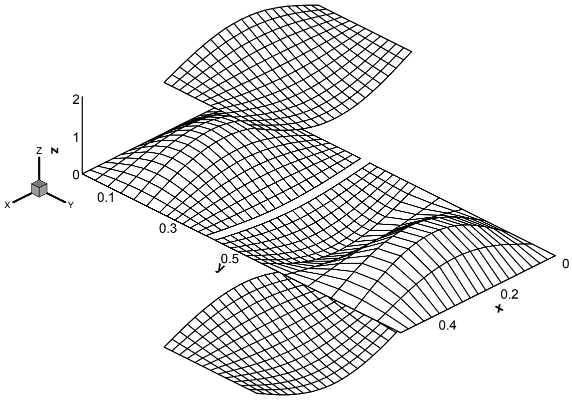
b) Second Mode with $\overline{\Omega}_2=\overline{\omega}_{12}$=116.324



c) Third Mode with $\overline{\Omega}_3=\overline{\omega}_{21}$=198.564



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=299.401

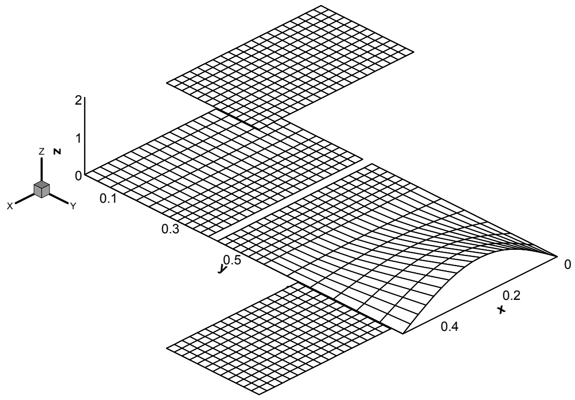


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=357.365

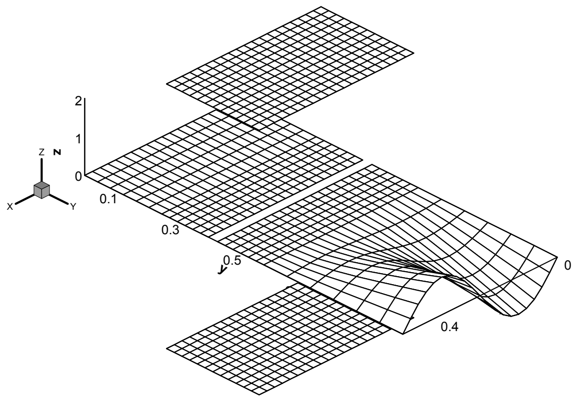
("Soft" Adhesive Case)

Fig.8.166 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

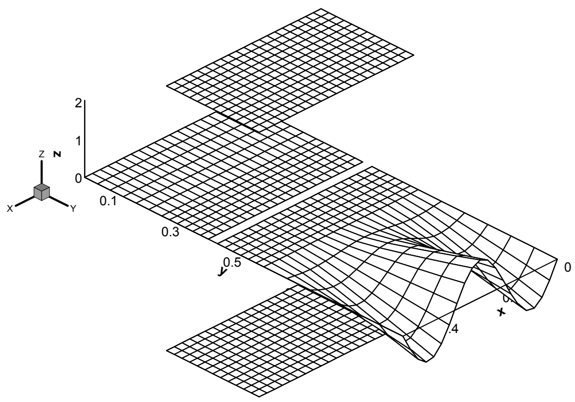
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFSFFSFF)



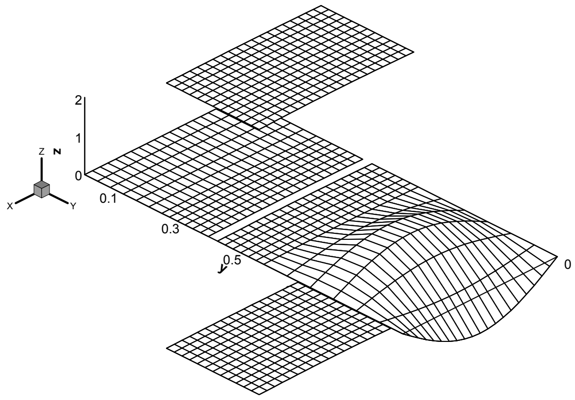
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 20.474$



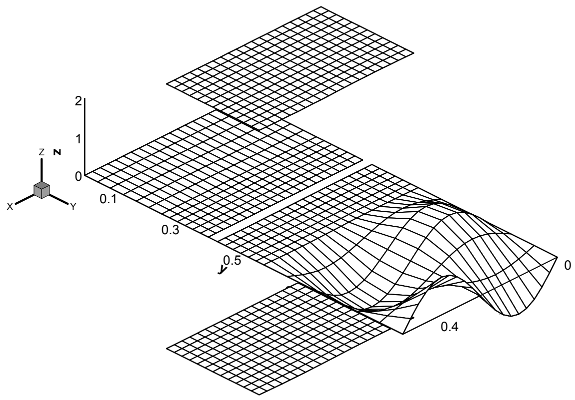
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 100.629$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 416.120$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=479.445

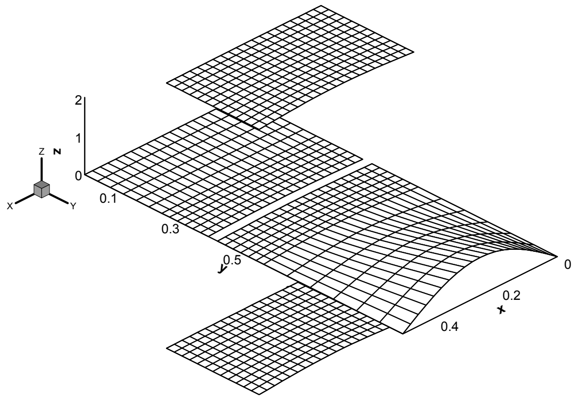


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=647.892

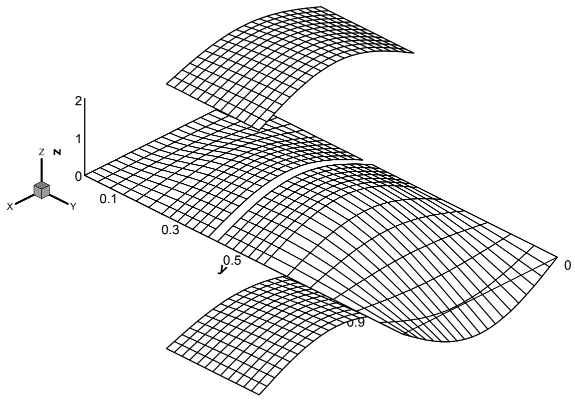
("Hard" Adhesive Case)

Fig.8.167 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

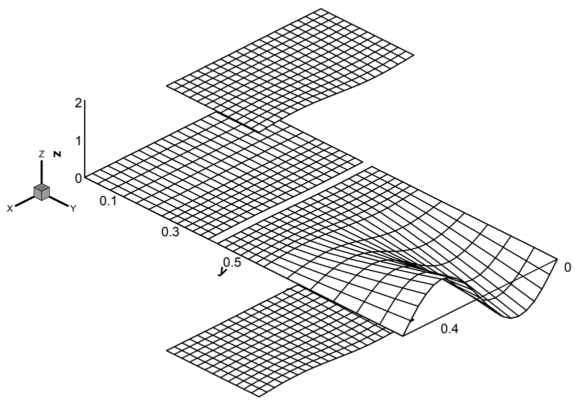
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFCFFFFF)



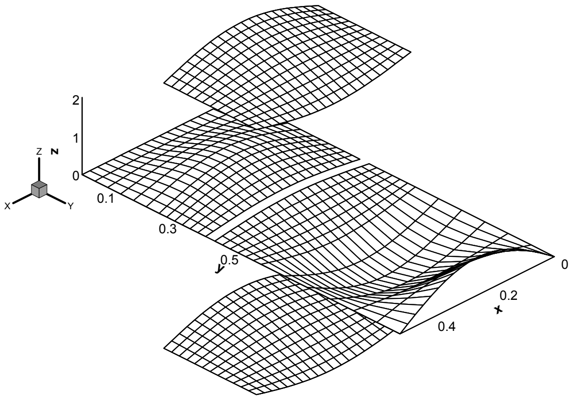
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=12.634



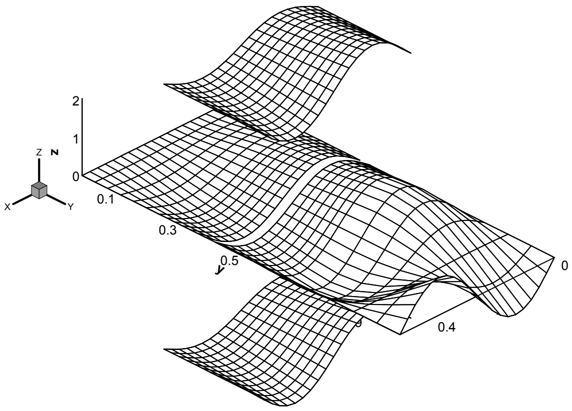
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=57.056



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=91.213



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=205.441

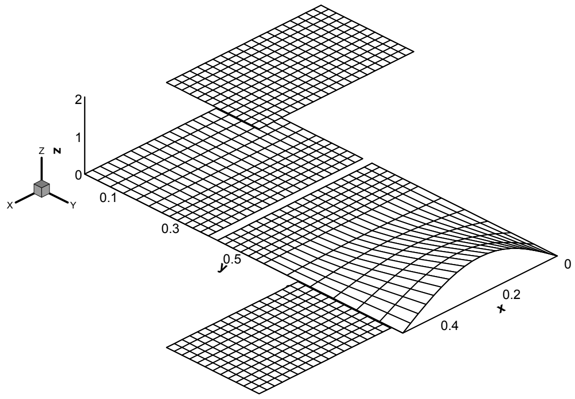


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=255.170

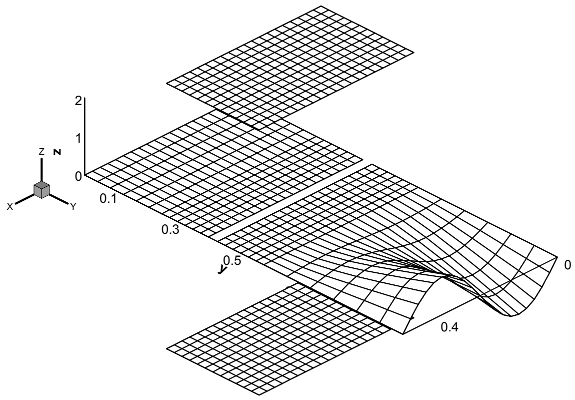
("Soft" Adhesive Case)

Fig.8.168 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

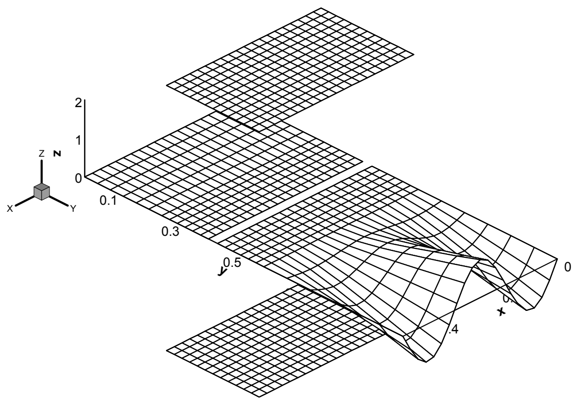
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFCFFFFF)



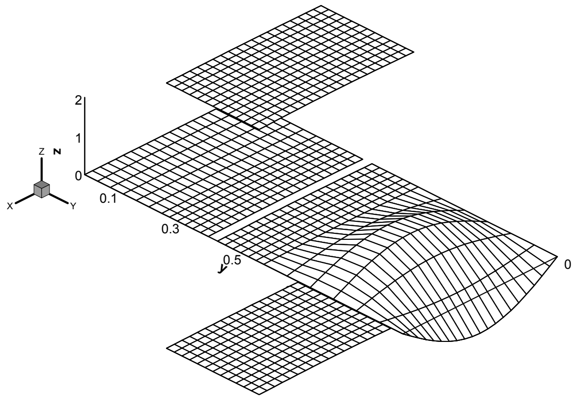
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=20.473



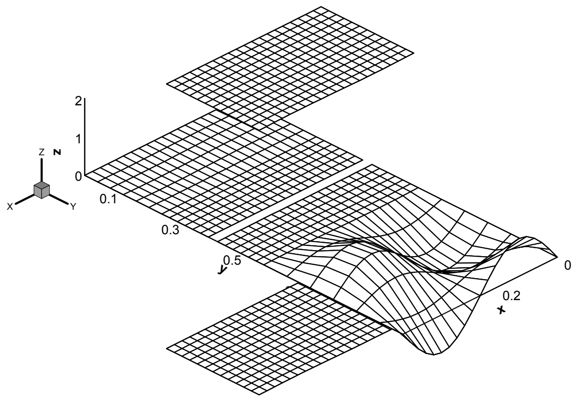
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=100.626



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=416.116



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=479.438

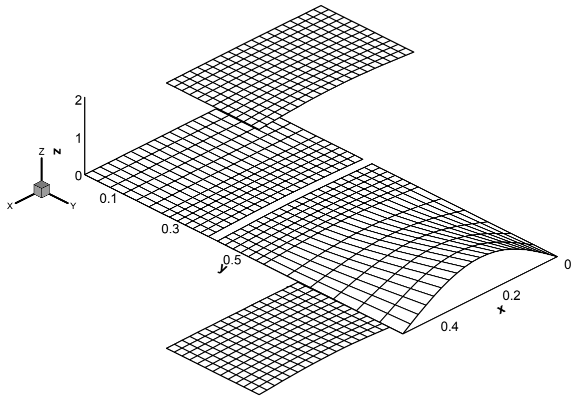


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=647.759

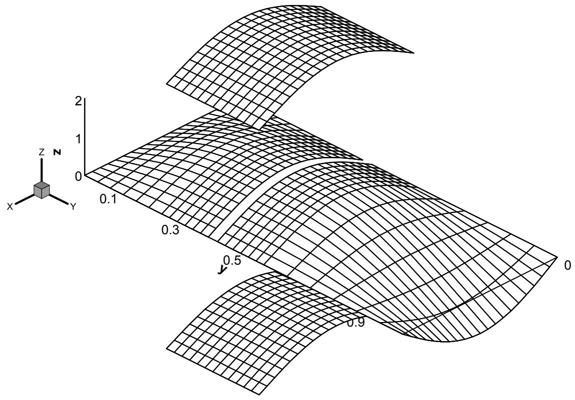
("Hard" Adhesive Case)

Fig.8.169 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

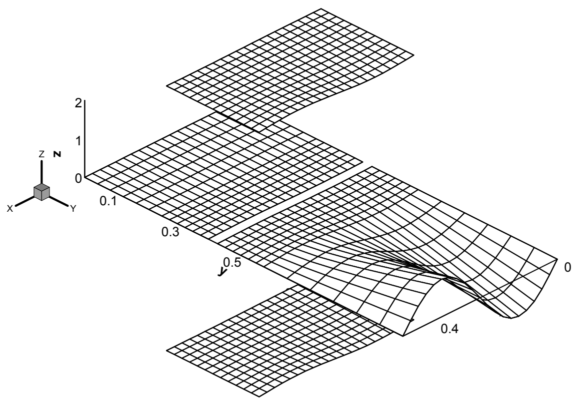
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFSFFFFF)



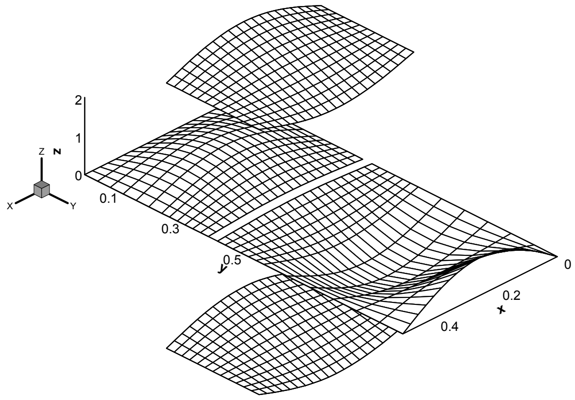
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=12.634



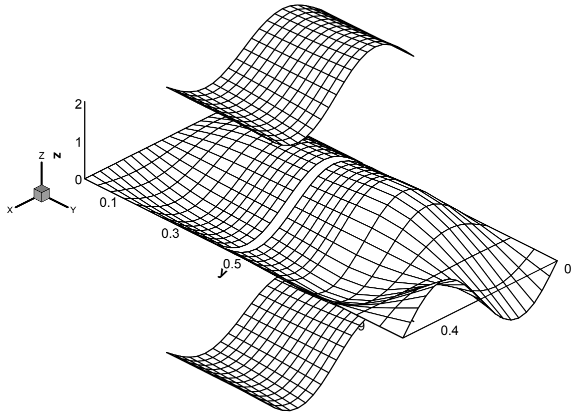
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=49.657



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=91.201



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=167.652

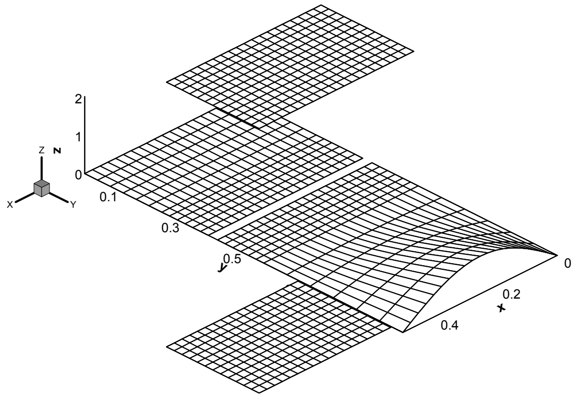


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=248.819

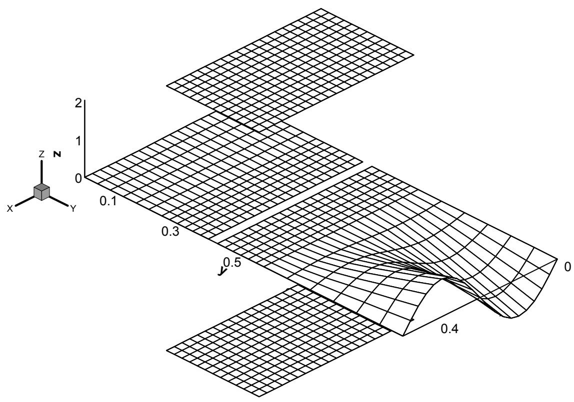
("Soft" Adhesive Case)

Fig.8.170 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

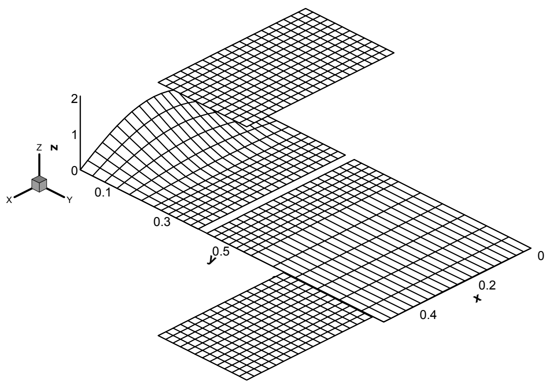
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I + \ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFSFFFFF)



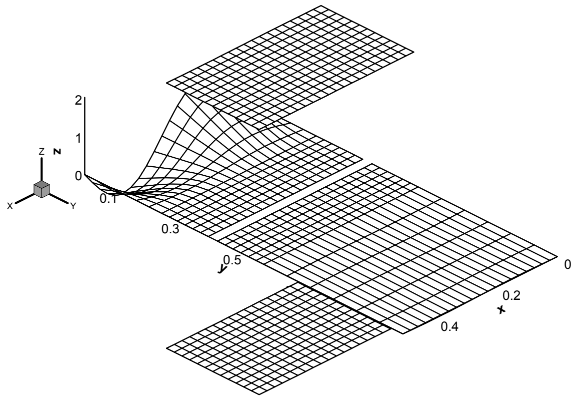
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 20.465$



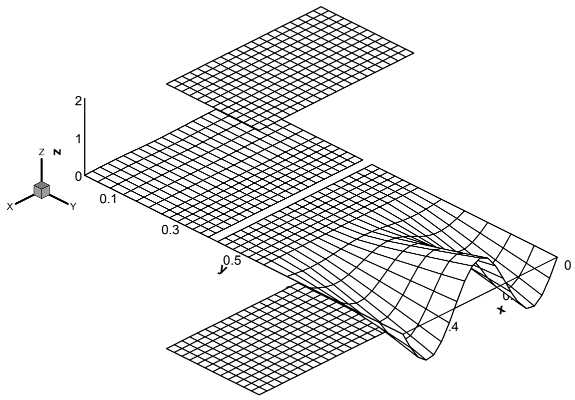
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 100.614$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{12} = 137.930$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=240.732

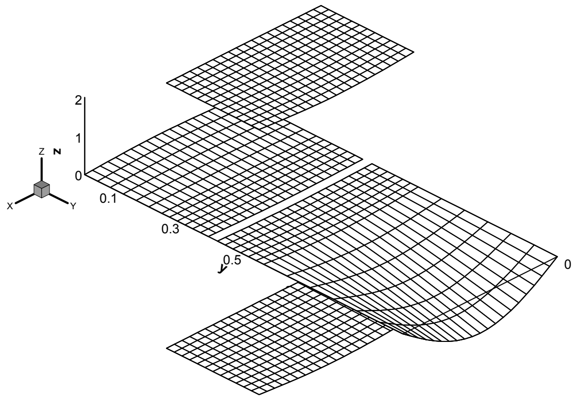


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{31}$=416.100

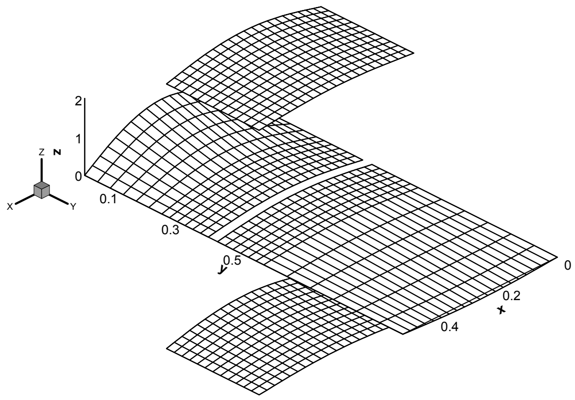
("Hard" Adhesive Case)

Fig.8.171 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

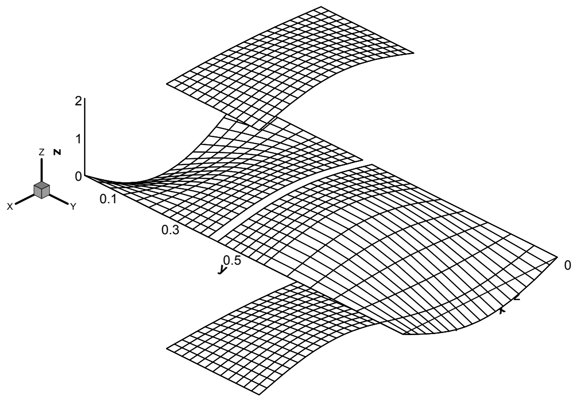
(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFFFFFFF)



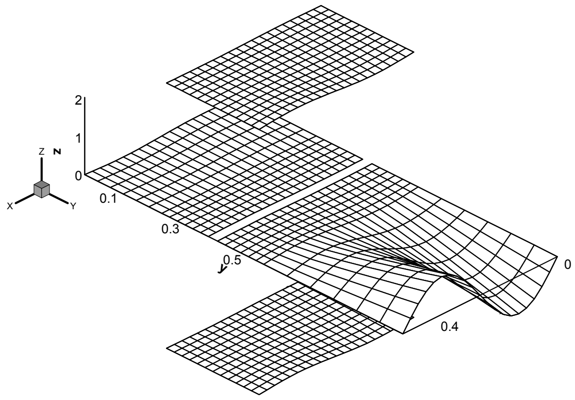
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=12.528



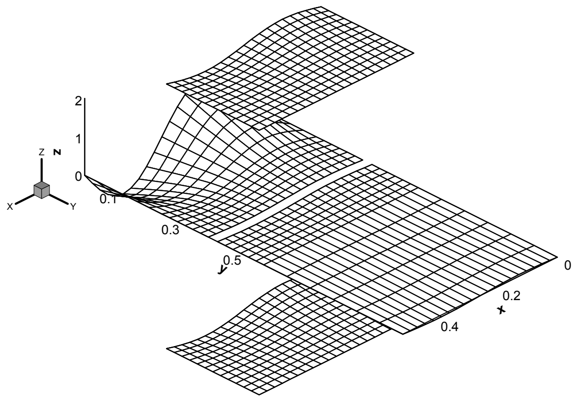
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=31.342



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{13}$=65.722



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{21}$=91.134



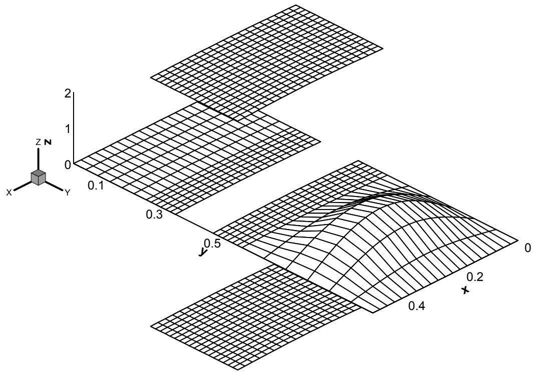
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=138.086

("<u>Soft</u>" Adhesive Case)

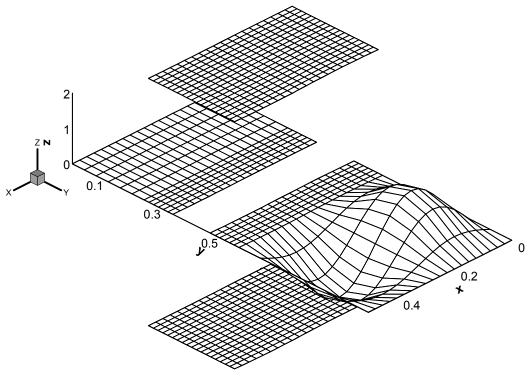
Fig.8.172 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $b_1=b_4$=0.3m, b_2=0.4m,b_3=0.6m, $\tilde{b}$ =0.4m, a=0.5 m L=1m)
(Boundary Conditions in y-direction FFFFFFFF)

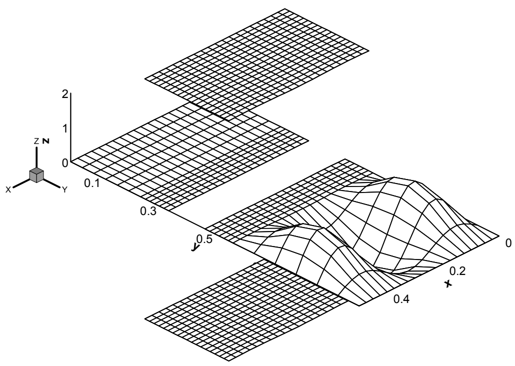
8.7.2 Natural Frequencies and Corresponding Mode Shapes for "Special Case of Main PROBLEM IIIb"



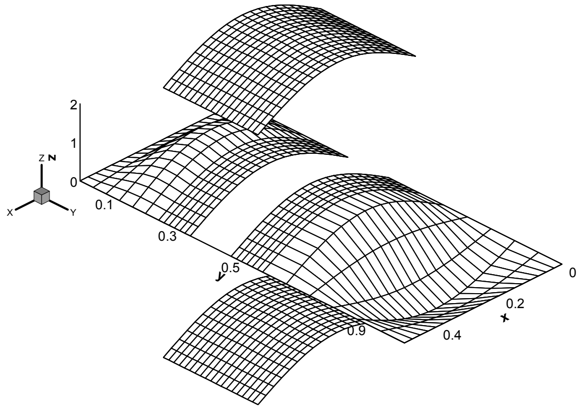
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=459.654



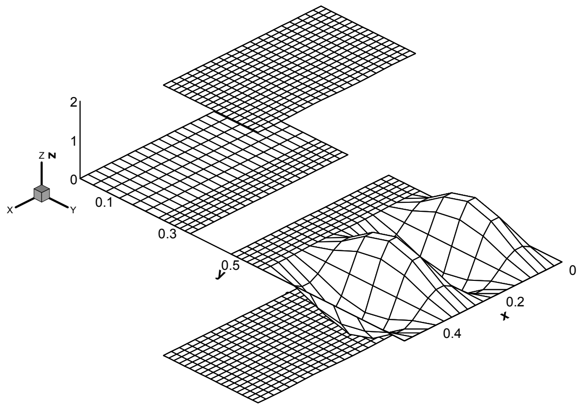
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21}$=579.969



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31}$=948.126



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=1702.401

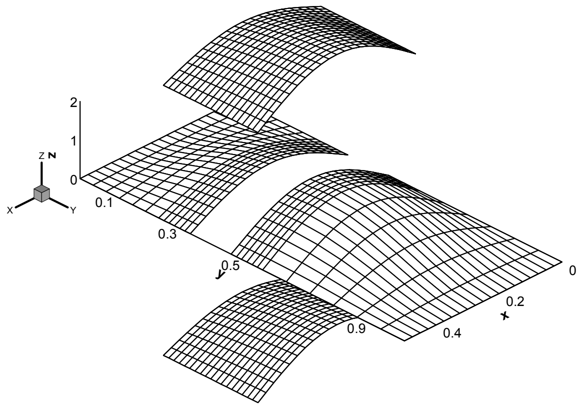


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{41}$=1833.595

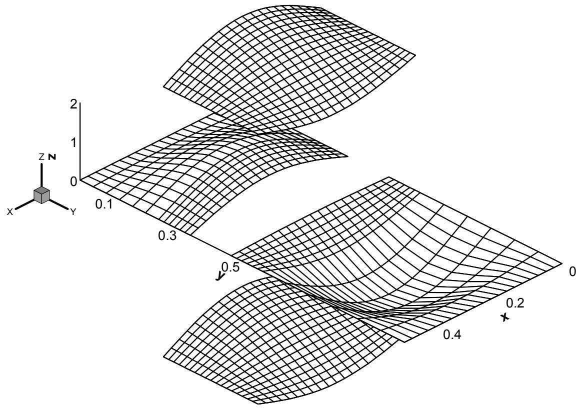
("Hard" Adhesive Case)

Fig 8.173 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

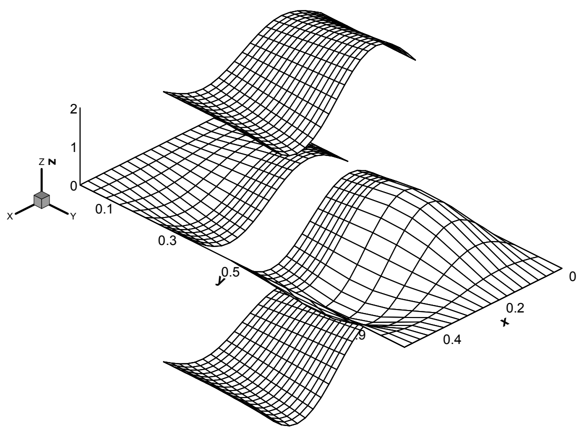
(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1m., b_1=0.3m, b_2=0.35m, b_3=0.55m
$\tilde{b}$ =0.5m, a=0.5m. L=1m)
(Boundary Conditions in y-direction FFCFFCFF)



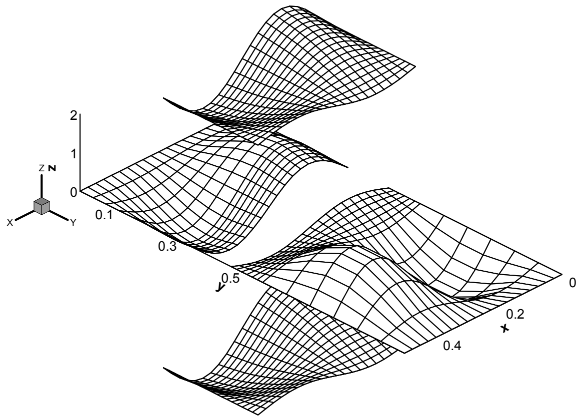
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=47.295



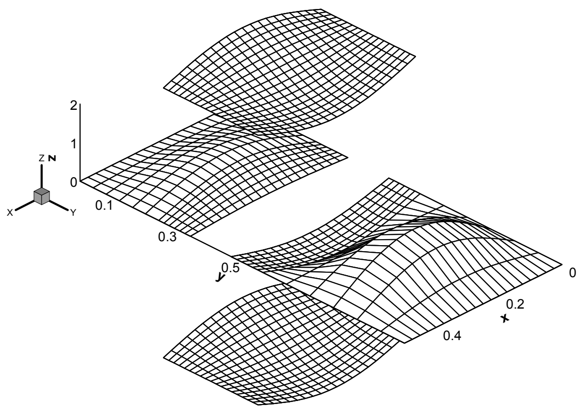
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=185.838



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=218.545



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{22}$=363.346

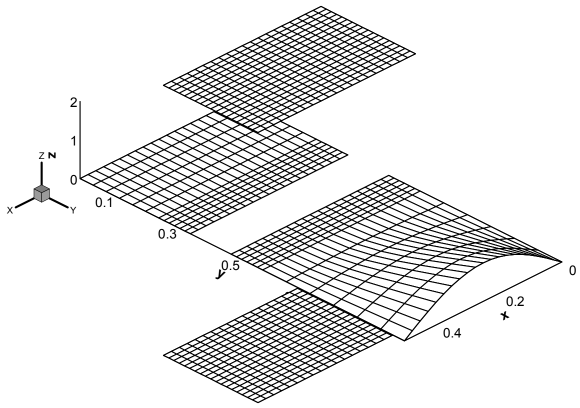


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{13}$=532.933

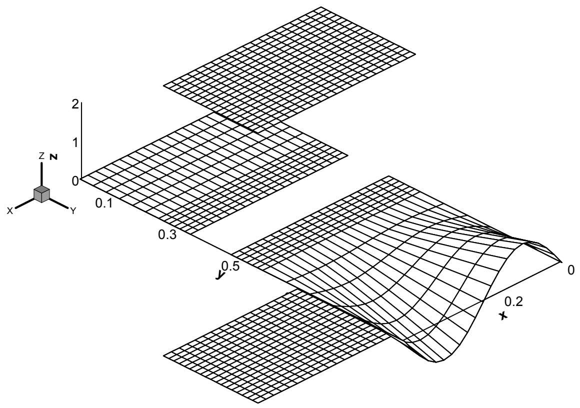
("Soft" Adhesive Case)

Fig 8.174 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

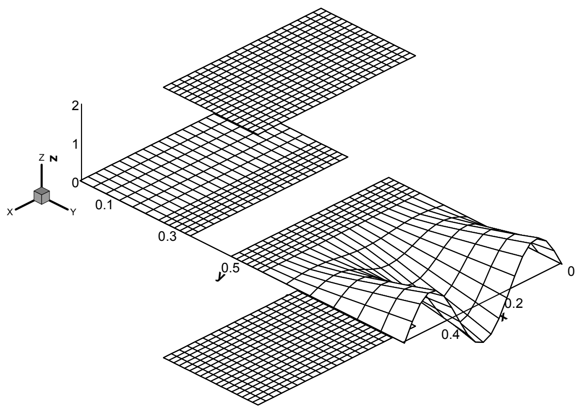
(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1m., b_1=0.3m, b_2=0.35m, b_3=0.55m
$\widetilde{b}$ =0.5m, a=0.5m. L=1m)
(Boundary Conditions in y-direction FFCFFCFF)



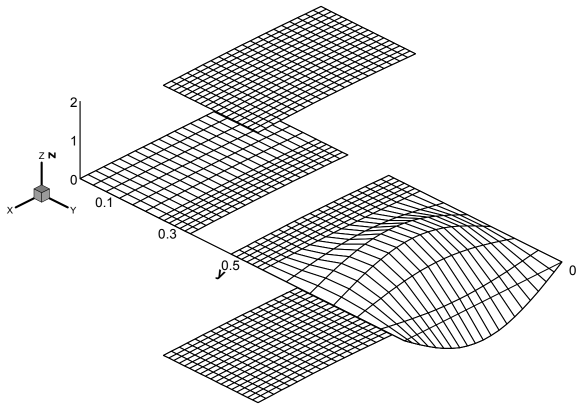
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11} = 20.423$



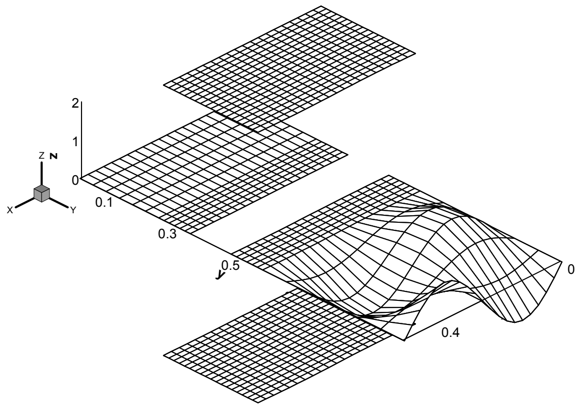
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{21} = 100.586$



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{31} = 416.085$



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{12}$=474.740

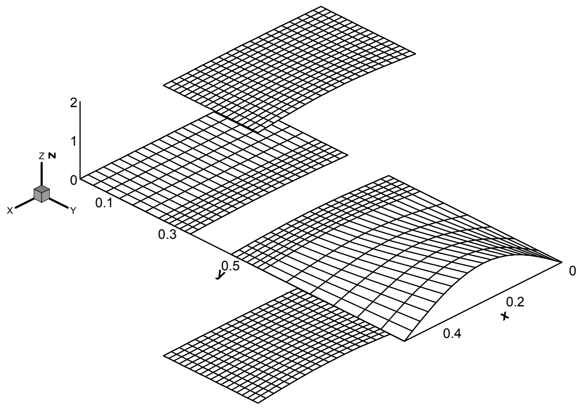


e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=646.943

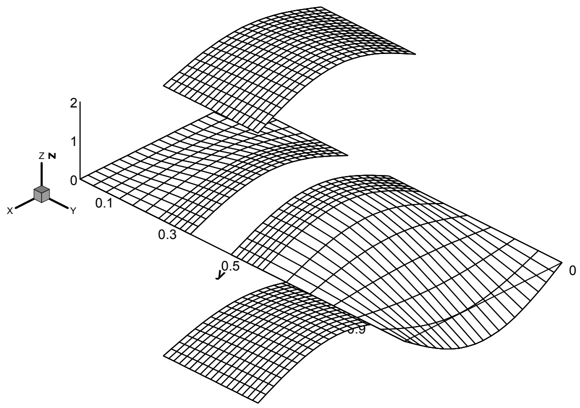
("Hard" Adhesive Case)

Fig 8.175 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

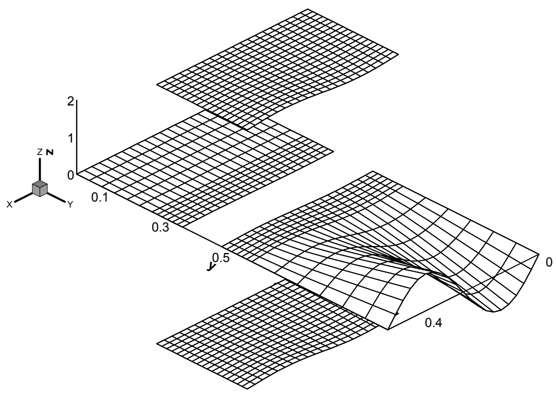
(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1m., b_1=0.3m, b_2=0.35m, b_3=0.55m
$\tilde{b}$ =0.5m, a=0.5m. L=1m)
(Boundary Conditions in y-direction FFCFFFFF)



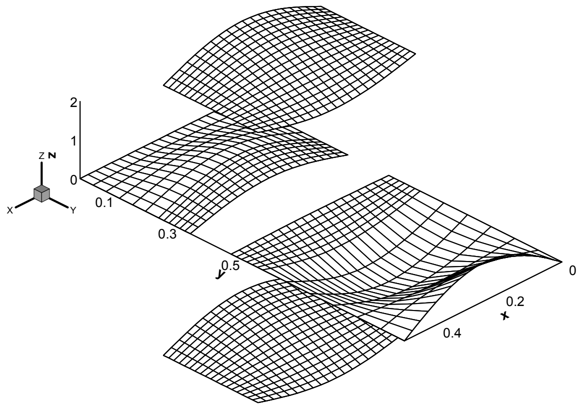
a) First Mode with $\overline{\Omega}_1 = \overline{\omega}_{11}$=11.949



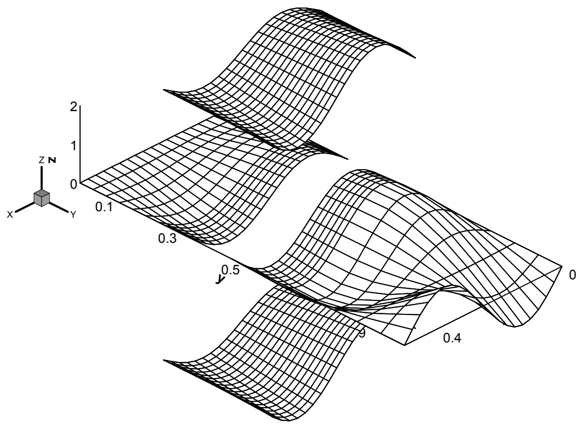
b) Second Mode with $\overline{\Omega}_2 = \overline{\omega}_{12}$=48.474



c) Third Mode with $\overline{\Omega}_3 = \overline{\omega}_{21}$=90.793



d) Fourth Mode with $\overline{\Omega}_4 = \overline{\omega}_{13}$=189.999



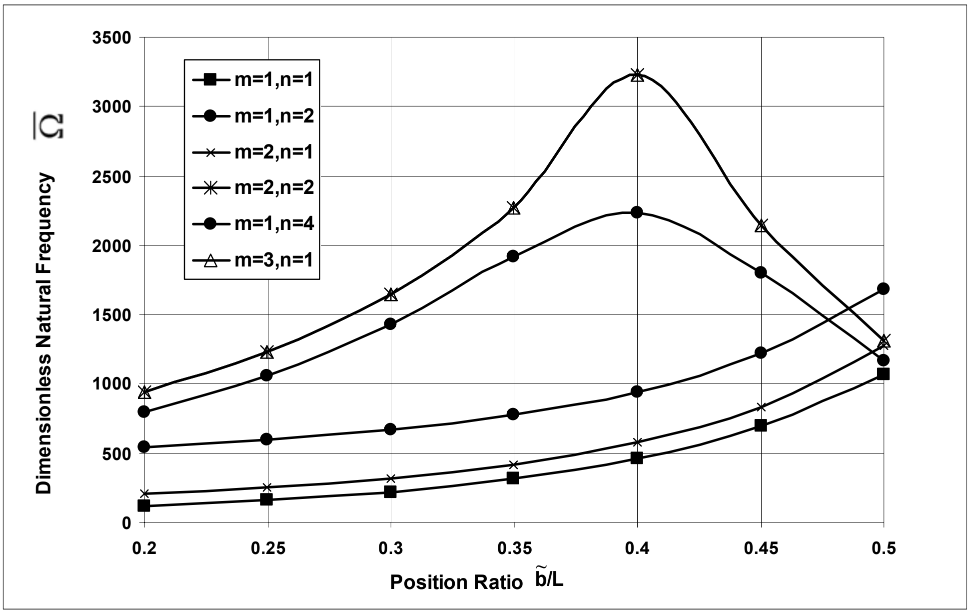
e) Fifth Mode with $\overline{\Omega}_5 = \overline{\omega}_{22}$=226.767

("Soft" Adhesive Case)

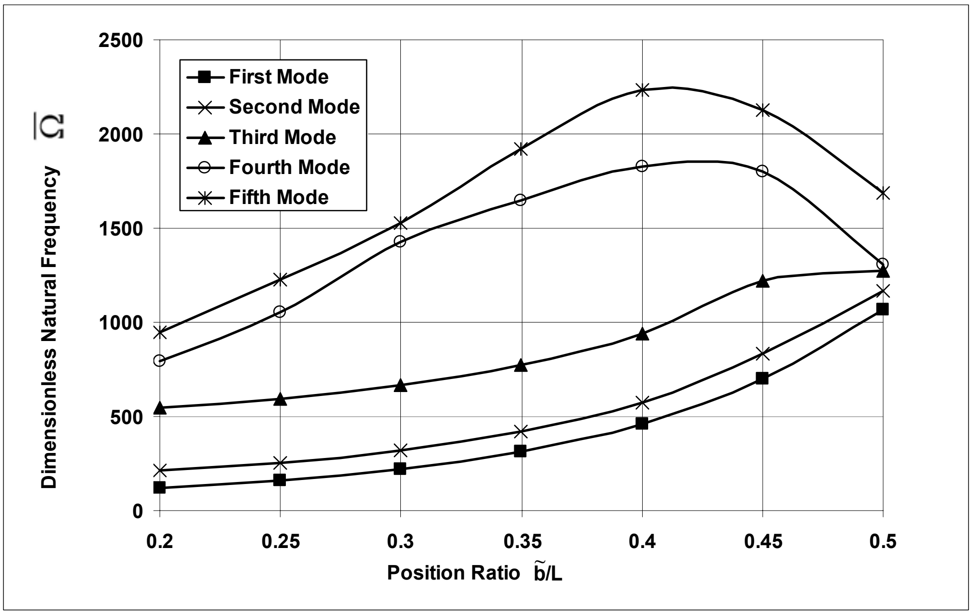
Fig 8.176 Mode Shapes and Dimensionless Natural Frequencies of "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint) with a Gap"

(Plate 1=Plate 4= Graphite-Epoxy, Plate2=Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I=0.1m, ℓ_{II}=0.1m., b_1=0.3m, b_2=0.35m, b_3=0.55m
$\tilde{b}$ =0.5m, a=0.5m. L=1m)
(Boundary Conditions in y-direction FFCFFFFF)

8.7.3 Some Parametric Studies for "Main PROBLEM IIIb"



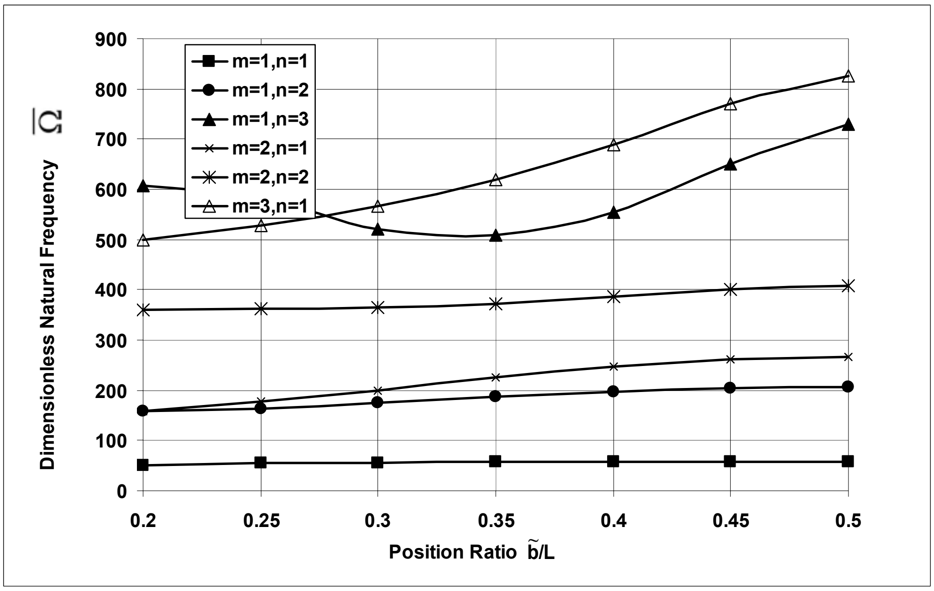
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''Hard'' Adhesive



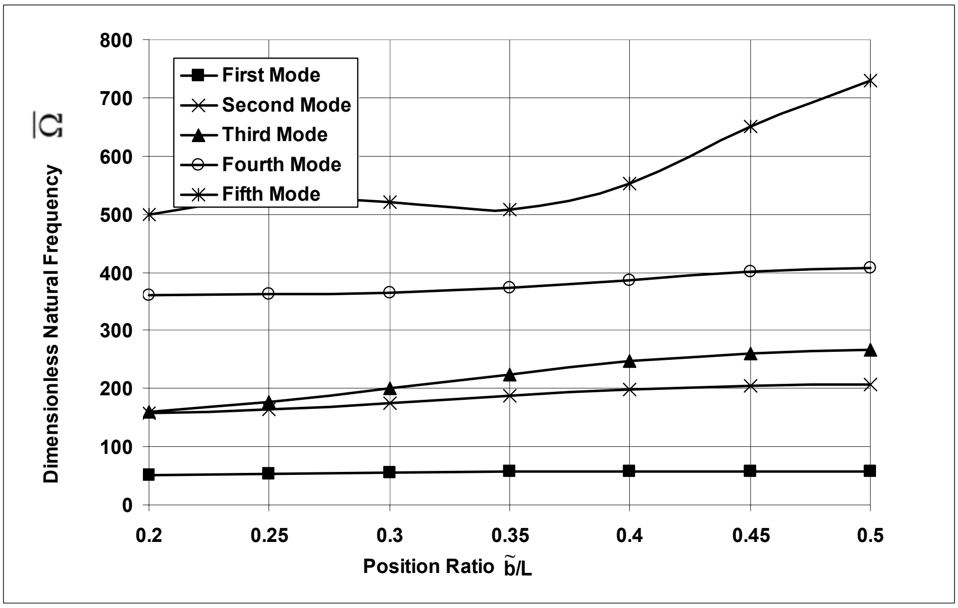
b) "Various Modes with (FFCFFCFF) B.C.'s, ''Hard'' Adhesive

Fig 8.177 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



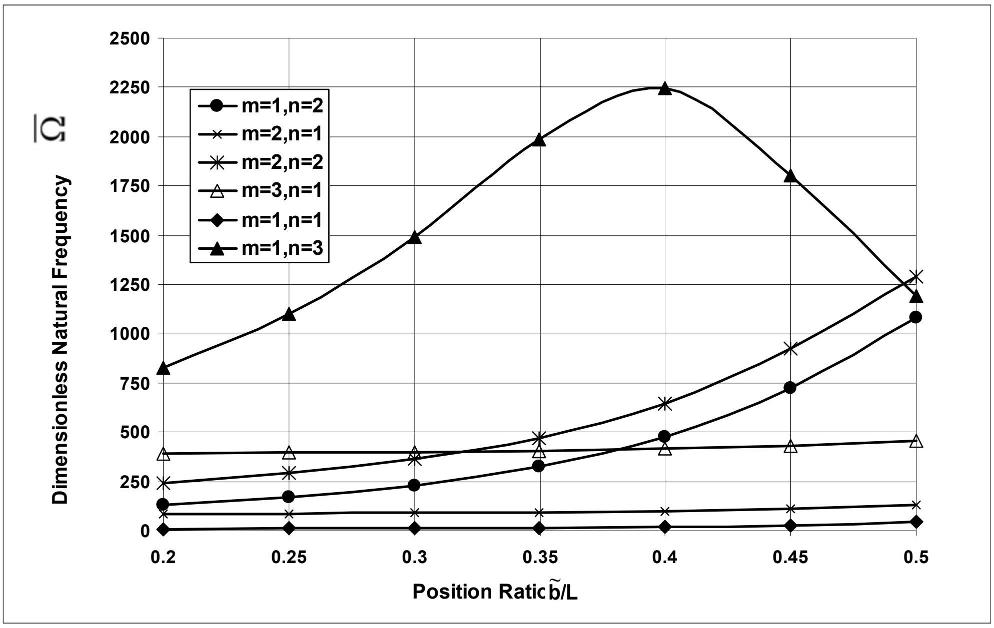
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''Soft'' Adhesive



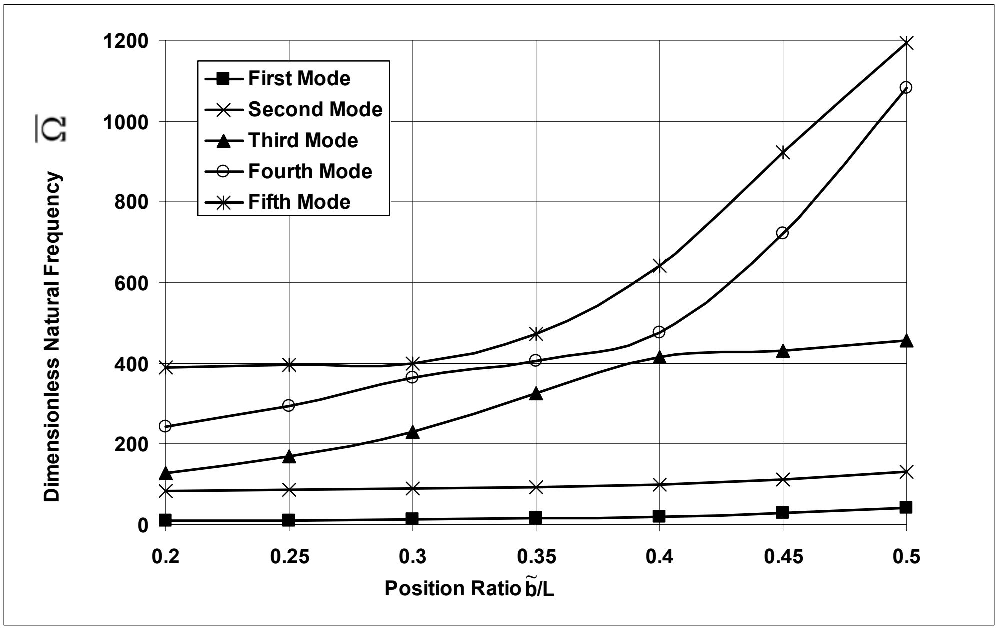
b) "Various Modes with (FFCFFCFF) B.C.'s, ''Soft'' Adhesive

Fig 8.178 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)



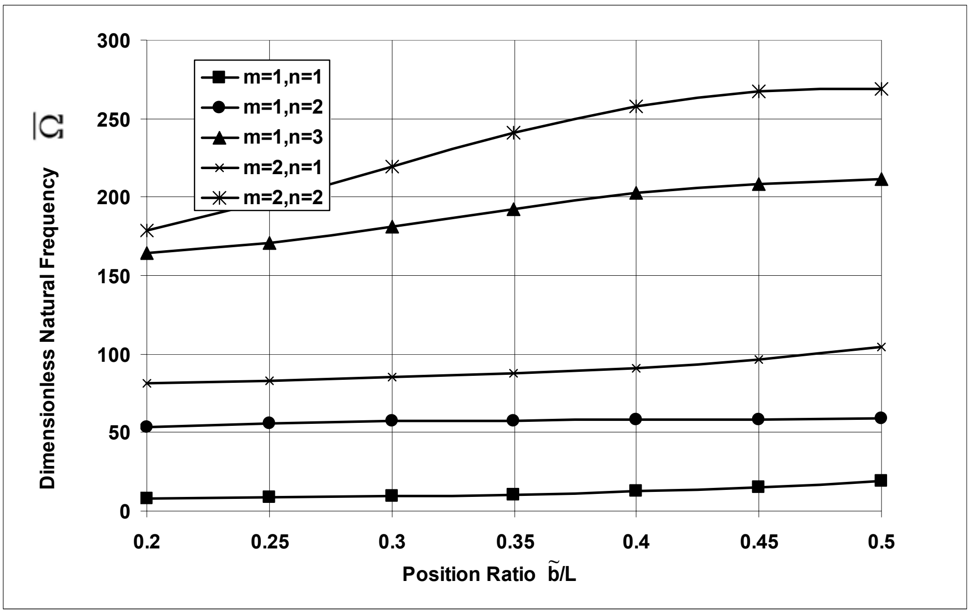
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''Hard'' Adhesive



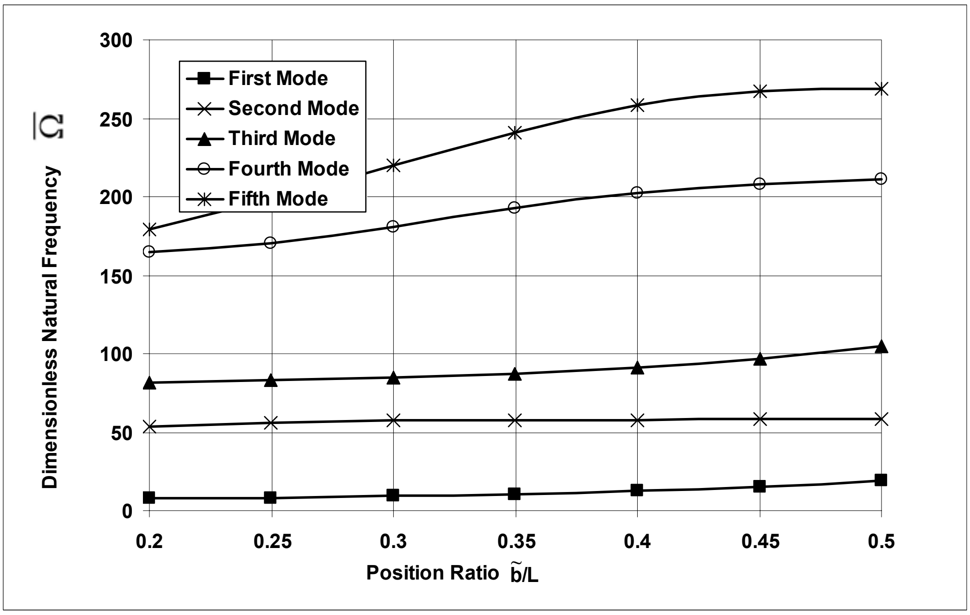
b) "Various Modes with (FFCFFFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.179 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



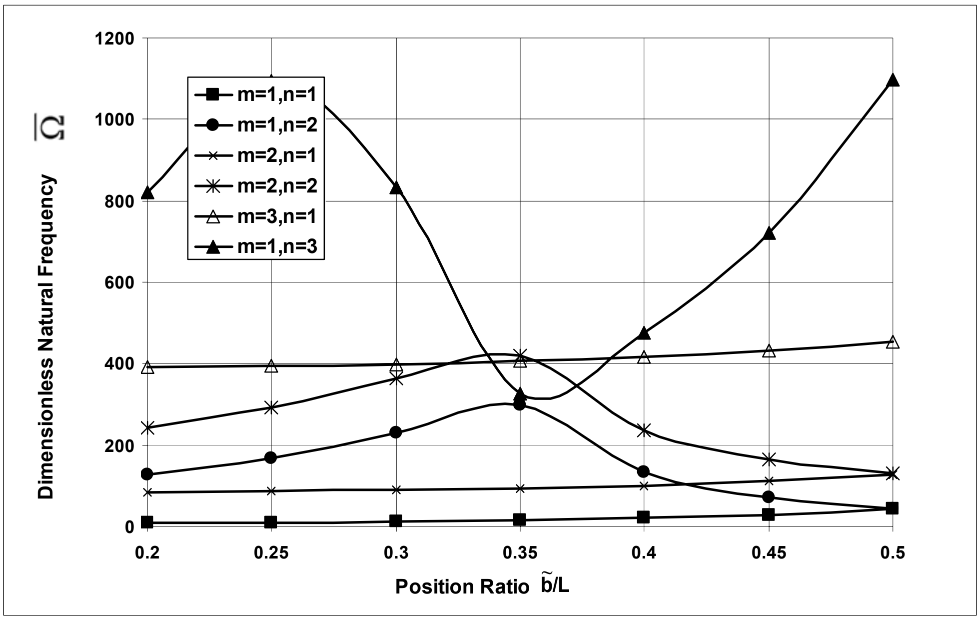
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''Soft'' Adhesive



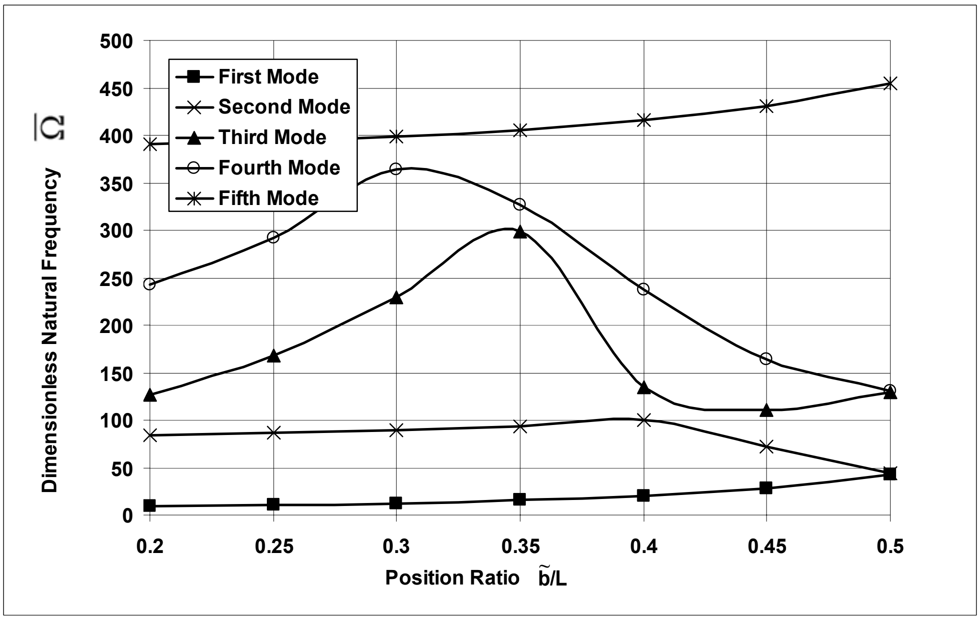
b) "Various Modes with (FFCFFFFF) B.C.'s, ''Soft'' Adhesive

Fig 8.180 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}$/L " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)



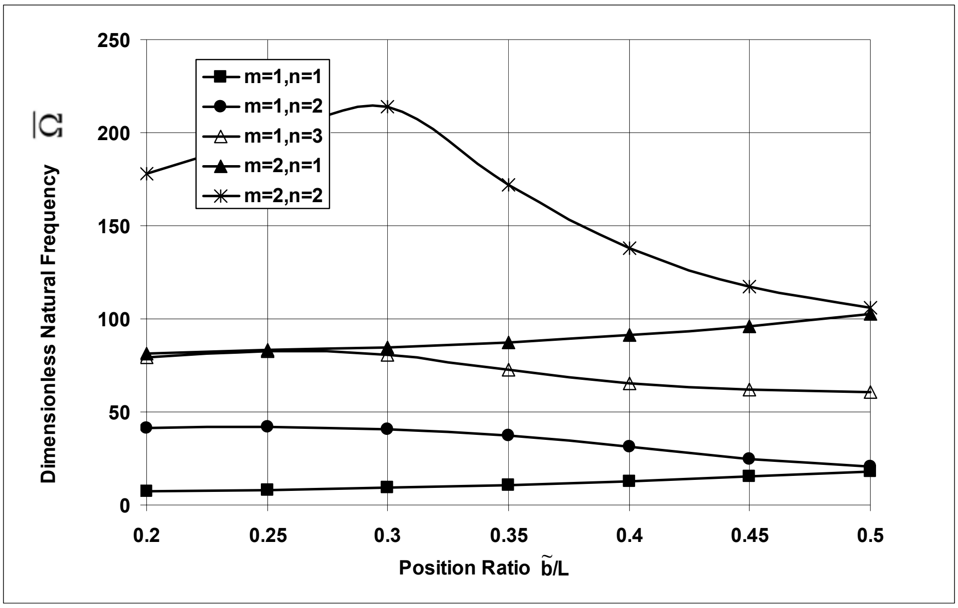
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''Hard' Adhesive



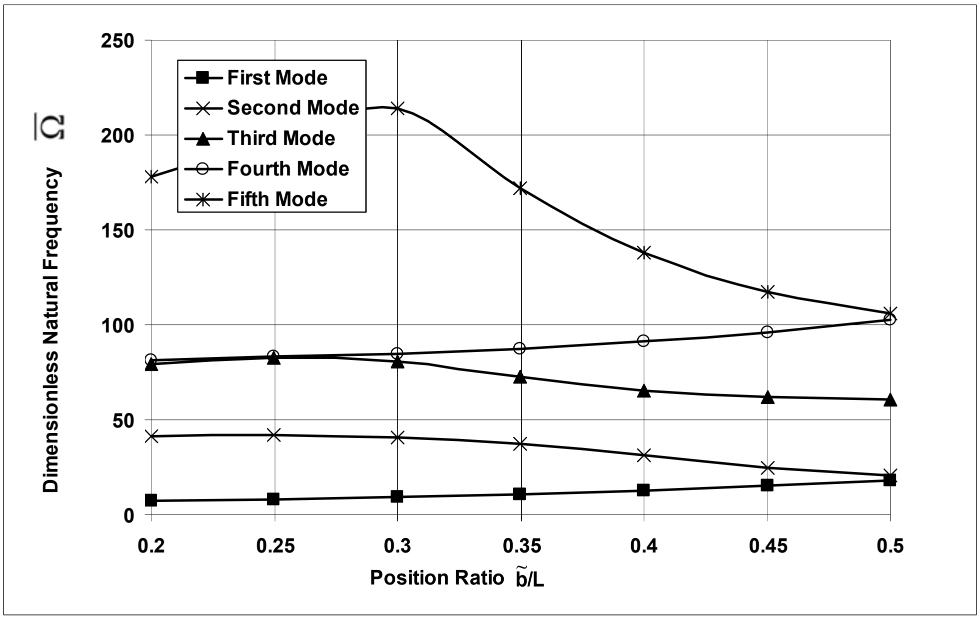
b) "Various Modes with (FFFFFFFF) B.C.'s, ''Hard'' Adhesive

Fig 8.181 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}/L$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =varies, a=0.5m. L=1m)
(Boundary Conditions in y-direction FFFFFFFF)



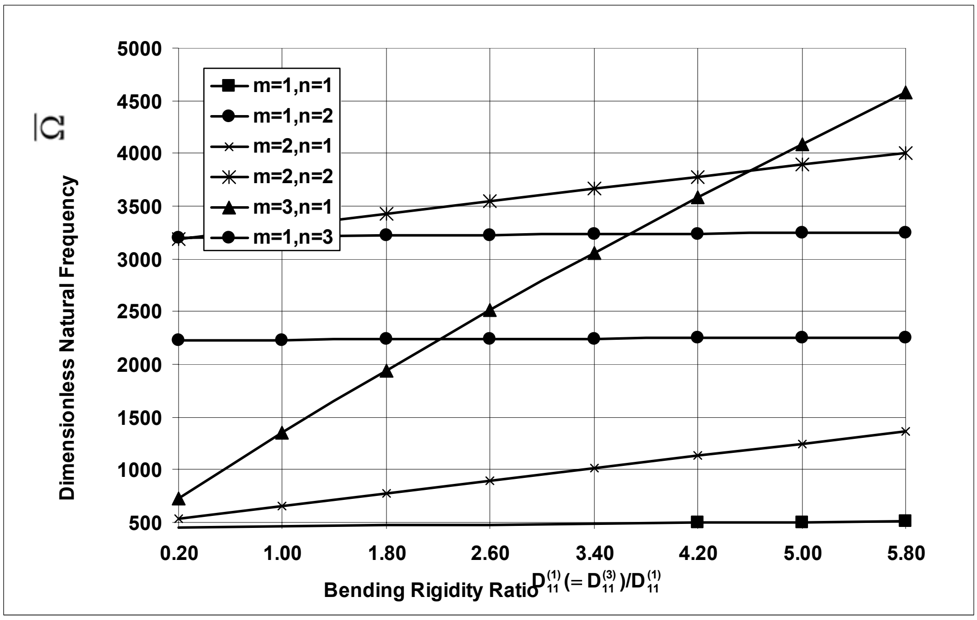
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, "Soft" Adhesive



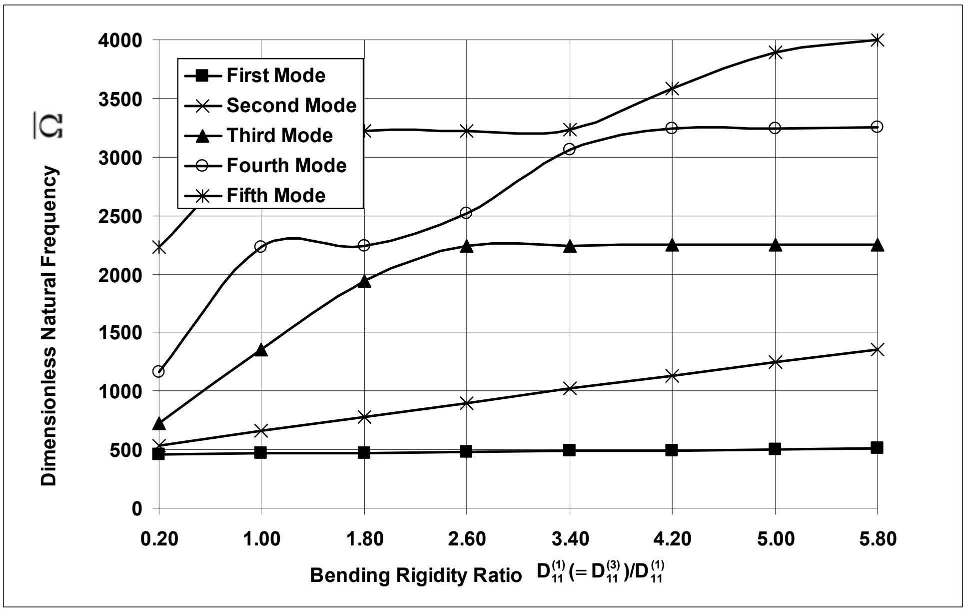
b) "Various Modes with (FFFFFFFF) B.C.'s, "Soft" Adhesive

Fig 8.182 "Dimensionless Natural Freq. ($\overline{\Omega}$)" versus "Position Ratio $\tilde{b}/L$ " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length (ℓ_I+ℓ_{II})=0.3m, $\tilde{b}$ =varies, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFFFFFFF)



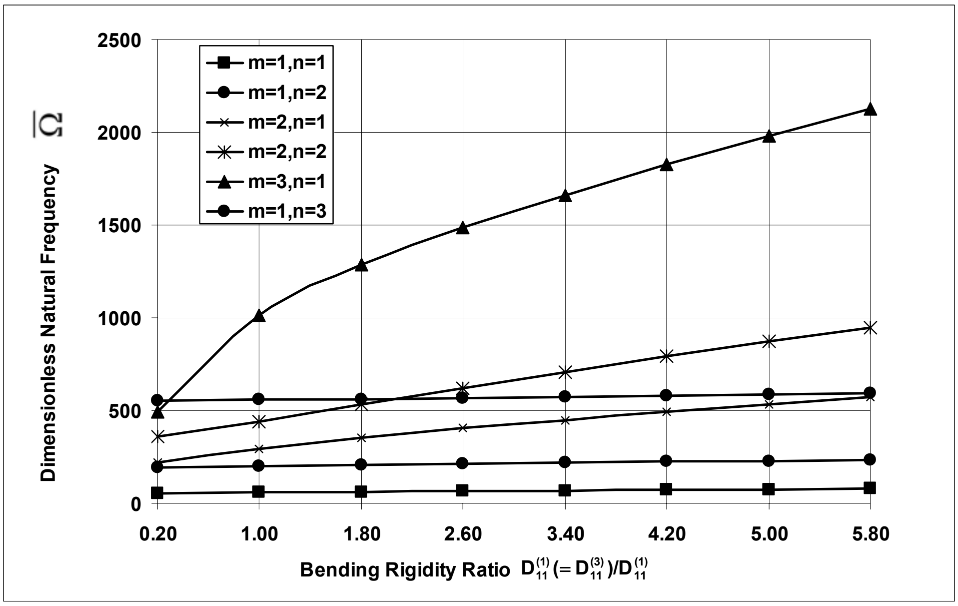
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''Hard'' Adhesive



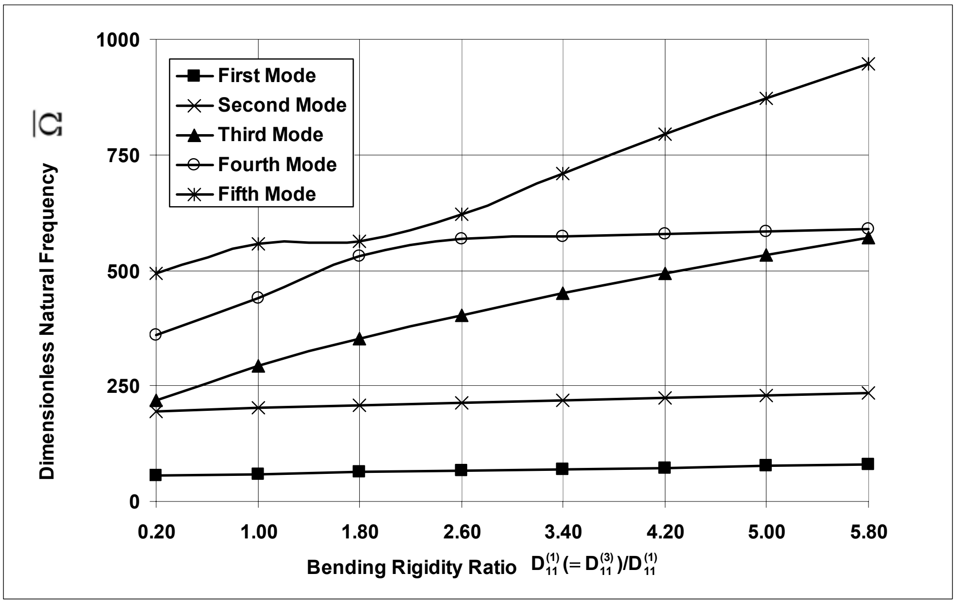
b) "Various Modes with (FFCFFCFF) B.C.'s, ''Hard'' Adhesive

Fig 8.183 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFCFFCFF)



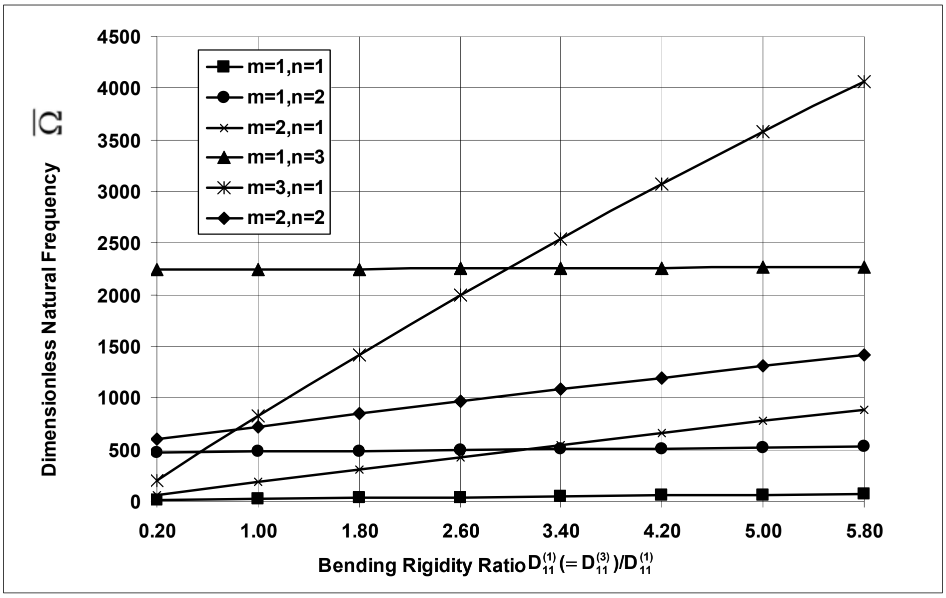
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s, ''<u>Soft</u>'' Adhesive



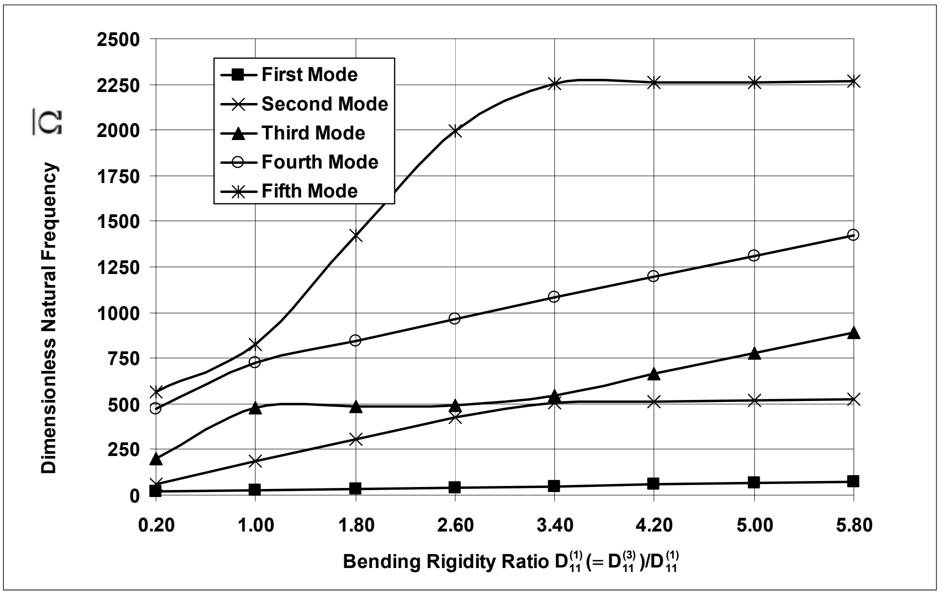
b) "Various Modes with (FFCFFCFF) B.C.'s, ''<u>Soft</u>'' Adhesive

Fig 8.184 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)}\left(=D_{11}^{(3)}\right)/D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFCFFCFF)



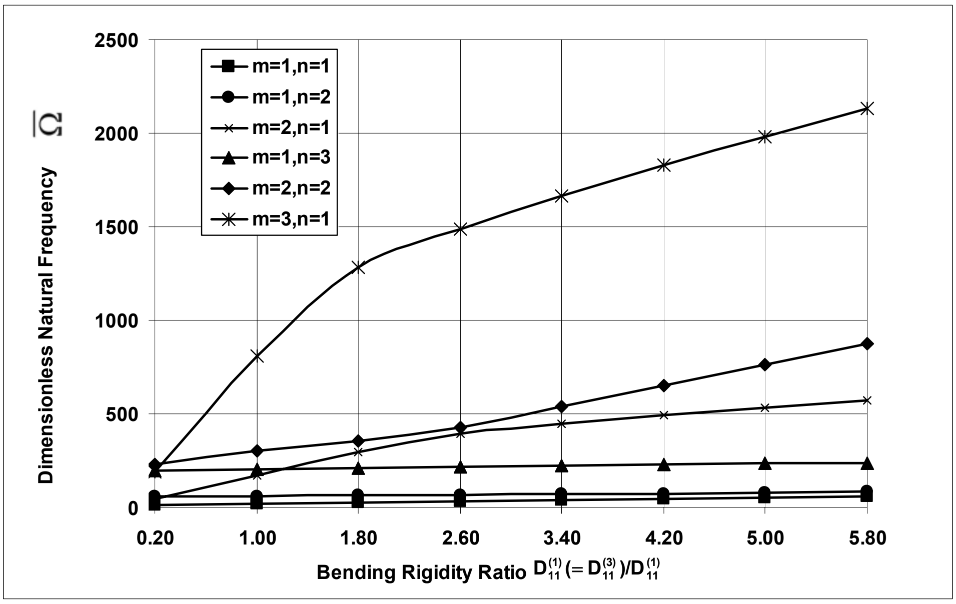
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, "Hard" Adhesive



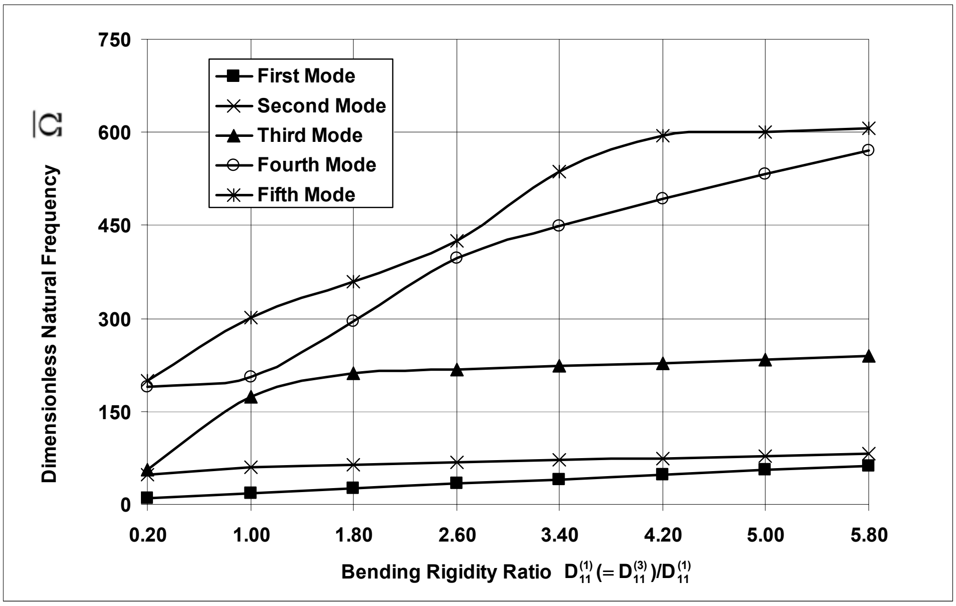
b) "Various Modes with (FFCFFFFF) B.C.'s, "Hard" Adhesive

Fig 8.185 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFCFFFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s, ''Soft'' Adhesive

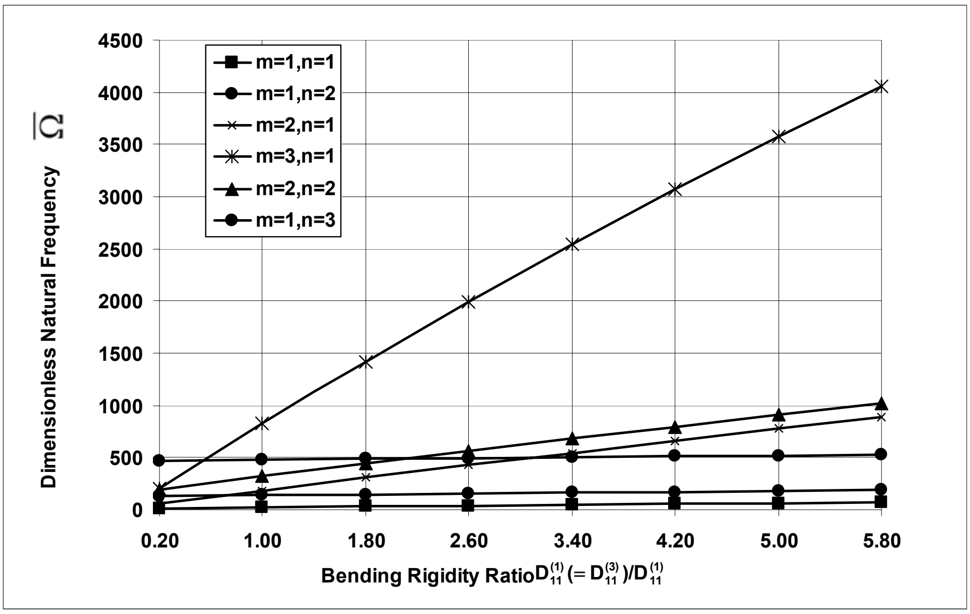


b) "Various Modes with (FFCFFFFF) B.C.'s, ''Soft'' Adhesive

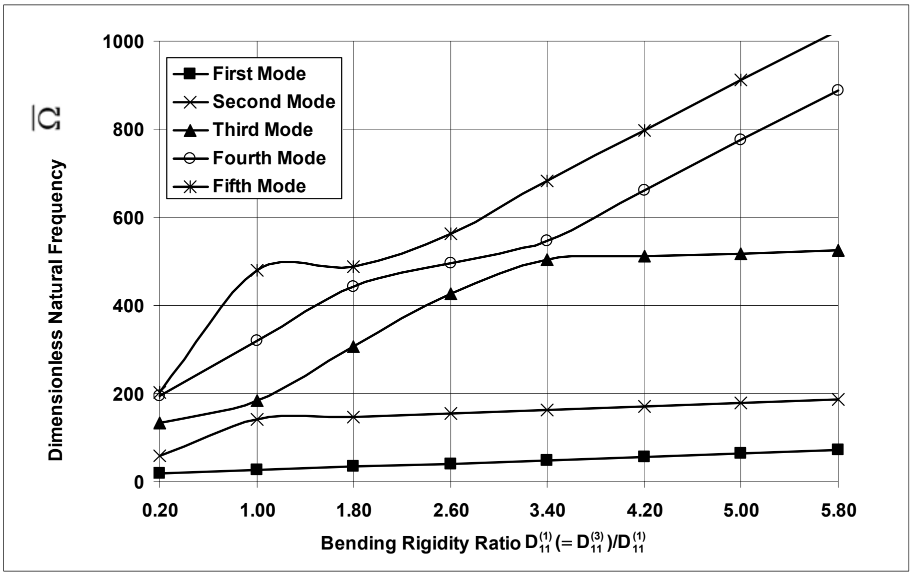
Fig 8.186 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$ " in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\widetilde{b}$ =0.4 m, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFCFFFFF)



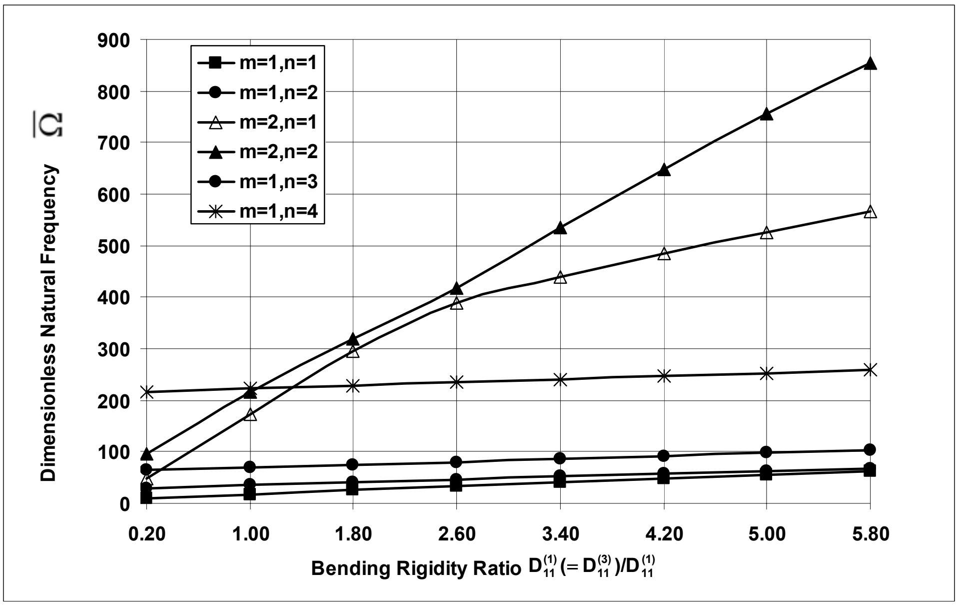
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive



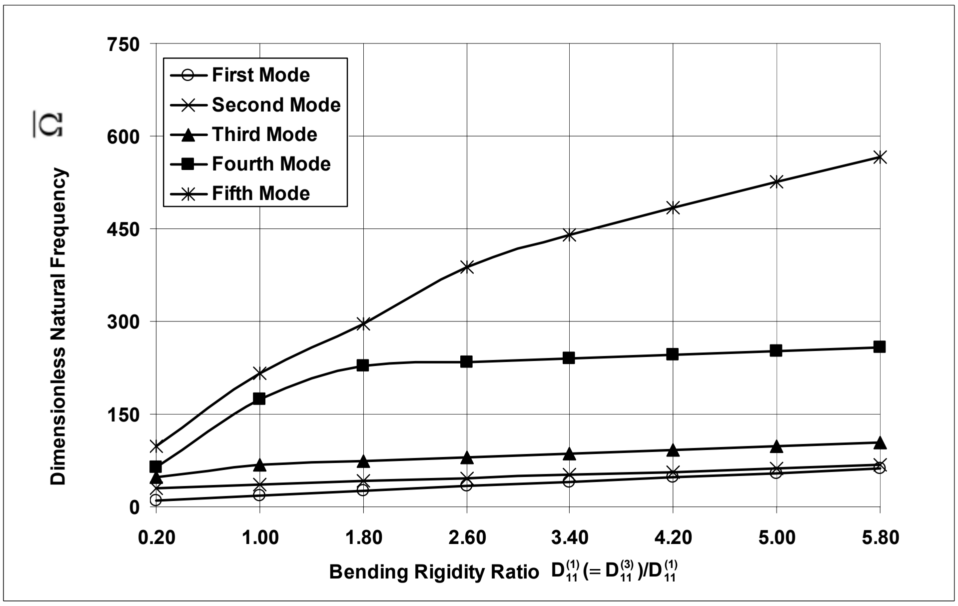
b) "Various Modes with (FFFFFFFF) B.C.'s, ''<u>Hard</u>'' Adhesive

Fig 8.187 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)
(Joint Length ($\ell_I + \ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFFFFFFF)



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s, ''Soft'' Adhesive

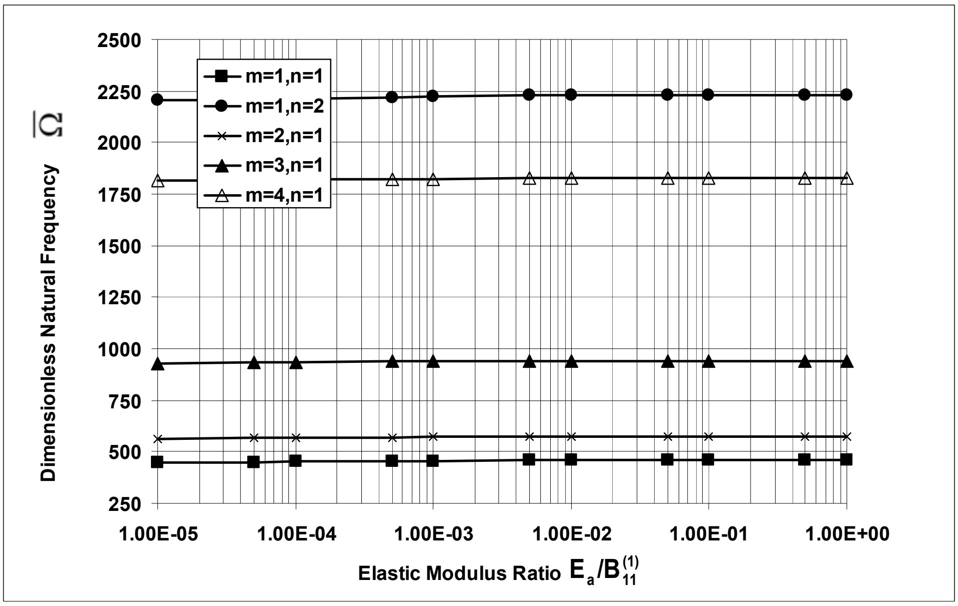


b) "Various Modes with (FFFFFFFF) B.C.'s, ''Soft'' Adhesive

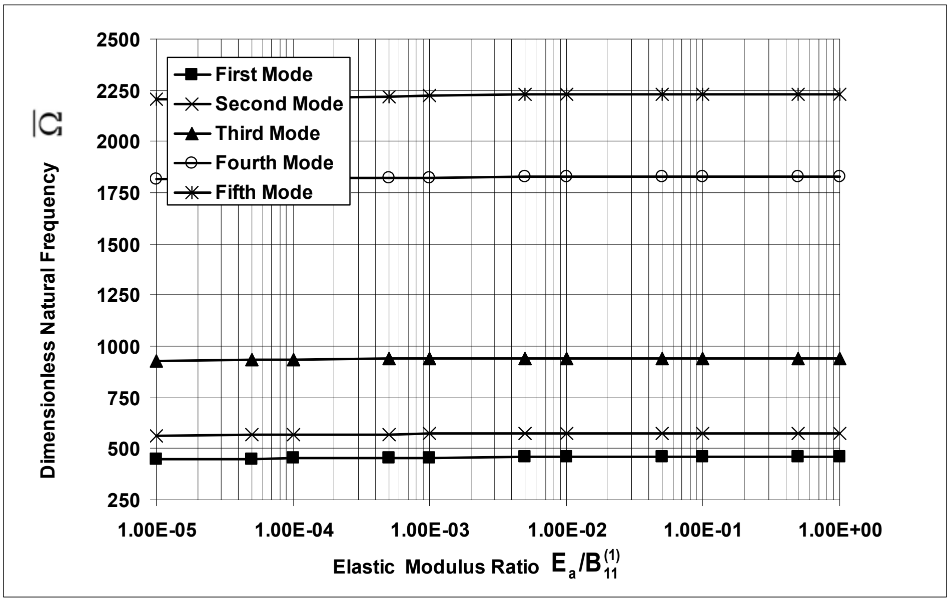
Fig 8.188 "Dimensionless Natural Frequencies ($\overline{\Omega}$)" versus "Bending Rigidity Ratio $D_{11}^{(2)} \left(=D_{11}^{(3)}\right)/ D_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (or Symmetric Double Doubler Joint)"

($D_{11}^{(2)}$ and $D_{11}^{(3)}$ increase while other stiffness constants are kept constant)

(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1m)
(Boundary Conditions in y-direction FFFFFFFF)



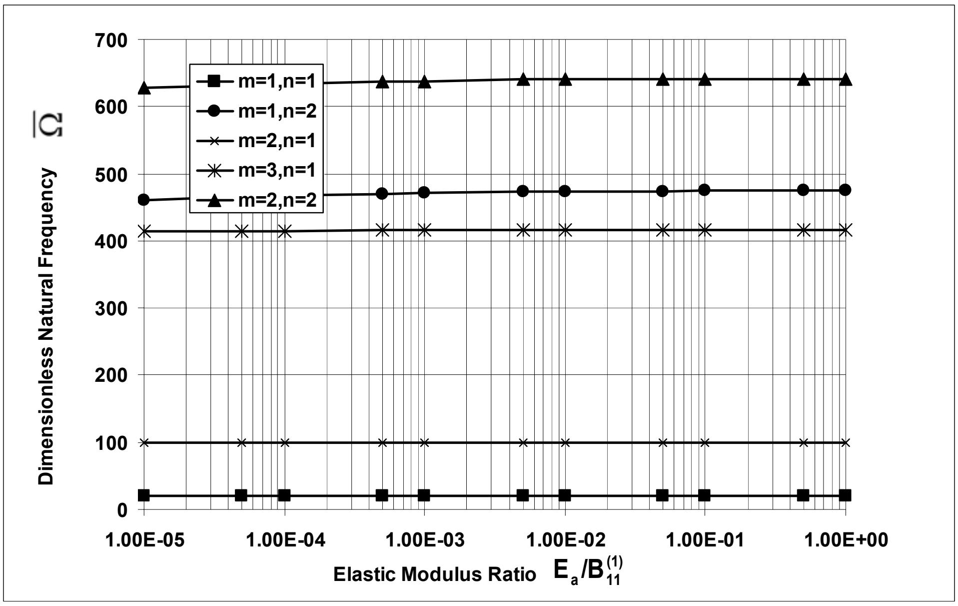
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s



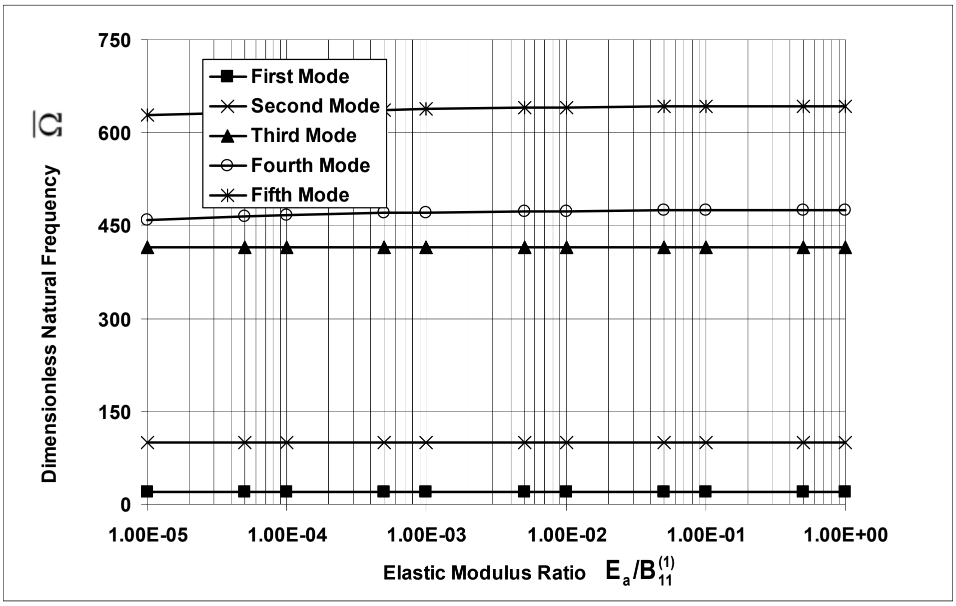
b) "Various Modes with (FFCFFCFF) B.C.'s

Fig 8.189 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_{a1}(=E_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)
Elastic Modulus Ratio axis is plotted in Log Scale



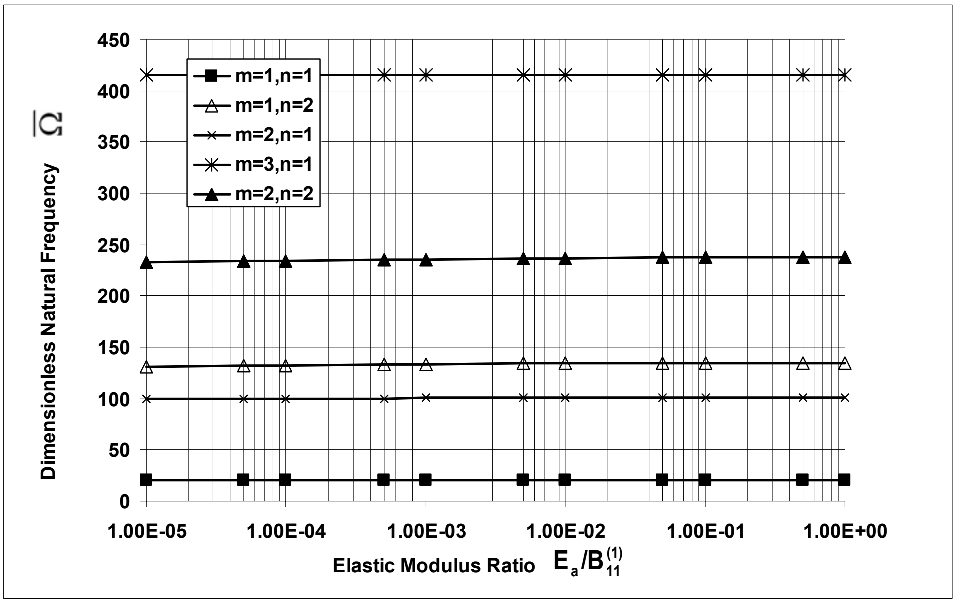
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s



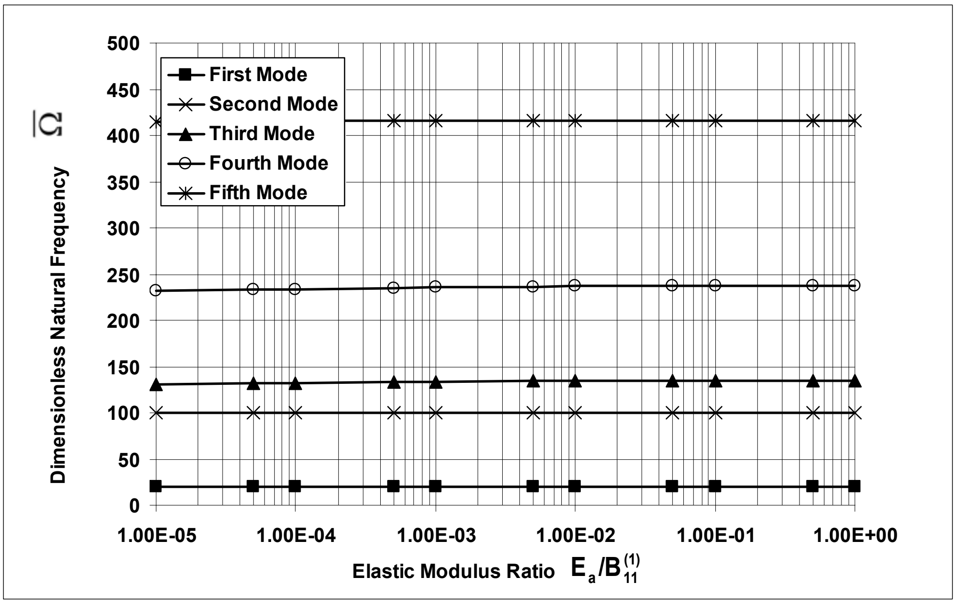
b) "Various Modes with (FFCFFFFF) B.C.'s

Fig 8.190 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_{a1}(=E_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



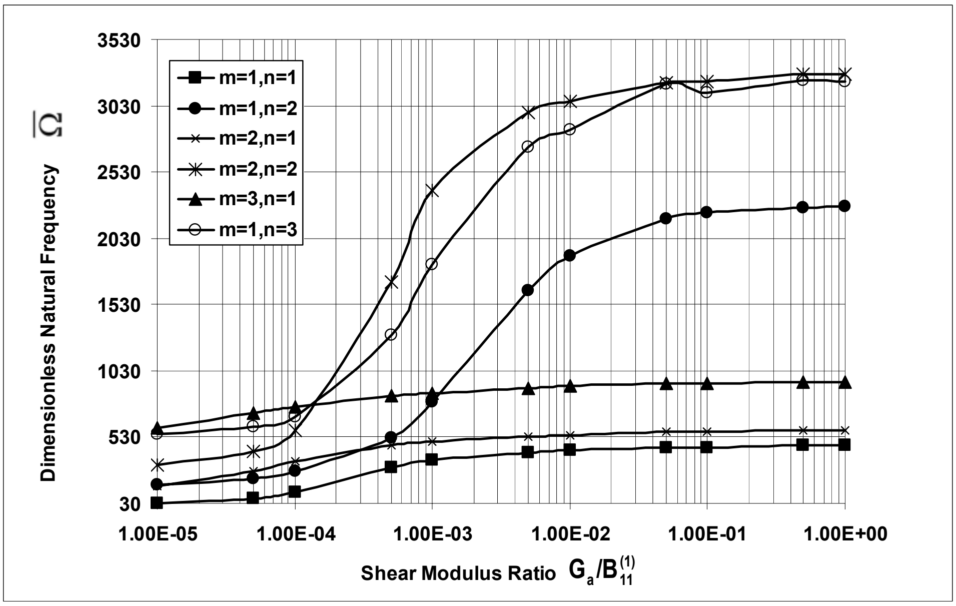
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s



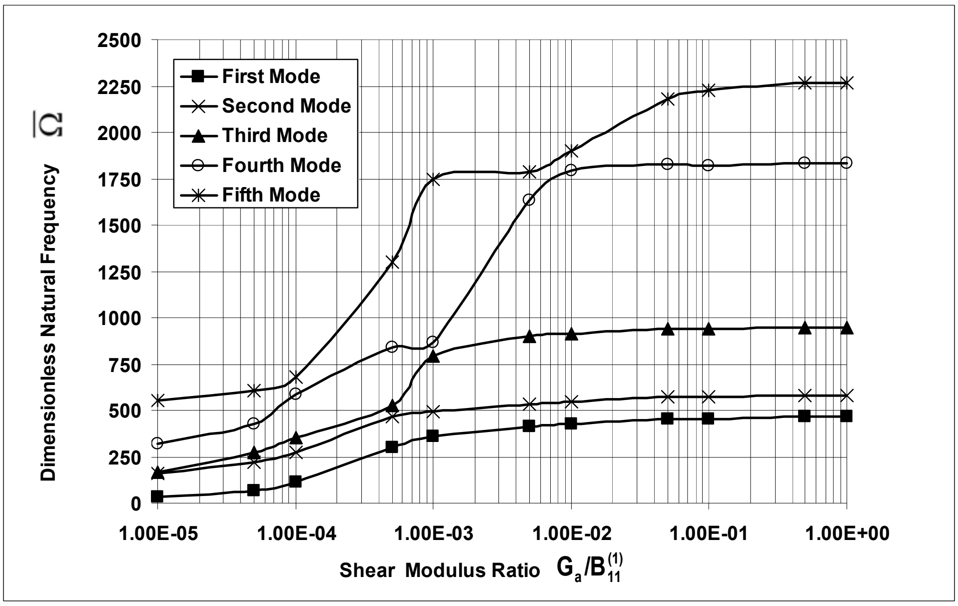
b) "Various Modes with (FFFFFFFF) B.C.'s

Fig 8.191 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Elastic Modulus Ratio $E_{a1}(=E_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)
Elastic Modulus Ratio axis is plotted in Log Scale



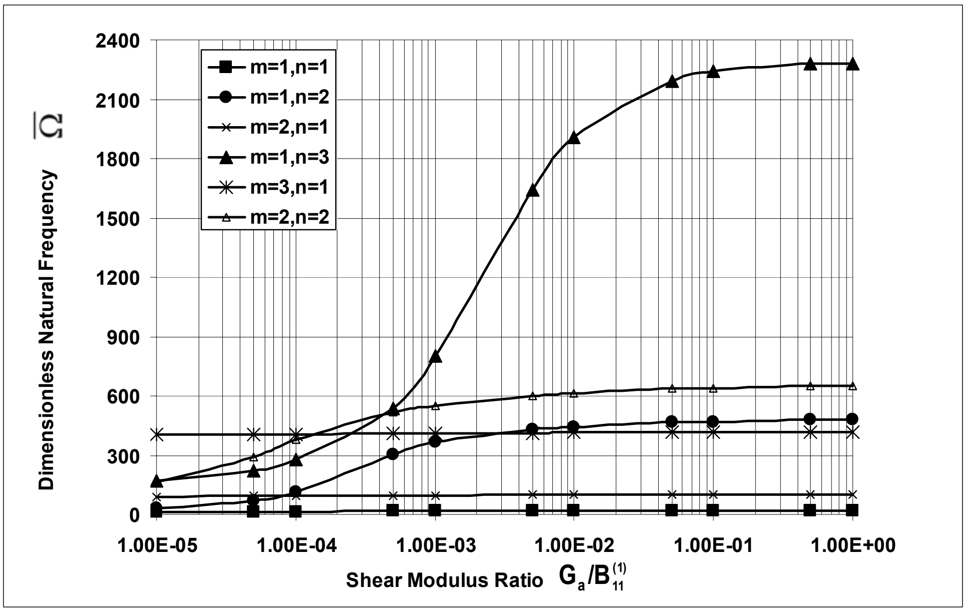
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFCFF) B.C.'s



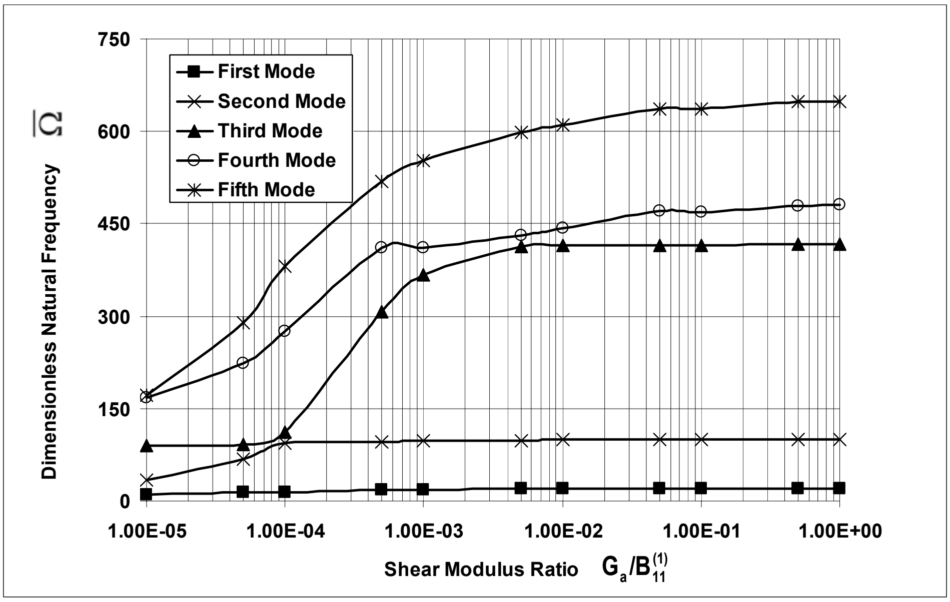
b) "Various Modes with (FFCFFCFF) B.C.'s

Fig 8.192 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_{a1}(=G_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFCFF)
Shear Modulus Ratio axis is plotted in Log Scale



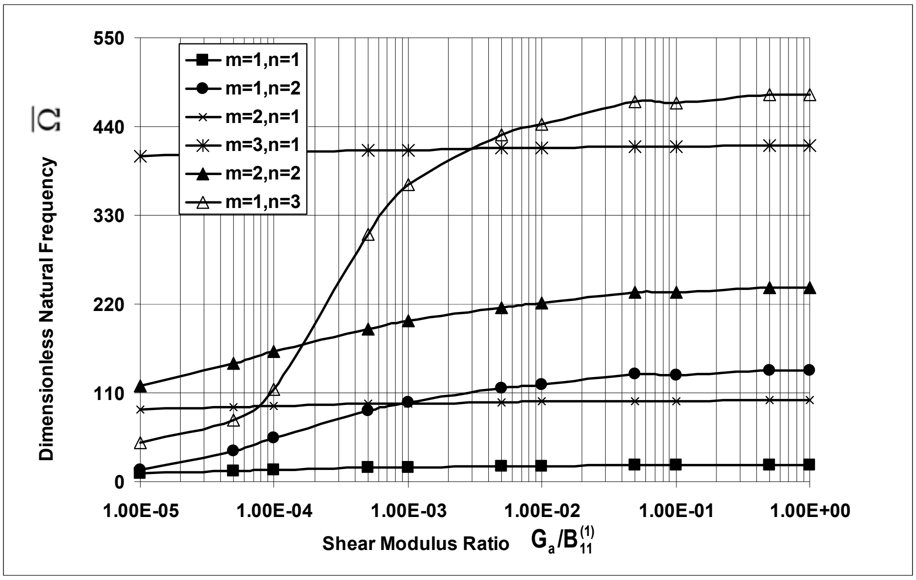
a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFCFFFFF) B.C.'s



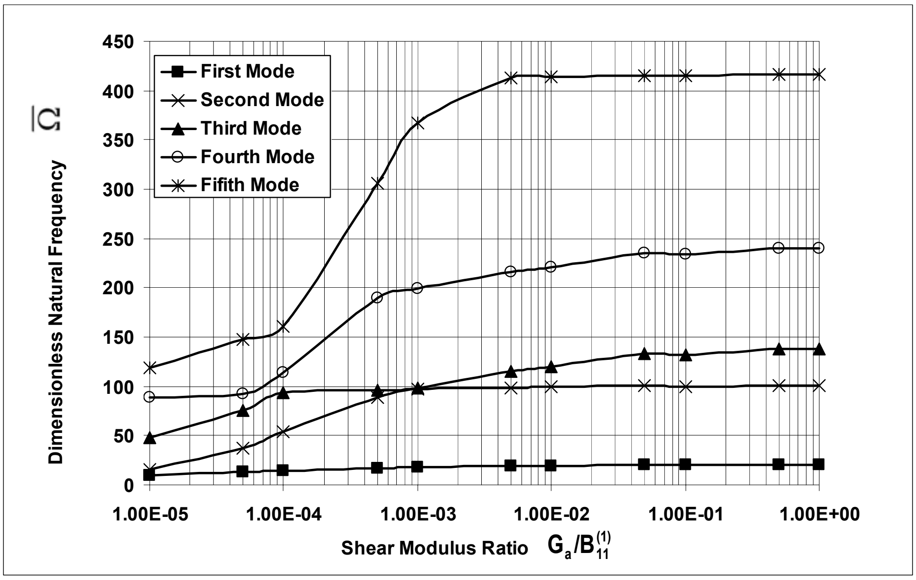
b) "Various Modes with (FFCFFFFF) B.C.'s

Fig 8.193 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_{a1}(=G_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFCFFFFF)
Shear Modulus Ratio axis is plotted in Log Scale



a) Dependency of natural frequency on the number of half waves in y- and x-direction with (FFFFFFFF) B.C.'s



b) "Various Modes with (FFFFFFFF) B.C.'s

Fig 8.194 "Dimensionless Nat. Freq's. ($\overline{\Omega}$)" versus "Shear Modulus Ratio $G_{a1}(=G_{a4})/B_{11}^{(1)}$" in "Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)"

(Plate 1=Plate 4=Graphite-Epoxy, Plate 2= Kevlar-Epoxy, Plate 3= Kevlar-Epoxy)
(Joint Length ($\ell_I+\ell_{II}$)=0.3m, $\tilde{b}$ =0.4 m, a=0.5 m. L=1 m)
(Boundary Conditions in y-direction FFFFFFFF)
Shear Modulus Ratio axis is plotted in Log Scale

8.7.4 Influences of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequencies"

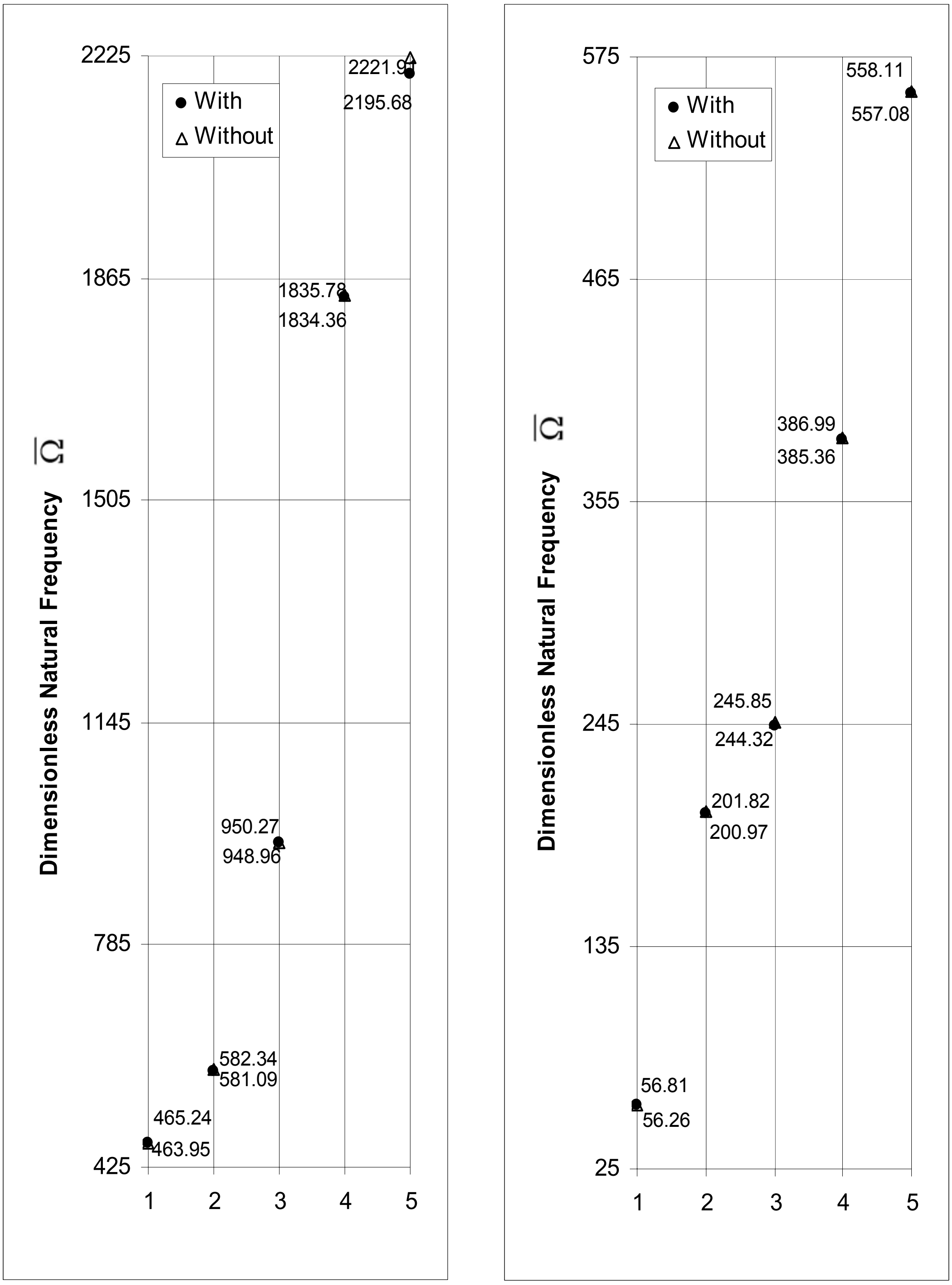
Table 8.9 Comparison of "Dimensionless Natural Frequencies" obtained by adding $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ terms to adhesive layer equations for "Main PROBLEM IIIb"

a) "Hard" Adhesive Case

Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	465.240	463.952	1.287
	2	582.342	581.094	1.249
	3	950.271	948.958	1.313
	4	1835.778	1834.358	1.419
	5	2221.909	2195.680	26.229
SFFS	1	240.558	239.941	0.617
	2	353.284	352.640	0.644
	3	719.927	719.203	0.724
	4	1604.417	1603.578	0.839
	5	1747.963	1733.026	14.936
CFFF	1	20.507	20.474	0.032
	2	100.675	100.629	0.046
	3	416.190	416.120	0.070
	4	480.790	479.445	1.345
	5	649.248	647.892	1.356
FFFF	1	20.497	20.465	0.032
	2	100.660	100.614	0.046
	3	138.489	137.930	0.558
	4	241.365	240.732	0.634
	5	416.170	416.100	0.070

b) "Soft" Adhesive Case

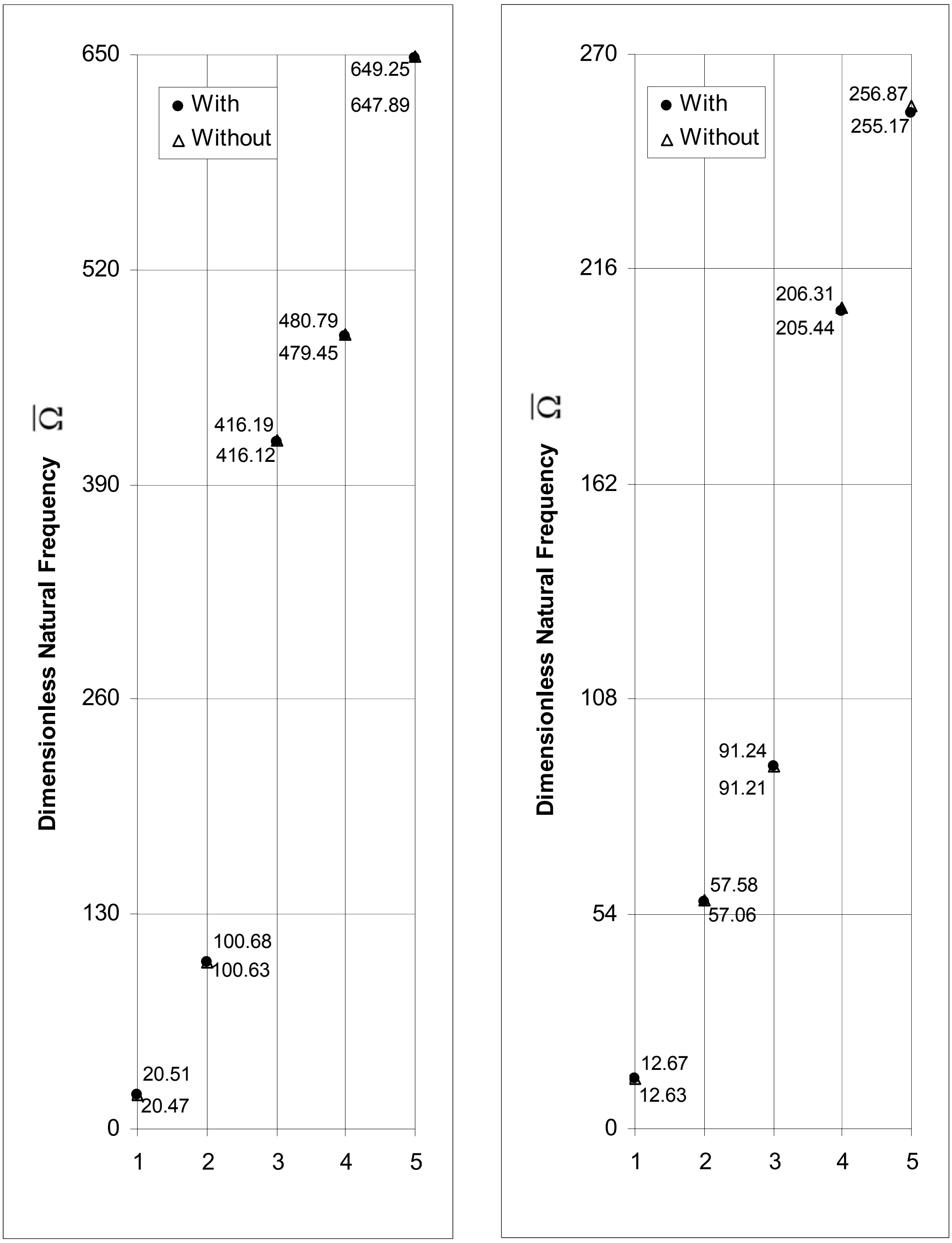
Boundary Condition		without $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	with $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$	Difference
CFFC	1	56.806	56.262	0.544
	2	201.818	200.966	0.852
	3	245.848	244.320	1.528
	4	386.993	385.357	1.636
	5	558.110	557.076	1.034
SFFS	1	43.382	42.914	0.467
	2	116.911	116.324	0.587
	3	199.325	198.564	0.761
	4	301.247	299.401	1.846
	5	358.402	357.365	1.038
CFFF	1	12.669	12.634	0.034
	2	57.580	57.056	0.524
	3	91.244	91.213	0.031
	4	206.305	205.441	0.865
	5	256.867	255.170	1.697
FFFF	1	12.564	12.528	0.036
	2	31.638	31.342	0.296
	3	66.172	65.722	0.450
	4	91.166	91.134	0.032
	5	138.365	138.086	0.279



a) "Hard" Adhesive Case **b) "Soft" Adhesive Case**

Figure 195 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on "Dimensionless Natural Frequency $\overline{\Omega}$" in "Composite, Orthotropic Plates and/or Panels with a <u>Non-Centrally</u> Bonded Symmetric Double Lap Joint"

(Boundary Conditions in y-direction FFCFFCFF)



a) “Hard” Adhesive Case

b) “Soft” Adhesive Case

Figure 196 Influence of $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ on “Dimensionless Natural Frequency $\overline{\Omega}$ ” in “Composite, Orthotropic Plates and/or Panels with a Non-Centrally Bonded Symmetric Double Lap Joint” (Boundary Conditions in y-direction FFCFFFFF)

CHAPTER 9

CONLUSIONS

9.1 Conclusions for "Main PROBLEM I.a" and "Main PROBLEM I.b"

The analytical modeling for the **''Free Flexural (Or Bending) Vibrations of Composite Orthotropic Mindlin Plates with a Bonded Single Lap Joint" (Main PROBLEM I)"** is based on the Mindlin Plate Theory... The solution technique employed is the "Modified Transfer Matrix Method (MTMM)" which is a combination of the ''Classical Levy's Method", the "Integrating Matrix Method" and the "Transfer Matrix Method". From the numerical results given in Chapter 8, the following conclusions are obtained:

- The analytical formulation of the problem in the "state vector" form and the present solution procedure (i.e. "Modified Transfer Matrix Method") is fairly general, and very efficient and can be extended to other composite plate bonded joint systems.

- "Classical Lévy's Solutions" were used in the formulation of the problem. Therefore, the only limitation in the solution technique is that the plate adherends are to be simply supported at two opposite edges, (i.e. the edges at x=0 and x=a), while other edges, (that is, the edges at y=0 and y=L), may have arbitrary boundary conditions.

- The mode shapes and the corresponding natural frequencies of the composite bonded plate or panel system are significantly affected by the "hardness" and the "softness" (i.e. elastic constants) of the adhesive layers. It is important to note here in the "hard" adhesive case, the "overlap region" and the left side plate adherend are almost stationary at least up to the fourth mode. In the "soft" adhesive case, however, the mode shapes and the natural frequencies are very much different and there is no almost stationary region in the left plate adherend.

- The position (Central or Non-Central) as well as the length (or wet area) of the "single lap joint" have drastic influences on the modes and the natural frequencies of the entire single lap joint system.

- Due to the transverse shear deformable nature of the Mindlin Plates, the adhesive elastic constant G_a, rather the E_a, affects the natural frequencies.

- As the constraining or the stiffenning effect of the boundary conditions in the y-direction increases so do the natural frequencies of the composite bonded plate or panel system.

9.2 Conclusions for "Main PROBLEM II.a" and "Main PROBLEM II.b"

The analytical modeling for the **"Free Flexural (or Bending) Vibrations of Composite Orthotropic Mindlin Plates or Panels with a Bonded Symmetric Single Lap Joint (Symmetric Doubler Joint)" (Main PROBLEM II)"** is based on the Mindlin's Plate Theory. The solution technique employed is the "Modified Transfer Matrix Method (MTMM)" which is a combination of the ''Classical Levy's Method", the "Integrating Matrix Method" and the "Transfer Matrix Method". From the numerical results given in Chapter 8, the following conclusions can be stated;

- The analytical formulation of the problem in the "state vector" form and the present solution procedure (i.e. "Modified Transfer Matrix Method") is fairly general, and very efficient. and can be extended to other composite plate bonded joint systems.

- "Classical Lévy's Solutions" were used in the formulation of the problem. Therefore, the only limitation in the solution technique is that the plate adherends are to be simply supported at two opposite edges, (i.e. the edges at x=0 and x=a), while other edges, (that is, the edges at y=0 and y=L), may have arbitrary boundary conditions.

- The mode shapes and the corresponding natural frequencies of the composite plate or panel system are significantly affected by the "hardness" and the "softness" (i.e. elastic constants) of the adhesive layers. The mode shapes in the "hard" adhesive cases, have an almost stationary area corresponding to the "Symmetric Double Lap Joint" region (in higher modes this may not necessarily be the case). In the "soft" adhesive cases in there is generally no almost stationary area in the mode shapes.

- The position (Central or Non-Central) and also the length of the "Symmetric Double Lap Joint" have serious effects on the mode shapes and the natural frequencies of the entire plate or panel system.

- The adhesive layer shear modulus G_a, rather than the elastic modulus E_a, affects the natural frequencies of the bonded plate or panel system.

- As the constraining or the stiffenning effect of the boundary conditions in the y-direction increases so do the natural frequencies of the composite plate or panel system.

9.3 Conclusions for "Main PROBLEM III.a" and "Main PROBLEM III.b"

The analytical formulation for the **"Free Flexural (or Bending) Vibrations of Composite Orthotropic Mindlin Plates and/or Panels with a Bonded Symmetric Double Lap Joint (Symmetric Double Doubler Joint)" (Main PROBLEM III)"** is based on the Mindlin Plate Theory. The solution technique employed is the "Modified Transfer Matrix Method (MTMM)" which is a combination of the ''Classical Levy's Method", the "Integrating Matrix Method" and the "Transfer Matrix Method". From the numerical results given in Chapter 8, one may state the following conclusions;

- The analytical formulation of the problem in the "state vector" form and the present solution procedure (i.e. "Modified Transfer Matrix Method (MTMM)") is fairly general, very efficient. and can be extended to other composite plate systems.

- "Classical Lévy's Solutions" were used in the formulation of the problem. Therefore, the only limitation in the solution technique is that the plate adherends are to be simply supported at two opposite edges, (i.e. the edges at x=0 and x=a), while other edges, (that is, the edges at y=0 and y=L), may have arbitrary boundary conditions.

- The mode shapes and the corresponding natural frequencies of the composite plate or panel system are significantly affected by the "hardness" and the "softness" (i.e. elastic constants) of the adhesive layers. The mode shapes in the "hard" adhesive cases, have an almost stationary area corresponding to the "Double Doubler Lap Joint" region (in higher modes this may not necessarily be the case). In the "soft" adhesive cases in there is generally no almost stationary area in the mode shapes.

- The location (Central, Non-Central) and also the length (or wet area) of the "Symmetric Double Doubler Joint" have considerable effect on the mode shapes and the natural frequencies of the entire bonded plate or panel system.

- The shear modulus G_a, rather than the elastic modulus E_a, influences the mode shapes and the natural frequencies of the entire bonded plate or panel system.

- As the constraining or the stiffenning effect of the boundary conditions in the y-direction increases so do the natural frequencies of the composite plate or panel system.

CHAPTER 10

RECOMMENDATIONS FOR FUTURE STUDY

In this section, some recommendations for future research will be considered. Various types of vibration problems may be solved for the composite bonded plate system with the present technique (which is the "Modified Transfers Matrix Method (MTMM)"). In the future the following problems may be solved by either extension of the present method or by some modifications in the method of solution.

- The damping effects in the plate system may be considered in terms of complex damping by employing some modifications in the present method of solution and the numerical procedures. Also, the adhesive layers in between the plate adherends and the doublers may be considered as viscoelastic layers with damping characteristics

- The elastic support conditions in terms of mechanical springs, that is, torsional and/or extensional springs, can be considered in the boundary conditions in the y-direction.

- Considering the supports in the y-direction stationary, one can obtain higher order "state vectors", and consequently increase the number of governing system of equations by adding u and v displacements and the membrane stress resultants. In this case, the solution procedure needs some modifications.

- The plate adherends may be replaced by a multi-layer composite plate. Then, the present solution technique will need considerable changes.

- For “Symmetric Single Doubler” or “Symmetric Double Doubler” cases, slightly curved plates or shallow shell panels may be considered instead of rectangular plates.

- The non-linear governing system of equations by using non-linear ‘‘strain-dislacement’’ and ‘‘strain-stress’’ relations (or Hooke’s Law) may be developed and investigated.

- The anisotropic plate adherends and the doublers which have arbitrary directions of orthotropy, can be analyzed.

- The extension of problems here into forced vibrations, flutter problems, etc., may be considered.

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