SOLUTION OF HELMHOLTZ-TYPE EQUATIONS BY DIFFERENTIAL QUADRATURE METHOD

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SOLUTION OF HELMHOLTZ-TYPE EQUATIONS BY DIFFERENTIAL QUADRATURE METHOD

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ABSTRACT

SOLUTION OF HELMHOLTZ-TYPE EQUATIONS BY DIFFERENTIAL QUADRATURE METHOD

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This thesis presents the Differential Quadrature Method (DQM) for solving Helmholtz, modified Helmholtz and Helmholtz eigenvalue-eigenvector equations. The equations are discretized by using Polynomial-based and Fourier expansion-based differential quadrature technique which use basically polynomial interpolation for the solution of differential equations.

The procedure is applied to several problems which are governed with Helmholtz or modified Helmholtz equations together with Dirichlet and/or Neumann type boundary conditions. Magnetohydrodynamic flow problem in a rectangular channel is also solved by reducing the coupled differential equations into two modified Helmholtz equations and then applying DQ method.

Solutions are presented in terms of graphics comparing with the exact solutions. It is found that Differential Quadrature Method exhibits high accuracy and efficiency with considerably small number of mesh points comparing to the other numerical methods.

DQM can also be used for obtaining eigenvalues of Helmholtz type eigenvalue-eigenvector problems with very high accuracy.

Keywords: Helmholtz equation, modified Helmholtz equation, Helmholtz eigenvalue-eigenvector problem, Differential Quadrature Method.

ÖZ

HELMHOLTZ TİPİ DENKLEMLERİN DİFERANSİYEL KUADRATÜR METODU İLE ÇÖZÜMÜ

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Bu tez, Helmholtz, modifiye edilmiş Helmholtz ve Helmholtz öz değer-öz vektör denklemlerini çözmek için Diferansiyel Kuadratür Metodunu sunmuştur. Denklemler, diferansiyel denklemlerin çözümü için temelde polinom interpolasyonunu kullanan polinoma ve Fourier genişlemesine dayalı diferansiyel kuadratür teknikleri kullanılarak ayrıklaştırılır.

Bu yöntem Dirichlet ve/veya Neumann sınır koşullarna sahip Helmholtz veya modifiye edilmiş Helmholtz denklemleri ile kontrol edilen bir çok probleme uygulanmıştır. Ayrıca magnetohidrodinamik kanal problemini de tanımlayan, diferansiyel denklem sistemi, modifiye edilmiş iki Helmholtz denklemine indirgendikten sonra Diferansiyel Kuadratür Metodu uygulanarak çözülmüştür.

Çözümler analitik çözümlerle karşılaştırılarak grafikler ile sunulmuştur. Diğer nümerik metodlarla karşılaştırıldığında, Diferansiyel Kuadratür Metodunun, oldukça az sayıda nokta ile yüksek doğruluk ve etkinlik gösterdiği bulunmuştur.

Diferansiyel Kuadratür Metodu aynı zamanda Helmholtz tipi öz değer-öz vektör problemlerinde, öz değerleri çok yüksek doğrulukla elde etmek için kullanılabilmektedir.

Anahtar Kelimeler : Helmholtz denklemi, modifiye edilmiş Helmholtz denklemi, Helmholtz öz değer-öz vektör problemi, Diferansiyel Kuadratür Metodu.

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Finally deepest regards and love to my parents and to my family, who have provided me the love, support and educational background that has enabled me to be successful both in my personal and academic life.

I would like to dedicate this thesis to my family.

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CHAPTER 1

INTRODUCTION

The Differential Quadrature Method (DQM) is a numerical solution technique for initial and/or boundary value problems. It was developed by the late Richard Bellmann and his associates in the early 70's and since then, the method has been successfully employed in a variety of problems in engineering and physical sciences. The method has been projected by its proponents as a potential alternative to the conventional numerical solution techniques such as the finite difference and finite element methods (Bert and Malik (1996)).

The DQ method, akin to the convential integral quadrature method, approximates the derivative of a function at any location by a linear summation of all the functional values along a mesh (grid) line. The key procedure in the DQ application lies in the determination of the weighting coefficients. The DQ method and its applications were rapidly developed after the late 1980's, thanks to the innovative work in the computation of the weighting coefficients by other researchers. As a result, the DQ method has emerged as a powerful numerical discretization tool in the past decade. (Shu and Richards (1990), Shu (2000)).

In 1996, Bert and Malik presented a comprehensive review of the chronological development and the application of the DQ method up to date. There is no abundant book which systematically describes both the theoretical analysis and the application of the DQ method. The DQ method was mentioned for the first time in a book written by Bellman and Roth in 1986. There are many innovative ideas contained in this book. The textbook of Shu (2000) represents the first comprehensive work on the DQ method and applications. Since there are many achievements in the DQ method, the number of reference books on the DQ method and its applications will increase.

In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Bellman (1971, 1972) introduced the method of DQ where a partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all grid points along that direction. Bellman (1972) suggested two methods to determine the weighting coefficients of the first order derivative. The first method solves an algebraic equation system. The second uses a simple algebraic formulations, but with the coordinates of grid points chosen as the roots of the shifted Legendre Polynomials. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill-conditioned. Thus, it is difficult to obtain the weighting coefficients for a large number of grid points using this method.

To further improve the computation of weighting coefficients, Quann and Chang(1989 a, b) applied Lagrange interpolated polynomials as test functions and obtained explicit formulations to calculate the weighting coefficients for the discretization of the first and second order derivatives.

Shu and Richards (1990), and Shu (1991), generalized all the current methods for determination of the weighting coefficients under the analysis of a high order polynomial approximation and the analysis of a linear vector space. The weighting coefficients of the first order derivative are determined by a simple algebraic formulation whereas the weighting coefficients of the second and higher order derivatives are determined by a recurrence relationship.

There are two versions of DQ method. One is based on the high order polynomial approximation, which is regarded as polynomial-based differential quadrature (PDQ) approach, the other is based on the Fourier series expansion, which is noted as Fourier expansion-based differential quadrature (FDQ) approach. The PDQ approach is the original DQ method. The FDQ approach was developed by Shu and Chew (1997), and Xue (1997). In the development of FDQ, a linear vector space containing trigonometric functions analysis is also employed. From these processes, it is evident that the mathematical fundamentals of PDQ and FDQ lie in the analysis of a linear vector space and the analysis of a function approximation.

Shu's (2000) general DQ approach depends on the approximation of the solution of a partial differential equation (PDE) by a polynomial of high degree. If the degree of the approximated polynomial is N-1 then it constitutes an

N-dimensional linear vector space V_N with the operation of vector addition and scalar multiplication, and can be expressed in different forms. Base polynomials in this approximation can be monomials x^{k-1},

$$f(x) = \sum_{k=1}^{N} c_k x^{k-1}$$

or Lagrange interpolated polynomials $r_k(x)$,

$$f(x) = \sum_{k=1}^{N} r_k(x) f(x_k)$$

for a distributed grid points x_k, $k = 1, ..., N$.

If one set of base polynomials satisfies a linear operator so does another set of base polynomials according to the property of a linear vector space.

So, the coefficients in the derivatives of the approximated function $f(x)$ can be easily obtained with the help of these two sets of base polynomials.

Similarly, in Fourier expansion-based differential quadrature two sets of base polynomials are used in approximating the solution of a (PDE). One set of base polynomials is

$$1, \sin \pi x, \cos \pi x, \sin 2\pi x, ..., \sin(\frac{N\pi x}{2}), \cos(\frac{N\pi x}{2})$$

and the other set of base polynomials are Lagrange interpolated base polynomials in terms of these trigonometric polynomials.

The one-dimensional PDQ and FDQ formulations can be directly extended to the multi-dimensional case if the discretization domain is regular (Shu (1991)). For the irregular physical domain, one has to first perform a coordinate transformation to map the irregular physical domain into a regular computational domain. Imposition of boundary conditions and ordering of the unknowns in the final algebraic system is also important for obtaining a non-singular coefficient matrix. Weighting coefficient matrices can also be modified for the imposition of higher order derivative boundary conditions (Fung (2003)).

Applications of DQ method may be found in the available literature include biosciences, transport processes fluid mechanics, static and dynamic structural mechanics, acoustic, waveguide analysis, static aeroelasticity and lubrication mechanics. It has been claimed that the DQ method has the capability of producing highly accurate solutions with minimal computational effort (Bert and Malik (1996)). Acoustic waves and microwaves can be simulated by the Helmholtz equation.

The Helmholtz and modified Helmholtz equation can be solved numerically by using several methods such as finite difference method, finite element method and boundary element method (BEM). Among all the methods, the finite element method is extensively used to obtain the solution of Helmholtz equation. It is well known that the numerical phase accuracy of the finite element solutions deteriorate rapidly as the wave number is increased. This is due to the use of low order polynomial approximation to highly oscillatory wave propagation solutions (Shu and Xue, 1999).

Chang (1990) uses a least-squares finite element method for the numerical solution of the Helmholtz equation with the homogenous Dirichlet boundary conditions. He converted the Helmholtz equation into a system of first order equations by identifying the derivatives of the solution as additional unknowns. It was possible to achieve $O(h^2)$ accuracy when using the C^0-piecewise linear vector functions over a triangulation.

Harari and Hughes (1991) presented finite element method for the radiation problem governed by the Helmholtz equation in an exterior domain. Exterior boundary conditions for the computational problem over a finite domain were derived from an exact relation between the solution and its derivatives on that boundary. Galerkin, Galerkin/ least squares and Galerkin/ gradient least squares finite element methods were evaluated by comparing errors pointwise and in integral norms. The Galerkin/ least squares method was shown to exhibit superior behavior for this class of problems.

El-Sayed, and Kaya (2004) implemented a relatively new numerical technique, A domain's decomposition method for solving the linear Helmholtz partial differential equations. They demonstrated that the new method is quite accurate and rapidly implemented than the finite-difference method.

Harris (1992) considered a numerical method for solving the exterior Helmholtz problem using the boundary integral formulation derived from

Green's theorem. Computational difficulties in evaluating singular integrals were overcome by employing special quadrature rules and the numerical results showed that it is possible to obtain accurate numerical solution to a wide range of problems using this scheme.

Chen, et al (2002) derived the dual integral formulation for the modified Helmholtz equation in solving the propagation of oblique incident wave passing a thin barrier (a degenerate boundary). Fundamental solution of Laplace equation was made use of since it is dual reciprocity BEM procedure and improper integrals are converted to regular integrals by using Gaussian quadrature rules. A dual boundary element method program was developed to solve the water scattering problem passing a barrier.

As compared to the conventional low order finite difference and finite element methods, the DQ method can obtain very accurate numerical results using a considerably smaller number of grid points and hence requiring relatively little computational effort.

Shu and Xue (1999) applied the polynomial-based differential quadrature and the Fourier expansion-based differential quadrature methods to solve the two dimensional Helmholtz equation. They demonstrated through some sample example that the accurate numerical solution can be obtained by using only 2 to 3 mesh points per wavelenght. It was found that the FDQ approach can generally obtain more accurate numerical results than the PDQ approach.

Shu (2000,a) presented a new approach for elliptical waveguide analysis. This approach applies the global method of a Differential Quadrature to discretize the Helmholtz equation and then reduces it into an eigenvalue equation system. It is demonstrated that the DQ results are in excellent agreement with theoretical values using just a few grid points and, thus requiring very small computational effort, that is DQ is a very efficient method for elliptical waveguide analysis.

In this thesis, DQM is applied to the Helmholtz and modified Helmholtz equations in two-dimensional space. The equations are supplemented with Dirichlet or Dirichlet-Neumann type boundary conditions to form a boundary value problem. The DQM is demonstrated for the one-dimensional case and then extended to two-dimensions in the applications. Differential Quadrature

Method approximates the derivatives in the partial differential equations by a linear weighted summation of all the functional values (unknown function values).

As a result, a linear algebraic system of equations is obtained and solved by using LU factorization. Application of modified Helmholtz equation is a magnetohydrodynamic (MHD) channel flow problem which is a actually a system of coupled partial differential equations in terms of velocity and the magnetic field. Thus, the solution of this problem implements the solution of an important physical problem.

Finally, Helmholtz equation is solved as an eigenvalue problem keeping both the function and the constant in the equation as unknowns. Therefore, the thesis gives differential quadrature solution of Helmholtz type equations and Helmholtz eigenvalue-eigenvector problems.

1.1 Plan of The Thesis

In Chapter 2, first the theory of the Differential Quadrature Method is given in one dimensional case and then the extension to the two dimensional case and the application to Helmholtz, modified Helmholtz and Helmholtz eigenvalue-eigenvector problems is explained. Implementation of Dirichlet and Neumann type boundary conditions to the final algebraic linear system of equations is also presented. Choice of the grid points is also explained in details.

Chapter 3 presents test problems which are solved by DQ method using PDQ and FDQ approaches. These problems are defined in terms of Helmholtz equations which are well defined boundary value problems. Applications to magnetohydrodynamic flow and Helmholtz eigenvalue-eigenvector problem are also given. MHD fluid flow problem in a rectangular channel is solved by reducing the coupled equations first to the decoupled modified Helmholtz equations and then applying again DQM to these equations.

CHAPTER 2

DIFFERENTIAL QUADRATURE METHOD FOR HELMHOLTZ-MODIFIED HELMHOLTZ EQUATIONS AND EIGENVALUE-EIGENVECTOR PROBLEMS

This Chapter presents the application of Differential Quadrature Method (DQM) to Helmholtz, modified Helmholtz equations and then to Helmholtz type eigenvalue-eigenvector problems.

The two-dimensional Helmholtz equation for a field problem can be written as

$$\nabla^2\phi(x,y) + k^2\phi(x,y) = f(x,y) \qquad (x,y) \in \Omega \tag{2.1}$$

supplemented by one of the boundary conditions,

1. Dirichlet type : $\phi(x,y) = f_1(x,y)$ on $\partial\Omega$
2. Neumann type : $\frac{\partial\phi}{\partial n}(x,y) = f_2(x,y)$ on $\partial\Omega$
3. Mixed type : $\alpha\phi + \beta\frac{\partial\phi}{\partial n} = \gamma$ on $\partial\Omega$

where ∇^2 is the Laplacian operator, given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\phi(x,y)$ and the constant k are potential(unknown) function and the given wave number respectively, f is a given continuous source function, f_1, f_2 are known continuous functions defined on the boundary $\partial\Omega$ of the region Ω, α, β, γ are known constants.

The two-dimensional modified Helmholtz equation is defined as

$$\nabla^2\phi(x,y) - k^2\phi(x,y) = f(x,y) \tag{2.2}$$

and a well defined boundary value problem can be obtained similarly with proper boundary conditions defined above in a two dimensional domain.

In a Helmholtz type eigenvalue-eigenvector problem, the wave number is also unknown and there is no source function

$$\begin{aligned} \nabla^2\phi(x,y) \mp k^2\phi(x,y) = 0 \qquad (x,y) \in \Omega \\ \phi(x,y) = 0 \qquad (x,y) \in \partial\Omega. \end{aligned} \tag{2.3}$$

In this thesis to demonstrate the high efficiency and accuracy of the PDQ and FDQ method, we study the Helmholtz field problems on a rectangular domain. The efficiency and accuracy of PDQ and FDQ approaches are validated by their application to some test problems, which have exact solutions.

2.1 Differential Quadrature Method

The Differential Quadrature Method was presented by R. E. Bellman and his associates in early 1970's and it is a numerical discretization technique for the approximation of derivatives. In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Belmann (1971, 1972) introduced the method of differential quadrature, where a partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all grid points along that direction. The key to DQ method is to determine the weighting coefficients for the discretization of a derivative of any order.

A major breakthrough in computing the weighting coefficients was made by Shu and Richards (1990) in which all the current methods for determination of the weighting coefficients are generalized under the analysis of a high order polynomial approximation and the analysis of a linear vector space. In Shu's approach, the weighting coefficients of the first order derivative are determined by a simple algebraic formulation without any restriction on the choice of grid points, whereas the weighting coefficients of the second and higher order derivatives are determined by a recurrence relationship. Clearly, all the above work is based on the polynomial approximation, and accordingly, the related DQ method can be considered as the polynomial-based differential quadrature (PDQ) method. Recently, Shu and Chew (1997) and Shu and Xue (1997) have developed explicit formulations for computing the weighting coefficients of the first and second order derivatives in the DQ approach when the function or the solution of a partial differential equation (PDE) is approximated by a Fourier series expansion. These formulations are different from PDQ and the approach can be termed as the Fourier expansion-based differential quadrature (FDQ) method.

2.1.1 One Dimensional Polynomial-Based Differential Quadrature Method

For simplicity, the one-dimensional problem is chosen to demonstrate the PDQ Method. When a structured grid is used, the one dimensional results can be directly extended to the multi-dimensional cases and thus to our problem in two dimensions.

Following the idea of integral quadrature, the Differential Quadrature Method approximates the derivative of a smooth function at a grid point by a linear weighted summation of all the functional values in the whole computational domain (Shu, 2000). For example, the first and second order derivatives of a function $u(x)$ at a point x_i are approximated by

$$u_x(x_i) = \frac{du}{dx}|_{x_i} = \sum_{j=1}^{N} a_{ij} u(x_j), \qquad i = 1, 2, ..., N \tag{2.4}$$

$$u_{xx}(x_i) = \frac{du^2}{dx^2}|_{x_i} = \sum_{j=1}^{N} b_{ij} u(x_j), \qquad i = 1, 2, ..., N \tag{2.5}$$

where a_{ij}, b_{ij} are the weighting coefficients and N is the number of grid points in the whole domain. It should be noted that the weighting coefficients a_{ij} (and b_{ij}) are different at different location of x_i since they depend on coordinates of the points. The important procedure in DQ approximation is to determine the weighting coefficients a_{ij} and b_{ij} efficiently.

When the function $u(x)$ is approximated by a high order polynomial, one needs some explicit formulations to compute the weighting coefficients within the scope of a high order polynomial approximation and a linear vector space. In accordance with the Weierstrass polynomial approximation theorem, it is known that the solution of a one dimensional differential equation is approximated by a (N-1)th degree polynomial

$$u(x) = \sum_{k=0}^{N-1} c_k x^k \tag{2.6}$$

where c_k's are constants. The polynomial of degree less than or equal to N-1 constitutes an N-dimensional linear vector space V_N with respect to the operation of vector addition and scalar multiplication.

Obviously, in the linear vector space V_N, a set of vectors (monomials) 1,x,x^2,...,x^{N-1} is linearly independent. Thus,

$$s_k(x) = x^{k-1}, \qquad k = 1, 2, ..., N \tag{2.7}$$

is a basis of V_N.

For the numerical solution of a differential equation, we need to find out the solution at certain discrete points. Now, it is supposed that in a closed interval $[a, b]$, there are N distinct grid points with the coordinates $a = x_1, x_2, ..., x_N = b$, and the functional value at a grid point x_i is $u(x_i)$. Then the constants in equation (2.6) can be determined from the following system of equations

$$\begin{aligned} &c_0 + c_1x_1 + c_2x_1^2 + ... + c_{N-1}x_1^{N-1} = u(x_1) \\ &c_0 + c_1x_2 + c_2x_2^2 + ... + c_{N-1}x_2^{N-1} = u(x_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &c_0 + c_1x_N + c_2x_N^2 + ... + c_{N-1}x_N^{N-1} = u(x_N). \end{aligned} \tag{2.8}$$

The matrix equation (2.8) is of Vandermonde form which is not singular. Thus the equation can give unique solutions for constants $c_0, c_1, ..., c_{N-1}$. Once these are determined, the approximated polynomial is obtained. However, when N is large, the matrix is highly ill-conditioned and its inversion is very difficult to find. Then it is hard to determine the constants c_0,c_1,...,c_{N-1}.

Here, if $r_k(x)$, $k = 1, 2, ..., N$ are the base polynomials in V_N, $u(x)$ can then be expressed by

$$u(x) = \sum_{k=1}^{N} d_k r_k(x). \tag{2.9}$$

Clearly, if all the base polynomials satisfy a linear constrained relationship such as equation (2.4) or equation (2.5), so does $u(x)$. In the linear vector space, there may exist several sets of base polynomials. Each set of base polynomials can be expressed uniquely by another set of base polynomials. This means that every set of base polynomials would give the same weighting coefficients. However, the use of different sets of base polynomials will result in different approaches to compute the weighting coefficients. Since there are many sets of base polynomials in the linear vector space, we have many approaches to compute the weighting coefficients.

The property of linear vector space also gives us the ability to apply the weighting coefficients for the discretization of a differential equation. Remembering that the solution of a differential equation is approximated by a polynomial of degree (N-1) which constitutes the N-dimensional vector space, the actual expression of the polynomial contains the unknown constants c_k's which are to be determined. On the other hand, in the linear vector space, the set of base polynomials can be chosen to be independent of the solution. From the property of a linear vector space, if one set of base polynomials satisfies a linear operator so does any polynomial in the space. This indicates that the solution of the partial differential equation also satisfies the linear operator.

For generality, two sets of base polynomials are used to determine the weighting coefficients (Shu (2000)). The first set of base polynomials is chosen as the Lagrange interpolated polynomials,

$$r_k(x) = \frac{(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...(x-x_N)}{(x_k-x_0)(x_k-x_1)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_N)}$$

in interpolating the function $u(x)$ as

$$u(x) = \sum_{k=1}^{N} u(x_k) r_k(x).$$

Here, polynomials $r_k(x)$ are given by

$$r_k(x) = \frac{M(x)}{(x-x_k)M^{(1)}(x_k)}, \qquad k = 1, 2, ..., N \tag{2.10}$$

with the property $r_k(x_i) = \delta_{ik}$ where,

$$M(x) = (x-x_1)(x-x_2)...(x-x_N) \tag{2.11}$$

and

$$M^{(1)}(x_k) = \prod_{j=1, j\neq k}^{N} (x_k - x_j) \tag{2.12}$$

being the derivative of $M(x)$.

Here $x_1, x_2, ...x_N$ are the coordinates of grid points and may be chosen arbitrarily but distinct.

For obtaining an efficient procedure to compute the polynomials $r_k(x)$ at discrete points we make use of Kronecker operator δ_{ij} as

$$M(x) = N(x, x_k)(x - x_k), \qquad k = 1, 2, ..., N \tag{2.13}$$

with

$$N(x_i, x_j) = M^{(1)}(x_i)\delta_{i,j}. \tag{2.14}$$

Using equation (2.13), equation (2.10) can be simplified to

$$r_k(x) = \frac{N(x, x_k)}{M^{(1)}(x_k)}, \qquad k = 1, 2, ..., N \tag{2.15}$$

and at the point x_i $(i = 1, 2, ..., N)$

$$r_k(x_i) = \frac{N(x_i, x_k)}{M^{(1)}(x_k)}, \qquad k = 1, 2, ..., N. \tag{2.16}$$

From equation (2.14), we can obtain the following expression as

$$N(x_i, x_k) = \begin{cases} 0 & \text{if} \quad i \neq k \\ M^{(1)}(x_i) & \text{if} \quad i = k \end{cases} \tag{2.17}$$

giving $r_k(x_i) = 0$ if $i \neq k$ and $r_k(x_i) = 1$ for $i = k$. Using this property of $r_k(x)$ when $i = k$ in the equation (2.9) at the point x_i, we obtain

$$u(x_i) = \sum_{k=1}^{N} d_k r_k(x_i) = d_i. \tag{2.18}$$

Then $u(x_i)$ takes the form

$$u(x_i) = \sum_{k=1}^{N} u(x_k) r_k(x_i). \tag{2.19}$$

Thus the first and second order derivatives of $u(x)$ with respect to x at the point x_i are

$$u_x(x_i) = \sum_{k=1}^{N} r'_k(x_i) u(x_k) \tag{2.20}$$

$$u_{xx}(x_i) = \sum_{k=1}^{N} r''_k(x_i) u(x_k). \tag{2.21}$$

From equation (2.4) and equation (2.5), the coefficients a_{ij} and b_{ij} in the first and second order derivatives of $u(x)$ at the point x_i become

$$r'_k(x_i) = a_{ik} \tag{2.22}$$

$$r''_k(x_i) = b_{ik}. \tag{2.23}$$

Thus the coefficients a_{ij} and b_{ij} can be computed by taking first and second order derivatives of $r_k(x)$ as follows,

$$r'_j(x_i) = \frac{N^{(1)}(x_i, x_j)}{M^{(1)}(x_j)} = a_{ij} \qquad i, j = 1, ..., N \tag{2.24}$$

$$r''_j(x_i) = \frac{N^{(2)}(x_i, x_j)}{M^{(1)}(x_j)} = b_{ij} \qquad i, j = 1, ..., N \tag{2.25}$$

where $N^{(1)}(x, x_j)$ and $N^{(2)}(x, x_j)$ are the first and second order derivatives of the function $N(x, x_j)$.

$M^{(1)}(x_j)$ can be easily computed by equation (2.12), to evaluate $N^{(1)}(x_i, x_j)$ and $N^{(2)}(x_i, x_j)$ we successively differentiate equation (2.13) with respect to x and obtain the following recurrence formulation

$$M^{(m)}(x) = N^{(m)}(x, x_k)(x - x_k) + mN^{(m-1)}(x, x_k)$$
$$\text{for} \quad m = 1, 2, ..., N-1, \quad k = 1, 2, ..., N \tag{2.26}$$

where $M^{(m)}(x)$ and $N^{(m)}(x, x_k)$ indicate the m^{th} order derivative of $M(x)$ and $N(x, x_k)$ respectively.

From the equation (2.26), we can easily obtain

$$N^{(1)}(x_i, x_j) = \frac{M^{(1)}(x_i)}{x_i - x_j}, \qquad i \neq j \tag{2.27}$$

$$N^{(1)}(x_i, x_i) = \frac{M^{(2)}(x_i)}{2}. \tag{2.28}$$

Similarly, using equation (2.26) for $m = 2$ gives

$$N^{(2)}(x_i, x_j) = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_j)}{x_i - x_j}, \qquad i \neq j \tag{2.29}$$

$$N^{(2)}(x_i, x_i) = \frac{M^{(3)}(x_i)}{3}. \tag{2.30}$$

Substituting equation (2.27) and (2.28) into equation (2.24) and equation (2.29) and (2.30) into equation (2.25), we finally obtain the coefficients a_{ij} and b_{ij}

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \qquad i \neq j \tag{2.31}$$

$$a_{ii} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)} \tag{2.32}$$

$$b_{ij} = M^{(2)}(x_i) - \frac{2N^{(1)}(x_i, x_j)}{(x_i - x_j)M^{(1)}(x_j)}, \qquad i \neq j \tag{2.33}$$

$$b_{ii} = \frac{M^{(3)}(x_i)}{3M^{(1)}(x_i)}. \tag{2.34}$$

Finally by substituting equations (2.31) and (2.32) into equations (2.33) and (2.34) we get a relationship between a_{ij} and b_{ij},

$$b_{ij} = 2a_{ij}(a_{ii} - \frac{1}{x_i - x_j}), \qquad i \neq j. \tag{2.35}$$

It is observed that, if x_i is given, it is easy to compute $M^{(1)}(x_i)$ from equation (2.12), and hence a_{ij} and b_{ij} for $i \neq j$ (equations (2.31) and (2.35)). However the calculation of a_{ii} and b_{ii} (equations (2.32) and (2.34)) involve the computation of the second order derivative $M^{(2)}(x_i)$ and the third order derivative $M^{(3)}(x_i)$ which are not easy task. This difficulty can be removed by the property of the linear vector space.

According to the theory of a linear vector space, one set of base polynomials can be expressed uniquely by another set of polynomials. Thus if one set of base polynomials satisfies a linear constrained relationship, say equation (2.4) or (2.5), so does another set of base polynomials. As a consequence, equations (2.4) and (2.5) should also be satisfied by the second set of base polynomials x^{k-1}, $k = 1, 2, ..., N$. Thus, a_{ii} and b_{ii} satisfy the following equation which is obtained by the base polynomial x^{k-1} when $k = 1$, for $i = 1, 2, ..., N$

$$\sum_{j=1}^{N} a_{ij} = 0 \qquad \text{or} \qquad a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij} \tag{2.36}$$

and

$$\sum_{j=1}^{N} b_{ij} = 0 \qquad \text{or} \qquad b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}. \tag{2.37}$$

From above equations, a_{ii} and b_{ii} can be determined from a_{ij}, b_{ij} $(i \neq j)$.

2.1.2 One Dimensional Fourier Expansion-Based Differential Quadrature Method

The polynomial approximation is suitable for most engineering problems. However, for some problems, especially for those with periodic behaviours such as the Helmholtz problems, polynomial approximations may not be the best choice for the accurate solution. In contrast, Fourier series expansion for the unknown function could be better approximation.

Fourier expansion-based differential quadrature method is also going to be explained in one dimensional case which can be easily extended to two dimensional problems. The FDQ approximation for the discretization of derivatives is similar to the polynomial-based differential quadrature approximation where the approximation is in terms of trigonometric polynomials. The only difference between FDQ method and PDQ method is in the computations of the weighting coefficients (Shu (2000)). Consider a one-dimensional continuous function $u(x)$ over an interval $[a, b]$ with length L and suppose that there are N grid points in the whole domain with coordinates $x_1, x_2, ..., x_N$. Now it is supposed that $u(x)$ is approximated by a Fourier series expansion of the form

$$u(x) = c_0 + \sum_{k=1}^{N/2} (c_k \cos \frac{k\pi x}{L} + d_k \sin \frac{k\pi x}{L}). \tag{2.38}$$

Similar to PDQ, it is easy to show that $u(x)$ in equation (2.38) constitutes a $(N{+}1)$ dimensional linear vector space with respect to the operation of addition and multiplication. Here, if r_k, $k = 0, 1, ..., N$, $r_0 = 1$ are the base functions, any function in the space can be expressed as a linear combination of r_k, $k = 0, 1, ..., N$, $r_0 = 1$.

For generality, two sets of base functions are used to derive explicit formulations to compute the weighting coefficients of the first and second order derivatives in FDQ. The first set of base functions are chosen as Lagrange interpolated trigonometric polynomials,

$$r_k(x) = \frac{\sin \frac{x-x_0}{2L}\pi ... \sin \frac{x-x_{k-1}}{2L}\pi \sin \frac{x-x_{k+1}}{2L}\pi ... \sin \frac{x-x_N}{2L}\pi}{\sin \frac{x_k-x_0}{2L}\pi ... \sin \frac{x_k-x_{k-1}}{2L}\pi \sin \frac{x_k-x_{k+1}}{2L}\pi ... \sin \frac{x_k-x_N}{2L}\pi}, \tag{2.39}$$

for $\quad k = 0, 1, 2, ..., N.$

For simplicity, we set

$$M(x) = \prod_{k=0}^{N} \sin \frac{x - x_k}{2L}\pi = N(x, x_k) \sin \frac{x - x_k}{2L}\pi \tag{2.40}$$

where

$$N(x_i, x_i) = \prod_{k=0,k\neq i}^{N} \sin\frac{x_i - x_k}{2L}\pi = P(x_i) \tag{2.41}$$

$$N(x_i, x_j) = N(x_i, x_i)\delta_{ij}, \tag{2.42}$$

δ_{ij} is again the Kronecker operator.

Equation (2.39) can then be reduced to

$$r_k(x) = \frac{N(x, x_k)}{P(x_k)}, \qquad k = 0, ..., N \tag{2.43}$$

with the property $r_k(x_i) = \delta_{ik}$. Using the same fashion as in PDQ, we let all the base functions given by (2.43) satisfy two linear constrained relations (2.4) and (2.5), and obtain

$$r_j'(x_i) = \frac{N^{(1)}(x_i, x_j)}{P(x_j)} = a_{ij}, \qquad i, j = 1, ..., N \tag{2.44}$$

$$r_j''(x_i) = \frac{N^{(2)}(x_i, x_j)}{P(x_j)} = b_{ij} \qquad i, j = 1, ..., N \tag{2.45}$$

where $N^{(1)}(x, x_k)$, $N^{(2)}(x, x_k)$ are respectively the first and second order derivatives of $N(x, x_k)$. It is observed from equation (2.44) and (2.45) that the computation of a_{ij} and b_{ij} is equivalent to evaluation of $N^{(1)}(x_i, x_j)$ and $N^{(2)}(x_i, x_j)$ since $P(x_j)$ can be easily calculated by equation (2.41). To evaluate $N^{(1)}(x_i, x_j)$, $N^{(2)}(x_i, x_j)$, we successively differentiate equation (2.40) and then obtain

$$M^{(1)}(x) = N^{(1)}(x, x_k)\sin\frac{x - x_k}{2L}\pi + \frac{\pi}{2L}N(x, x_k)\cos\frac{x - x_k}{2L}\pi \tag{2.46}$$

$$M^{(2)}(x) = N^{(2)}(x, x_k)\sin\frac{x - x_k}{2L}\pi + \frac{\pi}{L}N^{(1)}(x, x_k)\cos\frac{x - x_k}{2L}\pi$$
$$-(\frac{\pi}{2L})^2 N(x, x_k)\sin\frac{x - x_k}{2L}\pi \tag{2.47}$$

$$M^{(3)}(x) = N^{(3)}(x, x_k)\sin\frac{x-x_k}{2L}\pi + 3(\frac{\pi}{2L})N^{(2)}(x, x_k)\cos\frac{x-x_k}{2L}\pi$$
$$-3(\frac{\pi}{2L})^2 N^{(1)}(x, x_k)\sin\frac{x-x_k}{2L}\pi - (\frac{\pi}{2L})^3 N(x, x_k)\cos\frac{x-x_k}{2L}\pi. \tag{2.48}$$

Applying these equations at the grid points, we get

$$N^{(1)}(x_i, x_j) = \frac{\pi P(x_i)}{2L\sin\frac{x_i-x_j}{2L}\pi}, \qquad j \neq i \tag{2.49}$$

$$N^{(1)}(x_i, x_i) = L\frac{M^{(2)}(x_i)}{\pi} \tag{2.50}$$

$$N^{(2)}(x_i, x_j) = \frac{M^{(2)}(x_i) - \frac{\pi}{L}N^{(1)}(x_i, x_j)\cos\frac{x_i-x_j}{2L}\pi}{\sin\frac{x_i-x_j}{2L}\pi}, \qquad j \neq i \tag{2.51}$$

$$N^{(2)}(x_i, x_i) = \frac{2L}{3\pi}[M^{(3)}(x_i) + \frac{\pi^3}{8L^3}N(x_i, x_i)] \tag{2.52}$$

substituting equations (2.49), (2.50) into equation (2.44), we obtain

$$a_{ij} = \frac{\pi}{2L}\frac{P(x_i)}{\sin\frac{x_i-x_j}{2L}\pi P(x_j)}, \qquad j \neq i \tag{2.53}$$

$$a_{ii} = \frac{LM^{(2)}(x_i)}{\pi P(x_i)}. \tag{2.54}$$

Similarly, substituting equations (2.51), (2.52) into equation (2.45) and using equations (2.53), (2.54), we obtain

$$b_{ij} = a_{ij}[2a_{ii} - \frac{\pi}{L}\cot\frac{x_i - x_j}{2L}\pi], \qquad j \neq i \tag{2.55}$$

$$b_{ii} = \frac{2L}{3\pi}[\frac{M^{(3)}(x_i)}{P(x_i)} + \frac{\pi^3}{8L^3}]. \tag{2.56}$$

From equations (2.53), (2.55), a_{ij} and b_{ij} can be easily computed. However the calculation of a_{ii} (equation (2.54) and b_{ii} (equation (2.56)) involve the computation of $M^{(2)}(x_i)$ and $M^{(3)}(x_i)$ which are not easy to compute. This difficulty can be removed using again the property of a linear vector space and substituting the second set of the base functions in FDQ which can be observed from equation (2.38),

$$1, \sin \pi x, \cos \pi x, \sin 2\pi x, ..., \sin(\frac{N\pi x}{2}), \cos(\frac{N\pi x}{2}).$$

Among the set of base vectors, we only apply the vector 1. Let the vector 1 satisfy the equations (2.4) and (2.5), we get for $i = 1, 2, ..., N$

$$\sum_{j=0}^{N} a_{ij} = 0 \qquad \text{or} \qquad a_{ii} = -\sum_{j=0, j\neq i}^{N} a_{ij} \tag{2.57}$$

and

$$\sum_{j=0}^{N} b_{ij} = 0 \qquad \text{or} \qquad b_{ii} = -\sum_{j=0, j\neq i}^{N} b_{ij}. \tag{2.58}$$

From the above equations, a_{ii} and b_{ii} can be easily calculated from a_{ij} $(i \neq j)$ and b_{ij} $(i \neq j)$.

2.2 Application of Differential Quadrature Method to Helmholtz and Modified Helmholtz Equations

The Helmholtz equation is frequently encountered in various fields of engineering and physics. For example, the wave guide problems in an electromagnetic field, the vibration of membranes, and the water wave diffraction problems in offshore structure engineering are governed by the Helmholtz equation. It is also used for analizing acoustics and elastic wave problems. The numerical solution of the Helmholtz equation can be obtained by using several techniques such as the finite difference technique (R. P. Shaw (1974)), boundary integral technique (P. J. Harris (1992), S. Amini and S. M. Kirkup (1995), F. Q. Hu (1995)) and the finite element technique (C. I. Goldstein (1982), J. Haslinger and P. Neittaanmaki (1984), A. Bayliss, C. I. Goldstein and E. Turkel (1985), C. I. Goldstein (1986), C. L. Chang (1990), I.Harari and T. J. R. Hughes (1991), L. L. Thompson and P. M. Pinsky (1995)). It is well known that in all these method solutions deteriorate rapidly as the wave number (k) is increased. This is due to the use of low order polynomial approximation to highly oscillatory wave propagation solutions. To obtain an acceptable level of accuracy more than 10 elements per wavelength should be used. For large wave numbers, refining the mesh or its boundary to this requirement may become prohibitively expensive. Thus the global method of differential quadrature is an efficient approach to solve the Helmholtz equation (Shu and Xue (1999)).

In previous Section Differential Quadrature Method was explained in approximating derivatives for one-dimensional case. In this Section the DQM is applied to two-dimensional Helmholtz and modified Helmholtz equations.

The two-dimesional Helmholtz equation for a field problem

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} + k^2\phi(x,y) = f(x,y) \qquad (x,y) \in \Omega \tag{2.59}$$

where ϕ and k are the potential and the wave number defined in the two-dimensional domain Ω surrounded by the boundary $\partial\Omega$, f is a source function. The boundary condition considered is one of the following types,

Dirichlet type: $\phi(x,y) = f_1(x,y)$ on $\partial\Omega$

Neumann type: $\frac{\partial\phi}{\partial n}(x,y) = f_2(x,y)$ on $\partial\Omega$

By applying the PDQ or FDQ method, equation (2.59) can be discretized in the Cartesian coordinate system as

$$\sum_{k=1}^{N} w_{ik}^{(2)}\phi_{kj} + \sum_{k=1}^{M} \bar{w}_{jk}^{(2)}\phi_{ik} + k^2\phi_{ij} = f_{i,j} \tag{2.60}$$

where N, M are the number of grid points in the x and y directions, $w_{ik}^{(2)}$ and $\bar{w}_{jk}^{(2)}$

are the weighting coefficients in the x and y directions, respectively. When PDQ method is used , $w_{ik}^{(2)}$ and $\bar{w}_{jk}^{(2)}$ are computed by equations (2.31), (2.35), (2.36), (2.37), while for the FDQ approach , $w_{ik}^{(2)}$ and $\bar{w}_{jk}^{(2)}$ are computed by equation (2.53), (2.55), (2.57), (2.58). Similarly, the derivative in the Neumann boundary condition can be discretized by the PDQ or FDQ approach.

Applying equation (2.60) at all interior points leads to the following algebraic equation system (since i and j indicate any point (x_i, y_j) in the region)

$$[A]\{\phi\} = \{b\} \tag{2.61}$$

where ϕ is a vector of unknowns which consists of functional values at all interior points, $\{b\}$ is a known vector, $[A]$ is the coefficient matrix resulting from the differential quadrature discretization of second order derivatives in Laplacian of ϕ.

The two-dimensional modified Helmholtz equation for a field problem written as

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} - k^2\phi(x,y) = f(x,y) \qquad (x,y) \in \Omega \tag{2.62}$$

where ϕ and k are again the potential and the wave number defined in the two-dimensional domain Ω surrounded by the boundary $\partial\Omega$, f is a source function.

This kind of problem would arise in the analysis of the small displacement of a tightly stretched membrane, if one adds the condition that membrane be elastically restrained. The minus sign is not the only difference between the Helmholtz and modified Helmholtz equation. When the sign is changed, the Helmholtz equation is being an important physical problem.

Similar to Helmholtz equation, modified Helmholtz equation can be discretized in the Cartesian coordinate system as

$$\sum_{k=1}^{N} w_{ik}^{(2)} \phi_{kj} + \sum_{k=1}^{M} \bar{w}_{jk}^{(2)} \phi_{ik} - k^2 \phi_{ij} = f_{i,j} \tag{2.63}$$

The weighting coefficients $w_{ik}^{(2)}$, $\bar{w}_{jk}^{(2)}$ and $w_{ik}^{(1)}$, $\bar{w}_{jk}^{(1)}$ are computed using PDQ method or FDQ method from the same equations with the Helmholtz equation. Applying equation (2.63) at all interior points we obtain the same algebraic equation system (2.61)

$$[A]\{\phi\} = \{b\}$$

The coefficient matrix $[A]$ involves the differences coming from the Differential Quadrature discretization of the modified Helmholtz equation.

2.3 Application of Differential Quadrature Method to Eigenvalue - Eigenvector Problems

Eigenvalue problems frequently arise in oscillation problems and in stability analysis. For example a biological system involving an investigation of a chemical substance's instability or a problem involving the oscillation of a membrane give rise to an eigenvalue problem.

A two-dimensional eigenvalue-eigenvector problem is in general in the form

$$\nabla^2 u(x,y) + \lambda f(x,y) u(x,y) = 0 \qquad (x,y) \in \Omega \tag{2.64}$$

$$u(x,y) = 0 \qquad (x,y) \in \partial\Omega$$

where $f(x,y)$ is a given positive function, $\Omega \subset R^2$, $u(x,y)$ and λ are the eigenvectors and eigenvalues which are sought.

Then one solution $u(x, y)$ of the problem is given by the so-called trivial solution $u \equiv 0$; we ask now whether we can also find a nontrivial solution to this problem. We note that if there does exist such a nontrivial solution, then because of the homogenity of the equation and of the boundary condition, any multiple of that solution is also a solution. Helmholtz eigenvalue-eigenvector problem involves the computation of the eigenvalues-eigenvectors of the Helmholtz-modified Helmholtz equation,

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = \mp \lambda^2 \phi(x,y) \qquad (x,y) \in \Omega \tag{2.65}$$

$$\phi(x,y) = 0 \qquad \text{or} \qquad \frac{\partial \phi(x,y)}{\partial n} = 0 \qquad (x,y) \in \partial\Omega. \tag{2.66}$$

By applying the DQ method, equation (2.65) can be discretized in the Cartesian coordinate system as

$$\sum_{k=1}^{N} w_{ik}^{(2)} \phi_{kj} + \sum_{k=1}^{M} \bar{w}_{jk}^{(2)} \phi_{ik} = \mp \lambda^2 \phi_{ij} \tag{2.67}$$

where N and M are respectively the numbers of grid points in the x and y directions, $w_{ik}^{(2)}$, $\bar{w}_{jk}^{(2)}$ are the weighting coefficients of the second order derivatives with respect to x and y respectively. Applying equation (2.67) all the interior points with the given boundary conditions gives the following eigenvalue equation system

$$[A]\{\phi\} = \mp \lambda^2 \{\phi\}. \tag{2.68}$$

From the equation (2.65), the (λ) values (wavenumbers) can be obtained from the eigenvalues of the matrix $[A]$.

To compute the eigenvalues of the matrix $[A]$, routine EVCRG in Fortran is used. The routine computes the eigenvalues and eigenvectors of a real matrix. The matrix is first balanced. Orthogonal similarity trnasformations are used to reduce the balanced matrix to a real upper Hessenberg matrix. The implicit double-shifted QR algorithm is used to compute the eigenvalues and the eigenvectors of the Hessenberg matrix. The balancing routine is based on EISPACK routine BALANC. The reduction routine is based on the EISPACK routine ORTHES and ORTRAN. The QR algorithm is based on the EISPACK routine HQR2.

2.4 Choice of Grid Points

Since the weighting coefficients $w_{ik}^{(2)}$, $\bar{w}_{jk}^{(2)}$ and $w_{ik}^{(1)}$, $\bar{w}_{jk}^{(1)}$ corresponding to the discretization of the second and first order derivatives respectively, contain grid points x_i, y_i's, the choice of these grid points becomes quite important. Equally spaced grid points, due to the their obvious convenience, have been in use by most investigators. However, uniquely spaced grid points especially the zeros of orthogonal polynomials like Legendre and Chebyshev polynomials usually give more accurate solutions then the equally spaced grid points.

The naturally choice for the grid points is the equally spaced points which is given by

$$x_i = \frac{i-1}{N-1}a, \qquad i = 1, 2, ..., N \tag{2.69}$$

and

$$y_j = \frac{j-1}{M-1}b, \qquad j = 1, 2, ..., M \tag{2.70}$$

in the x and y directions, respectively for a region $[0, a] \times [0, b]$. For this uniform grid (equally spaced) with step sizes Δx and Δy in x and y directions respectively, one can obtain

$$x_k - x_i = (k-i)\Delta x, \qquad y_k - y_j = (k-j)\Delta y$$

$$M^{(1)}(x_i) = (-1)^{N-1}(\Delta x)^{N-1}(i-1)!(N-i)! \qquad i = 1, 2, ..., N$$

$$M^{(1)}(y_j) = (-1)^{M-1}(\Delta y)^{M-1}(j-1)!(N-j)! \qquad j = 1, 2, ..., M$$

and the coefficients for the first order derivatives reduce to

$$w_{ij}^{(1)} = (-1)^{i+j}\frac{(i-1)!(N-i)!}{\Delta x(i-j)(j-1)!(N-j)!} \qquad i, j = 1 \dots N, \quad i \neq j$$

$$\bar{w}_{ij}^{(1)} = (-1)^{i+j}\frac{(i-1)!(M-i)!}{\Delta y(i-j)(j-1)!(M-j)!} \qquad i, j = 1 \dots M, \quad i \neq j.$$

The so-called Chebyshev-Gauss-Lobatto point distribution offer a better choice and have been found consistently better than the equally spaced, Legendre and Chebyshev points in a variety of problems (Bert and Malik (1996)).

These points are the Chebyshev collection points which are the roots of $|T_N(x)| = 1$ and given by (Shu (2000))

$$x_k = \cos(\frac{k-1}{N-1}\pi) \qquad 1 \leq k \leq N$$

for an interval $[-1, 1]$. $T_N(x)$ is the N'th degree Chebyshev polynomial. For a region on $[0, a] \times [0, b]$

$$x_i = \frac{1}{2}[1 - \cos(\frac{i-1}{N-1})\pi]a, \qquad i = 1, 2, ..., N \tag{2.71}$$

and

$$y_j = \frac{1}{2}[1 - \cos(\frac{j-1}{M-1})\pi]b, \qquad j = 1, 2, ..., M \tag{2.72}$$

in the x and y directions respectively.

For the Chebyshev-Gauss-Lobatto points we have (in x direction)

$$M^{(1)}(x_i) = (-1)^{i+1}N^2$$

$$M^{(2)}(x_i) = (-1)^{i}N^2\frac{x_i}{1-x^2}$$

and the corresponding weighting coefficients simplify greatly.

2.5 Implementation of Boundary Conditions

Proper implementation of boundary conditions is very important for the accurate solution. The insertion of Dirichlet type boundary conditions is straightforward since these known values contribute to the right hand side vector $\{b\}$ in the system (2.61). If the boundary condition involves normal derivatives of the unknown function ϕ then these derivatives can also be approximated by the Differential Quadrature Method. In this section implementation of boundary conditions are explained for the Helmholtz and the modified Helmholtz equations.

Dirichlet Type Boundary Condition:

For imposing the Dirichlet Type boundary conditions, equations (2.60) and (2.63) should only be applied at the interior points since the solution at the boundary grid points is known. Thus equations (2.60) and (2.63) can be rewritten as

$$\sum_{k=2}^{N-1} w_{ik}^{(2)} \phi_{kj} + \sum_{k=2}^{M-1} \bar{w}_{jk}^{(2)} \phi_{ik} \mp k^2 \phi_{ij} = f_{i,j} - s_{ij} \tag{2.73}$$

where $2 \leq i \leq N-1$, $2 \leq j \leq M-1$ and

$$s_{ij} = (w_{i1}^{(2)} \phi_{1j} + w_{iN}^{(2)} \phi_{Nj} + \bar{w}_{j1}^{(2)} \phi_{i1} + \bar{w}_{jM}^{(2)} \phi_{iM}).$$

Equation (2.73) is set of DQ algebraic equations which can be written in matrix form

$$[A]\{\phi\} = \{b\} - \{s\} \tag{2.74}$$

where $\{\phi\}$ is a vector of unknown functional values at all the interior points given by

$$\{\phi\} = \{\phi_{22}, \phi_{23}, ..., \phi_{2,M-1}, \phi_{32}, ..., \phi_{3,M-1}, ..., \phi_{N-1,2}, ..., \phi_{N-1,M-1}\}^T$$

and $\{b\}$ is a vector and $\{s\}$ vector contains known values of ϕ at the boundary grid points. The size of the matrix $[A]$ is $(N-2) \times (M-2)$.

We can show from the equation (2.73) that there isn't any difference in implementation of the Dirichlet type boundary conditions to Helmholtz or modified Helmholtz equation. However the elements of the matrix $[A]$ in equation (2.74) change because of the sign of (k^2).

In this work, the solution of equation system (2.74) can be obtained by using the routine LSARG in Fortran. The routine solves a system of linear algebraic equations having a real general coefficient matrix. It first uses the routine LFCRG to compute an LU factorization of the coefficient matrix and to estimate the condition number of the matrix. The solution of the linear system is then found using the iterative refinement routine LFIRG. LSARG fails if U, the upper triangular part of the factorization, has a zero diagonal element or if the iterative refinement algorithm fails to converge. These error occur only if $[A]$ is singular or very close to a singular matrix.

If the estimated condition number is greater than $1/\varepsilon$ (where ε is machine precision), a warning error is issued. This indicates that very small changes in $[A]$ can cause very large changes in the solution x. Iterative refinement can sometimes find the solution to such a system. LSARG solves the problem that is represented in the computer; however, this problem may differ from the problem whose solution is desired.

Neumann Type Boundary Condition:

For the Neumann conditions the normal derivatives on the boundary should also be discretized by Differential Quadrature Method.

The normal derivative of ϕ can be written as

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y \tag{2.75}$$

and $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are discretized by using PDQ or FDQ method.

Now,

$$\frac{\partial \phi_{ij}}{\partial x} = \sum_{k=1}^{N} w_{ik}^{(1)} \phi_{kj}, \qquad i = 1, 2, ..., N \tag{2.76}$$

and

$$\frac{\partial \phi_{ij}}{\partial y} = \sum_{k=1}^{M} w_{jk}^{(1)} \phi_{ik}, \qquad j = 1, 2, ..., M \tag{2.77}$$

where $w_{ik}^{(1)}$ and $\bar{w}_{jk}^{(1)}$ are the weighting coefficients with respect to x and y directions and obtained analogous in the one dimensional case. (equation (2.31) and equation (2.32)).

Thus,

$$w_{ik}^{(1)} = \frac{M_{(x_i)}^{(1)}}{(x_i - x_k) M^{(1)}(x_k)}, \qquad i \neq k \tag{2.78}$$

$$w_{ii}^{(1)} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)} \tag{2.79}$$

$$\bar{w}_{jk}^{(1)} = \frac{M^{(1)}(y_j)}{(y_j - y_k) M^{(1)}(y_k)}, \qquad j \neq k \tag{2.80}$$

$$\bar{w}_{jj}^{(1)} = \frac{M^{(2)}(y_j)}{2M^{(1)}(y_j)} \tag{2.81}$$

Assuming $\partial\phi_{i1}/\partial y = c_i$ $(i = 1, 2, ..., N)$ and $\partial\phi_{i1}/\partial x = 0$ are given on one part of the boundary, we can write

$$\frac{\partial \phi_{i1}}{\partial y} = \sum_{k=1}^{M} \bar{w}_{1k}^{(1)} \phi_{ik} = c_i. \tag{2.82}$$

Rewriting equation (2.82) as

$$\bar{w}_{11}^{(1)} \phi_{i1} + \sum_{k=2}^{M} \bar{w}_{1k}^{(1)} \phi_{ik} = c_i \tag{2.83}$$

ϕ_{i1} is easily obtained as a value on the boundary

$$\phi_{i1} = \frac{1}{\bar{w}_{11}^{(1)}}(c_i - \sum_{k=2}^{M} \bar{w}_{1k}^{(1)} \phi_{ik}), \qquad i = 1, 2, ..., N. \tag{2.84}$$

These N equations for the unknowns ϕ_{i1}, $(i = 1, 2, ..., N)$ are going to be added to the DQ system of equations (2.60) and (2.63) for the Helmholtz equation and modified Helmholtz equation which is written for $j \neq 1$, $i = 1, 2, ..., N$ for the case of Neumann type of boundary conditions $\frac{\partial \phi_{i1}}{\partial y} = c_i$ on $y = y_1$, ($j = 1$ case). When normal boundary conditions are implemented at all the related grid points, the final system of DQ equations will be again in the form of equation (2.74)

$$[A]\{\phi\} = \{b\} - \{s\}$$

where the vector $\{s\}$ contains known values of ϕ at the boundary grid points.

Thus, equations found by discretizing the normal derivatives of ϕ on the boundary are updated using interior ϕ values which are not known yet.

The mixed type boundary conditions which are combinations of the Dirichlet and Neumann conditions are implemented in a similar fashion.

CHAPTER 3

PROBLEMS AND RESULTS

In this Chapter, application of Differential Quadrature Method for Helmholtz, modified Helmholtz and eigenvalue problems, described in Chapter 2 as equations (2.1), (2.2) and (2.3) respectively, are given. First three problems are Helmholtz equations with Dirichlet type boundary conditions. The fourth problem is the solution of a modified Helmholtz equation with Dirichlet-Neumann type boundary conditions. The fifth problem gives the solution of magnetohydrodynamic flow in a channel which is transformed to two modified Helmholtz equations together with Dirichlet boundary conditions. This problem is very attractive from the physical point of view since the velocity and induced magnetic field inside the channel can be computed numerically with very small number of grid points with Differential Quadrature Method, compared to the other domain discretization methods. As a last application of DQ method the solution of Helmholtz eigenvalue-eigenvector problem if the constant (k) is also considered as an unknown, is presented. The computational domain is a rectangle for all problems considered and the discretizations of the regions are performed by using Chebyshev-Gauss-Lobatto grid distribution.

Solutions of Helmholtz or modified Helmholtz equations show critical behaviours close to the boundaries. Thus, using equally spaced grid points is not suitable for these type of equations but Chebyshev-Gauss-Lobatto grid distribution is the best choice since the points are clustered through the boundaries.

All the problems are solved by using both PDQ and FDQ formulations and the results are compared in terms of graphics and tables. Also the effect of the variation of the constant (k) in solving the equations is studied.

When Helmholtz or modified Helmholtz equation is discretized with DQM, the resulting system of algebraic equations is solved by using subroutine LSARG in Fortran. From the DQ application of differential eigenvalue-eigenvector equations an algebraic eigenvalue-eigenvector equation system is obtained and again subroutine EVCRG in Fortran package is made use of for obtaining the smallest eigenvalue.

Computer programs are written in Fortran Language and run in PC platform. All the graphics are obtained using MATLAB graphic program.

3.1 Problem 1

This problem is the Laplace equation ($k = 0$ in Helmholtz equation) with Dirichlet type boundary conditions

$$\nabla^2 u = 0 \qquad (x, y) \in (0, \pi) \times (0, \pi) \tag{3.1}$$

$$\phi(0, y) = -y^2, \quad \text{at} \quad x = 0 \tag{3.2}$$

$$\phi(\pi, y) = \pi^2 - y^2, \quad \text{at} \quad x = \pi \tag{3.3}$$

$$\phi(x, 0) = x^2, \quad \text{at} \quad y = 0 \tag{3.4}$$

$$\phi(x, \pi) = x^2 - \pi^2, \quad \text{at} \quad y = \pi. \tag{3.5}$$

in which the constant k is taken as zero in the Helmholtz equation.

The exact solution of the problem is given as

$$\phi = x^2 - y^2. \tag{3.6}$$

The number of grid points in x and y directions is taken as equal ($M = N$) and it is varied starting from small values to larger to see the effect on the solution. From $N = 5$ to $N = 23$ several values are tested and it is found that $N = 15$ is the right number of grid points since agreement with the exact solution is excellent with that value of N and there is no need to increase it further. Even with a small value of N (e.g. N=9, 11) the agrement with exact solution is obtained. But $N = 15$ gives very smooth behavior in the domain.

PDQ and FDQ approaches are compared for the solution of Laplace equation and it is found that these two formulations are equivalently good and they both require around 15 number of grid points. Figures (1) and (2) show the closeness of the two solutions even for very small N and Figure(3)-Figure(4) show that very well agreement of the DQ method solution of Laplace equation with the exact solution if PDQ and FDQ approaches are used respectively.

The performance of PDQ and FDQ approaches is measured by maximum error, $\triangle\phi_{max}$, which is defined as

$$\triangle\phi_{max} = max|\phi_{i,j} - \phi(x_i, y_j)| \tag{3.7}$$

where $\phi_{i,j}$ is the numerical solution at mesh points (x_i, y_j) and $\phi(x_i, y_j)$ is the exact solution at the same mesh point.

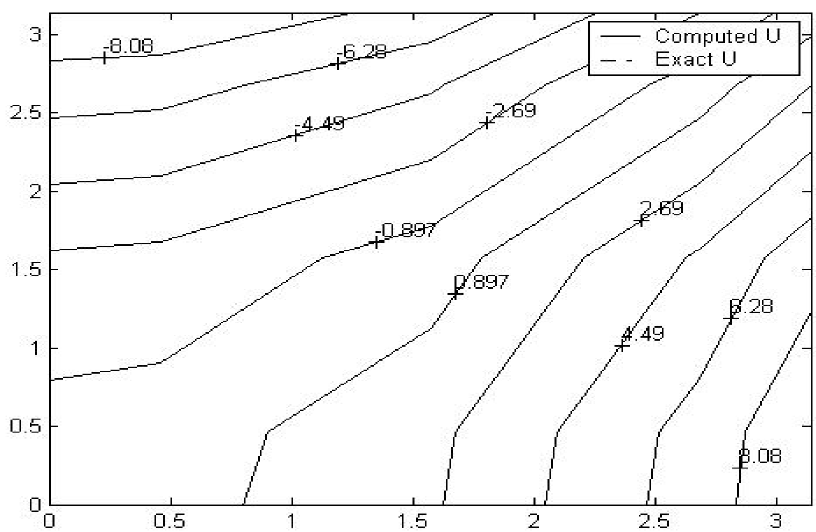

Figure 3.1: Problem 1 with PDQ, $N = 5$

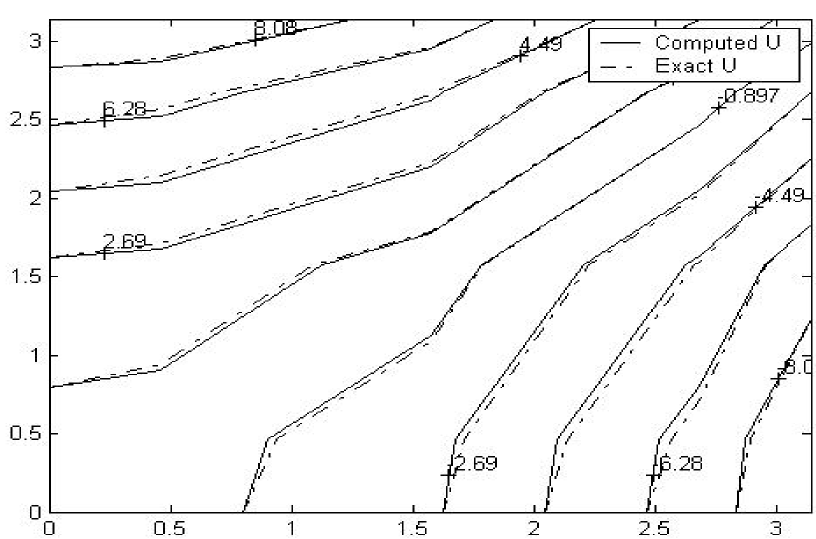

Figure 3.2: Problem 1 with FDQ, $N = 5$

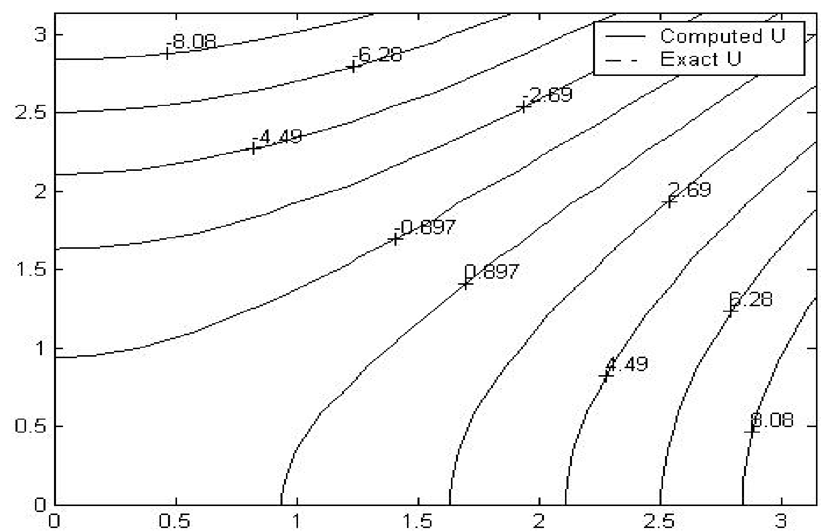

Figure 3.3: Problem 1 with PDQ, $N = 15$

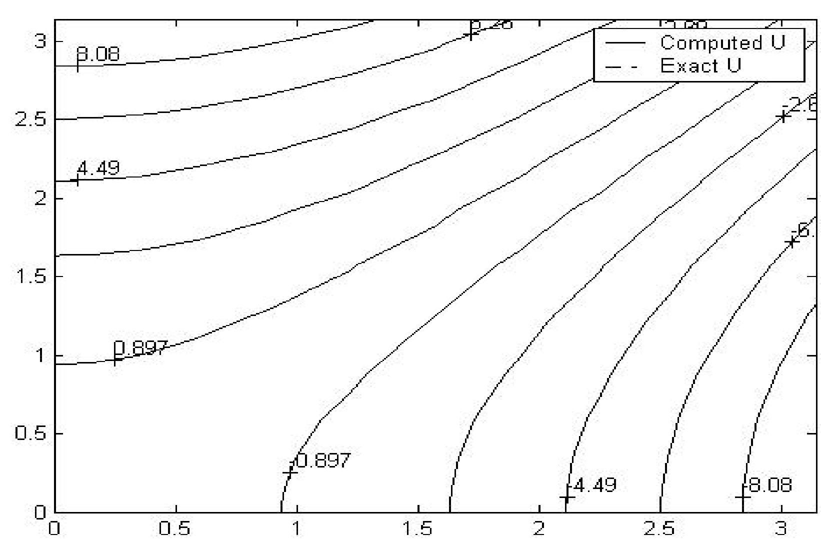

Figure 3.4: Problem 1 with FDQ, $N = 15$

3.2 Problem 2

This application is the solution of nonhomogeneous Helmholtz equation with Dirichlet boundary conditions

$$\nabla^2\phi + 0.5\phi = (-2\pi^2 + 0.5)\sin(\pi x)\sin(\pi y) \qquad (x, y) \in (0, 1) \times (0, 1) \tag{3.8}$$

$$\phi = 0 \qquad x = 0,\ x = 1;\ \ y = 0,\ y = 1 \tag{3.9}$$

where the exact solution is

$$\phi = \sin(\pi x)\sin(\pi y). \tag{3.10}$$

Since $k \neq 0$ in this problem a larger number of grid points is required as $N \geq 19$, $N = 23$ is found to be the right number of grid points. Figures (5), (6), (7) and (8) show the improvement of the solution with the increase of N. It can be seen that there is no need to increase N further. When $k \neq 0$ FDQ formulation is a better choice since it is suitable for problems with oscillatory behaviours. Thus, FDQ method requires a smaller number of grid points than PDQ method. In this problem $N = 19$ or $N = 21$ gives the accuracy as obtained from PDQ method with $N = 23$. This can be observed from figures (9) and (10).

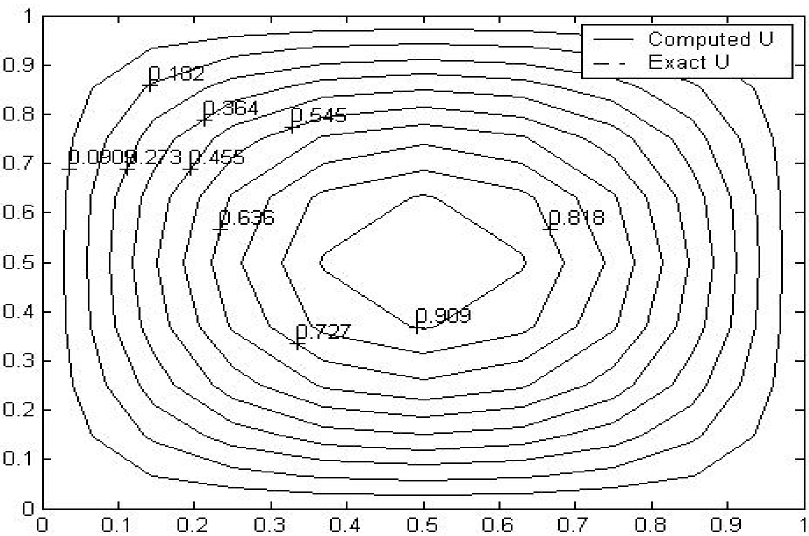

Figure 3.5: Problem 2 with PDQ, $N = 13$

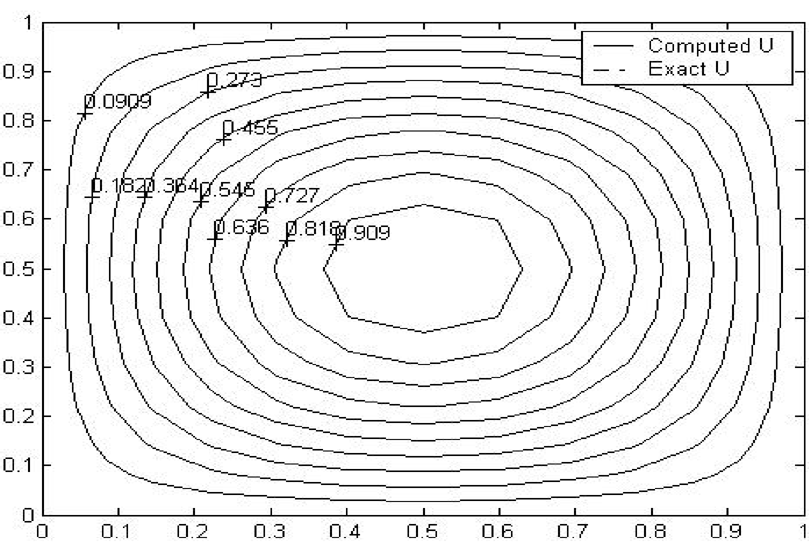

Figure 3.6: Problem 2 with PDQ, $N = 17$

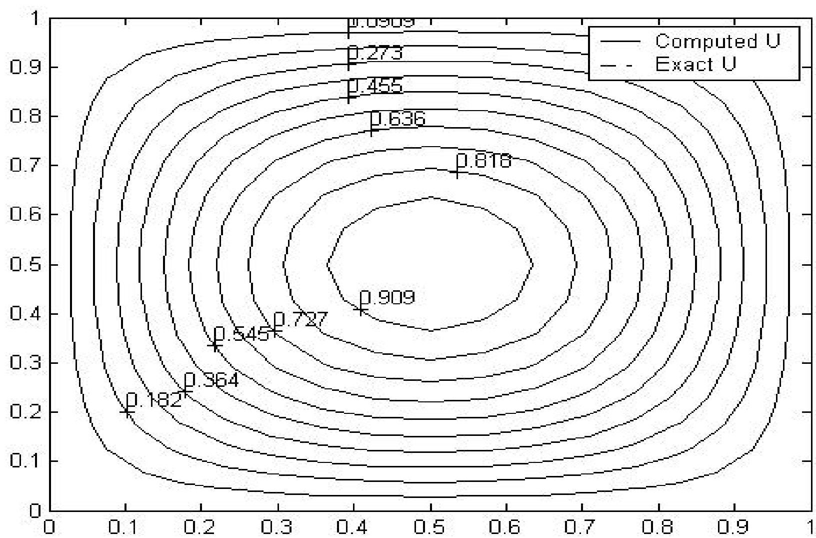

Figure 3.7: Problem 2 with PDQ, $N = 23$

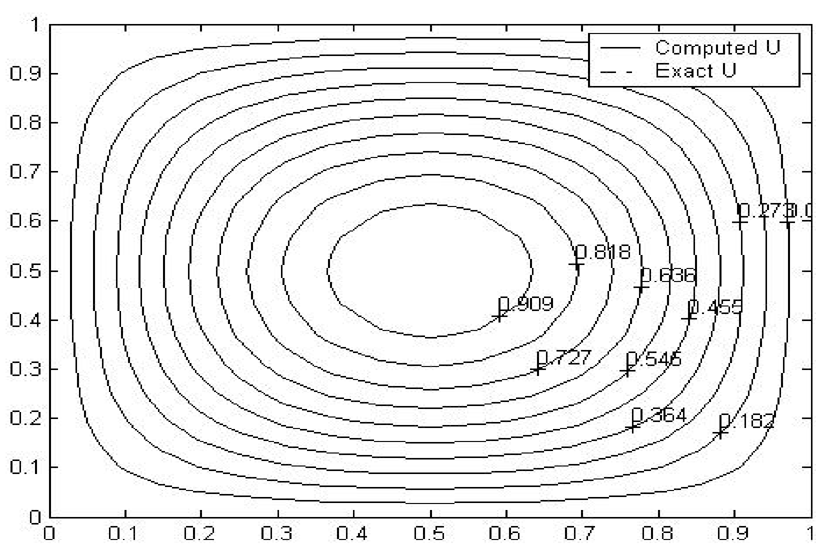

Figure 3.8: Problem 2 with PDQ, $N = 25$

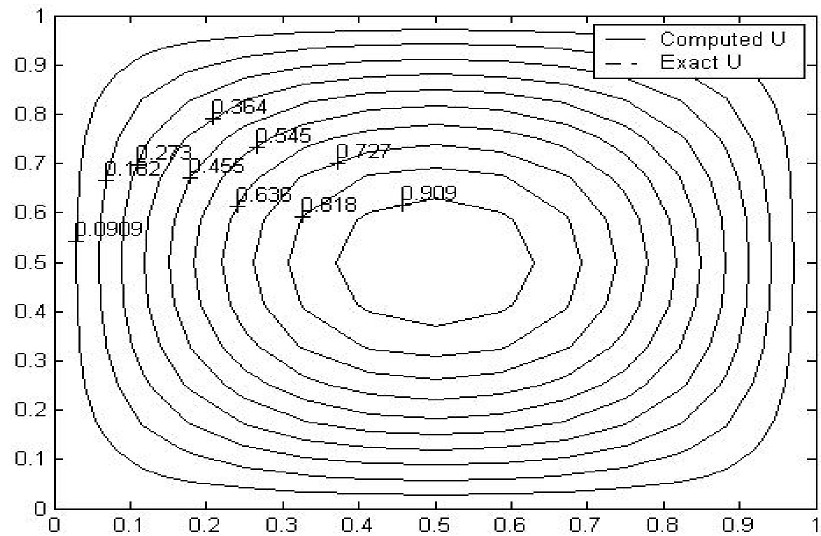

Figure 3.9: Problem 2 with FDQ, $N = 19$

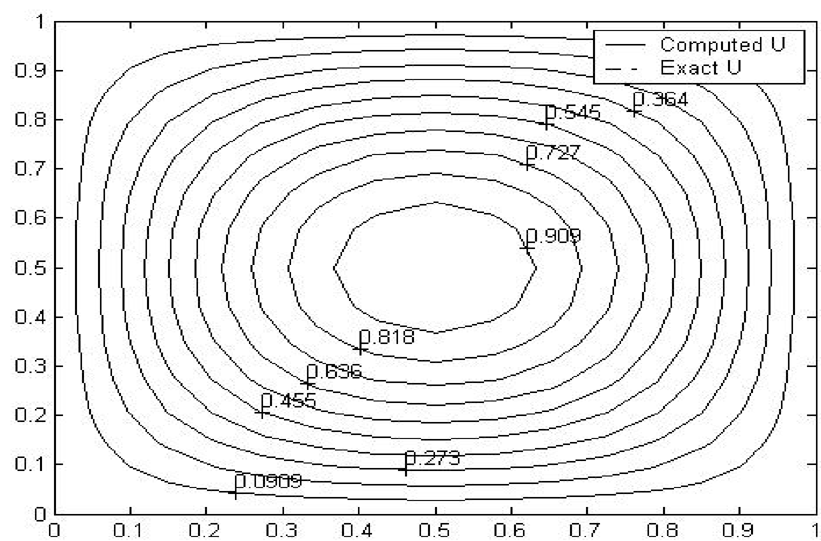

Figure 3.10: Problem 2 with FDQ, $N = 21$

3.3 Problem 3

This problem is also nonhomogeneous Helmholtz equation with Dirichlet boundary conditions but the constant k is varied to see the effect on the solution.

$$\nabla^2\phi + k^2\phi = -k^2 \sin(kx)\sin(ky) \qquad (x,y) \in (0,\pi)\times(0,\pi) \tag{3.11}$$

$$\phi = 0 \qquad x=0,\ x=\pi;\ \ y=0,\ y=\pi \tag{3.12}$$

in which the exact solution is given as

$$\phi = \sin(kx)\sin(ky). \tag{3.13}$$

For this problem as N increases both PDQ and FDQ formulations give better results in terms of maximum error. As (k) increases N must be increased to get good accuracy. These effects of N and (k) are listed in Table (3.1).

FDQ method gives better accuracy then the PDQ method as expected from the results of problem 2 also, since $k \neq 0$. Thus, especially for large (k) FDQ method is recommended since it requires less number of grid points than PDQ method.

Table (3.1) displays the maximum absolute errors between the numerical results and the exact solution for various meshes and $k = 1,2,3,4,5,6$. It can be observed from table (3.1) that for a fixed (k), as the mesh size is increased, the PDQ results are gradually improved. However, the FDQ results seem to be improved suddenly. Figures (11), (12) and (13) show the solutions for this nonhomogeneous problem for k=1, 2 and 3 respectively.

In Figures (14) and (15) we present distribution of the solutions along the line $y = \frac{\pi}{2}$ for $k = 7$ respectively for PDQ and FDQ formulations. Figure (14) shows that PDQ formulation with $N = 15$ gives better result than with $N = 13$

in comparison with the exact solution. Figure (15) implies that FDQ method is better than PDQ method showing the very well agreement with the exact solution by using $N = 15$.

Table 3.1: Comparison of $\triangle\phi_{max}$ for problem 3

$k=1$	3×3	5×5	7×7	9×9	11×11	13×13
PDQ	$6.100{\times}10^{-1}$	$1.017{\times}10^{-2}$	$2.447{\times}10^{-5}$	$1.512{\times}10^{-7}$	$5.929{\times}10^{-7}$	$2.483{\times}10^{-6}$
FDQ	$1.192{\times}10^{-7}$	$2.118{\times}10^{-7}$	$1.236{\times}10^{-7}$	$7.118{\times}10^{-7}$	$9.988{\times}10^{-8}$	$1.188{\times}10^{-6}$
$k=2$	5×5	7×7	9×9	11×11	13×13	15×15
PDQ	$1.909{\times}10^{-1}$	$3.935{\times}10^{-3}$	$1.312{\times}10^{-4}$	$4.522{\times}10^{-6}$	$1.965{\times}10^{-6}$	$1.511{\times}10^{-6}$
FDQ	$2.661{\times}10^{-7}$	$4.212{\times}10^{-7}$	$1.945{\times}10^{-7}$	$2.286{\times}10^{-7}$	$1.715{\times}10^{-6}$	$1.773{\times}10^{-6}$
$k=3$	5×5	7×7	9×9	11×11	13×13	15×15
PDQ	$4.830{\times}10$	$2.138{\times}10^{-1}$	$8.864{\times}10^{-3}$	$1.164{\times}10^{-4}$	$6.861\text{x}10^{-6}$	$1.477{\times}10^{-6}$
FDQ	$5.924{\times}10^{0}$	$1.828{\times}10^{-6}$	$7.891{\times}10^{-7}$	$4.851{\times}10^{-7}$	$2.170{\times}10^{-6}$	$1.498{\times}10^{-6}$
$k=4$	7×7	9×9	11×11	13×13	15×15	17×17
PDQ	$2.623{\times}10^{-1}$	$4.394{\times}10^{-2}$	$2.766{\times}10^{-3}$	$2.578{\times}10^{-4}$	$1.274{\times}10^{-5}$	$1.085{\times}10^{-6}$
FDQ	$1.824{\times}10^{-1}$	$1.557{\times}10^{-6}$	$6.797{\times}10^{-7}$	$1.549{\times}10^{-6}$	$1.096{\times}10^{-6}$	$3.053{\times}10^{-6}$
$k=5$	9×9	11×11	13×13	15×15	17×17	19×19
PDQ	$5.295{\times}10^{-1}$	$4.287{\times}10^{-2}$	$4.860{\times}10^{-3}$	$1.412{\times}10^{-4}$	$2.143{\times}10^{-5}$	$3.442{\times}10^{-6}$
FDQ	$9.944{\times}10^{-1}$	$1.869{\times}10^{-6}$	$1.791{\times}10^{-6}$	$1.306{\times}10^{-6}$	$2.499{\times}10^{-6}$	$1.335{\times}10^{-6}$
$k=6$	11×11	13×13	15×15	17×17	19×19	21×21
PDQ	$1.258\text{x}10^{-1}$	$1.931{\times}10^{-2}$	$2.054{\times}10^{-3}$	$1.903{\times}10^{-4}$	$1.379{\times}10^{-5}$	$1.511{\times}10^{-6}$
FDQ	$6.177{\times}10^{-2}$	$3.949{\times}10^{-6}$	$1.543{\times}10^{-6}$	$2.584{\times}10^{-6}$	$4.721{\times}10^{-6}$	$1.476{\times}10^{-6}$

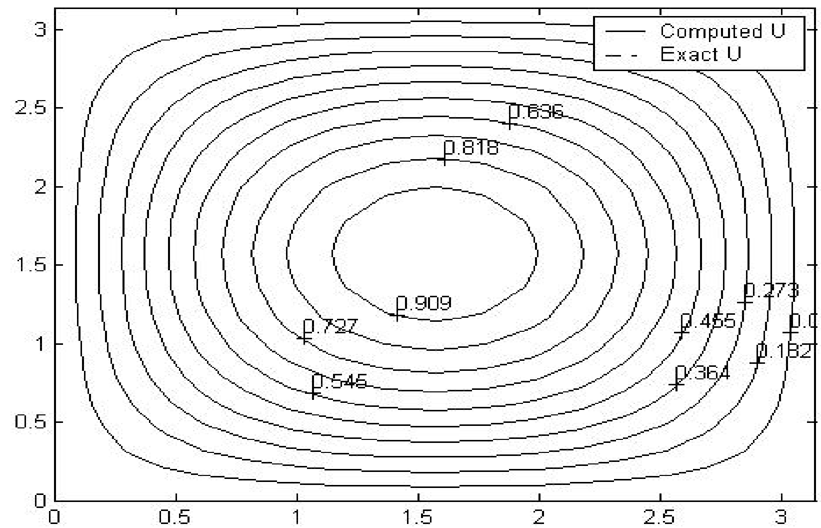

Figure 3.11: Problem 3 with PDQ, $k = 1$, $N = 25$

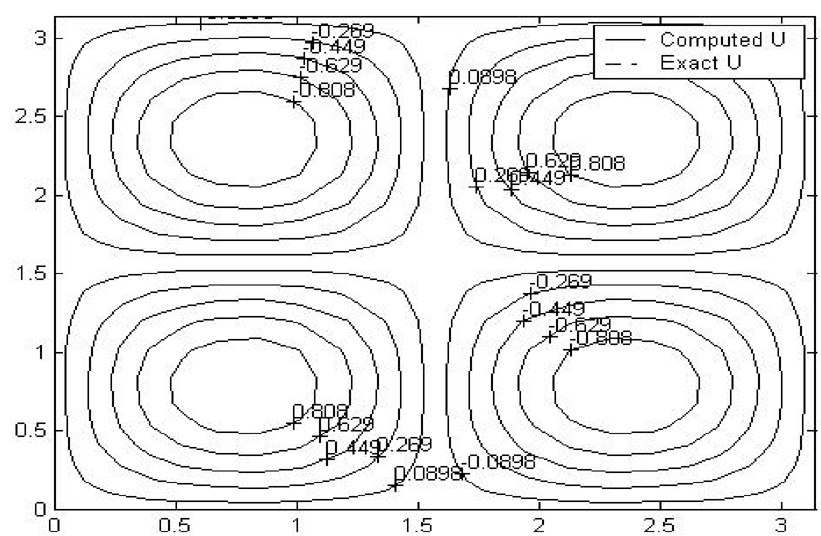

Figure 3.12: Problem 3 with PDQ, $k = 2$, $N = 27$

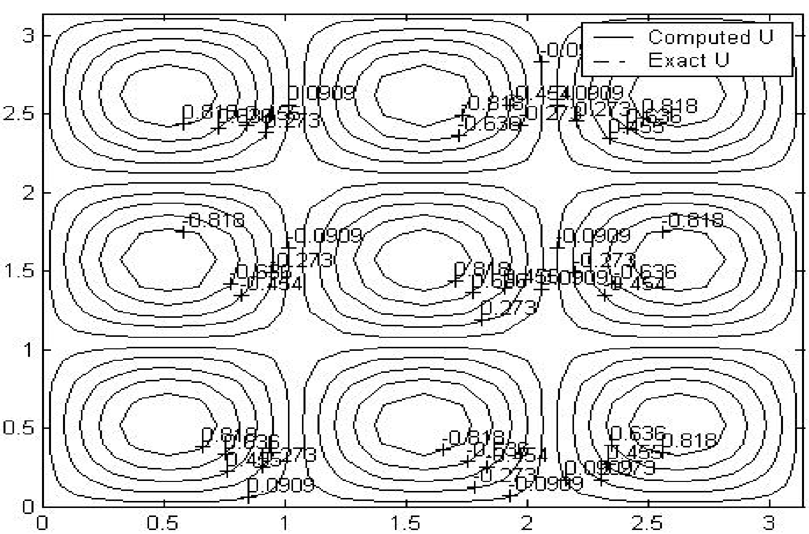

Figure 3.13: Problem 3 with PDQ, $k = 3$, $N = 31$

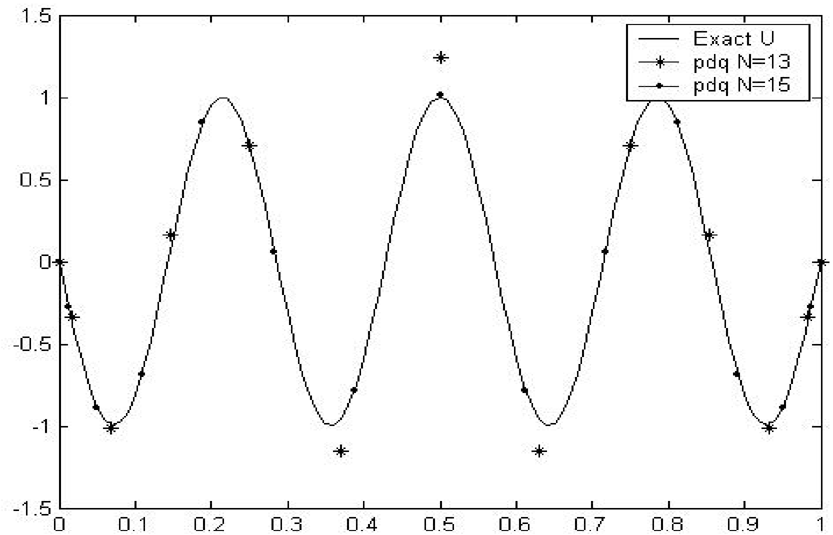

Figure 3.14: Problem 3 with PDQ, $k = 7$, $N = 13$ and $N = 15$, $y = \frac{\pi}{2}$

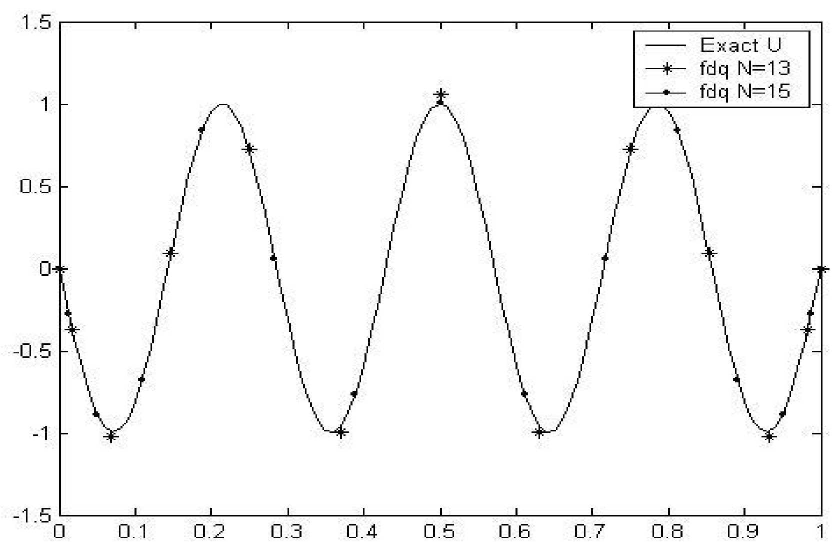

Figure 3.15: Problem 3 with FDQ, $k = 7$, $N = 13$ and $N = 15$, $y = \frac{\pi}{2}$

3.4 Problem 4

As a fourth test problem we consider homogeneous modified Helmholtz equation

$$\nabla^2 u - u = 0 \qquad 0 \le x, y \le 1$$

with Dirichlet-Neumann type boundary conditions

$$u(0, y) = y$$

$$u(1, y) = ey + cosh(y)$$

$$u(x, 1) = e^x + xcosh(1)$$

$$\frac{\partial u}{\partial y}(x, 0) = e^x$$

which has the exact solution

$$\phi(x, y) = ye^x + xcosh(y).$$

DQ discretization of this problem requires the discretization of the modified Helmholtz equation and the derivative condition at $y = 0$

$$\sum_{k=1}^{N} w_{ik}^{(2)} u_{kj} + \sum_{k=1}^{N} \bar{w}_{jk}^{(2)} u_{ik} - u_{ij} = 0 \qquad i = 1, ..., N; \;\; j = 2, ..., N$$

$$\sum_{k=1}^{N} \bar{w}_{jk}^{(1)} u_{ik} = 0 \qquad i = 1, ..., N.$$

Thus, the resulting system of algebraic linear equations contain the last N-rows from the Neumann boundary condition.

Figures (3.16) and (3.17) show the DQ solution with N=11 using PDQ and FDQ approaches respectively. The agrement with the exact solution is very well and, since $k = 1$ in this problem, the number of grid points (N) does not have to be large as has been observed in problem 3.

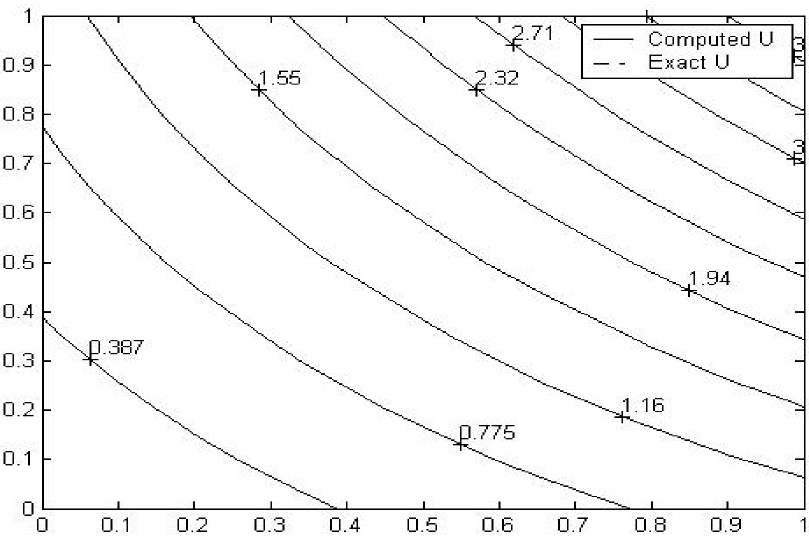

Figure 3.16: Problem 4 with PDQ , $N = 11$

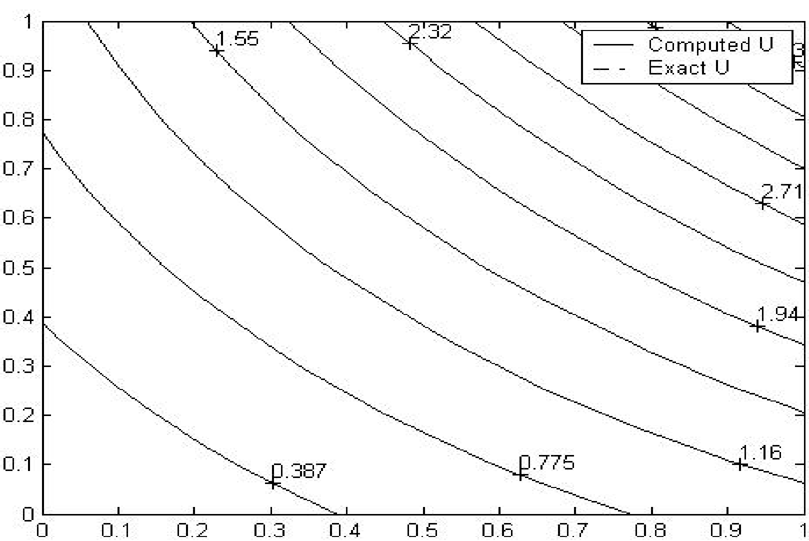

Figure 3.17: Problem 4 with FDQ , $N = 11$

3.5 Problem 5

In this example we will deal with the well known Maxwell equations of electromagnetism and the basic equations of fluid mechanics which lead to the coupled system of equations in the velocity and magnetic field. These equations of steady, laminar, fully developed flow of viscous, incompressible and electrically conducting fluid in a rectangular duct Ω, subjected to a constant and uniform applied magnetic field B_0, can be put in the following non-dimensional form

$$
\begin{aligned}
&\nabla^2 V + M\frac{\partial B}{\partial x} = -1 \\
&\nabla^2 B + M\frac{\partial V}{\partial x} = 0 \qquad \text{in} \quad \Omega
\end{aligned} \tag{3.14}
$$

with the boundary conditions

$$
V = B = 0 \quad \text{on} \quad \partial\Omega \tag{3.15}
$$

meaning that the boundaries of the duct are insulating.

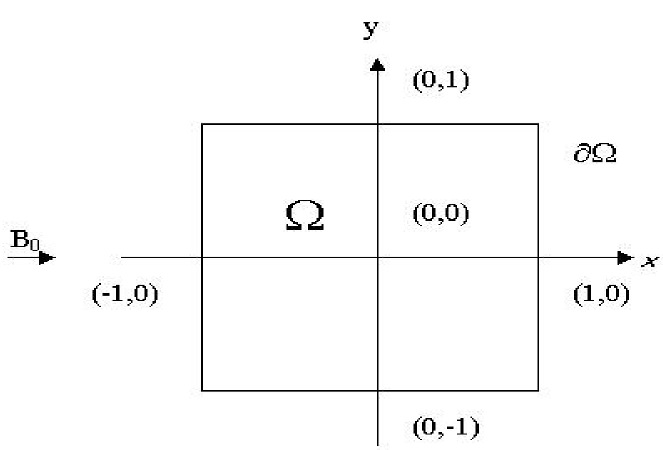

Figure 3.18: Physical problem

$V(x,y)$, $B(x,y)$ are the velocity and the induced magnetic field respectively, M is the Hartmann number. Here it is assumed that the applied magnetic field B_0 is parallel to x-axis, $V(x,y)$, $B(x,y)$ are in the z-direction which is the axis of the duct, and the fluid is driven down the duct by means of a constant pressure gradient.

Equations (3.14) may be decoupled by the change of variables

$$u_1 = V + B \ , \ u_2 = V - B \tag{3.16}$$

as

$$\begin{aligned} \nabla^2 u_1 + M\frac{\partial u_1}{\partial x} &= -1 \\ \nabla^2 u_2 - M\frac{\partial u_2}{\partial x} &= -1 \qquad \text{in} \qquad \Omega \end{aligned} \tag{3.17}$$

with the boundary conditions

$$u_1 = u_2 = 0 \qquad \text{on} \qquad \partial\Omega. \tag{3.18}$$

Furthermore, if we define a transformation

$$\begin{aligned} U^1 &= e^{\frac{M}{2}x} u_1 \\ U^2 &= e^{-\frac{M}{2}x} u_2 \end{aligned} \tag{3.19}$$

equations (3.17) become

$$\begin{aligned} \nabla^2 U^1 &= \frac{M^2}{4} U^1 - e^{\frac{M}{2}x} \\ \nabla^2 U^2 &= \frac{M^2}{4} U^2 - e^{-\frac{M}{2}x} \qquad \text{in} \qquad \Omega \end{aligned} \tag{3.20}$$

with

$$U^1 = U^2 = 0 \qquad \text{on} \qquad \partial\Omega. \tag{3.21}$$

When the solutions U^1 and U^2 are obtained by solving the corresponding system of linear equations, it is possible to go back to the original unknowns $V(x,y)$ and $B(x,y)$, through the equations (3.19) and (3.16). The discretization of equations (3.20) by using PDQ method will result

$$\sum_{k=1}^{N} w_{ik}^{(2)} U_{kj}^{1} + \sum_{k=1}^{N} \bar{w}_{jk}^{(2)} U_{ik}^{1} - \frac{M^2}{4} U_{ij}^{1} = -e^{\frac{M}{2}x_i}$$

$$\sum_{k=1}^{N} w_{ik}^{(2)} U_{kj}^{2} + \sum_{k=1}^{N} \bar{w}_{jk}^{(2)} U_{ik}^{2} - \frac{M^2}{4} U_{ij}^{2} = -e^{-\frac{M}{2}x_i} \tag{3.22}$$

and

$U^1 = U^2 = 0$ on the boundary points.

Calculated results of $V(x,y)$ and $B(x,y)$ with PDQ method are compared with the Shercliff's (1953) exact solution which may be written as

$$V = \frac{1}{2}(1-y^2) - \sum_{k=1}^{\infty} A_k \frac{\sinh m_1 \cosh m_2 x + \sinh m_2 \cosh m_1 x}{\sinh(m_1+m_2)} \cos \omega_k y \tag{3.23}$$

$$B = \sum_{k=1}^{\infty} A_k \frac{\sinh m_1 \sinh m_2 x - \sinh m_2 \sinh m_1 x}{\sinh(m_1+m_2)} \cos \omega_k y \tag{3.24}$$

where

$$\alpha = M/2$$

and

$$m_1 = -\alpha + \mu_k \quad , \quad m_2 = \alpha + \mu_k \tag{3.25}$$

$$\omega_k = \frac{(2k-1)\pi}{2} \quad , \quad \mu_k = (\alpha^2 + \omega^2)^{1/2}$$

$$A_k = \frac{16}{\pi^3} \frac{(-1)^{k+1}}{(2k-1)^3}.$$

In the computations, domain is defined by $|x| \leq 1$, $|y| \leq 1$ taking the origin at the center of the section and axes parallel to the sides. In the PDQ formulation we choose 16×16 uniform mesh. The values of the velocity $V(x, y)$ and the induced magnetic field $B(x, y)$ are obtained as a result of PDQ method application at the interior points.

In Figures (3.19) - (3.20), (3.21) - (3.22) and (3.23) - (3.24), (3.25) - (3.26) we present equal velocity and current contours respectively for $M = 2$, $M = 5$, $M = 10$ and $M = 15$. One can see that agreement with the exact values are excellent for these moderate values of Hartmann number M. For higher values of M we can still obtain the values but increase the number of grid points N. For example for M =20 we need to take N as about 40. Figures (3.27) and (3.28) exhibit equal velocity and current lines for M=20.

Also, we notice that as M increases boundary layer formulation starts close to the walls which is the well known behavior of MHD flow.

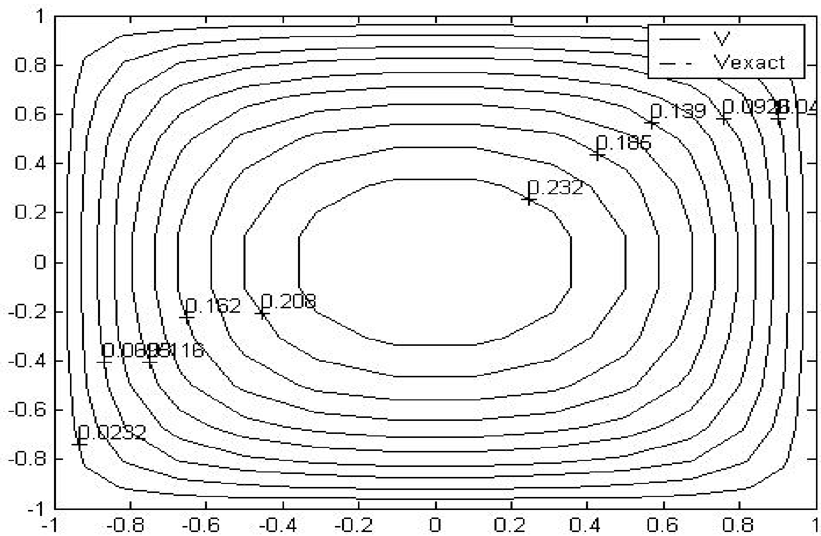

Figure 3.19: Equal velocity lines for $M = 2$

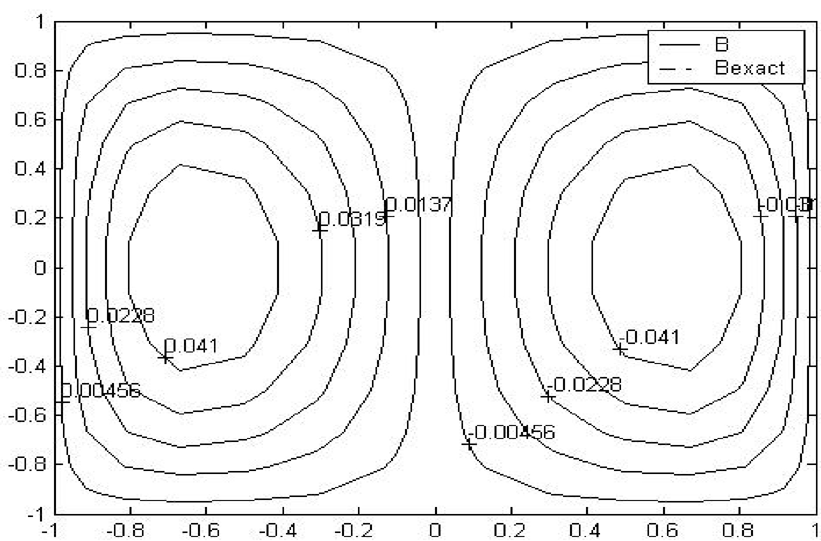

Figure 3.20: Current lines for $M = 2$

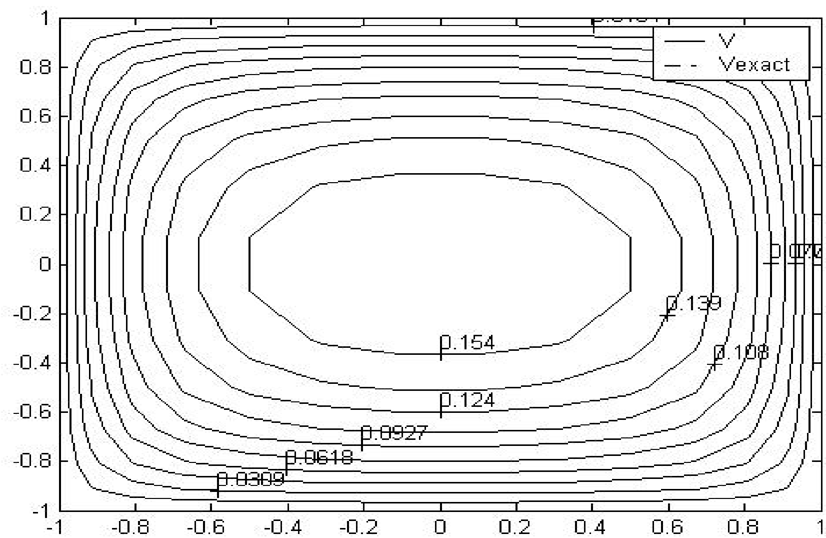

Figure 3.21: Equal velocity lines for $M = 5$

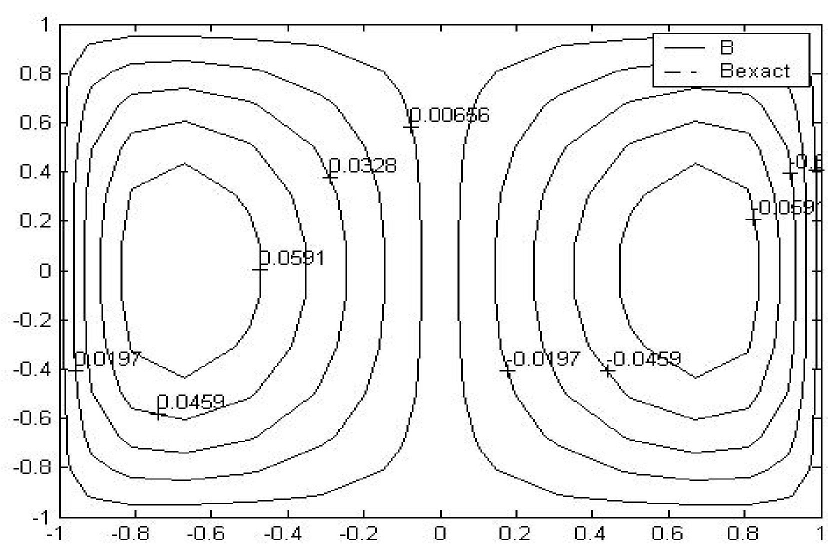

Figure 3.22: Current lines for $M = 5$

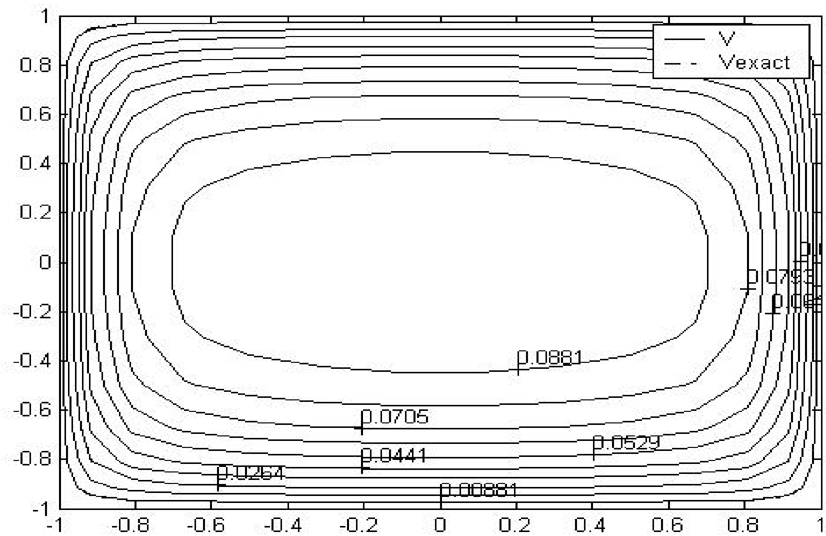

Figure 3.23: Equal velocity lines for $M = 10$

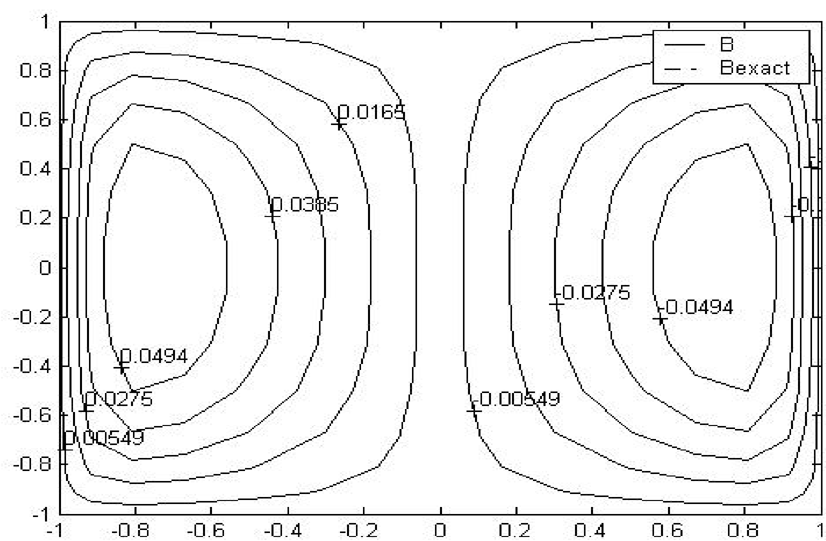

Figure 3.24: Current lines for $M = 10$

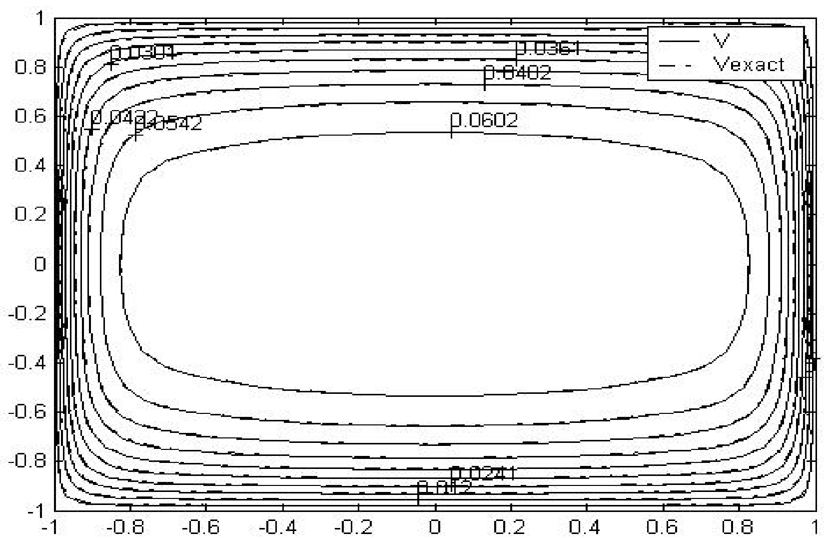

Figure 3.25: Equal velocity lines for $M = 15$

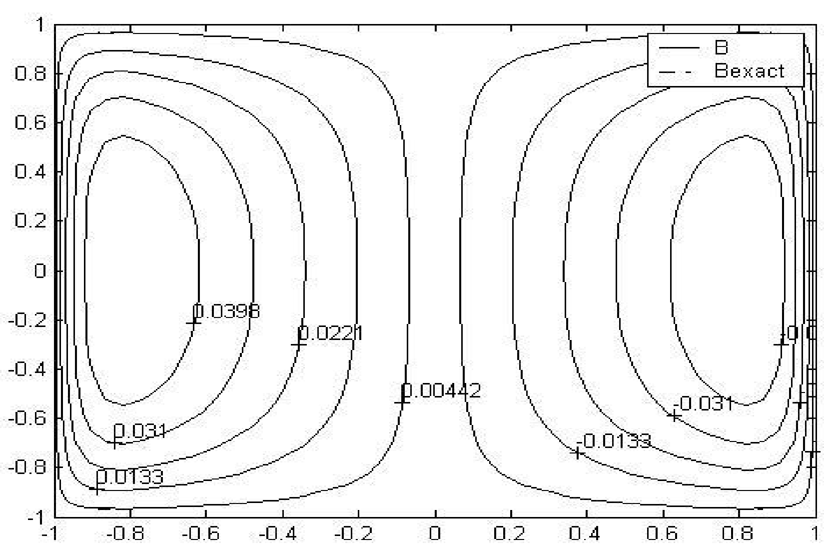

Figure 3.26: Current lines for $M = 15$

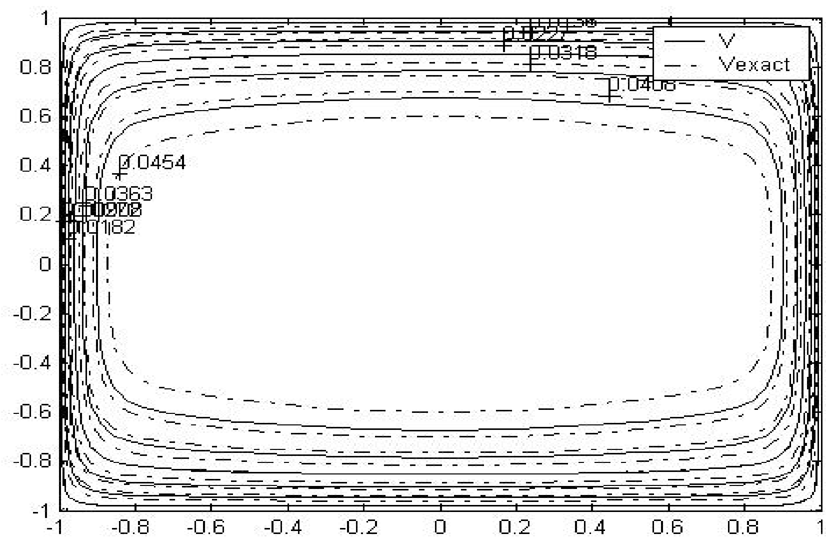

Figure 3.27: Equal velocity lines for $M = 20$

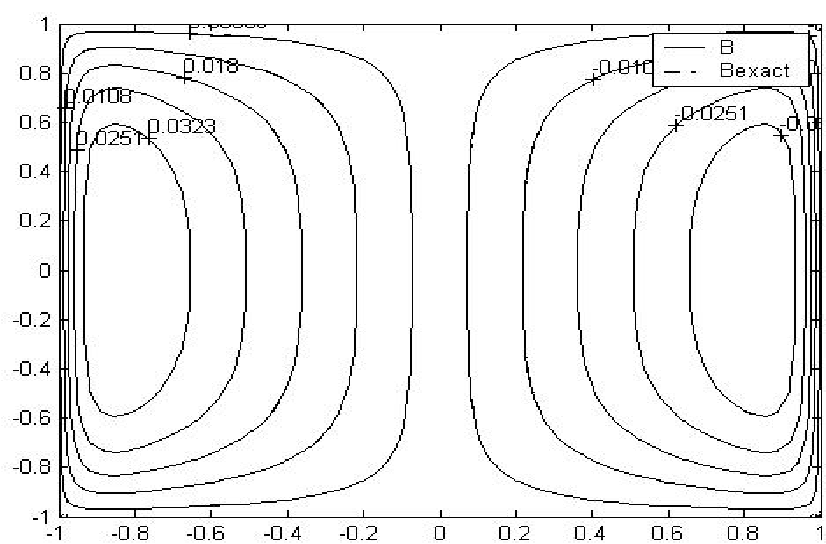

Figure 3.28: Current lines for $M = 20$

3.6 Problem 6

Last example is the application of differential quadrature method to the differential eigenvalue problem which is

$$\nabla^2 u = \lambda u$$
$$u = 0 \tag{3.26}$$

in a rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$. Here λ is the eigenvalue and u is the corresponding eigenvector.

The discretization with the PDQ method will result

$$\sum_{k=1}^{N} w_{ik}^{(2)} u_{kj} + \sum_{k=1}^{M} \bar{w}_{jk}^{(2)} u_{ik} - \lambda u_{ij} = 0. \tag{3.27}$$

This is now an algebraic eigenvalue-eigenvector problem

$$AU = \lambda U \tag{3.28}$$

where A is the coefficient matrix U and λ's are eigenvectors and eigenvalues of the matrix A.

In this problem the smallest eigenvalue of the matrix $[A]$ is obtained for several sizes of $[A]$ which means different number of points are taken in the region. Table (3.2) represents the smallest eigenvalues for different sizes and compares with the exact value of the smallest eigenvalue of $[A]$

$$\lambda_{mn} = \pi\sqrt{\frac{m^2}{4} + n^2}, \qquad m, n = 1, 2, 3, ... \qquad . \tag{3.29}$$

One can see that as the mesh is increased to approximately 11x11 or 15x15 the accuracy is increased but there is no need to increase the size further since the accuracy is not increased further, probably due to the ill-conditioned behaviour of coefficient matrix. The smallest eigenvalue is obtained with an accuracy 10^{-8} by only using 11×11 mesh.

It is possible to obtain all the eigenvalues of the problem with DQ method together with the corresponding eigenvectors. Thus, DQ method is very effective in calculating these eigenvalue-eigenvector pairs.

Table 3.2: Variation of the smallest eigenvalue with the mesh discretization

Mesh	Smallest eigenvalue of $[A]$
3x3	3.162277660168380
4x4	3.651483708915398
5x5	3.504280415679706
6x6	3.511994561313923
7x7	3.512421409571974
8x8	3.512411780273009
9x9	3.512407710743190
10x10	3.512406992741640
11x11	3.512407468976294
12x12	3.512407153284894
13x13	3.512406974091619
14x14	3.512408815127528
15x15	3.512407994467541
exact	3.512407463262003

CHAPTER 4

CONCLUSION

In this thesis, the Differential Quadrature Method is used for solving Helmholtz, modified Helmholtz and Helmholtz type eigenvalue-eigenvector problems. The DQ method discretizes the domain of the problem by using considerably small number of grid points and approximates the derivatives of the solution at any location by a linear summation of all the functional values along a grid line. Both, polynomials and Fourier series are made use of in these approximations since for some problems (e.g. with oscillatory behaviour) Fourier expansion based polynomials are much suitable for approximating the solutions. When the differential equations or differential eigenvalue-eigenvector problems are discretized by using DQ technique, the resulting system of algebraic linear equations or algebraic eigenvalue-eigenvector problems are solved with the known solution techniques. For the discretization of the domain Chebyshev-Gauss-Lobatto points are used in which the roots are clustered through the boundaries.

The numerical prosedure presented is applied to several Helmholtz, modified Helmholtz equations with Dirichlet, Diriclet-Neumann type boundary conditions. The solutions are presented in terms of contours for several values of parameter (k) in the equations.

It is observed that when the value of (k) is increased, the number of grid points N must also be increased. FDQ method needs less number of grid points than PDQ method for solving Helmholtz equation. An important application from the physical point of view is the solution of magnetohydrodynamic flow in a rectangular region. This problem is governed with coupled differential equations for the velocity and induced magnetic field. The equations are transformed first to decoupled modified Helmholtz equations and then solved by PDQ method. Solutions are obtained with very high accuracy for moderate values of Hartmann number M ($M \leq 50$). The DQ method is also applied for finding the smallest eigenvalue of a Helmholtz type eigenvalue-eigenvector problem.

Thus, DQM has the advantage of saving computational time and space compared to the other numerical methods since it gives very well accuracy using quite a small number of grid points for discretization.

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