ISOMORPHISMS OF ℓ-KÖTHE SPACES

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

ISOMORPHISMS OF ℓ-KÖTHE SPACES

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In this thesis, we study on ℓ-Köthe spaces. By the help of interpolation theory, we use linear topological invariants to get isomorphisms of Cartesian products of ℓ-power series spaces. We also see that multirectangular n-equivalent characteristics is linear toplogical invariant for power ℓ-Köthe spaces of first type.

Keywords: ℓ-Köthe Spaces, ℓ-Power Series Spaces.

ÖZ

ℓ-Köthe uzaylarının eş yapıları

KARAPINAR, Erdal
Doktora, Matematik Bölümü
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Bu tezde, ℓ-Köthe uzayları üstünde çalıştık. Enterpolasyon teori yardımıyla, topolojik değişmezler kullanıp ℓ-Köthe uzaylarının çarpımlarının eş yapılarını gözlemledik. Ayrıca, n-denkli çoklu dikdörtgenseller karakterlerin, birinci tipte karışmış uzayların topolojik değişmezi olduğunu gözlemledik.

Anahtar Kelimeler: ℓ-Köthe Uzayları, ℓ-Kuvvetli Seri Uzayları

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CHAPTER 1

INTRODUCTION

In chapter 2, we not only collect the necessary background for the sequence spaces with unconditional bases and monotone norm, but also obtain some of their simple but important properties. By using the method of analytic scale we observe that they have a very nice interpolation property (see Lemma 2.2.9) which enables us to study the isomorphic classification of Cartesian products of ℓ-power series spaces.

In chapter 3, we recall definition of ℓ-Köthe spaces and give the modifications of necessary background to be able to use topological invariants as well as modifications of some their basics. As one application we obtain a criteria for quasidiagonal isomorphism of ℓ-power series spaces. Besides that we observe the usual Köthe space $K^{l_p}(A)$ is nuclear when it is complenmentedly embedded in $K^{l_q}(B)$ for $1 \leq p < q < \infty$ with $p < 2$ or $1 < q < p \leq \infty$ with $p > 2$. Finally, we use both results J. Prada and V.P. Zahariuta and get that any stable complemented subspace of $E_0^{l_2}(a) \times E_\infty^{l_2}(b)$, with a_i or b_i tends to infinity, is basic.

In chapter 4, we consider properties $d_1 - d_2$ on ℓ-Köthe spaces. We study on isomorphisms of Cartesian products of ℓ-power series spaces of finite and infinite type.

In chapter 5, we recall the first type power ℓ-Köthe spaces and give some basic properties. After that, using n-equivalent multirectangular characteristic invariants we consider the problem of quasidiagonal isomorphisim of the first type power ℓ-Köthe spaces. We see that the system of all m-rectangle charactersitics μ_m is a complete quasidiagonal invariant on the class of all first type ℓ-Köthe spaces.

CHAPTER 2

PRELIMINARIES

2.1 Banach Sequence Spaces

A basis for a Banach space X is a sequence $(x_n)_{n=1}^{\infty}$ of vectors in X such that every vector in X has a unique representation of the form $\sum_{n=1}^{\infty} \alpha_n x_n$ with each α_n a scalar and where the sum converges in the norm topology. The mapping $x \mapsto \alpha_n$ then defines for each n a linear functional x'_n on X. It is easy to check that the expression $\|x\| := \sup_n |\sum_{k=1}^{n} x'_k(x)x_k|$ defines a stronger complete norm on X, so that $|\cdot|$ and $\|\cdot\|$ are equivalent by the open mapping theorem. We deduce from this that the biorthogonal functionals for a basis are necessarily continuous. Infact, all bases in Banach spaces are Schauder bases, that is, all biorthogonal functionals for a basis are continuous. Moreover, the biorthogonal functionals are a basic sequence in X^*; that is, they form a basis for their closed linear span. When it is useful to specify the biorthogonal functionals, we refer to the "basis" $(x_n, x'_n)_{n=1}^{\infty}$.

A series $\sum_{n=1}^{\infty} x_n$ in a Banach space is said to *converge unconditionally* provided every rearrangement of the series converges. This is equivalent to

(a) $\sum_{n=1}^{\infty} x_{k_n}$ converges for each subsequence (k_n), and also to

(b) $\sum_{n=1}^{\infty} \theta_n x_n$ converges for any sequence $(\theta_n)_{n=1}^{\infty}$ with $\theta_n = \pm 1$, $n = 1, 2, ...$

A basis $(x_n)_{n=1}^{\infty}$ in a Banach space X is said to be an *unconditional basis* provided that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges unconditionally whenever it converges. This is equivalent to saying that every permutation of $(x_n)_{n=1}^{\infty}$ is also a basis.

For any two sequences, $x = (\xi_i), y = (\eta_i)$, we use the notation $xy := (\xi_i \eta_i)$. Similarly, for any three sequences, $x = (\xi_i), y = (\eta_i), z = (\zeta_i)$, we notate $xyz := (\xi_i \eta_i \zeta_i)$. Let X be Banach sequence space. The norm $\|.\|_X$ is called *monotone* (see [14]) (*unconditionally monotone* (see [18])), if the following implication holds: for any $x = (\xi_n), y = (\eta_n) \in X$, $|\xi_n| \leq |\eta_n|$, $n \in \mathbb{N}$ implies $\|x\|_X \leq \|y\|_X$.

It is known that every Banach space X with an unconditional basis (x_n) has a monotone norm $\|\cdot\|_X$ which is equivalent to its original norm $|\cdot|$. Indeed, it is enough to put

$$\|x\|_X = \sup_{|\beta_n| \leq 1} \left| \sum_n \beta_n x_n'(x) x_n \right|$$

Throughout this work we denote by ℓ a Banach sequence space in which the canonical system (e_n) is an unconditional basis, with a monotone norm $\|\cdot\|$ satisfying $\|e_n\| = 1$ for each n. Let Λ be the class of all such spaces; in particular, l_p, c_0 are in this class.

Proposition 2.1.1. *Let $(X, \|.\|_0)$ be a Banach space with an unconditional basis $(x_n, x_n')_{n=1}^{\infty}$. Then there exists a space $\ell \in \Lambda$ such that the map*

$$T : X \to \ell$$

$$x := \sum_{n=1}^{\infty} x_n'(x) x_n \to \xi = (x_n'(x))_{n=1}^{\infty}, \qquad x \in X$$

is an isomorphism.

Proof. Let ℓ be the space of all sequences $(x_n'(x))_{n=1}^{\infty}$, $x \in X$, endowed with the norm

$$\|(x_n'(x))\|_\ell := \sup \left\{ \left\| \sum_{n=1}^{\infty} x_n'(x) \alpha_n x_n \right\|_0, |\alpha_n| \leq 1 \right\}$$

In X, we have a norm $\|\cdot\|_1$ which is equivalent to the original norm $\|\cdot\|_0$ (see [14], [18]) and defined as

$$\|x\|_1 := \sup\left\{ \left\|\sum_{n=1}^{\infty} x'_n(x)\alpha_n x_n\right\|_0, |\alpha_n| \leq 1\right\}, \qquad x \in X.$$

We want to show that the map

$$T : X \to \ell$$

$$x = \sum_{n=1}^{\infty} x'_n(x)x_n \mapsto \xi = (x'_n(x))_{n=1}^{\infty}, \qquad x \in X$$

is an isomorphism.

Since ℓ is defined as the image of T, then the map T is onto. It is also clear that T is one-to-one. Now, it is sufficient to show that T and its inverse are continuous. Since

$$\|Tx\|_\ell = \left\|T\left(\sum_{n=1}^{\infty} x'_n(x)x_n\right)\right\|_\ell = \|(x'_n(x))\|_\ell = \sup_{|\alpha_n|\leq 1}\left\|\sum_{n=1}^{\infty} x'_n(x)\alpha_n x_n\right\|_0 = \|x\|_1,$$

T is isometry.

The norms $\|.\|_0$ and $\|.\|_1$ are equivalent in X, so there exists $C > 0$ such that

$$\frac{1}{C}\|x\|_0 \leq \|x\|_1 \leq C\|x\|_0, \tag{2.1}$$

that is,

$$\frac{1}{C}\|x\|_0 \leq \|Tx\|_\ell \leq C\|x\|_0. \tag{2.2}$$

Hence T is an isomorphism. Since ℓ is isomorphic to X, then ℓ is a Banach space.

To see that the norm $\|.\|_\ell$ is monotone; we take $\xi = (\xi_n), \eta = (\eta_n) \in \ell$. Since T is onto, there exist $x, y \in X$ such that $(\xi_n) = (x'_n(x))$ and

$(\eta_n) = (x'_n(y))$. If $|\xi_n| \leq |\eta_n|$ holds for each n, then there there exists $\theta = (\theta_n),\ \theta_n = \pm 1$ such that $x'_n(x) \leq \theta_n x'_n(y)$ which satisfies

$$\|\xi\|_\ell = \|(x'_n(x))\|_\ell = \sup_{|\alpha_n|\leq 1} \left\{ \left\| \sum_{n=1}^{\infty} x'_n(x)\alpha_n x_n \right\|_0 \right\}$$

$$= \sup_{|\alpha_n|\leq 1} \left\{ \left\| \sum_{n=1}^{\infty} x'_n(x)\frac{x'_n(y)}{x'_n(y)}\alpha_n x_n \right\|_0 \right\} \leq \sup_{|\alpha_n|\leq 1} \left\{ \left\| \sum_{n=1}^{\infty} x'_n(y)\alpha_n \theta_n x_n \right\|_0 \right\}$$

$$= \sup_{|\beta_n|\leq 1} \left\{ \left\| \sum_{n=1}^{\infty} x'_n(y)\beta_n x_n \right\|_0 \right\} = \|(x'_n(y))\|_\ell = \|\eta\|_\ell.$$

Canonical system $(e_n)_{n=1}^{\infty}$ is a basis in ℓ . Indeed, since T is an isomorphism, the image of the basis $(x_n)_{n=1}^{\infty}$ in X, which is $(e_n)_{n=1}^{\infty}$, is a base for ℓ. □

Hereafter, we prefer to use $\|\cdot\|$ instead of $\|\cdot\|_\ell$.

Lemma 2.1.2. *Let $\ell \in \Lambda$. If $x = \sum_{n=1}^{\infty} e'_n(x)e_n \in \ell$ and $\alpha = (\alpha_n)$ is such that $|\alpha_n| \leq 1 \forall n$, then $\alpha x := \sum_{n=1}^{\infty} e'_n(x)\alpha_n e_n$ is also in ℓ.*

Proof. Let $S_m(x) := \sum_{n=1}^{m} e'_n(x)e_n$ and $\tilde{S}_m(x) := \sum_{n=1}^{m} e'_n(x)\alpha_n e_n$.

From Proposition 2.1.1 we have $\|x\| = \sup_{|\alpha_n|\leq 1} \left\| \sum_{n=1}^{\infty} e'_n(x)\alpha_n e_n \right\|_0$. For $k < m$

$$\|\tilde{S}_m(x) - \tilde{S}_k(x)\| = \left\| \sum_{n=k+1}^{m} e'_n(x)\alpha_n e_n \right\|$$

$$= \sup_{|\beta_n|\leq 1} \left\| \sum_{n=k+1}^{m} e'_n(x)\beta_n \alpha_n e_n \right\|_0 \leq \sup_{|\gamma_n|\leq 1} \left\| \sum_{n=k+1}^{m} e'_n(x)\gamma_n e_n \right\|_0$$

$$= \left\| \sum_{n=k+1}^{m} e'_n(x)e_n \right\| = \|S_m(x) - S_k(x)\|$$

where $\gamma_n := \beta_n \alpha_n$ for each n.

Since the series $\sum_{n=1}^{\infty} e'_n(x) e_n$ converges (i.e. $\|S_m(x) - S_k(x)\| \to 0$, as $m, k \to \infty$), the sequence $(\tilde{S}_m(x))_{n=1}^{\infty}$ is a Cauchy sequence. Since ℓ is complete, $(\tilde{S}_m(x))_{n=1}^{\infty}$ converges in ℓ, say to αx. □

Lemma 2.1.3. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a convex and non-decreasing function. If $\varphi(t_1) = \varphi(t_2)$ for $t_1 < t_2$, then $\varphi(0) = \varphi(t)$ for $0 \leq t \leq t_2$.*

Proof. Assume the contrary, that is, for $t_1 < t_2$, $\varphi(t_1) = \varphi(t_2)$ but $\varphi(0) \neq \varphi(t_0)$ for some $t_0 \in [0, t_2]$. Since φ is non-decreasing this means that $\varphi(0) < \varphi(t_0) \leq \varphi(t_2)$. Thus $\varphi(0) < \varphi(t_2)$.

By convexity of φ, we have $\varphi(t) \leq (1 - \lambda)\varphi(0) + \lambda\varphi(t_2)$ where $0 < \lambda < 1$ and $0 < t < t_2$. If we take $\lambda = \frac{t_1}{t_2}$ and $t = t_1$, we obtain that

$$\varphi(t_1) \leq (1 - \frac{t_1}{t_2})\varphi(0) + (\frac{t_1}{t_2})\varphi(t_2).$$

Since $\varphi(0) < \varphi(t_2)$ we obtain that $\varphi(t_1) < (1 - \frac{t_1}{t_2})\varphi(t_2) + (\frac{t_1}{t_2})\varphi(t_2) = \varphi(t_2)$ which contradicts our assumption that $\varphi(t_1) = \varphi(t_2)$. Hence $\varphi(0) = \varphi(t)$ for all $t \in [0, t_2]$. □

Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis of ℓ in Λ. For $x \in \ell$ define

$$x_t := te_1 + \sum_{n=2}^{\infty} e'_n(x) e_n$$

for all t in $\mathbb{R}$. Since the series $\sum_{n=1}^{\infty} e'_n(x) e_n$ converges to x in ℓ, then $e'_1(x) te_1 + \sum_{n=2}^{\infty} e'_n(x) e_n$ converges to in ℓ.

Lemma 2.1.4. *Let $\ell \in \Lambda$. If $x \neq 0$ is in ℓ and $\alpha = (\alpha_n)$, $|\alpha_n| < 1, \forall n$, then $\|\alpha x\| < \|x\|$.*

Proof. By monotonicity of the norm, it is sufficient to prove that $\|\alpha x\| \neq \|x\|$. Assume the contrary that

$$\|\alpha x\| = \|x\|. \tag{2.3}$$

Let $r_m(x) := \sum_{i=m+1}^{\infty} e_i'(x)e_i$ and

$$\varphi_m(t) := \|e_t\| = \|te_m'(x)e_m + r_{m+1}(x)\|$$

for $m = 1, 2, ...$

First, we show that $\varphi_m(t)$ is convex. For $\lambda \in (0,1)$ it follows from the triangular inequality that

$$\varphi_m\left(\lambda t + (1-\lambda)\tilde{t}\right) := \|\left(\lambda t + (1-\lambda)\tilde{t}\right) e_m'(x)e_m + r_{m+1}(x)\|$$

$$= \|\lambda te_m'(x)e_m + \lambda r_{m+1}(x) + (1-\lambda)\tilde{t}e_m'(x)e_m + (1-\lambda)r_{m+1}(x)\|$$

$$\leq \lambda\|te_m'(x)e_m + r_{m+1}(x)\| + (1-\lambda)\|\tilde{t}e_m'(x)e_m + r_{m+1}(x)\|$$

$$\leq \lambda\varphi_m(t) + (1-\lambda)\varphi_m(\tilde{t}).$$

It is clear that $\varphi_m(t)$ is non-decreasing. Indeed, if $t \leq \tilde{t}$ then

$$|te_m'(x)e_m| \leq |\tilde{t}e_m'(x)e_m|$$

By monotonicity of the norm it follows that $\varphi_1(t) \leq \varphi_1(\tilde{t})$.

We want to prove by induction that for every positive integer n,

$$\|r_n(x)\| = \|x\|.$$

In the first step, we need to show that this assumption holds for $n = 1$. From the definition,

$$\varphi_1(1) = \|x\|. \tag{2.4}$$

By monotonicity of the norm and by the assumption $|\alpha_n| < 1$ for each n, we get that

$$\varphi_1(\alpha_1) = \|\alpha_1 e_1'(x)e_1 + r_2(x)\| \geq \|\sum_{n=1}^{\infty} e_n'(x)\alpha_n e_n \alpha_n\| = \|\alpha x\| \tag{2.5}$$

If we combine (2.3), (2.4), (2.5) we obtain that

$$\varphi_1(1) = \|x\| = \|\alpha x\| \leq \varphi_1(\alpha_1) \leq \|x\|$$

which implies that $\varphi_1(1) = \varphi_1(\alpha_1)$. Since $\varphi_1(t)$ is non-decreasing and convex, then by Lemma 2.1.3 we have

$$\varphi_1(0) = \varphi_1(1) \tag{2.6}$$

From (2.6) we conclude that $\|r_1(x)\| = \|x\|$.

Now, assume that the property is true for some integer n, i.e.

$$\|r_n(x)\| = \|x\| \tag{2.7}$$

We must prove that it is also true for $n+1$ i.e. $\|r_{n+1}(x)\| = \|x\|$

Let $y = r_n(x)$. By (2.7), since $x \neq 0$, $y \neq 0$. It is obvious that

$$\varphi_{n+1}(1) = \|y\| \tag{2.8}$$

By monotonicity

$$\|\varphi_{n+1}(\alpha_{n+1})\| = \|e'_{n+1}(x)\alpha_{n+1}e_1 + r_{n+2}(x)\|$$

$$\geq \| \sum_{k=n+1}^{\infty} e'_k(x)\alpha_k e_k \alpha_k\| = \|\alpha y\| \tag{2.9}$$

If we combine (2.7), (2.8) and (2.9) we obtain that

$$\varphi_{n+1}(1) = \|y\| = \|\alpha y\| \leq \varphi_{n+1}(\alpha_{n+1}) \leq \|y\|$$

which implies that $\varphi_{n+1}(1) = \varphi_{n+1}(\alpha_{n+1})$. Since $\varphi_{n+1}(t)$ is convex and non-decreasing, then by Lemma 2.1.3 we have $\varphi_{n+1}(1) = \varphi_{n+1}(0)$. Hence $\|r_{n+1}(x)\| = \|x\|$. Thus by induction $\|r_n(x)\| = \|x\|$ for all positive integer n. Since (e_n) is a basis, $\|r_n(x)\| \to 0$. But this contradicts the assumption that $x \neq 0$. Hence we have proved that $\|x\| > \|\alpha x\|$. □

With Lemma 2.1.4, we can conclude that the monotone norm is also *strictly monotone*, that is, $\|\alpha x\| < \|x\|$ for all $x \in \ell$ and $\alpha = (\alpha_n)$, such that $|\alpha_n| < 1$, $n \in \mathbb{N}$.

Proposition 2.1.5. *If $\ell_1, \ell_2 \in \Lambda$ then there exists a Banach sequence space ℓ, satisfying 2.1.2 such that $\ell \simeq \ell_1 \times \ell_2$, where the norm in the Cartesian product of ℓ_1 and ℓ_2 is defined as follows:*

$$\|(x,y)\|_{\ell_1 \times \ell_2} := \max\{\|(x)\|_{\ell_1}, \|(y)\|_{\ell_2}\}, \tag{2.10}$$

$x \in \ell_1$ and $y \in \ell_2$.

Proof. Let $x = (\xi_k) \in \ell_1$ and $y = (\eta_k) \in \ell_2$. Then we define $z = (\zeta_n)$ such that

$$\zeta_n = \begin{cases} \xi_k & \text{if} \quad n = 2k-1, \\ \eta_k & \text{if} \quad n = 2k. \end{cases} \tag{2.11}$$

Let ℓ be the space of all such sequences (ζ_n). Define $I : \ell_1 \times \ell_2 \to \ell$ such that $(x,y) \mapsto z$. Define a norm in ℓ as:

$$\|(\zeta_n)\|_{\ell} := \|I((\xi_k, \eta_k))\|_{\ell} = \max\{\|(\xi_k)\|_{\ell_1}, \|(\eta_k)\|_{\ell_2}\}$$

By definition, I is one-to-one and onto. Since

$$\|I((\xi_k, \eta_k))\|_{\ell} = \max\{\|(\xi_k)\|_{\ell_1}, \|(\eta_k)\|_{\ell_2}\} = \|(x,y)\|_{\ell_1 \times \ell_2},$$

I is isometry. Hence we get a canonical isomorphism $\ell_1 \times \ell_2 \simeq \ell$. Let $(e_i) \in \ell_1$ and $(\tilde{e}_j) \in \ell_2$ be the bases of the related spaces. Then the sequence, we want to denote it again by (e_n), which is just the image of the $(e_{2k-1}, \tilde{e}_{2k})$ under I, becomes a basis in ℓ.

Let $\alpha = (\alpha_n)$, and $\beta = (\beta_n)$, be sequences of constants such that $|\alpha_n| \leq 1$ and $|\beta_n| \leq 1$ for all n. Then by monotonicity of the norms $\|.\|_{\ell_1}$ and $\|.\|_{\ell_2}$ we get

$$\begin{aligned} \|(\alpha x, \beta y)\|_{\ell_1 \times \ell_2} &:= \max\{\|(\alpha x)\|_{\ell_1}, \|(\beta y)\|_{\ell_2}\} \\ &\leq \max\{\|(x)\|_{\ell_1}, \|(y)\|_{\ell_2}\} = \|(x,y)\|_{\ell_1 \times l_2} \end{aligned}$$

Thus $\|.\|_{\ell}$ satisfies Lemma 2.1.2. □

2.2 Methods of Interpolation

In this section, we want to collect the necessary background from the Interpolation Theory which is necessary to obtain some interpolative property of linear maps between ℓ-Köthe spaces. Our main reference is [16].

2.2.1 Interpolation Spaces

A Banach couple (A, B) is two Banach spaces A and B algebraically and topologically imbedded in a separated topological linear space $\mathcal{S}$. With any Banach pair we may associate a couple of imbedded Banach spaces:

1. The space $A \cap B$ consists of the elements common to A and B; the norm is $\|x\|_{A \cap B} = \max\{\|x\|_A, \|x\|_B\}$, where $x \in A \cap B$.

2. The space $A + B$ consists of the elements of the form $x = u + v$, where $u \in A$ and $v \in B$ and norm is $\|x\|_{A+B} = \inf\{\|u\|_A + \|v\|_B\}$ where the infimum is taken over all elements $u \in A$ and $v \in B$ whose sum is equal to x.

The first of these spaces is called the *intersection* of the spaces of the Banach couple, and the second is called the *sum* of the spaces of the Banach couple.

The Banach space E is said to be *intermediate* for the spaces of Banach couple (A, B) if the imbeddings $A \cap B \subset E \subset A + B$ hold. It is easy to see that any space $L_p(0, 1)$ of p-integrable real (complex) valued functions on the interval $(0, 1)$, is an intermediate space between $L_{p_0}(0, 1)$ and $L_{p_1}(0, 1)$ $(p_0 < p_1)$ if $(p_0 \leq p \leq p_1)$.

Let (A, B) and (C, D) be two Banach couples. A linear mapping T acting from the space $A + B$ to $C + D$ is called a *bounded operator from the couple* (A, B) to (C, D) if the restrictions of T to the spaces A and B are bounded operators from A to C and B to D, respectively.

We denote by $L(AB, CD)$ the linear space of all bounded operators from the couple (A, B) to the couple (C, D). This is a Banach space equipped with the norm

$$\|T\|_{L(AB,CD)} = \max\{\|T\|_{A\to C}, \|T\|_{B\to D}\}.$$

Indeed, if the operators T_n form a Cauchy sequence in $L(AB, CD)$, then their restrictions to A and B converge in $L(A, C)$ and $L(B, D)$ to operators T' and T'', respectively, which obviously coincide on $A \cap B$. Then the sequence T_n converges in $L(AB, CD)$ to a uniquely defined operator T acting from $A + B$ to $C + D$ according to the formula $Tx = T'u + T''v$ $(x = u + v, u \in A, v \in B)$.

Definition 2.2.1. *Let (A, B) and (C, D) be two Banach couples, and E (respectively F) be intermediate for the spaces of Banach couple (A, B) (respectively (C, D)). The triple (A, B, E) is called an interpolation triple relative to (C, D, F) if every bounded operator from (A, B) to (C, D) maps E to F.*

Lemma 2.2.2. *If a triple (A, B, E) is an interpolation triple relative to (C, D, F), then there exists a constant $c = c(E, F) > 0$ (interpolation constant) such that*

$$\|T\|_{E\to F} \le c\|T\|_{L(AB,CD)}$$

for any $T \in L(AB, CD)$.

Definition 2.2.3. *The triple (A, B, E) is said to be interpolation triple of type α $(0 \le \alpha \le 1)$ relative to (C, D, F) if it is an interpolation triple and the following inequality holds:*

$$\|T\|_{E\to F} \le c\|T\|_{A\to B}^{1-\alpha} \cdot \|T\|_{C\to D}^{\alpha},$$

for some interpolation constant c.

If the spaces A, B, E coincide with C, D, F, respectively, then E is said to be an *interpolation space of type* α between A and B. If the constant

c is equal to one, then (A, B, E) is said to be a *normalized interpolation triple relative to* (C, D, F).

Theorems which establish that one triple of Banach spaces is an interpolation triple relative to the another are called *interpolation theorems*. Historically, the first interpolation theorem was obtained by M.Riesz and G. O. Thorin, and the whole theory of interpolation for linear operators began to develop in the direction of generalizations of this theorem. Here we give a general formulation of this first interpolation theorem. (See [16], Chapter I.)

Theorem 2.2.4 (Riesz-Thorin). *Let* $(\Omega_1, \sum_1, \mu_1)$*, and* $(\Omega_2, \sum_2, \mu_2)$ *be two measure spaces, and let* $L_p(\Omega_i)(i = 1, 2; p \geq 1)$ *be Banach spaces of complex-valued functions, p-th power summable with respect to* μ_i*. Then* $(L_{p_0}(\Omega_1), L_{p_1}(\Omega_1), L_p(\Omega_1))$ *is a normalized interpolation triple of Banach spaces of type* α *relative to the triple* $(L_{q_0}(\Omega_2), L_{q_1}(\Omega_2), L_q(\Omega_2))$*, if* $\frac{1}{p} = \frac{(1-\alpha)}{p_0} + \frac{(\alpha)}{p_1}, \frac{1}{q} = \frac{(1-\alpha)}{q_0} + \frac{(\alpha)}{q_1},$ $(0 \leq \alpha \leq 1.)$

Definition 2.2.5. *An interpolation functor is a functor* $\Im$ *acting from the category of Banach couples into the category of Banach spaces and assigning to every Banach couple* (A, B) *a Banach space* $\Im(A, B)$ *which is intermediate for the spaces of Banach couple* (A, B)*:* $A \cap B \subset \Im(A, B) \subset A + B$ *and to every operator* $T \in L(AB, CD)$ *assigning its restriction* $\Im(T)$ *to* $\Im(A, B)$*.*

If the above correspondence is a functor, then the operator $\Im(T)$ belongs to the set of morphisms $Mor(\Im(A, B), \Im(C, D))$, i.e. is a bound linear operator from $\Im(A, B)$ to $\Im(C, D)$. In other words, the triples $(A, B, \Im(A, B))$ and $(C, D, \Im(C, D)$ must be interpolation triples for any Banach couples (A, B) and (C, D).

An interpolation functor $\Im$ is said to be normalized if the interpolation constant for any triples $(A, B, \Im(A, B))$ and $(C, D, \Im(C, D))$ is not greater than one. Simple examples of normalized interpolation functors from the category of Banach couples into the category of Banach spaces are the functors of the sum and intersection of spaces in a Banach couple.

2.2.2 Calderon's Complex Method

The modern approach to the Riesz-Thorin theorem proceeds via a construction called the *complex method*. (See [16], Chapter IV.)

Let A be a complex Banach space. We denote with $\mathcal{F}(A)$, the collection of all analytic functions inside $\Pi : 0 \leq Rez \leq 1$, with values in A, and continuous and bounded in the closed strip.

Let (A_0, A_1) be a Banach couple of complex spaces. By $\mathcal{F}(A_0, A_1)$ we denote the linear space consisting of all functions $f(z)$ defined in the strip Π with values in the space $A_0 + A_1$ and having the following properties:

1. $f(z)$ is continuous and bounded in the norm of $A_0 + A_1$ in the closed strip Π.

2. $f(z)$ is analytic relative to the norm of $A_0 + A_1$ inside the strip.

3. $f(i\tau)$ assumes values in the space A_0 and is continuous and bounded in the norm of this space, while $f(1 + i\tau)$ assumes values in A_1 and is continuous and bounded in the norm of A_1.

The norm of $\mathcal{F}(A_0, A_1)$ is defined by

$$\|f\|_{\mathcal{F}(A_0,A_1)} := \max \ \{\sup_{\tau} \ \ \|f(i\tau)\|_{A_0}, \sup_{\tau} \ \ \|f(1+i\tau)\|_{A_1}\}.$$

We note that, by maximum principle for an analytic function, we have

$$\begin{aligned} \|f\|_{A_0+A_1} &\leq \max\{\sup_{\tau} \ \|f(i\tau)\|_{A_0+A_1}, \sup_{\tau} \ \|f(1+i\tau)\|_{A_0+A_1}\} \\ &\leq \max\{\sup_{\tau} \ \|f(i\tau)\|_{A_0}, \sup_{\tau} \ \|f(1+i\tau)\|_{A_1}\} = \ \|f\|_{\mathcal{F}(A_0,A_1)} \qquad (2.12) \end{aligned}$$

for $f \in \mathcal{F}(A_0, A_1)$. From this it follows in particular that $\|f\|_{\mathcal{F}(A_0,A_1)} = 0$ implies $f(z) = 0$, and that $\mathcal{F}(A_0, A_1)$ is complete.

We denote by $[A_0, A_1]_\alpha$ $(0 \leq \alpha \leq 1)$ the collection of all elements $x \in A_0 + A_1$ representable in the form $x = f(\alpha)$ for some function $f \in \mathcal{F}(A_0, A_1)$.This collection is linear. The space $[A_0, A_1]_\alpha$ is endowed with the norm:

$$\|x\|_{[A_0,A_1]_\alpha} = \ \|x\|_\alpha := \inf_{x=f(\alpha)} \ \ \|f\|_{\mathcal{F}(A_0,A_1)}$$

It follows from (2.12) that the linear manifold N_α of all functions from $\mathcal{F}(A_0, A_1)$ vanishing at $z = \alpha$, is a closed subspace of $\mathcal{F}(A_0, A_1)$. The above definition of the norm shows that $[A_0, A_1]_\alpha$ is isometric to the quotient space of $\mathcal{F}(A_0, A_1)$ modulo N_α, and consequently it is a Banach spaces. Also with the help of (2.12), we have:

Theorem 2.2.6 (Interpolation Theorem). *Let (A_0, A_1) and (B_0, B_1) be two Banach couples. The triple $(A_0, A_1, [A_0, A_1]_\alpha)$ of Banach spaces is a normalized interpolation triple of type α relative to the triple $(B_0, B_1, [B_0, B_1]_\alpha)$ i.e. for $T \in L(A_0A_1, B_0B_1)$*

$$\|Tx\|_{[B_0,B_1]_\alpha} \leq \ \|T\|_{A_0 \to B_0}^{1-\alpha} \ \|T\|_{A_1 \to B_1}^{\alpha} \ \|x\|_{[A_0,A_1]_\alpha}.$$

Theorem 2.2.7. *The space $A_0 \cap A_1$ is densely imbedded in any space $[A_0, A_1]_\alpha$ for $0 \leq \alpha \leq 1$.*

2.2.3 Method of Analytic Scales

We need to introduce the notion of the analytic scale of spaces. Let M be a normed linear space in which a family of linear operators $T(z)$ acts in such a way that the following conditions are satisfied:

(i) For every $x \in M$ the function $T(z)x$ is an entire function of the complex variable z.

(ii) The function $\|T(z)x\|_M$ is bounded on every straight line parallel to the imaginary axis.

(iii) $T(0)x = x$.

(iv) $\sup\limits_{\mu,\nu} \|T(\alpha + i\mu)T(\beta + i\nu)x\|_M \leq \sup\limits_{\tau} \|T(\alpha + \beta + i\tau)x\|_M$

(v)
$$T(i\mu)\frac{T(z + \Delta z)x - T(z)x}{\Delta z} \to T(i\mu)(T(z)x)'$$
uniformly in μ as $\Delta z \to 0$.

We define the family of norms

$$\|x\|_\alpha := \sup_{-\infty<\tau<\infty} \|T(\alpha + i\tau)x\|_M$$

in the space M and complete M to a Banach space E_α in each of these norms. The family E_α, $(-\infty < \alpha < \infty)$, of Banach spaces will be called an *analytic scale of spaces*.

Property (iv) can be written as

$$\|T(\beta + i\nu)x\|_{E_\alpha} \leq \|x\|_{\alpha+\beta} \tag{2.13}$$

Also, from property (iii), it follows that

$$\|x\|_M = \|T(0)x\|_M \leq \|x\|_{E_0} \tag{2.14}$$

As an example of an analytic scale; we consider the set of all continuous functions on [0,1], equal to zero in some neighborhood of zero which may vary with the function. On this set we define the family of operators $T(z)x(t) = t^{-z}x(t)$. The operators $T(z)$ will be considered as linear operators in the space M equipped with the norm of $L_p(0,1)$ $(1 \leq p < \infty)$. We denote by L_p^α $(-\infty < \alpha < \infty)$ the scale of spaces constructed from these operators. The space L_p^α consists of measurable functions for which

$$\|x\|_{L_p^\alpha} = \left(\int_0^1 |t^{-\alpha}x(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

Theorem 2.2.8. *Let* E_α, $(0 \leq \alpha \leq 1)$, *be an analytic scale of spaces, and let* E_1 *be normally imbedded in* E_0. *The space* E_α *coincides isometrically with* $[E_0, E_1]_\alpha$.

Proof. Let x belong to the set M from which the scale E_α is constructed. The function $f(z) := T(\alpha - z)x$ is analytic in the norm of E_0, by the property (v). Also by using the property (iv) we get

$$\|f(z)\|_{E_0} = \|T(\alpha - z)x\|_{E_0} = \sup_\tau \|T(i\tau)T(\alpha - z)x\|_M$$

$$\leq \sup_{\mu} \|T(\alpha - Rez + i\mu)x\|_M \leq \max\{\|x\|_{E_\alpha}, \|x\|_{E_{\alpha-1}}\},$$

for $0 \leq Rez \leq 1$, and so $\|f(z)\|_{E_0}$ is bounded in the strip Π. So by (2.13), on the boundary Π we have

$$\|f(i\tau)\|_{E_0} = \|T(\alpha - i\tau)x\|_{E_0} \leq \|x\|_{E_\alpha}$$
$$\|f(1+i\tau)\|_{E_0} = \|T(\alpha - 1 - i\tau)x\|_{E_1} \leq \|x\|_{E_\alpha}$$

Finally, by property (iii), we have $f(\alpha) = T(0)x = x$. Consequently, $x \in [E_0, E_1]_\alpha$ and $\|x\|_{[E_0,E_1]_\alpha} \leq \|x\|_{E_\alpha}$.

For the reverse inequality: For $x \in M$ we may construct a function $f(z) = \sum_{k=1}^N a_k(z)x_k$ such that a_k is complex-valued function which is analytic inside Π, continuous and bounded in the closed strip; $x_k \in M$, $f(\alpha) = x$ and

$$\|f\|_{\mathcal{F}(E_0,E_1)} \leq \|x\|_{[E_0,E_1]_\alpha} + \epsilon. \tag{2.15}$$

Set $\Psi(z) := T(z + i\mu)f(z)$, where μ is a fixed real number. This function is analytic in E_0 inside Π and continuous and bounded in the closed strip. We further have

$$\|\Psi(i\tau)\|_{E_0} \leq \sup_{t,\tau} \|T(it)f(i\tau)\|_{E_0} \leq \sup_{\tau} \|f(i\tau)\|_{E_0} \leq \|f\|_{\mathcal{F}(E_0,E_1)}$$

$$\|\Psi(1+i\tau)\|_{E_0} \leq \sup_{t,\tau} \|T(1+it)f(1+i\tau)\|_{E_0} \leq \sup_{\tau} \|f(1+i\tau)\|_{E_1} \leq \|f\|_{\mathcal{F}(E_0,E_1)}.$$

By the maximum principle, taking account of (2.15), we obtain

$$\|\Psi(\alpha)\|_{E_0} = \|T(\alpha + i\mu)f(\alpha)\|_{E_0} = \|T(\alpha + i\mu)x\|_{E_0} \leq \|x\|_{[E_0,E_1]_\alpha} + \epsilon.$$

Finally, from (2.14) it follows that

$$\|x\|_{E_\alpha} = \sup_{\mu} \|T(\alpha + i\mu)x\|_M \leq \sup_{\mu} \|T(\alpha + i\mu)x\|_{E_0} \leq \|x\|_{[E_0,E_1]_\alpha} + \varepsilon.$$

Thus $\|x\|_{[E_0,E_1]_\alpha} = \|x\|_{E_\alpha}$ on M. On the other hand, since M is dense in all spaces E_α and also in $[E_0, E_1]_\alpha$ by Theorem 2.2.7, the theorem is proved. □

For a sequence $a = (a_i)$, we use notation $a^\alpha := (a_i^\alpha)$.

Lemma 2.2.9. *Let $\ell \in \Lambda$ and $a^{(0)}, a^{(1)}$ be positive sequences of numbers. Then $\ell(a^\alpha) = [\ell(a^{(0)}), \ell(a^{(1)})]_\alpha$ where $a^\alpha := (a^{(0)})^{1-\alpha}(a^{(1)})^\alpha,\ 0 \le \alpha \le 1$.*

This result is obtained from the interpolation theory by using the method of the analytic scale. In that case, it is enough to take the normed linear space M as a dense subspace of $\ell(a^{(0)})$, that is,

$$M := \{x = (\xi_k) \in \ell(a^{(0)}) : \exists k_0 = k_0(x), \xi_k = 0, k \ge k_0\}.$$

We define an operator $T(z) : M \longrightarrow M$ such that $T(z)x := \left(\xi_k \left(\frac{a_k^{(1)}}{a_k^{(0)}}\right)^z\right)$ where $x = (\xi_k)$. Clearly the conditions $(i) - (v)$ in the definition of the analytic scale are satisfied.

$$\|x\|_\alpha := \sup_{-\infty<\tau<\infty} \|T(\alpha + i\tau)x\|_{\ell(a^{(0)})} = \sup_{-\infty<\tau<\infty} \|\left(\xi_k \left(\frac{a_k^{(1)}}{a_k^{(0)}}\right)^{\alpha+i\tau} a_k^{(0)}\right)\|$$

$$= \sup_{-\infty<\tau<\infty} \|(\xi_k (a_k^{(0)})^{1-\alpha-i\tau}(a_k^{(1)})^{\alpha+i\tau})\| = \sup_{-\infty<\tau<\infty} \|T(i\tau)x\|_{\ell(\ (a^{(0)})^{1-\alpha}\ (a^{(1)})^\alpha)}$$

$$= \sup_{-\infty<\tau<\infty} \|x\|_{\ell(\ (a^{(0)})^{1-\alpha}\ (a^{(1)})^\alpha)}.$$

With this observation, we obtained that $E_\alpha := \ell(a^\alpha) = \ell(\ (a^{(0)})^{1-\alpha}\ (a^{(1)})^\alpha)$. In the same way, we get that $E_0 := \ell(a^{(0)}), E_1 := \ell(a^{(1)})$. After getting the analytic scale E_α in this way, we conclude the result just by Theorem 2.2.8.

CHAPTER 3

ℓ-KÖTHE SPACES

3.1 $\ell-$Köthe Spaces

A matrix $A := (a_{i,n})_{i,n\in\mathbb{N}}$ of real numbers is called a *Köthe matrix* if $0 \leq a_{i,n} \leq a_{i,n+1}$ for each $i, n \in \mathbb{N}$; and for each $i \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $a_{i,n} > 0$.

Definition 3.1.1. *Let $\ell \in \Lambda$. The* ℓ-Köthe space $K^\ell(A)$, defined by the Köthe matrix $A = (a_{i,n})_{i,n\in\mathbb{N}}$, is a Fréchet space of number sequences $\xi = (\xi_i)$ such that $(\xi_i a_{i,n}) \in \ell$, for each n, with the topology generated by the system of seminorms $\{|(\xi_i)|_n := \|(\xi_i a_{i,n})\| : n \in \mathbb{N}\}$.

Note that $|(e_i)|_n = \|(e_i a_{i,n})\| = a_{i,n}$. Hereafter the notation $e = (e_i)_{i\in\mathbb{N}}, e_i := (\delta_{i,k})_{k\in\mathbb{N}}$, will be always used for the canonical basis of $K^\ell(A)$ regardless of a matrix A.

When ℓ is an l_p, we obtain the usual Köthe space

$$K^{l_p}(A) = \{(\xi) = (\xi_i) : |(\xi_i)|_n = \left(\sum_{i=1}^{\infty} |\xi_i|^p (a_{i,n})^p\right)^{\frac{1}{p}} < +\infty, \forall n \in \mathbb{N}\}.$$

In some sources, usual Köthe spaces are also denoted by $\lambda^p(A)$.

Due to [19], it is known that every Fréchet space with an absolute basis is isomorphic to some ℓ_1-Köthe space.

Let $A := (a_{i,n})_{i,n\in\mathbb{N}}$ and $B := (b_{j,n})_{j,n\in\mathbb{N}}$ be Köthe matrices. Then the Cartesian product of ℓ-Köthe spaces $K^{\ell_1}(A)$ and $K^{\ell_2}(B)$ is naturally isomorphic to the space $K^\ell(C)$ where $\ell \simeq \ell_1 \times \ell_2,\ \ell_1, \ell_2 \in \Lambda$ (see Proposition 2.1.5) and $C = (c_{k,n})_{k,n\in\mathbb{N}}$ such that c_{kn} is equal to $a_{i,n}$ if $k = 2i-1$ and $b_{i,n}$ if $k = 2i$.

For a given sequence of positive real numbers $a = (a_i)_{i\in\mathbb{N}}$ and $\lambda_n \to \alpha$, $-\infty < \alpha \leq \infty$, we call the ℓ-Köthe space

$$E_\alpha^\ell(a) := K^\ell(\exp(\lambda_n a_i)),$$

ℓ-power series space of finite (respectively, *infinite*) *type* if $\alpha < \infty$ (respectively, $\alpha = \infty$).

The sequences $a = (a_n)$ and $\tilde{a} = (\tilde{a}_n)$ of positive numbers are weakly equivalent ($a \asymp \tilde{a}$) if there is a $C > 0$ such that $\frac{1}{C}a_n \leq \tilde{a}_n \leq Ca_n$ for each n.

For any set S, we denote by $|S|$ the number of elements in S if it is finite and the symbol ∞ if S is infinite.

Let $X = K^\ell(A)$ and $\tilde{X} = K^\ell(\tilde{A})$ be ℓ-Köthe spaces with the canonical bases (e_i). We say that X is quasidiagonally isomorphic to $\tilde{X}$ and write $X \overset{qd}{\simeq} \tilde{X}$ if there exists $T : X \to \tilde{X}$ such that

$$Te_i := t_i e_{\varphi(i)}, \quad i \in \mathbb{N}, \tag{3.1}$$

is an isomorphism, where t_i is a sequence of numbers and $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection. Also, we denote by $X \overset{qd}{\hookrightarrow} \tilde{X}$, a quasidiagonal isomorphic imbedding, for which φ in (3.1) is an injection.

The following statement is proved in [37] (see also [21]), for Köthe spaces.

Lemma 3.1.2. *Let X and $\tilde{X}$ be ℓ-Köthe spaces with $X \overset{qd}{\hookrightarrow} \tilde{X}$ and $\tilde{X} \overset{qd}{\hookrightarrow} X$ then $X \overset{qd}{\simeq} \tilde{X}$.*

Proof. Let $(e_i)_{i\in\mathbb{N}}$ and $(e_j)_{j\in\mathbb{N}}$ be bases of X and $\tilde{X}$, respectively. Let the quasidiagonal embeddings $X \overset{qd}{\hookrightarrow} \tilde{X}$ and $\tilde{X} \overset{qd}{\hookrightarrow} X$ be defined respectively by (r_i), $\varphi : \mathbb{N} \to \mathbb{N}$, and (t_j), $\psi : \mathbb{N} \to \mathbb{N}$. By Cantor-Bernstein theorem, there exist complementary subsets $I_1, I_2 \subset \mathbb{N}$ and $J_1, J_2 \subset \mathbb{N}$ such that $\varphi(I_1) = J_1$ and $\psi(J_2) = I_2$. Since, any part of an unconditional basis is a basis in its closed linear span, and any permutation of an unconditional basis is also basis, then $(e_{\varphi(i)})_{i\in I_1} \cup (e_{\psi^{-1}(i)})_{i\in I_2}$ is a basis in $\tilde{X}$. We define the quasidiagonal isomorphism T between X and $\tilde{X}$ as

$$Te_i = \begin{cases} r_i e_{\varphi(i)} & \text{if } \ i \in I_1, \\ t^{-1}_{\psi^{-1}(i)} e_{\psi^{-1}(i)} & \text{if } \ i \in I_2. \end{cases}$$ □

Let $a = (a_n)$ where $a_n \geq 1$. In [20], [21] Mitiagin investigated isomorphism of some non-Montel power series spaces by using the following counting function:

$$\mu_a(t,\tau) := |\{n \in \mathbb{N} : \tau < a_n \leq t\}|, \quad 0 < \tau < t < \infty.$$

We use the notation $\mu_a \approx \mu_{\tilde{a}}$ if both $\mu_a(t,\tau) \leq \mu_{\tilde{a}}(\Delta t, \frac{\tau}{\Delta})$ and $\mu_{\tilde{a}}(t,\tau) \leq \mu_a(\Delta t, \frac{\tau}{\Delta})$ hold for some constant $\Delta > 0$.

The following two propositions are proved in the survey [37].

Proposition 3.1.3. *Let the number sequences $a = (a_i), b = (b_j)$ be such that $a_i \geq 1, b_j \geq 1$, $\lim_{i\to\infty} a_i = \infty, \lim_{j\to\infty} b_j = \infty$, and satisfy the following condition:*

$$\mu_a(t,\tau) \leq \mu_b(\Delta t, \frac{\tau}{\Delta}) \qquad 1 \leq \tau \leq t < \infty \tag{3.2}$$

with some constant $\Delta > 1$. Then there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the inequalities

$$\frac{1}{\Delta} a_i \leq b_{\sigma(i)} \leq \Delta a_i \ , \forall i \in \mathbb{N}. \tag{3.3}$$

hold.

Lemma 3.1.4. *Let for arbitrary sequences $a = (a_i)$,$b = (b_j)$, $a_i \geq 1, b_j \geq 1$, the condition (3.2) hold. Then there exists an injection $\varphi : \mathbb{N} \to \mathbb{N}$ such that the inequalities*

$$\frac{1}{\Delta^2} a_i \leq b_{\varphi(i)} \leq \Delta^2 a_i \ , \ i \in \mathbb{N}, \tag{3.4}$$

hold.

Proof. Let us define following sets ($s \in \mathbb{Z}_+$) :

$$N_s = \{i : \Delta^{s-1} < a_i \leq \Delta^s\}$$

$$\tilde{N}_s = \{i : \Delta^{s-2} < a_i \le \Delta^{s+1}\}$$

$$M_s = \{j : \Delta^{s-1} < b_j \le \Delta^{s}\}$$

$$\tilde{M}_s = \{j : \Delta^{s-2} < b_j \le \Delta^{s+1}\}$$

Let $S := \{s \in \mathbb{Z}_+ : |M_s| = \infty\}$ and $I := \bigcup_{s\in S} \tilde{N}_s$, $J := \bigcup_{s\in S} M_s$

Then both the sequences $\tilde{a} := (a_i)_{i\in\mathbb{N}\setminus I}$ and $\tilde{b} := (b_j)_{j\in\mathbb{N}\setminus J}$ have no limit points and satisfy the condition

$$\mu_{\tilde{a}}(t,\tau) \le \mu_{\tilde{b}}(\Delta t, \frac{\tau}{\Delta}) \tag{3.5}$$

So by the Proposition 3.1.3, there exists an injection $\lambda : \mathbb{N}\setminus I \to \mathbb{N}\setminus J$ such that

$$\frac{1}{\Delta}a_i < b_{\lambda(i)} \le \Delta a_i, \quad i \in \mathbb{N}\setminus I.$$

On the other hand, we construct an injection $\gamma_s : \tilde{N}_s \to M_s$ for any $s \in S$. By the same taken we have got the many-valued mapping:

$$\gamma(i) := \{j \in \mathbb{N} : \exists s,\ j = \gamma_s(i)\},\ i \in I$$

such that $\gamma(i) \neq \emptyset, i \in I$ and $\gamma(i) \cap \gamma(i') = \emptyset,\ i \neq i'$. Therefore, we can obtain an injection $\mu : I \to J$ simply by choosing of one element in each set $\gamma(i), i \in I$. So it satisfies the condition $\mu(\tilde{N}_s) \subset M_s$ which implies that

$$\frac{1}{\Delta^2}a_i < b_{\mu(i)} \le \Delta^2 a_i.$$

Thus the injection $\sigma : \mathbb{N} \to \mathbb{N}$, defined as μ on I and λ on $\mathbb{N}\setminus I$ is that what required. □

Lemma 3.1.5. *If $a = (a_k)$ and $b = (b_k)$ are sequences of positive numbers satisfying (3.2), then $E^\ell_\nu(a)$ can be quasidiagonally imbedded into $E^\ell_\nu(b)$, where $\nu = 0$ or ∞.*

Proof. Because of the similarity, we restrict ourselves to the case $\nu = \infty$. By Lemma 3.1.4, there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ satisfying

$$\frac{1}{\Delta^2}a_i < b_{\sigma(i)} \le \Delta^2 a_i, \quad \forall i. \tag{3.6}$$

Define an operator $T : E_\infty^\ell(a) \to E_\infty^\ell(b)$ by $e_i \mapsto e_{\sigma(i)}$.

By (3.6), we obtain that $\exp\big(p\frac{1}{\Delta^2}a_i\big) \le \exp\big(pb_{\sigma(i)}\big) \le \exp\big(\lambda\Delta^2 a_i\big)$. By monotonicity of the norm , we get that

$$\|\big(\exp\big(p\frac{1}{\Delta^2}a_i\big)\big)\xi_i\| \le \|\big(\exp\big(pb_{\sigma(i)}\big)\big)x_{\sigma(i)}\| \le \|\big(\exp\big(\lambda\Delta^2 a_i\big)\big)\xi_i\|$$

From which continuity of T and $T^{-1}|_{R(T)}$ follow.

□

Proposition 3.1.6. *If a sequence $a = (a_n)$ of positive numbers is bounded, then $E_\nu^\ell(a) \overset{qd}{\cong} \ell$, where $\nu = 0$ or ∞ .*

Proof. It is sufficient to show that identity operator from ℓ to $E_0^\ell(a)$ is quasidiagonal isomorphism. Let I be identity operator from

$$\ell = \{x = (\xi_n) : \|(\xi_n)\|_\ell < \infty\}$$

to

$$E_0^\ell(a) = \{x = (\xi_n) : \|(\xi_n \exp(\frac{-a_n}{p}))\|_\ell < \infty\}$$

Let $x = (\xi_n) \in E_0^\ell(a)$. Since $a = (a_n)$ is bounded, then there exists $C_1, C_2 > 0$ such that $C_1 \le (\exp(\frac{-a_n}{p})) \le C_2$. By monotonicity of the norm, we get $C_1\|(\xi_n)\|_\ell \le \|(\xi_n \exp(\frac{-a_n}{p}))\|_\ell \le C_2\|(\xi_n)\|_\ell$ from which continuity of I and I^{-1} follow. I is quasidiagonal, because $\sigma(i) = i$ and $t_i = 1$ for each $i \in I$ in (3.1). □

3.2 Some Geometric Invariant Characteristics

Let $\mathcal{X}$ be a class of locally convex spaces and Γ a set with an equivalence relation $\sim$. We say that $\gamma : \mathcal{X} \to \Gamma$ is a *linear topological invariants* if $X \simeq \tilde{X} \Rightarrow \gamma(X) \sim \gamma(\tilde{X}),\ X, \tilde{X} \in \mathcal{X}$.

In the isomorphic classification of Fréchet spaces, A.N.Kolmogorov and A. Pelczynski gave the first examples of linear topological invariants. They introduced the approximative dimensions and used them to show that the spaces of analytic functions $A(H), A(G)$ where $H \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ are not isomorphic if $n \neq m$ [15] and $A(D^n) \not\simeq A(\mathbb{C}^n)$ where D^n is the unit polydisc in $\mathbb{C}^n$ [23]. Later C. Bessaga, A. Pelczynski, S. Rolewicz [3] and B.S. Mitiagin [19] considered the diametral dimensions which are more convenient for investigation of Köthe spaces (see also M.M. Dragilev [12], [13], T.Terzioğlu [26]). But both approximative and diametral dimensions are not enough powerful to distinguish non-regular Köthe spaces. In ([20],[21]) B.S. Mitiagin used counting function μ_a to investigate the quasiequivalence property of bases in non-Montel power series spaces $E_\alpha(a)$. $\mu_a(t,\tau)$ is the strongest invariant in the category of power series spaces (see [35], [22]). In [29] V.P. Zahariuta introduced some general invariant characteristics as generalizations of Mitiagin's invariants ([20],[21]). These characteristics are convenient especially in the isomorphic classification of Cartesian or tensor products of Köthe spaces. In ([33], [35]) new geometric invariants are considered. Those are based on the asymptotic behaviour of n-diameters of pairs of *synthetic* neighborhoods of zero, built, in an invariant way, by neighborhoods taken from a given fundamental system of neighborhoods of zero; for example $d_n(\lambda_1 U_1 \cap \lambda_3 U_3, \lambda_2 U_2), d_n(\lambda_2 U_2, \overline{conv}(\lambda_1 U_1 \cup \lambda_3 U_3))$ $d_n(\lambda_1 U_1 \cap \lambda_3 U_3, \overline{conv}(\lambda_2 U_2 \cup \lambda_4 U_4))$, ... where $\overline{conv} L$ denotes the closed convex hull of the set L. In some sense, these geometric invariants are equivalent to the previous characteristics for Köthe spaces and this equivalence gives the desired invariance. For more details concerning the general theory of linear topological invariants we refer the reader to [37].

Suppose E is a linear space, U and V are absolutely convex sets in E and $\mathcal{E}_V$ is the set of all finite dimensional subspaces of E that are spanned on elements of V. Then the characteristic $\beta(V,U)$ is

$$\beta(V,U) := \sup \ \{\dim \ L : L \in \mathcal{E}_V, L \cap U \subset V\}.$$

It is obvious that $\tilde{V} \subset V$ and $U \subset \tilde{U}$ implies

$$\beta(\tilde{V}, \tilde{U}) \leq \beta(V, U). \tag{3.7}$$

and that

$$\beta(CV, U) = \beta(V, \frac{1}{C}U). \tag{3.8}$$

holds for any positive constant C.

Lemma 3.2.1. *If T is an injective linear operator on E then*

$$\beta(T(V), T(U)) = \beta(V, U). \tag{3.9}$$

Proof. Let

$$\beta(V, U) = sup\{\dim L : L \in \mathcal{E}_V, \quad L \cap U \subset V\}$$

$$\beta(T(V), T(U)) = sup\{\dim M : M \in \mathcal{E}_{T(V)}, \quad M \cap T(U) \subset T(V)\}$$

$$\mathcal{E}_V = \{\sum_{i=1}^{n} c_i \xi_i : c_i \in \mathbb{K}, \xi_i \in V\}$$

$$\mathcal{E}_{T(V)} = \{\sum_{i=1}^{n} c_i y_i : c_i \in \mathbb{K}, y_i \in T(V)\}$$

$$= \{\sum_{i=1}^{n} c_i y_i : c_i \in \mathbb{K}, y_i = T(\xi_i), \xi_i \in V\}$$

Since the operator T is injective, it sends finite sets to finite sets and hence $\dim M = \dim T(L) = \dim L$. □

Lemma 3.2.2. *Let F be a subspace of a linear space E. Then*

$$\beta_F(V \cap F, U \cap F) \leq \beta_E(V, U).$$

Proof. Let U, V be absolutely convex sets in E. Set $\tilde{U} := U \cap F$ and $\tilde{V} := V \cap F$. Since $\tilde{L} \in \mathcal{E}_{\tilde{V}}$ implies $\tilde{L} \in \mathcal{E}_V$, then

$$\{\dim\ \tilde{L} : \tilde{L} \in \mathcal{E}_{\tilde{V}}, \tilde{L} \cap \tilde{U} \subset \tilde{V}\} \subset \{\dim\ L : L \in \mathcal{E}_V, L \cap U \subset V\}.$$

Hence,

$$\beta_F(\tilde{V}, \tilde{U}) := \sup \ \{\dim \ \tilde{L} : \tilde{L} \in \mathcal{E}_{\tilde{V}}, \tilde{L} \cap \tilde{U} \subset \tilde{V}\}$$
$$\leq \sup \ \{\dim \ L : L \in \mathcal{E}_V, L \cap U \subset V\} =: \beta_E(V, U).$$

□

Let E be ℓ-Köthe space and ω_+ be the set of all sequences with positive terms. For any $a, b \in \omega_+$ and $\alpha \in (0,1)$ we set

$$a \cdot b := (a_i b_i), \qquad a^\alpha := (a_i^\alpha), \qquad a \wedge b := (\min \ \{a_i, b_i\}),$$
$$a \vee b := (\max \ \{a_i, b_i\}).$$

For any $x = (\xi_i) \in E$ and $a \in \omega_+$ we put $\|x\|_a = \|(x_i a_i)\|$ where $\|\cdot\|$ is the monotone norm on E and $B_a^\ell = \{x \in E \ : \ \|(x_i a_i)\| \leq 1\}$.

Lemma 3.2.3. *Let $a, b \in \omega_+$, then*

$$B_{a\vee b}^\ell \subset B_a^\ell \cap B_b^\ell \subset 2B_{a\vee b}^\ell. \tag{3.10}$$

Proof. Let $x \in B_{a\vee b}^\ell = \{x \in E \ : \ \|x\|_{a\vee b} < 1\}$. By definition $\|x\|_{a\vee b} = \|(\xi_i \max\{a_i, b_i\})\|$. Since $\|.\|$ is monotone and $\max \ \{a_i, b_i\} \geq a_i$, $\max \ \{a_i, b_i\} \geq b_i$, then, $\|x\|_{a\vee b} \geq \|x\|_a$ and $\|x\|_{a\vee b} \geq \|x\|_b$. Hence $x \in B_a^\ell$ and $x \in B_b^\ell$ i.e. $x \in B_a^\ell \cap B_b^\ell$.

To show the second inclusion, assume $x \in B_a^\ell \cap B_b^\ell$, i.e. $\|x\|_a < 1$ and $\|x\|_b < 1$. Since $\|.\|$ is monotone and

$$\max\{a_i, b_i\} \leq a_i + b_i$$

for each $i \in \mathbb{N}$, we obtain

$$\|(\xi_i \max\{a_i, b_i\})\| \leq \|(\xi_i a_i + \xi_i b_i)\| \leq \|x\|_a + \|x\|_b.$$

So we get, $\|x\|_{a\vee b} < 2$ which implies $x \in 2B_{a\vee b}^\ell$. Hence, $B_a^\ell \cap B_b^\ell \subset 2B_{a\vee b}^\ell$. □

Lemma 3.2.4. *Let $a, b \in \omega_+$, then*

$$\frac{1}{2}B^{\ell}_{a\wedge b} \subset conv(B^{\ell}_a \cup B^{\ell}_b) \subset B^{\ell}_{a\wedge b} \tag{3.11}$$

Proof. Let us show the first inclusion. Let $I := \{i \in \mathbb{N} : a_i \leq b_i\}$ and $J := \mathbb{N} \setminus I$. Let $x = (\xi_i)_{i\in\mathbb{N}} \in B^{\ell}_{a\wedge b}$. We define $u = (u_i)$ and $v = (v_i)$ as folows: $u_i = \begin{cases} \xi_i & \text{if} \quad i \in I \\ 0 & \text{if} \quad otherwise \end{cases}, \qquad v_i = \begin{cases} \xi_i & \text{if} \quad i \in J \\ 0 & \text{if} \quad otherwise \end{cases}.$

From the construction of u, v and from the monotonicity of the norm, we obtain that $\|u\|_a = \|u\|_{a\wedge b} \leq \|u+v\|_{a\wedge b} = \|x\|_{a\wedge b}$ and $\|v\|_b = \|v\|_{a\wedge b} \leq \|u+v\|_{a\wedge b} = \|x\|_{a\wedge b}$. Hence $u, v \in B^{\ell}_a \cup B^{\ell}_b$ and $\frac{1}{2}x = \frac{1}{2}u + \frac{1}{2}v \in conv(B^{\ell}_a \cup B^{\ell}_b)$.

Let us show the second inclusion. Let $x \in conv(B^{\ell}_a \cup B^{\ell}_b)$, i.e. $x = \alpha u + (1-\alpha)v$ where $u, v \in B^{\ell}_a \cup B^{\ell}_b$ and $\alpha \in (0,1)$. From monotonicity of the norm, we have, $\|u\|_{a\wedge b} \leq \|u\|_a$ and $\|u\|_{a\wedge b} \leq \|u\|_b$. Similarly, $\|v\|_{a\wedge b} \leq \|v\|_a$ and $\|v\|_{a\wedge b} \leq \|v\|_b$. Hence

$$\|x\|_{a\wedge b} = \|\alpha u + (1-\alpha)v\|_{a\wedge b} \leq \alpha\|u\|_{a\wedge b} + (1-\alpha)\|v\|_{a\wedge b} \leq \alpha + (1-\alpha) = 1,$$

that is, $x \in B^{\ell}_{a\wedge b}$. □

Lemma 3.2.5. *Let ℓ be a Banach sequence space with the monotone norm $\|.\|$. If $a, b \in \omega_+$ then*

$$\beta(B^{\ell}_a, B^{\ell}_b) = |\{i : \frac{a_i}{b_i} \leq 1\}|.$$

Proof. Let $I = \{i : a_i \leq b_i\}$ and $Px := \sum\limits_{i\in I} \xi_i e_i$. Let M be the linear span of the set $\{e_i : i \in I\}$. For $x \in M$, by the monotonicity of the norm, we have

$$\|x\|_a = \|(x_i a_i)\| \leq \|(x_i b_i)\| = \|x\|_b$$

which implies that $M \cap B^{\ell}_b \subset B^{\ell}_a$ and $\beta(B^{\ell}_a, B^{\ell}_b) \geq \dim M = |I|$.

Conversely, suppose L is a finite dimensional subspace in E, satisfying $L \cap B^{\ell}_b \subset B^{\ell}_a$, that is, $\|x\|_a \leq \|x\|_b$ for all $x \in L$. If $\dim L > |I|$, then there exists $x = \sum_{i=1}^{\infty} \xi_i e_i \in L, \quad x \neq 0$ such that $Px = 0$. But then $\xi_i = 0$ for

$i \in I$ and $\xi_i \neq 0$ for some $i \notin I$. Since $a_i > b_i$ for $i \notin I$, and the norm is strictly monotone (by 2.1.4) , we get $\|(\xi_i a_i)\| > \|(\xi_i b_i)\| \Rightarrow \|x\|_a > \|x\|_b$ which is a contradiction. Hence $\beta(B_a^\ell, B_b^\ell) = |I|$. □

Lemma 3.2.6. *Suppose E and $\tilde{E}$ are ℓ-Köthe spaces, (e_i) and $(\tilde{e}_i)$ are their canonical bases, and $T : E \to \tilde{E}$ is a linear operator. If $a^{(\nu)}, \tilde{a}^{(\nu)} \in \omega_+$ where $\nu = 0, 1$ and*

$$T(B^\ell_{a^{(0)}}) \subset B^\ell_{\tilde{a}^{(0)}} \quad , \quad T(B^\ell_{a^{(1)}}) \subset B^\ell_{\tilde{a}^{(1)}}$$

then for any $\alpha \in (0, 1)$ we have

$$T((B^\ell_{a^{(0)}})^{1-\alpha}(B^\ell_{a^{(1)}})^\alpha) \subset ({B^\ell}_{\tilde{a}^{(0)}})^{1-\alpha}({B^\ell}_{\tilde{a}^{(1)}})^\alpha.$$

This result follows from the Interpolation Theorem 2.2.6 and Lemma 2.2.9.

3.3 Isomorphisms of ℓ-Power Series Spaces

Proposition 3.3.1. *Let $a = (a_i)$ and $\tilde{a} = (\tilde{a}_i)$ be sequences of positive numbers such that $a_i \geq 1$ and $\tilde{a}_i \geq 1$, for all i. Then the following statements are equivalent:*

(i) $E^\ell_\nu(a) \hookrightarrow E^\ell_\nu(\tilde{a})$;

(ii) there exists $\Delta > 0$ such that for $t > \tau > 1$ we have

$$\mu_a(\tau, t) \leq \mu_{\tilde{a}}(\frac{\tau}{\Delta}, \Delta t);$$

(iii) there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\exists \Delta : \quad \frac{1}{\Delta^2} \leq \tilde{a}_{\sigma(i)} \leq \Delta^2 a_i;$$

(iv) $E^\ell_\nu(a) \overset{qd}{\hookrightarrow} E^\ell_\nu(\tilde{a})$,

where $\nu = 0, \infty$.

Proof. The implications $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$ are obvious. By Lemma 3.1.4 we have $(ii) \Rightarrow (iii)$. To complete the proof, it is sufficient to show $(i) \Rightarrow (ii)$.

Because of the similarity, we restrict ourselves to the case $\nu = \infty$. Suppose that $T : E^\ell_\nu(a) \to E^\ell_\nu(\tilde{a})$ is an embedding. Let (U_p) and (V_p) be the systems of unit balls in $E^\ell_\nu(a)$ and $E^\ell_\nu(\tilde{a})$, respectively.

Let $W_p := V_p \cap R(T)$, where $R(T)$ denotes the range of T. Choose indices

$$p_2 \leq p \leq p_1 \leq q_2 \leq q \leq q_1 \leq r_2 \leq r \leq r_1$$

so that

$$C_{p_2} W_{p_2} \supset C_p T(U_p) \supset C_{p_1} W_{p_1} \supset C_{q_2} W_{q_2}$$
$$\supset C_q T(U_q) \supset C_{q_1} W_{q_1} \supset C_{r_2} W_{r_2} \supset C_r T(U_r) \supset C_{r_1} W_{r_1}$$

Then, from the elementary properties of the characteristic β and Lemma 3.2.1, Lemma 3.2.2 it follows that

$$\beta(e^{-\tau} U_p \cap e^t U_r, U_q) = \beta(e^{-\tau} T(U_p) \cap e^t T(U_r), T(U_q))$$
$$\leq \beta(C_{p_2} e^{-\tau} W_{p_2} \cap C_{r_2} e^t W_{r_2}, C_{q_1} W_{q_1})$$
$$= \beta(C_{p_2} e^{-\tau}[V_{p_2} \cap R(T)] \cap C_{r_2} e^t [V_{r_2} \cap R(T)], C_{q_1}[V_{q_1} \cap R(T)])$$
$$\leq \beta(C_{p_2} e^{-\tau} V_{p_2} \cap C_{r_2} e^t V_{r_2}, C_{q_1} V_{q_1})$$
$$= \beta(\frac{C_{p_2}}{C_{q_1}} e^{-\tau} V_{p_2} \cap \frac{C_{r_2}}{C_{q_1}} e^t V_{r_2}, V_{q_1})$$

Choose $C := \max\{\frac{C_{p_2}}{C_{q_1}}, \frac{C_{r_2}}{C_{q_1}}\}$ then we observe that

$$\beta(e^{-\tau} U_p \cap e^t U_r, U_q) \leq \beta(C(e^{-\tau} V_{p_2} \cap e^t V_{r_2}), V_{q_1}) \tag{3.12}$$

Using Lemma 3.2.5, Lemma 3.2.3 and Lemma 3.2.4 we estimate both sides of the inequality (3.12) from below and above respectively, and obtain

$$|\{i : \frac{\max\{\exp(\tau + pa_i), \exp(-t + ra_i)\}}{\exp(qa_i)} \leq 1\}|$$
$$\leq |\{i : \frac{\max\{\exp(\tau + p_2\tilde{a}_i), \exp(-t + r_2\tilde{a}_i)\}}{\exp(q_1\tilde{a}_i)} \leq 2C\}| \qquad (3.13)$$

This inequality is equivalent to

$$|\{i : \exp(\tau + (p - q)a_i) \leq 1, \exp(-t + (r - q)a_i) \leq 1\}|$$
$$\leq |\{i : \exp(\tau + (p_2 - q_1)\tilde{a}_i) \leq 2C, \exp(-t + (r_2 - q_1)\tilde{a}_i) \leq 2C\}| \qquad (3.14)$$

Taking logarithms we obtain that

$$|\{i : \frac{\tau}{q - p} \leq a_i \leq \frac{t}{r - q}\}| \leq |\{i : \frac{\tau - \log 2C}{q_1 - p_2} \leq \tilde{a}_i \leq \frac{t + \log 2C}{r_2 - q_1}\}|. \qquad (3.15)$$

Set $\tau' := \frac{\tau}{q-p}$ and $t' := \frac{t}{r-q}$. Then, from the equation (3.15) we get that

$$|\{i : \tau' \leq a_i \leq t'\}| \leq |\{i : \frac{\tau(q - p) - \log 2C}{q_1 - p_2} \leq \tilde{a}_i \leq \frac{t(r - q) + \log 2C}{r_2 - q_1}\}| \qquad (3.16)$$

From the upper bound of the right hand side of the inequality (3.16) we get the following estimation:

$$\frac{t(r - q) + \log 2C}{r_2 - q_1} \leq \frac{t'[(r - q)^2 + \log 2C]}{r_2 - q_1} \leq \Delta t'$$

From that we conclude that our Δ must satisfy

$$\Delta \geq \frac{(r - q)^2 + \log 2C}{r_2 - q_1}. \qquad (3.17)$$

From the lower bound of the right hand side of the inequality (3.16) we get the following estimation: If $\frac{\tau'(q-p)^2}{2} \geq \log 2C$ then we have

$$\frac{\tau(q - p) - \log 2C}{q_1 - p_2} = \frac{\tau'(q - p)^2 - \log 2C}{q_1 - p_2} \geq \frac{\tau'(q - p)^2}{2(q_1 - p_2)} \geq \frac{\tau'}{\Delta}.$$

Thus, we get

$$\Delta \geq \frac{2(q_1 - p_2)}{(q - p)^2}. \qquad (3.18)$$

If not, i.e. $\frac{\tau'(q-p)^2}{2} < \log 2C$ then lower bound is less than 0. So only we need Δ must satisfy $\frac{\tau'}{\Delta} < 1$. This is possible when $\frac{2\log 2C}{\Delta(q-p)^2} < 1$ or

$$\Delta \geq \frac{2\log 2C}{(q-p)^2}. \tag{3.19}$$

If we combine the estimations (3.17), (3.18), and (3.19), we choose

$$\Delta := \max\{\frac{(r-q)^2 + \log 2C}{r_2 - q_1}, \frac{2(q_1 - p_2)}{(q-p)^2}, \frac{2\log 2C}{(q-p)^2}\}. \tag{3.20}$$

Thus, by (3.20) the inequality (3.16) terminates the proof :

$$\mu_a(\tau', t') \leq \mu_{\tilde{a}}(\frac{\tau'}{\Delta}, \Delta t')$$

□

Corollary 3.3.2. *Let $a = (a_i)$ and $\tilde{a} = (\tilde{a}_i)$ be sequences of positive numbers such that $a_i \geq 1$ and $\tilde{a}_i \geq 1$, for all i. Then the following statements are equivalent:*

(i) $E^{\ell}_{\nu}(a) \simeq E^{\ell}_{\nu}(\tilde{a})$;

(ii) there exists $\Delta > 0$ such that for $t > \tau > 1$ we have

$$\mu_a(\tau, t) \leq \mu_{\tilde{a}}(\frac{\tau}{\Delta}, \Delta t), \quad \mu_{\tilde{a}}(\tau, t) \leq \mu_a(\frac{\tau}{\Delta}, \Delta t);$$

(iii) there exists an bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\exists \Delta : \quad \frac{1}{\Delta^2} \leq \tilde{a}_{\sigma(i)} \leq \Delta^2 a_i;$$

(iv) $E^{\ell}_{\nu}(a) \overset{qd}{\simeq} E^{\ell}_{\nu}(\tilde{a})$,

where $\nu = 0, \infty$.

3.4 Complemented Imbeddings

Lemma 3.4.1. *Let $a, b \in \omega_+$ and one of the cases*

(i) $1 \le p < q < \infty$ *with* $p < 2$

(ii) $1 < q < p \le \infty$ *with* $p > 2$

be held. If the diagram

$$\begin{array}{ccc} \ell_p(a) & & \\ \downarrow T & \overset{Id}{\searrow} & \\ \ell_q & \xrightarrow[S]{} & \ell_p(b) \end{array}$$

commutes and all operators are linear and continuous, then
$\left(\dfrac{b_n}{a_n}\right) \le \left(\dfrac{1}{n}\right)^{\frac{1}{p}-\frac{1}{q_1}}$ *where* $q_1 := \min(2, q)$.

Proof. Let us consider the case (i). From the diagram, we have

$$|Sy|_{l_p(b)} \le C_1 \|y\|_{l_q} \quad \forall y \in l_q. \tag{3.21}$$

$$\|Tx\|_{l_q} \le C_2 |x|_{l_p(a)} \quad \forall x \in l_p(a). \tag{3.22}$$

That is, $|STx|_{l_p(b)} \le C_1 \|Tx\|_{l_q} \le C_1 C_2 |x|_{l_p(a)}$

Without loss of generality we assume that $(\frac{b_n}{a_n})$ is decreasing. (If not, one can reorder it.) Using (3.21) , (3.22) and the fact that the space l_q is of the type $q_1 = min(2, q)$ [see [18], Vol.2, p.72] we obtain for any n and $(\theta_i)_{i=1}^n$, $\theta_i = \pm 1$ such that

$$\left(\frac{b_n}{a_n} n^{\frac{1}{p}}\right) \le \left(\sum_{i=1}^n \left(\frac{b_i}{a_i}\right)^p\right)^{\frac{1}{p}} \le average\left\{\left|\sum_{i=1}^n \theta_i \frac{STe_i}{a_i}\right|_{l_p(b)} : \theta_i = \pm 1\right\}$$

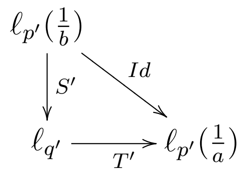
$$\le average\left\{\left|\sum_{i=1}^n \theta_i \frac{Te_i}{a_i}\right|_{l_q} : \theta_i = \pm 1\right\} \le MC_1\left(\sum_{i=1}^n \left(\frac{\|Te_i\|_{l_q}}{a_i}\right)^{q_1}\right)^{\frac{1}{q_1}}$$

$$\le MC_1C_2\left(\sum_{i=1}^n \left(\frac{|e_i|_{l_p(a)}}{a_i}\right)^{q_1}\right)^{\frac{1}{q_1}} \le MC_1C_2 n^{\frac{1}{q_1}} \tag{3.23}$$

Thus $\left(\frac{b_n}{a_n}\right) \leq \left(\frac{1}{n}\right)^{\frac{1}{p}-\frac{1}{q_1}}$.

For the case(ii), we use duality argument. It is easy to get $(l_p(a))' = l_{p'}(\frac{1}{a}), (l_q)' = l_{q'}, (l_p(b))' = l_{p'}(\frac{1}{b})$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

From the above diagram one can obtain the following diagram which commutes:



Then $\frac{\frac{1}{a_n}}{\frac{1}{b_n}} = \frac{b_n}{a_n}$ is estimated as in (3.23). $\square$

From the above lemma, we conclude also that $\frac{1}{a_n} \in l_r$ for any $r > \frac{1}{p} - \frac{1}{q_1}$ and $n \in \mathbb{N}$.

Lemma 3.4.2. *$K^{l_p}(a_{nk})$ is nuclear if and only if*

$$\exists r \forall k \exists \sigma(k) \ : \ \sum_{i=1}^{\infty} \left(\frac{a_{nk}}{a_{n\sigma(k)}}\right)^r < \infty \tag{3.24}$$

where $\sigma : \mathbb{N} :\rightarrow \mathbb{N}$ is an injection map.

Proof. It is well known fact that (see [25]) $K^{l_p}(a_{nk})$ is nuclear if and only if

$$\sum_{i=1}^{\infty} \frac{a_{nk}}{a_{n\phi(k)}} < \infty. \tag{3.25}$$

So it is sufficient to show that (3.24) implies (3.25). Assume that (3.24) holds. Fix $r \in \mathbb{N}$, for any $k \in \mathbb{N}$ there exists an injective map $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{i=1}^{\infty} \left(\frac{a_{nk}}{a_{n\phi(k)}}\right)^r < \infty$, that is, $\left(\frac{a_{nk}}{a_{n\phi(k)}}\right) \in l_r$. In the same way, for $\phi(k)$, there exist $\phi^2(k)$ such that $\sum_{i=1}^{\infty} \left(\frac{a_{n\phi(k)}}{a_{n\phi^2(k)}}\right)^r < \infty$, that is, $\left(\frac{a_{n\phi(k)}}{a_{n\phi^2(k)}}\right) \in l_r$.

If we determine recursively $\phi^s(k)$, for $s = 1, ..., r$ so that $\sum_{i=1}^{\infty} \left(\frac{a_{n\phi^{s-1}(k)}}{a_{n\phi^s(k)}}\right)^r < \infty$, that is, $\left(\frac{a_{n\phi^{s-1}(k)}}{a_{n\phi^s(k)}}\right) \in l_r$.

So we have, $\left(\frac{a_{nk}}{a_{n\phi^r(k)}}\right) = \left(\frac{a_{nk}}{a_{n\phi(k)}}\right) . \left(\frac{a_{n\phi}(k)}{a_{n\phi^2(k)}}\right) ... \left(\frac{a_{n\phi^{r-1}(k)}}{a_{n\phi^r(k)}}\right)$

By generalized Hölder's inequality

$$\sum_{n=1}^{\infty} \left| \left(\frac{a_{nk}}{a_{n\phi^r(k)}}\right) \right| = \sum_{n=1}^{\infty} \left| \left(\frac{a_{nk}}{a_{n\phi(k)}}\right) . \left(\frac{a_{n\phi}(k)}{a_{n\phi^2(k)}}\right) ... \left(\frac{a_{n\phi^{r-1}(k)}}{a_{n\phi^r(k)}}\right) \right|$$
$$\leq C \left\| \left(\frac{a_{nk}}{a_{n\phi(k)}}\right) \right\|_{l_r} . \left\| \left(\frac{a_{n\phi}(k)}{a_{n\phi^2(k)}}\right) \right\|_{l_r} ... \left\| \left(\frac{a_{n\phi^{r-1}(k)}}{a_{n\phi^r(k)}}\right) \right\|_{l_r}$$

Since all factors are finite, product is finite. □

Theorem 3.4.3. *Let $A = (a_{in}), B = (b_{in})$ be Köthe matrices and one of the cases*

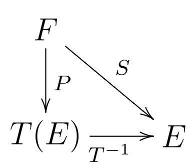
(i) $1 \leq p < q < \infty$ *with* $p < 2$

(ii) $1 < q < p \leq \infty$ *with* $p > 2$

be held. If $K^{l_p}(A)$ complementedly imbedded in $K^q(B)$ then $K^{l_p}(A)$ is nuclear.

Proof. It is sufficient to check whether the criteria of the Lemma 3.4.2 is satisfied.

Let $E := K^{l_p}(a_{in})$ and $F := K^{l_q}(b_{in})$ with the systems of seminorms $(|.|_n)_{n\in\mathbb{N}}$ and $(\|.\|_m)_{m\in\mathbb{N}}$, respectively. Let $T : E \to F$ be a complemented isomorphic embedding, that is, $F = T(E) \oplus M$. Let $P : F \to T(E)$ be a continuous projection and $S = T^{-1}P : F \to E$ be the left inverse of T i.e. $ST = T^{-1}PT = Id_E$, that is, $kerS = M$ and $F/kerS \simeq E$.



By continuity of T and S we get for each k, there exists m, k_1, C_1, C_2 such that $|Sy|_k \leq C_1\|y\|_m$, $y = Tx$ and $\|Tx\|_m \leq C_2|x|_{k_1}$.

Hence, for the weights $a^k := (a_{ik})$, $a^{k_1} := (a_{ik_1})$ and $b^m := (a_{im})$ the following diagram commutes:

$$\begin{array}{ccc} \ell_p(a^{k_1}) & & \\ \downarrow T & \overset{Id}{\searrow} & \\ \ell_q(b^m) & \xrightarrow[S]{} & \ell_p(a^k) \end{array}$$

Hence, if we apply the Lemma 3.4.1, we get that $\left(\frac{a_{nk}}{a_{nk_1}}\right) \leq \left(\frac{1}{n}\right)^{\frac{1}{p}-\frac{1}{q_1}}$ where $q_1 = min(2, p)$

□

3.5 Complemented Subspaces of $E_0^{l_2}(a) \times E_\infty^{l_2}(b)$

In this section, we use the result [24] (or see e.g. [28]) of J.Prada about complemented subspaces of Cartesian products of Fréchet spaces and the result [30] of V.P. Zahariuta about isomorphisms of Cartesian products of locally convex spaces to observe that any stable complemented subspace of $E_0^{l_2}(a) \times E_\infty^{l_2}(b)$ with $a_i \to \infty$ (or $b_i \to \infty$) is isomorphic to the product of basic supspaces of $E_0^{l_2}(a)$ and $E_\infty^{l_2}(b)$, respectively.

Definition 3.5.1. *Let X and Y be topological vector spaces. A linear operator $T : X \to Y$ is called a compact (respectively, bounded) if there exists a neighborhood U in X such that its image $T(U)$ is precompact (respectively, bounded) in Y.*

We say that an ordered pair of locally convex spaces (X, Y) satisfies condition $\mathcal{K}(respectively, \mathcal{B})$, $(X, Y) \in \mathcal{K}(respectively, (X, Y) \in \mathcal{B})$, if every linear continuous operator $T : X \to Y$ is compact (respectively, bounded). We also say that (X, Y) has the *compact* (respectively, *bounded*) *factorization* property and write $(X, Y) \in \mathcal{KF}$ (respectively, $(X, Y) \in \mathcal{BF}$) if each linear continuous operator $T : X \to X$ that factors over Y (That is, $T = S_1 S_2$ such that $S_2 : X \to Y$ and $S_1 : Y \to X$) is compact (respectively, bounded).

Definition 3.5.2. *Let X and Y be locally convex spaces. A linear operator $T : X \to Y$ is called a near-isomorphism if T is an open map with finite* *dimensional kernel and finite codimensional closed range. X and Y are said to be nearly isomorphic ($X \approx Y$) if there exists a near-isomorphism T from X onto Y.*

Proposition 3.5.3. *Let X, Y be locally convex spaces and $(X, Y) \in \mathcal{K}$. Then $(X_0, Y_0) \in \mathcal{K}$ for every subspace X_0 which is topologically complemented in X, and any subspace Y_0 of Y.*

Proof. Let $T : X_0 \to Y_0$ be an arbitrary linear, continuous operator. By the assumption, there exists a subspace X_1 in X such that $X = X_0 \oplus X_1$. Let $T : X \to Y$ be the linear, continuous operator $T(x) = \begin{cases} T_0 x & \text{if} \quad x \in X_0, \\ 0 & \text{if} \quad x \in X_1. \end{cases}$

Since $(X, Y) \in \mathcal{K}$, this operator is compact and therefore T_0 is compact too. Hence $(X_0, Y_0) \in \mathcal{K}$.

Proposition 3.5.4. *([24]) A complemented subspace of $X_1 \times X_2$, where X_1, X_2 are Fréchet spaces such that $(X_1, X_2) \in \mathcal{K}$ is isomorphic to a subspace of $X_1 \times X_2$ of the form $L_1 \times L_2$, where L_1, respectively L_2, is complemented subspace of X_1, respectively X_2 .*

Proposition 3.5.5. *([30]) Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ be locally convex spaces and $(X_1, Y_2) \in \mathcal{K}$, $(Y_1, X_2) \in \mathcal{K}$. Then $X \approx Y$ if and only if $X_1 \approx Y_1$ and $X_2 \approx Y_2$.*

Theorem 3.5.6. *Let L be a complemented subspace of $E_0^{l_2}(a) \times E_\infty^{l_2}(b)$, where $a, b \in \omega_+$. If $L = L^2$ and $a_i \to \infty$ (or $b_i \to \infty$), then L is isomorphic to basic subspace of $E_0^{l_2}(a) \times E_\infty^{l_2}(b)$.*

Proof. Let $X_1 = E_0^{l_2}(a)$ and $X_2 = E_\infty^{l_2}(b)$. By Proposition 3.5.4 we have that if L is complemented subspaces of $X_1 \times X_2$, then there exists L_1, L_2 such that L_ν are complemented subspace of X_ν, where $\nu = 1, 2$.

Due to Mitiagin [21], we know that any complemented subspace of a finite type power series space is basic, that is, $L_1 \simeq E_0^{l_2}(\tilde{a})$ where $\tilde{a} := (a_{i_n})$.

Since $a_i \to \infty$, we have $(X_1, X_2) \in \mathcal{K}$ (Theorem 3 [30]) and by Proposition 3.5.3, we get $(L_1, L_2^2) \in \mathcal{K}$, $(L_1^2, L_2) \in \mathcal{K}$.

Taking into account Proposition 3.5.5, we obtain that $L \approx L^2$ iff $L_1 \approx L_1^2$ and $L_2 \approx L_2^2$.

By Wagner [27] (see also [2]), if $L_2 \approx L_2^2$ then L_2 has a basis. Hence $L_2 = E_\infty^{l_2}(\tilde{b})$, where $\tilde{b} := (b_{j_k})$. □

CHAPTER 4

CARTESIAN PRODUCTS OF ℓ-POWER SPACES

4.1 $d_1 - d_2$ properties

Invariant of the type $\mathcal{D}_1, \mathcal{D}_2$ were used in various forms by Dragilev, Zahariuta, Dubinsky, Robinson, Vogt, Wagner. Definition of the Dragilev's invariants were for Fréchet spaces with regular basis. In 1973, Zahariuta defined the classes d_1, d_2 for Fréchet spaces with an absolute basis. Later, Zahariuta (1974, 1978, 1980) improved the definion of d_1, d_2 for Fréchet spaces without adding an extra condition to the bases. Here, we will use the notation of Zahariuta (1980). Vogt (1982), Vogt and Wagner (1980), Wagner (1980) used DN for $\mathcal{D}_1$, $\underline{DN}$ for Ω_2 , Ω for Ω_1, and $\overline{\Omega}$ for $\mathcal{D}_2$.

Definition 4.1.1. *(cf [30],[36]) Let X be a Fréchet space, $\{|x|_p : p \in \mathbb{N}\}$ a system of norms defining the topology of X, and*

$$|x|_p^* := \sup\{|x'(x)| : x \in X, |x|_p \leq 1\} \tag{4.1}$$

a polar system of norms in the strong dual X^*. We define four classes of spaces by means of relations:

$$X \in \mathcal{D}_1 := \exists p\ \forall q\ \exists r \exists C : \quad {|x|_q}^2 \leq C|x|_p|x|_r \ , x \in X, \tag{4.2}$$

$$X \in \mathcal{D}_2 := \forall p\ \exists q\ \forall r \exists C : \quad (|x'|_q^*)^2 \leq C|x'|_p^*|x'|_r^* \ , x' \in X^* \tag{4.3}$$

$$X \in \Omega_1 := \forall p\ \exists q\ \forall r \exists C \exists \alpha > 0 : \quad |x'|_q^* \leq C(|x'|_p^*)^{1-\alpha}(|x'|_r^*)^{\alpha} \ , x' \in X^* \tag{4.4}$$

$$X \in \Omega_2 := \exists p\ \forall q\ \exists \alpha < 1 \exists r \exists C : \quad |x|_q \leq C(|x|_p)^{1-\alpha}(|x|_r)^{\alpha} \ , x \in X, \tag{4.5}$$

Theorem 4.1.2. *Let $A = (a_{kn})_{k,n\in\mathbb{N}}$ be an Köthe matrix and $\ell \in \Lambda$. $K^\ell(a_{kp}) \in \mathcal{D}_1$ if and only if one of the following conditions is satisfied*

(i) $\exists p\ \forall q\ \exists r\ \exists C: \quad {a_{kq}}^2 \leq C a_{kp} a_{kr}$

(ii) $\exists p\ \forall q\ \exists r\ \exists C: \quad a_{kq} \leq C a_{kp}^{1/2} a_{kr}^{1/2}$

(iii) $\exists p\ \forall q\ \forall \alpha > 0\ \exists\ \ \exists r\ \exists C: \quad a_{kq} \leq C a_{kp}^{1-\alpha} a_{kr}^{\alpha}$

Proof. It is clear that $(i), (ii)$ and (iii) are equivalent.

Necessity. Take $x = (e_k)_{k\in\mathbb{N}}$. Since $|e_k| = a_{kp}$ the definition 4.2 gives (i).

Sufficiency. Without loss of generality, $a_{kn} \neq 0 (\forall k, n)$. Assume (iii) holds. By monotonicity of the norm in ℓ, we get that

$$\|x\|_{\ell(a_{kq})} \leq C \|x\|_{\ell(a_{kp}^{1/2} a_{kr}^{1/2})}. \tag{4.6}$$

Now we consider an analytic scale of spaces $G_\alpha := \ell(a_{kp}^{1-\alpha} a_{kr}^{\alpha})$ (see section 1.2 and section 1.3) with the norm

$$\|x\|_\alpha = \|x\|_{G_\alpha} = \sup_{-\infty<\tau<\infty} \|(\xi_k)(\frac{a_{kr}}{a_{kr}})^{\alpha+i\tau}(a_{kp})\|.$$

By Theorem 2.2.6 and Theorem 2.2.8 we have

$$\|x\|_\alpha \leq C(\|x\|_0)^{1-\alpha}(\|x\|_1)^\alpha. \tag{4.7}$$

Combinin the equation (4.7) and the equation (4.6) we obtain that

$$\|x\|_{\ell(a_{kq})} \leq C\|x\|_{\ell(a_{kp}^{1-\alpha} a_{kr}^{\alpha})} \leq C(\|x\|_{\ell(a_{kp})})^{1-\alpha}(\|x\|_{\ell(a_{kr})})^\alpha.$$

Taking $\alpha = 1/2$ and usuing the definition of the seminorm, $|x|_n = \|x\|_{\ell(a_{kn})}$, we get that $|x|_q \leq C(|x|_p)^{1/2}(|x|_r)^{1/2}$. Hence, $K^\ell(a_{kp}) \in \mathcal{D}_1$. □

Let $\ell \in \Lambda$. Define

$$\ell^* := \{x' = (\xi'_i) : \|x'\|_{\ell^*} < \infty\}$$

with the norm

$$\|x'\|_{\ell^*} := \sup\left\{\left|\sum_{i=1}^{\infty}\xi_i'\xi_i\right| < C(x)\|x\|_{\ell}, \ ,\forall x = (\xi_i) \in \ell\right\}$$

The class of such ℓ^* spaces denoted by Λ^*.

Proposition 4.1.3. *Let $\ell^* \in \Lambda^*$. Then the norm $\|.\|_{\ell^*}$ is monotone.*

Proof. Let $x' = (\xi_i')$ and $y' = (\eta_i') \in \ell^*$ with $\eta_i' := \alpha_i\xi_i'$, $|\alpha_i| \leq 1$. It is sufficient to show that $\|y'\|_{\ell^*} \leq \|x'\|_{\ell^*}$.

$$\|y'\|_{\ell^*} := \left\{\left|\sum_{i=1}^{\infty}\eta_i'\eta_i\right| : \|(\eta_i)\|_{\ell} \leq 1\right\} = \left\{\left|\sum_{i=1}^{\infty}\alpha_i\xi_i'\eta_i\right| : \|(\eta_i)\|_{\ell} \leq 1\right\}$$

$$= \left\{\left|\sum_{i=1}^{\infty}\xi_i'\zeta_i\right| : \zeta_i = \alpha_i\eta_i, \quad \|(\eta_i)\|_{\ell} \leq 1\right\}$$

By Lemma 2.1.2

$$\leq \left\{\left|\sum_{i=1}^{\infty}\xi_i'\zeta_i\right| : \|(\zeta_i)\|_{\ell} \leq 1\right\} = \|x'\|_{\ell^*}.$$

□

Theorem 4.1.4. *Let $A = (a_{kn})_{k,n\in\mathbb{N}}$ be an Köthe matrix and $\ell \in \Lambda$. $K^{\ell}(a_{kp}) \in \mathcal{D}_2$ if and only if one of the following conditions is satisfied*

(i) $\forall p\ \exists q\ \forall r\ \exists C: \quad a_{kp}a_{kr} \leq C{a_{kq}}^2$

(ii) $\forall p\ \exists q\ \forall r\ \exists C: \quad \frac{1}{a_{kq}} \leq C\left(\frac{1}{a_{kp}}\right)^{1/2}\left(\frac{1}{a_{kr}}\right)^{1/2}$

(iii) $\exists p\ \forall q\ \forall\alpha > 0\ \exists\ \ \exists r\ \exists C: \quad \frac{1}{a_{kq}} \leq C\left(\frac{1}{a_{kp}}\right)^{1-\alpha}\left(\frac{1}{a_{kr}}\right)^{\alpha}$

The theorem is proved by using a simple duality argument and Proposition 4.1.3.

4.2 Cartesian Products of ℓ-Power Series Spaces

In this section we obtain a complete isomorphic classification of the Cartesian products of the kind $E_0^\ell(a) \times E_\infty^\ell(b)$, where $E_0^\ell(a)$ is a finite ℓ-power series space and $E_\infty^\ell(b)$ is an infinite ℓ-power series space. In the case where at least one of the Cartesian factors is a Schwartz space, such a classification is known by the result of Zahariuta obtained in [30] by using the theory of Fredholm operators. If both Cartesian factors are non-Schwartz spaces we use the method of generalized linear topological invariants developed in [29], [32], [37] as a generalization of the classical invariants [3], [15],[19], [23].

Theorem 4.2.1. *Let $\ell \in \Lambda$. If $E_0^\ell(a^{(0)}) \times E_\infty^\ell(a^{(1)}) \simeq E_0^\ell(\tilde{a}^{(0)}) \times E_\infty^\ell(\tilde{a}^{(1)})$, then the following relations hold:*

$$\exists M, \tau_0 > 0 : |\{k : \tau \leq a_k^{(0)} \leq t\}| \leq |\{k : \frac{\tau}{M} \leq \tilde{a}_k^{(0)} \leq Mt\}|, \ \tau \geq \tau_0; \quad (4.8)$$

$$\exists M, \tau_0 > 0 : |\{k : \tau \leq a_k^{(1)} \leq t\}| \leq |\{k : \frac{\tau}{M} \leq \tilde{a}_k^{(1)} \leq Mt\}|, \ \tau \geq \tau_0. \quad (4.9)$$

where $a^{(\nu)}, \tilde{a}^{(\nu)} \in \omega_+, \ \nu = 0, 1$

Proof. The Cartesian product $E_0^\ell(a^{(0)}) \times E_\infty^\ell(a^{(1)})$ and $E_0^\ell(\tilde{a}^{(0)}) \times E_\infty^\ell(\tilde{a}^{(1)})$ are naturally isomorphic to ℓ-Köthe spaces $X = K^\ell(c_{ip})$ and $\tilde{X} = K^\ell(d_{ip})$ where

$$c_{ip} = \begin{cases} \exp(-\frac{a_k^{(0)}}{p}) & \text{if} \quad i = 2k-1 \\ \exp(p a_k^{(1)}) & \text{if} \quad i = 2k \end{cases} \quad d_{ip} = \begin{cases} \exp(-\frac{\tilde{a}_k^{(0)}}{p}) & \text{if} \quad i = 2k-1 \\ \exp(p \tilde{a}_k^{(1)}) & \text{if} \quad i = 2k \end{cases}$$

Assume that X and $\tilde{X}$ are isomorphic and that $T : X \to \tilde{X}$ is an isomorphism. Let (U_p) and (V_p) be the systems of unit balls in X and $\tilde{X}$, respectively. We choose indices

$$p_2 < p < p_1 < q_2 < q < q_1 < r_2 < r < r_1 < s_2 < s < s_1,$$

for which we have the following properties $2p_1 < q_2, \quad 2q_1 < r_2$ and also

$$C_{p_2}V_{p_2} \supset T(U_p), \quad C_pT(U_p) \supset V_{p_1}, \quad C_{p_1}V_{p_1} \supset V_{q_2}, \quad C_{q_2}V_{q_2} \supset T(U_q),$$

$$C_qT(U_q) \supset V_{q_1}, \quad C_{q_1}V_{q_1} \supset V_{r_2}, \quad C_{r_2}V_{r_2} \supset T(U_r), \quad C_rT(U_r) \supset V_{r_1},$$

$$C_{r_1}V_{r_1} \supset V_{s_2}, \quad C_{s_2}V_{s_2} \supset T(U_s), \quad C_sT(U_s) \supset V_{s_1},$$

for some constants $C_p, C_{p_1}, C_{p_2}, C_q, C_{q_1}, C_{q_2}, C_r, C_{r_1}, C_{r_2}, C_s, C_{s_1}, C_{s_2}$. Since T is an isomorphism, we use (3.9) and we consider the system of unit balls (U_p) instead of $T(U_p)$ in the following estimations:

$$U_q \cap e^tU_s := B^\ell_{(c_{iq})} \cap e^tB^\ell_{(c_{is})} = B^\ell_{(c_{iq})} \cap B^\ell_{(e^{-t}c_{is})}.$$

If we apply the Lemma 3.2.3 we obtain that

$$B^\ell_{(c_{iq} \vee e^{-t}c_{is})} \subset B^\ell_{(c_{iq})} \cap e^tB^\ell_{(c_{is})} \subset 2B^\ell_{(c_{iq} \vee e^{-t}c_{is})}.$$

In the same way, from

$$conv(U_q \cup U_p^{\frac{1}{2}}U_r^{\frac{1}{2}} \cup e^\tau U_r) := conv(B^\ell_{(c_{iq})} \cup B^\ell_{(c_{ip}^{\frac{1}{2}}c_{ir}^{\frac{1}{2}})} \cup e^\tau B^\ell_{(c_{ir})})$$

$$= conv(B^\ell_{(c_{iq})} \cup B^\ell_{(c_{ip}^{\frac{1}{2}}c_{ir}^{\frac{1}{2}})} \cup B^\ell_{(e^{-\tau}c_{ir})}),$$

by Lemma 3.2.4 we get that

$$\frac{1}{2}B^\ell_{(c_{iq} \wedge c_{ip}^{\frac{1}{2}}c_{ir}^{\frac{1}{2}} \wedge e^{-\tau}c_{ir})} \subset conv(B^\ell_{(c_{iq})} \cup B^\ell_{(c_{ip}^{\frac{1}{2}}c_{ir}^{\frac{1}{2}})} \cup e^\tau B^\ell_{(c_{ir})}) \subset B^\ell_{(c_{iq} \wedge c_{ip}^{\frac{1}{2}}c_{ir}^{\frac{1}{2}} \wedge e^{-\tau}c_{ir})}.$$

From our choice, we observe the following:

$$B^\ell_{(c_{iq})} \cap e^tB^\ell_{(c_{is})} \subset C_{iq_2}B^\ell_{(d_{iq_2})} \cap C_{is_2}e^tB^\ell_{(d_{is_2})}, \tag{4.10}$$

$$conv(B^\ell_{(c_{iq})} \cup B^\ell_{(c_{ip}^{\frac{1}{2}}c_{ir}^{\frac{1}{2}})} \cup e^\tau B^\ell_{(c_{ir})})$$

$$\supset conv(\frac{1}{C_{q_1}} B^{\ell}_{(d_{iq_1})} \cup \frac{1}{\sqrt{C_{p_1} C_{r_1}}} B^{\ell}_{(d^{\frac{1}{2}}_{ip_1} d^{\frac{1}{2}}_{ir_1})} \cup \ \frac{1}{C_{r_1}} e^{\tau} B^{\ell}_{(d_{ir_1})}). \tag{4.11}$$

If we combine (3.7) with the above observations (4.10) and (4.11)

$$\beta(B^{\ell}_{(c_{iq})} \cap e^t B^{\ell}_{(c_{is})}, conv(B^{\ell}_{(c_{iq})} \cup B^{\ell}_{(c^{\frac{1}{2}}_{ip} c^{\frac{1}{2}}_{ir})} \cup e^{\tau} B^{\ell}_{(c_{ir})}))$$

$$\leq \beta(C_{iq_2} B^{\ell}_{(d_{iq_2})} \cap C_{is_2} e^t B^{\ell}_{(d_{is_2})}, conv(\frac{1}{C_{q_1}} B^{\ell}_{(d_{iq_1})} \cup \frac{1}{\sqrt{C_{p_1} C_{r_1}}} B^{\ell}_{(d^{\frac{1}{2}}_{ip_1} d^{\frac{1}{2}}_{ir_1})} \cup \ \frac{1}{C_{r_1}} e^{\tau} B^{\ell}_{(d_{ir_1})})). \tag{4.12}$$

Set $C_{min} := \min \ (C_{q_1}, C_{p_1}, C_{r_1})$ and $C_{max} := \max \ (C_{iq_2}, C_{is_2})$ then

$$\beta(B^{\ell}_{(c_{iq})} \cap e^t B^{\ell}_{(c_{is})}, conv(B^{\ell}_{(c_{iq})} \cup B^{\ell}_{(c^{\frac{1}{2}}_{ip} c^{\frac{1}{2}}_{ir})} \cup e^{\tau} B^{\ell}_{(c_{ir})}))$$

$$\leq \beta(C_{max} B^{\ell}_{(d_{iq_2})} \cap e^t B^{\ell}_{(d_{is_2})}, conv(\frac{1}{C_{min}} B^{\ell}_{(d_{iq_1})} \cup B^{\ell}_{(d^{\frac{1}{2}}_{ip_1} d^{\frac{1}{2}}_{ir_1})} \cup e^{\tau} B^{\ell}_{(d_{ir_1})})) \tag{4.13}$$

If we set $C := C_{max}. \ C_{min}$ and use (3.8) then we get more simple form, as follows

$$\beta(B^{\ell}_{(c_{iq})} \cap e^t B^{\ell}_{(c_{is})}, conv(B^{\ell}_{(c_{iq})} \cup B^{\ell}_{(c^{\frac{1}{2}}_{ip} c^{\frac{1}{2}}_{ir})} \cup e^{\tau} B^{\ell}_{(c_{ir})})) \leq$$

$$\beta(C B^{\ell}_{(d_{iq_2})} \cap e^t B^{\ell}_{(d_{is_2})}, conv(B^{\ell}_{(d_{iq_1})} \cup B^{\ell}_{(d^{\frac{1}{2}}_{ip_1} d^{\frac{1}{2}}_{ir_1})} \cup e^{\tau} B^{\ell}_{(d_{ir_1})})) \tag{4.14}$$

$$\beta(B^{\ell}_{(c_{iq} \vee e^{-t} c_{is})}, B^{\ell}_{(c_{iq} \wedge c^{\frac{1}{2}}_{ip} c^{\frac{1}{2}}_{ir} \wedge e^{-\tau}(c_{ir})}) \leq \beta(2 B^{\ell}_{(d_{iq_2} \vee e^{-t} d_{is_2})}, \frac{1}{2} B^{\ell}_{(d_{iq_1} \wedge d^{\frac{1}{2}}_{ip_1} d^{\frac{1}{2}}_{ir_1} \wedge e^{-\tau}(d_{ir_1})}) \tag{4.15}$$

In that step, if we use the Lemma 3.2.5, then we observe

$$|\{i : \frac{\max{(c_{iq}, e^{-t} c_{is})}}{\min{(c_{iq}, c^{\frac{1}{2}}_{ip} c^{\frac{1}{2}}_{ir}, e^{-\tau} c_{ir})}} \leq 1\}| \leq |\{i : \frac{\max{(d_{iq_2}, e^{-t} d_{is_2})}}{\min{(d_{iq_1}, d^{\frac{1}{2}}_{ip_1} d^{\frac{1}{2}}_{ir_1}, e^{-\tau} d_{ir_1})}} \leq 4C\}| \tag{4.16}$$

If we proceed the estimation as omitting the trivial inequalities, we get

$$|\{i : \frac{c_{iq}}{c_{ip}^{\frac{1}{2}} c_{ir}^{\frac{1}{2}}} \leq 1, \frac{c_{iq}}{e^{-\tau} c_{ir}} \leq 1, \frac{e^{-t} c_{is}}{c_{iq}} \leq 1\}|$$

$$\leq |\{i : \frac{d_{iq_2}}{d_{ip_1}^{\frac{1}{2}} d_{ir_1}^{\frac{1}{2}}} \leq 4C, \frac{d_{iq_2}}{e^{-\tau} d_{ir_1}} \leq 4C, \frac{e^{-t} d_{is_2}}{d_{iq_1}} \leq 4C\}| \tag{4.17}$$

If we repeat all steps for the following

$$\beta \ (U_q \cap U_p^{\frac{1}{2}} U_r^{\frac{1}{2}} \cap e^t U_r, conv \ (U_q \cup e^{\tau} U_s))$$

$$\leq \beta \ (cV_{q_2} \cap V_{p_2}^{\frac{1}{2}} V_{r_2}^{\frac{1}{2}} \cap e^t V_{r_2}, conv \ (V_{q_1} \cup e^t V_{s_1}), \tag{4.18}$$

we similarly obtain the following;

$$|\{i : \frac{\max (c_{iq}, c_{ip}^{\frac{1}{2}} c_{ir}^{\frac{1}{2}}, e^{-\tau} c_{ir})}{\min (c_{iq}, e^{-\tau} c_{is})} \leq 1\}|$$

$$\leq |\{i : \frac{\max \ (d_{iq_2}, d_{ip_2}^{\frac{1}{2}} d_{ir_2}^{\frac{1}{2}}, e^{-\tau} d_{ir_2})}{\min (d_{iq_1}, e^{-\tau} d_{is_1})} \leq 4c\}|, \tag{4.19}$$

and

$$|\{i : \frac{c_{ip}^{\frac{1}{2}} c_{ir}^{\frac{1}{2}}}{c_{iq}} \leq 1, \frac{c_{iq}}{e^{-\tau} c_{is}} \leq 1, \frac{e^{-t} c_{ir}}{c_{iq}} \leq 1\}|$$

$$\leq |\{i : \frac{d_{ip_2}^{\frac{1}{2}} d_{ir_2}^{\frac{1}{2}}}{d_{iq_1}} \leq 4C, \frac{d_{iq_2}}{e^{-t} d_{is_1}} \leq 4C, \frac{e^{-\tau} d_{ir_2}}{d_{iq_1}} \leq 4C\}|. \tag{4.20}$$

Let us analyze these. The left-hand side of (4.17) (respectively (4.20)) is equal to the left-hand side of (4.16) (respectively (4.19)). The right-hand side of (4.17) (respectively (4.20)) is greater than or equal to the right-hand side of (4.16) (respectively (4.19)). Also (4.17) implies the relation (4.16) . The first inequality in the left-hand side of (4.17) is $c_{iq} \leq c_{ip}^{\frac{1}{2}} c_{ir}^{\frac{1}{2}}$. For the odd indices $i = 2k - 1$ this is equivalent to the inequality $(-\frac{1}{q} + \frac{1}{2p} + \frac{1}{2r}) a_k^{(0)} \leq 0$, which is impossible because $q > 2p$. For the even indices $i = 2k$ this is equivalent to $(2q-p-r)a_k^{(1)} \leq 0$, which is always true since $r > 2q$. Therefore the left-hand side of (4.17) equals

$$|\{k : \frac{\tau}{r-q} \leq a_k^{(1)} \leq \frac{t}{s-q}\}| \tag{4.21}$$

Consider the right-hand side of (4.17) .

The first inequality there is $d_{iq_2} \leq 4Cd_{ip_1}^{\frac{1}{2}} d_{ir_1}^{\frac{1}{2}}$. For odd indices $i = 2k-1$ this is equivalent to the inequality

$$\tilde{a}_k^{(0)} \leq \tau_1 := \frac{\log 4C}{-\frac{1}{q_2} + \frac{1}{2p_1} + \frac{1}{2r_1}}$$

In this case the other two inequalities imply

$$\frac{\tau - \log 4C}{\frac{1}{q_2} - \frac{1}{r_1}} \leq \tilde{a}_k^{(0)} \leq \frac{t + \log 4C}{\frac{1}{q_1} - \frac{1}{s_2}}.$$

Therefore, for $\tau > \tau_2 := \tau(\frac{1}{q_2} - \frac{1}{r_1} + \log 4C)$, the triple of inequalities in the right-hand side of (4.17) does not hold for odd indices.
For the even indices $i = 2k$, the first inequality in the right-hand side of (4.17) is equivalent to the inequality $(2q_2 - p_1 - r_1)\tilde{a}_k^{(1)} \leq \log 4C$, which is always true because $r_1 > 2q_2$ (we can assume without loss of generality that C is bigger than 1). Thus for $\tau > \tau_2$ the right-hand side of (4.17) equals

$$|\{k : \frac{\tau - \log 4C}{r_1 - q_2} \leq \tilde{a}_k^{(1)} \leq \frac{t + \log 4C}{s_2 - q_1}\}| \tag{4.22}$$

Since for $\tau > \tau_2$ the expression (4.21) is less than the expression (4.22), there exist a constant $M > 0$ and a $\tau_0 > \tau_2$ such that the relation (4.9) holds. In the same way, (4.20) implies (4.8). $\square$

Due to Proposition 3.3.1, Theorem 4.2.1 implies that $span\{e_k : a_k^{(\nu)} \geq \tau_0\} \overset{qd}{\hookrightarrow} span\{e_k : \tilde{a}_k^{(\nu)} \geq \frac{\tau_0}{M}\}$ where $\nu = 0, 1$. That is X_ν is quasidiagonally imbedded in $\tilde{X}_\nu$ ($\nu = 0, 1$) up to Banach subspaces.

4.3 Infinite Dimensional Complemented Banach Subspaces Of ℓ-Köthe Spaces

In this section we follow [10] and [1] and try to get the ℓ-Köthe space analogue of Proposition 13 in [1].

Definition 4.3.1. *Let $U(X)$ denote a basis of absolutely convex neighborhoods of the origin in a locally convex space X. X has the property of smallness up to a complemented Banach subspace (the SCBS property) if for each bounded subset M of X, for each $U \in U(X)$ and for each $\varepsilon > 0$, there are complementary subspaces B and E of X such that B is a Banach space and $M \subset B + \varepsilon U \cap E$.*

ℓ-Köthe spaces satisfy $SCBS$ property [1]. For the sake of completeness we present a proof of:

Lemma 4.3.2. *If $X = K^{\ell}(a_{i,n})$ is an ℓ-Köthe space and $M \subset X$ is a bounded set, then for any n_0 and any $\varepsilon > 0$, there exists a Banach basic subspace B such that $M \subset B + \varepsilon U_{n_0}$ where $U_{n_0} := \{x \in X : \|x\|_{n_0} \leq 1\}$.*

Proof. Let $X = K^{\ell}(a_{i,n})$ be an ℓ-Köthe space. Since $M \subset X$ is a bounded subset of X, then there exists a sequence of positive numbers (c_n) such that

$$M \subset \{x \in X : \|x\|_n := \|(\xi_i a_{i,n})_i\| \leq c_n, n = 1, 2, ...\}$$

We can chose (c_n) big enough to satisfy that $(\frac{a_{i,n}}{c_n})$ tends to zero, for each i. We set $\gamma_i := \sup_n \{\frac{1}{2^n}\frac{a_{i,n}}{c_n}\}$ and $\gamma := (\gamma_i)$. Since $\sup_n \{\frac{1}{2^n}\frac{a_{i,n}}{c_n}\} \leq \sum_n \left(\frac{1}{2^n}\frac{a_{i,n}}{c_n}\right)$, then by monotonicity of the norm in ℓ, for any $x \in M$ we obtain

$$\|\gamma x\| = \left\| \left(\sup_n \{\frac{1}{2^n}\frac{a_{i,n}}{c_n}\} \right) x \right\| \leq \left\| \left(\sum_n \frac{1}{2^n}\frac{a_{i,n}}{c_n} \right) x \right\| \leq \sum_n \frac{1}{2^n} \left\| \left(\frac{a_{i,n}}{c_n} \right) x \right\| \leq 1$$

Hence $M \subset \bigcap_n B^\ell(\frac{a_{i,n}}{c_n}) \subset B^\ell(\gamma)$.

Take any $\varepsilon > 0, n_0 \in \mathbb{N}$ and set $B := [e_i : \varepsilon\gamma_i \leq a_{i,n_0}]$ and

$E := [e_i : \varepsilon\gamma_i > a_{i,n_0}]$ where the square brackets denote the closed linear span of the corresponding vectors. For $x \in B$ and $n \in \mathbb{N}$, since

$$|\xi_i a_{i,n_0}| \geq |\xi_i \gamma_i \varepsilon| \geq |\xi_i \frac{1}{2^n} \frac{a_{i,n}}{c_n} \varepsilon|$$

for each n, then by monotonicity of the norm we get

$$\|(\xi_i a_{i,n_0})\| \geq \|(\xi_i \gamma_i \varepsilon)\| \geq \left\|(\xi_i \frac{1}{2^n} \frac{a_{i,n}}{c_n} \varepsilon)\right\|$$

Hence $\|(\xi_i a_{i,n_0})\| \varepsilon 2^n c_n \geq \|(\xi_i a_{i,n})\| = \|x\|_n$.

Let $x \in M \cap E$. Since $|\xi_i a_{i,n_0}| \leq |\xi_i \gamma_i \varepsilon|$ then by monotonicity of the norm we obtain that $\|x\|_{n_0} = \|(\xi_i a_{i,n_0})_i\| \leq \|(\xi_i \gamma_i \varepsilon)_i\| < \varepsilon$

Thus, all norms are equivalent and so B is Banach basic subspace. This completes the proof. □

Proposition 4.3.3. *Let X be a ℓ-Köthe space. If $T : X \to X$ is a bounded operator (respectively, compact) operator, then there exists complementary basic subspaces B and E such that :*

(i) B is a Banach (repectively, finite dimensional) space; and

(ii) if π_E and ι_E are the canonical projection onto E and embedding into X, respectively, then the operator $1_E - \pi_E T \iota_E$ is an automorphism on E.

Proof. Let $\{\|.\|_p : p \in \mathbb{N}\}$ be a fundamental system of norms in X. Since T is a bounded operator there exists a k_0 such that $T(U_{k_0})$ is a bounded set in X, i.e.

$$\forall k \exists C_k : \|Tx\|_k \leq C_k \|x\|_{k_0}$$

By Lemma 4.3.2, there exists a Banach (respectively, finite dimensional) basic subspace B such that $T(U_{k_0}) \subset B + \frac{1}{2} U_{k_0}$. Let E be the basic subspace that is complementary to B. Then, setting $T_1 := \pi_E T \iota_E : E \to E$, we obtain that

$$\|Tx\|_k \leq \frac{1}{2}\|x\|_{k_0}, \forall x \in E.$$

It is clear that the operator $1_E - T_1$ is an automorphism. Indeed, for any $x \in E$ consider the series

$$Sx = x + T_1 x + T_1^2 x + T_1^3 x + ... + T_1^m x + ... \tag{4.23}$$

This series convergent in E because, for any k, we have

$$\|T_1^m x\|_k \leq C_k \|T_1^{m-1} x\|_{k_0} \leq C_k \left(\frac{1}{2}\right)^{m-1} \|x\|_{k_0},\ m = 1, 2, ...$$

and so, by Banach-Stheinhaus theorem, (4.23) defines a linear continuous operator $S : E \to E$. Since $(1_E - T_1)Sx = S(1_E - T_1)x = x$, the linear operator S is inverse to the operator $1_E - T_1$. $\square$

Lemma 4.3.4. *([10]) Let X_1, X_2, Y_1 and Y_2 be topological vector spaces. If $T := (T_{ij}) : X_1 \times X_2 \to Y_1 \times Y_2$ (where $T_{ij} : X_j \to Y_i$) is an isomorphism such that $T_{11} : X_1 \to Y_1$ is also an isomorphism, then $X_2 \simeq Y_2$.*

Proof. Let $T^{-1} := (S_{ij}) : Y_1 \times Y_2 \to X_1 \times X_2$ and $R : X_2 \to Y_2$ such that $R = T_{22} - T_{21} T_{11}^{-1} T_{12}$.

Consider $T_{11}S_{12} + T_{12}S_{22} = 0$ which implies that $RS_{22} = T_{22}S_{22} - T_{21}T_{11}^{-1}T_{12}S_{22} = T_{22}S_{22} + T_{21}S_{12} = I_{Y_2}$. We can also esaily see that $S_{21}T_{11} + S_{22}T_{21} = 0$ implies that $S_{22}R = S_{22}T_{22} - S_{22}T_{21}T_{11}^{-1}T_{12} = S_{22}T_{22} + S_{21}T_{12} = I_{X_2}$. Thus, the spaces X_2 and Y_2 are isomorphic. $\square$

Now, we state a theorem which is a modification of generalized Douady Lemma in [30] (section 6).

Theorem 4.3.5. *Let X_1 be a ℓ-Köthe space and X_2, Y_1, Y_2 be topological vector sapces If $X_1 \times X_2 \simeq Y_1 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$, then there exists* *complementary basic subspaces E and B in X_1 and complementary subspaces F and G of Y_1 such that B is Banach space and*

$$F \simeq E, \qquad B \times X_2 \simeq G \times Y_2.$$

In addition, if $(X_2, Y_1) \in \mathcal{BF}$, then G is Banach space.

Proof. Let $T : (T_{ij}) : X_1 \times X_2 \to Y_1 \times Y_2$ such that $T_{ij} : X_j \simeq Y_i$ be an isomorphism, and let $T^{-1} = (S_{ij})$. It is easy to get $S_{11}T_{11} + S_{12}T_{21} = I_{X_1}$. Observe that $(X_1, Y_2) \in \mathcal{BF}$ implies that the operator $S_{12}T_{21}$ is bounded. Due to Proposition 4.3.3 and boundedness of $S_{12}T_{21}$, there exist complementary basic subspaces E and B of X_1 such that B is a Banach space and the operator $A = \pi_E S_{11} T_{11} \iota_E$ is an automorphism of E. It is clear that the operator $P = T_{11}A^{-1}\pi_E S_{11}$ is a projection on Y_1. Define now,

$$F = P(Y_1), \qquad G = P^{-1}(0).$$

Easily we observe that $F = T_{11}(E)$ and the restriction of T_{11} on E is an isomorphism between E and F. Due to Lemma 4.3.4, we get that

$$B \times X_2 \simeq G \times Y_2.$$

If, in addition, each operator acting in Y_1 that factors through X_2 is bounded, then each operator acting in G that factors through X_2 is also bounded. Let $H = (H_{ij}) : G \times Y_2 \to B \times X_2$ be an isomorphism and $H^{-1} = (R_{ij})$. Then, easily we get that $R_{11}H_{11} + R_{12}H_{21} = I_G$. Observe that $R_{12}H_{21}$ is bounded because it factors through X_2 and the operator $R_{11}H_{11}$ since it factors through the Banach space B. Thus the identity operator I_G is bounded. Therefore G is a Banach space. □

For $\ell = l_p$, we have some useful facts:

(i) Each infinite-dimensional complemented subspace of l_p ($1 \leq p < \infty$) is isomorphic to l_p, (*ii*) Each infinite-dimensional basic Banach subspace of an ℓ-Köthe space is isomorphic to l_p.

Due to the facts (*i*) and (*ii*), in [10] Djakov, Terzioğlu, Yurdakul, Zahariuta showed that infinite-dimensional Banach complemented subspace of an l_p-Köthe space is isomorphic to l_p. Since the facts (*i*) and (*ii*) are not true for an arbitary $\ell \in \Lambda$, we can not generalize the above result to the ℓ-Köthe spaces. We state the following result, instead:

Let $\mathcal{C}(\ell)$ be the set of all complemented subspaces of ℓ.

Proposition 4.3.6. *Let X be an ℓ-Köthe space, and let F and G be complementary subspaces in X. If G is an infinite-dimensional Banach space then $G \simeq L$ where $L \in \mathcal{C}(\ell)$.*

Proof. Let $X = K^{\ell}(A) = F \oplus G$ and G be a Banach space. So we have $X \times \{0\} \simeq F \times G$. By Theorem 4.3.5 there exists complementary basic subspaces E and B in X and complementary subspaces F_1 and G_1 in F such that B is a Banach space and

$$F_1 \simeq E, \qquad B \simeq B \times \{0\} \simeq G_1 \times G.$$

Hence G is complemented subspace of B. Since B is the basic Banach subspace of X, and so ℓ, then there exists $L \in \mathcal{C}(\ell)$ such that $G \simeq L$. □

CHAPTER 5

POWER ℓ-KÖTHE SPACES OF FIRST TYPE

In this chapter, we consider the first type power ℓ-Köthe spaces and give some basic properties. After that, using n-equivalent multirectangular characteristic invariants we investigate quasidagonal isomorphisms of the first type power ℓ-Köthe spaces.

5.1 Power ℓ-Köthe Spaces of First Type

Let $\mathcal{E}$ be the class of the ℓ-Köthe Spaces of the kind

$$E^\ell(\lambda, a) := K^\ell\left(\exp\left[\left(-\frac{1}{p} + \lambda_i p\right) a_i\right]\right) \tag{5.1}$$

where $a = (a_i)_{i\in\mathbb{N}},\ a_i > 0,\ \lambda = (\lambda_i),\ \ 0 < \lambda_i \leq 1$. Spaces of that kind will be called *power ℓ- Köthe spaces of first type.*

We easily observe that if $\tilde{a}_i = 1 + a_i \quad \tilde{\lambda}_i = \max\{\lambda_i, \frac{1}{\tilde{a}_i}\}$ where $\tilde{a} = (\tilde{a}_i)$ and $\tilde{\lambda} = (\tilde{\lambda}_i)$ then $X := E^\ell(\lambda, a) \simeq E^\ell(\tilde{\lambda}, \tilde{a}) =: \tilde{X}$.

Indeed, if $\tilde{\lambda}_i = \frac{1}{\tilde{a}_i}$ then $\tilde{\lambda}_i\tilde{a}_i = 1$ then we can get the following inequalities

$$\left(-\frac{1}{p} + \lambda_i p\right) a_i \leq \left(-\frac{1}{p} + \tilde{\lambda}_i p\right) \tilde{a}_i$$

$$= \left(-\frac{\tilde{a}_i}{p} + \tilde{\lambda}_i\tilde{a}_i p\right) = \left(-\frac{1}{p} - \frac{a_i}{p} + p\right) \leq \left(-\frac{1}{p} + \lambda_i p\right) a_i + p$$

for any $i, p \in \mathbb{N}$. By monotonocity of the norm, for any p we obtain that

$$\|x\|_X = \left\| \left(\xi_i \exp\left[\left(-\frac{1}{p} + \lambda_i p\right) a_i\right]\right) \right\|_\ell \leq \left\| \left(\xi_i \exp\left[-\frac{\tilde{a}_i}{p} + \tilde{\lambda}_i \tilde{a}_i p\right]\right) \right\|_\ell = \left\| Ix \right\|_{\tilde{X}}$$

$$\leq \left\| \left(\xi_i \exp\left[\left(-\frac{1}{p} + \lambda_i p\right) a_i + p\right]\right) \right\|_\ell \leq C \left\| \left(\xi_i \exp\left[\left(-\frac{1}{p} + \lambda_i p\right) a_i\right]\right) \right\|_\ell = C\|x\|_X$$

i.e.

$$\|x\|_X \leq \|Ix\|_{\tilde{X}} \leq C\|x\|_X$$

where $C = \exp(p)$. By inverse mapping theorem I^{-1} is continuous. Thus I and I^{-1} are continuous, hence I is an isomorphism. So, we obtain $E^\ell(\lambda, a) \simeq E^\ell(\tilde{\lambda}, \tilde{a})$.

The space $E^\ell(\lambda, a)$ is said to be *finite*, *infinite* or *mixed* respectively, if it satisfies the following cases:

(i) $\lambda_i \to 0$;

(ii) $\underline{\lim} \lambda_i > 0$;

(iii) $\underline{\lim} \lambda_i = 0, \overline{\lim} \lambda_i > 0$.

The corresponding classes of the spaces to the cases $(i), (ii), (iii)$ denoted by $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$.

Proposition 5.1.1. (a) *The case* (i) *is equivalent to* $E^\ell(\lambda, a) \simeq E_0^\ell(a)$,

(b) *The case* (ii) *is equivalent to* $E^\ell(\lambda, a) \simeq E_\infty^\ell(a)$.

Proof. Since (a) and (b) are proved in a similar way, we restrict ourselves to prove only (a). It is sufficent to show that identity operator I is an isomorphism between $E^\ell(\lambda, a)$ and $E_0^\ell(a)$. Assume that $\lambda_i \to 0, i \in \mathbb{N}$, then there exists i_0 such that for all $i > i_0$, $0 < \lambda_i < \frac{1}{2p^2}$

$$-\frac{1}{p} a_i \leq (-\frac{1}{p} + \lambda_i p) a_i \leq (-\frac{1}{p} + \frac{1}{2p^2} p) a_i = (-\frac{1}{2p}) a_i.$$

By monotonicity of the norm, for any p we obtain that

$$\|(\xi_i \exp[-\frac{1}{p}a_i])\|_\ell \leq \|(\xi_i \exp[(-\frac{1}{p} + \lambda_i p)a_i])\|_\ell$$

$$\leq \|(\xi_i \exp[(-\frac{1}{p} + \frac{2}{p^2}p)a_i])\|_\ell = \|(\xi_i \exp[-\frac{1}{2p}a_i])\|_\ell$$

□

For any given infinite set $L = \{i_k\} \subset \mathbb{N}$ we shall consider the corresponding basic subspace as follows:

$$E_L^\ell(\lambda, a) := \overline{span}\{e_j : j \in L\} \simeq E^\ell(\lambda_L, a_L),$$

where $\lambda_L = (\lambda_{i_k}), a_L = (a_{i_k})$. From this definition, we get the following:

(i) $\lambda_{i_k} \to 0 \Leftrightarrow E_L^\ell(\lambda, a) \simeq E_0^\ell(a_L)$;

(ii) $\underline{\lim}\lambda_i > 0 \Leftrightarrow E_L^\ell(\lambda, a) \simeq E_\infty^\ell(a_L)$

(iii) $a_{i_k} \leq C < \infty (\forall i, k) \Rightarrow E_L^\ell(\lambda, a) \simeq E_0^\ell(a_L) \simeq E_\infty^\ell(a_L) \simeq \ell$.

Proposition 5.1.2. *Let $a = (a_i), a_i \nearrow \infty$ and $b = (b_i), b_i \nearrow \infty$, are given $\tau = 0, \infty$. Then $E_0^\ell(a) \not\simeq E_\infty^\ell(b)$ always and*

$$E_\tau^\ell(a) \simeq E_\tau^\ell(b) \Leftrightarrow E_\tau^\ell(a) \overset{e}{\simeq} E_\tau^\ell(b) \Leftrightarrow a_i \asymp b_i.$$

Lemma 5.1.3. *The space $E^\ell(\lambda, a)$ is isomorphic to $E_0^{\ell_1}(c) \times E_\infty^{\ell_2}(d)$ for some c, d and $\ell \simeq \ell_1 \times \ell_2$ if an only if there exist $N_1, N_2 \subset \mathbb{N}$ such that $\mathbb{N} = N_1 \cup N_2, N_1 \cap N_2 = \emptyset$ and $\lambda_i \geq \delta > 0$ for $i \in N_1; \lambda_i \to 0$ if $i \to \infty, i \in N_2$.*

Let $\overline{\mathcal{E}}$ be the class of all locally convex spaces, which are isomorphic to some spaces in the class $\mathcal{E}$. The spaces in $\overline{\mathcal{E}}$ are called *first type power spaces*. In the same way, the classes $\overline{\mathcal{E}}_1, \overline{\mathcal{E}}_2, \overline{\mathcal{E}}_3$ corresponds to $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$. It is clear that $E_0^\ell(a) \times E_\infty^\ell(b) \in \overline{\mathcal{E}}_3$, for arbitrary a, b tending to infinity.

Lemma 5.1.4. *The following conditions are equivalent:*

(i) $E^\ell(\lambda, a) \overset{p}{\simeq} E^\ell(\mu, b)$;

(ii) $E^\ell(\lambda, a) \overset{qd}{\simeq} E^\ell(\mu, b)$

(iii) there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ *and a constant* $\alpha > 1$ *such that*

$$\frac{a_i}{\alpha} \le b_{\sigma(i)} \le a_i\alpha \tag{5.2}$$

and for any subsequence i_k

$$\lambda_{i_k} \to 0 \Leftrightarrow \mu_{\sigma(i_k)} \to 0. \tag{5.3}$$

Proof. $(i) \Rightarrow (ii)$ is obvious. Let us show $(iii) \Rightarrow (i)$. We observe that $E^\ell(\lambda, a) = E^\ell(\lambda, b)$, since $a_i \asymp b_i$. We need to prove that under conditions $a = b$ and $\sigma(i) \equiv i$ the corresponding operator $I : E^\ell(\lambda, a) \to E^\ell(\mu, b)$ is an isomorphism. Let us see that I is continuous, i.e.

$$\forall p \exists q \exists C: \ \exp\left(-\frac{1}{p} + \mu_i p\right) b_i \le C \exp\left(-\frac{1}{q} + \lambda_i q\right) a_i. \tag{5.4}$$

From (5.3) there exists a function $\varphi : (0,1] \to (0,1]$ such that $\varphi(t) \downarrow 0$ as $t \downarrow 0$ and for any $\delta \in (0,1]$ $\mu_i \ge \delta \Rightarrow \lambda_i \ge \varphi(\delta)$. Fix an p and set

$$\mathbb{N}_1 = \mathbb{N}_1(p) := \{i : \mu_i \ge \frac{1}{2p^2}\}, \quad \mathbb{N}_2 := \mathbb{N} \setminus \mathbb{N}_1.$$

For $q = q(p) = \max\{p/\epsilon, 2p\}$, where $\epsilon = \varphi(1/2p^2)$, we obtain the following inequalities

$$-\tfrac{1}{p} + \mu_i p \le -\tfrac{1}{p} + p \le -\tfrac{1}{q} + q\epsilon \le -\tfrac{1}{q} + q\lambda_i \text{ for } i \in \mathbb{N}_1;$$

$$-\tfrac{1}{p} + \mu_i p \le -\tfrac{1}{2p} \le -\tfrac{1}{q} \le -\tfrac{1}{q} + q\lambda_i \text{ for } i \in \mathbb{N}_2.$$

which imply (5.4) even with $C = 1$, hence I is continuous. Since the condition (iii) is symmetric with respect to (a_i) and (b_i), the same arguments give us that the operator I^{-1} is continuous, . This conclude that $(iii) \Rightarrow (i)$.

Let us see that $(ii) \Rightarrow (iii)$. Suppose that there exist a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and sequence of numbers $\gamma_i \neq 0$ such that the linear operator $T : E^\ell(\lambda, a) \to E^\ell(\mu, b)$ defined by $Te_i = \gamma_i e_{\sigma(i)}$ is an isomorphism. We will show that (iii) holds with the same σ. If we assume that (5.3) does not hold for some subsequence i_k, then we obtain the existence of a subsequence $j_s = i_{k_s}$ such that $a_{j_s} \to \infty$ and either $\lambda_{j_s} \to 0$ but $\mu_{\sigma(j_s)} \geq \delta > 0$ or $\mu_{\sigma(j_s)} \to 0$ but $\lambda_{j_s} \geq \delta > 0$.
It is enough to consider the first case. Let us put $L = \{j_s\}, M = \sigma(L)$; then

$$E_0^\ell(a_L) \simeq E_L^\ell(\lambda, a) \simeq E_M^\ell(\mu, b) \simeq E_\infty^\ell(b_M)$$

where $a_L = (a_{j_s}), b_M = (b_{\sigma(j_s)})$.
This is a contradiction, since by Proposition 5.1.2 in the Montel case finite and infinite power series spaces are not isomorphic. Hence (5.3) is proved for any subsequence (i_k).

To conclude the proof, we need to show $a_i \asymp b_{\sigma(i)}$. Assume that $a_i \asymp b_{\sigma(i)}$ is not true. Then there exists a subsequence (i_k) such that

(1) either $a_{i_k}/b_{\sigma(i_k)} \to \infty$, or $b_{\sigma(i_k)}/a_{i_k} \to \infty$

(2) either $\lambda_{i_k} \to 0$, or $\lambda_{i_k} \geq \delta > 0$.

We consider one of the four possible cases we get since the rest can be treated analogously. Let $a_{i_k}/b_{\sigma(i_k)} \to \infty$ and $\lambda_{i_k} \to 0$. Then it follows $\mu_{\sigma(i_k)} \to 0$ and we have for $L = \{i_k\},\ \mu = \sigma(L)$

$$E_0^\ell(a_L) \simeq E_L^\ell(\lambda, a) \simeq E_M^\ell(\mu, b) \simeq E_0^\ell(b_M)$$

where $a_L = (a_{i_k}), b_M = (b_{\sigma(i_k})$. But by Proposition 5.1.2 the isomorphism $E_0^\ell(a_L) \simeq E_0^\ell(b_M)$ implies $a_i \asymp b_{\sigma(i)}$ contrary the assumption. □

5.2 Multirectangular Characteristics and Compound Invariants

In this section we consider the problem of quasidiagonal isomorphism of first type ℓ-power Köthe spaces [31], [37]: $E^\ell(\lambda, a)$ where $a = (a_i)_{i\in\mathbb{N}}$ and $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ are sequences of positive numbers. From [7],[5],[6] it is known that the system of all m-rectangular charactersitics μ_m (see 5.6 below) is a complete quasidiagonal invariant on the class of all first type power Köthe spaces, if the relation of equivalency of systems (μ_m^X) and $(\mu_m^{\tilde{X}})$ be defined by some natural estimates with constants independent of m. Here we prove that the system of all m-rectangular charactersitics μ_m is also a complete quasidiagonal invariant on the class of all first type ℓ-Köthe spaces under the same conditions as above.

Dealing with spaces (5.1) we always assume without loss of generality that

$$a_i > 1, \quad \frac{1}{a_i} \le \lambda_i \le 1, \quad i \in \mathbb{N}. \tag{5.5}$$

For given $a = (a_i)_{i\in\mathbb{N}}$, $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ with (5.5) and $m \in \mathbb{N}$ we introduce *m-rectangular characteristic* of pair (λ, a) as the function

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) := \left| \bigcup_{k=1}^{m} \left\{ i : \delta < \lambda_i \le \varepsilon_k,\ \tau < a_i \le t_k \right\} \right|, \tag{5.6}$$

defined for

$$\delta = (\delta_k),\ \varepsilon = (\varepsilon_k),\ \tau = (\tau_k), t = (t_k),$$
$$0 \le \delta_k < \varepsilon_k,\ 0 \le \tau_k < t_k < \infty, \quad k = 1, 2, ..., m. \tag{5.7}$$

The function (5.6) calculates how many points (λ_i, a_i) are contained in the union of m-rectangles:

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) = \left| \bigcup_{k=1}^{m} \left\{ i : (\lambda_i, a_i) \in P_k \right\} \right| = \left| \left\{ i : (\lambda_i, a_i) \in \bigcup_{k=1}^{m} P_k \right\} \right|, \tag{5.8}$$

where $P_k := (\delta_k, \varepsilon_k] \times (\tau_k, t_k]$, $k = 1, 2, ..., m$.

Let $\tilde{a} = (\tilde{a}_i)_{i\in\mathbb{N}}$, $\tilde{\lambda} = (\tilde{\lambda}_i)_{i\in\mathbb{N}}$ be another couple of positive sequences and m a fixed natural number. Then the functions $\mu_m^{(\lambda,a)}$ and $\mu_m^{(\tilde{\lambda},\tilde{a})}$ are *equivalent* (or $\mu_m^{(\lambda,a)} \approx \mu_m^{(\tilde{\lambda},\tilde{a})}$) if there exists a strictly increasing function $\varphi : [0,2] \to [0,1]$, $\varphi(0) = 0, \varphi(2) = 1$, and positive constant α such that the following inequalities

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) \leq \mu_m^{(\tilde{\lambda},\tilde{a})}(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t), \tag{5.9}$$

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta,\varepsilon;\tau,t) \leq \mu_m^{(\lambda,a)}(\varphi(\delta),\varphi^{-1}(\varepsilon);\frac{\tau}{\alpha},\alpha t), \tag{5.10}$$

hold with $\varphi(\delta) = (\varphi(\delta_k))$, $\varphi^{-1}(\varepsilon) = (\varphi^{-1}(\varepsilon_k))$, $\frac{\tau}{\alpha} = (\frac{\tau_k}{\alpha})$, $\alpha t = (\alpha t_k)$ for all collections of parameters (5.7) with $\varepsilon_k \leq 1, \tau_k \geq 1, k = 1,2,...,m$ (in line with our agreement (5.5) we shall suppose always that the parameters (5.7) satisfy this conditions.) If $X = E^\ell(\lambda,a)$, we write also μ_m^X in place of $\mu_m^{(\lambda,a)}$.

Recall that $\gamma : \mathcal{X} \to \Gamma$ is called *linear topological invariants* if $X \simeq \tilde{X} \Rightarrow \gamma(X) \sim \gamma(\tilde{X})$, $X, \tilde{X} \in \mathcal{X}$, where $\mathcal{X}$ is a class of locally convex spaces and Γ is a set with an equivalence relation $\sim$.

Lemma 5.2.1. *Let $X = E^\ell(\lambda,a)$, $\tilde{X} = E^\ell(\tilde{\lambda},\tilde{a}), \ell \in \Lambda$ $n \in \mathbb{N}$. If $X \simeq \tilde{X}$, then there exists an increasing function $\varphi : [0,2] \to [0,1]$, $\varphi(0) = 0, \varphi(2) = 1$, a decreasing function $M : (0,1] \to (0,\infty)$ and a constant $\alpha > 1$ such that the inequality*

$$\mu_m^X(\delta,\varepsilon;\tau,t) \leq \mu_m^{\tilde{X}}(\varphi(\delta) - \frac{M(\delta)}{\tau},\varphi^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau};\frac{\tau}{\alpha},\alpha t) \tag{5.11}$$

holds for each $m \in \mathbb{N}$ and the inequalities (5.9) and (5.10) hold for all collection of parameters (5.7) satisfying the following additional condition: among all the numbers $\delta_1,\delta_2,...,\delta_m$ there are no more than n different.

Proof. Let $T : \tilde{X} \to X$ be an isomorphism. Consider two unconditional bases of the space X:

(i) the canonical basis $e = (e_i)_{i\in\mathbb{N}}$, and

(ii) T-image of the basis of $\tilde{X}$: $\tilde{e} = (\tilde{e}_i)_{i\in\mathbb{N}}, \tilde{e}_i = Te_i$, for each i.

Hence, for each $x \in X$, we have two representation:

$$x = \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \eta_i \tilde{e}_i.$$

For each $x \in X$, the system of norms

$$\|x\|_p = \|(\eta_i \tilde{a}_{ip})\| \quad (p \in \mathbb{N}),$$

is equivalent to the original system of norms in X :

$$|x|_p = \|(\xi_i a_{ip})\| \quad (p \in \mathbb{N}),$$

where

$$a_{ip} := \exp\Big(\big[-\frac{1}{p} + \lambda_i p\big] a_i\Big),$$

$$\tilde{a}_{ip} := \exp\Big(\big[-\frac{1}{p} + \tilde{\lambda}_i p\big] \tilde{a}_i\Big).$$

We define the weights

$$\alpha_p := (a_{ip}) \quad , \quad \tilde{\alpha}_p := (\tilde{a}_{ip}),$$

to get the balls

$$B^e(\alpha_p) \quad , \quad B^{\tilde{e}}(\tilde{\alpha}_p). \tag{5.12}$$

We will use these weighted balls to build two pairs of synthetic neighborhoods U, V and $\tilde{U}, \tilde{V}$ in the form of certain compound geometrical and interpolational constructions to provide the inclusions

$$U \supset \tilde{U} \quad , \quad V \subset \tilde{V} \tag{5.13}$$

and also, to make sure that the following estimations will be satisfied:

$$\mu_m^X(\delta, \varepsilon; \tau, t) \leq \beta(V, \frac{1}{n}U,), \tag{5.14}$$

$$\beta(\tilde{V}, \frac{1}{n}\tilde{U},) \leq \mu_m^{\tilde{X}}(\varphi(\delta) - \frac{M(\delta)}{\tau}, \varphi^{-1}(\varepsilon) + \frac{M(\delta)}{\tau}; \frac{\tau}{\alpha}, \alpha t). \tag{5.15}$$

Due to (3.7), the inclusion (5.13) implies the assertion (5.11) of the Lemma 5.2.1.

Construction of synthetic neighborhoods. Let $n, m \in \mathbb{N}$ such that $n \leq m$. Now, regarding the equivalence of the systems of norms, we can choose an infine chain of positive integers

$$r_i < p_i < s_i < r_{i+1},\ i = 0, 1, ..., m+1;$$

$$s_{m+1} < q_j < q_{j+1},\ j = 1, 2, ... \tag{5.16}$$

in a way that each consequent number of the chain is four times larger than the preceding one and $4s_0 q_j < q_{j+1}$, and that the following inclusions

$$B^{\tilde{e}}(\tilde{\alpha}_{s_k}) \subset C(k)B^e(\alpha_{p_k}); B^e(\alpha_{p_k}) \subset C(k)B^{\tilde{e}}(\tilde{\alpha}_{r_k})k = 0, 1, ..., m+1; \tag{5.17}$$

$$B^{\tilde{e}}(\tilde{\alpha}_{q_{j+1}}) \subset C_{q_j}B^e(\alpha_{q_j}); B^e(\alpha_{q_{j+1}}) \subset C_{q_j}B^{\tilde{e}}(\tilde{\alpha}_{q_j}),\ j = 1, 2, ... \tag{5.18}$$

are satisfied with some constants $(C(k))_{k=0}^{m+1}, (C_{q_j})_{j=1}^{\infty}$.

Let all different values of $(\delta_k)_{k=1}^m$ be represented as a non-decreasing finite sequence $\sigma_1 < \sigma_2 < ... < \sigma_l < ...$ Due to that, let us redefine the numbers

$$p_k := p_{l_k},\quad r_k := r_{l_k},\quad s_k := s_{l_k}$$

where l_k is such that $\delta_k = \sigma_{l_k},\ k = 1, ..., m$. Obviously, the inclusions (5.17) are still valid. Beside these, we consider also the sequence

$$\zeta_0 := 1,\quad \zeta_j := \frac{1}{q_j},\quad j = 1, 2, ... \tag{5.19}$$

and choose indices ι_k and j_k so that

$$\zeta_{\iota_k} \leq \delta_k < \zeta_{\iota_k - 1},\quad \zeta_{j_{k+1}} \leq \varepsilon_k < \zeta_{j_k}\ ,\quad k = 1, 2, ..., m. \tag{5.20}$$

Due to the construction above, we define the sets supplying as elementary blocks in order to build the sets $U, V, \tilde{U}, \tilde{V}$. The first couple of the sets U, V is built with the blocks $(k = 1, 2, ..., m)$

$$W_l^{(k)} = B^e(w_l^{(k)}),\quad l = 1, 2\ ;\quad \bar{W}_l^{(k)} = B^{\tilde{e}}(\bar{w}_l^{(k)})\quad l = 1, 2, 3, 4 \tag{5.21}$$

where each weighted-sequence will be responsible for certain inequality for (λ_i) or (a_i) in (5.6). We begin with the following blocks:

$$w_1^{(k)} = \bar{w}_1^{(k)} = \alpha_{p_k} \quad , \quad k = 1, 2, ..., m.$$

The estimations of λ_i from below and above in (5.11), (5.6) are linked with the following two series of "interpolational" weights ($k = 1, 2, ..., m$):

$$w_2^{(k)} = \alpha_{p_0}^{\frac{1}{2}} \alpha_{q_{\iota_k}}^{\frac{1}{2}} \quad , \quad \bar{w}_2^{(k)} = \begin{cases} \alpha_{p_0}^{\frac{1}{2}} \alpha_{q_{j_{k-1}}}^{\frac{1}{2}} & \text{if} \quad j_k > 3, \\ \alpha_{p_0} & \text{if} \quad j_k \leq 3. \end{cases}$$

On the other hand, the estimations of a_i by the parameters τ_k and t_k in (5.11), we require the following series:

$$\bar{w}_3^{(k)} = \exp\left(\frac{\tau_k}{2p_0}\right)\alpha_{p_0} \quad , \quad \bar{w}_4^{(k)} = \exp\big(-2p_{m+1}t_k\big)\alpha_{p_{m+1}} \; , \quad k = 1, 2, ..., m.$$

Finally, construct second couple of the sets $\tilde{U}, \tilde{V}$. For this goal, we use the corresponding series of blocks which are balls with respect to the T-image basis $\tilde{e}$: $(k = 1, 2, ..., m)$

$$\tilde{W}_l^{(k)} = B^e(\tilde{w}_l^{(k)}), \quad l = 1, 2 \; ; \quad \tilde{\bar{W}}_l^{(k)} = B^{\tilde{e}}(\tilde{\bar{w}}_l^{(k)}) \quad l = 1, 2, 3, 4.$$

We define their weights by the same formulae as for the balls (5.21) but with the following rules of substitution: to obtain weight $\tilde{w}_l^{(k)}$ and $\tilde{\bar{w}}_l^{(k)}$ we replace α_{p_k} with $\frac{1}{C(k)}\tilde{\alpha}_{s_k}$ (or, respectively $C(k)\tilde{\alpha}_{r_k}$) and replace $\alpha_{q_{\iota_k}}$ (or, respectively $\alpha_{q_{j_{k-1}}}$) with $\frac{1}{C_{q_{\iota_k}}}\tilde{\alpha}_{q_{\iota_k}}$ (or, respectively $C_{q_{j_{k-2}}}\tilde{\alpha}_{q_{j_{k-2}}}$).

For $k = 1, 2, ..., m$ we put

$$U^{(k)} = conv\Big(\bigcup_{l=1}^{2} W_l^{(k)}\Big) \quad , \quad V^{(k)} = \bigcap_{l=1}^{4} \bar{W}_l^{(k)},$$

$$\tilde{U}^{(k)} = conv\Big(\bigcup_{l=1}^{2} \tilde{W}_l^{(k)}\Big) \quad , \quad \tilde{V}^{(k)} = \bigcap_{l=1}^{4} \tilde{\bar{W}}_l^{(k)},$$

to define the sets

$$U = \bigcap_{k=1}^{m} U^{(k)}\ ,\quad \tilde{U} = \bigcap_{k=1}^{m} \tilde{U}^{(k)}\ ,\quad V = conv\Big(\bigcup_{k=1}^{m} V^{(k)}\Big)\ ,\quad \tilde{V} = conv\Big(\bigcup_{k=1}^{m} \tilde{V}^{(k)}\Big).$$

On account of 5.17 we observe the inclusions $(k = 1, 2, ..., m)$

$$W_l^{(k)} \supset \bar{W}_l^{(k)}\ ,\quad l = 1, 2;$$

$$\bar{W}_l^{(k)} \subset \bar{\tilde{W}}_l^{(k)},\quad l = 1, 2, 3, 4.$$

which imply the inclusion (5.13).

Approximation of sets $U, V, \tilde{U}, \tilde{V}$ with the weighted ℓ-balls. Since the sets $U, V, \tilde{U}$ and $\tilde{V}$ are not weighted balls, we cannot use Lemma 3.2.5 directly to calculate $\beta(U, V)$ and $\beta(\tilde{U}, \tilde{V})$. Nevertheless, using the Lemma 3.2.4 and Lemma 3.2.3, we derived some suitable weighted balls from these sets. In the direction of this thought, we consider the sequences: $(k = 1, 2, ..., m)$

$$c^{(k)} = (c_i^{(k)})\ ,\quad \tilde{c}^{(k)} = (\tilde{c}_i^{(k)})\ ,\quad d^{(k)} = (d_i^{(k)})\ ,\quad \tilde{d}^{(k)} = (\tilde{d}_i^{(k)});$$

and the sequences

$$c = (c_i)\ ,\quad \tilde{c} = (\tilde{c}_i)\ ,\quad d = (d_i)\ ,\quad \tilde{d} = (\tilde{d}_i),$$

defined in the following way:

$$c_i^{(k)} := \min\{w_{i,l}^{(k)} : l = 1, 2\} \qquad \tilde{c}_i^{(k)} := \min\{\tilde{w}_{i,l}^{(k)} : l = 1, 2\},$$

$$d_i^{(k)} := \max\{\bar{w}_{i,l}^{(k)} : l = 1, 2\} \qquad \tilde{d}_i^{(k)} := \max\{\bar{\tilde{w}}_{i,l}^{(k)} : l = 1, 2\},$$

$$c_i := \min\{d_i^{(k)} : k = 1, 2, ..., m\} \qquad \tilde{c}_i := \min\{\tilde{d}_i^{(k)} : k = 1, 2, ..., m\},$$

$$d_i := \max\{c_i^{(k)} : k = 1, 2, ..., m\} \qquad \tilde{d}_i := \max\{\tilde{c}_i^{(k)} : k = 1, 2, ..., m\}.$$

Taking into account of Lemma 3.2.4 and Lemma 3.2.3, and the above con-strcution, we obtain the following relation:

$$B^e(c^{(k)}) = U^{(k)}\ ,\quad B^{\tilde{e}}(\tilde{c}^{(k)}) = \tilde{U}^{(k)}\ ,\quad B^e(d^{(k)}) \subset V^{(k)}\ ,\quad \tilde{V}^{(k)} \subset 4B^{\tilde{e}}(\tilde{d}^{(k)}).$$

From the construction of $(\delta_k)_{k=1}^m$, $\delta_k = \delta_\iota$ implies that $j_k = j_\iota$, $p_k = p_\iota$, $w_{i,l}^{(k)} = w_{i,l}^{(\iota)}, l = 1, 2$, $c_i^{(k)} = c_i^{(\iota)}$, $i \in \mathbb{N}$. By Lemma 3.2.4, Lemma 3.2.3 and the fact that there are no more than n different among the sets $U^{(k)}$, $k = 1, 2, ..., m$, we conclude that

$$B^e(c) \subset V \ , \quad U \subset nB^e(d) \ , \quad \tilde{V} \subset 4B^{\tilde{e}}(\tilde{c}) \ , \quad B^{\tilde{e}}(\tilde{d}) \subset \tilde{U}.$$

Hence, owing to 3.7 and 3.8, we obtain

$$\beta(B^e(c), B^e(d)) \leq \beta(V, \frac{1}{n}U), \tag{5.22}$$

$$\beta(\tilde{V}, \frac{1}{n}\tilde{U}) \leq \beta(4nB^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})). \tag{5.23}$$

Estimation of (5.14). After the above construction of synthetic neighborhoods U, V to prove the estimation (5.14) we shall prove the following inequality:

$$\mu_m^X(\delta, \varepsilon; \tau, t) \leq \beta(B^e(c), B^e(d)). \tag{5.24}$$

By aid of 5.22, this gives the desired result.

Lemma 3.2.5 and the definitions of the sequences c and d implies that

$$\beta(B^e(c), B^e(d)) = \left| \bigcup_{k=1}^m \bigcup_{\nu=1}^m \{i : d_i^{(k)} \leq c_i^{(\nu)}\} \right|.$$

In the sequel,

$$\beta(B^e(c), B^e(d)) \geq \left| \bigcup_{k=1}^m \{i : d_i^{(k)} \leq c_i^{(k)}\} \right|. \tag{5.25}$$

Taking into account of the definitions $d^{(k)}$ and $c^{(k)}$, $(k = 1, 2, ..., m)$, we obtain

$$\{i : d_i^{(k)} \leq c_i^{(k)}\} = \{i : \max_{1 \leq l \leq 4} \bar{w}_{i,l}^{(k)} \leq \min_{1 \leq l \leq 2} w_{i,l}^{(k)}\}. \tag{5.26}$$

Due to the fact that $\bar{w}_{i,1}^{(k)} = w_{i,1}^{(k)}$, we can write the right-hand side of 5.26 as follows:

$$\{i : d_i^{(k)} \leq c_i^{(k)}\} = \bigcap_{l=2}^4 \big(\{i : \bar{w}_{i,l}^{(k)} \leq w_{i,1}^{(k)}\} \cap \{i : \bar{w}_{i,1}^{(k)} \leq w_{i,2}^{(k)}\}\big). \tag{5.27}$$

We shall prove the following four inclusions to get (5.24) : $(k = 1, 2, ..., m)$

$$\{i : \lambda_i \le \varepsilon_k\} \subset \{i : \bar{w}_{i,2}^{(k)} \le w_{i,1}^{(k)}\}, \tag{5.28}$$

$$\{i : \lambda_i > \varepsilon_k\} \subset \{i : \bar{w}_{i,1}^{(k)} \le w_{i,2}^{(k)}\}, \tag{5.29}$$

$$\{i : a_i > \tau_k\} \subset \{i : \bar{w}_{i,3}^{(k)} \le w_{i,1}^{(k)}\}, \tag{5.30}$$

$$\{i : a_i \le t_k\} \subset \{i : \bar{w}_{i,4}^{(k)} \le w_{i,1}^{(k)}\}, \tag{5.31}$$

Let us show first (5.28). Due to the definition $\bar{w}_2^{(k)}$, we need to check two cases: $j_k \le 3$ and $j_k > 3$. Since the case $j_k \le 3$ is trivial, let us consider the case $j_k > 3$: By definitions of the weights and (5.12) we observe that the left-hand side of (5.28) is equivalent to

$$\lambda_i(\frac{1}{2}q_{j_{k-1}} + \frac{1}{2}p_0 - p_k) \le \frac{1}{2p_0} + \frac{1}{2q_{j_{k-1}}} - \frac{1}{p_k}. \tag{5.32}$$

The construction of the chains (5.16), (5.19) and (5.20) imply that the left-hand side of (5.32) is larger than $\frac{1}{4p_0}$. In the same way, we observe that rihgt-hand side is less than

$$\lambda_i \frac{q_{j_k}}{4p_0} = \lambda_i \frac{1}{4p_0\zeta_{j_k}} \le \lambda_i \frac{1}{4p_0\varepsilon_k}.$$

Thus we obtain the desired result (5.28). Analogously we can observe (5.29).

Now, let us consider the inclusion (5.30). By definitions of the weights and (5.12) we observe that the left-hand side of (5.30) is equivalent to

$$\frac{\tau_k}{2p_0} \le \frac{(p_k - p_0)(1 + \lambda_i p_0 p_k)}{p_0 p_k} a_i.$$

It is easy to see that

$$\frac{1}{2p_0} \le \frac{(p_k - p_0)(1 + \lambda_i p_0 p_k)}{p_0 p_k}$$

which implies the (5.30). Analogously we can observe (5.31).

After this estimation, we can conclude that (5.27)-(5.31) implies that

$$\{i : d_i^{(k)} \leq c_i^{(k)}\} \supset \{i : \delta_k < \lambda_i \leq \varepsilon_k, \tau_k < a_i \leq t_k\}.$$

Regarding (5.25), we observe (5.24) which implies (5.14).

Estimation of (5.15). Taking account of (5.23), to estimate (5.15) we check the following inequality:

$$\beta(4nB^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})) \leq \mu_m^{\tilde{X}}(\varphi(\delta) - \frac{M(\delta)}{\tau}, \varphi^{-1}(\varepsilon) + \frac{M(\delta)}{\tau}; \frac{\tau}{\alpha}, \alpha t). \qquad (5.33)$$

Lemma 3.2.5 and the definitions of the sequences $\tilde{c}$ and $\tilde{d}$ implies that

$$\beta(4nB^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})) = \Big|\bigcup_{k=1}^{m}\bigcup_{\nu=1}^{m}\{i : \tilde{d}_i^{(k)} \leq 4n\tilde{c}_i^{(\nu)}\}\Big|. \qquad (5.34)$$

Taking into account of the definitions $\tilde{d}^{(k)}$ and $\tilde{c}^{(k)}$, $(k, \nu = 1, 2, ..., m)$, we obtain

$$\{i : \tilde{d}_i^{(k)} \leq 4n\tilde{c}_i^{(\nu)}\} \subset \bigcap_{l=1}^{4}\{i : \tilde{\tilde{w}}_{i,l}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)}\} \cap \{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leq 4n\tilde{w}_{i,2}^{(\nu)}\}. \qquad (5.35)$$

Due to the definition $\tilde{\tilde{w}}_2^{(k)}$, we need to check two cases: $j_k \leq 3$ and $j_k > 3$. Let us consider first case, $j_k > 3$: By definitions of the weights and (5.12) we observe that the inequality

$$\tilde{\tilde{w}}_{i,2}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)} \qquad (5.36)$$

is equivalent to following

$$\big[\big(\frac{1}{2}r_0 - \frac{1}{2}q_{j_{k-2}} - s_\nu\big)\tilde{\lambda}_i - \big(\frac{1}{2r_0} + \frac{1}{2q_{j_{k-2}}} - \frac{1}{s_\nu}\big)\big]\tilde{a}_i \leq R(C\sqrt{CC_{j_{k-2}}}) \quad (5.37)$$

where $R(C) := \ln(4n(C))$. The construction of the chains (5.16), (5.19) and (5.20) imply that

$$\frac{q_{j_{k-2}}}{4} < \frac{1}{2}r_0 - \frac{1}{2}q_{j_{k-2}} - s_\nu,$$

$$\frac{1}{2r_0}+\frac{1}{2q_{j_{k-2}}}-\frac{1}{s_\nu}<\frac{1}{r_0}.$$

By the definition $\zeta_{j_{k-3}}=\dfrac{1}{q_{j_{k-3}}}>\dfrac{4}{r_0q_{j_{k-2}}}$, we obtain that

$$\{i:\tilde{\bar{w}}^{(k)}_{i,2}\le 4n\tilde{w}^{(\nu)}_{i,1}\}\subset\big\{i:\tilde{\lambda}_i\le\zeta_{j_{k-3}}+\frac{4\zeta_{j_{k-2}}R(C^2_{j_{k-2}})}{\tilde{a}_i}\big\}. \tag{5.38}$$

Now, tet us consider second case, $j_k\le 3$: In that case, by definitions of the weights and (5.12) we get that the inequality (5.36) is equivalent to

$$\big[\big(r_0-s_\nu\big)\tilde{\lambda}_i-\big(\frac{1}{r_0}-\frac{1}{s_\nu}\big)\big]\tilde{a}_i\le R(C^2). \tag{5.39}$$

The construction of the chains (5.16), (5.19) and (5.20) imply that

$$\{i:\tilde{\bar{w}}^{(k)}_{i,2}\le 4n\tilde{w}^{(\nu)}_{i,1}\}\subset\{i:\tilde{\lambda}_i\le\zeta_0\}. \tag{5.40}$$

Analogously we can observe

$$\{i:\tilde{\bar{w}}^{(k)}_{i,1}\le 4n\tilde{w}^{(\nu)}_{i,2}\}\subset\big\{i:\tilde{\lambda}_i\le\zeta_{\iota_\nu+2}+\frac{4\zeta_{\iota_\nu+1}R(C^2_{\iota_\nu})}{\tilde{a}_i}\big\}. \tag{5.41}$$

At this step, we will prove that (5.11) is guaranteed if we take a constant α, an increaing function $\varphi:[0,2]\to[0,1]$ and a decreasing function $M:(0,1]\to(0,\infty)$, satisfying the following conditions:

$$\alpha>\max\{4p_{m+1}R(C^2),8p_{m+1}s_{m+1}\}; \tag{5.42}$$

$$\varphi(0)=0,\ \varphi(2)=1,\ \varphi(\zeta_j)=\zeta_{j+4},\quad j=0,1,...; \tag{5.43}$$

$$M(\zeta_j)\ge\alpha\zeta_{j+2}R(C^2_{j+1}),\quad j=0,1,2,3; \tag{5.44}$$

$$M(\zeta_j)\ge\alpha\max\{\zeta_{j+2}R(C^2_{j+1}),4\zeta_{j-2}R(C^2_{j-2})\},\quad j=4,5,.... \tag{5.45}$$

When we combine (5.38) - (5.41), we obtain

$$\{i:\tilde{\bar{w}}^{(k)}_{i,2}\le 4n\tilde{w}^{(\nu)}_{i,1}\}\subset\{i:\tilde{\lambda}_i\le\varphi^{-1}(j_{k+1})+\frac{M(\zeta_{j_k})}{\alpha\tilde{a}_i}\}$$

$$\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leq 4n\tilde{w}_{i,2}^{(\nu)}\} \subset \{i : \tilde{\lambda}_i \geq \varphi(\iota_{\nu-2}) - \frac{M(\zeta_{\iota_k})}{\alpha\tilde{a}_i}\}$$

Due to the definitions in (5.20), the above two inclusions becomes

$$\{i : \tilde{\tilde{w}}_{i,2}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)}\} \subset \{i : \tilde{\lambda}_i \leq \varphi^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\alpha\tilde{a}_i}\}, \tag{5.46}$$

$$\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leq 4n\tilde{w}_{i,2}^{(\nu)}\} \subset \{i : \tilde{\lambda}_i \geq \varphi(\delta_\nu) - \frac{M(\delta_\nu)}{\alpha\tilde{a}_i}\}. \tag{5.47}$$

To complete the estimation (5.35), we have to check also the following inclusions:

$$\{i : \tilde{\tilde{w}}_{i,3}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)}\} \subset \{i : \tilde{a}_i > \frac{\tau_k}{\alpha}\}, \tag{5.48}$$

$$\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)}\} \subset \{i : \tilde{a}_i > \alpha t_k\}. \tag{5.49}$$

It is sufficient to prove only the inclusion (5.48), because the inclusion (5.48) will be proved in the same way. Due to definitions of the weights, the left hand side of (5.48) is equivalent to the following inequality:

$$\frac{\tau}{2p_0} \leq R(C^2) + [\frac{1}{r_0} - \frac{1}{s_\nu} + \tilde{\lambda}_i(s_\nu - r_0)]\tilde{a}_i. \tag{5.50}$$

If we consider (5.5), (5.20) and (5.42), we see that $\frac{\tau}{2p_0} \leq \alpha\tilde{a}_i$. Hence, this ensure (5.48).

By the aid of (5.46), (5.47), (5.48) and (5.49) the right-hand side of (5.35) becomes

$$\bigcap_{l=2}^{4}\{i : \tilde{\tilde{w}}_{i,l}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)}\} \cap \{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leq 4n\tilde{w}_{i,2}^{(\nu)}\} \subset S_{k,\nu}, \tag{5.51}$$

where

$$S_{k,\nu} = \{i : \varphi(\delta_\nu) - \frac{M(\delta_\nu)}{\tau_k} < \tilde{\lambda}_i \leq \varphi^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k}; \frac{\tau_k}{\alpha} < \tilde{a}_i \leq \alpha t_k\}. \tag{5.52}$$

Regarding the definitions of the sequences $\tilde{\tilde{w}}_{i,1}^{(k)}, \tilde{w}_{i,1}^{(\nu)}$ and (5.12) we obtain that

$$\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leq 4n\tilde{w}_{i,1}^{(\nu)}\} \subset T_{k,\nu}, \tag{5.53}$$

where

$$T_{k,\nu} = \big\{ i : [(\frac{1}{s_\nu} - \frac{1}{r_k}) + \tilde{\lambda}_i (r_k - s_\nu)]\tilde{a}_i \leq R(C^2) \big\}. \tag{5.54}$$

If we bring to mind (5.34), (5.35), (5.51) and (5.53) we get

$$\beta(4nB^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})) \leq \left| \bigcup_{k=1}^{m} \bigcup_{\nu=1}^{m} (S_{k,\nu} \cap T_{k,\nu}) \right| \tag{5.55}$$

Due to (5.16), for $\nu < k$ we have

$$(\frac{1}{s_\nu} - \frac{1}{r_k}) + \tilde{\lambda}_i (r_k - s_\nu) > \frac{1}{2s_\nu} > \frac{1}{4p_{m+1}}.$$

Hence,

$$T_{k,\nu} = \big\{ i : \tilde{a}_i 4p_{m+1} \leq R(C^2) \big\} \; if \; \nu < k.$$

By (5.42), this is equivalent to ($\nu < k$)

$$T_{k,\nu} \subset \{ i : \tilde{a}_i \leq \alpha \}. \tag{5.56}$$

To complete the proof of the lemma, it is sufficient to show that for $k, \nu = 1, 2, ..., m$, the following inclusions are true:

$$S_{k,\nu} \cap T_{k,\nu} \subset S_{k,k}. \tag{5.57}$$

For the case $k = \nu$,$(k, \nu = 1, 2, ..., m)$ the inclusions are trival. Thus, we need to check the cases $k < \nu, k > \nu$. Let us consider first, $k < \nu$. So we have $\delta_k < \delta_\nu$. Due to the definitions of the functions M and φ we observe that $M(\delta_k) \geq M(\delta_\nu)$ and $\varphi(\delta_k) \leq \varphi(\delta_\nu)$. Under this observation, for $k < \nu$ definition of $S_{k,\nu}$ implies (5.57). Now, let us check the last case, $k > \nu$. By (5.5), we have $\tilde{\lambda}_i \geq \frac{1}{\tilde{a}_i}$ for all $i \in \mathbb{N}$. With the aid of this, we oberve from (5.56) that $\tilde{\lambda}_i \geq \frac{1}{\alpha}$. So, we obtain

$$S_{k,\nu} \cap T_{k,\nu} \subset \{ i : \frac{1}{\alpha} \leq \tilde{\lambda}_i \leq \varphi^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k}; \frac{\tau_k}{\alpha} < \tilde{a}_i \leq \alpha t_k \}. \tag{5.58}$$

If we take into account the definitions of δ, φ, and (5.43) we observe that

$$\varphi(\delta_k) - \frac{M(\delta_k)}{\tau_k} < \varphi(\delta_k) < \varphi(\zeta_0) = \zeta_4 = \frac{1}{q_4}. \tag{5.59}$$

The constant depends on n, for this reason we may assume that the number q_4 chosen in the following way

$$\frac{1}{q_4} \leq \frac{1}{\alpha}. \tag{5.60}$$

Combining (5.58),(5.59) and (5.59), we see the that (5.57) is valid. Due to (5.55) it implies the inequality

$$\beta(4nB^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})) \leq \left| \bigcup_{k=1}^{m} S_{k,k} \right|.$$

Thus, the estimation (5.15) is obtained, due to the (5.11) which completes the proof.

□

Proposition 5.2.2. *Let $X = E^\ell(\lambda, a)$, $\tilde{X} = E^\ell(\tilde{\lambda}, \tilde{a})$, $m \in \mathbb{N}$. If $X \simeq \tilde{X}$, then $\mu_m^X \approx \mu_m^{\tilde{X}}$.*

Systems of characteristics $\left(\mu_m^X\right)_{m\in\mathbb{N}}$ and $\left(\mu_m^{\tilde{X}}\right)_{m\in\mathbb{N}}$ are equivalent if the function φ and the constant α can be chosen so that the inequalities (5.9), (5.10) hold for all $m \in \mathbb{N}$. We denote this equivalence by $\left(\mu_m^X\right) \approx \left(\mu_m^{\tilde{X}}\right)$.

Proposition 5.2.3. *For spaces $X = E^\ell(\lambda, a)$ and $\tilde{X} = E^\ell(\tilde{\lambda}, \tilde{a})$, the following statment are equivalent:*

(i) $X \overset{qd}{\simeq} \tilde{X}$;

(ii) $\left(\mu_m^X\right) \approx \left(\mu_m^{\tilde{X}}\right)$.

Proof. Let us show first (i) implies (ii). Let the spaces X and $\tilde{X}$ be quasidiagonally isomorphic. Then the condition the condition (c) of Lemma

5.1.4 holds. It follows from (5.3) that there is a strictly increasing function $\varphi : (0, 1] \to (0, 1], \varphi(t) \downarrow 0$ as $t \downarrow 0$, such that the following inclusions

$$\{i : \lambda_i \geq \delta\} \subset \{i : \tilde{\lambda}_{\sigma(i)} \geq \varphi(\delta)\},$$

$$\{i : \tilde{\lambda}_{\sigma(i)} \geq \delta\} \subset \{i : \lambda_i \geq \varphi(\delta)\} \tag{5.61}$$

hold for any $\delta \in (0, 1]$. Without loss of generality we can assume that $\varphi(1) < 1$. Let us extend the function φ on the segment $[0, 2]$ so that it will be a strictly increasing function and $\varphi(0) = 0, \varphi(2) = 1$. Then the function φ has the inverse function φ^{-1} on the segment $[0, 1]$ and $\varphi^{-1}(0) = 0$, $\varphi^{-1}(1) = 2$. It follows from (5.4) (5.61) that the inclusions

$$\{i : \delta < \lambda_i \leq \varepsilon, \tau < a_i \leq t\} \subset \{i : \varphi(\delta) < \tilde{\lambda}_{\sigma(i)} \leq \varphi^{-1}(\varepsilon), \frac{\tau}{\alpha} < \tilde{a}_{\sigma(i)} \leq \alpha t\},$$

$$\{i : \delta < \tilde{\lambda}_{\sigma(i)} \leq \varepsilon, \tau < \tilde{a}_{\sigma(i)} \leq t\} \subset \{i : \varphi(\delta) < \lambda_i \leq \varphi^{-1}(\varepsilon), \frac{\tau}{\alpha} < a_i \leq \alpha t\},$$

holds for any parameters $\delta, \varepsilon, \tau, t (0 < \delta < \varepsilon \leq 1)$. Hence, taking into account that σ is the bijection, we obtain the (ii).

To complete proof, let us show that (ii) implies (i). The condition (ii) means that the inequalities (5.9) and (5.10) hold for all collections of parameters $\delta, \varepsilon, \tau$ and t.

Let us define a multiple-valued function $S : \mathbb{N} \to \mathbb{N}$ by the rule:

$$S(i) := \{j : \varphi(\lambda_i) < \tilde{\lambda}_j \leq \varphi^{-1}(\lambda_i), \frac{a_i}{\alpha} \leq \tilde{a}_j \leq a_i \alpha\},\ i \in \mathbb{N}$$

It follows from (5.9) that the map S satisfies all conditions of the Hall-König Theorem. By this theorem there exists an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(i) \in S(i), i \in \mathbb{N}$. Therefore the operator $T : X \to \tilde{X}$ defined by $Te_i = e_{\sigma(i)}$ is a quasidiagonal embedding. By repeating this argument with (5.10), we obtain that $\tilde{X} \overset{qd}{\hookrightarrow} X$.Then by the Lemma 3.1.2 we have $X \overset{qd}{\simeq} \tilde{X}$. □

We dont know whether the statment of the Proposition 5.2.3 remain true if $\overset{qd}{\simeq}$ is replaced by $\simeq$, in other words, is the quasidiagonal invariant

$\gamma(X) := (\mu_m^X)_{m\in\mathbb{N}}$ also a linear topological invariant on the class $\mathcal{E}$ with the above notion of equivalence? Nevertheless it is shown in [6] that it is possible to get new linear topological invariants, essentially stronger than any invariant (5.6), simply by taking the same map $\gamma(X)$ but introducing new equivalence relations on the set $\Gamma := \{(\mu_m^X)_{m\in\mathbb{N}} : X \in \mathcal{E}\}$.

Definition 5.2.4. *Let $n \in \mathbb{N}$. We say that systems of characteristics $(\mu_m^X)_{m\in\mathbb{N}}$ and $(\mu_m^{\tilde{X}})_{m\in\mathbb{N}}$ are n-equivalent, and we write $(\mu_m^X) \overset{n}{\approx} (\mu_m^{\tilde{X}})$, if there is a strictly increasing function $\varphi : [0,2] \to [0,1], \varphi(0) = 0, \varphi(2) = 1$ and a positive constant α such that for arbitrary $m \in \mathbb{N}$, the inequalities (5.9) and (5.10) holds for all collection of parameters (5.7) satisfying the following additional condition: among all the numbers $\delta_1, \delta_2, ..., \delta_m$ there are no more than n different.*

We consider the maps γ_n from $\mathcal{X}$ onto the set with equivalence $(\Gamma, \overset{n}{\approx})$ which all coincide with the map γ if considered as set maps, $n \in \mathbb{N}$. The following theorem shows that the map γ_n is a linear topological invariant.

Theorem 5.2.5. *Let the spaces $X = E^\ell(\lambda, a)$ and $\tilde{X} = E^\ell(\tilde{\lambda}, \tilde{a})$ be isomorphic. Then $(\mu_m^X) \overset{n}{\approx} (\mu_m^{\tilde{X}})$ for each $n \in \mathbb{N}$.*

Proof. We apply Lemma 5.2.1 to establish the estimates (5.9), (5.10) for each $m \in \mathbb{N}$ and arbitrary collections of parameters (5.7) satisfying the condition: among the numbers $\delta_1, \delta_2, ..., \delta_m$, there are no more than n different. Therewith the function φ will be chosen in the end of our proof, while the constant α will be the same as in (5.11).

Because of symmetric we need to prove only the inequality (5.9). Let us rewrite this estimate, using (5.8) in the form:

$$\left|\left\{i : (\lambda_i, a_i) \in \bigcup_{k=1}^m P_k\right\}\right| \le \left|\left\{i : (\tilde{\lambda}_i, \tilde{a}_i) \in \bigcup_{k=1}^m Q_k\right\}\right|,$$

where

$$Q_k := \Big(\varphi(\delta_k), \varphi^{-1}(\varepsilon_k)\Big] \times \Big(\frac{\tau_k}{\alpha}, \alpha t_k\Big],\ k = 1, 2, ..., m.$$

We cover each rectangle P_k by an appropriate couple of nonintersecting rectangles P'_k and P''_k (some of them may be empty) and the apply Lemma 5.2.1. For construction of above-mentioned rectangles we need to define the decreasing function $\Psi : (0,1] \to \mathbb{R}_+$, so that

$$\Psi(\xi) > \frac{M(\xi)}{\gamma(\xi)}, \quad 0 < \xi \leq 1, \tag{5.62}$$

where M and γ are as in Lemma 5.2.1. We are acting in a different way for each of three cases:

$$(a)\tau_k \leq \Psi(\delta_k), \quad (b)\tau_k < \Psi(\delta_k) < t_k, \quad (c)t_k \leq \Psi(\delta_k).$$

Setting the notation

$$\tau'_k := \max\{\Psi(\delta_k), \tau_k\}, \quad t'_k := \min\{\Psi(\delta_k), t_k\}, \quad \varepsilon'_k := \begin{cases} \Psi^{-1}(\tau_k) & \text{if} \quad \tau_k \geq \Psi(1), \\ 1 & otherwise, \end{cases}$$

we put

$$P'_k = \begin{cases} (\delta_k, \varepsilon_k] \times (\tau'_k, t_k] & in \ \ the \ \ cases \quad (a) \ and \ (b), \\ \emptyset & otherwise, \end{cases}$$

and

$$P''_k = \begin{cases} \emptyset & in \ \ the \ \ case \ (a), \\ (\delta_k, \varepsilon'_k] \times (\tau_k, t'_k] & otherwise. \end{cases}$$

Applying Lemma 5.2.1, we get

$$\left| \left\{ i : (\lambda_i, a_i) \in \bigcup_{k=1}^{m} (P'_k \cup P''_k) \right\} \right| \leq \left| \left\{ i : (\tilde{\lambda}_i, \tilde{a}_i) \in \bigcup_{k=1}^{m} (\tilde{P}'_k \cup \tilde{P}''_k) \right\} \right|,$$

with

$$\tilde{P}'_k = \begin{cases} (\Delta'_k, E'_k] \times (\frac{\tau'_k}{\alpha}, \alpha t_k] & in \ \ the \ \ cases \quad (a) \ and \ (b), \\ \emptyset & otherwise, \end{cases}$$

and

$$\tilde{P}''_k = \begin{cases} \emptyset & \textit{in the case } (a), \\ (\Delta_k, E''_k] \times (\frac{\tau_k}{\alpha}, \alpha t'_k] & \textit{otherwise}, \end{cases}$$

where

$$\Delta'_k = \gamma(\delta_k) - \frac{M(\delta_k)}{\tau'_k}, \qquad E'_k = \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau'_k},$$

$$\Delta_k = \gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \qquad E''_k = \gamma^{-1}(\varepsilon'_k) + \frac{M(\varepsilon'_k)}{\tau_k}.$$

It follows from (5.62) and the definition of the numbers τ'_k that

$$\Delta'_k \geq \frac{1}{2}\gamma(\delta_k).$$

Since $\gamma(\xi) \leq \gamma^{-1}(\xi)$ when $\xi \in [0,1]$, we obtain also the estimate

$$E'_k \leq \frac{3}{2}\gamma^{-1}(\varepsilon_k).$$

From $\tilde{\lambda}_i \geq \frac{1}{\tilde{a}_i}$ and (5.62) it follows that

$$\{i : (\tilde{\lambda}_i, \tilde{a}_i) \in \tilde{P}''_k\} \subset \left\{ i : (\tilde{\lambda}_i, \tilde{a}_i) \in \left(\frac{1}{2\alpha\Psi(\delta_k)}, E''_k \right] \times \left(\frac{\tau_k}{\alpha}, \alpha t_k \right] \right\}.$$

We can always assume that $\tau_k \geq \frac{1}{2\varepsilon_k},\ k = 1, 2, ..., m.$ So, taking into account (5.62), the definition of the numbers ε_k and the estimate $\gamma(\xi) \leq \gamma^{-1}(\xi)$, $\xi \in [0,1]$, we obtain that

$$E''_k \leq \frac{3}{2}\gamma^{-1}(\varepsilon'_k) \leq \frac{3}{2}\gamma^{-1}\left(\Psi^{-1}\left(\frac{1}{2\varepsilon_k}\right)\right).$$

Now we choose an increasing function $\varphi : [0,2] \to [0,1],\ \varphi(2) = 1,\ \varphi(0) = 0,$ so that

$$\varphi(\xi) \leq \min\left\{ \frac{1}{2}\gamma(\xi), \gamma\left(\frac{2}{3}\xi\right), \frac{1}{2\alpha\Psi(\xi)}, \frac{1}{2\Psi(\gamma(\frac{2}{3}\xi))} \right\}, \xi \in [0,1].$$

Then the estimate (5.9) holds for each $m \in \mathbb{N}$ and any collection of parameters (5.7) satisfying condition: among the numbers $\delta_1, \delta_2, ..., \delta_m$, there are more than n different. This completes the proof.

$\square$

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