

NONLINEAR ESTIMATION TECHNIQUES APPLIED TO ECONOMETRIC
PROBLEMS

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ABSTRACT

NONLINEAR ESTIMATION TECHNIQUES APPLIED TO ECONOMETRIC

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This thesis considers the filtering and prediction problems of nonlinear noisy econometric systems. As a filter/predictor, the standard tool Extended Kalman Filter and new approaches Discrete Quantization Filter and Sequential Importance Resampling Filter are used. The algorithms are compared by using Monte Carlo Simulation technique. The advantages of the new algorithms over Extended Kalman Filter are shown.

Keywords: Discrete Quantization Filter, Sequential Importance Resampling Filter, Extended Kalman Filter, Stochastic Calculus, Monte Carlo

ÖZ

DOĞRUSAL OLMAYAN KESTİRME ALGORİTMALARININ EKONOMETRİK PROBLEMLERE UYGULANMASI

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Bu tez doğrusal olmayan ekonometrik gürültülü sistemlerde gürültüden arındırma ve tahmin yapma üzerinedir. Filtre ve öngörücü olarak standart gereç olan Geliştirilmiş Kalman Algoritması ve yeni yaklaşımlar olan Zamanda Ayrık Nicemleme Filtresi ve Ardışık Önem Tekrarlı Örnekleme Filtresi kullanıldı. Algoritmalar birbiriyle Monte Carlo Similasyon tekniğiyle karşılaştırıldı. Algoritmaların Geliştirilmiş Kalman Algoritmasından avantajlı yanları gösterildi.

Anahtar Kelimeler: Geliştirilmiş Kalman Filtresi, Zamanda Ayrık Nicemleme Filtresi, Ardışık Önem Tekrarlı Örnekleme Filtresi, Olasılıksal Analiz, Monte Carlo.

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CHAPTER 1

INTRODUCTION

In electrical and electronics engineering, there are many dynamical system representations, which include noise components. In such cases, one of the primary goals is to remove the noise component from the measurement and/or try to predict the next state of the system. If the problem at hand can be written in a linear Gaussian system form, then the Kalman Filter emerges as the optimal methodology. However, when the system has nonlinear characteristics, then the Extended Kalman Filter (EKF henceforth) should be employed¹. However, the estimation of nonlinear systems should by no means be restricted to using only EKF. As discussed in [3] and [25] in details, Discrete Quantization Filter and Sequential Importance Resampling (SIR) filters can conveniently be used

Based on the above discussion, there are three features of this study:

- 1) Control theory approaches are applied to the field of econometrics:
- 2) Stochastic Calculus has been introduced within the context of a stochastic Ito process, which is used basically in the asset pricing models
- 3) Two relatively unknown algorithms, DQF and SIR filter are introduced and compared with the EKF. As it will be clear, some cases where these two alternatives are superior to EKF are presented.

To begin with, state space representations are convenient to describe the dynamical systems. It is straightforward to obtain the dynamics of the model from the differential equation representation of the nonnoisy systems. [4] However, in noisy systems, the description of the system is generally introduced within the state space representation. In such systems, the noise is assumed to be generally additive. But thanks to the stochastic calculus, we have a more general perspective.

¹ For a detailed discussion on Kalman Filter see [1] and [2]

The assumptions of the noise structure can be made directly in a differential equation form. Then, the differential equations with noise are solved by employing stochastic calculus. For a general treatment of the subject stochastic calculus, the reader is referred to [5].

Consequently, the study briefly described above is composed of the following parts:

- 1) Introduction
- 2) First Model – From Continuous Time to Discrete Time
- 3) Stochastic Calculus and Application to the Case Study
- 4) DQF Algorithm
- 5) EKF Algorithm
- 6) Application of EKF to the Econometric Problem
- 7) Particle Filters-SIR Algorithm
- 8) Simulations of the First model
- 9) Simulations of the Second Model

The second section explains how the discrete-time nonlinear noisy system representation of the first model is obtained from continuous time representation. The dynamical system is based on obtaining the expression for the price of one risky asset (stock). Since the primary goal of this study is not to focus on the details of the derivation, just a succinct introduction is given in this section.

In the third section named “Stochastic Calculus and Application to the Case Study”, the necessary background to understand the basics of Stochastic Calculus is explained. Then it is shown, how the continuous time solution has been obtained from the stochastic differential equations.

In the fourth section, named “DQF Algorithm”, the basics of the algorithms proposed by Demirbaş [3] are explained. Although not widely known, this algorithm is capable to lead to better results than the standard EKF algorithm. The next two sections briefly discuss the EKF algorithm. In the former, EKF for discrete time cases is explained. In the latter section, the algorithm is applied to the nonlinear dynamic system of the model derive in the previous sections.

The seventh section elaborates the particle filters. It gives the derivation of the Sequential Importance Sampling (SIS) filter along with its proof. Then, the SIR filter, which is also employed in this study, is easily derived from SIS filter.

The eighth section displays the simulation results of the filtering and prediction algorithms of the benchmark model introduced before.

The next section presents the simulation results of the filtering algorithm of the second model, which basically examines productivity shocks and capital accumulation within a stochastic growth model. The final section concludes.

CHAPTER 2

FIRST MODEL – FROM CONTINUOUS TIME TO DISCRETE TIME

In a conventional asset pricing model, the price of one share of the risky asset (stock) is described by the following stochastic differential equation (2.1).

$$dX(t) = c X(t) dt + \delta X(t) dB(t) \quad (2.1)$$

$X(t)$: The price $X(t)$ is called *risky asset*. It is also named as *stock*.

c : The constant c is the *mean rate of return*.

δ : The parameter δ denotes *volatility*.

B : $B = (B(t), t \geq 0)$ represents the Brownian motion. [6]

The differential formula in equation (2.1) is similar to the expressions used in differential calculus with the important exception that there is a term $dB(t)$, which represents the stochastic part. Such a term makes the usage of the stochastic calculus necessary, which has also been widely used in the field of econometrics.

In the third section, a necessary background for a thorough understanding of the above mentioned subject is given. As the third section shows in details, the solution to the above given formula for $\mathbf{X}(t)$ is given by:

$$X(t) = f(t, B(t)) = X(0)e^{(c-0.5\delta^2)t + \delta B(t)} \quad (2.2)$$

(2.2) is an exponential function which is randomly perturbed. It is also named as *Geometric Brownian Motion*. $X(0)$ is assumed to be independent of B . If the

volatility term $\delta = 0$, then (2.2) reduces to the usual deterministic exponential function, which is given by (2.3):

$$X(t) = X(0)e^{ct} \quad (2.3)$$

2.1 Discretization of the Model

In this section, the continuous time expression specified in (2.2) is transformed into the discrete time domain. One basic approach is to sample (2.2) with uniform time interval ΔT . Then a recursive relation between $X(t + \Delta T)$ and $X(t)$ is obtained as follows:

$$\begin{aligned}
X(t + \Delta T) &= X_0 e^{(c-0.5\delta^2)(t+\Delta T) + \delta B(t+\Delta T)} \\
&= X_0 e^{(c-0.5\delta^2)(t+\Delta T)} e^{\delta B(t+\Delta T)} \\
&= X_0 e^{(c-0.5\delta^2)t} e^{(c-0.5\delta^2)\Delta T} e^{\delta B(t+\Delta T)} \\
&= X_0 e^{(c-0.5\delta^2)t} e^{(c-0.5\delta^2)\Delta T} e^{\delta B(t+\Delta T)} (e^{\delta B(t)} e^{-\delta B(t)}) \\
&= X_0 e^{(c-0.5\delta^2)t} e^{\delta B(t)} e^{(c-0.5\delta^2)\Delta T} e^{\delta B(t+\Delta T)} e^{-\delta B(t)} \\
&= [X_0 e^{(c-0.5\delta^2)t} e^{\delta B(t)}] e^{(c-0.5\delta^2)\Delta T} e^{\delta B(t+\Delta T)} e^{-\delta B(t)} \\
&= [X_0 e^{(c-0.5\delta^2)t + \delta B(t)}] e^{(c-0.5\delta^2)\Delta T} e^{\delta B(t+\Delta T)} e^{-\delta B(t)} \\
&= [X(t)] e^{(c-0.5\delta^2)\Delta T} e^{\delta B(t+\Delta T)} e^{-\delta B(t)} \\
&= e^{(c-0.5\delta^2)\Delta T} X(t) e^{\delta B(t+\Delta T)} e^{-\delta B(t)} \\
&= e^{(c-0.5\delta^2)\Delta T} X(t) e^{\delta B(t+\Delta T) - \delta B(t)} \\
&= e^{(c-0.5\delta^2)\Delta T} X(t) e^{\delta [B(t+\Delta T) - B(t)]} \\
&= e^{(c-0.5\delta^2)\Delta T} X(t) e^{\delta w(\Delta T)}
\end{aligned}$$

Where $w(\Delta T)$ represents Gaussian noise with variance ΔT .

Hence (2.4) can be written as.

$$X(t + \Delta T) = e^{(c-0.5\delta^2)\Delta T} X(t)e^{\delta w(\Delta T)} \quad (2.4)$$

In order to obtain (2.4), the fact that the random variables $B(t) - B(s)$ have a normal distribution $N(0, t - s)$ for $s < t$, is used in the last equation. [8]

Summarizing the solution (2.4), equation (2.5) can be obtained as:

$$X(t + \Delta T) = LX(t)e^{\delta w(\Delta T)} \quad (2.5)$$

where $L = e^{(c-0.5\delta^2)\Delta T}$, which is assumed to be constant.

Hence starting at $X(0)$ and using (2.5) it is possible to obtain the values of $\{X(\Delta T), X(2\Delta T), X(3\Delta T), \dots, X(k\Delta T), \dots\}$.

The notations $X(\Delta T)$, $X(2\Delta T)$, $X(3\Delta T)$, $X(k\Delta T)$ are written as $X(1)$, $X(2)$, $X(3)$, $X(k)$ respectively. As a result (2.5) has the alternate representation as:

$$X(k + 1) = LX(k)e^{\delta w(k)} \quad (2.6)$$

where we still have $L = e^{(c-0.5\delta^2)\Delta T}$.

Since the next state of the system is calculated according to (2.6), it emerges as the state equation of the system.

The measurement equation, on the other hand is given by (2.7):

$$Z(k) = X(k) + v(k) \quad (2.7)$$

where $v(k)$ represents the *speculators noise or traders noise*.

The above derivation and presentation of the asset pricing model is fairly standard and can be conveniently used to evaluate the performances of several filters designed to be used in non-linear cases.

CHAPTER 3

STOCHASTIC CALCULUS and APPLICATION TO THE CASE STUDY

This section briefly introduces stochastic differential calculus. Before discussing the implications it should be mentioned that, the reader, who is not familiar with stochastic processes is referred to [9].

First, it is important to remind the definition and the interpretation of stochastic differential equations. The integral that is written in the form of (3.1)

$$\int_0^t b(s, X(s))dB(s) \tag{3.1}$$

is called Ito Integral. This integral representation differs in two ways from its counterparts:

- 1) *We have $dB(s)$ instead of ds .*
- 2) *B is not a deterministic function but a stochastic one.*

The formal definition of (3.1) is given in the subsection 3.8.

The subsection 3.2 “The Riemann Integral” reminds the reader the formal definition of Riemann Integral. The subsection 3.3 named “The Riemann-Stieltjes Integral” is a simple extension of Riemann Integral. The reason to introduce this integral type is the connection between the Ito Integral and Riemann-Stieltjes Integral

The subsections 3.4 and 3.5, which are “ L^p Convergence and Mean Square Convergence” and “Simple Processes” respectively, define convergence types of stochastic processes, which are used in the definition of Ito integral,

The subsection 3.7, namely the “Ito Integral of Simple Processes” first defines the Ito Integral for a restricted class of processes. Then the subsection 3.8 “Ito Integral of General Processes” enlarges the definition of the Ito Integral for a wider class of processes. At this stage, the theoretical implication of the Ito integral is concluded with a brief discussion. After this section, practical aspects of stochastic calculus are introduced and applied to econometrics, which was explained in the second section.

The subsection 3.9 Ito Lemma is the basic tool, which is needed to solve the stochastic differential equations.

The subsection 3.10 is an application of the Ito Lemma to the case study, which is a standard stochastic growth model.

Finally, the subsection 3.11 discusses the uniqueness problem of the solution, which is obtained in the section 3.10

3.1 STOCHASTIC DIFFERENTIAL EQUATIONS

In stochastic calculus, a basic expression takes the following form:

$$dX(t) = a(t, X(t))dt + b(t, x(t))dB \quad (3.2)$$

where t is the independent variable, $a(t, x)$ and $b(t, x)$ are deterministic functions, and $B = (B(t), t \geq 0)$ is the Brownian motion. It is easy to see that, for $b(t, x) = 0$, (3.2) reduces to the usual deterministic differential equation.

In the stochastic calculus (3.2), can also be interpreted as:

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dB(s) \quad , \quad 0 \leq t \leq T \quad (3.3)$$

In this notation, $\int_0^t a(s, X(s))ds$ is the usual Riemann integral. The integrand function $b(s, X(s))$ and the integrating function $B(s)$ are stochastic processes.

Integrals of the form $\int_0^t b(s, X(s))dB(s)$ should be interpreted differently. Ito process offers such an interpretation. Since Ito interpretation is the most commonly used process, it will be employed throughout the study. However, it should be reminded that there are also other interpretations like the Stratonovich interpretation. In some cases, it is better to use Stratonovich interpretation, as expressed in [10]

The next section Riemann integral is defined formally.

3.2 The Riemann Integral

The classical integration technique is the Riemann integral. The standard notation is shown in (3.4).

$$\int_{x=0}^{x=T} f(t)dt \quad (3.4)$$

where t denotes the independent variable. f is the function to be integrated. 0 and T are the limits of integration. The definition of Riemann Integral is as follows:

1) First the interval $[0, T]$ is partitioned as:

Let $\tau_n : 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$ be the mentioned partition of the interval $[0, T]$.

2) Choose $y_i, t_{i-1} \leq y_i \leq t_i$ for $i = 1, \dots, n$.

3) Let $S_n = \sum_{i=1}^n f(y_i)(t_i - t_{i-1})$

4) If $\lim_{n \rightarrow \infty} S_n = S$ exists, for some S and for each possible partition τ_n and y_i as

$(t_i - t_{i-1}) \rightarrow 0$, then $S = \int_{x=0}^{x=T} f(t)dt$.

While Riemann integration technique is used extensively in several areas it is not appropriate for analyzing the dynamics of the systems with random terms. For that purpose, Ito's Stochastic Integral offers a solution.

The Riemann-Stieltjes Integral, which is a simple extension of Riemann Integral, is discussed in the next section.

3.3 The Riemann-Stieltjes Integral

Definition [11]

1) Let $\tau_n : 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$ be a partition of the interval $[0,1]$.

2) Choose $y_i, t_{i-1} \leq y_i \leq t_i$ for $i = 1, \dots, n$.

3) Let f and g be two real-valued functions on $[0,1]$ and define (3.5)

$$\Delta_i g = g(t_i) - g(t_{i-1}), i=1, \dots, n. \quad (3.5)$$

4) The Riemann-Stieltjes sum corresponding to τ_n and is given by (3.6)

$$S_n = S_n(\tau_n) = \sum_{i=1}^n f(y_i) \Delta_i g = \sum_{i=1}^n f(y_i) [g(t_i) - g(t_{i-1})]. \quad (3.6)$$

5) If the limit $\lim_{n \rightarrow \infty} S_n$ exists and S is independent of the choice of the partitions τ_n and y_i as $(t_i - t_{i-1}) \rightarrow 0$, then S is called the Riemann-Stieltjes integral of f with respect to g on $[0,1]$. Symbolically, it is written as (3.7).

$$S = \int_0^1 f(t) dg(t). \quad (3.7)$$

Riemann-Stieltjes integral is just an extension of the classical Riemann integral. It is reduced to the Riemann integral for $g(t)=t$. This integration technique is also used in the probability theory. In these cases g is to be chosen as F , which is the distribution function of a random variable. For example, (3.8) represents such a form:

$$P\{x \in A\} = \int_{x \in A} dF(x) \quad (3.8)$$

Although Ito's stochastic integral is different from the Riemann-Stieltjes integral, as the proceeding sections will show, there are connections as well. These connections are shown after the definition of the Ito stochastic integral. In the next section, some mathematical terms, which are necessary to understand the definition of the Ito integral, are introduced.

3.4 L^p Convergence and Convergence in Mean Square

If X_n and X belong to L^p and $E|X_n - X|^p \rightarrow 0$, then $\{X_n\}$ is said to converge in p 'th mean to X . For $p = 1$, it is simply called convergence in mean; for $p = 2$, on the other hand, the process is generally termed as convergence in mean square or in quadratic mean.[12] This process is employed for the definition of Ito Integral.

In the above definition, E is the usual expectations operator while L^p is the space of functions, for which the L^p norm is finite.

3.5 Simple Processes

The stochastic process $C = (C(t), t \in [0, T])$ is said to be “simple process” if it satisfies the following properties:

There exists a partition: $\tau_n : 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$, and a sequence $(Z_i, i = 1, \dots, n)$ of random variables such that: [13]

$$C(t) = Z_n \text{ if } t = T,$$

$$C(t) = Z_i \text{ if } t_{i-1} \leq t < t_i, \quad i = 1, \dots, n.$$

3.6 Outline for the Definition of the Ito Stochastic Integral

As mentioned before, the Ito Integral employs the mean square convergence. For the definition of the Ito stochastic integral, the following way will be tracked:

1. *The integrand function is first assumed to be a simple process.*
2. *The Ito integral of the simple processes is given.*
3. *The integrand function is assumed to be a general process.*
4. *The Ito integral of the general process is defined via the simple processes.*

3.7 The Ito Stochastic Integral of Simple Processes

The Ito stochastic integral of a simple process C on $[0, T]$ is given by (3.9).

$$\int_0^T C(s)dB(s) := \sum_{i=1}^n C(t_{i-1})(B(t_i) - B(t_{i-1})) = \sum_{i=1}^n Z_i \Delta_i B \quad (3.9)$$

where $Z_i = C(t_{i-1})$ and $\Delta_i B = (B(t_i) - B(t_{i-1}))$.

The Ito stochastic integral of a simple process C on $[0,t]$, $t_{k-1} \leq t \leq t_k$, is given by (3.10).

$$\int_0^t C(s)dB(s) := \int_0^T C(s)I_{[0,t]}(s)dB(s) = \sum_{i=1}^{k-1} Z_i \Delta_i B + Z_k (B(t) - B(t_{k-1})) \quad (3.10)$$

where $\sum_{i=1}^0 Z_i \Delta_i B = 0$, $I_{[0,t]}(s) = 1$ for $s \in [0,t]$ and $I_{[0,t]}(s) = 0$ otherwise.

When comparing (3.9) with the Riemann-Stieltjes sum notation (3.6), it is easily seen that $y_i = t_{i-1}$. Furthermore, “g” corresponds to Brownian motion. Hence the Ito stochastic integral is the Riemann-Stieltjes sum, where the intermediate points y_i are chosen at the left end points of the intervals $[t_{i-1}, t_i]$.

This concludes the definition of the Ito integral for simple processes, which can be regarded as the corner stone for the general stochastic integration.

3.8 The Ito Stochastic Integral of General Processes

The Ito stochastic integral is defined for simple processes in section 3.7. In this section, however, the integrand process will be from a wider class of function space. But the following restrictions are put for this class:

Assumptions on the Integrand Process C:

1. C is adapted to Brownian motion on $[0,T]$, i.e. $C(t)$ is a function of $B(s)$ $s \leq t$.
2. The integral (3.11) is finite.

$$\int_0^T EC(s)^2 ds \quad (3.11)$$

Let C be a process satisfying the above additional assumptions. Then, one can find a sequence $(C^{(n)})$ of simple processes such that the following statement holds. See [14], for the proof of this statement.

$$\int_0^T E[C(s) - C(s)^{(n)}]^2 ds \rightarrow 0 \quad (3.12)$$

Hence, the simple processes $C^{(n)}$ converge in mean square sense to the integrand process C . Using (3.10), we can calculate the Ito integrals $I_t(C^{(n)})$ of these simple processes. Hence we have a new sequence $\{I_t(C^{(1)}), I_t(C^{(2)}), \dots\}$. This new sequence converges to a unique process, which can be conveniently called as $I(C)$ on $[0, T]$. Then, we can write (3.13) as [15]:

$$E \sup_{0 \leq t \leq T} [I_t(C) - I_t(C^{(n)})]^2 \rightarrow 0. \quad (3.13)$$

The mean square limit $I(C)$ is called the Ito stochastic integral of C . It is denoted by (3.14) [14]

$$I_t(C) = \int_0^t C(s) dB(s), \quad t \in [0, T]. \quad (3.14)$$

This ends the formal definition for the Ito stochastic integral. In order to apply the integral, however, the Ito Lemma should be introduced.

3.9 Ito's Lemma

In this subsection, the Ito's Lemma is introduced in order to apply the stochastic integral that is defined above.

Ito's Lemma:

Let $f(t, x)$ be a function whose second order partial derivatives are continuous. Then (3.15) is true.

$$f(t, B(t)) - f(s, B(s)) = \int_s^t [f_1(x, B(x)) + \frac{1}{2} f_{22}(x, B(x))] dx + \int_s^t f_2(x, B(x)) dB(x) \quad (3.15)$$

where $s < t$. For the details of the lemma, please refer to [16].

$$\int_s^t [f_1(x, B(x)) + \frac{1}{2}f_{22}(x, B(x))]dx \quad (3.16)$$

$$\int_s^t f_2(x, B(x))dB(x) \quad (3.17)$$

The first integral (3.16) is a simple deterministic integral. The second integral (3.17) is the aforementioned stochastic integral in section 3.8. Furthermore f_1, f_2 and f_{22} are the partial derivatives.

3.10 Solution of the Stochastic Differential Equation

In subsection 3.1, it is explained that (3.18) is equivalent to the following representation of (3.19).

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB \quad (3.18)$$

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dB(s), \quad 0 \leq t \leq T \quad (3.19)$$

That is, (3.18) is interpreted as (3.19) in stochastic calculus. As it is mentioned in the second chapter, the price of one share of the risky asset (stock) is described by the stochastic differential equation (3.20).

$$dX(t) = cX(t)dt + \delta X(t)dB(t) \quad (3.20)$$

where

$X(t)$: The price $X(t)$ is called risky asset. It is also called as stock.

c : The constant c is called mean rate of return

δ : The δ is called volatility.

B : $B = (B(t), t \geq 0)$ is Brownian motion.

When the equations (3.20) and (3.18) are compared, it is seen that (3.20) is a special case of (3.18) with $a(t, X(t)) = cX(t)$ and $b(t, X(t)) = \delta X(t)$.

Hence (3.20) is equivalent to the following:

$$X(t) = X(0) + c \int_0^t X(s)ds + \delta \int_0^t X(s)dB(s), \quad 0 \leq t \leq T \quad (3.21)$$

More generally, the stochastic differential equations in the following form of (3.22), where c_i and σ_i , $i = 1, 2$ are constants, can be termed as linear stochastic differential equations. Hence (3.20) is from the family of linear stochastic differential equations.

$$X(t) = X(0) + \int_0^t (c_1 X(s) + c_2) ds + \int_0^t (\sigma_1 X(s) + \sigma_2) dB(s) \quad , \quad 0 \leq t \leq T \quad (3.22)$$

The introduced stochastic integral in (3.21) can be solved according to the Ito's Lemma, where the solution is given in [17].

Solution:

Suppose $X(t)$ is described as (3.23).

$$X(t) = f(t, B(t)) = e^{(c-0.5\delta^2)t + \delta B(t)} \quad (3.23)$$

where c and $\delta > 0$ are constants. It is proved below that (3.23) is indeed a solution. According to (3.23), equations (3.24) to (3.27) are valid.

$$f(t, B) = e^{(c-0.5\delta^2)t + \delta B(t)} \quad (3.24)$$

$$f_1(t, B) = (c - 0.5\delta^2)f(t, B) \quad (3.25)$$

$$f_2(t, B) = \delta f(t, B) \quad (3.26)$$

$$f_{22}(t, B) = \delta^2 f(t, B) \quad (3.27)$$

where f_1, f_2 and f_{22} are again the partial derivatives with respect to first and second variables.

Let $f(t, x)$ be a function whose second order partial derivatives are continuous. Then (3.28) is true.

$$f(t, B(t)) - f(s, B(s)) = \int_s^t [f_1(x, B(x)) + \frac{1}{2}f_{22}(x, B(x))]dx + \int_s^t f_2(x, B(x))dB(x)$$

, $s < t$

(3.28)

Using the equations (3.24) to (3.27), equation (3.29) is obtained.

$$f(t, B(t)) - f(s, B(s)) = c \int_s^t X(y) dy + \delta \int_s^t X(y) dB(y) \quad (3.29)$$

If one substitutes $s = 0$ in (3.29), and uses $X(t) = f(t, B(t))$, (3.30) is obtained, which is nothing but (3.21).

$$X(t) = X(0) + c \int_0^t X(y) dy + \delta \int_0^t X(y) dB(y) \quad (3.30)$$

Hence to summarize, it is shown that $X(t) = e^{(c-0.5\delta^2)t+\delta B(t)}$ is a solution for (3.21).

3.11 Uniqueness of the Solution of the Stochastic Differential Equation

In the solution of the stochastic differential equation, it was proven that (3.31) is a solution for the stochastic integral equation (3.32).

$$X(t) = e^{(c-0.5\delta^2)t+\delta B(t)} \quad (3.31)$$

$$X(t) = X(0) + c \int_0^t X(s)ds + \delta \int_0^t X(s)dB(s) \quad (3.32)$$

This section proves that (3.31) is also the unique solution. The following Lemma is given which can be applied to our case.

Lemma

Assume that the general stochastic integral equation is described by (3.33).

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dB(s) \quad (3.33)$$

Assume that the initial condition $X(0)$ has a finite second moment: $E[X(0)^2] < \infty$, and is independent of $(B(t), t \geq 0)$. Assume that, for all $t \in [0, T]$ and $x, y \in \mathbb{R}$, the coefficient functions $a(t, x)$ and $b(t, x)$ satisfy the following conditions:

1) *They are continuous.*

2) *They satisfy a Lipschitz condition (3.34) with respect to the second variable:*

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y| \quad (3.34)$$

Then the Ito stochastic differential equation (3.33) has a unique strong solution X on $[0, T]$. [18] For more information on the strongness of the solution, the reader can refer to [19].

Application of the Lemma

After defining the Ito's Lemma, it is straightforward to apply the above statement to (3.32)

Since $a(t, x) = cx$ and $b(t, x) = x\delta$, then $a(t, x)$ and $b(t, x)$ are continuous.

Furthermore:

$$\begin{aligned} |a(t, x) - a(t, y)| &= c|x - y| \\ |b(t, x) - b(t, y)| &= \delta|x - y| \\ |a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| &= (c + \delta)|x - y| \end{aligned} \tag{3.35}$$

Comparing (3.34) and (3.35), for $K = (c + \delta) + 1$, $a(t, x)$ and $b(t, x)$ satisfy the Lipschitz condition. As a result, (3.32) has a unique strong solution X on $[0, T]$. This unique strong solution is given in (3.31).

CHAPTER 4

DQF ALGORITHM

4.1 DQF

Let the system be represented by (4.1) and (4.2).

$$X(k+1) = f(X(k), k, w(k)) \quad (4.1)$$

$$Z(k) = g(X(k), k, v(k)) \quad (4.2)$$

where $X(k)$ is state, $X(k+1)$ is next state, $Z(k)$ is measurement, $v(k)$, $w(k)$ are noises. Finally, f and g are nonlinear functions in general.

The problem can be stated as:

- Given $Z(1), \dots, Z(k)$, find $X(1), \dots, X(k)$ in the filtering algorithm
- Given $Z(1), \dots, Z(k)$, find $X(k+1)$ in the prediction algorithm

This problem does not have optimal solutions. However, the EKF emerges as a suboptimal solution to the problems given above.

The idea of DQF algorithm proposed by Demirbaş [3] is to quantize the state space and quantize the continuous random variables. The quantized state space representation is analyzed in order to find an optimal solution.

Two approximation techniques are used:

- The space will be separated by ‘gates’. All the elements in the ‘gates’ will be approximated with the center of the ‘gates’.
- $X(0)$, $w(0)$, $w(1)$, ... will be approximated by discrete random variables.

Using these two assumptions, Demirbaş obtains a trellis diagram for the states. The algorithm composes of 3 main steps.

- 1) *Obtaining a trellis diagram*
- 2) *Assigning metrics to the nodes of the trellis diagram*
- 3) *Choosing the highest metric for the determination of the best path*

4.1.1 Obtaining a Trellis Diagram

The above mentioned algorithm states that the first step is to obtain a Trellis Diagram. Note that the algorithm uses discrete random variables instead of the continuous random variables. That is $X(0)$, $w(1)$, $w(2)$, ... are the discrete versions of the original continuous random variables. The trellis diagram is the set of all possible values of $\{X(1), X(2), \dots\}$.

$$X(1) = f(X(0), 0, w(0)) \quad (4.3)$$

Putting all possible values $X(0)$ and $w(0)$ to (4.3), all possible values of $X(1)$ are obtained. The next step is similar:

$$X(2) = f(X(1), 1, w(1)) \quad (4.4)$$

Similarly putting all possible values $X(1)$ and $w(1)$ to (4.4), all possible values of $X(2)$ are obtained. As a result, all possible values of $X(1)$, $X(2)$, ... , $X(N)$ are obtained via (4.4).

The trellis diagram is just the composition of these values. The diagram consists of columns, each of which indicates the time 0, 1, 2... N . Also, each column composes from the possible values of the state $X(0)$, $X(1)$, .. , $X(N)$

4.1.2 Assigning Metrics to the Nodes of the Trellis Diagram

The second step of the algorithm is the most crucial one. First, the metrics of the first column are assigned. Let the possible values of $X(0)$ be $X_{q_1}(0)$, $X_{q_2}(0)$, $X_{q_3}(0)$, .. , $X_{q_m}(0)$

Let $M(X_{q_1}(0)), M(X_{q_2}(0)), M(X_{q_3}(0)), \dots, M(X_{q_m}(0))$ denote the metrics of $X_{q_1}(0), X_{q_2}(0), \dots, X_{q_m}(0)$ respectively.

The values of the Metrics are calculated according to (4.5).

$$M(X_{q_i}(0)) = \ln(P(X(0) \in X_{q_i}(0))) \text{ where } i \in \{1, 2, \dots, m\} \quad (4.5)$$

The next step is to determine the metrics of the second column nodes.

There are mainly three steps to be followed to find these metrics.

1) *Transition metrics will be calculated from each node of the first column to each node of the next column. For example, $M(X_{q_i}(0) \Rightarrow X_{q_l}(1))$ denotes the transition metric from the i 'th node of the first column to the l 'th node of the second column. The calculation of this transition metric is according to (4.9), which is explained in the subsection 4.1.2.1.*

2) *For the determination of the metrics of the second column nodes, the following procedure must be followed: Let $X_{q_l}(1)$ denote the l 'th node of the next column.*

The sum (4.6) must be calculated for each node of the first column.

$$M(X_{q_l}(1)) = M(X_{q_i}(0)) + M(X_{q_i}(0) \Rightarrow X_{q_l}(1)) \quad (4.6)$$

3) *The real Metric $M(X_{q_l}(1))$ is the largest among the calculated metrics according to (4.6). The calculation of the metrics of the third column just requires the utilization of the same procedure.*

4.1.2.1 Calculation of the transition metric:

Let the system be described as (4.7) and (4.8).

$$X(k+1) = f(k, X(k), w(k)) \quad (4.7)$$

$$Z(k) = g(k, X(k)) + v(k) \quad (4.8)$$

where $X(0)$ is a Gaussian random vector with mean m_0 and covariance R_0 ; $Z(k)$ is the measurement data; $w(k)$ is a Gaussian disturbance noise vector with zero mean and variance σ_w^2 ; f and g are nonlinear functions in general; $v(k)$ is a Gaussian noise with zero mean and variance σ_v^2 . Moreover, the random variables $X(0)$, $w(j)$, $w(k)$, $v(l)$ and $v(m)$ are assumed to be independent for all j, k, l and m . Then the transition metric is calculated according to the following statement (4.9):

$$M\{X_{qi}(k-1) \Rightarrow X_{qk}(k)\} = \ln \Pi_k^i - \ln \{\sqrt{2\pi\delta_v^2}\} - \frac{[z(k) - g(k, X_{qi}(k))]^2}{2\delta_v^2} \quad (4.9)$$

where Π_k^i is the transition probability from the node. For the calculation of transition metric of the multidimensional system, see [20].

4.1.3 Choosing the Highest Metric for the Determination of the Best Path

Using the procedure described in section 4.1.2, the following list is obtained

1. *A trellis diagram,*
2. *Each nodes metrics has been calculated*
3. *The previous node from which we came to that node*

Then the output of the algorithm is found as follows:

1. *Look at the last column*
2. *Find the largest metric*
3. *Looking at the associated previous node, trace one step back*
4. *Go up to the first column*
5. *The output is just the composition of these nodes.*

4.2 Quantization of the random variable

DQF algorithm uses the quantized version of continuous random variables. In this study, since the noises are assumed to be Gaussian, the probability density function of the Gaussian random variable with mean μ and variance δ^2 is described by (4.10):

$$f(x) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(x-\mu)^2}{2\delta^2}} \quad (4.10)$$

Where (4.10) is approximated according to (4.11).

$$f_y(x) = \sum_{i=1}^n P_i \delta(x - y_i) \quad (4.11)$$

where y_i 's are the discrete possible values of the continuous random variable and P_i 's are the associated probability values.

Let $F_x(x)$ and $F_y(x)$ denote the probability distribution functions of the continuous and discrete time random variables. Then $F_y(x) = p_i$ for $y_i \leq x < y_{i+1}$,

where $p_i = \sum_{k=1}^i P_k$.

In order to find y_i and p_i , the following cost function is defined:

$$J(F_y(.)) = \int_{-\infty}^{\infty} [F_x(a) - F_y(a)]^2 da \quad (4.12)$$

The best discrete candidate for the probability distribution function is assumed to minimize the cost function described in (4.12).

According to Demirbaş [26], minimization of (4.12) is equivalent to solving the system of nonlinear equations from (4.13) to (4.16).

$$p_1 = 2F(y_1) \quad (4.13)$$

$$p_i + p_{i+1} = 2F(y_{i+1}), \quad i = 1, 2, 3, \dots, n-2 \quad (4.14)$$

$$1 + p_{n-1} = 2F(y_n) \quad (4.15)$$

$$p_i(y_{i+1} - y_i) = \int_{y_i}^{y_{i+1}} F(a) da \quad i = 1, 2, 3, \dots, n-1 \quad (4.16)$$

where F denotes the probability distribution function.

In this study, a recursive Matlab program is written in order to find y_i and p_i .

The algorithm of the program is as follows:

- 1) Assume an initial set of probabilities p_i
- 2) Use (4.13) to (4.15) to find y_i
- 3) Update p_i using (4.16)
- 4) Go to step 2

Integral in the equation (4.16) is calculated numerically using the trapezoidal rule. Using this algorithm, the system is solved up to $n = 40$.

CHAPTER 5

EKF

The Kalman Filter approach to the filtering and estimation problems is one of the standard tools in estimation theory. In the linear case, Kalman Filter works in the optimal sense. But in nonlinear problems, where the next state is a nonlinear function of current state and noise, Extended Kalman Filter is used. Let the system be described as:

$$X(k+1) = \Phi(x(k), k) + \Gamma(x(k), k)w(k) \quad (5.1)$$

$$Z(k) = h(x(k), k) + v(k) \quad (5.2)$$

Let

- $E X(0) = \mu_0$, $Variance \{ X(0) \} = V_x (0)$
- $Cov\{w(k), w(j)\} = V_w(k)\delta[k - j]$
- $Ew(k) = Ev(k) = 0$
- $Cov\{v(k), v(j)\} = V_v(k)\delta[k - j]$
- $Cov\{v(k), w(j)\} = Cov\{v(k), X(k)\} = Cov\{w(k), X(0)\} = 0$

In this stage $x(k+1)$ and $z(k)$ will be approximated according to (5.3) and (5.4).

$$X(k+1) \approx \Phi(X_p(k), k) + \frac{\partial \phi(X_p(k), k)}{\partial X_p(k)} [x(k) - X_p(k)] + \Gamma(X_p(k), k)w(k) \quad (5.3)$$

$$Z(k) \approx h(X_p(k), k) + \frac{\partial h(X_p(k), k)}{\partial X_p(k)} [x(k) - X_p(k)] + v(k) \quad (5.4)$$

Since (5.3) and (5.4) constitute a linear system, the standard KF can be utilized. As a result the following equations are used for filtering and prediction [22]:

$1) V_{\tilde{x}}(k+1 k) = \frac{\partial \phi(X_p(k), k)}{\partial X_p(k)} V_{\tilde{x}p}(k) \frac{\partial \phi^T(X_p(k), k)}{\partial X_p(k)} + \Gamma(X_p(k), k) V_w(k) \Gamma^T(X_p(k), k)$	(5.5))
$2) X_p(k+1 k) = \Phi[X_p(k), k]$	(5.6))
$3) V_{\tilde{x}p}(k+1) = V_{\tilde{x}p}(k+1) - V_{\tilde{x}p}(k+1) \frac{\partial h^T(X_p(k+1 k), k+1)}{\partial(X_p(k+1 k))} \left\{ \frac{\partial h(X_p(k+1 k), k+1)}{\partial(X_p(k+1 k))} * V_{\tilde{x}p}(k+1 k) * \frac{\partial h^T(X_p(k+1 k), k+1)}{\partial(X_p(k+1 k))} + V_v(k+1) \right\}^{-1} + V_x(k+1 k)$	(5.7))
$4) K(k+1) = V_{\tilde{x}}(k+1) \frac{\partial h^T(X_p(k+1 k), k+1)}{\partial(X_p(k+1 k))} [V_v(k+1)]^{-1}$	(5.8))
$5) X_p(k+1) = X_p(k+1 k) + K(k+1) \{ z(k+1) - h(X_p(k+1 k), k+1) \}$	(5.9))

In the above equations, the initial conditions are $E(X_p(0)) = \mu_0$ and $V_{\tilde{x}}(0) = V_{x(0)}$.

CHAPTER 6

APPLICATION OF EKF TO ECONOMETRICS

The nonlinear system is described with the following two equations, which are (6.1) and (6.2).

$$X(k+1) = LX(k)e^{\delta w(k)} \quad (6.1)$$

where $L = e^{(c-0.5\delta^2)\Delta T}$, which is a constant.

$$Z(k) = X(k) + v(k) \quad (6.2)$$

In the fifth section, the system representation of EKF was:

$$X(k+1) = \Phi(X(k)) + \Gamma(X(k), k)w(k) \quad (6.3)$$

$$Z(k) = h(X(k), k) + v(k) \quad (6.4)$$

Comparing (6.3) and (6.4) with (6.1) and (6.2), we get the following result:

$$h(X(k), k) = X(k) \quad (6.5)$$

Equation (6.1) and (6.3) cannot be equal for any choice of Γ and Φ . In order to use EKF, the linearization technique will be used for (6.1).

$e^{\delta w(k)}$ is represented in the Taylor expansion form (6.6).

$$e^{\delta w(k)} = 1 + \delta w(k) + \frac{[\delta w(k)]^2}{2} + \dots \quad (6.6)$$

$$LX(k)e^{\delta w(\Delta T)} = LX(k)[1 + \delta w(k) + \frac{[\delta w(k)]^2}{2} + \dots]$$

$$LX(k)e^{\delta w(k)} = LX(k) + LX(k)\delta w(k) + \dots$$

Equation (6.7) is obtained by taking a first order approximation for Taylor series

$$LX(k)e^{\delta w(\Delta T)} \approx LX(k) + L\delta w(\Delta T) X(k) \quad (6.7)$$

Instead (6.1), equation (6.7) is used to be able to apply EKF. Comparing (6.7) with (6.3), equations (6.8) and (6.9) are obtained.

$$\Phi(X(k), k) = LX(k) \quad (6.8)$$

$$\Gamma(X(k), k) = L\delta X(k) \quad (6.9)$$

Replacing the values of h , ϕ , Γ in (5.5) to (5.9), the following set of equations is obtained.

1) $V_{\hat{x}}(k+1 k) = L V_{\hat{x}_p}(k) L + [L\delta X_p(k)] V_w(k) [L\delta X_p(k)]$	(6.10)
2) $X_p(k+1 k) = LX_p(k)$	(6.11)
3) $V_{\hat{x}_p}(k+1) = V_{\hat{x}_p}(k+1) - V_{\hat{x}_p}(k+1) * \{ V_{\hat{x}_p}(k+1 k) + V_v(k+1) \}^{-1} * V_x(k+1 k)$	(6.12)
4) $K(k+1) = V_x(k+1) [V_v(k+1)]^{-1}$	(6.13)
5) $X_p(k+1) = X_p(k+1 k) + K(k+1) \{ z(k+1) - (X_p(k+1 k)) \}$	(6.14)

Finally, with the help of the equations (6.10) to (6.14), it is possible to smooth the noisy data and make a prediction.

CHAPTER 7

PARTICLE FILTERS-SIR ALGORITHM

SIR algorithm is derived from a group of algorithms, which are known as particle filters. The algorithm is a Monte Carlo method that can be applied to recursive Bayesian filtering problems. It is first proposed by Gordon et.al. [23].

SIR algorithm can be applied to systems represented by (7.1), (7.2)

$$x(k+1) = f_k(x(k), w(k)) \quad (7.1)$$

$$z(k) = h_k(x(k), v(k)) \quad (7.2)$$

The conditions for the state and measurement equations are weak. The state and measurement noises are white noise sequences, where f_k and h_k are nonlinear functions in general.

The task is to update (7.3) recursively.

$$p(x(k) | z(1), \dots, z(k)) \quad (7.3)$$

The main idea is to approximate the conditional probability density function according to (7.4).

$$p(x_k | z_{1:k}) \approx \sum_{i=1}^{N_s} \omega_k^i \delta(x - x_k^i) \quad (7.4)$$

Here, x_k^i represents the randomly generated particles and w_k^i 's are the associated probability values. $\delta(x)$ is the usual impulse function. $z_{1:k}$ denotes the set $\{z(1), z(2), \dots, z(k)\}$. Also $x_{1:k}$ denotes the set $\{x(1), x(2), \dots, x(k)\}$.

The SIR filter is composed of three steps:

- 1) Prediction
- 2) Update
- 3) Resampling

Prediction

At the prediction step, given, $p(x_{k-1}|z_{1:k-1})$ the task is to obtain $p(x_k|z_{1:k-1})$

Each particle will be updated using the following equation (7.5).

$$x_k(i) = f_{k-1}(x_{k-1}(i), w_{k-1}(i)) \quad (7.5)$$

Each $w_{k-1}(i)$ is drawn from the probability density function (pdf) of $p(w_{k-1})$.

Update

The weights of $x_k(i)$ in (7.4) are calculated according to (7.6).

$$w_k(i) = \frac{p(z_k|x_k(i))}{\sum_{j=1}^{N_s} p(z_k|x_k(j))} \quad (7.6)$$

Resampling

When the update is completed, $p(x_k|z_{1:k})$ from $p(x_{k-1}|z_{1:k-1})$ is achieved. The particles x_k^i and the associated probability values w_k^i represent the conditional value in the sense of (7.4). If the resampling step was to be skipped, then w_k^i would become more and more skewed. After a while, only one particle with nonzero probability value will exist [25]. To avoid this degeneracy, the resampling step is proposed. As a result, the new pdf is represented by (7.7).

$$p(x_k|z_{1:k}) = \frac{1}{N} \sum_{i=1}^N N_t^i \delta(x - x_k^i) \quad (7.7)$$

where

$$\sum_{i=1}^N N_t^i = N \quad (7.8)$$

For more information about the resampling stage, the reader is referred to [24].

7.1 Derivation of the SIS and SIR algorithm

The sequential importance sampling (SIS) algorithm is a Monte Carlo (MC) method that forms the basis for most sequential MC filters developed over the past decades. SIR is also one of these filters.

Suppose $p(x) \propto \pi(x)$, is a probability density function from which it is difficult to draw samples, but for which $\pi(x)$ can be evaluated [as well as up to proportionality]. In addition, let $x^i \sim q(x), i = 1, \dots, N_s$ be samples that are easily generated from a proposal $q(x)$ called *importance density*. Then, a weighted approximation to the density $p(x)$ is given by

$$p(x) \approx \sum_{i=1}^{N_s} w^i \delta(x - x^i) \quad (7.9)$$

where

$$w^i \propto \frac{\pi(x^i)}{q(x^i)} \quad (7.10)$$

(7.10) is the normalized weight of the i 'th particle [24].

If the samples are drawn from $q(x_{0:k} | z_{1:k})$, then the weights will be (7.11).

$$w_k^i \propto \frac{p(x_{0:k}^i | z_{1:k})}{q(x_{0:k}^i | z_{1:k})} \quad (7.11)$$

Let the importance density be chosen as:

$$q(x_{0:k} | z_{1:k}) = q(x_k | x_{0:k-1}, z_{1:k}) q(x_{0:k-1} | z_{1:k-1}) \quad (7.12)$$

Using the Bayes rule:

$$p(x_{0:k} | z_{1:k}) = \frac{p(z_k | x_{0:k}, z_{1:k-1}) p(x_{0:k} | z_{1:k-1})}{p(z_k | z_{1:k-1})} \quad (7.13)$$

$$p(x_{0:k} | z_{1:k}) = \frac{p(z_k | x_{0:k}, z_{1:k-1}) p(x_k | x_{0:k-1}, z_{1:k-1}) p(x_{0:k-1} | z_{1:k-1})}{p(z_k | z_{1:k-1})} \quad (7.14)$$

Since the state is actually a Markov process (due to whiteness of the process noise)

$$p(x_k | x_{0:k-1}, z_{1:k-1}) = p(x_k | x_{k-1}) \quad (7.15)$$

and since the measurements are a static function of the last state and the measurement noise is white

$$p(z_k | x_{0:k}, z_{1:k-1}) = p(z_k | x_k) \quad (7.16)$$

$$p(x_{0:k} | z_{1:k}) = \frac{p(z_k | x_k) p(x_k | x_{k-1}) p(x_{0:k-1} | z_{1:k-1})}{p(z_k | z_{1:k-1})} \quad (7.17)$$

$$p(x_{0:k} | z_{1:k}) \propto p(z_k | x_k) p(x_k | x_{k-1}) p(x_{0:k-1} | z_{1:k-1}) \quad (7.18)$$

Putting (7.12) and (7.18) into (7.11), the resulting equations (7.19) and (7.20) are obtained.

$$w_k^i \propto \frac{p(z_k | x_k^i) p(x_k^i | x_{k-1}^i) p(x_{0:k-1}^i | z_{1:k-1})}{q(x_k^i | x_{0:k-1}^i, z_{1:k}) q(x_{0:k-1}^i | z_{1:k-1})} \quad (7.19)$$

$$w_k^i \propto w_{k-1}^i \frac{p(z_k | x_k^i) p(x_k^i | x_{k-1}^i)}{q(x_k^i | x_{0:k-1}^i, z_{1:k})} \quad (7.20)$$

Hence, the general framework for SIS filters is concluded.

The SIR filter is easily derived by choosing

$$q(x_k | x_{0:k-1}^i, z_{1:k}) = p(x_k | x_{k-1}^i) \quad (7.21)$$

Then,

$$w_k^i \propto w_{k-1}^i p(z_k | x_k^i) \quad (7.22)$$

Hence the weights are updated according to the (7.22) and x^i 's are updated according to the (7.21).

CHAPTER 8

SIMULATIONS OF FILTERING AND PREDICTION EXPERIMENTS

8.1 General Outline

In this section, the econometric system is analyzed from several aspects. The system was described by (8.1) and (8.2).

$$X(k+1) = LX(k)e^{\delta w(k)} \quad (8.1)$$

where $L = e^{(c-0.5\delta^2)\Delta T}$, which is a constant. (8.1) is the standard state equation. In real world applications, it corresponds to the real price update in terms of current real price and noise.

The observation equation is defined by (8.2).

$$Z(k) = X(k) + v(k) \quad (8.2)$$

where $v(k)$ is the measurement noise or speculators noise or, alternatively, the traders noise. (8.1) and (8.2) constitute together the dynamic system representation.

In this section, it is assumed that the measurement data is given and the task is to filter the noise and expect the current and the next real price

8.2 Filtering Tools

The previous literature on filtering generally views the *Extended Kalman Filter* as a, standard analysis tool for filtering of noise for nonlinear dynamic systems. In this study, an alternative method DQF, as mentioned before in details, is used. Although the theory of DQF is discussed in section 4.1, the applications of DQF to several cases need to be worked out. Therefore, this study is an attempt to apply DQF to a new area, Mathematical Finance, which became increasingly popular in the recent years.

8.3 Prediction Tools

Needless to mention, prediction is crucially important in economic systems. In this context, predicting the most probable next value of the stock emerges as an interesting question to answer. In order to predict the next value of the stock, which is described by the system representation (8.1) and (8.2), SIR filter and EKF are used.

8.4 Error Criteria and Monte Carlo Simulations

When there is great deal of difficulty in analyzing the systems in several branches of science, Monte Carlo simulations are employed. Therefore, in this study, to compare the performances of EKF and DQF, this method is preferred. The method can simply be summarized as making many experiments and using the results of these experiments to predict the performance of the system.

In each simulation experiment, the followed procedure is similar:

- 1) *One parameter is varied (or chosen a specific value).*
- 2) *For each value of the parameter, 20 experiments are done.*
- 3) *Errors for each experiment are calculated for*
 - *EKF output*

- *DQF output*
- 4) *Average Error EKF and DQF are calculated.*

8.4.1 Filtering Error

8.4.1.1 Error in R^N Spaces:

Let $x \in R^N$ be defined by (8.3).

$$x = [x(1), x(2), \dots, x(N)] \quad (8.3)$$

L_2 norm of x is calculated according to (8.4)

$$\|x\|_2 = \sqrt{\sum_{i=1}^N [x(i)]^2} \quad (8.4)$$

Let $x, y \in R^N$, then the distance between the vectors x and y is given by (8.6)

$$\|x - y\|_2 \quad (8.6)$$

The distance concept defined in (8.6) is used for the definition of error. Let x be the “true” value then the $error_y$ is defined by (8.7)

$$error_y = \|x - y\|_2 \quad (8.7)$$

8.4.1.2 Errors in the Experiments

In order to calculate the errors, the notations (8.8) to (8.11) are used to represent real data, EKF output, and DQF output respectively.

$$X = [X(1), X(2), \dots, X(N)] \quad (8.8)$$

$$EKF = [EKF(1), EKF(2), \dots, EKF(N)] \quad (8.10)$$

$$DQF = [DQF(1), DQF(2), \dots, DQF(N)] \quad (8.11)$$

The corresponding error values of EKF and DQF are calculated according to (8.12) and (8.13)

$$error_{EKF} = \|EKF - X\|_2 = \sqrt{\sum_{i=1}^N [EKF(i) - X(i)]^2} \quad (8.12)$$

$$error_{DQF} = \|DQF - X\|_2 = \sqrt{\sum_{i=1}^N [DQF(i) - X(i)]^2} \quad (8.13)$$

Having obtained the errors for each experiment, the average of the errors of both EKF and DQF are calculated according to the following formula in (8.14).

$$average_error = \frac{1}{K} \sum_{i=1}^K error(i) \quad (8.14)$$

where K is the number of experiments done in the Monte Carlo simulation.

8.4.2 Prediction Error

For a single experiment, the error calculations of prediction algorithms of interest, which are EKF and SIR particle filter, are (8.15) and (8.16).

$$error_{EKF} = |EKF(N+1) - X(N+1)| \quad (8.15)$$

$$error_{SIR} = |SIR(N+1) - X(N+1)| \quad (8.16)$$

Having obtained the errors for each experiment, then the average of the errors of both EKF and SIR particle filter are calculated according to the formula (8.17).

$$average_error = \frac{1}{K} \sum_{i=1}^K error(i) \quad (8.17)$$

where K is the number of experiments done in one Monte Carlo simulation.

8.5 Simulations

After the simulation results are obtained, the following regularities have been observed:

- 1) The principle of EKF algorithm is the linearization of the system equations. When the linearization condition is violated, the error performance of EKF becomes poor.
- 2) In order to obtain good error performance for DQF algorithm, the quantization levels must be adequately chosen. When this condition is satisfied, DQF performs better compared to EKF.
- 3) The computation time performance of the DQF algorithm is bad when it is compared to EKF.

Quantization Level

As implied above and the results show, the DQF algorithm has exponential complexity, which stands out as its biggest disadvantage. It is hard to implement it in real time applications, where the computation time performance is the crucial point. In econometrics, there are several cases where the computation time is not the primary concern. Therefore, if the space must be adequately quantized, then the DQF algorithm may be preferred. However, the state equation (8.1) contains an exponential term (8.18)

$$e^{\delta\omega(k)} \quad (8.18)$$

With the term (8.18), it is more difficult to adequately describe the system in the quantized domain. It requires a huge number of quantization level.

Linearization Condition

(8.18) is expressed in Taylor expansion form as (8.19).

$$e^{\delta w(k)} = 1 + \delta w(k) + \left[\frac{\delta w(k)^2}{2} \right] + \dots \quad (8.19)$$

Then, a first order approximation (8.20) has been made.

$$e^{\delta w(k)} \approx 1 + \delta w(k) \quad (8.20)$$

In order (8.20) to be an adequate approximation of (8.19), (8.21) must be satisfied.

$$[\delta w(k)]^2 \ll \delta w(k) \quad (8.21)$$

(8.21) is also equivalent to (8.22).

$$\delta w(k) \ll 1 \quad (8.22)$$

8.5.1 Filtering

8.5.1.1

When the simulations are run and the results are obtained, Table 1 shows the error performances versus the change of variance of the state equation noise. For each variance value, 20 experiments are done. Initial conditions are: quantization level of $X(0)$ and w are 3, Variance of v is 4, δ is 2, mean of $X(0)$ is 1, Variance of $X(0)$ is 0.1, and time index N is 1. The variance of w is changed from 1 to 6. c is 0.1.

Table 1 Monte Carlo results-Errors of the algorithms versus change of the variance of w

EKF	0.5403	1.9134	4.4610	7.6728	11.2825	15.4363
DQF	0.7294	1.6048	3.9705	11.3385	28.5365	64.8481

It is clear from Table 1 that while the variance of w is increased, the errors of EKF and DQF algorithms are increased. This is expected, since according to (8.22), the approximation of the exponential function by first order terms is not right. Hence the linearization of the system with the first order is not adequate. The linearization of the system is the basic assumption of EKF, which is simply false. DQF is assumed to be an optimal solution when the gate size approaches to zero and quantization level approaches to infinity. From the errors, it is also concluded that the quantization level is not high enough.

8.5.1.2

The results that are displayed in the preceding section is not satisfactory in terms of the the errors of the DQF. It was assumed that the quantization levels were not high enough. For that purpose, in this experiment, the number of quantization level of w is taken as 40 and the quantization level of $X(0)$ is taken as 10. The errors of DQF algorithm are decreased drastically, but the crucial point is that, it performed better than the EKF. For example, when the variance of w is 5, it performed

approximately 8 times better than EKF. Table 2 shows the experimental results of the second Monte Carlo simulation. The second observation from Table 2 is that, with the increase of variance of w , the error of DQF is also increased. This is also expected since the range of possible values of w is increased, but for each variance value of w , the same quantization level is used.

Table 2 Monte Carlo results-Errors of the algorithms versus change of the variance of w

EKF	0.5403	1.9134	4.4610	7.6728	11.2825	15.4363
DQF	0.5601	0.9251	0.7533	0.7340	1.3653	2.2599

8.5.1.3

In this experiment, the error performances of the algorithms are shown when the volatility constant is decreased to 0.1. All initial conditions are same with the previous experiment except the constant volatility. Again the variance of w is varied between 1 and 6. Table 3 shows the Monte Carlo Simulation results.

Table 3 Monte Carlo results-Errors of the algorithms versus change of the variance of w

DQF	0.4027	0.6686	0.8989	1.0730	1.1917	1.2651
EKF	0.3846	0.6743	0.9137	1.0931	1.2556	1.2727

As it is seen from Table 3, EKF is stable in this experiment. This is also expected, since the term $\delta w(k)$ is decreased 20 times for the same values of w compared to the experiment (8.5.1.1).

DQF is also stable. This is also expected since the increase due to exponential term is now limited. The range of possible values of X is decreased. This range is also adequately quantized with the chosen quantization levels.

8.5.1.4

In the experiment 8.5.1.2, it was shown that with the increase of the quantization level, the error performance of DQF increased substantially. In this experiment, the increase of performance is studied in details. Initial conditions are same with the second experiment, but the quantization level of w is varied from 1 to 40. The quantization level of $X(0)$ is assumed to be 10 in all simulations. The noise variances are assumed to be 5 and 4, for w and v respectively. Figure 1, which displays the plot of quantization level versus the error terms shows the improvement of errors when the quantization level is increased. It approaches to a limiting value as it can be easily seen from the figure.

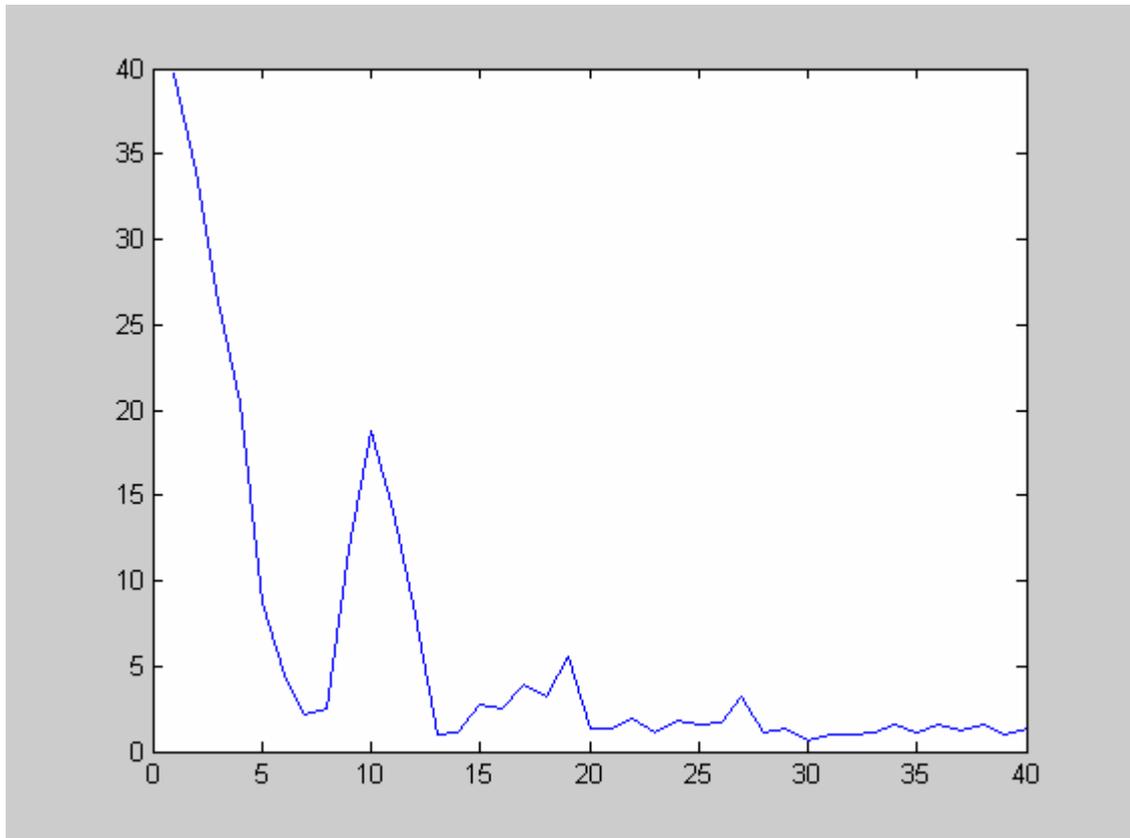


Figure 1

8.5.1.5

As mentioned above, one important problem with the DQF and DQP was the complexity of the algorithms. Compared to EKF, computation time performance of the algorithm is also worse for large quantization levels. In this experiment the time index will be incremented one level. To take reasonable error performance and computation time, quantization level is chosen as 20 for w and 10 for $X(0)$. All other initial conditions are same as the second experiment but N is 2. Consistent with the previous experiment, 20 experiments are performed for Monte Carlo simulation.

Table 4 Monte Carlo results-Errors of the algorithms for the variance of $w=5$

EKF	55.8102
DQF	11.7439

The results are remarkable, which indicate that the error performance of the algorithm is approximately 5 times better than the conventional EKF. But it should be noted that computation time of DQF increases.

8.5.2 Prediction

The prediction part differs from the smoothing part. The system described by (8.23) and (8.24) is transformed to another equivalent form. The reason for that is the number of experiments in the Monte Carlo simulation required to compare DQP and EKF is too high. However, after transforming to the new equivalent form, better results have been obtained. Furthermore, the time index could be incremented to even higher numbers, which was not possible before for the case of DQF algorithm.

$$X(k + 1) = LX(k)e^{\delta w(k)} \quad (8.23)$$

$$Z(k) = X(k) + v(k) \quad (8.24)$$

When taking logarithm of both sides, (8.23) is rewritten as (8.25)

$$\ln(X(k + 1)) = \ln(L) + \ln(x(k)) + \delta w(k) \quad (8.25)$$

Furthermore if

$$Y(k) = \ln(X(k)) \quad (8.26)$$

$$Y(k + 1) = \ln(L) + Y(k) + \delta w(k) \quad (8.27)$$

As a result the new set of equations are (8.27) and (8.28).

$$Z(k) = e^{Y(k)} + v(k) \quad (8.28)$$

The equation (7.27) is linear, but the equation (7.28) is nonlinear. Furthermore the probability density function of $Y(0)$ is not Gaussian. In fact, it is obtained from the Gaussian noise by the transformation (8.26). Hence the standard assumption of EKF for the initial noise does not hold. Even in this case, the new set of nonlinear equation is implemented by SIR and EKF algorithm.

Mean and Variance of $Y(0)$

$Y(k)$ can take values 0 and negative, so $Y(0)$ can have complex values according to (8.26). For that reason, all the values for $Y(k) \leq 0$ is approximated by $Y(k) = 0.001$. This makes sense, since the mean of $X(0)$ is 1 and its variance is taken as 0.1. Under these assumptions, mean and variance of $Y(0)$ is obtained experimentally. For that purpose (8.29) and (8.30) are used.

$$\mu_{Y(0)} = \frac{\sum_{i=1}^N Y_i}{N} \quad (8.29)$$

$$s_{Y(0)}^2 = \frac{\sum_{i=1}^N (Y_i - \mu_{Y(0)})^2}{N - 1} \quad (8.30)$$

where Y_i is the randomly generated number. These estimates are known to converge to the true mean and variance values as $N \rightarrow \infty$.

Experiments

In this experiment, δ is 2, the mean of $X(0)$ is 1, the variance of $X(0)$ is 0.1, and the time index N is 5. The variance of w and v are set to unity. For this simulation, 1000 experiments are done.

Table 5 Monte Carlo results-Errors of the algorithms SIR and EKF

EKF	3.7928
SIR	2.9726

As it can be seen from Table 5, SIR error performance is better than the one for EKF.

CHAPTER 9

SIMULATIONS OF FILTERING EXPERIMENTS FOR STOCHASTIC GROWTH MODELS

Beginning with the standard stochastic growth models, for which the typical solutions are offered in Hansen [27],[28], modeling both productivity shocks and capital stock accumulation became critical issues. Although, the relationship between the two variables can be explained in a standard way by using the “Solow residuals”, there are also some recent studies, which explain the dynamics within the context of more compact models. However, there are only a few of these studies that attempt to explore the issue in a non-linear framework. However, as it is evidently discussed in Novales et al [29], the productivity shocks may carry a non-linear nature, for which the standard Kalman filtering algorithms fail to be appropriate. In this context, the non-linear state space model can be solved either by extended Kalman filter or Particle filter. Following Novales et al [29], the state space model can be written as:

$$\log(\theta(t)) = \rho \log(\theta(t-1)) + \varepsilon(t) \quad (9.1)$$

$$k(t) = c + \alpha \theta(t) + N(t) \quad (9.2)$$

The first equation (9.1) models the productivity shocks as being first order autoregressive process, where the disturbance term is assumed to be independent and identically distributed. If the parameter rho gets closer to unity, then the productivity shocks follow random walk processes, where any shock to the equation will have permanent effects.

The second equation (9.2) relates the importance of these productivity shocks on the change of the capital stock. Based on the neo-classical growth models, an increase in the productivity will increase the return to the capital, which will create

an extra incentive for the firms to accumulate further capital. Using the parameters that are obtained from the calibration of the micro-based fundamentals, the following values for the parameters are used.

$$c = 0.73 \quad (9.3)$$

$$\alpha = 1.75 \quad (9.4)$$

$$\rho = 1 \quad (9.5)$$

In terms of estimation, the first equation, which is the state equation in the model, has a non-linear nature, where the performance of extended Kalman filter and Particle filter can be tested.

By use of the transformation (9.6), the equations (9.7) and (9.8) are obtained.

$$X(t) = \ln(\theta(t)) \quad (9.6)$$

$$X(t) = X(t-1) + \varepsilon(t) \quad (9.7)$$

$$k(t) = c + \alpha e^{X(t)} + N(t) \quad (9.8)$$

The mean and variance are calculated experimentally.

Mean and Variance of $X(0)$

As before, $\theta(0)$ can take values 0 and negative, so $X(0)$ can have complex values according to (9.6). For that reason, all the values for $\theta(0) \leq 0$ is approximated by $\theta(0) = 0.001$. This makes sense, since the mean of $\theta(0)$ is 1 and its variance is taken as 0.1. Under these assumptions, the mean and the variance of $X(0)$ is obtained experimentally.

$$\mu_{X(0)} = \frac{\sum_{i=1}^N X_i}{N} \quad (9.9)$$

$$s_{X(0)}^2 = \frac{\sum_{i=1}^N (X_i - \mu_{X(0)})^2}{N-1} \quad (9.10)$$

where X_i is the randomly generated number. These estimates are known to converge to the true mean and variance values as $N \rightarrow \infty$.

Experiment

In this experiment, filtering performance of the algorithms of SIR and EKF are compared. Mean of $X(0)$ is 1, variance of $X(0)$ is 0.1, and the time index N is 5. The variance of w is 1. The variance of v is 1. For this simulation, again, 1000 experiments are done.

Table 6 Monte Carlo results-Errors of the algorithms SIR and EKF

EKF	2.0875
SIR	1.2836

As seen from Table 6, error performance of the SIR algorithm is better than the EKF algorithm.

Therefore, the last two experiments clearly show that when the SIR algorithm is compared with the conventionally used Extended Kalman Filter, it has better performances. Such a conclusion indicates that SIR can be a competing alternative to EKF in computation process of the econometrics problem.

CHAPTER 10

CONCLUSION

This study extends the previous work on the nonlinear estimation problems. For that purpose, Extended Kalman Filter (EKF), Discrete Quantization Filter and Sequential Importance Resampling (SIR) Filter are employed. . Since the already dense literature on nonlinear estimation has not evaluated the last two filters, this study can be viewed as a contribution in offering two alternative algorithms. Another primary concern in this thesis is to show the advantages of these two algorithms over EKF, which is the conventionally used algorithm in the field.

The main idea of DQF is to quantize the random variables. If sufficient quantization of the random variables are made, then DQF performs better than the EKF. However, its major disadvantage is the computation time. Hence there is a trade-off for performance versus computation time.

SIR filter is from a group of filters that are known as particle filters. It is the implementation of Monte Carlo techniques to estimation problems. Error performances of the SIR filter were better compared to EKF. The computation time performance was also reasonable compared to DQF. Therefore, the filter can be seen as a good alternative to theEKF.

The case studies were chosen from the field of econometrics. Hence, the control theory techniques have been applied to a different science branch.

Other than evaluating the performance of the above mentioned algorithms, another primary concern in this study is to promote the use of stochastic calculus, which enables us to have a more general perspective for nonlinear dynamical systems with noise. The classical approach in the system theory is to make assumptions for noise directly in the state space form. However, with the use of stochastic calculus, we can make assumptions also in the differential equations form.

Although the theory behind this subject requires understanding advanced mathematical concepts, its usage is fairly simple.

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APPENDIX

Comma Separated Form for the quantization of the Gaussian random variable upto 40 points. Quantized values for the random variable with mean 0 and variance 1.

1:0
2:-0.675,0.675
3:-1.0050,0,1.0050
4:.2190,-0.3550,0.3550,1.2190
5:.3760,-0.5920,0,0.5920,1.3760
6:-1.4990,-0.7670,-0.2420,0.2420,0.7670,1.4990
7:.5990,-0.9050,-0.4230,0,0.4230,0.9050,1.5990
8:-1.6830,-1.0180,-0.5670,-0.1830,0.1830,0.5670,1.0180,1.6830
9:-1.7581,-1.1173,-0.6884,-0.3310,-0.0001,0.3309,0.6882,1.1172,1.7580
10:-1.8221,-1.2005,-0.7900,-0.4534,-0.1482,0.1480,0.4532,0.7898,1.2003,1.8219
11:-1.8790,-1.2736,-0.8780,-0.5578,-0.2719,-0.0002,0.2716,0.5575,0.8778,1.2733,
1.8788
12:-1.9301,-1.3385,-0.9555,-0.6485,-0.3777,-0.1243,0.1239,0.3773,0.6481,0.9552,
1.3382,1.9299
13:-1.9765,-1.3969,-1.0245,-0.7285,-0.4699,-0.2308,-0.0002,0.2303,0.4694,0.7280,
1.0241,1.3966,1.9762
14:-2.0189,-1.4499,-1.0867,-0.8000,-0.5515,-0.3239,-0.1071,0.1065,0.3233,0.5509,
0.7994,1.0862,1.4495,2.0186
15:-2.0580,-1.4984,-1.1432,-0.8644,-0.6245,-0.4065,-0.2007,-0.0004,0.1999,0.4058,
0.6238,0.8638,1.1426,1.4978,2.0575
16:-2.0941,-1.5430,-1.1949,-0.9231,-0.6905,-0.4805,-0.2838,-0.0941,0.0933,
0.2829,0.4797,0.6897,0.9223,1.1942,1.5423,2.0935
17:-2.1277,-1.5842,-1.2425,-0.9768,-0.7506,-0.5475,-0.3585,-0.1776,-
0.0005,0.1766,
0.3574,0.5465,0.7497,0.9759,1.2416,1.5834,2.1271
18:-2.1591,-1.6226,-1.2865,-1.0264,-0.8058,-0.6086,-0.4261,-0.2527,-0.0841,
0.0829,0.2515,0.4249,0.6075,0.8047,1.0253,1.2856,1.6217,2.1584
19:-2.1886,-1.6584,-1.3275,-1.0723,-0.8567,-0.6648,-0.4880,-0.3208,-0.1594,-
0.0007,
0.1580,0.3194,0.4866,0.6634,0.8554,1.0711,1.3264,1.6573,2.1877
20:-2.2164,-1.6920,-1.3659,-1.1151,-0.9039,-0.7167,-0.5448,-0.3831,-0.2278,-
0.0761,
0.0745,0.2262,0.3815,0.5432,0.7151,0.9024,1.1136,1.3645,1.6908,2.2153
21:-2.2426,-1.7236,-1.4018,-1.1551,-0.9479,-0.7648,-0.5974,-0.4405,-0.2905,-
0.1447
, -0.0010,0.1428,0.2886,0.4386,0.5956,0.7630,0.9462,1.1534,1.4003,1.7222,2.2413
22:-2.2674,-1.7534,-1.4356,-1.1926,-0.9891,-0.8098,-0.6463,-0.4936,-0.3482,-
0.2076
, -0.0696,0.0675,0.2054,0.3461,0.4915,0.6442,0.8077,0.9871,1.1907,1.4339,1.7518,
2.2660

23:-2.2909,-1.7817,-1.4675,-1.2279,-1.0277,-0.8518,-0.6919,-0.5430,-0.4017,-
 0.2656
 ,-0.1326,-0.0012,0.1302,0.2631,0.3993,0.5406,0.6896,0.8496,1.0256,1.2258,1.4656
 ,1.7799,2.2894
 24:-2.3133,-1.8085,-1.4978,-1.2613,-1.0642,-0.8914,-0.7347,-0.5892,-0.4515,-
 0.3194,-0.1908,-
 0.0643,0.0615,0.1880,0.3166,0.4488,0.5865,0.7320,0.8888,1.0617,1.2589
 ,1.4956,1.8065,2.3116
 25:-2.3347,-1.8339,-1.5265,-1.2929,-1.0986,-0.9286,-0.7749,-0.6324,-0.4981,-
 0.3695
 ,-0.2448,-0.1226,-0.0016,0.1194,0.2417,0.3664,0.4950,0.6295,0.7720,0.9258,1.0959
 ,1.2903,1.5240,1.8318,2.3328
 26:-2.3551,-1.8582,-1.5537,-1.3228,-1.1312,-0.9638,-0.8128,-0.6731,-0.5418,-
 0.4163
 ,-0.2951,-0.1767,-0.0599,0.0564,0.1732,0.2917,0.4129,0.5384,0.6699,0.8096
 ,0.9607,1.1282,1.3200,1.5511,1.8558,2.3530
 27:-2.3747,-1.8814,-1.5797,-1.3514,-1.1622,-0.9972,-0.8486,-0.7116,-0.5829,-
 0.4603
 ,-0.3422,-0.2272,-0.1141,-
 0.0019,0.1102,0.2233,0.3384,0.4565,0.5792,0.7079,0.8451
 ,0.9938,1.1589,1.3482,1.5768,1.8788,2.3724
 28:-2.3934,-1.9036,-1.6046,-1.3786,-1.1917,-1.0290,-0.8827,-0.7480,-0.6217,-
 0.5018
 ,-0.3865,-0.2744,-0.1647,-
 0.0562,0.0519,0.1604,0.2702,0.3822,0.4976,0.6176,0.7439
 ,0.8788,1.0252,1.1880,1.3752,1.6013,1.9007,2.3909
 29:-2.4114,-1.9249,-1.6284,-1.4046,-1.2198,-1.0592,-0.9150,-0.7824,-0.6585,-
 0.5409
 ,-0.4281,-0.3189,-0.2121,-0.1068,-
 0.0023,0.1022,0.2074,0.3143,0.4236,0.5364,0.6541
 ,0.7781,0.9107,1.0551,1.2158,1.4008,1.6248,1.9217,2.4087
 30:-2.4288,-1.9453,-1.6511,-1.4295,-1.2466,-1.0880,-0.9458,-0.8153,-0.6934,-
 0.5780
 ,-0.4676,-0.3608,-0.2566,-0.1543,-0.0530,0.0479,0.1492,0.2516,0.3557,0.4626
 ,0.5731,0.6886,0.8105,0.9412,1.0836,1.2424,1.4254,1.6473,1.9419,2.4258,
 31:-2.4454,-1.9650,-1.6730,-1.4533,-1.2723,-1.1155,-0.9751,-0.8465,-0.7266,-
 0.6132
 ,-0.5049,-0.4004,-0.2987,-0.1990,-0.1006,-0.0028,0.0950,0.1935,0.2932,0.3950
 ,0.4996,0.6080,0.7214,0.8414,0.9702,1.1107,1.2677,1.4489,1.6689,1.9612,2.4422
 32:-2.4615,-1.9838,-1.6940,-1.4762,-1.2970,-1.1419,-1.0032,-0.8763,-0.7582,-
 0.6467
 ,-0.5404,-0.4379,-0.3385,-0.2411,-0.1453,-0.0503,0.0444,0.1394,0.2352,0.3326,
 0.4321,0.5346,0.6411,0.7526,0.8709,0.9979,1.1367,1.2920,1.4714,1.6895,1.9798,2.4
 580
 33:-2.4770,-2.0020,-1.7142,-1.4981,-1.3206,-1.1672,-1.0301,-0.9049,-0.7884,-
 0.6787
 ,-0.5742,-0.4736,-0.3762,-0.2811,-0.1876,-0.0951,-
 0.0032,0.0887,0.1811,0.2746,0.3698

,0.4673,0.5679,0.6726,0.7824,0.8990,1.0244,1.1616,1.3153,1.4931,1.7094,1.9977,2.4732
34:-2.4920,-2.0196,-1.7337,-1.5193,-1.3434,-1.1914,-1.0559,-0.9322,-0.8173,-0.7092
,-0.6064,-0.5076,-0.4120,-0.3189,-0.2276,-0.1375,-0.0481,0.0411,0.1305,0.2206
,0.3120,0.4052,0.5009,0.5997,0.7027,0.8109,0.9259,1.0498,1.1855,1.3377,1.5139,
1.7286,2.0149,2.4880
35:-2.5065,-2.0365,-1.7524,-1.5397,-1.3652,-1.2148,-1.0807,-0.9584,-0.8450
,-0.7384,-0.6371,-0.5400,-0.4462,-0.3549,-0.2655,-0.1775,-0.0904,-0.0037,
0.0830,0.1702,0.2582,0.3476,0.4390,0.5329,0.6301,0.7315,0.8382,0.9517,1.0742
,1.2085,1.3592,1.5339,1.7470,2.0316,2.5022
36:-2.5205,-2.0529,-1.7706,-1.5593,-1.3863,-1.2372,-1.1045,-0.9836,-0.8716,-0.7664
,-0.6666,-0.5710,-0.4788,-0.3892,-0.3016,-0.2156,-0.1305,-0.0461,0.0382,0.1226,0.2077
,0.2938,0.3814,0.4711,0.5634,0.6591,0.7590,0.8643,0.9765,1.0976,1.2305,1.3799,1.5532
,1.7648,2.0476,2.5160,
37:-2.5341,-2.0687,-1.7881,-1.5783,-1.4066,-1.2588,-1.1274,-1.0078,-0.8971,-0.7932
,-0.6948,-0.6007,-0.5100,-0.4219,-0.3360,-0.2517,-0.1686,-0.0862,-0.0042,0.0778,0.1602,
0.2434,0.3277,0.4137,0.5018,0.5926,0.6869,0.7854,0.8894,1.0003,1.1201,1.2518,1.3998
,1.5718,1.7820,2.0631,2.5292
38:-2.5472,-2.0841,-1.8050,-1.5966,-1.4262,-1.2797,-1.1494,-1.0311,-0.9216,-0.8190,
-0.7219,-0.6291,-0.5398,-0.4533,-0.3689,-0.2863,-0.2049,-0.1244,-0.0444,0.0354,0.1154
,0.1960,0.2774,0.3601,0.4445,0.5311,0.6205,0.7134,0.8107,0.9134,1.0231,1.1416,1.2721
1.4189,1.5897,1.7985,2.0781,2.5420
39:-2.5600,-2.0989,-1.8214,-1.6143,-1.4452,-1.2998,-1.1707,-1.0535,-0.9452,-0.8438,
-0.7479,-0.6564,-0.5684,-0.4832,-0.4003,-0.3192,-0.2395,-0.1607,-0.0826
,-0.0047,0.0731,0.1512,0.2300,0.3098,0.3910,0.4740,0.5593,0.6474,0.7390,0.8351,
0.9366,1.0451,1.1625,1.2919,1.4375,1.6070,1.8145,2.0926,2.5545
40: -2.5724,-2.1133,-1.8372,-1.6314,-1.4634,-1.3192,-1.1912,-1.0751,
-0.9679,-0.8677,-0.7729,-0.6826,-0.5958,-0.5120,-0.4304,-0.3507,-0.2725,
-0.1953,-0.1189,-0.0429,0.0329,0.1089,0.1853,0.2625,0.3408,0.4206,0.5022,0.5862,
0.6731,0.7636,0.8584,0.9589,1.0663,1.1826,1.3108,1.4554,1.6237,1.8300,2.1066,
2.5666

Comma Separated Form for the quantization of the Gaussian random variable upto 40 points. Associated probability values for the quantized values for the random variable with mean 0 and variance 1.

1:1
 2:0.5,0.5
 3: 0.3150,0.3700,0.3150
 4: 0.2230,0.2770,0.2770,0.2230
 5: 0.1690,0.2160,0.2300,0.2160,0.1690
 6: 0.1340,0.1750,0.1910,0.1910,0.1750,0.1340
 7: 0.1100,0.1450,0.1620,0.1660,0.1620,0.1450,0.1100
 8: 0.0930,0.1230,0.1390,0.1450,0.1450,0.1390,0.1230,0.0930
 9: 0.0787,0.1064,0.1209,0.1285,0.1308,0.1285,0.1210,0.1064,0.0788
 10: 0.0684,0.0931,0.1065,0.1142,0.1177,0.1177,0.1142,0.1066,0.0931,0.0685
 11:
 0.0602,0.0823,0.0948,0.1023,0.1064,0.1077,0.1064,0.1023,0.0948,0.0824,0.0603
 12:0.0536,0.0735,0.0851,0.0923,0.0967,0.0987,0.0988,0.0967,0.0923,0.0851,0.0736
 ,0.0536
 13:0.0481,0.0662,0.0769,0.0838,0.0883,0.0908,0.0916,0.0908,0.0883,0.0839,0.0769
 , 0.0663,0.0481
 14:0.0435,0.0601,0.0700,0.0766,0.0810,0.0837,0.0850,0.0850,0.0837,0.0810,0.0766
 , 0.0700,0.0601,0.0435
 15:0.0396,0.0548,0.0641,0.0703,0.0746,0.0775,0.0791,0.0796,0.0791,0.0775,0.0747
 ,
 0.0704,0.0641,0.0549,0.0396
 16:0.0363,0.0503,0.0590,0.0649,0.0691,0.0719,0.0738,0.0747,0.0747,0.0738,0.0720
 ,
 0.0691,0.0649,0.0590,0.0504,0.0363
 17:0.0334,0.0464,0.0545,0.0601,0.0641,0.0670,0.0690,0.0701,0.0705,0.0701,0.0690
 ,
 0.0671,0.0642,0.0602,0.0546,0.0465,0.0334
 18:0.0308,0.0430,0.0506,0.0559,0.0598,0.0626,0.0646,0.0659,0.0665,0.0665,0.0659
 ,
 0.0647,0.0627,0.0598,0.0560,0.0506,0.0431,0.0309
 19:0.0286,0.0400,0.0471,0.0522,0.0559,0.0587,0.0607,0.0621,0.0629,0.0632,0.0629
 ,
 0.0621,0.0608,0.0587,0.0560,0.0522,0.0472,0.0401,0.0287
 20:0.0267,0.0373,0.0440,0.0488,0.0524,0.0551,0.0572,0.0586,0.0595,0.0600,0.0600
 ,
 0.0596,0.0586,0.0572,0.0552,0.0525,0.0489,0.0441,0.0374,0.0267
 21:0.0249,0.0349,0.0413,0.0458,0.0493,0.0519,0.0539,0.0554,0.0564,0.0571,0.0573
 ,
 0.0571,0.0565,0.0555,0.0540,0.0520,0.0494,0.0459,0.0414,0.0350,0.0250
 22:0.0234,0.0328,0.0388,0.0431,0.0464,0.0490,0.0510,0.0525,0.0536,0.0543,0.0546
 ,
 0.0547,0.0543,0.0536,0.0526,0.0511,0.0491,0.0466,0.0433,0.0389,0.0329,0.0235
 23:0.0220,0.0309,0.0366,0.0407,0.0439,0.0464,0.0483,0.0498,0.0509,0.0517,0.0522
 ,
 0.0523,0.0522,0.0518,0.0510,0.0499,0.0484,0.0465,0.0440,0.0408,0.0367,0.0310,0.0
 221
 24:0.0207,0.0291,0.0345,0.0385,0.0415,0.0439,0.0459,0.0474,0.0485,0.0493,0.0499
 ,
 ,

0.0502,0.0502,0.0499,0.0494,0.0486,0.0474,0.0460,0.0441,0.0417,0.0386,0.0347,0.0292,0.0208
25:0.0196,0.0275,0.0327,0.0365,0.0394,0.0417,0.0436,0.0451,0.0462,0.0471,0.0477
>
0.0481,0.0482,0.0481,0.0478,0.0472,0.0463,0.0452,0.0437,0.0419,0.0396,0.0366,0.0328,0.0277,0.0197
26:0.0185,0.0261,0.0310,0.0346,0.0375,0.0397,0.0415,0.0430,0.0441,0.0450,0.0457
>
0.0461,0.0464,0.0464,0.0462,0.0458,0.0451,0.0442,0.0431,0.0417,0.0398,0.0376,0.0348,0.0312,0.0262,0.0186
27:0.0176,0.0248,0.0295,0.0329,0.0357,0.0378,0.0396,0.0410,0.0422,0.0431,0.0438
>
0.0443,0.0446,0.0447,0.0446,0.0443,0.0439,0.0432,0.0423,0.0412,0.0397,0.0380,0.0358,0.0331,0.0296,0.0249,0.0177
28:0.0167,0.0236,0.0281,0.0314,0.0340,0.0361,0.0378,0.0392,0.0404,0.0413,0.0420
>
0.0426,0.0429,0.0431,0.0431,0.0429,0.0426,0.0421,0.0414,0.0405,0.0394,0.0380,0.0363,0.0342,0.0316,0.0282,0.0237,0.0168
29:0.0159,0.0225,0.0267,0.0299,0.0325,0.0345,0.0362,0.0376,0.0387,0.0396,0.0404
>
0.0409,0.0413,0.0416,0.0416,0.0416,0.0414,0.0410,0.0405,0.0397,0.0388,0.0377,0.0364,0.0347,0.0327,0.0301,0.0269,0.0226,0.0160
30:0.0152,0.0214,0.0255,0.0286,0.0310,0.0330,0.0347,0.0360,0.0371,0.0381,0.0388
>
0.0394,0.0398,0.0401,0.0402,0.0403,0.0401,0.0399,0.0395,0.0389,0.0382,0.0373,0.0362,0.0348,0.0332,0.0312,0.0288,0.0257,0.0216,0.0153
31:0.0145,0.0205,0.0244,0.0274,0.0297,0.0316,0.0332,0.0346,0.0357,0.0366,0.0373
>
0.0379,0.0384,0.0387,0.0389,0.0390,0.0389,0.0388,0.0385,0.0380,0.0375,0.0367,0.0358,0.0347,0.0334,0.0319,0.0299,0.0276,0.0246,0.0207,0.0146
32:0.0138,0.0196,0.0234,0.0262,0.0285,0.0304,0.0319,0.0332,0.0343,0.0352,0.0359
>
0.0366,0.0370,0.0374,0.0376,0.0378,0.0378,0.0377,0.0375,0.0371,0.0367,0.0361,0.0354,0.0345,0.0334,0.0321,0.0306,0.0287,0.0265,0.0236,0.0198,0.0140
33:0.0132,0.0188,0.0224,0.0252,0.0274,0.0292,0.0307,0.0319,0.0330,0.0339,0.0346
>
0.0353,0.0358,0.0361,0.0364,0.0366,0.0366,0.0366,0.0365,0.0362,0.0359,0.0354,0.0348,0.0341,0.0332,0.0321,0.0309,0.0294,0.0276,0.0254,0.0226,0.0190,0.0134
34:0.0127,0.0180,0.0215,0.0242,0.0263,0.0280,0.0295,0.0307,0.0318,0.0327,0.0334
>
0.0340,0.0345,0.0349,0.0352,0.0354,0.0356,0.0356,0.0355,0.0353,0.0350,0.0347,0.0342,0.0336,0.0329,0.0320,0.0309,0.0297,0.0283,0.0265,0.0244,0.0218,0.0182,0.0128
35:0.0122,0.0173,0.0207,0.0232,0.0253,0.0270,0.0284,0.0296,0.0306,0.0315,0.0323
>
0.0329,0.0334,0.0338,0.0341,0.0344,0.0345,0.0346,0.0345,0.0344,0.0342,0.0339,0.0335,0.0330,0.0324,0.0317,0.0309,0.0298,0.0286,0.0272,0.0255,0.0235,0.0209,0.0175,0.0123

36:0.0117,0.0166,0.0199,0.0224,0.0244,0.0260,0.0274,0.0286,0.0296,0.0304,0.0312
,
0.0318,0.0323,0.0327,0.0331,0.0333,0.0335,0.0336,0.0336,0.0335,0.0334,0.0332,0.0
329,0.0325,0.0320,0.0314,0.0306,0.0298,0.0288,0.0276,0.0263,0.0246,0.0226,0.020
1,0.0169,0.0119
37:0.0113,0.0160,0.0192,0.0216,0.0235,0.0251,0.0264,0.0276,0.0286,0.0294,0.0301
,
0.0308,0.0313,0.0317,0.0321,0.0323,0.0325,0.0327,0.0327,0.0327,0.0326,0.0324,0.0
322,0.0318,0.0314,0.0309,0.0303,0.0296,0.0288,0.0278,0.0267,0.0253,0.0237,0.021
8,0.0194,0.0162,0.0114
38:0.0109,0.0154,0.0185,0.0208,0.0227,0.0242,0.0255,0.0266,0.0276,0.0284,0.0292
,
0.0298,0.0303,0.0307,0.0311,0.0314,0.0316,0.0318,0.0318,0.0318,0.0318,0.0317,0.0
315,0.0312,0.0309,0.0305,0.0300,0.0294,0.0287,0.0278,0.0269,0.0258,0.0245,0.022
9,0.0211,0.0187,0.0157,0.0110
39:0.0105,0.0149,0.0178,0.0201,0.0219,0.0234,0.0247,0.0257,0.0267,0.0275,0.0282
,
0.0288,0.0294,0.0298,0.0302,0.0305,0.0307,0.0309,0.0310,0.0310,0.0310,0.0309,0.0
308,0.0306,0.0303,0.0300,0.0296,0.0290,0.0285,0.0278,0.0270,0.0260,0.0249,0.023
7,0.0222,0.0204,0.0181,0.0151,0.0106
40:0.0101,0.0144,0.0172,0.0194,0.0211,0.0226,0.0238,0.0249,0.0258,0.0266,0.0273
,
0.0280,0.0285,0.0289,0.0293,0.0296,0.0299,0.0300,0.0302,0.0302,0.0303,0.0302,0.0
301,0.0300,0.0297,0.0295,0.0291,0.0287,0.0282,0.0276,0.0269,0.0261,0.0252,0.024
1,0.0229,0.0214,0.0197,0.0175,0.0146,0.0103