OSCILLATION OF SECOND ORDER MATRIX EQUATIONS ON TIME SCALES

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

OSCILLATION OF SECOND ORDER MATRIX EQUATIONS ON TIME SCALES

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The theory of time scales is introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis. In our thesis, by making use of the time scale calculus we study the oscillation of nonlinear matrix differential equations of second order. The first chapter is introductory in nature and contains some basic definitions and tools of the time scales calculus, while certain well-known results have been presented with regard to oscillation of the solutions of second order matrix equations and some new oscillation criteria for the same type equations have been established in the second chapter.

Keywords: Differential equation, Time scales, Riccati equation, Oscillation

ÖZ

ZAMAN SKALASI ÜZERİNDE MATRİS DENKLEMLERİN SALINIMI

Selçuk Aysun
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Zaman skalası teorisi 1988 yılında Stefan HILGER' in doktara tezinde sürekli ve ayrık analizi birlestirmek üzere ortaya konmustu. Tezimizde, zaman skalasının analizi kullanılarak lineer olmayan ikinci mertebeden denklemlerin salınımlılığını çalıstık. Öncelikle zaman skalasının analizindeki temel tanımları ve araçları verdik. Ikinci mertebeden matrix diferensiyel denklemlerinin çözümlerinin salınımlılığıyla ilgili iyi bilinen sonuçları sunduk. Son olarakta aynı tip denklemlerin salınımlılığı için yeni kriterler kurduk.

Anahtar Kelimeler: Diferensiyel denklem, Zaman skalasi, Riccati denklemi, Salınımlı çözüm, Salınımsız çözüm.

To my lovely husband

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CHAPTER 1

INTRODUCTION

1.1 Basic Definitions:

A time scale is an arbitrary nonempty closed subset of the real numbers. Thus, the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are $[0,1]\cup[2,3]$,$[0,1]\cup\mathbb{N}$ and the Cantor set, while the rational numbers, the irrational numbers, and the open interval between 0 and 1 are not time scales. We will denote a time scale by the symbol $\mathbb{T}$.

For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s < t : s \in \mathbb{T}\}.$$

If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$, we say t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

If $\mathbb{T}$ has a left-scattered maximum m, then by $\mathbb{T}^k$ we denote the set $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. Throughout the thesis we shall make use of the notation f^σ to mean that

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

Let us identify σ, ρ, and μ by considering specific time scales.

1. If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be $\mu(t) \equiv 0$ for all $t \in \mathbb{T}$.

2. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, ...\} = t+1$$

and

$$\rho(t) = \sup\{s \in \mathbb{Z} : s < t\} = \sup\{..., t-3, t-2, t-1\} = t-1$$

Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is $\mu(t) \equiv 1$ for all $t \in \mathbb{T}$.

3. If $\mathbb{T} = \{2^n : \ n \in \mathbb{Z}\} \cup \{0\}$; then

$$\sigma(2^n) = 2^{n+1}, \sigma(t) = 2t \quad \text{and} \quad \rho(2^n) = 2^{n-1}, \rho(t) = \frac{t}{2}$$

Hence every point $t \in \mathbb{T}$ is isolated.

4. If $\mathbb{T} = \left\{\frac{1}{n}: \quad n \in \mathbb{N}\right\} \cup \{0\}$; then

$$\sigma(\frac{1}{n}) = \frac{1}{n-1}, \sigma(t) = \frac{t}{1-t} \quad \text{and} \quad \rho(\frac{1}{n}) = \frac{1}{n+1}, \rho(t) = \frac{t}{1+t}$$

Hence every point $t \in \mathbb{T}$ is isolated.

1.2 Differentiation

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The derivative of f at t, denoted by $f^{\triangle}(t)$, is defined to be the number with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\triangle}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U$$

We call $f^{\triangle}(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\triangle} : \mathbb{T}^k \to \mathbb{R}$ is then called the derivative of f on $\mathbb{T}^k$.

For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\triangle\triangle}$ provided $f^{\triangle}$ is differentiable on $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$ with derivative $f^{\triangle\triangle} = (f^{\triangle})^{\triangle} : \mathbb{T}^{k^2} \to \mathbb{R}$.

The proof of the next theorem which can be found in [3] is based on the definition of the derivative.

Theorem 1.2.1. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

1. *If f is differentiable at t , then f is continuous at t.*

2. *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

3. *If t is right-dense, then f is differentiable at t iff the limit*

$$\lim_{s\to t^+}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case

$$f^{\triangle}(t)=\lim_{s\to t^+}\frac{f(t)-f(s)}{t-s}$$

4. *If f is differentiable at t, then*

$$f(\sigma(t))=f(t)+\mu(t)f^{\triangle}(t).$$

For instance, if $f(t)=t$, then $f^{\triangle}(t)=1$ since

$$f^{\triangle}(t)=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}=\frac{\sigma(t)-t}{\sigma(t)-t}=1$$

and if $f(t)=t^2$ then

$$f^{\triangle}(t)=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}=\frac{(\sigma(t))^2-t^2}{\sigma(t)-t}=\sigma(t)+t.$$

Let $f,g:\mathbb{T}\to\mathbb{R}$ are differentiable. It is not difficult to see that the sum $f+g$ is differentiable with

$$(f+g)^{\triangle}(t)=f^{\triangle}(t)+g^{\triangle}(t);$$

for any constant $\alpha, \alpha f$ is differentiable with

$$(\alpha f)^{\triangle}(t)=\alpha f^{\triangle}(t);$$

the product fg is differentiable with

$$(fg)^{\triangle}(t)=f^{\triangle}(t)g(t)+f(\sigma(t))g^{\triangle}(t)=f(t)g^{\triangle}(t)+f^{\triangle}(t)g(\sigma(t));$$

$1/f$ is differentiable whenever $f(t)f(\sigma(t)) \neq 0$ with

$$(\frac{1}{f})^{\triangle}(t) = -\frac{f^{\triangle}(t)}{f(t)f(\sigma(t))};$$

f/g is differentiable whenever $g(t)g(\sigma(t)) \neq 0$ with

$$(\frac{f}{g})^{\triangle}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}.$$

1.3 Integration

In order to describe classes of functions that are "integrable", the following two concepts are defined:

Definition 1.3.1 ([3]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 1.3.2 ([3]). A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$$

Theorem 1.3.1 ([3]). *Let $f : \mathbb{T} \to \mathbb{R}$. Then the following statements are true:*

1. *If f is continuous , then f is rd-continuous.*
2. *If f is rd-continuous, then f is regulated.*

3. *The jump operator σ is rd-continuous.*

4. *If f is regulated or rd-continuous , then so is f^{σ}.*

5. *Assume f is continuous . If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous , then $f \circ g$ is regulated or rd-continuous, respectively.*

The indefinite integral of a regulated function f is defined as

$$\int f(t)\triangle(t) = F(t) + C$$

where C is an arbitrary constant and F is called a pre-antiderivative of f. The Cauchy integral is defined by

$$\int_r^s f(t)\triangle(t) = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\triangle}(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^k.$$

Theorem 1.3.2 ([3]). *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) = \int_{t_0}^{t} f(\tau)\triangle\tau \quad \textit{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

Morever, If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then

$$\int_t^{\sigma(t)} f(\tau)\triangle\tau = \mu(t)f(t)$$

To see this we note that

$$\begin{aligned}\int_t^{\sigma(t)} f(\tau)\triangle\tau &= F(\sigma(t)) - F(t) \\ &= \mu(t)F^{\triangle}(t) \\ &= \mu(t)f(t),\end{aligned}$$

where the second equality holds because of the last part of Theorem 1.2.1.

Theorem 1.3.3 ([3]). *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

1. $\int_a^b [f(t) + g(t)]\triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t$,
2. $\int_a^b (\alpha f)(t)\triangle t = \alpha \int_a^b f(t)\triangle t$,
3. $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t$,
4. $\int_a^a f(t)\triangle t = 0$,
5. $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t$,
6. *If* $|f(t)| \leq g(t)$ *on* $[a, b)$, *then*

$$|\int_a^b f(t)\triangle t| \leq \int_a^b g(t)\triangle t,$$

7. *If* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\triangle t \geq 0$,

Theorem 1.3.4. *If $a, b \in \mathbb{T}$, and $f, g \in C_{rd}$, and f, g are differentiable, then*

1. $\int_a^b f(\sigma(t))g^{\triangle}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(t)\triangle t$,
2. $\int_a^b f(t)g^{\triangle}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t$,

Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$. Then

$$\int_a^b f(t)\triangle t = \int_a^b f(t)dt, \quad \text{if} \quad \mathbb{T} = \mathbb{R}.$$

If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h, & \text{if } a > b. \end{cases}$$

If $f(t) = 1$, $F(t) = t$ is an antiderivative of 1, since $F^{\triangle}(t) = 1 = f(t)$. Hence

$$\int_a^t \triangle s = F(t) - F(a) = t - a$$

Let us evaluate $\int_0^t s\triangle s$ for $t \in \mathbb{T} = \mathbb{Z}$

$$\int_0^t s\triangle s = \sum_{s=0}^{t-1} s = \frac{(t-1)t}{2} = \frac{t^2}{2} - \frac{t}{2}.$$

1.4 Regressive Matrices

Let A be an $m \times n$ matrix valued function on $\mathbb{T}$. The matrix A is said to be rd-continuous on $\mathbb{T}$ if each element is so, and the class of all such rd-continuous $m \times n$ matrix valued functions defined on $\mathbb{T}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}^{m\times n}).$$

As in the classical case we say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$, and in this case we put

$$A^{\triangle}(t) = (a_{ij}^{\triangle}(t))_{1\le i\le m, 1\le j\le n}, \quad A = (a_{ij})_{1\le i\le m, 1\le j\le n}.$$

If A is differentiable at $t \in \mathbb{T}^k$, then $A^{\sigma}(t) = A(t) + \mu(t)A^{\triangle}(t)$. We observe that at t,

$$\begin{aligned} A^\sigma &= (a_{ij}^\sigma) \\ &= (a_{ij} + \mu a_{ij}^\triangle) \\ &= (a_{ij}) + \mu(a_{ij}^\triangle) \\ &= A + \mu A^\triangle. \end{aligned}$$

Theorem 1.4.1 ([3]). *Suppose A and B are differentiable $n \times n$ matrices. Then*

1. $(A + B)^\triangle = A^\triangle + B^\triangle$;
2. $(\alpha A)^\triangle = \alpha A^\triangle$ *if α is constant;*
3. $(A.B)^\triangle = A^\triangle B^\sigma + AB^\triangle = A^\sigma B^\triangle + A^\triangle B$;
4. $(A^{-1})^\triangle = -(A^\sigma)^{-1} A^\triangle A^{-1} = -A^{-1} A^\triangle (A^\sigma)^{-1}$ *if AA^σ is invertible;*
5. $(AB^{-1})^\triangle = (A^\triangle - AB^{-1}B^\triangle)(B^\sigma)^{-1} = (A^\triangle - (AB^{-1})^\sigma B^\triangle))B^{-1}$ *if BB^σ is invertible.*

An $n \times n$ matrix valued function A on a time scale $\mathbb{T}$ is called regressive provided

$$I + \mu(t)A(t) \tag{1.1}$$

is invertible for all $t \in \mathbb{T}^k$, and the class of all such regressive and rd-continuous functions is denoted by

$$\Re = \Re(\mathbb{T}) = \Re(\mathbb{T}, \mathbb{R}^{n\times n}).$$

Lemma 1.4.1 ([3]). *An $n \times n$ matrix valued function A is regressive if and only if the eigenvalues $\lambda_i(t)$ of $A(t)$ are regressive for all $1 \leq i \leq n$.*

Definition 1.4.1 ([3]). Let $\mathbb{T}$ be a time scale and $\mathbb{X}$ be a Banach space. A function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is called rd continuous if $g(t) = f(t, x(t))$ is rd continuous for any continuous function $x : \mathbb{T} \to \mathbb{X}$, it is called regressive at

$t \in \mathbb{T}^k$, if the mapping

$$I + \mu(t) f(t, .) : \mathbb{X} \to \mathbb{X} \quad \text{is invertible}$$

(where I is the identity function), and f is called regressive on $\mathbb{T}^k$, if f is regressive at each $t \in \mathbb{T}^k$.

Theorem 1.4.2 ([3]). *(A global existence and uniqueness theorem) Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd continuous and regressive. Suppose that there exists $L(t, x) > 0$ such that the Lipschitz condition*

$$|f(t, x_1) - f(t, x_2)| \leq L(t, x)|x_1 - x_2| \quad \textit{for all} \quad (t, x_1), (t, x_2) \in \mathbb{T} \times R^n$$

holds. Then the IVP

$$x^{\triangle} = f(t, x), \quad x(t_0) = x_0 \tag{1.2}$$

has exactly one solution defined on $\mathbb{T}$.

Theorem 1.4.3 ([3]). *Let $A \in \Re$ be an $n \times n$ matrix valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\triangle} = A(t)y + f(t), \quad y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$.

It follows that The matrix initial value problem

$$Y^{\triangle} = A(t)Y, \quad Y(t_0) = Y_0, \tag{1.3}$$

where Y_0 is a constant $n \times n$ matrix, has a unique solution Y.

Definition 1.4.2 ([3]). Let $t_0 \in \mathbb{T}$ and assume that $A \in \Re$ is an $n \times n$ matrix valued function. The unique matrix valued solution of the IVP

$$Y^{\triangle} = A(t)Y, \quad Y(t_0) = I,$$

where I denotes as usual the $n \times n$ identity matrix, is called the matrix exponential function, and it is denoted by $e_A(., t_0)$.

Definition 1.4.3. Assume A and B are regressive $n \times n$ matrix valued functions on $\mathbb{T}$. Then we define $A \oplus B$ by

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t) \quad \text{for all} \quad t \in \mathbb{T}^k,$$

and $\ominus A$ by

$$(\ominus A)(t) = -A(t)[I + \mu(t)A(t)]^{-1} \quad \text{for all} \quad t \in \mathbb{T}^k.$$

Definition 1.4.4. If the matrix valued functions A and B are regressive on $\mathbb{T}$, then we define $A \ominus B$ by

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t) \quad \text{for all} \quad t \in \mathbb{T}^k.$$

If A is a matrix , then we let A^* denote its conjugate transpose.

Theorem 1.4.4 ([3]). *If $A, B \in \Re$ are matrix valued functions on $\mathbb{T}$, then*

1. $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$;
2. $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;
3. $e_A^{-1}(t, s) = e^*_{\ominus A^*}(t, s)$;
4. $e_A(t, s) = e_A^{-1}(s, t) = e^*_{\ominus A^*}(s, t)$;
5. $e_A(t, s)e_A(s, r) = e_A(t, r)$;
6. $e_A(t, s)e_B(t, s) = e_{A \oplus B}(t, s)$ *if* $e_A(t, s)$ *and* $B(t)$ *commute.*

Proof We only prove (1) and (5). If we use Theorem 1.1,

$$\begin{aligned} e_A(\sigma(t), s) &= e_A(t, s) + \mu(t)e_A^{\triangle}(t, s) \\ &= e_A(t, s) + \mu(t)A(t)e_A(t, s) \\ &= (I + \mu(t)A(t))e_A(t, s) \end{aligned}$$

To prove (5), consider $Y(t) = e_A(t,s)e_B(t,s)$ and assume that $e_A(t,s)$ and $B(t)$ commute. By using Theorem 1.4.1

$$\begin{aligned} Y^{\triangle}(t) &= e_A^{\triangle}(t,s).e_B^{\sigma}(t,s) + e_A(t,s)e_B^{\triangle}(t,s) \\ &= A(t)e_A(t,s)(I+\mu(t)B(t))e_B(t,s) + e_A(t,s)B(t)e_B(t,s) \\ &= A(t)(I+\mu(t)B(t))e_A(t,s)e_B(t,s) + B(t)e_A(t,s)e_B(t,s) \\ &= [A(t)(I+\mu(t)B(t)) + B(t)]e_A(t,s)e_B(t,s) \\ &= (A\oplus B)(t)e_A(t,s)e_B(t,s) \\ &= (A\oplus B)(t)Y(t) \end{aligned}$$

Also $Y(s) = e_A(s,s)e_B(s,s) = I.I = I$. So Y is a unique solution of the IVP

$$Y^{\triangle} = (A\oplus B)(t)Y, \quad Y(s) = I,$$

and therefore we have $e_{A\oplus B}(t,s) = Y(t) = e_A(t,s)e_B(t,s)$.

Theorem 1.4.5 ([3]). *If $A \in \Re$ and $a,b,c \in \mathbb{T}$, then*

$$[e_A(c,.)]^{\triangle} = -[e_A(c,.)]^{\sigma}A$$

and

$$\int_a^b e_A(c,\sigma(t))A(t)\triangle t = e_A(c,a) - e_A(c,b).$$

We shall also consider the nonhomogeneous equation

$$y^{\triangle} = A(t)y + f(t) \tag{1.4}$$

where $f : \mathbb{T} \to \mathbb{R}^n$ is a vector valued function. If $f(t) \equiv 0$, then

$$y^{\triangle} = A(t)y \tag{1.5}$$

is called the homogeneous equation corresponding 1.4.

Theorem 1.4.6 ([3]). *(Variation of constants formula) Let $A \in \Re$ be an $n\times n$ matrix valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd continuous. Let*

$t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem

$$y^{\triangle} = A(t)y + f(t), \quad y(t_0) = y_0 \tag{1.6}$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(r))f(r)\triangle r. \tag{1.7}$$

In Theorem 1.4.4, we assume that e_A and B commute. Now, we ask under what conditions the two matrices e_A and B commute.

Theorem 1.4.7 ([3]). *Suppose $A \in \Re$ and C is differentiable. If C is a solution of the dynamic equation,*

$$C^{\triangle} = A(t)C - C^{\sigma}A(t),$$

then

$$C(t)e_A(t, s) = e_A(t, s)C(s).$$

Let us consider adjoint equation of (1.5)

$$x^{\triangle} = -A^*(t)x^{\sigma} \tag{1.8}$$

Theorem 1.4.8 ([3]). *Let $A \in \Re$ be an $n \times n$ matrix valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the IVP*

$$x^{\triangle} = -A^*(t)x^{\sigma} + f(t), \quad x(t_0) = x_0 \tag{1.9}$$

has a unique solution $x : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by

$$x(t) = e_{\ominus A^*}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus A^*}(t, r)f(r)\triangle r. \tag{1.10}$$

1.5 Constant Coefficients

In this part, we consider the vector dynamic equation

$$x^{\triangle} = Ax \tag{1.11}$$

where $A \in \Re$ is a real constant $n \times n$ matrix.

Theorem 1.5.1 ([3]). *If λ_0, ξ is an eigenpair for A, then $x(t) = e_{\lambda_0}(t, t_0)\xi$ is a solution of 1.11 on $\mathbb{T}$.*

EXAMPLE 1.5.1. Let consider the vector dynamic equation

$$x^{\triangle} = \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix} x \tag{1.12}$$

on time scale satisfies $\mu(t) \neq \frac{1}{2}$. The eigenvalues of 1.12 are $\lambda_1 = -2$ and $\lambda_2 = 3$. Since $1 - 2\mu(t) \neq 0$ and $1 + 3\mu(t) \neq 0$ for all $t \in \mathbb{T}^k$ the vector equation 1.12 is regressive for any time scale. Eigenvectors corresponding to λ_1 and λ_2 are

$$\xi_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

respectively. Hence, the general solution of 1.12 is

$$x(t) = c_1 e_{-2}(t, t_0) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e_3(t, t_0) \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Theorem 1.5.2 ([3]). *(Putzer Algorithm) Let $A \in \Re$ be a constant $n \times n$ matrix. Suppose $t_0 \in \mathbb{T}$. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, then*

$$e_A(t, t_0) = \sum_{i=0}^{n-1} r_{i+1}(t) P_i, \tag{1.13}$$

where $r(t) = (r_1(t), r_2(t), ...r_n(t))^T$ *is the solution of the IVP*

$$r^{\triangle} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 1 & \lambda_2 & 0 & \dots \\ 0 & 1 & \lambda_3 & \dots \\ \vdots & \vdots & \ddots & \ddots \\ \dots & 0 & 1 & \lambda_n \end{pmatrix} r, \quad r(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.14}$$

and the P matrices $P_0, P_1, ..., P_n$ *are recursively defined by* $P_0 = I$ *and*

$$P_{k+1} = (A - \lambda_{k+1} I) P_k \quad \textit{for} \quad 0 \le k \le n-1.$$

1.6 Self-Adjoint Matrix Equations

Let P and Q be Hermitian $n \times n$ matrix valued functions on a time scale $\mathbb{T}$ such that $P(t)$ is invertible for all $t \in \mathbb{T}$. We shall consider the self -adjoint second order matrix differential equation

$$LX = 0, \quad \text{where} \quad LX = (PX^{\triangle})^{\triangle} + Q(t)X^{\sigma} \tag{1.15}$$

on $\mathbb{T}^{k^2}$.

By a solution of (1.15) we mean a matrix X defined on $\mathbb{T}$ such that X is differentiable on $\mathbb{T}^k$ and $(PX^{\triangle})^{\triangle}$ is rd continuous on $\mathbb{T}^{k^2}$, and satisfies (1.15) on $\mathbb{T}$. If X satisfies above conditions then we write $X \in \mathbb{D}$.

Definition 1.6.1 ([3]). The unique solution of the initial value problem

$$LX = 0, \quad X(a) = 0, \quad X^{\triangle}(a) = P^{-1}(a)$$

is called the principal solution of (1.15)(at a), while the unique solution of the initial value problem

$$LX = 0, \quad X(a) = -I, \quad X^{\triangle}(a) = 0$$

is said to be the associated solution of (1.15)(at a).

DEFINITION 1.6.2 ([3]). If $X, Y \in \mathbb{D}$, then we define the Wronskian matrix of X and Y by

$$W(X,Y)(t) = X^*(t)P(t)Y^{\triangle}(t) - [P(t)X^{\triangle}(t)]^*Y(t)$$

for $t \in \mathbb{T}^k$.

Theorem 1.6.1 ([3]). *(Lagrange Identity) If $X, Y \in \mathbb{D}$, then*

$$X^*(\sigma(t))LY(t) - [LX(t)]^*Y(\sigma(t)) = [W(X,Y)]^{\triangle}(t)$$

for $t \in \mathbb{T}^{k^2}$.

Proof Let $X, Y \in \mathbb{D}$, then

$$\begin{aligned} W^{\triangle}(X,Y) &= \{X^*PY^{\triangle} - (PX^{\triangle})^*Y\}^{\triangle} \\ &= (X^*)^{\sigma}(PY^{\triangle})^{\triangle} + (X^*)^{\triangle}PY^{\triangle} - (PX^{\triangle})^*Y^{\triangle} - \{(PX^{\triangle})^{\triangle}\}^*Y^{\sigma} \\ &= (X^*)^{\sigma}(PY^{\triangle})^{\triangle} - \{(PX^{\triangle})^{\triangle}\}^*Y^{\sigma} \\ &= (X^*)^{\sigma}\{(PY^{\triangle})^{\triangle} + QY^{\sigma}\} - \{(PX^{\triangle})^{\triangle} + QX^{\sigma}\}^*Y^{\sigma} \\ &= (X^*)^{\sigma}LY - (LX)^*Y^{\sigma} \end{aligned}$$

on $t \in \mathbb{T}^{k^2}$.

Corollary 1.6.1 ([3]). *(Abel's Formula) If X and Y are solutions of* (1.15) *on $\mathbb{T}$, then*

$$W(X,Y)(t) \equiv C$$

for $t \in \mathbb{T}^k$, where C is a constant matrix.

Proof Assume X and Y are solutions of (1.15) on $\mathbb{T}$. By the Lagrange identity

$$W^{\triangle}(X,Y)(t) = 0 \quad \text{for all} \quad t \in \mathbb{T}^{k^2}$$

Hence, $W(X,Y)$ is a constant matrix for all $t \in \mathbb{T}^k$.

DEFINITION 1.6.3 ([3]). If X is a solution of (1.15) satisfying

$$W(X, X)(t) \equiv 0 \quad \text{for} \quad t \in \mathbb{T}^k,$$

then we say that X is a prepared solution (or conjoined solution or isotropic solution) of (1.15), and if X and Y are two conjoined solutions with

$$W(X, Y)(t) \equiv I \quad \text{for} \quad t \in \mathbb{T}^k,$$

then we say that X and Y are normalized conjoined bases of (1.15).

Theorem 1.6.2 ([3]). *Assume that X is a solution of* (1.15) *on $\mathbb{T}$. Then the following statements are equivalent:*

1. *X is a prepared solution;*
2. *$X^*(t)P(t)X^\triangle(t)$ is Hermitian for all $t \in \mathbb{T}^k$;*
3. *$X^*(t_0)P(t_0)X^\triangle(t_0)$ is Hermitian for some $t_0 \in \mathbb{T}^k$.*

Proof Assume that X is a prepared solution of (1.15) on $\mathbb{T}$. Then it satisfies

$$W(X, X)(t) = X^*(t)P(t)X^\triangle(t) - [P(t)X^\triangle(t)]^*X(t) = 0$$

for $t \in \mathbb{T}^k$. This implies that X is a prepared solution iff $X^*(t)P(t)X^\triangle(t)$ is Hermitian for all $t \in \mathbb{T}^k$, and iff $X^*(t_0)P(t_0)X^\triangle(t_0)$ is Hermitian for some $t_0 \in \mathbb{T}^k$.

Lemma 1.6.1 ([3]). *Let X be a solution of* (1.15). *If X is a prepared, then*

$$X^*(\sigma(t))P(t)X(t) \quad \textit{is Hermitian for all} \quad t \in \mathbb{T}^k.$$

Conversely, if there is $t_0 \in \mathbb{T}^k$ such that $\mu(t_0) = \sigma(t_0) - t_0 > 0$ and $X^*(\sigma(t_0))P(t_0)X(t_0)$ is Hermitian, then X is a prepared solution of (1.15). *Also, if X is a nonsingular prepared solution, then*

$$P(t)X(\sigma(t))X^{-1}(t), \quad P(t)X(t)X^{-1}(\sigma(t)), \quad \textit{and} \quad Z(t) = P(t)X^\triangle(t)X^{-1}(t)$$

are Hermitian for all $t \in \mathbb{T}^k$.

Lemma 1.6.2 ([3]). *Assume that* X *is a prepared solution of* (1.15) *on* $\mathbb{T}$. *Then the following statements are equivalent:*

1. $X^*(\sigma(t))P(t)X(t) > 0$ *on* $\mathbb{T}^k$;

2. $X(t)$ *is nonsingular and*

$$P(t)X(\sigma(t))X^{-1}(t) > 0$$

   on $\mathbb{T}^k$;

3. $X(t)$ *is nonsingular and*

$$P(t)X(t)X^{-1}(\sigma(t)) > 0$$

   on $\mathbb{T}^k$.

In this part, we shall consider the matrix Riccati dynamic equation

$$RZ = 0, \quad \text{where} \quad RZ = Z^{\triangle} + Q(t) + Z^*\{P(t) + \mu(t)Z\}^{-1}Z. \tag{1.16}$$

Theorem 1.6.3 ([3]). *(Riccati Equation) If the self -adjoint matrix equation* (1.15) *has a prepared solution* X *such that* $X(t)$ *is invertible for all* $t \in \mathbb{T}$, *then* Z *defined by the Riccati substitution*

$$Z(t) = P(t)X^{\triangle}(t)X^{-1}(t), \tag{1.17}$$

$t \in \mathbb{T}^k$, *is a Hermitian solution of the matrix Riccati equation* (1.16) *on* $\mathbb{T}^k$. *Conversely, if* (1.16) *has a Hermitian solution* Z *on* $\mathbb{T}^k$, *then there exists a prepared solution* X *of* (1.15) *such that* $X(t)$ *is invertible for all* $t \in \mathbb{T}$ *and relation* (1.17) *holds.*

Theorem 1.6.4 ([3]). *The self -adjoint matrix equation* (1.15) *has a prepared solution* X *on* $\mathbb{T}$ *with* $X^*(\sigma(t))P(t)X(t) > 0$ *on* $\mathbb{T}^k$ *iff the matrix Riccati equation*

(1.16) *has a Hermitian solution* Z *on* $\mathbb{T}^k$ *satisfying*

$$P(t) + \mu(t)Z(t) > 0$$

for all $t \in \mathbb{T}^k$.

Theorem 1.6.5 ([3]). *(Picone's Identity) Let* $\alpha \in \mathbb{R}^n$ *and suppose* X *and* Y *are normalized conjoined bases of* (1.15) *such that* X *is invertible on* $\mathbb{T}^k$. *We put*

$$Z = PX^{\triangle}X^{-1} \quad \text{and} \quad D = X(X^{\sigma})^{-1}P^{-1} \quad \text{on} \quad \mathbb{T}^k.$$

Let $t \in \mathbb{T}^k$ *and assume that* $u : \mathbb{T} \to \mathbb{R}^n$ *is differentiable at* t. *Then we have at* t

$$\begin{aligned}(u^*Zu \;&+\; 2\alpha^*X^{-1}u - \alpha^*X^{-1}Y\alpha)^{\triangle} = (u^{\triangle})^*Pu^{\triangle} - (u^{\sigma})^*Qu^{\sigma} \\ &-\; \{Pu^{\triangle} - Zu - (X^{-1})^*\alpha\}^*D\{Pu^{\triangle} - Zu - (X^{-1})^*\alpha\}.\end{aligned}$$

Let denote the set of all continuous functions whose derivatives are piecewise rd-continuous by C^1_{prd}.

Definition 1.6.4 ([3]). The quadratic functional

$$\mathcal{F}(u) = \int_a^b \{(u^{\triangle})^*Pu^{\triangle} - (u^{\sigma})^*Qu^{\sigma}\}(t)\triangle t$$

is called positive definite (we write $\mathcal{F} > 0$) provided

$$\mathcal{F}(u) > 0 \quad \text{for all} \quad u \in C^1_{prd}([a,b],\mathbb{R}^n)\backslash\{0\} \quad \text{with} \quad u(a) = u(b) = 0.$$

Lemma 1.6.3 ([3]). *If* $u \in C^1_{prd}$ *and*

$$\{(Pu^{\triangle})^{\triangle} + Qu^{\sigma}\}(t) = 0 \quad \text{for all} \quad t \in [a,b]^{k^2},$$

then

$$\int_a^{\rho(b)} \{(u^{\triangle})^*Pu^{\triangle} - (u^{\sigma})^*Qu^{\sigma}\}(t)\triangle t = \{u^*Pu^{\triangle}\}(\rho(b)) - \{u^*Pu^{\triangle}\}(a).$$

Definition 1.6.5 ([3]). A conjoined solution of (1.15) is said to have no focal points in $(a, b]$ provided it satisfies

$$X \quad \text{invertible on} \quad (a, b] \quad \text{and} \quad X(X^{\sigma})^{-1}P^{-1} \geq 0 \quad \text{on} \quad [a, b]^{k}.$$

Theorem 1.6.6 ([3]). *(Sufficient Condition for Positive Definiteness) A sufficient condition for $\mathcal{F} > 0$ is that there exist normalized conjoined bases X and Y of (1.15) such that X has no focal points in $(a, b]$.*

Definition 1.6.6 ([3]). We say that equation (1.15) is disconjugate on $[a, b]$ if the principal solution $\widetilde{X}$ of (1.15) satisfies

$$\widetilde{X} \quad \text{invertible on} \quad (a, b] \quad \text{and} \quad \widetilde{X}(\widetilde{X}^{\sigma})^{-1}P^{-1} > 0 \quad \text{on} \quad (a, b]^{k}.$$

We can conclude from definition of 1.6.6, (1.15) is disconjugate iff the principal solution of (1.15) has no focal points in $(a, b]$.

Theorem 1.6.7 ([3]). *(Jacobi's Condition) $\mathcal{F} > 0$ iff (1.15) is disconjugate.*

Definition 1.6.7 ([3]). We call a solution X of (1.15) a basis whenever

$$\text{rank} \begin{pmatrix} X(a) \\ P(a)X^{\triangle}(a) \end{pmatrix} = n$$

Theorem 1.6.8 ([3]). *(Sturm's Separation Theorem) Suppose there exists a conjoined basis of (1.15) with no focal points in $(a, b]$. Then equation (1.15) is disconjugate on $[a, b]$.*

Proof Let X be a conjoined basis of (1.15) with no focal points in $(a, b]$. Since X is a basis

$$K = X^{*}(a)X(a) + (X^{\triangle})^{*}(a)P^{2}(a)X(a) \quad \text{is invertible.}$$

Let Y be the solution of (1.15) satisfying

$$Y(a) = -P(a)X^{\triangle}(a)K^{-1}, \quad Y(a) = P^{-1}(a)X(a)K^{-1}.$$

Then Y satisfies from Wronskian identity,

$$\begin{aligned}\{Y^* \quad P \quad Y^{\triangle} - (Y^{\triangle})^* PY\} &\equiv \{Y^* P Y^{\triangle} - (Y^{\triangle})^* PY\}(a) \\ &= -(K^{-1})^*(X^{\triangle})^*(a)P(a)X(a)K^{-1} + (K^{-1})^* X^*(a)P(a)X^{\triangle}(a)K^{-1} \\ &= (K^{-1})^*\{X^* P X^{\triangle} - (X^{\triangle})^* PX\}(a)K^{-1} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\{X^* \quad P \quad Y^{\triangle} - (X^{\triangle})^* PY\} &\equiv \{X^* P Y^{\triangle} - (X^{\triangle})^* PY\}(a) \\ &= X^*(a)X(a)K^{-1} + (X^{\triangle})^*(a)P^2(a)X^{\triangle}(a)K^{-1} \\ &= I.\end{aligned}$$

and hence X and Y are normalized conjoined bases of (1.15).

Now we shall also consider the equation

$$[\widetilde{P}(t)X^{\triangle}]^{\triangle} + \widetilde{Q}(t)X^{\sigma} = 0, \tag{1.18}$$

where $\widetilde{P}$ and $\widetilde{Q}$ satisfy the same assumptions as P and Q.

Theorem 1.6.9. *(Sturm's Comparison Theorem) Suppose we have for all $t \in \mathbb{T}$*

$$\widetilde{P}(t) \leq P(t) \quad \text{and} \quad \widetilde{Q}(t) \geq Q(t).$$

If (1.18) *is disconjugate, then* (1.15) *is also disconjugate.*

Proof Suppose (1.18) is disconjugate. Then by Jacobi's Condition

$$\widetilde{\mathcal{F}}(u) = \int_a^b \{(u^{\triangle})^* \widetilde{P} u^{\triangle} - (u^{\sigma})^* \widetilde{Q} u^{\sigma}\}(t) \triangle t > 0$$

for all nontrivial $u \in C^1_{prd}$ with $u(a) = u(b) = 0$. For such u we also have

$$\begin{aligned}\mathcal{F}(u) &= \int_a^b \{(u^\triangle)^* P u^\triangle - (u^\sigma)^* Q u^\sigma\}(t)\triangle t \\ &\geq \int_a^b \{(u^\triangle)^* \widetilde{P} u^\triangle - (u^\sigma)^* \widetilde{Q} u^\sigma\}(t)\triangle t \\ &= \widetilde{\mathcal{F}}(u) > 0\end{aligned}$$

Hence $\mathcal{F} > 0$ and thus (1.15) is disconjugate by Jacobi's Condition.

CHAPTER 2

OSCILLATION CRITERIA OF DYNAMIC EQUATIONS

In this chapter we first state and prove some well-known results with regard to oscillation of matrix dynamic equations on time scales, further results, see [3, 11, 15, 4, 8, 18, 2, 6]. Next we provide some new oscillation criteria for solutions of such equations. Roughly speaking a matrix solution $X(t)$ of a matrix dynamic equation is called nonoscillatory if $\det X(t) \neq 0$ for all $t \geq t_0$ for some sufficiently large $t_0 > 0$. Otherwise, it is called oscillatory. As in the continuous and discrete cases the definition require that the time scale under consideration be unbounded from above. That is, we assume that

$$\sup \mathbb{T} = \infty.$$

In the theorems that follow we employ some basic facts from Linear Algebra. For instance, if A is an Hermitian $n \times n$ matrix, then all eigenvalues are real. We shall also use the convention that if $\lambda_i(A)$ denotes the i-th eigenvalue of A, then

$$\lambda_{\max}(A) = \lambda_1(A) \geq ... \geq \lambda_n(A) = \lambda_{\min}(A).$$

By tr$A$ we denote the trace of an $n \times n$ matrix A , i.e., the sum of all diagonal elements of A.

An important and useful tool in comparing the eigenvalueas of the sum of two Hermitian matrices A and B is the Weyl's inequality, which states that

$$\lambda_i(A) + \lambda_{\max}(B) \geq \lambda_i(A+B) \geq \lambda_i(A) + \lambda_{\min}(B). \tag{2.1}$$

2.1 Some known oscillation theorems

We shall consider the self -adjoint second order matrix dynamic equations of the form

$$(P(t)X^{\triangle})^{\triangle} + Q(t)X^{\sigma} = 0, \quad t \geq a, \quad t \in \mathbb{T}^{k^2}, \tag{2.2}$$

where $a \in \mathbb{T}$ is fixed, P and Q are Hermitian $n \times n$ matrix valued functions defined on $\mathbb{T}$. It is also assumed that the matrix $P(t)$ is invertible.

In the special case $P(t) = I$ the above equation reduces to

$$X^{\triangle\triangle} + Q(t)X^{\sigma} = 0 \tag{2.3}$$

A matrix solution (2.2) is called nonoscillatory on $[a, \infty)$ provided there exist a prepared solution X of (2.2) and $t_0 \in [a, \infty)$ such that

$$X^*(\sigma(t)P(t)X(t) > 0, \quad t \geq t_0$$

Otherwise we say that (2.2) is oscillatory on $[a, \infty)$.

Theorem 2.1.1 ([6]). *Assume that for a given $t_0 \in [a, \infty)$ there exist $a_0, b_0 \in [t_0, \infty)$ such that $\mu(a_0) > 0, \mu(b_0) > 0$, and*

$$\lambda_{\max}\left[\int_{a_0}^{b_0} Q(t)\triangle t\right] \geq \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)} \tag{2.4}$$

Then (2.3) is oscillatory on $[a, \infty)$.

Proof Assume that equation (2.3) is nonoscillatory on $[a, \infty)$. Then there is a $t_0 \in [a, \infty)$ and a prepared solution $X(t)$ of (2.3) satisfying

$$X^*(\sigma(t))X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) = X^{\triangle}(t)X^{-1}(t)$$

for $t \in [t_0, \infty)$, then by Theorem 1.6.4 we get that

$$Z^{\triangle} = -Q - Z[I + \mu(t)Z(t)]^{-1}Z, \quad I + \mu(t)Z(t) > 0$$

on $[t_0, \infty)$. By the hypothesis of the theorem there exist $t_0 \leq a_0 < b_0$ such that $\mu(a_0) > 0, \mu(b_0) > 0$ and inequality (2.4) holds. Integrating both sides of the Riccati equation from a_0 to $t > a_0$ we obtain

$$\begin{aligned}
Z(t) &= Z(a_0) - \int_{a_0}^{t} Q(s)\triangle s - \int_{a_0}^{t} Z(s)[I + \mu(s)Z(s)]^{-1}Z(s)\triangle s \\
&= Z(a_0) - \int_{a_0}^{t} Q(s)\triangle s - \int_{a_0}^{\sigma(a_0)} Z(s)[I + \mu(s)Z(s)]^{-1}Z(s)\triangle s \\
&\quad - \int_{\sigma(a_0)}^{t} Z(s)[I + \mu(s)Z(s)]^{-1}Z(s)\triangle s \\
&= Z(a_0) - \mu(a_0)Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}Z(a_0) - \int_{a_0}^{t} Q(s)\triangle s \\
&\quad - \int_{\sigma(a_0)}^{t} Z(s)[I + \mu(s)Z(s)]^{-1}Z(s)\triangle s \\
&= Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}[I + \mu(a_0)Z(a_0) - \mu(a_0)Z(a_0)] \\
&\quad - \int_{a_0}^{t} Q(s)\triangle s - \int_{\sigma(a_0)}^{t} Z(s)[I + \mu(s)Z(s)]^{-1}Z(s)\triangle s \\
&= Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1} - \int_{a_0}^{t} Q(s)\triangle s \\
&\quad - \int_{\sigma(a_0)}^{t} Z(s)[I + \mu(s)Z(s)]^{-1}Z(s)\triangle s.
\end{aligned}$$

This implies that

$$Z(t) + \int_{a_0}^{t} Q(s)\triangle s \leq Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}. \tag{2.5}$$

Now let U be a unitary matrix (so $U^*U = I$) such that

$$Z(a_0) = U^*DU, \quad D = \operatorname{diag}(d_1, ..., d_n)$$

where $d_i = \lambda_i(Z(a_0))$ is the $i-th$ eigenvalue of $Z(a_0), i = 1, 2, ...$ Consider

$$\begin{aligned} Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1} &= U^*DU[I + \mu(a_0)U^*DU]^{-1} \\ &= U^*DU\{U^*[I + \mu(a_0)D]U\}^{-1} \\ &= U^*D[I + \mu(a_0)D]^{-1}U. \end{aligned}$$

Since $I + \mu(a_0)Z(a_0) > 0$ implies that

$$I + \mu(a_0)d_i > 0$$

and $h(x) = x/(1 + \mu(a_0)x)$ is increasing when $1 + \mu(a_0)x > 0$, we see that

$$\lambda_i(Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}) = \lambda_i(D[I + \mu(a_0)D]) = \frac{d_i}{1 + \mu(a_0)d_i}.$$

Using $h(x) = x/(1 + \mu(a_0)x) < 1/\mu(a_0)$ when $1 + \mu(a_0)x > 0$ we get

$$\lambda_i(Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}) = \frac{d_i}{1 + \mu(a_0)d_i} < \frac{1}{\mu(a_0)}. \tag{2.6}$$

Hence from equation (2.5) we obtain

$$\lambda_i(Z(t) + \int_{a_0}^{t} Q(s)\triangle s) \leq \lambda_i(Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}) < \frac{1}{\mu(a_0)}.$$

Applying Weyl's inequality we have

$$\frac{1}{\mu(a_0)} > \lambda_{max}(Z(t) + \int_{a_0}^{t} Q(s)\triangle s) \geq \lambda_{max}(\int_{a_0}^{t} Q(s)\triangle s) + \lambda_{min}(Z(t)).$$

If $t = b_0$, then

$$\lambda_{max}(\int_{a_0}^{b_0} Q(s)\triangle s) < \frac{1}{\mu(a_0)} - \lambda_{min}Z(b_0).$$

Since $I + \mu(b_0)Z(b_0) > 0$ implies $\lambda_{min}(Z(b_0)) > -1/\mu(b_0)$, it follows that

$$\lambda_{max}(\int_{a_0}^{b_0} Q(s)\triangle s) < \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)},$$

which is a contradiction.

Corollary 2.1.1. *Let $a \in \mathbb{T}$. If there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset [a, \infty)$ such that $\lim_{k\to\infty} t_k = \infty$ with $\mu(t_k) \geq K > 0$ for some $K > 0$ and such that*

$$\lim_{t_k \to \infty} \sup \lambda_{\max}(\int_a^{t_k} Q(s)\triangle s) = +\infty, \tag{2.7}$$

then (2.3) *is oscillatory.*

Proof Let $t_0 \in [a, \infty)$. Choose k_0 sufficiently large so that $a_0 = t_{k_0} \in [t_0, \infty)$. Using (2.7), we can pick $k_1 > k_0$ sufficiently large so that with $b_0 = t_{k_1}$ we have

$$\lambda_{\max}(\int_{a_0}^{b_0} Q(s)\triangle s) \geq \frac{2}{K} \geq \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}.$$

Theorem 2.1.2 ([6]). *Let $a \in \mathbb{T}$. Suppose that there is a strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset [a, \infty)$ such that $\mu(t_k) > 0$ for $k \in \mathbb{N}$, with $\lim_{k\to\infty} t_k = \infty$. Further assume that there is a sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset [a, \infty)$ such that $\sigma(\tau_k) > \tau_k \geq \sigma(t_k)$ for $k \in \mathbb{N}$ with*

$$\lambda_{\min}(\frac{P(t_k)}{\mu(t_k)} + \frac{P(\tau_k)}{\mu(\tau_k)} - \int_{t_k}^{\tau_k} Q(s)\triangle s) \leq 0$$

for $k \in \mathbb{N}$. Then (2.2) *is oscillatory on* $[a, \infty)$.

Proof Assume (2.2) is nonoscillatory on $[a, \infty)$. Then there is a prepared solution X(t) of (2.2) and a $t_0 \in [a, \infty)$ such that

$$X^*(\sigma(t))P(t)X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) = P(t)X^{\triangle}(t)X^{-1}(t)$$

for $t \in [a, \infty)$. Then by Theorem 1.6.4 we get that

$$P(t) + \mu(t)Z(t) > 0$$

on $[t_0, \infty)$ and Z(t) is a Hermitian solution of the Riccati equation $RZ = 0$ on $[t_0, \infty)$. Let $\{t_k\}, \{\tau_k\}$ be the sequences given in the statement of this theorem. Pick a fixed integer k so that $t_k \geq t_0$. Integrating both sides of the Riccati equation from t_k to τ_k we obtain

$$\begin{aligned}
Z(\tau_k) &= Z(t_k) - \int_{t_k}^{\tau_k} Q(t)\triangle t - \int_{t_k}^{\tau_k} F(t)\triangle t \quad \text{where,} \quad F = Z^*(P + \mu Z)^{-1}Z \\
&= Z(t_k) - \int_{t_k}^{\tau_k} Q(t)\triangle t - \int_{t_k}^{\sigma(t_k)} F(t)\triangle t - \int_{\sigma(t_k)}^{\tau_k} F(t)\triangle t \\
&= Z(t_k) - F(t_k)\mu(t_k) - \int_{t_k}^{\tau_k} Q(t)\triangle t - \int_{\sigma(t_k)}^{\tau_k} F(t)\triangle t \\
&= Z(t_k) - \mu(t_k)Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}Z(t_k) \\
&\quad - \int_{t_k}^{\tau_k} Q(t)\triangle t - \int_{\sigma(t_k)}^{\tau_k} F(t)\triangle t \\
&= Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}[P(t_k) + \mu(t_k)Z(t_k) - \mu(t_k)Z(t_k)] \\
&\quad - \int_{t_k}^{\tau_k} Q(t)\triangle t - \int_{\sigma(t_k)}^{\tau_k} F(t)\triangle t \\
&= Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(t)\triangle t \\
&\quad - \int_{\sigma(t_k)}^{\tau_k} F(t)\triangle t \\
&\leq Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(t)\triangle t.
\end{aligned}$$

But now we get that

$$\begin{aligned} Z(t_k)[P(t_k)+\mu(t_k)Z(t_k)]^{-1}P(t_k) &= Z(t_k)[X(t_k)X^{-1}(\sigma(t_k))P^{-1}(t_k)]P(t_k) \\ &= P(t_k)X^{\triangle}(t_k)X^{-1}(\sigma(t_k)) \\ &= \frac{P(t_k)}{\mu(t_k)}[X^{\sigma}(t_k)) - X(t_k)]X^{-1}(\sigma(t_k)) \\ &= \frac{P(t_k)}{\mu(t_k)} - \frac{1}{\mu(t_k)}P(t_k)X(t_k)X^{-1}(\sigma(t_k)) \\ &< \frac{P(t_k)}{\mu(t_k)} \end{aligned}$$

Hence, from above we have

$$Z(\tau_k) < \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{\tau_k} Q(t)\triangle t.$$

Using $P(\tau_k) + \mu(\tau_k)Z(\tau_k) > 0$, we have finally

$$\frac{P(\tau_k)}{\mu(\tau_k)} + \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{\tau_k} Q(t)\triangle t > 0,$$

which is a contradiction.

Corollary 2.1.2. *Let $a \in \mathbb{T}$. A necessary condition for* (2.2) *to be nonoscillatory on $[a,\infty)$ is that for any strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset [a,\infty)$ such that $\mu(t_k) > 0$ for $k \in \mathbb{N}$, with $\lim_{k\to\infty} t_k = \infty$, there is $N \in \mathbb{N}$ such that*

$$D_k := \frac{P(t_k)}{\mu(t_k)} + \frac{P(t_{k+1})}{\mu(t_{k+1})} - \int_{t_k}^{t_{k+1}} Q(s)\triangle s > 0 \quad \textit{for} \quad k \geq N.$$

Corollary 2.1.3. *Let $a \in \mathbb{T}$. Assume that there is a strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset [a,\infty)$ such that $\mu(t_k) > 0$ for $k \in \mathbb{N}$, with $\lim_{k\to\infty} t_k = \infty$. Furthermore assume that there are sequences $\{s_k\}_{k\in\mathbb{N}} \subset [a,\infty)$ and $\{\tau_k\}_{k\in\mathbb{N}} \subset [a,\infty)$ such that $\sigma(\tau_k) > \tau_k \geq \sigma(s_k) > s_k \geq \sigma(t_k), k \in \mathbb{N}$, with*

$$\int_{t_k}^{s_k} Q(t)\triangle t \geq \frac{P(t_k)}{\mu(t_k)}$$

and

$$\lambda_{\min}(\frac{P(\tau_k)}{\mu(\tau_k)} - \int_{s_k}^{\tau_k} Q(t)\triangle t) \leq 0$$

for $k \in \mathbb{N}$. *Then* (2.2) *is oscillatory on* $[a, \infty)$.

Proof It follows from Weyl's inequality that if A and B are Hermitian matrices, then

$$\lambda_{\min}(A - B) \leq \lambda_{\min}(A) - \lambda_{\min}(B).$$

We use this fact in the following chain of inequalities. Consider

$$\begin{aligned} \lambda_{\min} \quad (& \frac{P(t_k)}{\mu(t_k)} + \frac{P(\tau_k)}{\mu(\tau_k)} - \int_{t_k}^{\tau_k} Q(t)\triangle t) \\ = & \lambda_{\min}([\frac{P(\tau_k)}{\mu(\tau_k)} - \int_{s_k}^{\tau_k} Q(t)\triangle t] - [\int_{t_k}^{s_k} Q(t)\triangle t - \frac{P(t_k)}{\mu(t_k)}]) \\ \leq & \lambda_{\min}(\frac{P(\tau_k)}{\mu(\tau_k)} - \int_{s_k}^{\tau_k} Q(t)\triangle t) - \lambda_{\min}(\int_{t_k}^{s_k} Q(t)\triangle t - \frac{P(t_k)}{\mu(t_k)}) \\ \leq & 0. \end{aligned}$$

Hence the result follows from Theorem 2.1.2.

Corollary 2.1.4. *Let* $a \in \mathbb{T}$. *Suppose also that*

$$\lim_{t\to\infty} \lambda_{\min}(\int_{t_0}^{t} Q(s)\triangle s) = \infty \tag{2.8}$$

and that for each $T \in [a, \infty)$ *there is* $t \in [T, \infty)$ *such that* $\mu(t) > 0$ *and*

$$\lambda_{\min}(\frac{P(t)}{\mu(t)} - \int_{T}^{t} Q(s)\triangle s) \leq 0. \tag{2.9}$$

Then (2.2) *is oscillatory on* $[a, \infty)$.

Theorem 2.1.3 ([6]). *Let* $a \in \mathbb{T}$. *Suppose for each* $t_0 \geq a$ *there is a strictly increasing sequence* $\{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ *with* $\mu(t_k) > 0$ *and* $\lim_{k\to\infty} t_k = \infty$, *and* *that there are constants* K_1 *and* K_2 *such that* $0 < K_1 \leq \mu(t_k) \leq K_2$ *for* $k \in \mathbb{N}$ *with*

$$\lim_{k\to\infty} \lambda_{\max}(\int_{t_1}^{t_k} Q(t)\triangle(t)) \geq \frac{1}{\mu(t_1)}.$$

Further assume that there is a constant M *such that*

$$tr(\int_{t_1}^{t_k} Q(t)\triangle t) \geq M \quad for \quad k \in \mathbb{N}.$$

Then (2.3) *is oscillatory on* $[a, \infty)$.

Proof Assume (2.3) is nonoscillatory on $[a, \infty)$. This implies that there is a nontrivial prepared solution $X(t)$ of (2.3) and $t_0 \in [a, \infty)$ such that

$$X^*(\sigma(t))X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) = X^{\triangle}(t)X^{-1}(t)$$

for $t \in [t_0, \infty)$, then by Theorem 1.6.4 we get that

$$I + \mu(t)Z(t) > 0 \quad \text{for} \quad t \geq t_0,$$

and Z(t) is a Hermitian solution of the Riccati equation

$$Z^{\triangle}(t) + Q(t) + F(t) = 0$$

on $[t_0, \infty)$, where $F = Z^*(P + \mu Z)^{-1}Z$. Corresponding to t_0 let $\{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ be the sequence guaranteed in the statement of this theorem . Integrating both sides of the Riccati equation from t_1 to t_k , where $k > 1$ gives

$$\begin{aligned} Z(t_1) &= Z(t_k) + \int_{t_1}^{t_k} Q(t)\triangle(t) + \int_{t_1}^{t_k} F(t)\triangle t \\ &= Z(t_k) + \int_{t_1}^{t_k} Q(t)\triangle t + \int_{t_1}^{\sigma(t_1)} F(t)\triangle t + \int_{\sigma(t_1)}^{t_k} F(t)\triangle t \\ &= Z(t_k) + \int_{t_1}^{t_k} Q(t)\triangle t + F(t_1)\mu(t_1) + \int_{\sigma(t_1)}^{t_k} F(t)\triangle t \end{aligned}$$

Simplifying we get

$$Z(t_k) + \int_{t_1}^{t_k} Q(t)\triangle t + \int_{\sigma(t_1)}^{t_k} F(t)\triangle t = Z(t_1)(I + \mu(t_1)Z(t_1))^{-1}Z(t_1).$$

Hence,

$$\begin{aligned} \lambda_{\max} & \left(Z(t_k) + \int_{t_1}^{t_k} Q(t)\triangle t + \int_{\sigma(t_1)}^{t_k} F(t)\triangle t \right) \\ &= \lambda_{\max}(Z(t_1)(I + \mu(t_1)Z(t_1))^{-1})Z(t_1). \end{aligned} \tag{2.10}$$

By Weyl's inequality

$$\begin{aligned} \lambda_{\max} & (Z(t_1)(I + \mu(t_1)Z(t_1))^{-1}Z(t_1)) \\ &> \lambda_{\min}(Z(t_k)) + \lambda_{\max}(\int_{t_1}^{t_k} Q(t)\triangle t) + \lambda_{\min}(\int_{\sigma(t_1)}^{t_k} F(t)\triangle t) \\ &\geq \lambda_{\min}(Z(t_k)) + \lambda_{\max}(\int_{t_1}^{t_k} Q(t)\triangle t). \end{aligned} \tag{2.11}$$

Taking the limit of both sides as $k \to \infty$ and using

$$\lim_{k\to\infty} \lambda_{\min}(Z(t_k)) = 0, \tag{2.12}$$

which we will prove later, we get

$$\begin{aligned}\lambda_{\max} \quad (& \quad Z(t_1)(I+\mu(t_1)Z(t_1))^{-1}) \\ \geq & \quad \lim_{k\to\infty} \lambda_{\max}(\int_{t_1}^{t_k} Q(t)\triangle t) \\ \geq & \quad \frac{1}{\mu(t_1)}.\end{aligned}$$

But in the proof of Theorem 2.1.1 we show that $I + \mu(t_1)Z(t_1) > 0$ implies that

$$\lambda_{\max}(Z(t_1)(I+\mu(t_1)Z(t_1))^{-1}) < \frac{1}{\mu(t_1)}, \tag{2.13}$$

and this is a contradiction. Hence to complete the proof of this theorem it remains to prove that (2.12) holds. In fact , we shall prove that

$$\lim_{k\to\infty} \lambda_i(Z(t_k)) = 0$$

for $1 \leq i \leq n$, which includes (2.12) as a special case. To do this we first show that

$$\lim_{k\to\infty} \lambda_i(F(t_k)) = 0$$

for $1 \leq i \leq n$. Since $F(t) \geq 0$ implies $\mathrm{tr}(F(t)) \geq 0$ we have

$$\begin{aligned}\sum_{j=1}^{k} \mu(t_j)\lambda_i(F(t_j)) \quad & \leq \quad \sum_{j=1}^{k} \mu(t_j)\mathrm{tr}(F(t_j)) \\ & = \quad \sum_{j=1}^{k} \int_{t_j}^{\sigma(t_j)} \mathrm{tr}(F(t))\triangle t \\ & \leq \quad \int_{t_1}^{\sigma(t_k)} \mathrm{tr}(F(t))\triangle t \\ & \leq \quad \mathrm{tr}(\int_{t_1}^{\sigma(t_k)} F(t)\triangle t) \\ & \leq \quad n\lambda_{\max}(\int_{t_1}^{\sigma(t_k)} F(t)\triangle t)\end{aligned}$$

for all $k > 1$. We now show that the sequence

$$\lambda_{\max}(\int_{t_1}^{t_k} F(t)\triangle t),$$

$k \geq 1$ is bounded. From (2.11) and (2.13) we get

$$\frac{1}{\mu(t_1)} > \lambda_{\min}(Z(t_k)) + \lambda_{\max}(\int_{t_1}^{t_k} Q(t)\triangle t).$$

Using (2.10) and (2.13) and applying Weyl's inequality twice yields

$$\begin{aligned} \frac{1}{\mu(t_1)} &> \lambda_{\max}(Z(t_k) + \int_{t_1}^{t_k} Q(t)\triangle t \\ &+ \int_{\sigma(t_1)}^{t_k} F(t)\triangle t) \\ &\geq \lambda_{\min}(Z(t_k)) + \lambda_{\min}(\int_{t_1}^{t_k} Q(t)\triangle t) \\ &+ \lambda_{\max}(\int_{\sigma(t_1)}^{t_k} F(t)\triangle t). \end{aligned}$$

It follows that

$$\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)} > \lambda_{\min}(\int_{t_1}^{t_k} Q(t)\triangle t) + \lambda_{\max}(\int_{\sigma(t_1)}^{t_k} F(t)\triangle t). \tag{2.14}$$

From Theorem 2.1.1, we can, without loss of generality, assume that

$$\lambda_{\max}(\int_{t_1}^{t_k} Q(t)\triangle t) < \frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)} \tag{2.15}$$

holds for $k > 1$. Therefore,

$$\begin{aligned} M &\leq \operatorname{tr}(\int_{t_1}^{t_k} Q(t)\triangle t) \\ &= \sum_{i=1}^{n} \lambda_i(\int_{t_1}^{t_k} Q(t)\triangle t) \\ &= \lambda_{\min}(\int_{t_1}^{t_k} Q(t)\triangle t) + \sum_{i=1}^{n-1} \lambda_i(\int_{t_1}^{t_k} Q(t)\triangle t) \\ &< \lambda_{\min}(\int_{t_1}^{t_k} Q(t)\triangle t) + (n-1)(\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}). \end{aligned}$$

Solving for the last term on the right-hand side of (2.14) and using the above inequality we get

$$\lambda_{\max}(\int_{\sigma(t_1)}^{t_k} F(t)\triangle t) < n(\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}) - M.$$

Therefore,

$$\sum_{j=1}^{\infty} \mu(t_j)\operatorname{tr}(F(t_j)) < \infty$$

and so since

$$\lambda_i(F(t)) \leq \operatorname{tr}(F(t))$$

and $0 < K_1 \leq \mu(t_k) \leq K_2$ for $k = 1, 2, ...$, it follows that

$$\lim_{k\to\infty} \lambda_i(F(t_k)) = 0.$$

That is

$$\lim_{k\to\infty} \frac{(\lambda_i[Z(t_k)])^2}{1 + \mu(t_k)\lambda_i[Z(t_k)]} = 0$$

for $i = 1, 2, ..., n$. Similar to the argument to prove (2.6) in Theorem 2.1.1 we can show

$$\lambda_i(F(t_k)) = \frac{d_i^2}{1 + \mu(t_k)d_i}. \tag{2.16}$$

If we consider only the case $\lambda_i(Z(t_k)) \geq 0$, then

$$0 \leq \frac{(\lambda_i[Z(t_k)])^2}{1 + K_2\lambda_i[Z(t_k)]} \leq \frac{(\lambda_i[Z(t_k)])^2}{1 + \mu(t_k)\lambda_i[Z(t_k)]},$$

which implies

$$\lim_{k\to\infty} \frac{(\lambda_i[Z(t_k)])^2}{1 + K_2\lambda_i[Z(t_k)]} = 0$$

for $i = 1, 2, ..., n$, which in turn implies that

$$\lim_{k\to\infty} \lambda_i(Z(t_k)) = 0$$

for $i = 1, 2, ..., n$. This completes the proof.

2.2 New oscillation criteria

We are concerned with the oscillation of nonlinear matrix dynamic equation of the form

$$[P(t)X^{\triangle}]^{\triangle} + F(t, X, X^{\triangle})X^{\sigma} = 0 \tag{2.17}$$

where $X = (x_{ij})$, $F = (f_{ij})$ and P are $n \times n$ matrices. By $F = F(t, X, X^{\triangle})$ is meant $f_{ij} = f_{ij}(t, x_{11}, ..., x_{nn}, x_{11}^{\triangle}, ..., x_{nn}^{\triangle})$. The functions f_{ij} are assumed to be continuous for t on $[a, \infty), a \geq 0$, and for all values of the remaining variables. The matrix $F(t, X, X^{\triangle})$ is symmetric and positive definite for every t on $[a, \infty)$ and every X with $\det X \neq 0$, while the matrix $P(t)$ is continuous, symmetric and positive definite for every t on $[a, \infty)$.

Remark: If $\mathbb{T} = \mathbb{R}$ then the results in this section were proved by TOMASTIK [[17]].

Lemma 2.2.1. *If* $X(t)$ *is a prepared matrix solution of* (2.17) *such that* $\det X(t) \neq 0$ *on some interval* (b, ∞), *then det*$X^{\triangle}(t) \neq 0$ *on some interval* $[c, \infty), c > b$. *Fur-*

thermore, if $Z(t) = P(t)X^{\triangle}(t)X^{-1}(t)$*, then*

$$\begin{aligned} Z(t) &= Z(c) - \int_c^t F[x, X(x), X^{\triangle}(x)]\triangle x \\ &- \int_c^t Z(x)P^{-1}(x)Z(x)[P + \mu Z]^{-1}(x)P(x)\triangle x \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} Z^{-1}(t) &= Z^{-1}(c) + \int_c^t Z^{-1}F[x, X(x), X^{\triangle}(x)](Z^{\sigma}(x))^{-1}\triangle x \\ &+ \int_c^t \{P^{-1}Z[P + \mu Z]^{-1}P(Z^{\sigma})^{-1}\}(x)\triangle x \end{aligned} \tag{2.19}$$

for all $t \geq c$*.*

Proof Since $\det X(t) \neq 0$ on (b, ∞), $Z(t)$ is well defined on (b, ∞). Since $X(t)$ is a prepared solution i.e.,

$$X^* P X^{\triangle} = (X^{\triangle})^* P X, \tag{2.20}$$

it follows that $Z(t)$ is also symmetric. If we take the derivative of $Z(t) = P(t)X^{\triangle}(t)X^{-1}(t)$, we get

$$Z^{\triangle} = (PX^{\triangle})^{\triangle}(X^{-1})^{\sigma} + PX^{\triangle}(X^{-1})^{\triangle}.$$

Using (2.17), we have

$$\begin{aligned} Z^{\triangle} &= -FX^{\sigma}(X^{-1})^{\sigma} - PX^{\triangle}X^{-1}X^{\triangle}(X^{\sigma})^{-1} \\ &= -F - ZX^{\triangle}(X^{\sigma})^{-1} \\ &= -F - ZP^{-1}ZX(X^{\sigma})^{-1} \\ &= -F - ZP^{-1}Z(X^{\sigma}X^{-1})^{-1} \end{aligned}$$

Since $X^{\sigma}X^{-1} = I + (X^{\sigma} - X)X^{-1}$, we may write

$$\begin{aligned} Z^{\triangle} &= -F - ZP^{-1}Z[I + (X^{\sigma} - X)X^{-1}]^{-1} \\ &= -F - ZP^{-1}Z[I + \mu X^{\triangle}X^{-1}]^{-1} \\ &= -F - ZP^{-1}Z[I + \mu P^{-1}PX^{\triangle}X^{-1}]^{-1} \\ &= -F - ZP^{-1}Z[I + \mu P^{-1}Z]^{-1} \\ &= -F - ZP^{-1}Z\{P^{-1}(P + \mu Z)\}^{-1} \\ &= -F - ZP^{-1}Z[P + \mu Z]^{-1}P. \end{aligned}$$

Therefore,

$$\begin{aligned} Z^{\triangle}(t) &= -F[t, X(t), X^{\triangle}(t)] \\ &- Z(t)P^{-1}(t)Z(t)[P(t) + \mu(t)Z(t)]^{-1}P(t) \end{aligned} \tag{2.21}$$

The right hand-side of (2.21) is negative definite since F is positive definite and $ZP^{-1}Z = Z^{*}P^{-1}Z$ is positive semidefinite. Each characteristic root of $Z(t)$, then is strictly decreasing and $\det Z(t)$ can vanish at most n times. Then there exists $c > b$ such that $Z(t)$ and thus $X^{\triangle}(t)$ is not singular on $[c, \infty)$. If we take the derivative of the both sides of $Z^{-1}Z = I$, then we get

$$(Z^{-1})^{\triangle}Z^{\sigma} = -Z^{-1}Z^{\triangle}.$$

By using (2.21), we have

$$\begin{aligned} (Z^{-1})^{\triangle} &= -Z^{-1}Z^{\triangle}(Z^{\sigma})^{-1} \\ &= Z^{-1}F(Z^{\sigma})^{-1} + Z^{-1}ZP^{-1}Z[P + \mu Z]^{-1}P(Z^{\sigma})^{-1} \end{aligned} \tag{2.22}$$

If we integrate both sides of (2.22), then

$$\begin{aligned} Z^{-1}(t) &= Z^{-1}(c) + \int_c^t Z^{-1}F(Z^{\sigma})^{-1}\triangle x \\ &+ \int_c^t P^{-1}Z[P + \mu Z]^{-1}P(Z^{\sigma})^{-1}\triangle x. \end{aligned}$$

Clearly, (2.18) and (2.19) follow from (2.21) and (2.22), respectively.

Theorem 2.2.1. *Let* $P(t) = I$*, the identity matrix. If*

$$\lambda_{max}[\int_a^\infty F(t, A(t), A^\triangle(t))\triangle t] = \infty,$$

for every differentiable matrix $A(t)$ *such that* $\lambda_{min}[A^*(t)A(t)] \geq \epsilon > 0$ *for large* t*, then* (2.17) *is oscillatory.*

Proof Assume on the contrary that (2.17) is not oscillatory. Therefore, there exists $b \geq a$ and a prepared matrix solution $X(t)$ of (2.17) such that $\det X(t) \neq 0$ on (b, ∞). In view of Lemma 2.21, (2.18) and (2.19) are then satisfied. Using Weyl inequality, we see that

$$\begin{aligned}
\lambda_{max}\{Z^{-1}(t)\} &= \lambda_{max}\{Z^{-1}(c) + \int_c^t Z^{-1}F)(Z^\sigma)^{-1}\triangle x \\
&+ \int_c^t P^{-1}Z[P + \mu Z]^{-1}P(Z^\sigma)^{-1}\triangle x\} \\
&\geq \lambda_{min}\{Z^{-1}(c)\} + \lambda_{max}\{\int_c^t Z^{-1}F(Z^\sigma)^{-1}\triangle x\} \\
&+ \lambda_{min}\{\int_c^t P^{-1}Z[P + \mu Z]^{-1}P(Z^\sigma)^{-1}\triangle x\}
\end{aligned}$$

Hence;

$$\begin{aligned}
\lambda_{max}\{Z^{-1}(t)\} &\geq \lambda_{min}\{Z^{-1}(c)\} \\
&+ \lambda_{max}\{\int_c^t Z^{-1}F(Z^\sigma)^{-1}\triangle x\} \qquad (2.23)
\end{aligned}$$

Since the first integral in (2.19) is positive definite, all characteristic roots $Z^{-1}(t)$ have limits equal to $+\infty$. Thus $Z(t)$ is positive definite for large t and

$$\lim Z(t) = 0$$

Since $\det X(t) \neq 0$ on $[c, \infty)$, we can consider $X(t)$ to be a matrix solution of the linear equation

$$X^{\triangle}(t) = Z(t)X(t)$$

Then, we get

$$\begin{aligned}
[X^*X]^{\triangle} &= (X^{\triangle})^* X^{\sigma} + X^* X^{\triangle} \\
&= (X^{\triangle})^* X^{\sigma} + X^* Z X^{\triangle} \\
&= X^* Z X^{\sigma} + X^* Z X \qquad (2.24)
\end{aligned}$$

which is positive definite for large t since $Z(t)$ is positive definite for large t and $detX(t) \neq 0$. Thus each characteristic root of $X^*(t)X(t)$ is strictly increasing and $\lambda_{min}[X^*(t)X(t)] \geq \epsilon > 0$ for some ϵ and for large t. By our hypothesis, we conclude that

$$\lambda_{max}[\int_a^{\infty} F \triangle t] = \infty$$

If we use this in (2.18), we see that $Z(t)$ must be negative definite , which is a contradiction. Hence, (2.17) is oscillatory.

Corollary 2.2.1. *Suppose $F(t, U, U^{\triangle}) = U^{*k}Q(t)U^k$ where $Q(t)$ is positive definite on $[a, \infty)$ and k is some nonnegative integer. If the matrix $\int_a^t Q(x)\triangle x$ is unbounded, then* (2.17) *is oscillatory.*

Assume that $A(t)$ is differentiable matrix such that $\lambda_{min}[A^*(t)A(t)] \geq \epsilon > 0$ for large t. This implies immediately that $\lambda_{max}[A^{*k}(t)A^k(t)] \geq \epsilon^k$ for large t. Since $Q(t)$ is positive definite, it follows readily from the Courant-Hilbert min-max theorem that

$$\lambda_{max}[A^{*k}(t)Q(t)A^k(t)] \geq \epsilon^k \lambda_{max}[Q(t)] \qquad (2.25)$$

for large t. Since $Q(t)$ is positive definite and $\int_a^t Q(x)\triangle x$ is unbounded, $\lambda_{max}[\int_a^{\infty} Q(x)\triangle x] = \infty$. Now using this fact and inequality in (2.25) and Theorem 2.2.1, we conclude that $\lambda_{max}[\int_a^{\infty} A^{*k}(t)Q(t)A^k(t)]\triangle t = \infty$. Therefore (2.17) is oscillatory.

Theorem 2.2.2. *If F satisfies the same hypothesis as in Theorem* 2.2.1 *and if*

$P(t) = p(t)I$, where $p(t)$ is a positive scalar function such that $\int_a^\infty p^{-1}\triangle x = \infty$, then (2.17) *is oscillatory.*

The proof is the same as for 2.2.1, except that (2.24) becomes

$$\begin{aligned}[X^*X]^\triangle &= (X^\triangle)^*X^\sigma + X^*X^\triangle \\ &= [P^{-1}ZX]^*X^\sigma + X^*[P^{-1}ZX] \\ &= X^*ZP^{-1}X^\sigma + X^*P^{-1}ZX \\ &= X^*[ZP^{-1}X^\sigma X^{-1} + P^{-1}Z]X. \end{aligned} \tag{2.26}$$

Now, $ZP^{-1}X^\sigma X^{-1} + P^{-1}Z$ positive definite and so the proof proceeds as before.

Theorem 2.2.3. *Suppose that for every differentiable matrix $A(t)$ with* $\det A(t) \neq 0$ *for large t, we have*

$$\lambda_i[\int_a^\infty F[t, A(t), A^\triangle(t)]\triangle t] = \infty, \quad i = 1, ..., p \tag{2.27}$$

and

$$\lambda_i[\int_a^\infty P^{-1}(t)\triangle t] = \infty, \quad i = 1, ..., r \tag{2.28}$$

If $r + p > n$, then (2.17) *is oscillatory.*

Assume that (2.17) is not oscillatory. Then there exists $b \geq a$ and a prepared matrix solution $U(t)$ of (2.17) such that $\det U(t) \neq 0$ on (b, ∞). Lemma 2.2.1. applies and so (2.21) and (2.22) are satisfied. Using the Weyl inequality, we see from (2.18)

$$\begin{aligned}\lambda_i(Z(t)) &\leq \lambda_i(Z(c)) - \lambda_{min}\{\int_c^t F\triangle x\} \\ &- \lambda_{min}\{\int_c^t ZP^{-1}Z[P + \mu Z]^{-1}P\triangle x\}\end{aligned}$$

Using (2.27) that $Z(t)$ has p characteristic roots whose limits are $-\infty$. From (2.19) using (2.28), we see that $Z^{-1}(t)$ has r characteristic roots whose limits are ∞. This contradicts the fact that $r + p > n$. Hence, (2.17) is oscillatory.

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