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NUMERICAL MODELING OF RE-SUSPENSION AND TRANSPORT OF FINE SEDIMENTS IN COASTAL WATERS

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NUMERICAL MODELING OF RE-SUSPENSION AND TRANSPORT OF FINE SEDIMENTS IN COASTAL WATERS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science/Arts/Doctor of Philosophy.

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ABSTRACT

NUMERICAL MODELING OF RE-SUSPENSION AND TRANSPORT OF FINE SEDIMENTS IN COASTAL WATERS

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In this thesis, the theory of three dimensional numerical modeling of transport and re-suspension of fine sediments is studied and a computer program is developed for simulation of the three dimensional suspended sediment transport. The computer program solves the three dimensional advection-diffusion equation simultaneously with a computer program prepared earlier for the simulation of three dimensional current systems. This computer program computes the velocity vectors, eddy viscosities and water surface elevations which are used as inputs by the program of fine sediment transport. The model is applied to Bay of Izmir for different wind conditions.

Keywords: Mathematical Model, numerical model, finite difference, finite element, sediment transport

ÖZ

KIYI SULARINDA İNCE SEDİMENT TAŞINMININ VE ASILI HALE GELMESİNİN MATEMATİKSEL MODELLENMESİ

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Bu tez çalışmasında, ince sediment taşınımının ve askı durumuna gelmesinin üç boyutlu modelleme kuramı incelenmiş ve üç boyutlu ince sediment taşınımını öykünen bir bilgisayar programi hazırlanmıştır. Bu bilgisayar programı üç boyutlu ilerlemeli yayılma denklemini; üç boyutlu akıntıların hesaplanması için daha önceden hazırlanmış olan bir bilgisayar programı ile birlikte eş zamanlı olarak çözmektedir. Bu bilgisayar programı, sediment taşınımı programına girdi olarak kullanılan akıntı hız vektörlerini, iç sürtünme katsayılarını ve su yüzeyi değişimlerini hesaplamaktadır. Model eğişik rüzgar durumları için İzmir körfezine uygulanmıştır.

Anahtar Sözcükler: Matematiksel modelleme, sayısal modelleme, sonlu farklar, sonlu elemanlar, sediment taşınımı

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LIST OF SYMBOLS

B	:	Buoyancy production of the kinetic energy,
C	:	Suspended sediment concentration,
$C_{1\varepsilon}$	:	A universal empirical constant ($C_{1\varepsilon}$ = 1.44),
$C_{2\varepsilon}$	:	A universal empirical constant ($C_{2\varepsilon}$ = 1.92),
$C_{3\varepsilon}$	:	A non-universal empirical constant
C_d	:	Drag coefficient of air
C_f	:	Empirical coefficient for bottom friction
D, E	:	Sediment transport rates through deposition and re-suspension, respectively
D_h	:	Horizontal eddy diffusivity
D_s	:	Representative diameter of particles,
D_x, D_y, D_z	:	Turbulent diffusion coefficient in x, y and z directions, respectively
f	:	Corriolis coefficient,
g	:	Gravitational acceleration,
$h(x,y)$	:	Water depth measured from the undisturbed water surface to the sea bed
k	:	Kinetic energy,
P	:	Stress production due to the kinetic energy,
p	:	Pressure,
P_r	:	Turbulent Prandatl or Schmidth number
t	:	Time,
s	:	Ratio of densities of particles and water,
u, v, w	:	Velocity components in x, y and z directions at any grid locations in space, respectively

u_b, v_b	:	Horizontal velocity components at the grid point nearest to the sea
u_w,v_w	:	The wind velocity components (m/s) in x and y directions,
u_{*s}	:	Surface shear velocity,
w_s	:	Settling Velocity
x,y	:	Horizontal coordinates,
z	:	Vertical coordinate,
Δz_s	:	Distance from the surface to the first grid point below
ε	:	Rate of dissipation of kinetic energy,
η	:	Water surface elevation.
κ	:	von Karman constant ($\kappa = 0.42$),
ρ	:	*In situ* water density
ρ_o	:	Reference Density,
ρ_a	:	Air density,
σ_k	:	A universal empirical constant ($\sigma_k = 1$),
σ_ε	:	A universal empirical constant ($\sigma_\varepsilon = 1.3$),
υ	:	Kinematic molecular viscosity of water,
υ_h	:	Horizontal eddy viscosity
$\upsilon_x, \upsilon_y, \upsilon_z$	:	Eddy viscosity coefficients in x, y and z directions respectively,
υ_z	:	Vertical eddy viscosity
τ_{bx}, τ_{by}	:	The bottom shear stress components,
τ_d	:	Critical shear stress for deposition
τ_e	:	Critical shear stress for re-suspension

CHAPTER I

INTRODUCTION

1.1 Description of the Problem

Re-suspension and transport of fine sediment by waves and currents in estuarine and coastal seas is an important phenomenon for many reasons as they play a critical role in the functionality and health of these systems. Firstly the process may affect the quality of water column because of its links to biological and chemical processes. When bottom sediment is re-suspended, trace metals, nutrients and organic contaminants can be released into the water column, which in turn can limit the amount of light entering the water, reduce the water quality and suitability of habitats to numerous species. Also, with growing awareness of navigational and flooding problems, mechanism of sediment transport in estuarine and coastal seas has received considerable attention in recent years as it determines seabed morphology. The increased suspended sediment into the estuarine ecosystem may cause enormous economic burden for local communities for rectifying the situation and/or preserving the environment in a suitable state.

Numerical models that simulate transport of sediment are being constructed primarily to aid the development of management strategies as they predict distribution of concentrations of suspended sediments and their transport and fate in estuaries and coastal waters.

1.2 Scope and Extent of the Present Study

In this study, a three dimensional suspended sediment transport model is developed. The processes that the suspended sediment transport model considers are advective and diffusive transport, settling, deposition and re-suspension. Also the model may be advanced in order to take into account the processes of consolidation and flocculation.

Modeling suspended sediment transport correctly requires a chain of modules forming an integrated model. The suspended sediment transport model is utilized as a sub-model together with a comprehensive three dimensional hydrodynamic model that simulates sea currents and transport processes. The computer program of the hydrodynamic model (Balas and Ozhan 2001) is advanced by this study to include suspended sediment transport terms.

The hydrodynamic model computes sea level variations and water particle velocity distributions after simulating temperature and salinity variations and flows induced by wind, sea level differences and density gradients by solving the Navier-Stokes equations with the hydrostatic pressure distribution assumption and the Boussinesq approximation. As the turbulence model, the values of eddy viscosities and dispersion coefficients can be taken as constant or calculated utilizing the two-equation k-ε turbulence model. Hydrodynamic action is the most important mechanism involved in sediment transport. It advects the suspended sediments, provides the force needed to re-suspend from the bed and, through turbulence; plays a major role in the flocculation of cohesive sediments.

In this study, three-dimensional advection diffusion is solved simultaneously with three dimensional hydrodynamic model using implicit finite difference approximations. Velocity fields (distribution of advective velocities), eddy viscosities (and dispersion coefficients) and bottom shear stresses are provided from the hydrodynamic model over the period of time considered for computations of the suspended sediment transport model.

The hydrodynamic and suspended sediment transport models are applied to the Bay of Izmir for the prediction of circulation patterns, distribution of suspended sediments and the sites of deposition and re-suspension. Izmir Bay is located at the west Anatolia–eastern Aegean Sea. Izmir Bay has a highly disturbed environment due to the rapid increase of the population and development of industry. Because of the untreated domestic and industrial wastes, atmospheric pollution, agricultural pollution, shipping, dredging activities in the harbor and the disposal of the dredged material to the outer bay, especially in the bay where the city of Izmir is located, the water quality in whole bay is seriously endangered.

As a result, the computer program developed in this study is capable of computing the distribution of suspended sediment concentration over horizontal plane and along water depth, identifying re-suspension, deposition and equilibrium sites of the study area and evaluating possible changes in the sea bed morphology. Therefore, it is a predictive three dimensional model that can be used for contributing to management policies and implementations.

CHAPTER II

LITERATURE SURVEY

Although flow structures and sediment circulation in coastal waters and estuaries are complex phenomena due to vertical gradients of suspended sediment generated from the settling velocity and the water–bottom interaction and only a three dimensional model seems to be appropriate to simulate their motions and patterns, most applications have been restricted to transport processes in one dimension (1D) or in two-dimensions (2D) until recently because of the complexity of the computations.

One dimensional models have been frequently used to simulate sediment transport and large-scale morphological changes in rivers. Two dimensional models involve solving the depth integrated or depth averaged two-dimensional equations to describe governing suspended sediment transport processes (Lin and Falconer, 1996) and need much less data and computer resources in comparison with three-dimensional models. In general, the simpler models tend to be very useful for many practical engineering applications and economical but sometimes inaccurate because they have comparatively small run time and incorporate unrealistic simplifications. The main disadvantage of two-dimensional suspended sediment transport models is absence of the value of near bed reference concentration, which is required to compute the sediment deposition or erosion rates.

In two dimensional models, the value of near bed reference concentration, which is required to compute the sediment deposition or erosion rate, must be related to depth averaged concentration. A common assumption made to relate reference sediment concentration with depth averaged concentration is that the ratio of near bed sediment concentration to the depth averaged concentration is equal to the corresponding ratio in the equilibrium state. This implies that the vertical sediment transport profile adjusts instantly to the equilibrium profile, with this approach therefore being limited to situations where the differences between the local true sediment profile and the local equilibrium profile are relatively small (Lin and Falconer, 1996).

Three dimensional approaches are the most adequate for sediment transport modeling purposes. These models calculate local suspended sediment distribution using the advective-diffusion equation and directly relate the near bed concentration to the re-suspension or deposition of sediment.

Nowadays, even with inexpensive computers, three dimensional models are feasible and capable of simulating the tide, wind and density forcing (Cancino and Neves, 1999a). (Nicholson and O'Connor 1986, Teisson 1992, Lin and Falconer 1996, Cancino and Neves 1999, Wu, Gerritsen *et al*. 2000, Roger and Falconer 2000, Douiler *et al.* 2001, van Ledden 2001).

CHAPTER III

STATE OF THE ART OF NUMERICAL MODELING OF SUSPENDED SEDIMENT TRANSPORT

Three dimensional suspended sediment transport models solve transport equations for suspended sediments in addition to the hydrodynamic momentum and continuity equations. The model is capable of computing suspended sediment distributions, amount of eroded and deposited sediment together with water levels and water particle velocity distributions. The simplifying approximations of the hydrodynamic model are hydrostatic pressure distribution assumption and Boussinesq approximation. The main assumptions of the suspended sediment transport model are:

- Water motion is not affected by the suspended sediment concentrations,
- A uniform and time-independent bed roughness parameter;
- Time and space independent settling velocity;
- Uniform and time-independent sea bed factors that affect critical shear stresses.

3.1 Governing Equations

The governing three dimensional advection–diffusion equation (conservation equation) for suspended sediment where the vertical advection includes the particle settling velocity can be written as:

$$\underbrace{\frac{\partial C}{\partial t}}\ +\ \underbrace{\frac{\partial (u.C)}{\partial x}+\frac{\partial (v.C)}{\partial y}+\frac{\partial (w.C)}{\partial z}}_{Advection}-\underbrace{\frac{\partial (w_s.C)}{\partial z}}_{Settling}$$

$$=\underbrace{\frac{\partial}{\partial x}\left(D_x.\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y.\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(D_z.\frac{\partial C}{\partial z}\right)}_{Diffusion} \tag{3.1}$$

where,

C	:	Suspended sediment concentration,
t	:	Time,
x,y	:	Horizontal coordinates,
z	:	Vertical coordinate,
u, v, w	:	Velocity components in x, y and z directions at any grid locations in space respectively
w_s	:	Settling Velocity
D_x, D_y, D_z	:	Turbulent diffusion coefficient in x, y and z directions respectively.

The governing hydrodynamic equations in the three dimensional cartesian coordinate system are as follows

The continuity equation:

$$\frac{\partial w}{\partial z}=-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y} \tag{3.2}$$

The momentum equations in the orthogonal horizontal directions x and y:

$$\frac{\partial u}{\partial t}+u.\frac{\partial u}{\partial x}+v.\frac{\partial u}{\partial y}+w.\frac{\partial u}{\partial z}=-f.v-\frac{1}{\rho_0}.\frac{\partial P}{\partial x}+2.\frac{\partial}{\partial x}\left(\upsilon_x.\frac{\partial u}{\partial x}\right)$$

$$+\frac{\partial}{\partial y}\left(\upsilon_y.\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\upsilon_y.\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\upsilon_z.\frac{\partial u}{\partial z}\right) \tag{3.3}$$

$$\frac{\partial v}{\partial t}+u.\frac{\partial v}{\partial x}+v.\frac{\partial v}{\partial y}+w.\frac{\partial v}{\partial z}=-f.u-\frac{1}{\rho_0}.\frac{\partial p}{\partial y}+2.\frac{\partial}{\partial y}\left(\upsilon_y.\frac{\partial v}{\partial y}\right)$$

$$+\frac{\partial}{\partial x}\left(\upsilon_x.\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}\left(\upsilon_x.\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\upsilon_z.\frac{\partial v}{\partial z}\right) \qquad (3.4)$$

and in vertical direction *z*:

$$\frac{\partial \rho}{\partial z}=-\rho.g \qquad (3.5)$$

where,

f	:	Corriolis coefficient,
P	:	Pressure,
ρ_o	:	Reference density,
$\upsilon_x, \upsilon_y, \upsilon_z$	:	Eddy viscosity coefficients in *x, y* and *z* directions respectively,
g	:	Gravitational acceleration,
ρ	:	*In situ* water density.

Depth integrated continuity equation:

$$\frac{\partial \eta}{\partial t}=-\frac{\partial}{\partial x}\left(\int_{-h}^{\eta} u.dz\right)-\frac{\partial}{\partial y}\left(\int_{-h}^{\eta} v.dz\right) \qquad (3.6)$$

where,

η	:	Water surface elevation.
h(x,y)	:	Water depth measured from the undisturbed water surface.

3.2 The Turbulence Model

The values of eddy viscosities and turbulent diffusion coefficients can be used as constant or be obtained from the two equation *k-ε* turbulence modeling.

The model equations for the kinetic energy and dissipation of the kinetic energy are:

$$\frac{\partial k}{\partial t}+u.\frac{\partial k}{\partial x}+v.\frac{\partial k}{\partial y}+w.\frac{\partial k}{\partial z}$$

$$=\frac{\partial}{\partial z}\left(.\frac{\upsilon_z}{\sigma_k}.\frac{\partial k}{\partial z}\right)+P+B-\varepsilon+\frac{\partial}{\partial x}\left(D_x.\frac{\partial k}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y.\frac{\partial k}{\partial y}\right) \quad (3.7)$$

$$\frac{\partial \varepsilon}{\partial t}+u.\frac{\partial \varepsilon}{\partial x}+v.\frac{\partial \varepsilon}{\partial y}+w.\frac{\partial \varepsilon}{\partial z}=\frac{\partial}{\partial z}\left(.\frac{\upsilon_z}{\sigma_\varepsilon}.\frac{\partial \varepsilon}{\partial z}\right)$$

$$+C_{1\varepsilon}.\frac{\varepsilon}{k}.(P+C_{3\varepsilon}.B)-C_{2\varepsilon}.\frac{\varepsilon^2}{k}+\frac{\partial}{\partial x}\left(D_x.\frac{\partial \varepsilon}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y.\frac{\partial \varepsilon}{\partial y}\right) \quad (3.8)$$

where,

k : Kinetic energy,

ε : Rate of dissipation of kinetic energy,

P : Stress production due to the kinetic energy,

B : Buoyancy production of the kinetic energy,

σ_k : A universal empirical constant (σ_k = 1),

σ_ε : A universal empirical constant (σ_ε = 1.3),

$C_{1\varepsilon}$: A universal empirical constant ($C_{1\varepsilon}$ = 1.44),

$C_{2\varepsilon}$: A universal empirical constant ($C_{2\varepsilon}$ = 1.92),

$C_{3\varepsilon}$: A non-universal empirical constant (If $G>0$ $C_{3\varepsilon}$ = 1 and If $G<0$ $C_{3\varepsilon}$ = 0.2) (Balas,1998).

The buoyancy production of kinetic energy is defined by:

$$B=\frac{g}{\rho_o}.\frac{\upsilon_z}{P_r}.\frac{\partial \rho}{\partial z} \quad (3.9)$$

where; P_r is the turbulent Prandatl or Schmidth number. It is considered as a constant (P_r = 0.7) (Balas, 1998).

The stress production of the kinetic energy is defined by:

$$P = \upsilon_h.\left[2.\left(\frac{\partial u}{\partial x}\right)^2 + 2.\left(\frac{\partial v}{\partial y}\right)^2 + 2.\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

$$+ \upsilon_z.\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right] \qquad (3.10)$$

where,

υ_z : Vertical eddy viscosity

υ_h : Horizontal eddy viscosity

The ratio of the vertical length scale to the horizontal length scale is generally very slow and the vertical eddy viscosity-diffusivity terms are correspondingly several orders of magnitude larger than horizontal eddy viscosity-diffusivity terms. Therefore, in applying governing hydrodynamic and suspended sediment transport equations, it is important to prescribe the vertical eddy viscosities and diffusivities more accurately than for the corresponding horizontal terms (Lin and Falconer, 1996). The vertical eddy viscosity is calculated by:

$$\upsilon_z = C_\mu.\frac{k^2}{\varepsilon} \qquad (3.11)$$

where; C_μ is a universal empirical constant ($C_\mu = 0.09$) (Balas, 1998).

Horizontal eddy viscosity can be simulated by the Smagorinsky algebraic sub-grid scale turbulence model (Balas, 1998):

$$\upsilon_h = 0.01 \times \Delta x.\Delta y.\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}.\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right)^{1/2} \qquad (3.12)$$

The relationship between eddy diffusivities and viscosities are as follows:

$$D_h = \upsilon_h \tag{3.13}$$

$$D_z = \frac{\upsilon_z}{P_r} \tag{3.14}$$

where,

D_z : Vertical eddy diffusivity

Dh : Horizontal eddy diffusivity

3.3 Settling Velocity

Initially, the setting velocity of sediment (w_s) is assumed to be constant in time and space. Settling motion of particles is effected by gravitational forces, viscous drag on particles and interparticle interactions. Thus, it is related to the sand grain size, kinematic viscosity of water and ratio of densities of particle and water. Due to the ability of forming flocs, the settling velocity for mud is generally not constant in time and space but strongly depends on turbulent intensity and the mud concentration in the water column. It is recommended to include and analyse the effect of a time- and space-dependent settling velocity for mud in a following phase (van Ledden, 2001).

For a given type of particles, the settling velocity may be derived from Stoke's formula for non-cohesive particles with a diameter less than 100 μm as follows:

$$w_s = \frac{(s-1).g.D_s{}^2}{18.\upsilon} \tag{3.15}$$

w_s : Settling Velocity,

s : Ratio of densities of particles and water $\left(s = \frac{\rho_{sed}}{\rho_{water}} \right)$,

D_s : Representetive diameter of particles,

υ : Kinematic molecular viscosity of water,

g : Gravitational acceleration.

The particles that have representative diameter in the range of 64 μm and 250 μm is considered as very fine sand. If the particle diameter is less than 64 μm then, it is considered as cohesive mud (silt and clay). Cohesive sediments, subject to surface electrochemical forces and inter-particle collision because of their small size, will flocculate, i.e. particles cluster in aggregates. As a result of floc aggregation, cohesive sediments settle by flocs rather than by individual particles (Teisson, 1991). It was found that the settling velocity of the flocs depended strongly on the suspended cohesive sediment concentration (Wu, Roger and Falconer, 1999). At moderate concentrations, the settling velocity increases with concentration but, for higher concentration settling is hindered, because water has to be expelled through the interstitial spaces of the continuous network of aggregates.

3.4 Boundary Conditions

There are four types of boundaries; free surface, sea bed, open sea and coastal land boundaries. All boundary conditions are chosen to be time independent for reasons of simplicity.

3.4.1 Free Surface

The water velocity gradient below the sea surface is caused by wind induced shear stress at the sea surface:

$$\frac{\partial u}{\partial z}.\upsilon_z = \frac{\tau_{wind,x}}{\rho} \quad , \quad \frac{\partial v}{\partial z}.\upsilon_z = \frac{\tau_{wind,y}}{\rho} \tag{3.16}$$

where, $\tau_{wind,x}$ and $\tau_{wind,y}$ are wind shear components in x and y directions; ρ is water density.

The wind induced shear stress at the sea surface is expressed as:

$$\left[\tau_{wx}, \tau_{wy}\right] = \rho_a . C_d . \left[u_w, v_w\right] . \sqrt{u_w^{\ 2} + v_w^{\ 2}} \tag{3.17}$$

where,

u_w, v_w	:	Wind velocity components (m/s) in x and y directions,
ρ_a	:	Air density,
C_d	:	Drag coefficient of air.

The formulation for drag coefficient:

$$C_d = \begin{cases} 1.2 \times 10^{-3} & W < 11 m/s \\ \left(0.49 + 0.065 \times W\right) \times 10^{-3} & 11 m/s \le W \le 25 m/s \end{cases} \tag{3.18}$$

where, W is the wind velocity (m/s).

The sea surface boundary condition for kinetic energy and its rate of dissipation when there exists wind shear is as follows:

$$k_s = \frac{u_{*s}^{\ 2}}{\sqrt{C_\mu}} \quad ; \quad \varepsilon_s = \frac{\lfloor u_{*s} \rfloor^3}{\kappa . \Delta z_s} \tag{3.19}$$

where,

u_{*s}	:	Surface shear velocity,
κ	:	von Karman constant ($\kappa = 0.42$),
Δz_s	:	Distance from the surface to the first grid point below.

Shear velocity is defined as:

$$u_{*sx} = \sqrt{\frac{\tau_{wx}}{\rho}} \tag{3.20}$$

At the free surface the net vertical sediment flux was assumed to be zero, i.e. there is no exchange of particles through surface:

$$D_z \frac{\partial C}{\partial z} = \left[(w - w_s).C\right]_{surface} \tag{3.21}$$

3.4.2 Sea Bed

At the sea bed, the bottom shear stress is determined by matching velocities with the logarithmic law of wall:

$$\tau_{bx} = \left(\upsilon_z . \frac{\partial u}{\partial z}\right)_b = \rho_0 . C_f . u_b . \sqrt{u_b^{\ 2} + v_b^{\ 2}}$$

$$\tau_{by} = \left(\upsilon_z . \frac{\partial v}{\partial z}\right)_b = \rho_0 . C_f . v_b . \sqrt{u_b^{\ 2} + v_b^{\ 2}} \tag{3.22}$$

where,

τ_{bx}, τ_{by}	:	The bottom shear stress components,
u_b, v_b	:	Horizontal velocity components at the grid point nearest to the sea bottom,
C_f	:	Empirical coefficient for bottom friction.

The sea bed boundary condition for kinetic energy and its rate of dissipation when there exists wind shear is as follows:

$$k_b = \frac{u_{*b}^{\ 2}}{\sqrt{C_\mu}} \quad ; \quad \varepsilon_b = \frac{\lfloor u_{*b} \rfloor^3}{\kappa . \Delta z_b} \tag{3.23}$$

It is assumed that, when bottom friction is smaller than a critical value for deposition, there is addition of matter to the bottom, and, when the bottom shear is higher than a minimum value, erosion occurs. Between those values, erosion and deposition balance each other.

At the bottom the boundary condition is that the fluxes of particles between the sea floor and water column is as flows:

$$-w_s.C_{bed} - D_z \frac{\partial C}{\partial z} = (E - D) \tag{3.24}$$

where,

D, E	:	Sediment transport rates through deposition and re-suspension, respectively

3.4.2.1 The Deposition Model

Deposition is calculated as the product of the settling flux and the probability of a particle to remain on the bed:

For $\tau \le \tau_d$ $\quad \frac{\partial M_D}{\partial t} = D = w_s(b).C(b).\left(1 - \frac{\tau_b}{\tau_d}\right)$

For $\tau > \tau_d$ $\quad \frac{\partial M_D}{\partial t} = D = 0$ (3.25)

where,

τ_d	:	Critical shear stress for deposition,
$\left(1 - \frac{\tau_b}{\tau_d}\right)$	:	The probability of a settling particle becomes attached to the bed.

The formulation is based on the assumption that a particle reaching the bottom has a probability of remaining there that varies between 0 and 1 as the bottom shear stress varies between its upper limit for deposition and zero, respectively.

The critical shear stress for deposition depends mainly on size of particles/flocs. Bigger particles have higher probability of remaining on the bed than smaller particles. Nevertheless, previous works suggest that a constant value is a reasonable approximation (Cancino and Neves, 1999a).

3.4.2.2 The Re-suspension Model

Erosion or re-suspension of bottom sediments is one of the most important factors controlling the fine sediment transport in natural water bodies. Far from lateral sources and sinks of materials, and in the absence of biological production, erosion is the major source for suspended particles in the water column. Re-suspension is a common physical process that occurs everywhere in the marine environment, both in shallow coastal areas and in the deep sea. Re-suspension occurs when shear stress (friction of the water against the bottom), is high enough to lift the sediment particles. Thus, re-suspension also leads to a transport of particles along the sea floor with currents.

There is general agreement that bottom shear stress exerted by currents and waves are dominant forces causing re-suspension. Also, the site specific sediment characteristics like particle size distribution, particle density, cohesiveness, water content etc. control resistance to re-suspension. In the generally accepted re-suspension rate formulation of Partheniades, the site specific sediment characteristics are represented with erosion rate constant:

$$\text{For } \tau \geq \tau_e \qquad \frac{\partial M_E}{\partial t} = E = ke.\left(\frac{\tau_b}{\tau_e} - 1\right)$$

$$\text{For } \tau < \tau_e \qquad \frac{\partial M_E}{\partial t} = E = 0 \qquad (3.26)$$

Critical shear stress for erosion is a function of the degree of compaction of bottom sediments measured by the dry density of the bottom sediments: ratio between the mass of sediment after extraction of the interstitial water at 105 C and its initial volume (Nicholson and O'Connor, 1986). In some applications threshold current velocities are used instead of critical shear stresses. One of the main difficulties in sediment transport modeling is the method of obtaining the re-suspension and deposition thresholds and the erodability constant. Thus, in previous applications they are selected by trial and error in such a way that they produce the best fit of model results to observations, although parameters must be physically realistic.

3.4.3 Coastal Land Boundary and Open Sea Boundary

In the case of outgoing flux at the lateral boundaries and at bank boundaries, a Neumann condition is imposed:

$$\frac{\partial C}{\partial x} = 0 \quad ; \quad \frac{\partial C}{\partial y} = 0 \qquad (3.27)$$

For inflow conditions at the open boundaries, the concentrations have to be specified. At open sea boundaries, generally a value of concentration is imposed. At coastal land boundaries, in the case of inflow flux condition, an equilibrium sand concentration profile is used.

CHAPTER IV

NUMERICAL SOLUTION

4.1 Numerical Solution Scheme

The governing hydrodynamic equations and advective-diffusive transport equation for suspended sediments are solved by utilizing a composite finite differences and finite element method. The governing equations, written in the Cartesian co-ordinates, are solved by the Galerkin weighted residual method in the vertical plane and by finite difference approximations in the horizontal plane, without any co-ordinate transformation. The water depths are divided into the same number of layers following the bottom topography. Therefore, the vertical layer thickness is proportional to the local water depth at all nodal points (Balas and Özhan, 2000).

The solution domain is divided into finite elements along the water depth. The next step is to develop equations to approximate the solution for each element. First of all, an approximate function with unknown coefficients must be chosen. Secondly, the coefficients must be evaluated, so that the function approximates the solution with an optimal fashion, i.e. minimum errors. The simplest alternative is the use of first order polynomial:

$$G(z) = a_0 + a_1.z \tag{4.1}$$

where *G(z)* is the dependent variable, *z* is independent variable, a_0 and a_1 are the coefficients

This function must pass through the values of *G(z)* at the beginning *(z_1)* and *(z_2)* end points of the element.

$$G_1 = a_0 + a_1.z_1 \qquad G_2 = a_0 + a_1.z_2 \tag{4.2}$$

where, G_1 and G_2 are values of the function at the beginning and end of the element.

For the finite element method, the values of the dependent variable *G* are written in terms of discrete values of this variable at the vertical nodal points by using linear shape functions:

$$G \rightarrow \widetilde{G} \qquad \widetilde{G} = N_1.G_1^k + N_2.G_2^k \tag{4.3}$$

where,

$\widetilde{G}$	:	The approximation or shape function,
k	:	The number of the element
N_1, N_2	:	Interpolation functions

$$N_1 = \frac{z_2 - z}{l_k} \quad , \qquad N_2 = \frac{z - z_1}{l_k} \tag{4.4}$$

where, l_k is the length of k^{th} element $(l_k = z_2 - z_1)$.

The approximate expressions for variables are substituted into governing equations. Since Equation (4.3) is an approximation , not the exact solution, there will be residuals (errors) in each of the equations. Resulting residuals *(R)*, are minimized by using the Galerkin procedure as follows (Balas, 1998):

$$\int_{z_1}^{z_2} R.N_i.dz = 0 \qquad \textit{for i = 1,2} \tag{4.5}$$

When the integrals of the functions are to be performed, all the horizontal gradient terms are replaced by finite difference expressions. The finite difference scheme is illustrated below:

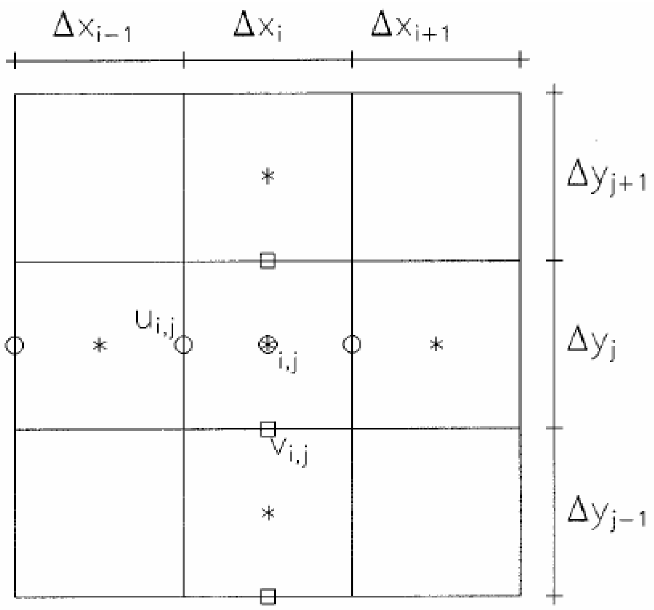

Figure 4.1. Finite difference molecule for computations of all other variables except horizontal velocities (Symbols represent the variables as follows: u ○, v □, all other variables *) (Balas, 1998).

4.2 Global Matrices

The governing equation for the dependent unknown variable G can be considered as follows:

$$\frac{\partial G}{\partial t} + F = 0 \tag{4.6}$$

In the above equation, the term *F* denotes the part of the equation that does not include time derivative terms. After the application of Galerkin weighted residual method for the vertical plane, the error is minimized as follows:

$$\frac{\partial \widetilde{G}}{\partial t} + \widetilde{F} = R \tag{4.7}$$

$$\int_{z_1}^{z_2} \frac{\partial \widetilde{G}}{\partial t}.N_i.dz + \int_{z_1}^{z_2} \widetilde{F}.N_i.dz = \int_{z_1}^{z_2} R.N_i.dz \cong 0 \qquad \textit{for i = 1,2} \tag{4.8}$$

$$\begin{aligned} \int_{z_1}^{z_2} \frac{\partial (G_1.N_1 + G_2.N_2)}{\partial t}.N_1.dz &= -\int_{z_1}^{z_2} \widetilde{F}.N_1.dz \\ \int_{z_1}^{z_2} \frac{\partial (G_1.N_1 + G_2.N_2)}{\partial t}.N_2.dz &= -\int_{z_1}^{z_2} \widetilde{F}.N_2.dz \end{aligned} \tag{4.9}$$

$$\begin{aligned} \frac{\partial G_1}{\partial t}.\int_{z_1}^{z_2} N_1^{\,2}.dz + \frac{\partial G_2}{\partial t}.\int_{z_1}^{z_2} N_1.N_2.dz &= -\int_{z_1}^{z_2} \widetilde{F}.N_1.dz \\ \frac{\partial G_1}{\partial t}.\int_{z_1}^{z_2} N_1.N_2.dz + \frac{\partial G_2}{\partial t}.\int_{z_1}^{z_2} N_2^{\,2}.dz &= -\int_{z_1}^{z_2} \widetilde{F}.N_2.dz \end{aligned} \tag{4.10}$$

After obtaining the integrals of shape functions and the replacement of derivative terms with respect to horizontal coordinates appearing in the equations with their finite difference approximations, the local matrices and vectors describing the non-linear equations for each of the element *k* over water depth are obtained as below:

$$\begin{aligned} \frac{l_k}{3}.\frac{\partial G_1^k}{\partial t} + \frac{l_k}{6}.\frac{\partial G_2^k}{\partial t} &= A(1,k) \\ \frac{l_k}{6}.\frac{\partial G_1^k}{\partial t} + \frac{l_k}{3}.\frac{\partial G_2^k}{\partial t} &= A(2,k) \end{aligned} \tag{4.11}$$

The local element matrices for all elements along the water depth are grouped together to form the global matrix equation for the unknown nodal time derivatives of the variables at a grid point on the horizontal plane. Every nodal point, except the nodal points at the sea bed and sea surface, is at the end point of the element k and at the beginning point of the element $k+1$.

For the boundary conditions, if the gradients of a variable are known at sea surface and sea bottom $\left(\frac{\partial G_1^1}{\partial z}, \frac{\partial G_2^m}{\partial z}\right)$ as for the advection – diffusion equation of suspended sediment transport, the system of equations are written as follows:

$$\frac{l_k}{3}.\frac{\partial G_1^1}{\partial t} + \frac{l_k}{6}.\frac{\partial G_2^1}{\partial t} = A(1,1) - D_z{}^1.\frac{\partial G_1^1}{\partial z}$$

$$\frac{l_k}{6}.\frac{\partial G_1^1}{\partial t} + \frac{l_k}{3}.\frac{\partial G_2^1}{\partial t} = A(2,1)$$

$$\frac{l_k}{3}.\frac{\partial G_1^2}{\partial t} + \frac{l_k}{6}.\frac{\partial G_2^2}{\partial t} = A(1,2)$$

$$\frac{l_k}{6}.\frac{\partial G_1^2}{\partial t} + \frac{l_k}{3}.\frac{\partial G_2^2}{\partial t} = A(2,2) \qquad (4.12)$$

$$.....$$

$$\frac{l_k}{3}.\frac{\partial G_1^m}{\partial t} + \frac{l_k}{6}.\frac{\partial G_2^m}{\partial t} = A(1,m)$$

$$\frac{l_k}{6}.\frac{\partial G_1^m}{\partial t} + \frac{l_k}{3}.\frac{\partial G_2^m}{\partial t} = A(2,m) + D_z{}^m.\frac{\partial G_2^m}{\partial z}$$

where, D_z represents eddy viscosity or turbulent diffusion constant, the subscripts show the matrix element location and superscripts show the layer for which the equation is written. In its general form, the global matrices are written as follows:

$$
\begin{bmatrix}
\frac{1}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{1}{3}
\end{bmatrix}
\times
\begin{bmatrix}
\frac{\partial G^1}{\partial t} \\
\frac{\partial G^2}{\partial t} \\
\frac{\partial G^3}{\partial t} \\
\frac{\partial G^4}{\partial t} \\
\cdot \\
\cdot \\
\cdot \\
\frac{\partial G^{m-3}}{\partial t} \\
\frac{\partial G^{m-2}}{\partial t} \\
\frac{\partial G^{m-1}}{\partial t} \\
\frac{\partial G^{m}}{\partial t}
\end{bmatrix}
=
\begin{bmatrix}
A(1,1) - D_z^{\,1}.\frac{\partial G^1}{\partial z} \\
A(2,1) + A(1,2) \\
A(2,2) + A(1,3) \\
A(2,3) + A(1,4) \\
\cdot \\
\cdot \\
\cdot \\
A(2,m-3) + A(1,m-2) \\
A(2,m-2) + A(1,m-1) \\
A(2,m-1) + A(1,m) \\
A(2,m) + D_z^{\,m}.\frac{\partial G^1}{\partial z}
\end{bmatrix}
\qquad (4.13)
$$

If the value of the variable is known at the sea bottom (G^1_1) and the gradient of the variable is known at the sea surface $\left(\frac{\partial G^m_2}{\partial z}\right)$ as the momentum equations in the orthogonal horizontal directions, then the global matrices are written as follow:

$$
\begin{bmatrix}
\frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \frac{1}{6} & \frac{1}{3}
\end{bmatrix}
\times
\begin{bmatrix}
\frac{\partial G^2}{\partial t} \\
\frac{\partial G^3}{\partial t} \\
\frac{\partial G^4}{\partial t} \\
\frac{\partial G^5}{\partial t} \\
\cdot \\
\cdot \\
\cdot \\
\frac{\partial G^{m-3}}{\partial t} \\
\frac{\partial G^{m-2}}{\partial t} \\
\frac{\partial G^{m-1}}{\partial t} \\
\frac{\partial G^{m}}{\partial t}
\end{bmatrix}
=
\begin{bmatrix}
A(1,2) + A(2,1) - \frac{1}{6}.\frac{\partial G^1}{\partial t} \\
A(2,2) + A(1,3) \\
A(2,3) + A(1,4) \\
A(2,4) + A(1,5) \\
\cdot \\
\cdot \\
\cdot \\
A(2,m-3) + A(1,m-2) \\
A(2,m-2) + A(1,m-1) \\
A(2,m-1) + A(1,m) \\
A(2,m) + D_z^{\,m}.\frac{\partial G^m}{\partial z}
\end{bmatrix}
\qquad (4.14)
$$

If the values of the variables at the sea surface and sea bottom $\left(G_1^1, G_2^m\right)$ as the model equations of k-epsilon turbulence model, then the global matrices are written as follow:

$$
\begin{bmatrix}
\frac{2}{3} & \frac{1}{6} & 0 & . & . & . & . & . & . & . & 0 \\
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & . & . & . & . & . & . & . \\
0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & . & . & . & . & . & . \\
. & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\
. & . & . & . & . & . & . & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
0 & . & . & . & . & . & . & . & 0 & \frac{1}{6} & \frac{2}{3}
\end{bmatrix}
\times
\begin{bmatrix}
\frac{\partial G^2}{\partial t} \\
\frac{\partial G^3}{\partial t} \\
\frac{\partial G^4}{\partial t} \\
\frac{\partial G^5}{\partial t} \\
. \\
. \\
. \\
. \\
\frac{\partial G^{m-3}}{\partial t} \\
\frac{\partial G^{m-2}}{\partial t} \\
\frac{\partial G^{m-1}}{\partial t}
\end{bmatrix}
=
\begin{bmatrix}
A(1,2) + A(2,1) - \frac{1}{6}.\frac{\partial G^1}{\partial t} \\
A(2,2) + A(1,3) \\
A(2,3) + A(1,4) \\
A(2,4) + A(1,5) \\
. \\
. \\
. \\
. \\
A(2,m-3) + A(1,m-2) \\
A(2,m-2) + A(1,m-1) \\
A(2,m-1) + A(1,m) - \frac{1}{6}.\frac{\partial G^m}{\partial t}
\end{bmatrix}
\qquad (4.15)
$$

After the values of terms that does not include time derivatives in the equation are obtained at each nodal point, the values of the unknown variables at the new time step are calculated by solving equations either explicitly or implicitly.

For the explicit solution, the system of non-linear equations are solved by the adaptive step size controlled Runge - Kutta Fehlberg Method (Chapra and Canale, 1989). The Runge - Kutta Fehlberg Method requires six function evaluations per time step. After defining a local error tolerance *(E)* and an initial step size, the maximum error in simultaneous equations is estimated. If the estimated maximum error is greater than the local error tolerance, the step size is reduced to its half value until the estimated maximum error falls below the local error tolerance. If the estimated maximum error is less than $\frac{E}{10}$, the step is doubled until the error is raised to within accepted range, i.e. $\frac{E}{10} < E_{\max} < E$.

For the implicit solution, the equations are solved by the Crank Nicholson Method which has second order accuracy in time. The Crank Nicholson Method develops difference approximations at the mid point of the time step. The temporal first derivative is approximated at $\left(t + \frac{1}{2}\right)$ and all other variables and derivatives at this time are determined by averaging the difference approximations at the beginning *(t)* and at the end *(t+1)* of the time increment. Resultant set of implicit equations are solved by an iterative method, which is controlled by under – relaxation in order to hasten convergence by dampening out oscillations (Balas, 1998).

4.3 The Computer Program

The computer program is written in Fortran 95 and Matlab R13 program languages. The program utilizes some of the most modern computer hardware and software and is available for personal computers with Windows 98, Windows NT, Windows XP and various workstations of UNIX and LINUX. Also, for preparing data and analyzing the results of simulation, a software, that is utilized for graphical presentation, is developed.

The program consists of three steps. Each step is executed consecutively and independently.

The first and last steps of the program are written in Matlab R13. Matlab enables the user to identify interesting concepts and useful techniques in scientific visualization. The first step takes the input values and calls initial values of all variables that are going to be computed at an advanced time step. The program can take the input data through a series of interactive execution or by preparing command files. In that step, the user can view the appropriate input data graphically to check them and give some of the input data by pointing the mouse on graphics. At the end of the first step, an input file, that contains sufficient data to start computations, is created.

The second step is computational part of the program and it is written in Fortran 95 because Fortran takes significantly less CPU time than Matlab does for calculations. After the termination of the program that takes input values (the first step), the computational part reads the input file and starts to make computations. The set of non-linear equations are solved depending on the specifications in the input data file. The output data is stored at desired time steps during the computations together with the values at end of final time. The block diagram of the computational program is illustrated in Figure 4.2.

The last step of the program provides the graphics of the data stored at desired time steps during the computations. After the termination of computational part of the program (the second step), the program reads the output files. Last step is interactive too. The user can modify the appearance of the graphics using various options. An example of obtaining input data and getting the output graphics utilizing interactive menu systems is shown in Appendix C.

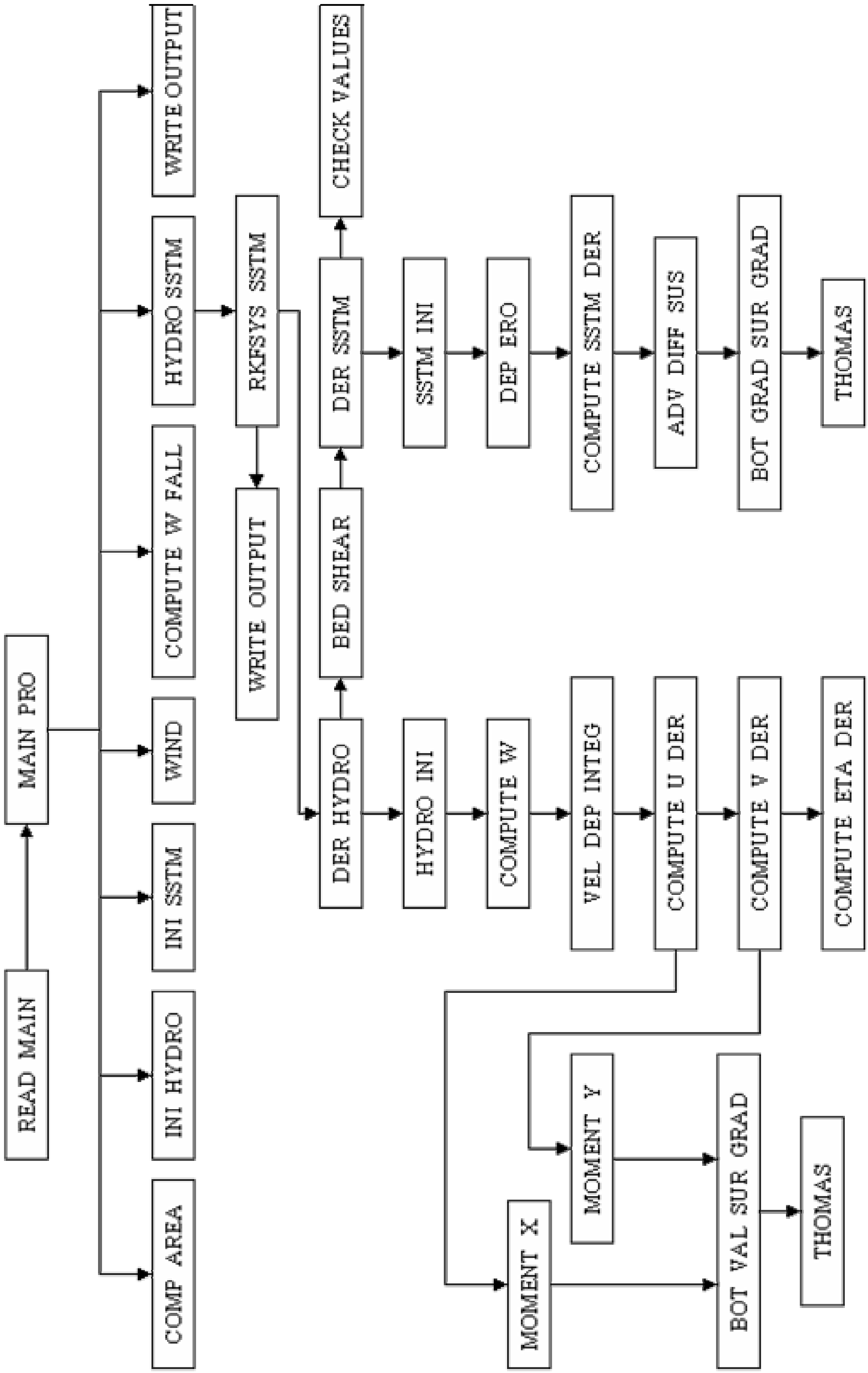

Figure 4.2. Block Diagram of the Computer Program

4.3.1 The Computational Part of the Program

The three dimensional mathematical model developed in this dissertation, consists of three sub-model problems: the hydrodynamic sub-model, the turbulence sub-model and suspended sediment transport sub-model.

The computer program prepared by Balas that computes the full spatial distribution of velocities of unsteady flow induced by wind, tide or water density differences forms the hydrodynamic sub-model (Balas, 1998). This program is modified to include suspended sediment transport component so that they will run simultaneously with each other.

The sub-programs of computational part of the program are as follows:

- *Subroutine READ_MAIN:* This routine reads the following data before calling the main program:
    - the bathymetry of coastal area,
    - the number of grid points on the horizontal plane and over the water depth,
    - the horizontal mesh lengths,
    - the start time, the final time and the time increment,
    - wind characteristics,
    - coriolis coefficient,
    - empirical coefficient for bottom friction,
    - critical shear stress for deposition and re-suspension,
    - re-suspension rate coefficient.
- *Subroutine MAIN_PRO:* This sub-routine calls following subroutines before starting to make computations and storing output data at desired time steps:
    - *Subroutine COMP_AREA*: Specifies the configuration of the coastal area

- *Subroutine INI_HYDRO:* Specifies the initial values of flow velocities and water elevations
- *Subroutine INI_SSTM:* Specifies initial values of suspended sediment concentration and the boundary condition of suspended sediment transport model at the sea bed.
- *Subroutine COMPUTE_W_FALL:* Gives the sediment fall velocity
- *Subroutine WIND:* Specifies the hydrodynamic boundary condition at the sea surface.

- *Subroutine RKFSYS_SSTM:* The set of nonlinear equations are solved by the adaptive step size controlled Runge Kutta Fehlberg Method explicitly. The flow chart of the program is shown in Figure 4.3. This program calls subroutines *DER_HYDRO, BED_SHEAR, DER_SSTM* and *CHECK_VALUES*.
- *Subroutine DER_HYDRO:* In this subroutine values of time derivatives of hydrodynamic terms are calculated.
- *Subroutine BED_SHEAR:* This subroutine gives bottom shear stress.
- *Subroutine DER_SSTM:* In this subroutine values of time derivatives of suspended sediment transport model terms are calculated.
- *Subroutines HYDRO_INI and SSTM_INI:* These subroutines sets the initial values of the variables, whose time derivatives are calculated, at the start of each time step.
- *Subroutine COMPUTE_W:* In this subroutine, the continuity equation (Equation 2.2) is solved to give vertical flow velocities.
- *Subroutine VEL_DEP_INTEG:* This subroutine gives the average values horizontal velocities along the water depth.
- *Subroutine DEP_ERO:* In this subroutine, values of re-suspension and deposition rates are calculated.
- *Subroutines COMPUTE_U_DER and COMPUTE_V_DER:* These subroutines call *MOMENT_X, MOMENT_Y, BOT_VAL_SUR_GRAD* and *THOMAS.*

- *Subroutines MOMENT_X and MOMENT_Y*: These subroutines, in order to calculate the values time derivative terms calculate the right hand side of the momentum equation in *x* direction (Equation 2.3) and the momentum equation in *y* direction (Equation 2.4). These are the elements of the local matrices $[A]$ for each of the element *k* over the water depth (Equation 4.11).
- *Subroutine ADV_DIFF_SUS*: This subroutine gives the right hand side of the conservation equation for suspended sediment (Equation 4.1).
- *Subroutine BOT_VAL_SUR_GRAD*: This function gives the elements of local vector on the left hand side of an equation if the value of the variable is known at the sea bed and the gradient of the variable is known at sea surface.
- *Subroutine BOT_GRAD_SUR_GRAD*: This function gives the elements of local vector on the left hand side of an equation if the gradients of the variable is known at sea surface and at the sea bed.
- *Subroutine THOMAS*: This subroutine calculates the values of unknown variables in equation using Thomas algorithm.

CHAPTER V

APPLICATION OF THE PROGRAM TO THE BAY OF IZMIR

5.1 General Description

Izmir Bay is a micro-tidal bay situated at the western coast of the Anatolian peninsula between latitudes of 38^{o}20' and 38^{o}42' N and longitudes of 29^{o}25' – 27^{o}10' E, and is connected to the Aegean Sea. The bay is roughly ''L'' shaped. The leg of the ''L shape'' is about 20 km wide and 40 km long, and the base of the ''L'' is about 5 – 7 km wide and 24 km long.

The Bay of Izmir has been divided into three areas according to their physical characteristics. These are Outer Bay, Middle Bay and Inner Bay. The Outer Bay is about 20 km wide at the east of Karaburun and extends 45 km in northwest-southeast direction. The Central Bay and Inner Bay extend in a west–east direction, and are together 24 km long and 6 km wide. The Outer Bay is further divided into three sub-regions, Outer I, Outer II and Outer III. There are a series of islands parallel to the west coast of the Bay. The narrow Mordogan Strait, which is situated between Uzunada Island and the west coast of the Bay (Outer II), has a sill depth of 14 m. From time to time, Aegean Sea surface water can flow in the surface layer through the narrow Mordogan Strait into the small Gulbahce Bay, which is situated at the southwest end of the Izmir Bay (Sayin, 2003). Another very important narrowness is the Yenikale Strait between the Inner Bay and the Middle Bay. The physical and chemical characteristics of water change drastically both sides of the Yenikale sill (Sayin, 2003). The inner bay, which is the shallowest part of the bay, exhibits a limited water exchange with the middle bay. The depth of the bay increases towards the middle and outer bays and reaches a maximum at the exit of the outer bay. The maximum depths of the inner, middle and outer bays are 20, 45 and 72 m, respectively (Figure 5.1).

The circulation of the surface water in Izmir Bay varies according to the prevailing winds. Tides are semidiurnal (range between 0.2 and 0.5 m) and do not constitute the major forcing agent that shape the water circulation in the bay. The eighty five percent of the energy necessary for the advection, mixing and dispersion is obtained from the wind, while the residual fifteen percent is supplied by tides (Duman *et. al.,* 2004).

The Bay of Izmir region is under the influence of northerly winds all around the year without any exceptional case. Actually the expected wind direction at this latitude is the westerly wind. However, the location and position of the Bay and the irregular distribution of land and sea are suitable for northerly winds to occur (Sayin, 2003). In summer and autumn, the surface water is driven by NW and WNW winds towards the southeast paralleling the coastline with speeds of about 0.4 $m.s^{-1}$. In winter, the winds are from the N and NE and the currents are directed towards the south with speeds of 0.3 $m.s^{-1}$. Sparse available data suggest that the bottom waters move more or less opposite to the surface waters (Duman *et. al.,* 2004).

Izmir Bay is an area of fine-grained and recently deposited sediments. The northern sea floor of the Outer Izmir Bay is floored by relict shore-face sands whilst at Izmir Bay it is entirely covered with recent sediments. Bottom sediments are relatively coarse in the west, with a general size decreasing towards to the east in the outer part of the bay. The western part of the outer bay is covered by silty sand and muddy sand, whilst the eastern part of the outer bay is covered with silt and mud. This general size decreasing pattern is reversed in the inner bay. Most of the inner bay is covered with sandy silt and the area between the central and inner bay is floored by silt (Figure 5.2) (Duman *et. al.,* 2004).

5.2 Wind Induced Currents in Izmir Bay

In this study, the coastal area is schematized by a rectangle grid as shown in Figure (5.3). The grid size is 637.14 m along x-axis (Δx) and along y-axis (Δy) it is 498.39 m. The model area was represented horizontally using a mesh of 67 x 105 grid rectangles. Computations are made at 7 elevations along the water depth. The bathymetry of coastal area is shown in Figure (5.4). The time step is 5 sec but, the time step can be reduced if the error found in the Runge-Kutta Fehlberg Method is greater than the tolerance value.

The water density is taken constant and is equal to 1025 kg/m^3. Horizontal eddy viscosities are used as 10 m^2/sec and vertical eddy viscosity is used as 0.1 m^2/sec, if "Constant Eddy Viscosities" is selected for the "Turbulence Model".

The water mass is subjected to the free surface shear induced by a uniform and steady wind with a speed 10 m/sec blowing from NNW. Steady state circulation pattern is established approximately 7 hours after beginning of the storm. For understanding the circulation pattern and suspended sediment transport in more detail, wind speed is increased to 20 m/sec. Steady state is reached approximately after 8 hours for this case. These steady state velocity vectors at the upper node of the sea bed are shown in Figure (5.5) and at the sea surface is shown in Figure (5.6). The depth averaged velocity vectors are presented in Figure (5.7). Figures (5.8) and (5.9) are the sketches of the vertical velocity profiles at different node points. Figure (5.10) shows the steady state water level changes.

The model results show flow patterns that are typical wind induced currents in the surface and bottom layers. The model being three dimensional allows observing counter flows at the near bed. However, the vertical variation horizontal velocities do not show characteristics of wind induced flows because of boundary effect at some parts of the bay due to narrowness.

As expected, at the sea surface the velocity vectors have the same direction with the wind. Steady state flow patterns show horizontal gyres at the east of Uzun Island and between Pelikan Lagoon and Hekim Island. These places are in Outer I Bay and the flow pattern turn to westward direction showing an anti-cyclonic movement.

At the upper node of the sea bed, the velocity vectors are generally in the opposite direction with the wind direction. A gyre is observed at the south of Uzun Island where topography is very narrow and flow pattern become parallel to the land boundaries. Another gyre is seen at the west of Pelikan and Homa Lagoons. There is limited water exchange between inner and middle bay. Maximum velocities occur at the east of Uzun Island and at the east of Hekim Island where water depth change drastically.

Model results showed that with northwesterly wind, at the Outer Izmir Bay, the depth averaged flow is inward to the bay i.e. north western direction for one hour after the beginning of the storm and gyres are observed at Outer I Bay and Central Bay. Then the flow pattern at the Outer Izmir starts to turn outward i.e. south eastern direction. When the steady state is reached, Outer Izmir Bay and its northern part are dominated by south eastern transport. In the northern part near the boundary with Aegean Sea, two gyres can be seen. One of them near Foca shows a cyclonic, the other one near Karaburun shows an anti-cyclonic movement. The water that entered the bay near Foca combine with the water flowing back into Aegean Sea at the northeast of Uzun Island and some part of it flows into the central bay. The water that entered the bay near

Karaburun flows into the Mordogan strait. Circulation of water is seen between Hekim Island and Pelikan Lagoon. The water from the Middle Bay and the water from the Mordogan strait combine with each other and flow back to the Aegean Sea as a compensation flow. Within the Central Izmir Bay transport vectors shows two main directions. Transport vectors east south east directed in the northern half of the central bay between Uzun Island and Homa lagoon. In the southern half of the central bay transport is mainly towards to northeast, north and west. Also there is limited water exchange between Outer I bay and Middle Bay. The currents directed to the Outer I Bay is much stronger in comparison to the currents directed into Middle Bay near the Pelikan Lagoon. Maximum values of depth averaged velocities are observed at the east Uzun Island.

5.3 Suspended Sediment Transport in Izmir Bay

Computations were made for different values of re-suspension rate coefficient and critical shear stresses for re-suspension and deposition:

- Increasing values of critical shear stress for re-suspension leads to greater deposition areas and smaller re-suspension zones due to the deposition flux formula whatever the value of re-suspension rate coefficient.
- The value of the re-suspension rate coefficient does not modify the type of flux (deposition, re-suspension or equilibrium) but, does affect the quantity of any particulate matter that is re-suspended. For one value of the re-suspension rate coefficient, re-suspension fluxes are produced which are insignificant compared to deposition. Another value of the re-suspension rate coefficient provides such large re-suspension fluxes that deposition is necessarily less.
- The settling velocity significantly affects the suspended sediment concentration. For a lower settling velocity, the decrease in suspended

sediment concentration is less sharp and vice versa. Also, for larger values of settling velocity, the model time step should be reduced in order to show the effect of settling velocity more accurately.

- The quantity of deposited sediment increase as the concentration of suspended sediment increases, even though the formula for deposition does not change. In the coastal area, if sediment is re-suspended in one area and deposited in another area, then the erosion rate will affect the quantity of deposited sediment due to its effect on suspended sediment concentration.
- At the surface layer advection by currents dominates and dispersion is smaller compared to the advection at the surface layer.
- Model results showed that with northwesterly wind, in the Middle and Inner Bay, the concentration of suspended sediment shows a continuous decrease. Fall velocity is the parameter that affects most significantly the profile of suspended sediment concentration. This is due to the limited water exchange with Outer Bay, low bottom shear stresses and lack of coastal land boundaries. Also, the model does not consider the mud content in the sea bed. The development of bed level and morphological behavior in time and space is determined by both sand and mud.
- In the Outer II Bay generally deposition occurs and concentration of suspended sediment shows a continuous decrease. This is mainly because of the flow pattern. The currents that creates smaller bottom shear stresses than critical shear stress for re-suspension, carries sediments from northern part of the bay through Mordogan strait to the south of Uzun Island. The water exchange between Mordogan strait and Gulbahce Bay is limited. The open sea boundary condition for suspended sediment transport and fall velocity are main aspects that control the profile of suspended sediment concentration.
- In the Outer I and Outer III Bays, the concentration of suspended sediment is maximum at west of Uzun and Hekim Islands. These areas are the sites where the re-suspension occurs most. Furthermore, the flow

pattern plays an effective role in increasing concentration of suspended sediment. Two currents carry sediments from Aegean Sea to that site additional to the sediment that is re-suspended. The sediment that entered the bay near the coastal land boundaries at the north of the bay follows two paths. The first one passes through Karaburun, Mordogan Passage and south of Hekim and Uzun Islands. The second one passes through west of Foca, Homa and Pelikan Lagoons. Then, two currents combine at the east of Uzun Island and flow towards to the Aegean Sea.

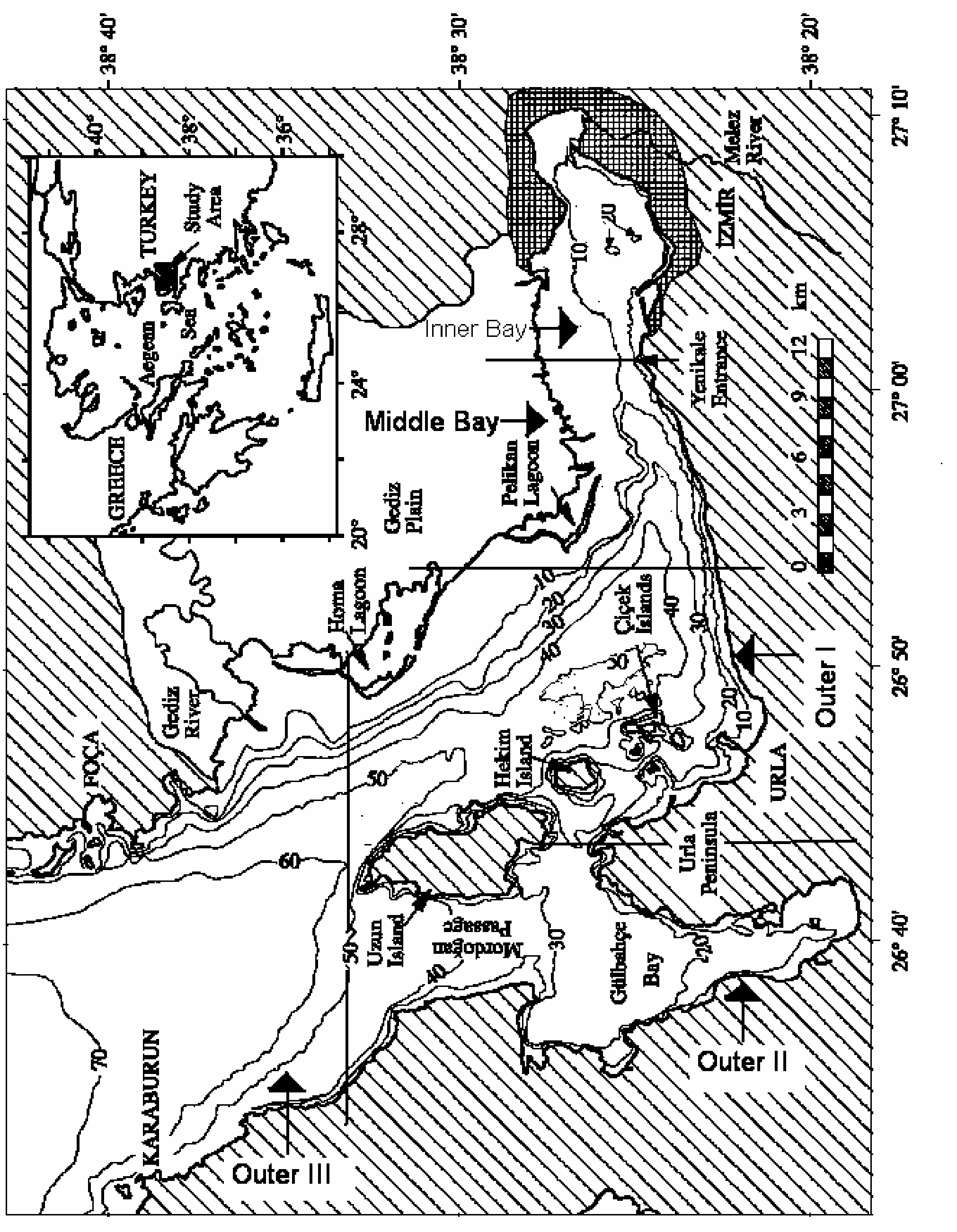

Figure 5.1. The regions, which have different types of water properties, and Izmir Bay topography (Sayin, 2003).

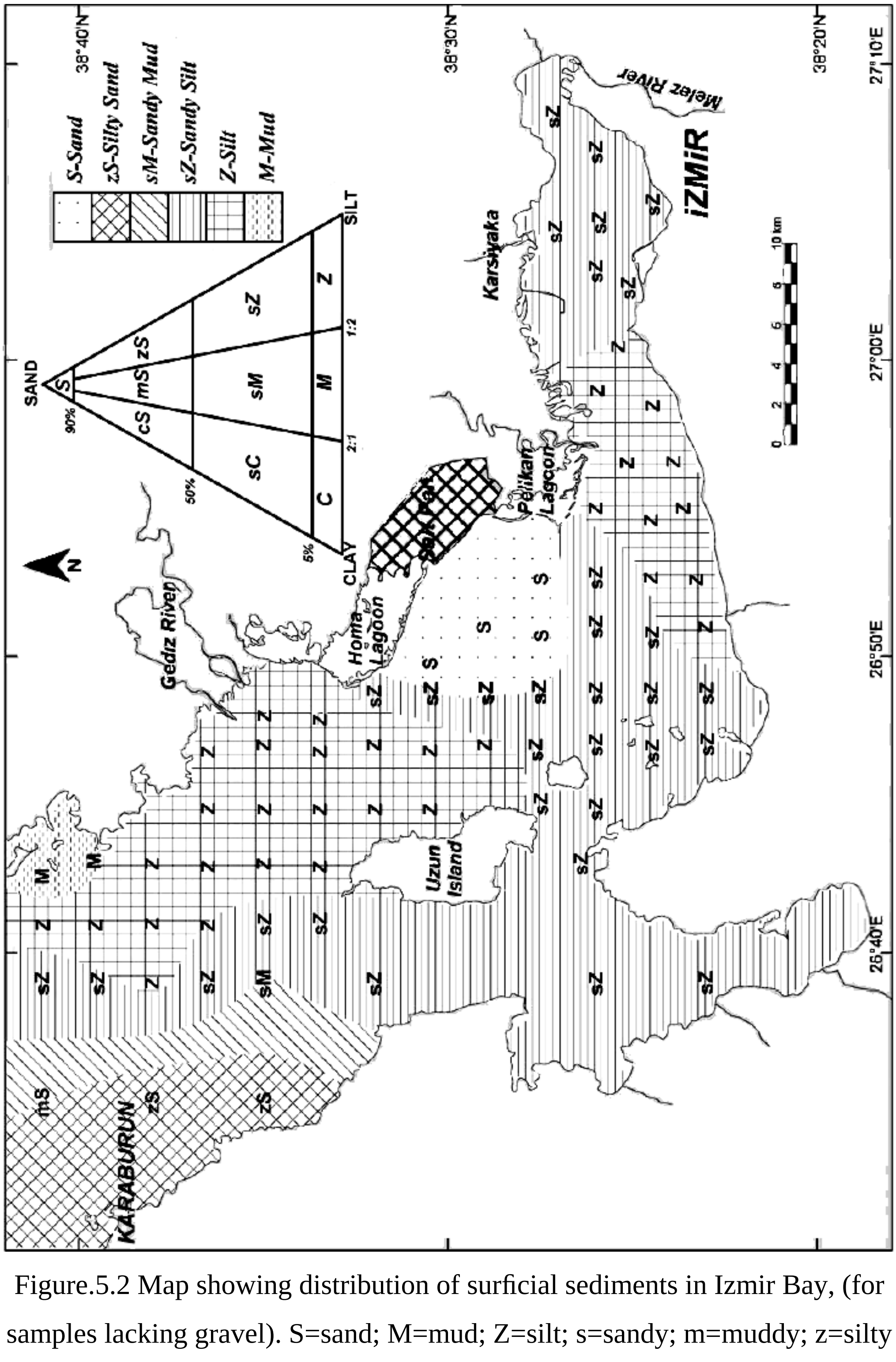

Figure.5.2 Map showing distribution of surficial sediments in Izmir Bay, (for samples lacking gravel). S=sand; M=mud; Z=silt; s=sandy; m=muddy; z=silty (Duman et. al., 2004).

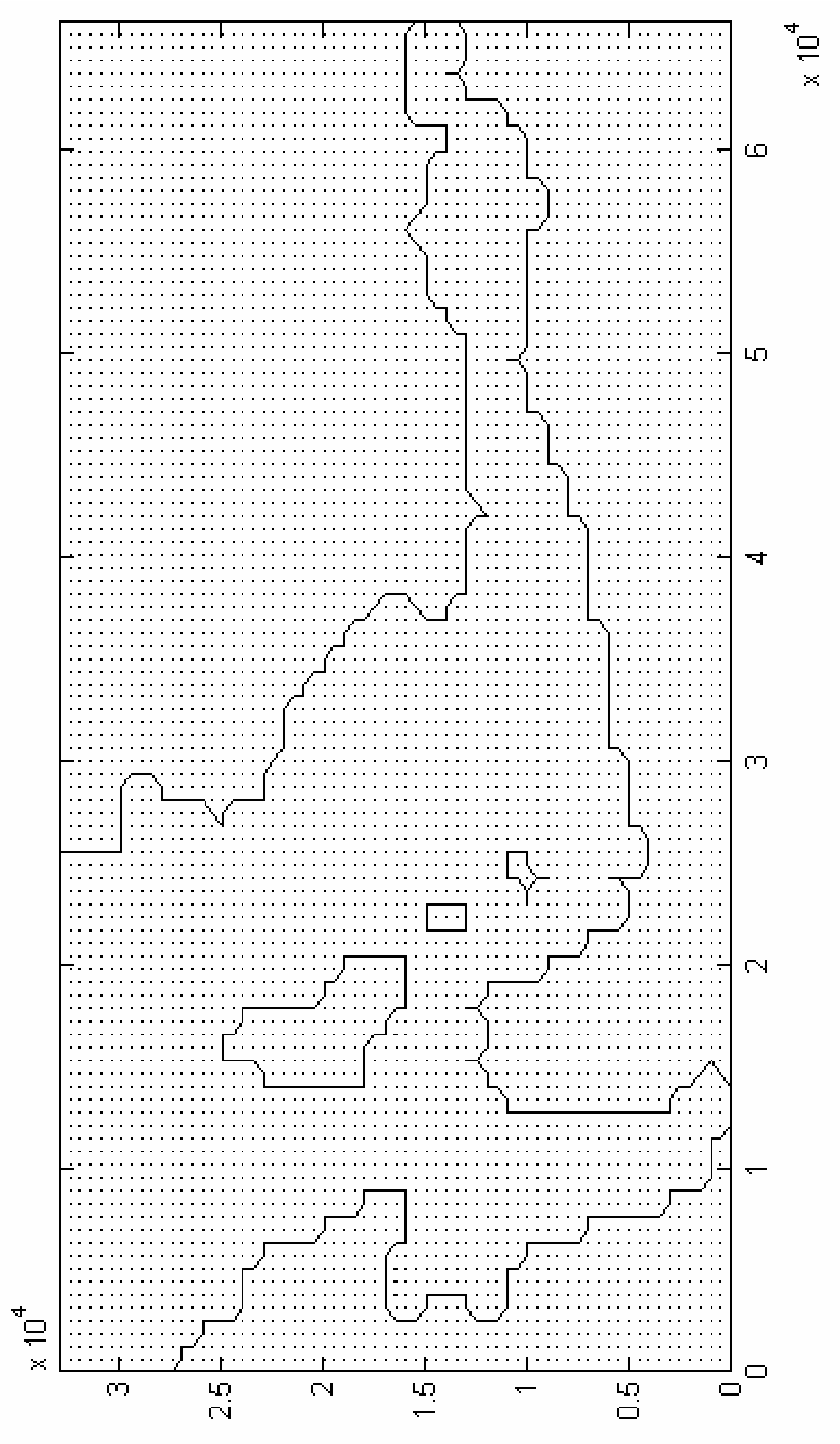

Figure 5.3 The schematization of the Bay of Izmir

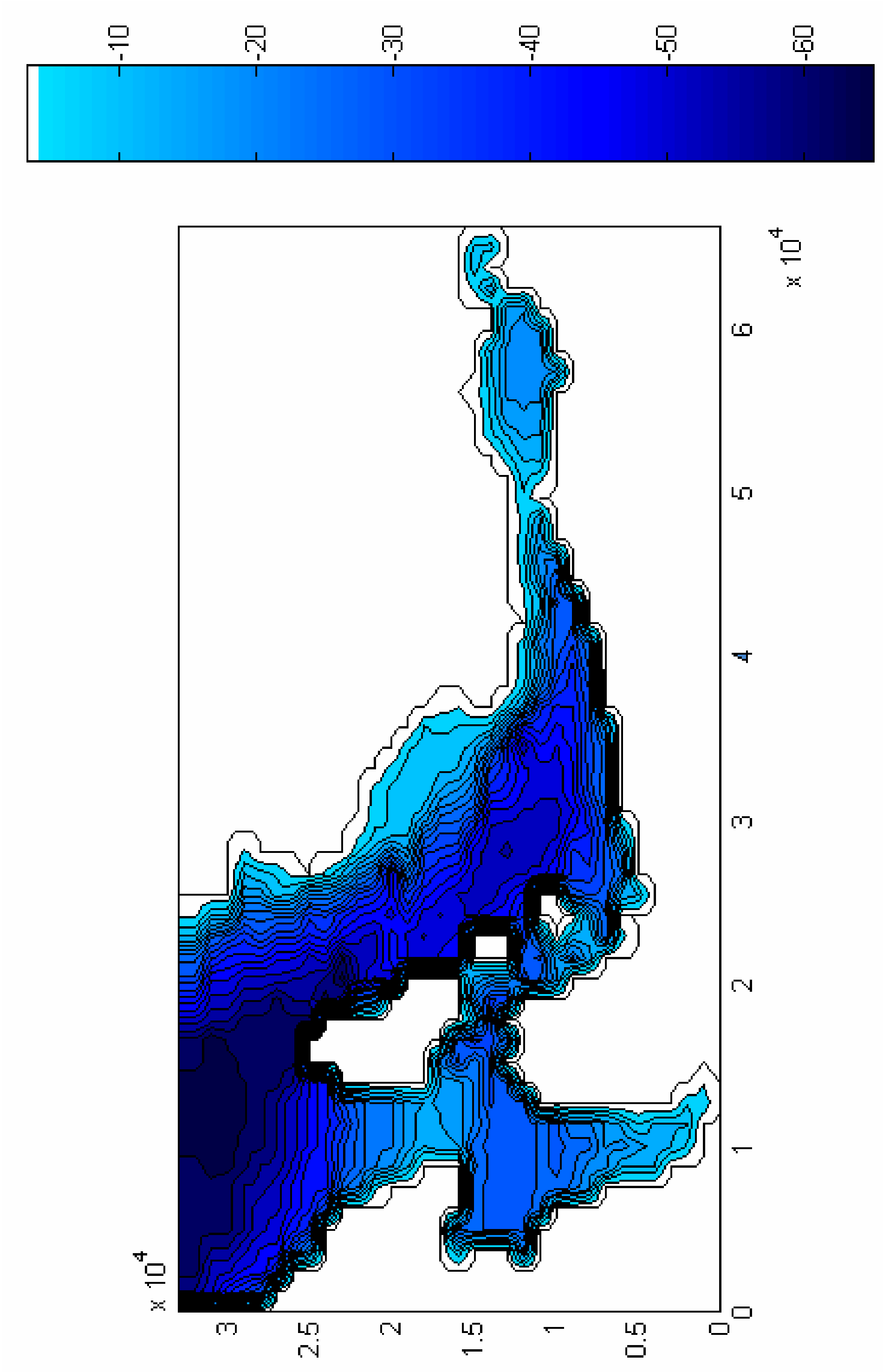

Figure 5.4 Bathymetry of Coastal Area

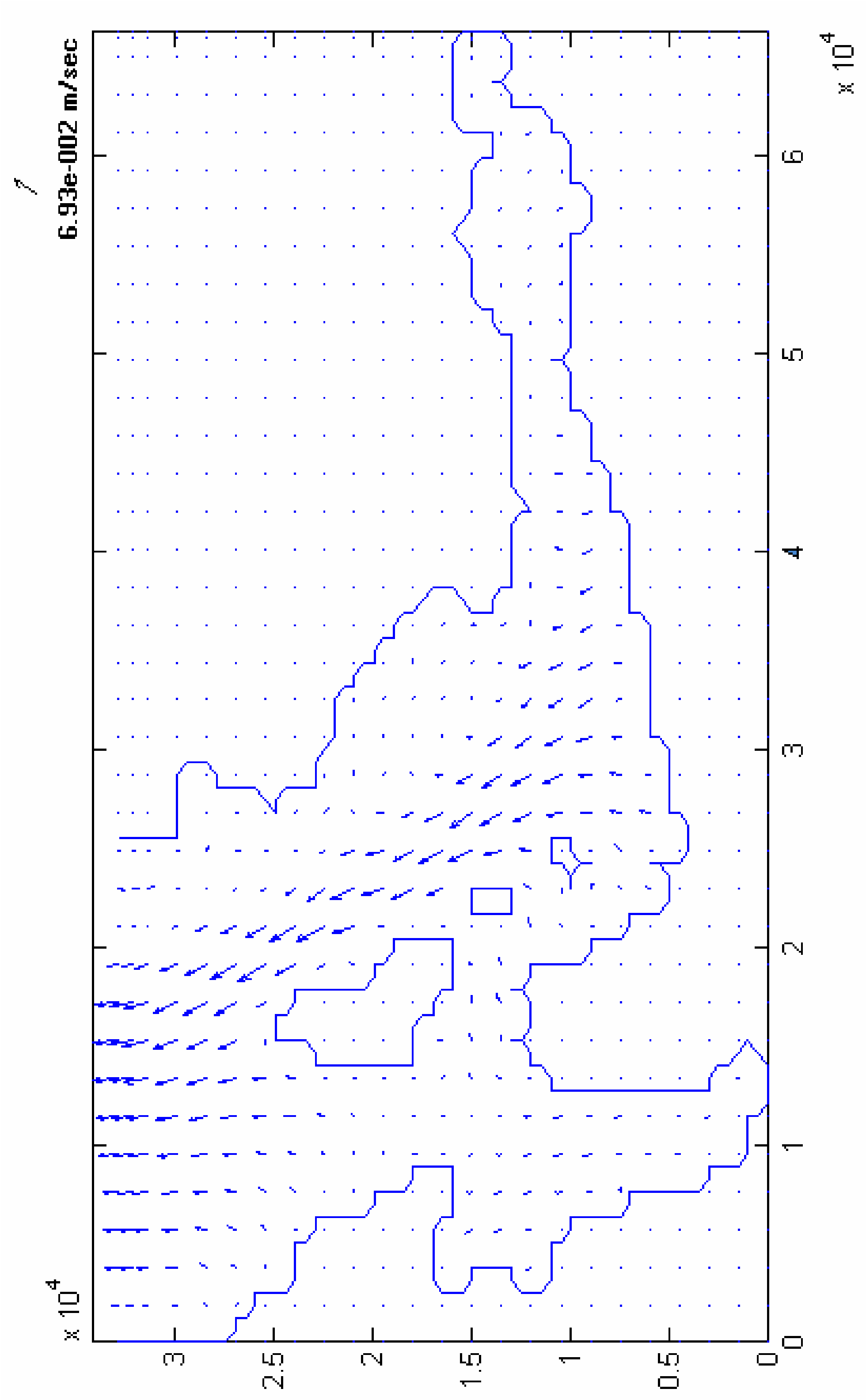

Figure 5.5 Current pattern at bottom layer (NNW wind speed: 10 m/sec) steady state condition

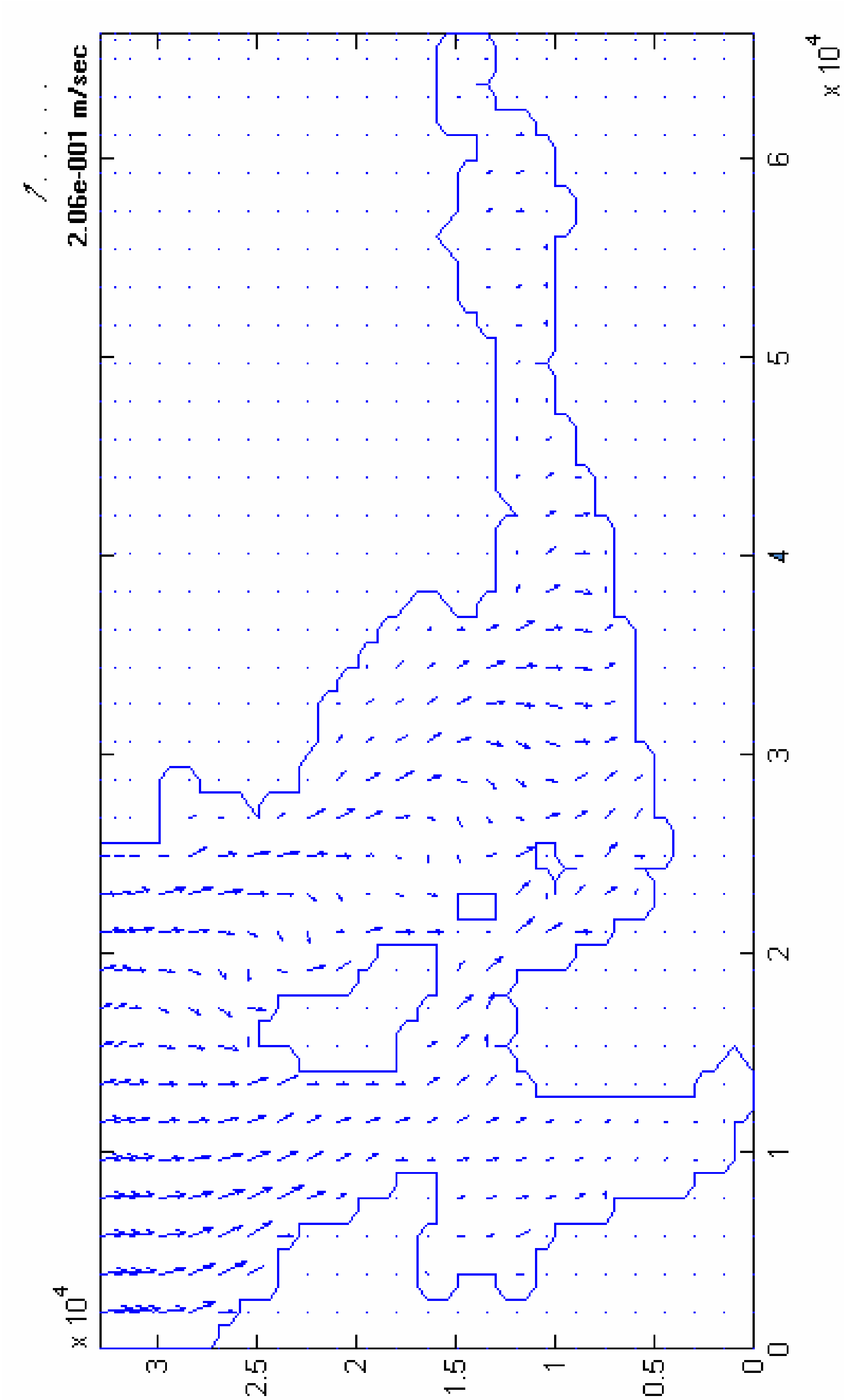

Figure 5.6 Current pattern at sea surface (NNW wind speed: 10 m/sec) steady state condition

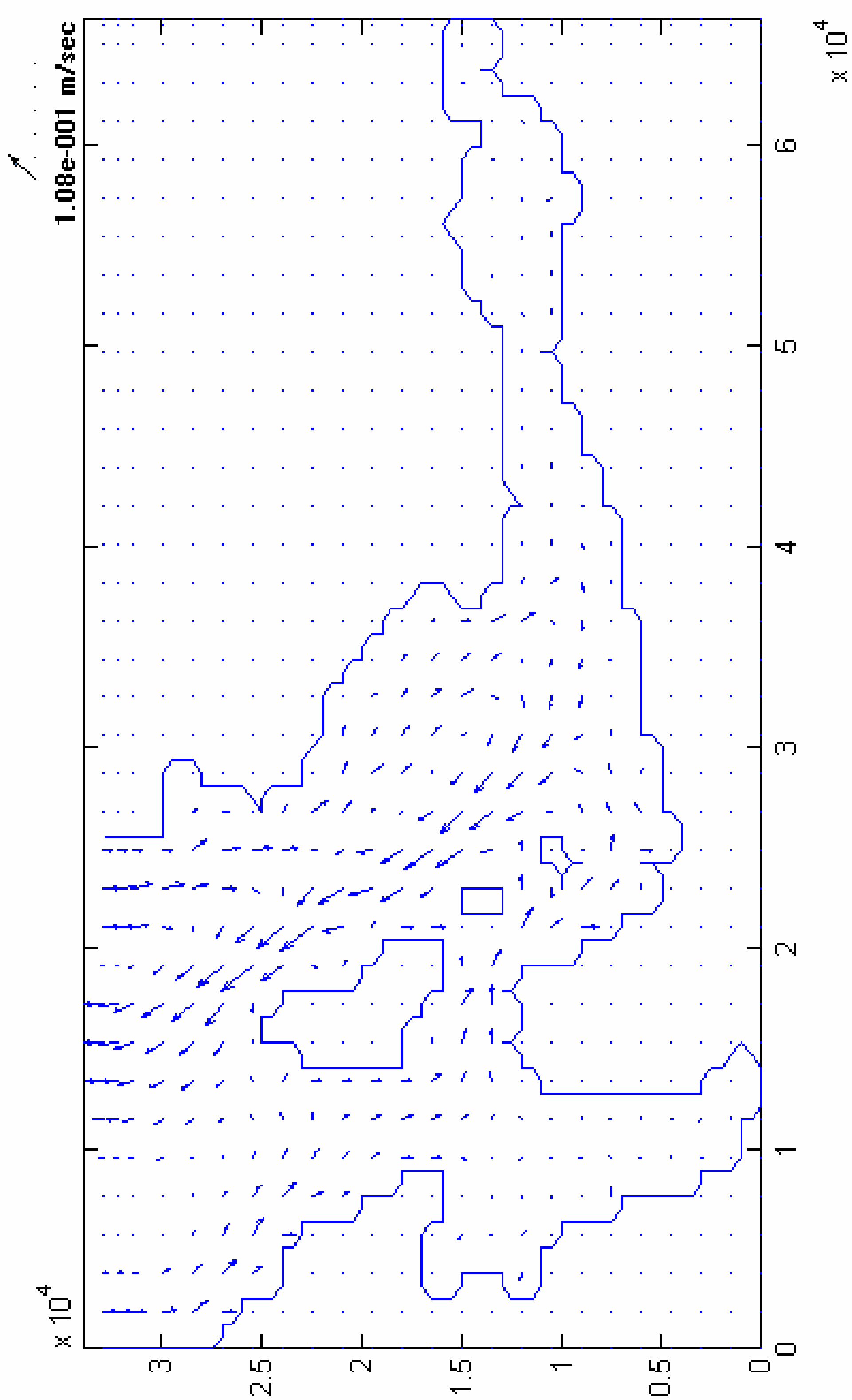

Figure 5.7 Depth average current pattern (NNW wind speed: 10 m/sec) steady state condition

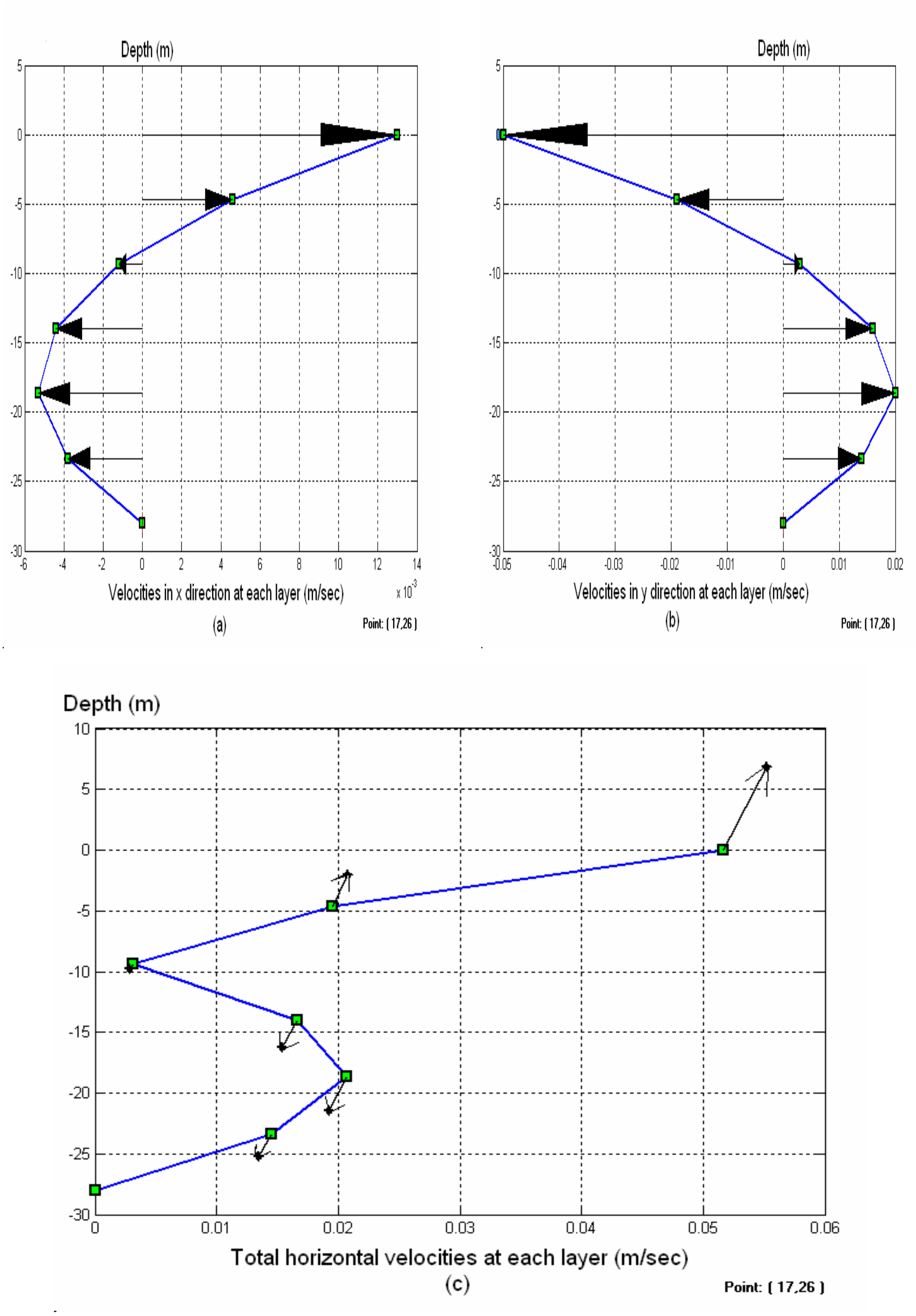

Figure 5.8 Vertical profiles of horizontal velocities at node (17,26), (NNW wind speed: 10 m/sec) steady state condition

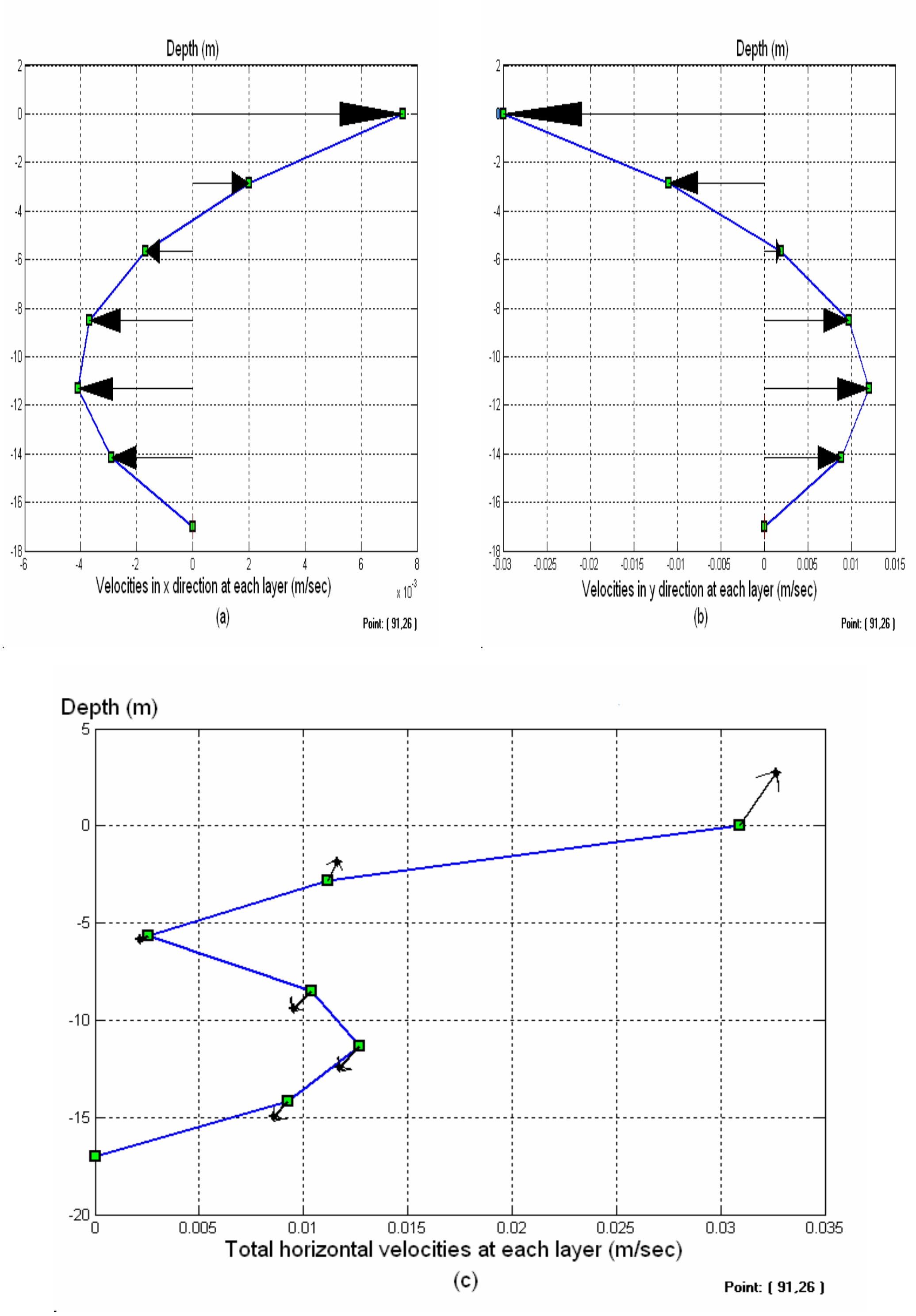

Figure 5.9 Vertical profiles of horizontal velocities at node (91,26), (NNW wind speed: 10 m/sec) steady state condition

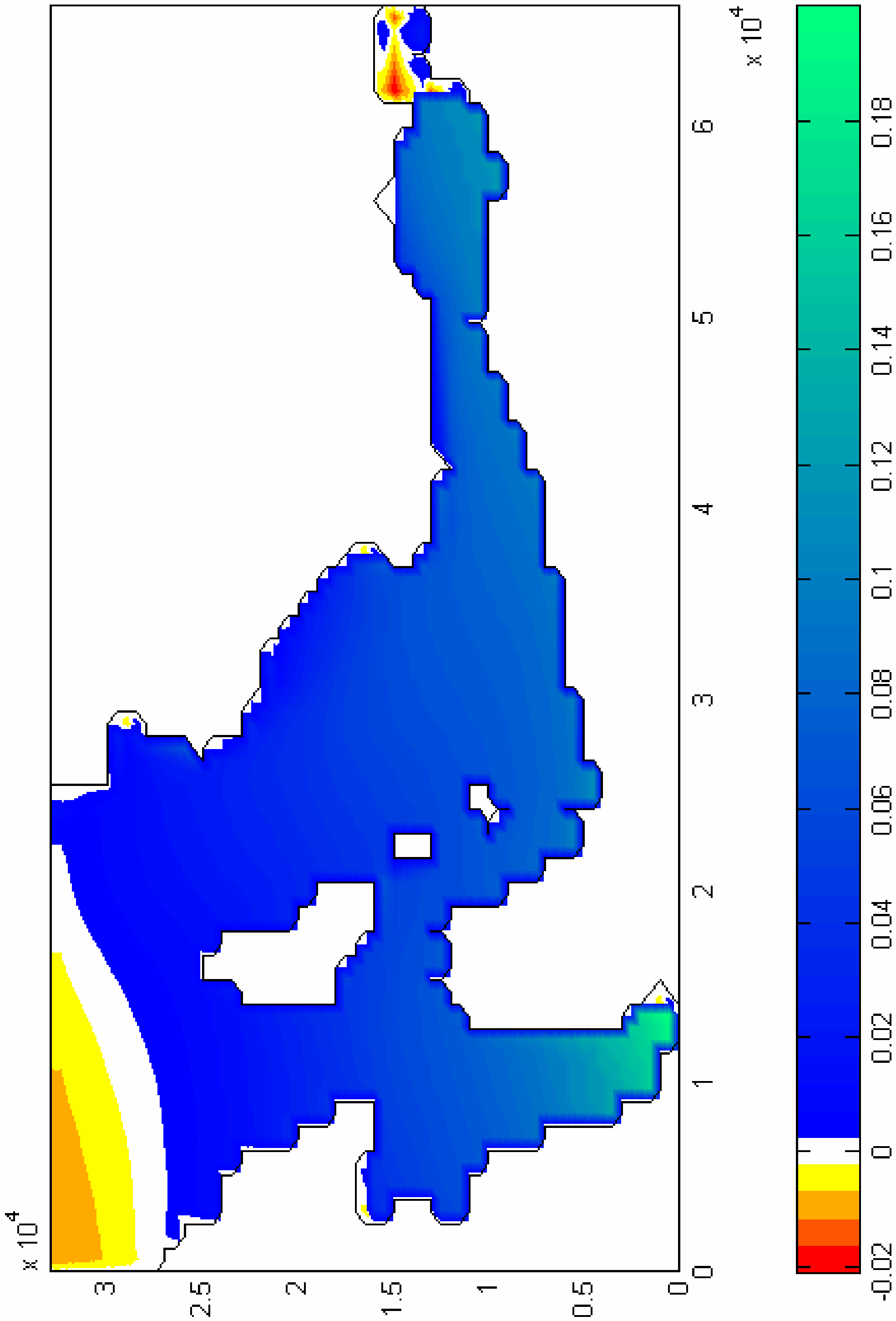

Figure 5.10 Steady state water level changes (NNW wind speed: 10 m/sec)

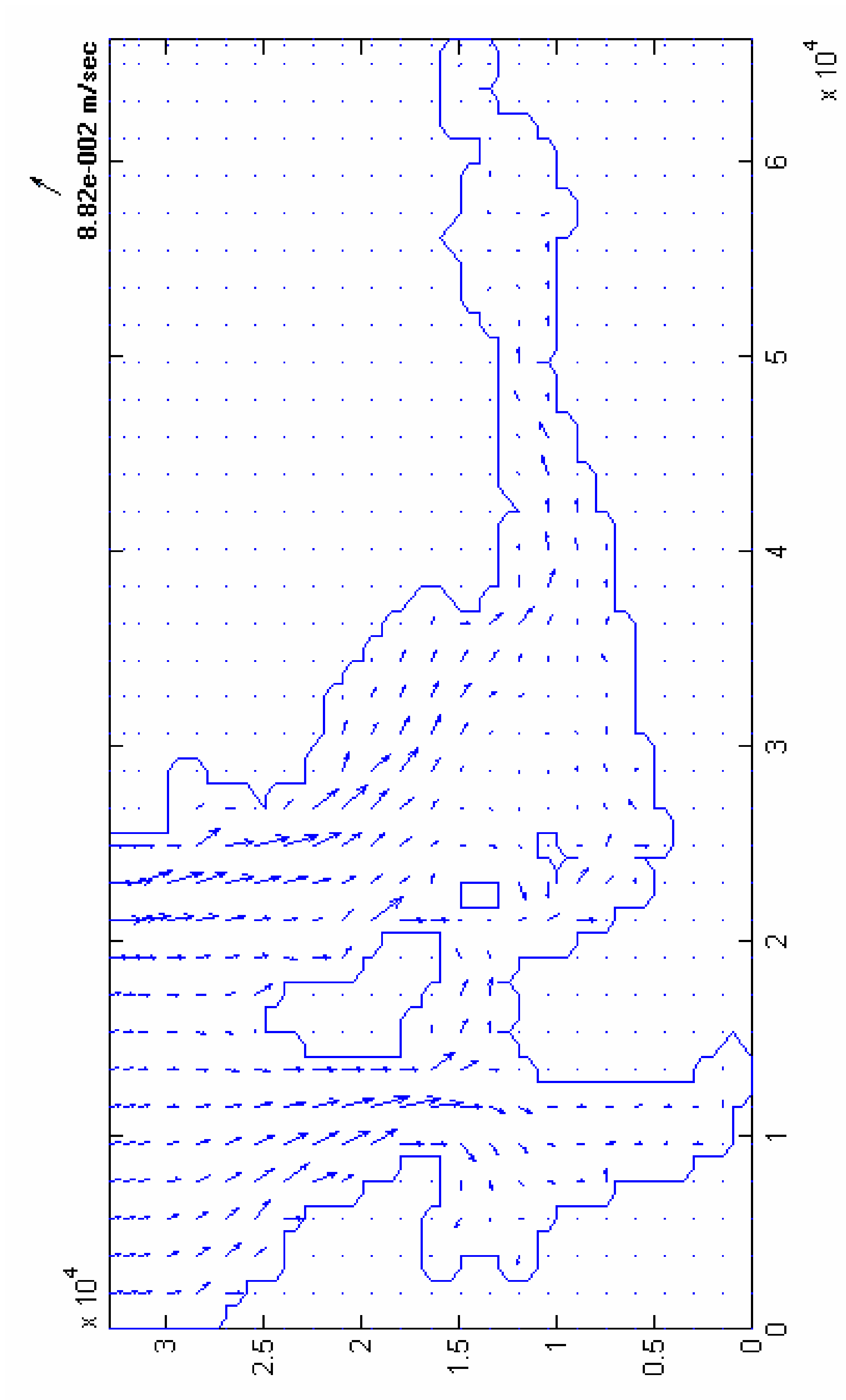

Figure 5.11 Depth average current pattern (NNW wind speed: 10 m/sec) 1 hour after beginning of the storm

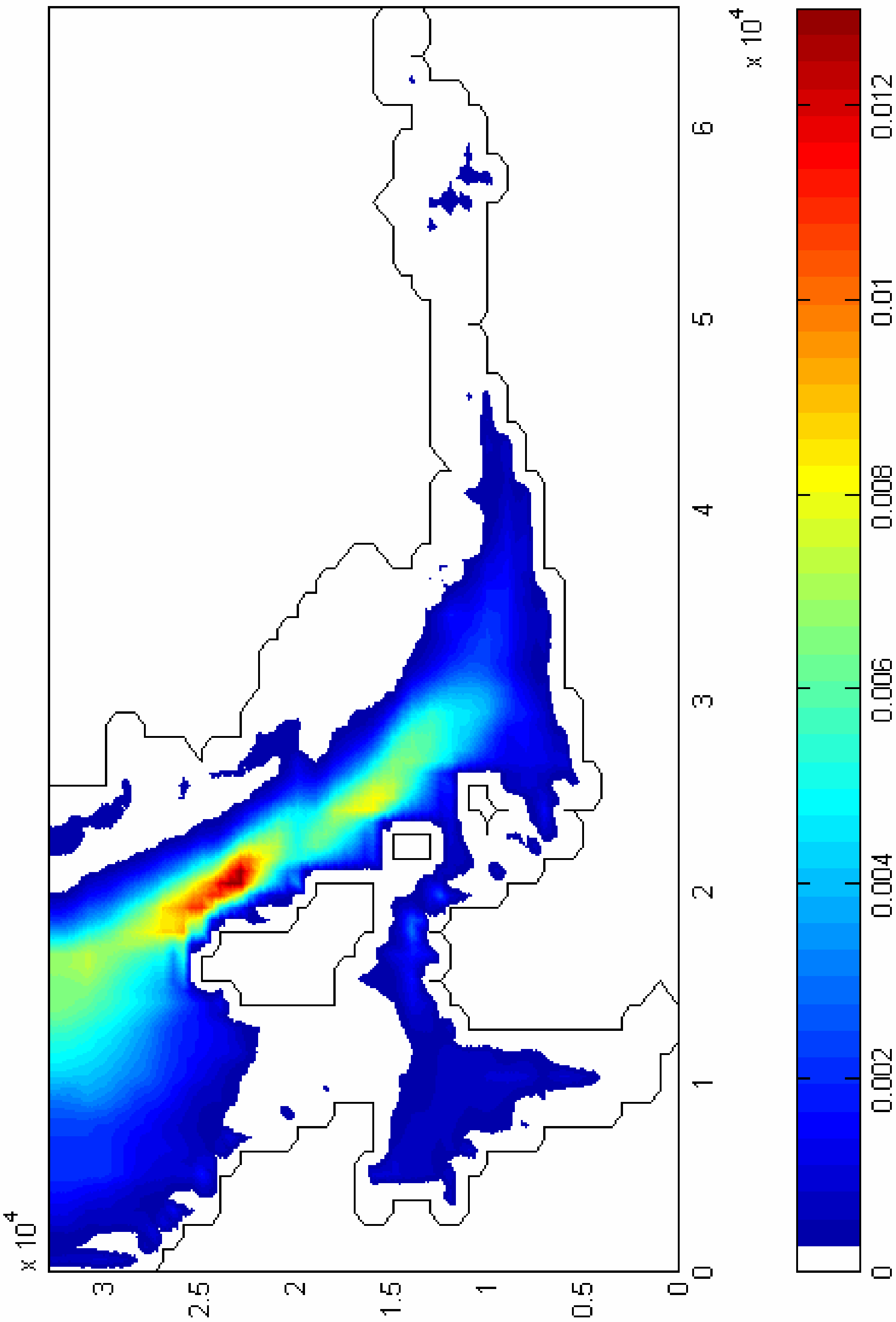

Figure 5.12 Values of shear stresses at the sea bed (NNW wind speed: 10 m/sec) steady state condition

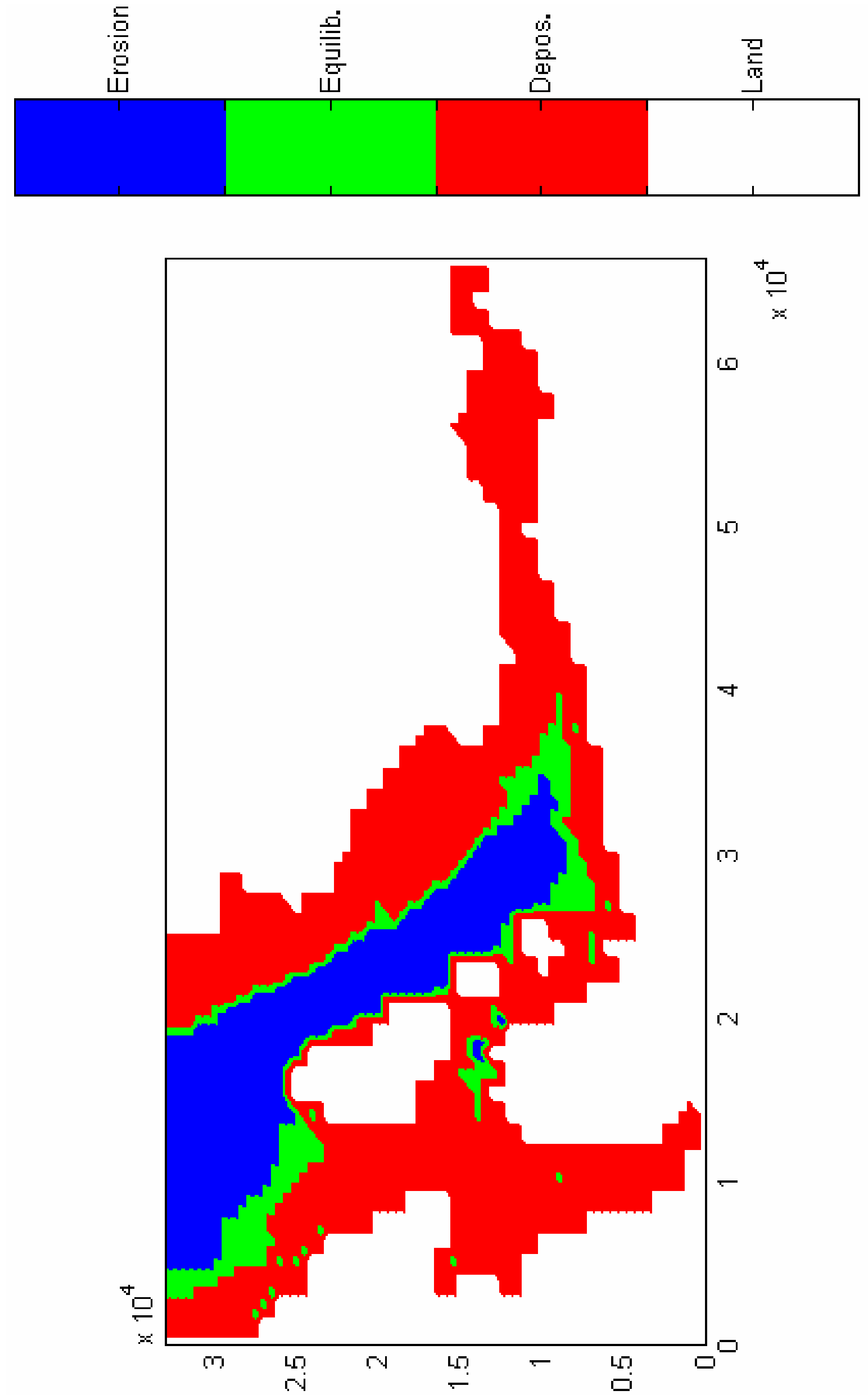

Figure 5.13 Deposition, re-suspension and equilibrium sites of the coastal area (NNW wind speed: 10 m/sec) steady state condition

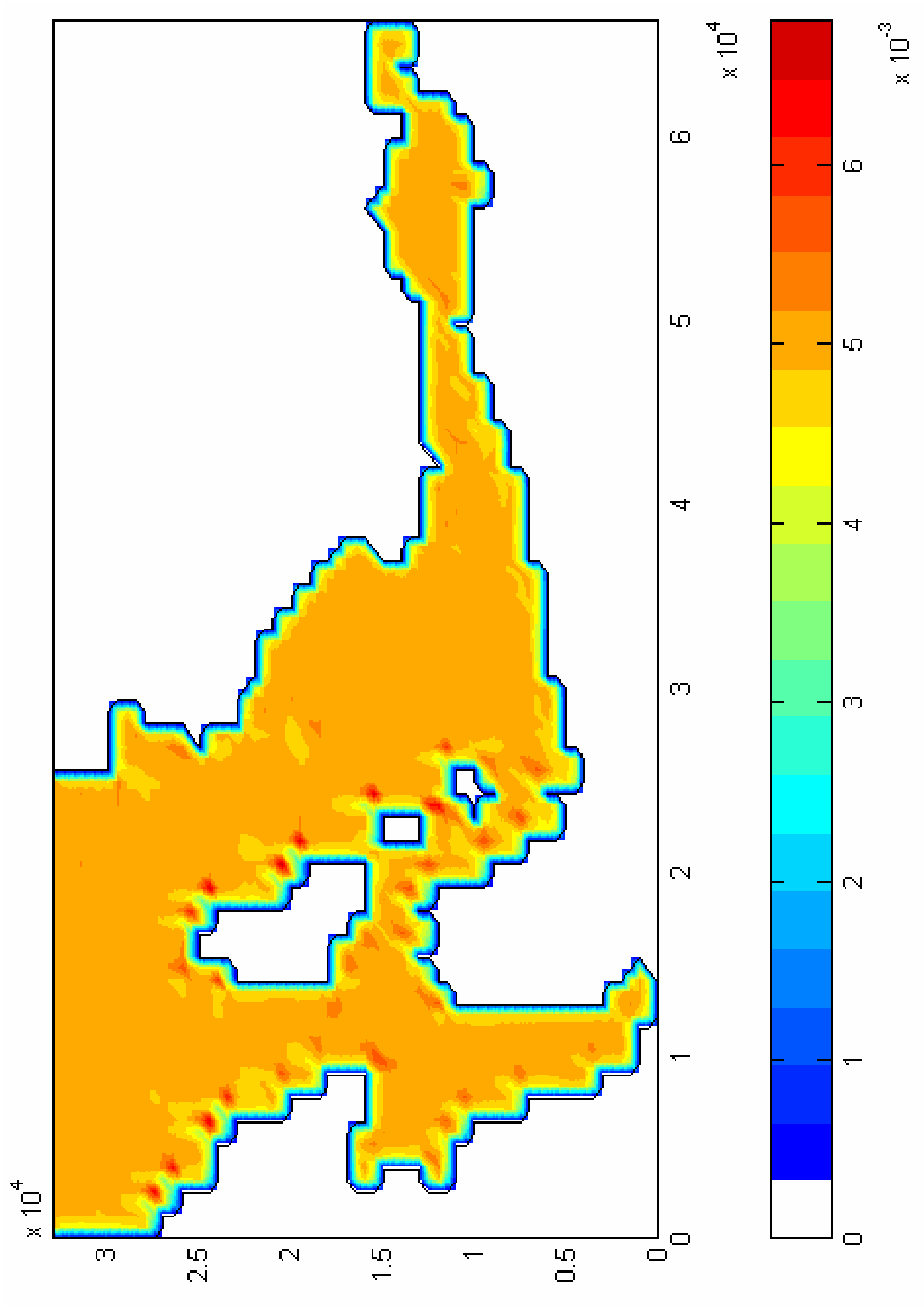

Figure 5.14 Distribution of suspended sediment concentration at the sea surface 1 hr after the storm (Initial value: 0.005 kg/m^3)

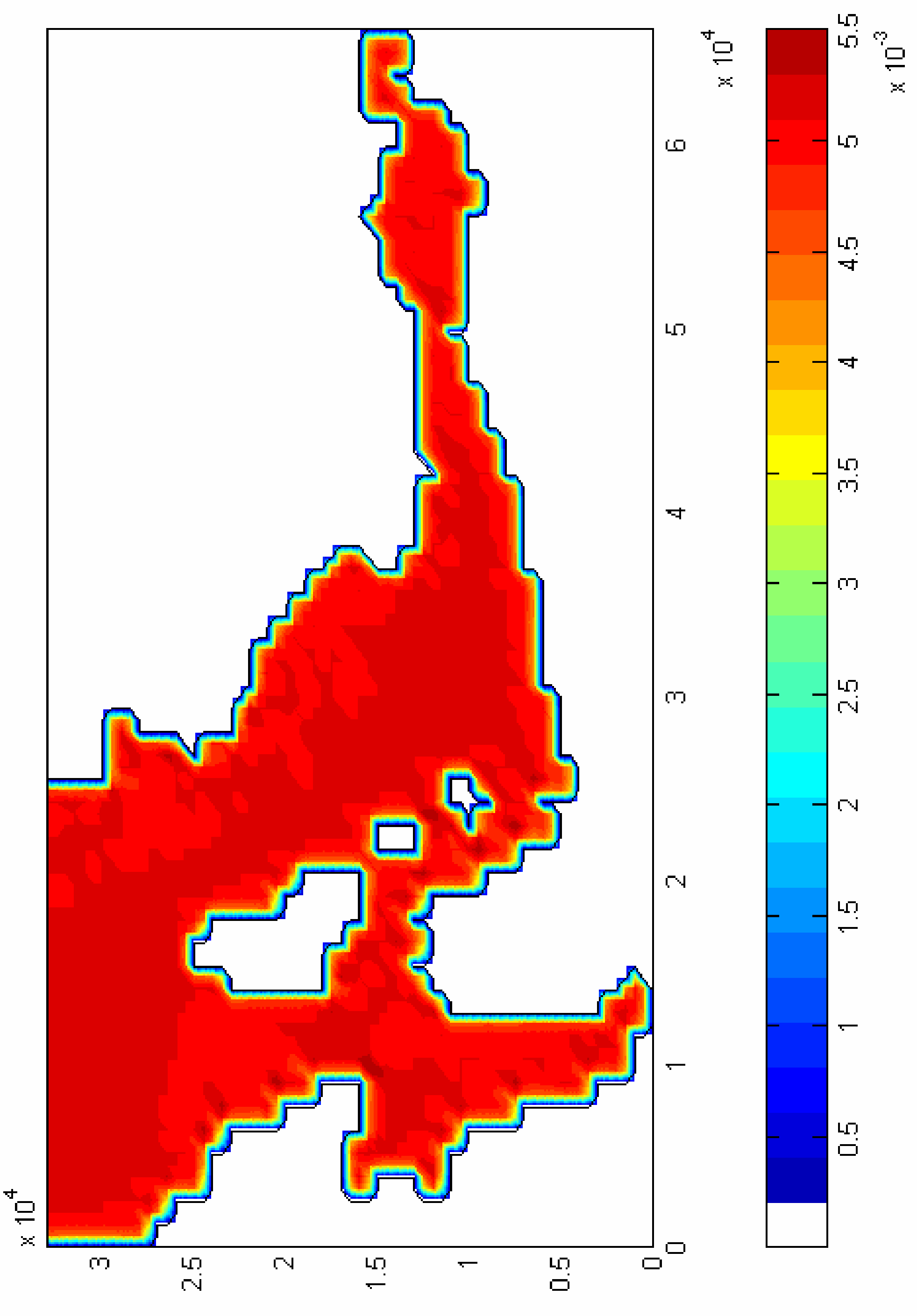

Figure 5.15 Distribution of suspended sediment concentration at the sea bed 1 hr after the storm (Initial value: 0.005 kg/m^3)

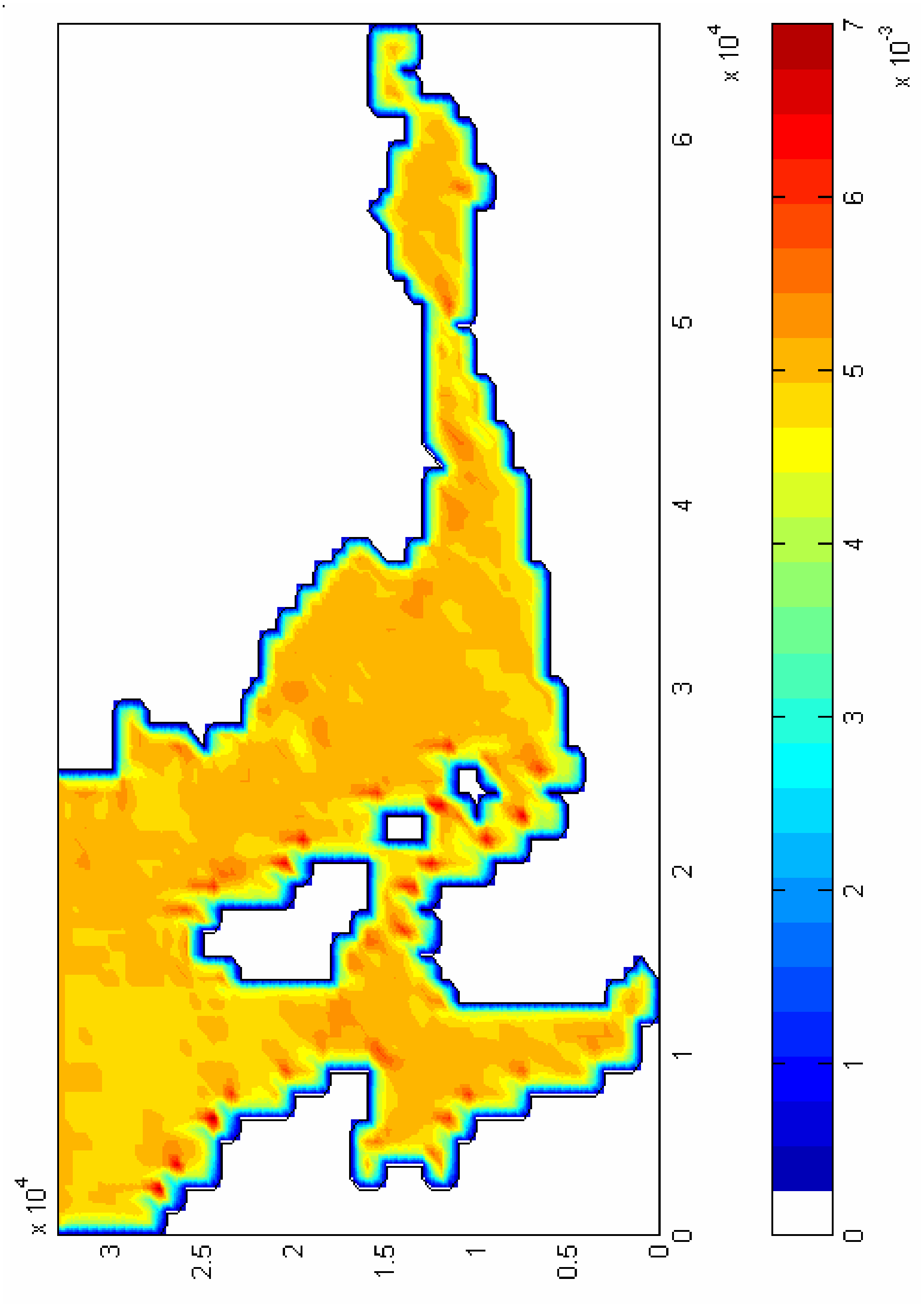

Figure 5.16 Distribution of suspended sediment concentration at the sea surface 2 hrs after the storm (Initial value: 0.005 kg/m^3)

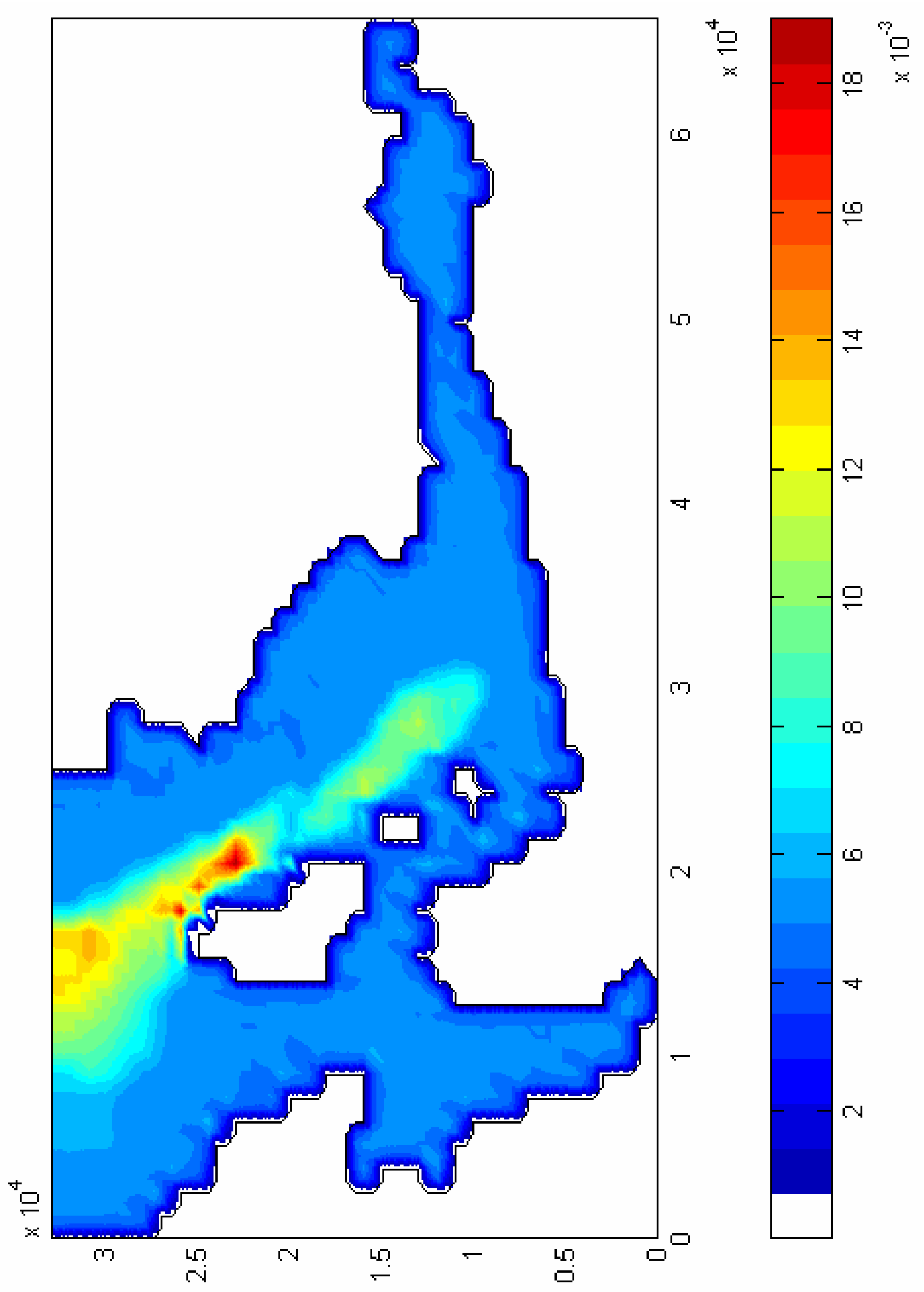

Figure 5.17 Distribution of suspended sediment concentration at the sea bed 2 hrs after the storm (Initial value: 0.005 kg/m^3)

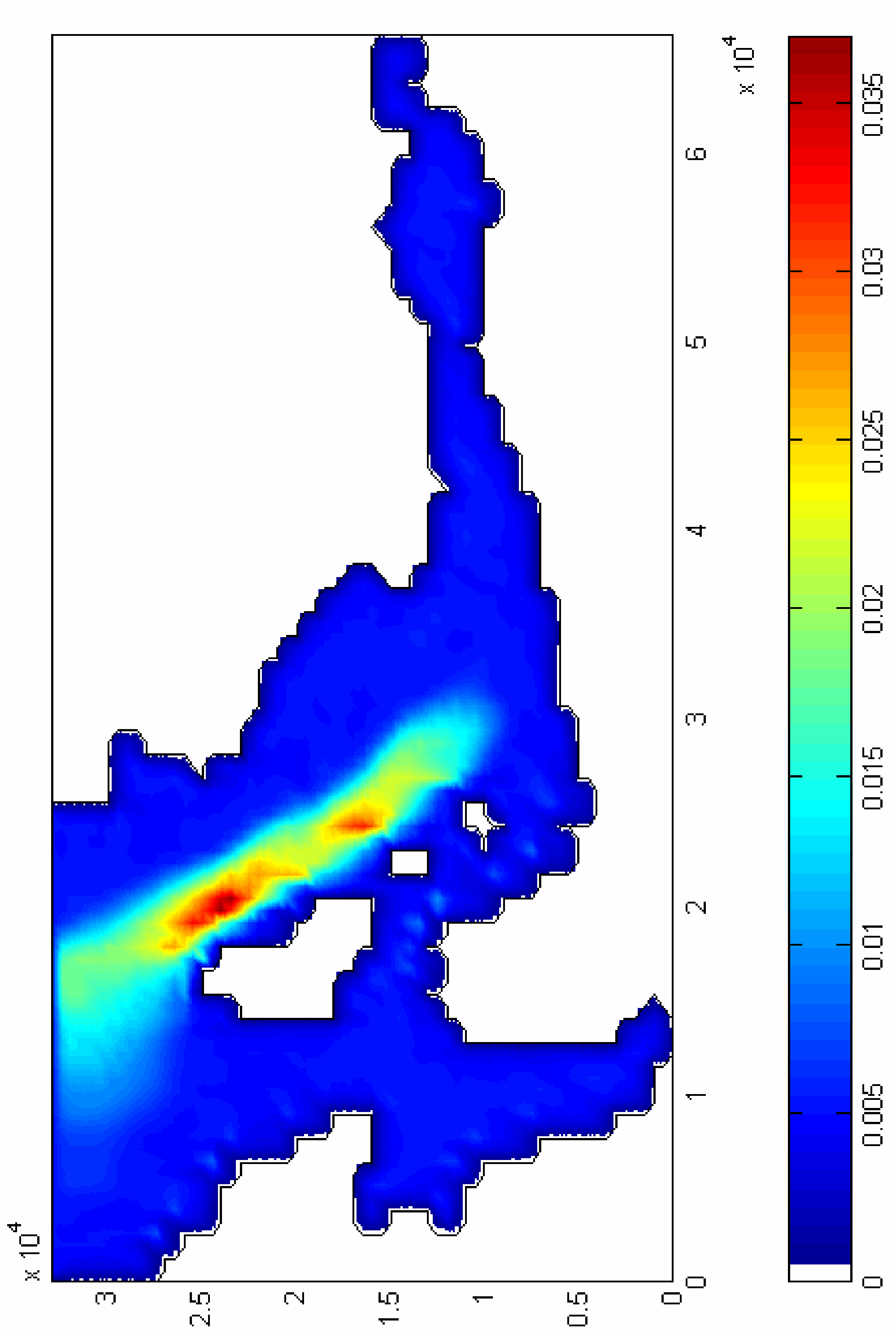

Figure 5.18 Distribution of suspended sediment concentration at the sea surface 5 hrs after the storm (Initial value: 0.005 kg/m^3)

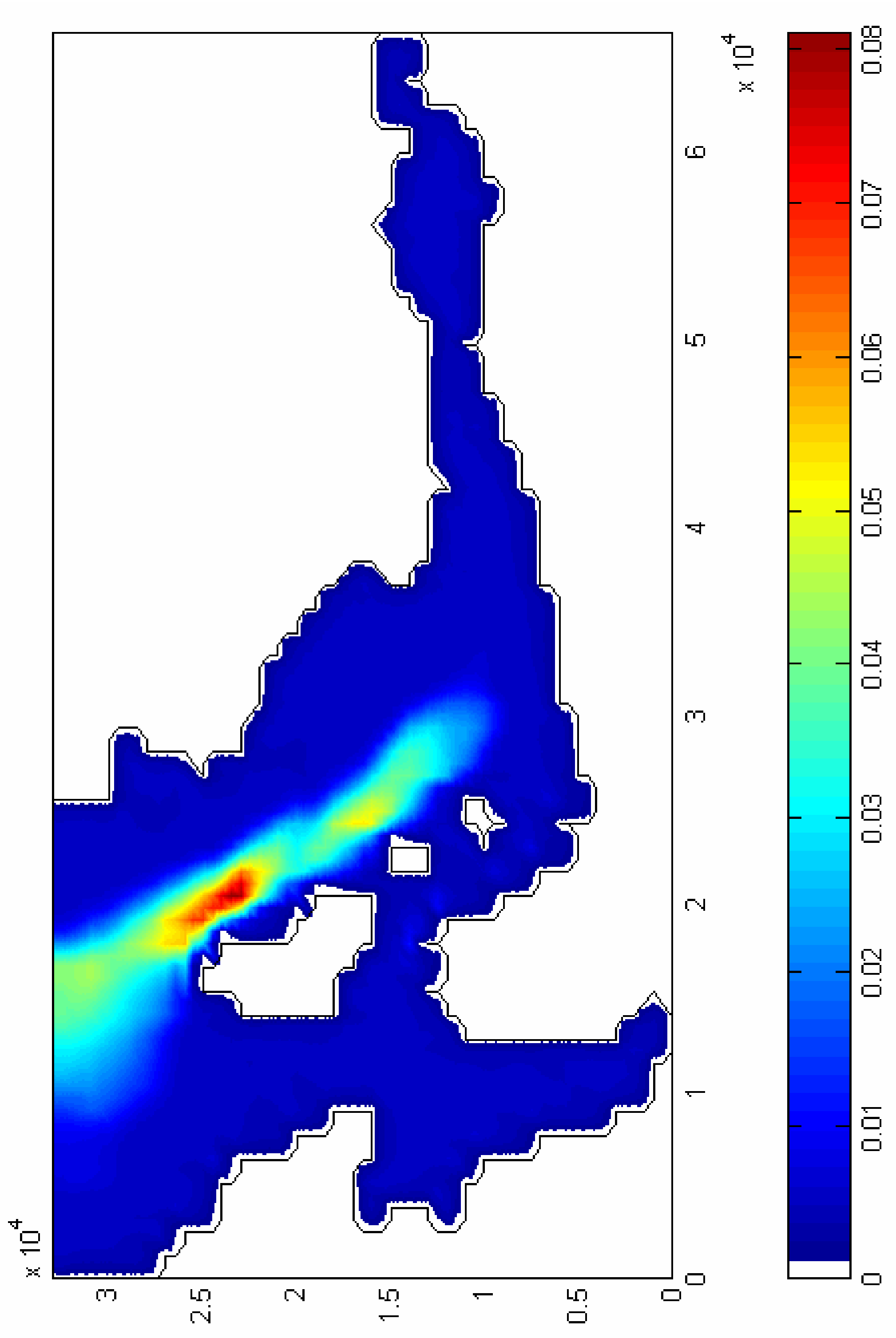

Figure 5.19 Distribution of suspended sediment concentration at the sea bed 5 hrs after the storm (Initial value: 0.005 kg/m^3)

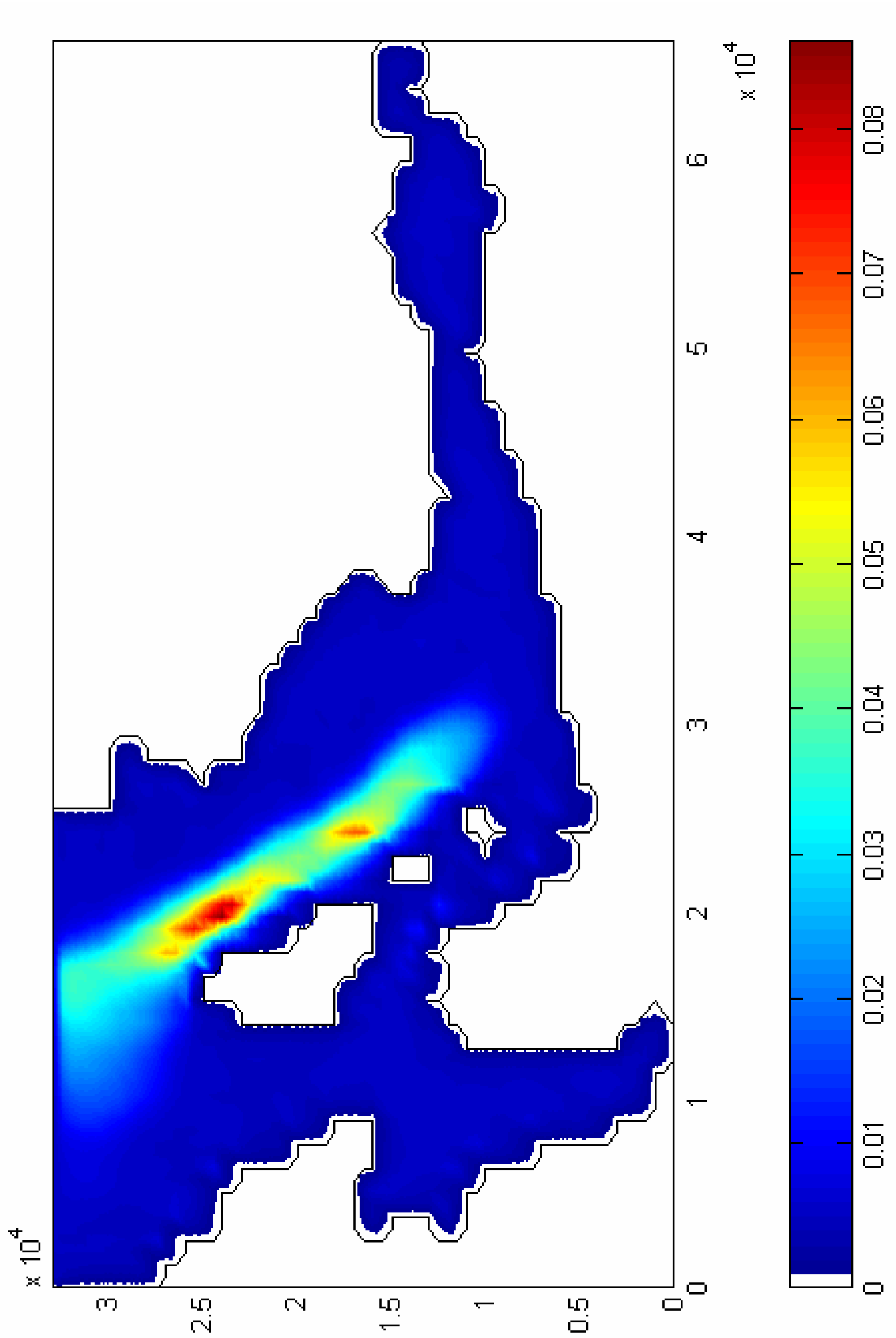

Figure 5.20 Distribution of suspended sediment concentration at the sea surface 8 hrs after the storm (Initial value: 0.005 kg/m^3)

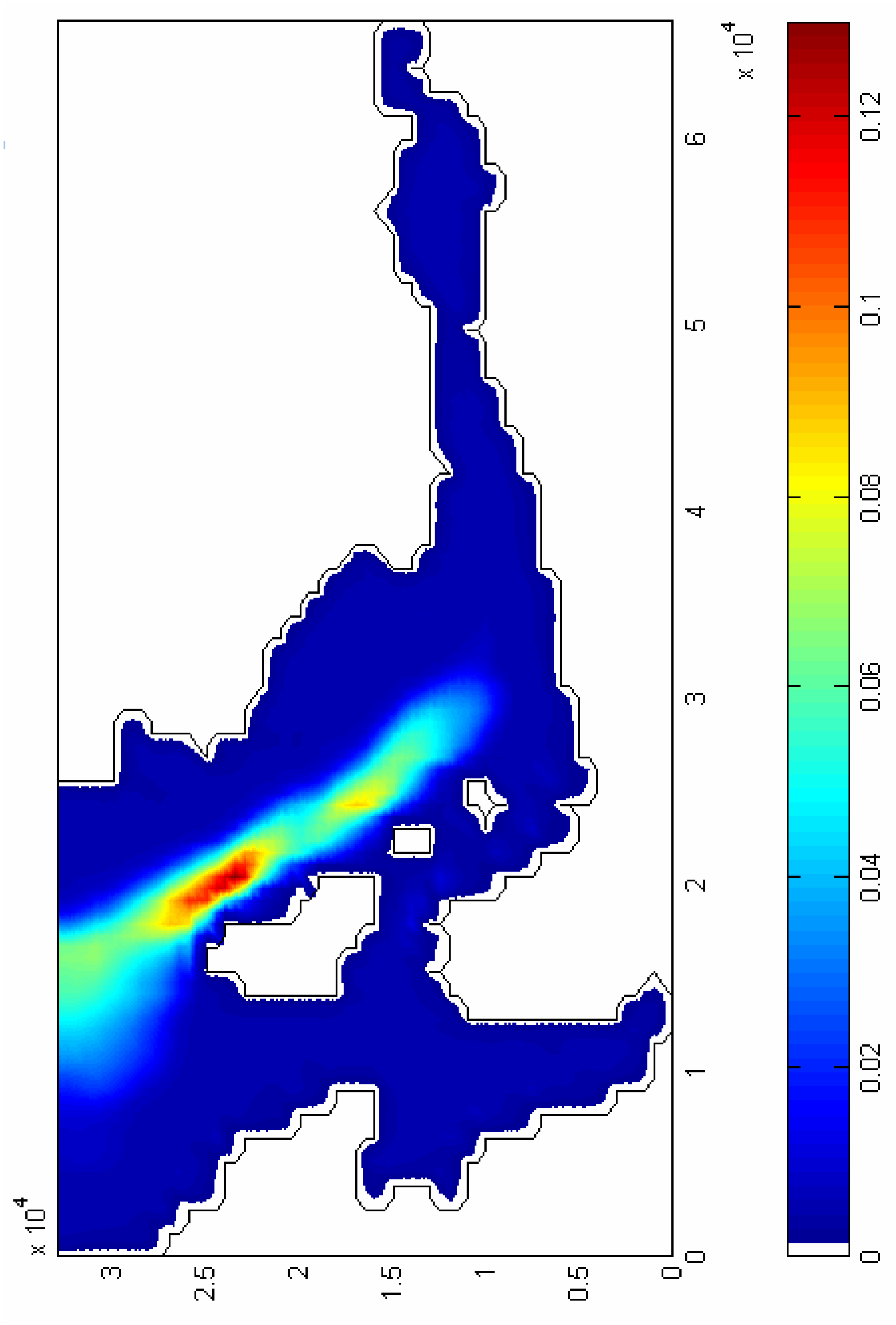

Figure 5.21 Distribution of suspended sediment concentration at the sea bed 8 hrs after the storm (Initial value: 0.005 kg/m^3)

CHAPTER VI

CONCLUSION

The followings are the main conclusions of this study:

1. The computer program prepared by Balas that computes the full spatial distribution of velocities of unsteady flow induced by wind, tide or water density differences is advanced to include suspended sediment transport component. The velocity components and eddy viscosities in *x, y, z* spaces together with water level fluctuations over the specified period of time are supplied by the hydrodynamic model component. The developed program computes the suspended sediment concentration profiles, determines the re-suspension, equilibrium, deposition sites of the coastal area and predicts the changes in sea bed morphology quantitatively. Using this program a sensitivity analysis providing a quantified assessment of the strengths and weaknesses of modeling and input data is derived and applied to the bay of Izmir.

2. The model consists of three components: hydrodynamic, turbulence and suspended sediment transport model components. In the hydrodynamic model hydrostatic pressure distribution assumption and Boussinesq approximation are used. The density gradient forces are introduced in the momentum equations through the hydrostatic pressure and horizontal variations of pressure (Balas, 1998). The suspended sediment transport model uses the assumptions of uniform and time independent bed

roughness parameter, time and space independent settling velocity and uniform and time independent sea bed factors that affect the critical shear stresses for re-suspension and deposition and re-suspension rate coefficient.

3. The mathematical program needs to be verified by using appropriate sediment sampling and concentration measurements because the parameters, that show the effect of settling velocity and bed roughness coefficient, can strongly vary in time and space. It is recommended to further analyze the model behavior by applying time and space dependent formulations for both the settling velocity for mud and the bed roughness coefficient.

4. Model results could only qualitatively be compared with field data. It is recommended to apply the model with more realistic boundary conditions and a realistic bathymetry in which sand and mud fraction was specified together with current and water surface elevation measurements and *in situ* erosion tests.

5. The numerical model can be utilized as a powerful design tool as it gives preoperational modeling simulations.

6. Further studies could be made on the following:
    i. Implementation of orthogonal curvilinear coordinate grids.
    ii. Implementation of higher order interpolating functions of Galerkin Weighted Residual Method.
    iii. The reflection of the effect of fluvial discharges and coastal land boundaries in the model.
    iv. The representation of flocculation by means of empirical formulations.

REFERENCES

Balas, L., Three Dimensional Numerical Modeling of Transport Processes in Coastal Water Bodies, PhD Thesis, The Graduate School of Natural and Applied Sciences, Middle East Technical University, Ankara, Turkey 1998.

Balas, L. and Özhan, E. "Application of a 3-D Numerical Model to Circulation in Coastal Waters", Coastal Engineering, Vol. 43, No. 2, pp. 99-120, 2001

Cancino L., Neves R., Hydrodynamic and sediment suspension modeling in estuarine systems Part I: Description of the numerical models, Journal of Marine Systems, Vol. 22, pp. 105–116, 1999.

Cancino L., Neves R., Hydrodynamic and sediment suspension modeling in estuarine systems Part II: Application to the Western Scheldt and Gironde Estuaries, Journal of Marine Systems, Vol. 22, pp. 117–131, 1999.

Douillet, P., Ouillon, S., Cordier, E., A numerical model for fine suspended sediment transport in the southwest lagoon of New Caledonia., Coral Reefs, Vol. 20, pp. 361 – 372, 2001.

Duman, M., Avcı, M., Duman, Ş., Demirkurt, E., Düzbastılar, M. K., Surficial sediment distribution and net sediment transport pattern in Izmir Bay, western Turkey, Continental Shelf Research, Vol. 24, pp. 965-981, 2004.

Lin, B. and Falconer, R. A., Numerical Modeling of Three-Dimensional Suspended Sediment For Estuarine and Coastal Waters, Journal of Hydraulic Research, IAHR, Vol. 34, No. 4, pp 435-456, 1996.

Nicholson J., O'Connor B. A., Cohesive Sediment Transport Model, J. Hydraulic Eng., Vol. 112, pp. 621-640, 1986.

Sanford, L. P.; Maa, J. P. Y., A unified erosion formulation for fine sediments, Marine Geology, Vol. 179, pp 9-23, 2001.

Sayin, E., Physical features of the Izmir Bay, Continental Shelf Research, Vol. 23, pp. 957-970, 2003.

Teisson, C., Cohesive suspended sediment transport: feasibility and limitations of numerical modeling, J. Hydraulic Res., Vol. 29, pp. 755-769, 1991.

Teisson, C., Ockenden, M., LeHir, P., Kranenburg, C., Hamm, L., Cohesive sediment transport processes, Coastal Engineering, Vol. 21, pp. 129-162, 1992.

van Ledden, M., Modelling of sand-mud mixtures Part II: A process-based sand-mud model, Delft Hydraulics (Z2840.00), The Netherlands, 2001.

Wu, Y., Falconer, R., A., A mass conservative 3-D numerical model for predicting solute fluxes in estuarine waters, Advances in Water Resources, Vol. 23, pp. 531-543, 1999.

APPENDIX A

FINITE ELEMENT APPROXIMATIONS

Application of the Galerkin Method is shown below where, $\widetilde{F},\widetilde{G},\widetilde{E}$ represent the variables of concern, as:

$$i = 1 \quad \int_{z_1}^{z_2} N_1.c.\widetilde{F}..dz \quad = c.\int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).dz$$

$$= c.\left(\int_{z_1}^{z_2} N_1^{\,2}.F_1^k.dz + \int_{z_1}^{z_2} N_1.N_2.F_2^k.dz\right) \quad = c.l_k\left(\frac{F_1^k}{3} + \frac{F_2^k}{6}\right) \tag{A.1}$$

$$i = 2 \quad \int_{z_1}^{z_2} N_2.c.\widetilde{F}..dz \quad = c.\int_{z_1}^{z_2} N_2.\left(N_1.F_1^k + N_2.F_2^k\right).dz$$

$$= c.\left(\int_{z_1}^{z_2} N_1.N_2.F_1^k.dz + \int_{z_1}^{z_2} N_1^{\,2}.F_2^k.dz\right) \quad = c.l_k\left(\frac{F_1^k}{6} + \frac{F_2^k}{3}\right) \tag{A.2}$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1.C.\frac{\partial \widetilde{F}}{\partial x}.dz = C.\left(\frac{\partial F_1^k}{\partial x}.\int_{z_1}^{z_2} N_1^{\,2}.dz + \frac{\partial F_2^k}{\partial x}.\int_{z_1}^{z_2} N_1.N_2.dz + \left(F_1^k - F_2^k\right).\int_{z_1}^{z_2} N_1.\frac{\partial N_1}{\partial x}.dz\right)$$

$$= C.\left(\frac{\partial F_1^k}{\partial x}.\frac{l_k}{3} + \frac{\partial F_2^k}{\partial x}.\frac{l_k}{6} + \left(F_1^k - F_2^k\right).\left[\frac{1}{6}.\frac{\partial l_k}{\partial x} + \frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\right]\right) \tag{A.3}$$

$$i = 2 \quad \int_{z_1}^{z_2} N_2.C.\frac{\partial \widetilde{F}}{\partial x}.dz \quad = C.\left(\frac{\partial F_1^k}{\partial x}.\int_{z_1}^{z_2} N_1.N_2.dz + \frac{\partial F_2^k}{\partial x}.\int_{z_1}^{z_2} N_2^{\,2}.dz + \left(F_1^k - F_2^k\right).\int_{z_1}^{z_2} N_2.\frac{\partial N_1}{\partial x}.dz\right)$$

$$= C.\left(\frac{\partial F_1^k}{\partial x}.\frac{l_k}{6} + \frac{\partial F_2^k}{\partial x}.\frac{l_k}{3} + \left(F_1^k - F_2^k\right).\left[\frac{1}{3}.\frac{\partial l_k}{\partial x} + \frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\right]\right) \tag{A.4}$$

$$i = 1 \qquad \int_{z_1}^{z_2} N_1.c.\frac{\partial^2 \tilde{F}}{\partial x^2}.dz \qquad = \int_{z_1}^{z_2} N_1.c.\frac{\partial}{\partial x}\left(\frac{\partial \tilde{F}}{\partial x}\right).dz \quad = c.\int_{z_1}^{z_2} N_1.\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(N_1.F_1^{\,k} + N_2.F_2^{\,k}\right)\right).dz$$

$$= c.\int_{z_1}^{z_2} N_1.\frac{\partial}{\partial x}\left(N_1.\frac{\partial F_1^{\,k}}{\partial x} + F_1^{\,k}.\frac{\partial N_1}{\partial x} + N_2.\frac{\partial F_2^{\,k}}{\partial x} + F_2^{\,k}.\frac{\partial N_2}{\partial x}\right).dz$$

$$= c.\int_{z_1}^{z_2} N_1.\left(\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{\,k}}{\partial x} + N_1.\frac{\partial^2 F_1^{\,k}}{\partial x^2} + \frac{\partial F_1^{\,k}}{\partial x}.\frac{\partial N_1}{\partial x} + F_1^{\,k}.\frac{\partial^2 N_1}{\partial x^2} + \frac{\partial N_2}{\partial x}.\frac{\partial F_2^{\,k}}{\partial x} + N_2.\frac{\partial^2 F_2^{\,k}}{\partial x^2} + \frac{\partial F_2^{\,k}}{\partial x}.\frac{\partial N_2}{\partial x} + F_2^{\,k}.\frac{\partial^2 N_2}{\partial x^2}\right).dz$$

$$= \quad c.\left\{ \quad \frac{\partial^2 F_1^{\,k}}{\partial x^2}.\int_{z_1}^{z_2} N_1^{\,2}.dz + \quad \frac{\partial^2 F_2^{\,k}}{\partial x^2}.\int_{z_1}^{z_2} N_1.N_2.dz \quad + \quad 2.\left(\frac{\partial F_1^{\,k}}{\partial x} - \frac{\partial F_2^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_1.\frac{\partial N_1}{\partial x}.dz \quad + \left(F_1^{\,k} - F_2^{\,k}\right).\int_{z_1}^{z_2} N_1.\frac{\partial^2 N_1}{\partial x^2}.dx \quad \right\}$$

$$= \quad c.\left\{ \quad l_k.\left(\frac{1}{3}.\frac{\partial^2 F_1^{\,k}}{\partial x^2} + \frac{1}{6}.\frac{\partial^2 F_2^{\,k}}{\partial x^2}\right) + 2.\left(\frac{\partial F_1^{\,k}}{\partial x} - \frac{\partial F_2^{\,k}}{\partial x}\right).\left[\frac{1}{6}.\frac{\partial l_k}{\partial x} + \frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\right]\right.$$

$$\left. + \left(F_1^{\,k} - F_2^{\,k}\right).\left\{\frac{1}{6}.\left(-\frac{2}{l_k}.\left(\frac{\partial l_k}{\partial x}\right)^2 + \frac{\partial^2 l_k}{\partial x^2}\right) + \frac{1}{2}.\left[\frac{z_2}{l_k}.\frac{\partial^2 z_1}{\partial x^2} - \frac{z_1}{l_k}.\frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial x} - \frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial x}\right) - \frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial x} - \frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial x}\right)\right]\right]\right\}\right\} \qquad \text{(A.5)}$$

$$i = 2 \qquad \int_{z_1}^{z_2} N_2.c.\frac{\partial^2 \tilde{F}}{\partial x^2}.dz$$

$$= c.\int_{z_1}^{z_2} N_2.\left(\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{\,k}}{\partial x} + N_1.\frac{\partial^2 F_1^{\,k}}{\partial x^2} + \frac{\partial F_1^{\,k}}{\partial x}.\frac{\partial N_1}{\partial x} + F_1^{\,k}.\frac{\partial^2 N_1}{\partial x^2} + \frac{\partial N_2}{\partial x}.\frac{\partial F_2^{\,k}}{\partial x} + N_2.\frac{\partial^2 F_2^{\,k}}{\partial x^2} + \frac{\partial F_2^{\,k}}{\partial x}.\frac{\partial N_2}{\partial x} + F_2^{\,k}.\frac{\partial^2 N_2}{\partial x^2}\right).dz$$

$$= \quad c.\left\{ \quad \frac{\partial^2 F_2^{\,k}}{\partial x^2}.\int_{z_1}^{z_2} N_2^{\;2}.dz + \quad \frac{\partial^2 F_1^{\,k}}{\partial x^2}.\int_{z_1}^{z_2} N_1.N_2.dz + \quad 2.\left(\frac{\partial F_1^{\,k}}{\partial x} - \frac{\partial F_2^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_2.\frac{\partial N_1}{\partial x}.dz \quad + \quad \left(F_1^{\,k} - F_2^{\,k}\right).\int_{z_1}^{z_2} N_2.\frac{\partial^2 N_1}{\partial x^2}.dx \quad \right\}$$

$$= c.\left\{ \quad l_k.\left(\frac{1}{6}.\frac{\partial^2 F_1^{\,k}}{\partial x^2} + \frac{1}{3}.\frac{\partial^2 F_2^{\,k}}{\partial x^2}\right) \quad + \quad 2.\left(\frac{\partial F_1^{\,k}}{\partial x} - \frac{\partial F_2^{\,k}}{\partial x}\right).\left[\frac{1}{3}.\frac{\partial l_k}{\partial x} + \frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\right]\right.$$

$$\left. + \left(F_1^{\,k} - F_2^{\,k}\right).\left\{\frac{1}{3}.\left(-\frac{2}{l_k}.\left(\frac{\partial l_k}{\partial x}\right)^2 + \frac{\partial^2 l_k}{\partial x^2}\right) + \frac{1}{2}\left[\frac{z_2}{l_k}.\frac{\partial^2 z_1}{\partial x^2} - \frac{z_1}{l_k}.\frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial x} - \frac{2.z_2}{l_k^{\;2}}.\frac{\partial l_k}{\partial x}\right) - \frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial x} - \frac{2.z_1}{l_k^{\;2}}.\frac{\partial l_k}{\partial x}\right)\right]\right\} \quad \right\} \qquad \text{(A.6)}$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1.c.\frac{\partial}{\partial y}\left(\frac{\partial \tilde{F}}{\partial x}\right).dz \quad = c.\int_{z_1}^{z_2} N_1.\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(N_1.F_1^{\,k} + N_2.F_2^{\,k}\right)\right).dz \quad = c.\int_{z_1}^{z_2} N_1.\frac{\partial}{\partial y}\left(N_1.\frac{\partial F_1^{\,k}}{\partial x} + F_1^{\,k}.\frac{\partial N_1}{\partial x} + N_2.\frac{\partial F_2^{\,k}}{\partial x} + F_2^{\,k}.\frac{\partial N_2}{\partial x}\right).dz$$

$$= c.\int_{z_1}^{z_2} N_1.\left(\frac{\partial N_1}{\partial y}.\frac{\partial F_1^{\,k}}{\partial x} + N_1.\frac{\partial}{\partial y}\left(\frac{\partial F_1^{\,k}}{\partial x}\right) + \frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial N_1}{\partial x} + F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right) + \frac{\partial N_2}{\partial y}.\frac{\partial F_2^{\,k}}{\partial x} + N_2.\frac{\partial}{\partial y}\left(\frac{\partial F_2^{\,k}}{\partial x}\right) + \frac{\partial F_2^{\,k}}{\partial y}.\frac{\partial N_2}{\partial x} + F_2^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial N_2}{\partial x}\right)\right).dz$$

$$= \quad c.\left\{ \quad \frac{\partial}{\partial y}\left(\frac{\partial F_1^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_1^{\;2}.dz \quad + \quad \frac{\partial}{\partial y}\left(\frac{\partial F_2^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_1.N_2.dz \right.$$

$$\left. + \quad \left(\frac{\partial F_1^{\,k}}{\partial y} - \frac{\partial F_2^{\,k}}{\partial y}\right).\int_{z_1}^{z_2} N_1.\frac{\partial N_1}{\partial x}.dz \quad + \quad \left(\frac{\partial F_1^{\,k}}{\partial x} - \frac{\partial F_2^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_1.\frac{\partial N_1}{\partial y}.dz \quad + \quad \left(F_1^{\,k} - F_2^{\,k}\right).\int_{z_1}^{z_2} N_1.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).dz \quad \right\}$$

$$= c.\left\{ \quad l_k.\left[\frac{1}{3}.\frac{\partial}{\partial y}\left(\frac{\partial F_1^{\,k}}{\partial x}\right)+\frac{1}{6}\frac{\partial}{\partial y}\left(\frac{\partial F_2^{\,k}}{\partial x}\right)\right] \quad + \quad \left(\frac{\partial F_1^{\,k}}{\partial y}-\frac{\partial F_2^{\,k}}{\partial y}\right).\left[\frac{1}{6}.\frac{\partial l_k}{\partial x}+\frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\right]\right.$$

$$+ \quad \left(\frac{\partial F_1^{\,k}}{\partial x}-\frac{\partial F_2^{\,k}}{\partial x}\right).\left[\frac{1}{6}.\frac{\partial l_k}{\partial y}+\frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial y}-\frac{z_1}{l_k}.\frac{\partial z_2}{\partial y}\right)\right]$$

$$\left.+\left(F_1^{\,k}-F_2^{\,k}\right).\left\{\frac{1}{6}.\left[-\frac{2}{l_k}.\frac{\partial l_k}{\partial x}.\frac{\partial l_k}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial l_k}{\partial x}\right)\right]+\frac{1}{2}.\left[\frac{z_2}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_1}{\partial x}\right)-\frac{z_1}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_2}{\partial x}\right)+\frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial y}-\frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)-\frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial y}-\frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)\right]\right\}\right\} \qquad \text{(A.7)}$$

$$i = 2 \quad \int_{z_1}^{z_2} N_2.c.\frac{\partial}{\partial y}\left(\frac{\partial \tilde{F}}{\partial x}\right).dz \quad = \quad c.\left\{ \quad \frac{\partial}{\partial y}\left(\frac{\partial F_2^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_2^{\,2}.dz \quad + \quad \frac{\partial}{\partial y}\left(\frac{\partial F_1^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_1.N_2.dz\right.$$

$$\left.+ \quad \left(\frac{\partial F_1^{\,k}}{\partial y}-\frac{\partial F_2^{\,k}}{\partial y}\right).\int_{z_1}^{z_2} N_2.\frac{\partial N_1}{\partial x}.dz \; + \; \left(\frac{\partial F_1^{\,k}}{\partial x}-\frac{\partial F_2^{\,k}}{\partial x}\right).\int_{z_1}^{z_2} N_2.\frac{\partial N_1}{\partial y}.dz \; + \; \left(F_1^{\,k}-F_2^{\,k}\right).\int_{z_1}^{z_2} N_2.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).dz \quad \right\}$$

$$= c.\left\{ \quad l_k.\left[\frac{1}{6}.\frac{\partial}{\partial y}\left(\frac{\partial F_1^{\,k}}{\partial x}\right)+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial F_2^{\,k}}{\partial x}\right)\right] \quad + \quad \left(\frac{\partial F_1^{\,k}}{\partial y}-\frac{\partial F_2^{\,k}}{\partial y}\right).\left[\frac{1}{3}.\frac{\partial l_k}{\partial x}+\frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\right]\right.$$

$$+ \quad \left(\frac{\partial F_1^{\,k}}{\partial x}-\frac{\partial F_2^{\,k}}{\partial x}\right).\left[\frac{1}{3}.\frac{\partial l_k}{\partial y}+\frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial y}-\frac{z_1}{l_k}.\frac{\partial z_2}{\partial y}\right)\right]$$

$$+\left(F_1^{\,k}-F_2^{\,k}\right).\left\{\frac{1}{3}.\left[-\frac{2}{l_k}.\frac{\partial l_k}{\partial x}.\frac{\partial l_k}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial l_k}{\partial x}\right)\right]+\frac{1}{2}.\left[\frac{z_2}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_1}{\partial x}\right)-\frac{z_1}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_2}{\partial x}\right)+\frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial y}-\frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)-\frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial y}-\frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)\right]\right\}$$

(A.8)

$$i=1\qquad \int_{z_1}^{z_2} N_1.\frac{\partial\tilde{F}}{\partial y}.\frac{\partial\tilde{G}}{\partial x}.dz \qquad =\int_{z_1}^{z_2} N_1.\frac{\partial}{\partial y}\left(N_1.F_1^{\,k}+N_2.F_2^{\,k}\right).\frac{\partial}{\partial x}\left(N_1.G_1^{k}+N_2.G_2^{k}\right).dz$$

$$=\int_{z_1}^{z_2} N_1.\left(N_1.\frac{\partial F_1^{\,k}}{\partial y}+F_1^{\,k}.\frac{\partial N_1}{\partial y}+N_2.\frac{\partial F_2^{\,k}}{\partial y}+F_2^{\,k}.\frac{\partial N_2}{\partial y}\right).\left(N_1.\frac{\partial G_1^{k}}{\partial x}+G_1^{\,k}.\frac{\partial N_1}{\partial x}+N_2.\frac{\partial G_2^{k}}{\partial x}+G_2^{\,k}.\frac{\partial N_2}{\partial x}\right).dz$$

$$=\int_{z_1}^{z_2} N_1.\left(N_1^{\,2}.\frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial G_1^{k}}{\partial x}+N_1.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{\,k}}{\partial y}.G_1^{k}+N_1.N_2.\frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial G_2^{k}}{\partial x}+N_1.\frac{\partial N_2}{\partial x}.\frac{\partial F_1^{\,k}}{\partial y}.G_2^{k}+N_1.\frac{\partial N_1}{\partial y}.F_1^{\,k}.\frac{\partial G_1^{k}}{\partial x}+\frac{\partial N_1}{\partial x}.\frac{\partial N_1}{\partial y}.F_1^{\,k}.G_1^{k}+N_2.\frac{\partial N_1}{\partial y}.F_1^{\,k}.\frac{\partial G_2^{k}}{\partial x}+\frac{\partial N_1}{\partial y}.\frac{\partial N_2}{\partial x}.F_1^{\,k}.G_2^{k}\right.$$

$$\left.N_1.N_2.\frac{\partial F_2^{\,k}}{\partial y}.\frac{\partial G_1^{k}}{\partial x}+N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^{\,k}}{\partial y}.G_1^{k}+N_2^{\,2}.\frac{\partial F_2^{\,k}}{\partial y}.\frac{\partial G_2^{k}}{\partial x}+N_2.\frac{\partial N_2}{\partial x}.\frac{\partial F_2^{\,k}}{\partial y}.G_2^{k}+N_1.\frac{\partial N_2}{\partial y}.F_2^{\,k}.\frac{\partial G_1^{k}}{\partial x}+\frac{\partial N_1}{\partial x}.\frac{\partial N_2}{\partial y}.F_2^{\,k}.G_1^{k}+N_2.\frac{\partial N_2}{\partial y}.F_2^{\,k}.\frac{\partial G_2^{k}}{\partial x}+\frac{\partial N_2}{\partial x}.\frac{\partial N_2}{\partial y}.F_2^{\,k}.G_2^{k}\right)..dz$$

$$=\int_{z_1}^{z_2}\left(\frac{\partial F_1^{k}}{\partial y}.\frac{\partial G_1^{k}}{\partial x}.N_1^{\,3}+\frac{\partial F_1^{k}}{\partial y}.G_1^{k}.N_1^{\,2}.\frac{\partial N_1}{\partial x}+N_1^{\,2}.N_2.\frac{\partial F_1^{k}}{\partial y}.\frac{\partial G_2^{k}}{\partial x}-N_1^{\,2}.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{k}}{\partial y}.G_2^{k}+N_1^{\,2}.\frac{\partial N_1}{\partial y}.F_1^{\,k}.\frac{\partial G_1^{k}}{\partial x}+N_1.\frac{\partial N_1}{\partial x}.\frac{\partial N_1}{\partial y}.F_1^{\,k}.G_1^{k}+N_1.N_2.\frac{\partial N_1}{\partial y}.F_1^{\,k}.\frac{\partial G_2^{k}}{\partial x}-N_1.\frac{\partial N_1}{\partial y}.\frac{\partial N_1}{\partial x}.F_1^{\,k}.G_2^{k}\right.$$

$$\left.N_1^{\,2}.N_2.\frac{\partial F_2^{k}}{\partial y}.\frac{\partial G_1^{k}}{\partial x}+N_1.N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^{k}}{\partial y}.G_1^{k}+N_1.N_2^{\,2}.\frac{\partial F_2^{k}}{\partial y}.\frac{\partial G_2^{k}}{\partial x}-N_1.N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^{k}}{\partial y}.G_2^{k}-N_1^{\,2}.\frac{\partial N_1}{\partial y}.F_2^{k}.\frac{\partial G_1^{k}}{\partial x}-N_1.\frac{\partial N_1}{\partial x}.\frac{\partial N_1}{\partial y}.F_2^{k}.G_1^{k}-N_1.N_2.\frac{\partial N_1}{\partial y}.F_2^{k}.\frac{\partial G_2^{k}}{\partial x}+N_1.\frac{\partial N_1}{\partial x}.\frac{\partial N_1}{\partial y}.F_2^{k}.G_2^{k}\right)..dz$$

$$=\qquad \frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial G_1^{\,k}}{\partial x}.\int_{z_1}^{z_2} N_1^{\,3}.dz \qquad + \qquad \left[\frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial G_2^{\,k}}{\partial x}+\frac{\partial F_2^{\,k}}{\partial y}.\left(\frac{\partial G_1^{\,k}}{\partial x}+\frac{\partial G_2^{\,k}}{\partial x}\right)\right].\int_{z_1}^{z_2} N_1^{\,2}.N_2.dz \qquad +\frac{\partial F_1^{\,k}}{\partial y}.\left(G_1^{k}-G_2^{k}\right).\int_{z_1}^{z_2} N_1^{\,2}.\frac{\partial N_1}{\partial x}.dz$$

$$+\quad \frac{\partial G_1^k}{\partial x}.\left(F_1^{\,k}-F_2^{\,k}\right).\int_{z_1}^{z_2} N_1^{\,2}.\frac{\partial N_1}{\partial y}.dz \quad + \quad \frac{\partial F_2^{\,k}}{\partial y}.\left(G_1^k-G_2^k\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz + \frac{\partial G_2^k}{\partial x}.\left(F_1^{\,k}-F_2^{\,k}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial y}.dz$$

$$+\quad \left(F_1^{\,k}-F_2^{\,k}\right)\!.\!\left(G_1^k-G_2^k\right).\int_{z_1}^{z_2} N_1.\frac{\partial N_1}{\partial x}.\frac{\partial N_1}{\partial y}.dz$$

$$=\quad \frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial G_1^k}{\partial x}.\frac{l_k}{4} \quad + \quad \left[\frac{\partial F_1^{\,k}}{\partial y}.\frac{\partial G_2^k}{\partial x}+\frac{\partial F_2^{\,k}}{\partial y}.\left(\frac{\partial G_1^k}{\partial x}+\frac{\partial G_2^k}{\partial x}\right)\right].\frac{l_k}{12}$$

$$+\quad \frac{\partial F_1^{\,k}}{\partial y}.\left(G_1^k-G_2^k\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial x}+\frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right) \quad + \quad \frac{\partial G_1^k}{\partial x}.\left(F_1^{\,k}-F_2^{\,k}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial y}+\frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial y}-\frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial y}\right)$$

$$+\quad \frac{\partial F_2^{\,k}}{\partial y}.\left(G_1^k-G_2^k\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial x}+\frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right) \quad + \quad \frac{\partial G_2^k}{\partial x}.\left(F_1^{\,k}-F_2^{\,k}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial y}+\frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial y}-\frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial y}\right)$$

$$+\left(F_1^{\,k}-F_2^{\,k}\right)\!.\!\left(G_1^k-G_2^k\right)$$

$$.\left\{\left(\frac{1}{12.l_k}.\frac{\partial l_k}{\partial x}.\frac{\partial l_k}{\partial y}\right)+\frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)\!.\!\left(\frac{z_2}{l_k^{\,2}}.\frac{\partial z_1}{\partial y}-\frac{z_1}{l_k^{\,2}}.\frac{\partial z_2}{\partial y}\right)+\frac{1}{6}.\left[\frac{\partial l_k}{\partial y}.\left(\frac{z_2}{l_k^{\,2}}.\frac{\partial z_1}{\partial x}-\frac{z_1}{l_k^{\,2}}.\frac{\partial z_2}{\partial x}\right)+\frac{\partial l_k}{\partial x}.\left(\frac{z_2}{l_k^{\,2}}.\frac{\partial z_1}{\partial y}-\frac{z_1}{l_k^{\,2}}.\frac{\partial z_2}{\partial y}\right)\right]\right\} \qquad \text{(A.9)}$$

$$i=2 \qquad \int_{z_1}^{z_2} N_2.\frac{\partial \widetilde{F}}{\partial y}.\frac{\partial \widetilde{G}}{\partial x}.dz$$

$$= \int_{z_1}^{z_2} N_2 . \left(N_1^{\;2} . \frac{\partial F_1^k}{\partial y} . \frac{\partial G_1^k}{\partial x} + N_1 . \frac{\partial N_1}{\partial x} . \frac{\partial F_1^k}{\partial y} . G_1^k + N_1 . N_2 . \frac{\partial F_1^k}{\partial y} . \frac{\partial G_2^k}{\partial x} + N_1 . \frac{\partial N_2}{\partial x} . \frac{\partial F_1^k}{\partial y} . G_2^k + N_1 . \frac{\partial N_1}{\partial y} . F_1^k . \frac{\partial G_1^k}{\partial x} + \frac{\partial N_1}{\partial x} . \frac{\partial N_1}{\partial y} . F_1^k . G_1^k + N_2 . \frac{\partial N_1}{\partial y} . F_1^k . \frac{\partial G_2^k}{\partial x} + \frac{\partial N_1}{\partial y} . \frac{\partial N_2}{\partial x} . F_1^k . G_2^k \right.$$

$$\left. N_1 . N_2 . \frac{\partial F_2^k}{\partial y} . \frac{\partial G_1^k}{\partial x} + N_2 . \frac{\partial N_1}{\partial x} . \frac{\partial F_2^k}{\partial y} . G_1^k + N_2^{\;2} . \frac{\partial F_2^k}{\partial y} . \frac{\partial G_2^k}{\partial x} + N_2 . \frac{\partial N_2}{\partial x} . \frac{\partial F_2^k}{\partial y} . G_2^k + N_1 . \frac{\partial N_2}{\partial y} . F_2^k . \frac{\partial G_1^k}{\partial x} + \frac{\partial N_1}{\partial x} . \frac{\partial N_2}{\partial y} . F_2^k . G_1^k + N_2 . \frac{\partial N_2}{\partial y} . F_2^k . \frac{\partial G_2^k}{\partial x} + \frac{\partial N_2}{\partial x} . \frac{\partial N_2}{\partial y} . F_2^k . G_2^k \right) .. dz$$

$$= \frac{\partial F_2^k}{\partial y} . \frac{\partial G_2^k}{\partial x} . \int_{z_1}^{z_2} N_2^{\;3} . dz \quad + \quad \left[\frac{\partial F_1^k}{\partial y} . \left(\frac{\partial G_1^k}{\partial x} + \frac{\partial G_2^k}{\partial x} \right) + \frac{\partial F_2^k}{\partial y} . \frac{\partial G_1^k}{\partial x} \right] . \int_{z_1}^{z_2} N_1^{\;2} . N_2 . dz \quad + \quad \frac{\partial F_2^k}{\partial y} . \left(G_1^k - G_2^k \right) . \int_{z_1}^{z_2} N_2^{\;2} . \frac{\partial N_1}{\partial x} . dz$$

$$+ \quad \frac{\partial G_2^k}{\partial x} . \left(F_1^k - F_2^k \right) . \int_{z_1}^{z_2} N_2^{\;2} . \frac{\partial N_1}{\partial y} . dz \quad + \quad \frac{\partial F_1^k}{\partial y} . \left(G_1^k - G_2^k \right) . \int_{z_1}^{z_2} N_1 . N_2 . \frac{\partial N_1}{\partial x} . dz \quad + \quad \frac{\partial G_1^k}{\partial x} . \left(F_1^k - F_2^k \right) . \int_{z_1}^{z_2} N_1 . N_2 . \frac{\partial N_1}{\partial y} . dz$$

$$+ \quad \left(F_1^k - F_2^k \right) . \left(G_1^k - G_2^k \right) . \int_{z_1}^{z_2} N_2 . \frac{\partial N_1}{\partial x} . \frac{\partial N_1}{\partial y} . dz$$

$$= \frac{\partial F_2^k}{\partial y} . \frac{\partial G_2^k}{\partial x} . \frac{l_k}{4} \quad + \quad \left[\frac{\partial F_1^k}{\partial y} . \left(\frac{\partial G_1^k}{\partial x} + \frac{\partial G_2^k}{\partial x} \right) + \frac{\partial F_2^k}{\partial y} . \frac{\partial G_1^k}{\partial x} \right] . \frac{l_k}{12}$$

$$+ \quad \frac{\partial F_2^k}{\partial y} . \left(G_1^k - G_2^k \right) . \left(\frac{1}{4} . \frac{\partial l_k}{\partial x} + \frac{1}{3} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial x} - \frac{1}{3} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial x} \right) \quad + \quad \frac{\partial G_2^k}{\partial x} . \left(F_1^k - F_2^k \right) . \left(\frac{1}{4} . \frac{\partial l_k}{\partial y} + \frac{1}{3} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial y} - \frac{1}{3} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial y} \right)$$

$$+ \quad \frac{\partial F_1^k}{\partial y} . \left(G_1^k - G_2^k \right) . \left(\frac{1}{12} . \frac{\partial l_k}{\partial x} + \frac{1}{6} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial x} - \frac{1}{6} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial x} \right) \quad + \quad \frac{\partial G_1^k}{\partial x} . \left(F_1^k - F_2^k \right) . \left(\frac{1}{12} . \frac{\partial l_k}{\partial y} + \frac{1}{6} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial y} - \frac{1}{6} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial y} \right)$$

$$+\left(F_1^k - F_2^k\right).\left(G_1^k - G_2^k\right)$$

$$.\left\{\left(\frac{1}{4.l_k}.\frac{\partial l_k}{\partial x}.\frac{\partial l_k}{\partial y}\right)+\frac{1}{2}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right).\left(\frac{z_2}{l_k^{\ 2}}.\frac{\partial z_1}{\partial y}-\frac{z_1}{l_k^{\ 2}}.\frac{\partial z_2}{\partial y}\right)+\frac{1}{3}.\left[\frac{\partial l_k}{\partial y}.\left(\frac{z_2}{l_k^{\ 2}}.\frac{\partial z_1}{\partial x}-\frac{z_1}{l_k^{\ 2}}.\frac{\partial z_2}{\partial x}\right)+\frac{\partial l_k}{\partial x}.\left(\frac{z_2}{l_k^{\ 2}}.\frac{\partial z_1}{\partial y}-\frac{z_1}{l_k^{\ 2}}.\frac{\partial z_2}{\partial y}\right)\right]\right\} \quad \text{(A.10)}$$

$$i=1 \qquad \int_{z_1}^{z_2} N_1.\frac{\partial \tilde{F}}{\partial x}.\frac{\partial \tilde{G}}{\partial x}.dz \qquad = \int_{z_1}^{z_2} N_1.\frac{\partial}{\partial x}\left(N_1.F_1^{\ k}+N_2.F_2^{\ k}\right).\frac{\partial}{\partial x}\left(N_1.G_1^k+N_2.G_2^k\right).dz$$

$$=\int_{z_1}^{z_2} N_1.\left(N_1.\frac{\partial F_1^{\ k}}{\partial x}+F_1^{\ k}.\frac{\partial N_1}{\partial x}+N_2.\frac{\partial F_2^{\ k}}{\partial x}+F_2^{\ k}.\frac{\partial N_2}{\partial x}\right).\left(N_1.\frac{\partial G_1^k}{\partial x}+G_1^k.\frac{\partial N_1}{\partial x}+N_2.\frac{\partial G_2^k}{\partial x}+G_2^k.\frac{\partial N_2}{\partial x}\right).dz$$

$$=\int_{z_1}^{z_2} N_1.\left(N_1^{\ 2}.\frac{\partial F_1^{\ k}}{\partial x}.\frac{\partial G_1^k}{\partial x}+N_1.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{\ k}}{\partial x}.G_1^k+N_1.N_2.\frac{\partial F_1^{\ k}}{\partial x}.\frac{\partial G_2^k}{\partial x}+N_1.\frac{\partial N_2}{\partial x}.\frac{\partial F_1^{\ k}}{\partial x}.G_2^k+N_1.\frac{\partial N_1}{\partial x}.F_1^{\ k}.\frac{\partial G_1^k}{\partial x}+\left(\frac{\partial N_1}{\partial x}\right)^2.F_1^{\ k}.G_1^k+N_2.\frac{\partial N_1}{\partial x}.F_1^{\ k}.\frac{\partial G_2^k}{\partial x}+\frac{\partial N_1}{\partial x}.\frac{\partial N_2}{\partial x}.F_1^{\ k}.G_2^k\right.$$

$$\left.N_1.N_2.\frac{\partial F_2^{\ k}}{\partial x}.\frac{\partial G_1^k}{\partial x}+N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^{\ k}}{\partial x}.G_1^k+N_2^{\ 2}.\frac{\partial F_2^{\ k}}{\partial x}.\frac{\partial G_2^k}{\partial x}+N_2.\frac{\partial N_2}{\partial x}.\frac{\partial F_2^{\ k}}{\partial x}.G_2^k+N_1.\frac{\partial N_2}{\partial x}.F_2^{\ k}.\frac{\partial G_1^k}{\partial x}+\frac{\partial N_1}{\partial x}.\frac{\partial N_2}{\partial x}.F_2^{\ k}.G_1^k+N_2.\frac{\partial N_2}{\partial x}.F_2^{\ k}.\frac{\partial G_2^k}{\partial x}+\left(\frac{\partial N_2}{\partial x}\right)^2.F_2^{\ k}.G_2^k\right)..dz$$

$$=\int_{z_1}^{z_2}\left(N_1^{\ 3}.\frac{\partial F_1^{\ k}}{\partial x}.\frac{\partial G_1^k}{\partial x}+N_1^{\ 2}.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{\ k}}{\partial x}.G_1^k+N_1^{\ 2}.N_2.\frac{\partial F_1^{\ k}}{\partial x}.\frac{\partial G_2^k}{\partial x}-N_1^{\ 2}.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^{\ k}}{\partial x}.G_2^k+N_1^{\ 2}.\frac{\partial N_1}{\partial x}.F_1^{\ k}.\frac{\partial G_1^k}{\partial x}+N_1.\left(\frac{\partial N_1}{\partial x}\right)^2.F_1^{\ k}.G_1^k+N_1.N_2.\frac{\partial N_1}{\partial x}.F_1^{\ k}.\frac{\partial G_2^k}{\partial x}-N_1.\left(\frac{\partial N_1}{\partial x}\right)^2.F_1^{\ k}.G_2^k\right.$$

$$\left.N_1^{\ 2}.N_2.\frac{\partial F_2^k}{\partial x}.\frac{\partial G_1^k}{\partial x}+N_1.N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^k}{\partial x}.G_1^k+N_1^{\ 2}.N_2.\frac{\partial F_2^k}{\partial x}.\frac{\partial G_2^k}{\partial x}-N_1.N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^k}{\partial x}.G_2^k-N_1^{\ 2}.\frac{\partial N_1}{\partial x}.F_2^k.\frac{\partial G_1^k}{\partial x}-N_1.\left(\frac{\partial N_1}{\partial x}\right)^2.F_2^k.G_1^k-N_1.N_2.\frac{\partial N_1}{\partial x}.F_2^k.\frac{\partial G_2^k}{\partial x}+N_1.\left(\frac{\partial N_1}{\partial x}\right)^2.F_2^k.G_2^k\right)..dz$$

$$= \qquad \frac{\partial F_1^k}{\partial x}.\frac{\partial G_1^k}{\partial x}.\int_{z_1}^{z_2} N_1^{\ 3}.dz \quad + \quad \left[\frac{\partial F_2^k}{\partial x}.\frac{\partial G_1^k}{\partial x}+\frac{\partial G_2^k}{\partial x}.\left(\frac{\partial F_1^k}{\partial x}+\frac{\partial F_2^k}{\partial x}\right)\right].\int_{z_1}^{z_2} N_1^{\ 2}.N_2.dz$$

$$
+ \quad \left[\frac{\partial F_1^k}{\partial x}.\left(G_1^k - G_2^k\right) + \frac{\partial G_1^k}{\partial x}.\left(F_1^k - F_2^k\right) \right].\int_{z_1}^{z_2} N_1^{\,2}.\frac{\partial N_1}{\partial x}.dz \quad + \quad \left[\frac{\partial G_2^k}{\partial x}.\left(F_1^k - F_2^k\right) + \frac{\partial F_2^k}{\partial x}.\left(G_1^k - G_2^k\right) \right].\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz
$$

$$
+ \quad \left[\left(F_1^k - F_2^k\right).\left(G_1^k - G_2^k\right)\right].\int_{z_1}^{z_2} N_1.\left(\frac{\partial N_1}{\partial x}\right)^2.dz
$$

$$
= \quad \frac{\partial F_1^k}{\partial x}.\frac{\partial G_1^k}{\partial x}.\frac{l_k}{4} \quad + \quad \left[\frac{\partial F_2^k}{\partial x}.\frac{\partial G_1^k}{\partial x} + \frac{\partial G_2^k}{\partial x}.\left(\frac{\partial F_1^k}{\partial x} + \frac{\partial F_2^k}{\partial x} \right) \right].\frac{l_k}{12}
$$

$$
+ \quad \left[\frac{\partial F_1^k}{\partial x}.\left(G_1^k - G_2^k\right) + \frac{\partial G_1^k}{\partial x}.\left(F_1^k - F_2^k\right) \right].\left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)
$$

$$
+ \quad \left[\frac{\partial F_2^k}{\partial x}.\left(G_1^k - G_2^k\right) + \frac{\partial G_2^k}{\partial x}.\left(F_1^k - F_2^k\right) \right].\left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)
$$

$$
+ \quad \left[\left(F_1^k - F_2^k\right).\left(G_1^k - G_2^k\right)\right].\left[\frac{1}{12.l_k}.\left(\frac{\partial l_k}{\partial x}\right)^2 + \frac{1}{2.l_k}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)^2 + \frac{1}{3}.\frac{\partial l_k}{\partial x}.\left(\frac{z_2}{l_k^{\,2}}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k^{\,2}}.\frac{\partial z_2}{\partial x} \right) \right] \qquad \text{(A.11)}
$$

$$
i = 2 \qquad \int_{z_1}^{z_2} N_2.\frac{\partial \tilde{F}}{\partial x}.\frac{\partial \tilde{G}}{\partial x}.dz
$$

$$= \int_{z_1}^{z_2} \left(N_1^{\,2}.N_2.\frac{\partial F_1^k}{\partial x}.\frac{\partial G_1^k}{\partial x} + N_1.N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^k}{\partial x}.G_1^k + N_1^{\,2}.N_2.\frac{\partial F_1^k}{\partial x}.\frac{\partial G_2^k}{\partial x} - N_1.N_2.\frac{\partial N_1}{\partial x}.\frac{\partial F_1^k}{\partial x}.G_2^k + N_1.N_2.\frac{\partial N_1}{\partial x}.F_1^k.\frac{\partial G_1^k}{\partial x} + N_2.\left(\frac{\partial N_1}{\partial x}\right)^2.F_1^k.G_1^k + N_2^{\,2}.\frac{\partial N_1}{\partial x}.F_1^k.\frac{\partial G_2^k}{\partial x} - N_2.\left(\frac{\partial N_1}{\partial x}\right)^2.F_1^k.G_2^k \right.$$

$$\left. N_1^{\,2}.N_2.\frac{\partial F_2^k}{\partial x}.\frac{\partial G_1^k}{\partial x} + N_2^{\,2}.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^k}{\partial x}.G_1^k + N_2^{\,3}.\frac{\partial F_2^k}{\partial x}.\frac{\partial G_2^k}{\partial x} - N_2^{\,2}.\frac{\partial N_1}{\partial x}.\frac{\partial F_2^k}{\partial x}.G_2^k - N_1.N_2.\frac{\partial N_1}{\partial x}.F_2^k.\frac{\partial G_1^k}{\partial x} - N_2.\left(\frac{\partial N_1}{\partial x}\right)^2.F_2^k.G_1^k - N_2^{\,2}.\frac{\partial N_1}{\partial x}.F_2^k.\frac{\partial G_2^k}{\partial x} + N_2.\left(\frac{\partial N_1}{\partial x}\right)^2.F_2^k.G_2^k \right)..dz$$

$$= \quad \frac{\partial F_2^k}{\partial x}.\frac{\partial G_2^k}{\partial x}.\int_{z_1}^{z_2} N_2^{\,3}.dz \quad + \quad \left[\frac{\partial F_1^k}{\partial x}.\left(\frac{\partial G_1^k}{\partial x} + \frac{\partial G_2^k}{\partial x}\right) + \frac{\partial F_2^k}{\partial x}.\frac{\partial G_1^k}{\partial x}\right].\int_{z_1}^{z_2} N_1^{\,2}.N_2\,dz$$

$$+ \quad \left[\frac{\partial F_1^k}{\partial x}.\left(G_1^k - G_2^k\right) + \frac{\partial G_1^k}{\partial x}\left(F_1^k - F_2^k\right)\right].\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz \quad + \quad \frac{\partial G_2^k}{\partial x}.\left(F_1^k - F_2^k\right) + \frac{\partial F_2^k}{\partial x}.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial N_1}{\partial x}.dz$$

$$+ \quad \left(F_1^k - F_2^k\right).\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_2.\left(\frac{\partial N_1}{\partial x}\right)^2.dz$$

$$= \quad \frac{\partial F_2^k}{\partial x}.\frac{\partial G_2^k}{\partial x}.\frac{l_k}{4} \quad + \quad \left[\frac{\partial F_1^k}{\partial x}.\left(\frac{\partial G_1^k}{\partial x} + \frac{\partial G_2^k}{\partial x}\right) + \frac{\partial F_2^k}{\partial x}.\frac{\partial G_1^k}{\partial x}\right].\frac{l_k}{12}$$

$$+ \quad \left[\frac{\partial F_1^k}{\partial x}.\left(G_1^k - G_2^k\right) + \frac{\partial G_1^k}{\partial x}\left(F_1^k - F_2^k\right)\right].\left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)$$

$$+ \quad \left[\frac{\partial F_2^k}{\partial x}.\left(G_1^k - G_2^k\right) + \frac{\partial G_2^k}{\partial x}.\left(F_1^k - F_2^k\right) \right].\left(\frac{1}{4}.\frac{\partial l_k}{\partial x} + \frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)$$

$$+ \quad \left(F_1^k - F_2^k\right).\left(G_1^k - G_2^k\right).\left[\frac{1}{4.l_k}.\left(\frac{\partial l_k}{\partial x}\right)^2 + \frac{1}{2.l_k}.\left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)^2 + \frac{2}{3}.\frac{\partial l_k}{\partial x}.\left(\frac{z_2}{l_k^{\ 2}}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k^{\ 2}}.\frac{\partial z_2}{\partial x} \right) \right] \qquad \text{(A.12)}$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1.\frac{\partial \tilde{F}}{\partial z}.\frac{\partial \tilde{G}}{\partial z}.dz \quad = \int_{z_1}^{z_2} N_1.\left(\frac{F_2^k - F_1^k}{l_k} \right).\left(\frac{G_2^k - G_1^k}{l_k} \right).dz \quad = \left(\frac{F_2^k - F_1^k}{l_k} \right).\left(\frac{G_2^k - G_1^k}{l_k} \right).\int_{z_1}^{z_2} N_1.dz$$

$$= \frac{1}{2.l_k}.\left(F_2^k - F_1^k\right).\left(G_2^k - G_1^k\right) \qquad \text{(A.13)}$$

$$i = 2 \quad \int_{z_1}^{z_2} N_2.\frac{\partial \tilde{F}}{\partial z}.\frac{\partial \tilde{G}}{\partial z}.dz \quad = \int_{z_1}^{z_2} N_2.\left(\frac{F_2^k - F_1^k}{l_k} \right).\left(\frac{G_2^k - G_1^k}{l_k} \right).dz \quad = \left(\frac{F_2^k - F_1^k}{l_k} \right).\left(\frac{G_2^k - G_1^k}{l_k} \right).\int_{z_1}^{z_2} N_2.dz$$

$$= \frac{1}{2.l_k}.\left(F_2^k - F_1^k\right).\left(G_2^k - G_1^k\right) \qquad \text{(A.14)}$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1.\tilde{F}.\frac{\partial \tilde{G}}{\partial x}.dz \quad = \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial x}\left(N_1.G_1^k + N_2.G_2^k\right).dz$$

$$= \int_{z_1}^{z_2} \left(N_1^{\ 2}.F_1^{\ k} + N_1.N_2.F_2^{\ k} \right) \cdot \left(N_1.\frac{\partial G_1^k}{\partial x} + G_1^k.\frac{\partial N_1}{\partial x} + N_2.\frac{\partial G_2^k}{\partial x} + G_2^k.\frac{\partial N_2}{\partial x} \right) .dz$$

$$= \int_{z_1}^{z_2} \left(F_1^{\ k}.\frac{\partial G_1^k}{\partial x}.N_1^{\ 3} + F_1^{\ k}.G_1^k.N_1^{\ 2}.\frac{\partial N_1}{\partial x} + F_1^{\ k}.\frac{\partial G_2^k}{\partial x}.N_1^{\ 2}.N_2 + F_1^{\ k}.G_2^k.N_1^{\ 2}.\frac{\partial N_2}{\partial x} + F_2^{\ k}.\frac{\partial G_1^k}{\partial x}.N_1^{\ 2}.N_2 + F_2^{\ k}.G_1^k.N_1.N_2.\frac{\partial N_1}{\partial x} + F_2^{\ k}.\frac{\partial G_2^k}{\partial x}.N_1.N_2^{\ 2} + F_2^{\ k}.G_2^k.N_1.N_2.\frac{\partial N_2}{\partial x} \right) dz$$

$$= F_1^{\ k}.\frac{\partial G_1^k}{\partial x}.\int_{z_1}^{z_2} N_1^{\ 3}.dz + F_1^{\ k}.G_1^k.\int_{z_1}^{z_2} N_1^{\ 2}.\frac{\partial N_1}{\partial x}.dz + F_1^{\ k}.\frac{\partial G_2^k}{\partial x}.\int_{z_1}^{z_2} N_1^{\ 2}.N_2.dz + F_1^{\ k}.G_2^k.\int_{z_1}^{z_2} N_1^{\ 2}.\left(-\frac{\partial N_1}{\partial x} \right) .dz$$

$$+ F_2^{\ k}.\frac{\partial G_1^k}{\partial x}.\int_{z_1}^{z_2} N_1^{\ 2}.N_2.dz + F_2^{\ k}.G_1^k.\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz + F_2^{\ k}.\frac{\partial G_2^k}{\partial x}.\int_{z_1}^{z_2} N_1^{\ 2}.N_2.dz + F_2^{\ k}.G_2^k.\int_{z_1}^{z_2} N_1.N_2.\left(-\frac{\partial N_1}{\partial x} \right) .dz$$

$$= \frac{\partial G_1^k}{\partial x}.\left(F_1^{\ k}.\int_{z_1}^{z_2} N_1^{\ 3}.dz + F_2^{\ k}.\int_{z_1}^{z_2} N_1^{\ 2}.N_2.dz \right) + \frac{\partial G_2^k}{\partial x}.\left(F_1^{\ k} + F_2^{\ k} \right) \cdot \left(\int_{z_1}^{z_2} N_1^{\ 2}.N_2.dz \right) + F_1^{\ k}.\left(G_1^k - G_2^k \right) \cdot \left(\int_{z_1}^{z_2} N_1^{\ 2}.\frac{\partial N_1}{\partial x}.dz \right) + F_2^{\ k}.\left(G_1^k - G_2^k \right) \cdot \left(\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz \right)$$

$$= \frac{\partial G_1^k}{\partial x}.l_k \left(\frac{F_1^k}{4} + \frac{F_2^k}{12} \right) + \frac{\partial G_2^k}{\partial x}.\left(F_1^{\ k} + F_2^{\ k} \right) .\frac{l_k}{12} \quad + \quad F_1^{\ k}.\left(G_1^k - G_2^k \right) \cdot \left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)$$

$$+ \quad F_2^k.\left(G_1^k - G_2^k \right) \cdot \left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right)$$

$$= \left(\frac{\partial l_k}{\partial x}.\left(G_1^k - G_2^k \right) + l_k.\frac{\partial G_2^k}{\partial x} \right) .\left(F_1^{\ k} + F_2^{\ k} \right) .\frac{1}{12} \quad + l_k.\frac{\partial G_1^k}{\partial x}.\left(\frac{1}{4}.F_1^{\ k} + \frac{1}{12}.F_2^{\ k} \right) \quad + \left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x} \right) .\left(G_1^k - G_2^k \right) .\left(\frac{F_1^k}{3} + \frac{F_2^k}{6} \right) \qquad \text{(A.15)}$$

$$i = 2 \quad \int_{z_1}^{z_2} N_2.\tilde{F}.\frac{\partial \tilde{G}}{\partial x}.dz \qquad = \int_{z_1}^{z_2} N_2.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial x}\left(N_1.G_1^k + N_2.G_2^k\right).dz$$

$$= \int_{z_1}^{z_2} \left(N_1.N_2.F_1^k + N_2^{\,2}.F_2^k\right).\left(N_1.\frac{\partial G_1^k}{\partial x} + G_1^k.\frac{\partial N_1}{\partial x} + N_2.\frac{\partial G_2^k}{\partial x} + G_2^k.\frac{\partial N_2}{\partial x}\right).dz$$

$$= \frac{\partial G_1^k}{\partial x}.\left(F_1^k + F_2^k\right).\left(\int_{z_1}^{z_2} N_1^{\,2}.N_2.dz\right) + \frac{\partial G_2^k}{\partial x}.\left(F_1^k.\int_{z_1}^{z_2} N_1^{\,2}.N_2.dz + F_2^k.\int_{z_1}^{z_2} N_2^{\,3}.dz\right) + F_1^k.\left(G_1^k - G_2^k\right).\left(\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz\right) + F_2^k.\left(G_1^k - G_2^k\right).\left(\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial N_1}{\partial x}.dz\right)$$

$$= \frac{\partial G_1^k}{\partial x}.\frac{l_k}{12}\left(F_1^k + F_2^k\right) + \frac{\partial G_2^k}{\partial x}.\left(\frac{F_1^k}{12} + \frac{F_2^k}{4}\right).l_k$$

$$+ \quad F_1^k.\left(G_1^k - G_2^k\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right) \quad + \quad F_2^k.\left(G_1^k - G_2^k\right).\left(\frac{1}{4}.\frac{\partial l_k}{\partial x} + \frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)$$

$$= \left(\frac{\partial l_k}{\partial x}.\left(G_1^k - G_2^k\right) + l_k.\frac{\partial G_2^k}{\partial x}\right).\left(\frac{F_1^k}{12} + \frac{F_2^k}{4}\right) \quad + \frac{l_k}{12}.\frac{\partial G_1^k}{\partial x}.\left(F_1^k + F_2^k\right) + \left(\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right).\left(G_1^k - G_2^k\right).\left(\frac{F_1^k}{6} + \frac{F_2^k}{3}\right) \qquad \text{(A.16)}$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1.\tilde{F}.\frac{\partial^2 \tilde{G}}{\partial x^2}.dz \qquad = \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial x}\left(\frac{\partial \tilde{G}}{\partial x}\right).dz \qquad = \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(N_1.G_1^k + N_2.G_2^k\right)\right).dz$$

$$= \int_{z_1}^{z_2} \left(N_1{}^2.F_1^k + N_1.N_2.F_2^k\right).\frac{\partial}{\partial x}\left(N_1.\frac{\partial G_1^k}{\partial x} + G_1^k.\frac{\partial N_1}{\partial x} + N_2.\frac{\partial G_2^k}{\partial x} + G_2^k.\frac{\partial N_2}{\partial x}\right).dz$$

$$= \int_{z_1}^{z_2} \left(N_1{}^2.F_1^k + N_1.N_2.F_2^k\right).\left(\frac{\partial N_1}{\partial x}.\frac{\partial G_1^k}{\partial x} + N_1.\frac{\partial^2 G_1^k}{\partial x^2} + \frac{\partial G_1^k}{\partial x}.\frac{\partial N_1}{\partial x} + G_1^k.\frac{\partial^2 N_1}{\partial x^2} + \frac{\partial N_2}{\partial x}.\frac{\partial G_2^k}{\partial x} + N_2.\frac{\partial^2 G_2^k}{\partial x^2} + \frac{\partial G_2^k}{\partial x}.\frac{\partial N_2}{\partial x} + G_2^k.\frac{\partial^2 N_2}{\partial x^2}\right).dz$$

$$= \int_{z_1}^{z_2} \left(N_1{}^2.F_1^k + N_1.N_2.F_2^k\right).\left[2.\frac{\partial N_1}{\partial x}.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right) + \frac{\partial^2 N_1}{\partial x^2}.\left(G_1^k - G_2^k\right) + N_1.\frac{\partial^2 G_1^k}{\partial x^2} + N_2.\frac{\partial^2 G_2^k}{\partial x^2}\right].dz$$

$$\begin{aligned}= \int_{z_1}^{z_2} \Bigg[& 2.F_1^k\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).N_1{}^2.\frac{\partial N_1}{\partial x} + F_1^k.\left(G_1^k - G_2^k\right).N_1{}^2.\frac{\partial^2 N_1}{\partial x^2} + F_1^k.\frac{\partial^2 G_1^k}{\partial x^2}.N_1{}^3 + F_1^k.\frac{\partial^2 G_2^k}{\partial x^2}.N_1{}^2.N_2 \\ & + 2.F_2^k.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).N_1.N_2.\frac{\partial N_1}{\partial x} + F_2^k.\left(G_1^k - G_2^k\right).N_1.N_2.\frac{\partial^2 N_1}{\partial x^2} + F_2^k.\frac{\partial^2 G_1^k}{\partial x^2}.N_1{}^2.N_2 + F_2^k.\frac{\partial^2 G_2^k}{\partial x^2}.N_1{}^2.N_2 \Bigg].dz\end{aligned}$$

$$\begin{aligned}= \; & F_1^k.\frac{\partial^2 G_1^k}{\partial x^2}.\int_{z_1}^{z_2} N_1{}^3.dz + \left[F_1^k.\frac{\partial^2 G_2^k}{\partial x^2} + F_2^k.\left(\frac{\partial^2 G_1^k}{\partial x^2} + \frac{\partial^2 G_2^k}{\partial x^2}\right)\right].\int_{z_1}^{z_2} N_1{}^2.N_2.dz + 2.F_1^k\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_1{}^2.\frac{\partial N_1}{\partial x}.dz \\ + \; & 2.F_2^k.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz + F_1^k.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_1{}^2.\frac{\partial^2 N_1}{\partial x^2}.dz + F_2^k.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial^2 N_1}{\partial x^2}.dz\end{aligned}$$

$$= F_1^k \cdot \frac{\partial^2 G_1^k}{\partial x^2} \cdot \frac{l_k}{4} + \left[F_1^k \cdot \frac{\partial^2 G_2^k}{\partial x^2} + F_2^k \cdot \left(\frac{\partial^2 G_1^k}{\partial x^2} + \frac{\partial^2 G_2^k}{\partial x^2} \right) \right] \cdot \frac{l_k}{12}$$

$$+ 2.F_1^k \left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x} \right) \cdot \left(\frac{1}{12} \cdot \frac{\partial l_k}{\partial x} + \frac{1}{3} \cdot \frac{z_2}{l_k} \cdot \frac{\partial z_1}{\partial x} - \frac{1}{3} \cdot \frac{z_1}{l_k} \cdot \frac{\partial z_2}{\partial x} \right) + 2.F_2^k \cdot \left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x} \right) \cdot \left(\frac{1}{12} \cdot \frac{\partial l_k}{\partial x} + \frac{1}{6} \cdot \frac{z_2}{l_k} \cdot \frac{\partial z_1}{\partial x} - \frac{1}{6} \cdot \frac{z_1}{l_k} \cdot \frac{\partial z_2}{\partial x} \right)$$

$$+ F_1^k . \left(G_1^k - G_2^k \right) . \left\{ \frac{1}{12} . \left(-\frac{2}{l_k} . \left(\frac{\partial l_k}{\partial x} \right)^2 + \frac{\partial^2 l_k}{\partial x^2} \right) + \frac{1}{3} . \left[\frac{z_2}{l_k} . \frac{\partial^2 z_1}{\partial x^2} - \frac{z_1}{l_k} . \frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_1}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_2}{\partial x} - \frac{2.z_2}{l_k^{\ 2}} . \frac{\partial l_k}{\partial x} \right) - \frac{\partial z_2}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_1}{\partial x} - \frac{2.z_1}{l_k^{\ 2}} . \frac{\partial l_k}{\partial x} \right) \right] \right\}$$

$$+ F_2^k . \left(G_1^k - G_2^k \right) . \left\{ \frac{1}{12} . \left(-\frac{2}{l_k} . \left(\frac{\partial l_k}{\partial x} \right)^2 + \frac{\partial^2 l_k}{\partial x^2} \right) + \frac{1}{6} . \left[\frac{z_2}{l_k} . \frac{\partial^2 z_1}{\partial x^2} - \frac{z_1}{l_k} . \frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_1}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_2}{\partial x} - \frac{2.z_2}{l_k^{\ 2}} . \frac{\partial l_k}{\partial x} \right) - \frac{\partial z_2}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_1}{\partial x} - \frac{2.z_1}{l_k^{\ 2}} . \frac{\partial l_k}{\partial x} \right) \right] \right\} \quad \text{(A.17)}$$

$$i = 2 \quad \int_{z_1}^{z_2} N_2 . \tilde{F} . \frac{\partial^2 \tilde{G}}{\partial x^2} . dz = \int_{z_1}^{z_2} N_2 . \left(N_1 . F_1^k + N_2 . F_2^k \right) . \frac{\partial}{\partial x} \left(\frac{\partial \tilde{G}}{\partial x} \right) . dz = \int_{z_1}^{z_2} N_2 . \left(N_1 . F_1^k + N_2 . F_2^k \right) . \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(N_1 . G_1^k + N_2 . G_2^k \right) \right) . dz$$

$$= \int_{z_1}^{z_2} \left(N_1 . N_2 . F_1^k + N_2^{\ 2} . F_2^k \right) . \frac{\partial}{\partial x} \left(N_1 . \frac{\partial G_1^k}{\partial x} + G_1^k . \frac{\partial N_1}{\partial x} + N_2 . \frac{\partial G_2^k}{\partial x} + G_2^k . \frac{\partial N_2}{\partial x} \right) . dz$$

$$= \int_{z_1}^{z_2} \left(N_1 . N_2 . F_1^k + N_2^{\ 2} . F_2^k \right) . \left[2 . \frac{\partial N_1}{\partial x} . \left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x} \right) + \frac{\partial^2 N_1}{\partial x^2} . \left(G_1^k - G_2^k \right) + N_1 . \frac{\partial^2 G_1^k}{\partial x^2} + N_2 . \frac{\partial^2 G_2^k}{\partial x^2} \right] . dz$$

$$
= \int_{z_1}^{z_2} \Bigg[2.F_1^{k}\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).N_1.N_2.\frac{\partial N_1}{\partial x} + F_1^{k}.\left(G_1^k - G_2^k\right).N_1.N_2.\frac{\partial^2 N_1}{\partial x^2} + F_1^{k}.\frac{\partial^2 G_1^k}{\partial x^2}.N_1^{\,2}.N_2 + F_1^{k}.\frac{\partial^2 G_2^k}{\partial x^2}.N_1^{\,2}.N_2
$$

$$
+ 2.F_2^{k}.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).N_2^{\,2}.\frac{\partial N_1}{\partial x} + F_2^{k}.\left(G_1^k - G_2^k\right).N_2^{\,2}.\frac{\partial^2 N_1}{\partial x^2} + F_2^{k}.\frac{\partial^2 G_1^k}{\partial x^2}.N_1^{\,2}.N_2 + F_2^{k}.\frac{\partial^2 G_2^k}{\partial x^2}.N_2^{\,3} \Bigg].dz
$$

$$
= \quad F_2^{k}.\frac{\partial^2 G_2^k}{\partial x^2}.\int_{z_1}^{z_2} N_2^{\,3}.dx \quad + \quad \left[F_1^{k}.\left(\frac{\partial^2 G_1^k}{\partial x^2} + \frac{\partial^2 G_2^k}{\partial x^2}\right) + F_2^{k}.\frac{\partial^2 G_1^k}{\partial x^2}\right].\int_{z_1}^{z_2} N_1^{\,2}.N_2.dx \quad + \quad 2.F_2^{k}.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial N_1}{\partial x}.dx
$$

$$
+ \quad 2.F_1^{k}\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dx \quad + \quad F_2^{k}.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial^2 N_1}{\partial x^2}.dx + \quad F_1^{k}.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial^2 N_1}{\partial x^2}.dx
$$

$$
= \quad F_2^{k}.\frac{\partial^2 G_2^k}{\partial x^2}.\frac{l_k}{4} \quad + \quad \left[F_1^{k}.\frac{\partial^2 G_2^k}{\partial x^2} + \frac{\partial^2 G_1^k}{\partial x^2}.\left(F_1^{k} + F_2^{k}\right)\right].\frac{l_k}{12}
$$

$$
+ \quad 2.F_2^{k}.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).\left(\frac{1}{4}.\frac{\partial l_k}{\partial x} + \frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right) \quad + \quad 2.F_1^{k}\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial x} + \frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x} - \frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right)
$$

$$
+ \quad F_2^{k}.\left(G_1^k - G_2^k\right).\left\{\frac{1}{4}.\left(-\frac{2}{l_k}.\left(\frac{\partial l_k}{\partial x}\right)^2 + \frac{\partial^2 l_k}{\partial x^2}\right) + \frac{1}{3}.\left[\frac{z_2}{l_k}.\frac{\partial^2 z_1}{\partial x^2} - \frac{z_1}{l_k}.\frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial x} - \frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial x}\right) - \frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial x} - \frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial x}\right)\right]\right\}
$$

$$
+ F_1^{k}.\left(G_1^k - G_2^k\right).\left\{\frac{1}{12}.\left(-\frac{2}{l_k}.\left(\frac{\partial l_k}{\partial x}\right)^2 + \frac{\partial^2 l_k}{\partial x^2}\right) + \frac{1}{6}.\left[\frac{z_2}{l_k}.\frac{\partial^2 z_1}{\partial x^2} - \frac{z_1}{l_k}.\frac{\partial^2 z_2}{\partial x^2} + \frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial x} - \frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial x}\right) - \frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial x} - \frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial x}\right)\right]\right\} \quad \text{(A.18)}
$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1.\tilde{F}.\frac{\partial}{\partial y}\left(\frac{\partial \tilde{G}}{\partial x}\right).dz \quad = \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial y}\left(\frac{\partial \tilde{G}}{\partial x}\right).dz \quad = \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(N_1.G_1^k + N_2.G_2^k\right)\right).dz$$

$$= \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\frac{\partial}{\partial y}\left(N_1.\frac{\partial G_1^k}{\partial x} + G_1^k.\frac{\partial N_1}{\partial x} + N_2.\frac{\partial G_2^k}{\partial x} + G_2^k.\frac{\partial N_2}{\partial x}\right).dz$$

$$= \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\left(\frac{\partial N_1}{\partial y}.\frac{\partial G_1^k}{\partial x} + N_1.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + \frac{\partial G_1^k}{\partial y}.\frac{\partial N_1}{\partial x} + G_1^k.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right) + \frac{\partial N_2}{\partial y}.\frac{\partial G_2^k}{\partial x} + N_2.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right) + \frac{\partial G_2^k}{\partial y}.\frac{\partial N_2}{\partial x} + G_2^k.\frac{\partial}{\partial y}\left(\frac{\partial N_2}{\partial x}\right)\right).dz$$

$$= \int_{z_1}^{z_2} N_1.\left(N_1.F_1^k + N_2.F_2^k\right).\left[\frac{\partial N_1}{\partial y}.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right) + \frac{\partial N_1}{\partial x}.\left(\frac{\partial G_1^k}{\partial y} - \frac{\partial G_2^k}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).\left(G_1^k - G_2^k\right) + N_1.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + N_2.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right].dz$$

$$= \int_{z_1}^{z_2} N_1.\left[N_1.\frac{\partial N_1}{\partial y}.F_1^k.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right) + N_1.\frac{\partial N_1}{\partial x}.F_1^k.\left(\frac{\partial G_1^k}{\partial y} - \frac{\partial G_2^k}{\partial y}\right) + N_1.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).F_1^k.\left(G_1^k - G_2^k\right) + N_1^{\,2}.F_1^k.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + N_1.N_2.F_1^k.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right.$$

$$\left. + N_2.\frac{\partial N_1}{\partial y}.F_2^k.\left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right) + N_2.\frac{\partial N_1}{\partial x}.F_2^k.\left(\frac{\partial G_1^k}{\partial y} - \frac{\partial G_2^k}{\partial y}\right) + N_2.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).F_2^k.\left(G_1^k - G_2^k\right) + N_1.N_2.F_2^k.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + N_2^{\,2}.F_2^k.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right].dz$$

$$
\begin{aligned}
= \quad & F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right).\int_{z_1}^{z_2} N_1^{\,3}.dz \quad + \quad \left[F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)+F_2^{\,k}.\left(\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right)\right].\int_{z_1}^{z_2} N_1^{\,2}.N_2.dz \\
+ \quad & F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right).\int_{z_1}^{z_2} N_1^{\,2}.\frac{\partial N_1}{\partial x}.dz \quad + \quad F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_1^{\,2}.\frac{\partial N_1}{\partial y}.dz \quad + \quad F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz \\
+ \quad & F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial y}.dz \quad + \quad F_1^{\,k}.\left(G_1^k-G_2^k\right).\int_{z_1}^{z_2} N_1^{\,2}.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).dz \quad + \quad F_2^{\,k}.\left(G_1^k-G_2^k\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).dz \\
= \quad & F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right).\frac{l_k}{4} \quad + \quad \left[F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)+F_2^{\,k}.\left(\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right)\right].\frac{l_k}{12} \\
+ \quad & F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial x}+\frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right) \quad + \quad F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial y}+\frac{1}{3}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial y}-\frac{1}{3}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial y}\right) \\
+ \quad & F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial x}+\frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial x}-\frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial x}\right) \quad + \quad F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right).\left(\frac{1}{12}.\frac{\partial l_k}{\partial y}+\frac{1}{6}.\frac{z_2}{l_k}.\frac{\partial z_1}{\partial y}-\frac{1}{6}.\frac{z_1}{l_k}.\frac{\partial z_2}{\partial y}\right) \\
+ \; & F_1^{\,k}.\left(G_1^k-G_2^k\right).\left\{\frac{1}{12}.\left[-\frac{2}{l_k}.\frac{\partial l_k}{\partial x}.\frac{\partial l_k}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial l_k}{\partial x}\right)\right]+\frac{1}{3}.\left[\frac{z_2}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_1}{\partial x}\right)-\frac{z_1}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_2}{\partial x}\right)+\frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial y}-\frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)-\frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial y}-\frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)\right]\right\} \\
+ \; & F_2^{\,k}.\left(G_1^k-G_2^k\right).\left\{\frac{1}{12}.\left[-\frac{2}{l_k}.\frac{\partial l_k}{\partial x}.\frac{\partial l_k}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial l_k}{\partial x}\right)\right]+\frac{1}{6}.\left[\frac{z_2}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_1}{\partial x}\right)-\frac{z_1}{l_k}.\frac{\partial}{\partial y}\left(\frac{\partial z_2}{\partial x}\right)+\frac{\partial z_1}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_2}{\partial y}-\frac{2.z_2}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)-\frac{\partial z_2}{\partial x}.\left(\frac{1}{l_k}.\frac{\partial z_1}{\partial y}-\frac{2.z_1}{l_k^{\,2}}.\frac{\partial l_k}{\partial y}\right)\right]\right\} \quad \text{(A1.9)}
\end{aligned}
$$

$$i = 2 \qquad \int_{z_1}^{z_2} N_2.\tilde{F}.\frac{\partial}{\partial y}\left(\frac{\partial \tilde{G}}{\partial x}\right).dz$$

$$\begin{aligned}
&= \int_{z_1}^{z_2} N_2.\left[N_1.\frac{\partial N_1}{\partial y}.F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right) + N_1.\frac{\partial N_1}{\partial x}.F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right) + N_1.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).F_1^{\,k}.\left(G_1^k - G_2^k\right) + N_1^{\,2}.F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + N_1.N_2.F_1^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right.\\
&\left. + N_2.\frac{\partial N_1}{\partial y}.F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right) + N_2.\frac{\partial N_1}{\partial x}.F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right) + N_2.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).F_2^{\,k}.\left(G_1^k - G_2^k\right) + N_1.N_2.F_2^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + N_2^{\,2}.F_2^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right].dz
\end{aligned}$$

$$\begin{aligned}
= \quad & F_2^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_2^{\,3}.dz \quad + \quad \left[F_1^{\,k}.\left(\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right) + F_2^{\,k}.\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right)\right].\int_{z_1}^{z_2} N_1^{\,2}.N_2.dz\\
+ \quad & F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right).\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial N_1}{\partial x}.dz \quad + \quad F_2^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial N_1}{\partial y}.dz \quad + \quad F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial y}-\frac{\partial G_2^k}{\partial y}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial x}.dz\\
+ \quad & F_1^{\,k}.\left(\frac{\partial G_1^k}{\partial x}-\frac{\partial G_2^k}{\partial x}\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial N_1}{\partial y}.dz \quad + \quad F_2^{\,k}.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_2^{\,2}.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).dz \quad + \; F_1^{\,k}.\left(G_1^k - G_2^k\right).\int_{z_1}^{z_2} N_1.N_2.\frac{\partial}{\partial y}\left(\frac{\partial N_1}{\partial x}\right).dz
\end{aligned}$$

$$= \quad F_2^k \cdot \frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right) \cdot \frac{l_k}{4} \quad + \quad \left[F_1^k . \left(\frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial G_2^k}{\partial x}\right)\right) + F_2^k \cdot \frac{\partial}{\partial y}\left(\frac{\partial G_1^k}{\partial x}\right)\right] \cdot \frac{l_k}{12}$$

$$+ \quad F_2^k . \left(\frac{\partial G_1^k}{\partial y} - \frac{\partial G_2^k}{\partial y}\right) . \left(\frac{1}{4} . \frac{\partial l_k}{\partial x} + \frac{1}{3} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial x} - \frac{1}{3} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial x}\right) \quad + \quad F_2^k . \left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right) . \left(\frac{1}{4} . \frac{\partial l_k}{\partial y} + \frac{1}{3} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial y} - \frac{1}{3} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial y}\right)$$

$$+ \quad F_1^k . \left(\frac{\partial G_1^k}{\partial y} - \frac{\partial G_2^k}{\partial y}\right) . \left(\frac{1}{12} . \frac{\partial l_k}{\partial x} + \frac{1}{6} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial x} - \frac{1}{6} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial x}\right) \quad + \quad F_1^k . \left(\frac{\partial G_1^k}{\partial x} - \frac{\partial G_2^k}{\partial x}\right) . \left(\frac{1}{12} . \frac{\partial l_k}{\partial y} + \frac{1}{6} . \frac{z_2}{l_k} . \frac{\partial z_1}{\partial y} - \frac{1}{6} . \frac{z_1}{l_k} . \frac{\partial z_2}{\partial y}\right)$$

$$+ \; F_2^k . \left(G_1^k - G_2^k\right) . \left\{\frac{1}{4} . \left[-\frac{2}{l_k} . \frac{\partial l_k}{\partial x} . \frac{\partial l_k}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial l_k}{\partial x}\right)\right] + \frac{1}{3} . \left[\frac{z_2}{l_k} . \frac{\partial}{\partial y}\left(\frac{\partial z_1}{\partial x}\right) - \frac{z_1}{l_k} . \frac{\partial}{\partial y}\left(\frac{\partial z_2}{\partial x}\right) + \frac{\partial z_1}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_2}{\partial y} - \frac{2.z_2}{l_k^{\,2}} . \frac{\partial l_k}{\partial y}\right) - \frac{\partial z_2}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_1}{\partial y} - \frac{2.z_1}{l_k^{\,2}} . \frac{\partial l_k}{\partial y}\right)\right]\right\}$$

$$+ \; F_1^k . \left(G_1^k - G_2^k\right) . \left\{\frac{1}{12} . \left[-\frac{2}{l_k} . \frac{\partial l_k}{\partial x} . \frac{\partial l_k}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial l_k}{\partial x}\right)\right] + \frac{1}{6} . \left[\frac{z_2}{l_k} . \frac{\partial}{\partial y}\left(\frac{\partial z_1}{\partial x}\right) - \frac{z_1}{l_k} . \frac{\partial}{\partial y}\left(\frac{\partial z_2}{\partial x}\right) + \frac{\partial z_1}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_2}{\partial y} - \frac{2.z_2}{l_k^{\,2}} . \frac{\partial l_k}{\partial y}\right) - \frac{\partial z_2}{\partial x} . \left(\frac{1}{l_k} . \frac{\partial z_1}{\partial y} - \frac{2.z_1}{l_k^{\,2}} . \frac{\partial l_k}{\partial y}\right)\right]\right\} \quad \text{(A.20)}$$

$$i = 1 \quad \int_{z_1}^{z_2} N_1 . \widetilde{F} . \frac{\partial \widetilde{G}}{\partial z} . dz \quad = \int_{z_1}^{z_2} N_1 . \left(N_1 . F_1^k + N_2 . F_2^k\right) . \frac{\partial \widetilde{G}}{\partial z} . dz \quad = \int_{z_1}^{z_2} \left(N_1^{\,2} . F_1^k + N_1 . N_2 . F_2^k\right) . \left(\frac{G_2^k - G_1^k}{l_k}\right) . dz$$

$$= \left(\frac{G_2^k - G_1^k}{l_k}\right) . \left(F_1^k . \int_{z_1}^{z_2} N_1^{\,2} . dz + F_2^k . \int_{z_1}^{z_2} N_1 . N_2 . dz\right) \quad = \quad \left(G_2^k - G_1^k\right) . \left(\frac{F_1^k}{3} + \frac{F_2^k}{6}\right) \qquad \text{(A.21)}$$

$$i=2\quad \int_{z_1}^{z_2} N_2.\tilde{F}.\frac{\partial \tilde{G}}{\partial z}.dz = \int_{z_1}^{z_2} N_2.\left(N_1.F_1^{\,k}+N_2.F_2^{\,k}\right).\frac{\partial \tilde{G}}{\partial z}.dz = \int_{z_1}^{z_2}\left(N_1.N_2.F_1^{\,k}+N_2^{\,2}.F_2^{\,k}\right).\left(\frac{G_2^k-G_1^k}{l_k}\right).dz$$

$$=\left(\frac{G_2^k-G_1^k}{l_k}\right).\left(F_1^{\,k}.\int_{z_1}^{z_2} N_1.N_2.dz+F_2^{\,k}.\int_{z_1}^{z_2} N_2^{\,2}.dz\right) = \left(G_2^k-G_1^k\right).\left(\frac{F_1^{\,k}}{6}+\frac{F_2^{\,k}}{3}\right) \tag{A.22}$$

$$i=1\quad \int_{z_1}^{z_2} N_1.\tilde{F}.\frac{\partial^2 \tilde{G}}{\partial z^2}.dz = N_1.\tilde{F}.\frac{\partial(\tilde{G})}{\partial z}\Bigg|_{z_1}^{z_2} - \int_{z_1}^{z_2}\frac{\partial(N_1.\tilde{F})}{\partial z}.\frac{\partial(\tilde{G})}{\partial z}.dz = 0-F_1^{\,k}.\frac{\partial G_1^k}{\partial z}-\int_{z_1}^{z_2}\left(\frac{\partial(N_1.\tilde{F})}{\partial z}\right).\left(\frac{G_2^k-G_1^k}{l_k}\right).dz$$

$$=-F_1^{\,k}.\frac{\partial G_1^k}{\partial z}+\left(\frac{G_1^k-G_2^k}{l_k}\right).\int_{z_1}^{z_2}\left(\frac{\partial(N_1.\tilde{F})}{\partial z}\right).dz = -F_1^{\,k}.\frac{\partial G_1^k}{\partial z}+\left(\frac{G_1^k-G_2^k}{l_k}\right).\left(N_1.\tilde{F}\right)\Big|_{z_1}^{z_2} = -F_1^{\,k}.\frac{\partial G_1^k}{\partial z}+\left(\frac{G_1^k-G_2^k}{l_k}\right).\left(0-F_1^{\,k}\right)$$

$$=-F_1^{\,k}.\frac{\partial G_1^k}{\partial z}+\left(\frac{G_2^k-G_1^k}{l_k}\right).F_1^{\,k} \tag{A.23}$$

$$i=2\quad \int_{z_1}^{z_2} N_2.\tilde{F}.\frac{\partial^2 \tilde{G}}{\partial z^2}.dz = N_2.\tilde{F}.\frac{\partial(\tilde{G})}{\partial z}\Bigg|_{z_1}^{z_2} - \int_{z_1}^{z_2}\frac{\partial(N_2.\tilde{F})}{\partial z}.\frac{\partial(\tilde{G})}{\partial z}.dz = F_2^{\,k}.\frac{\partial G_2^k}{\partial z}-\left(\frac{G_2^k-G_1^k}{l_k}\right).\left(N_2.\tilde{F}\right)\Big|_{z_1}^{z_2} = F_2^{\,k}.\frac{\partial G_2^k}{\partial z}-\left(\frac{G_2^k-G_1^k}{l_k}\right).\left(F_2^{\,k}-0\right)$$

$$=F_2^{\,k}.\frac{\partial G_2^k}{\partial z}-\left(\frac{G_2^k-G_1^k}{l_k}\right).F_2^{\,k} \tag{A.24}$$

APPENDIX B

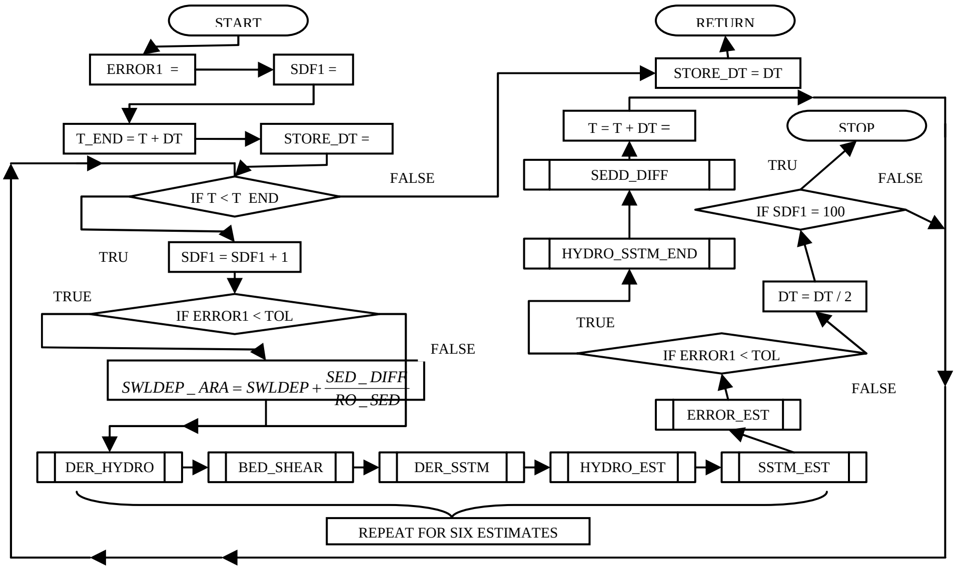

Figure B.1. Flowchart of RKFSYS_SSTM_1