2D CORRELATED DIFFUSION PROCESS FOR MOBILITY MODELING IN MOBILE NETWORKS

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ABSTRACT

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This thesis introduces a novel mobility model based on so called "2D correlated diffusion process". In this model, motion components over x and y axes are dependent. Joint density function of the process is derived. The expected exit time from an arbitrary domain is characterized by a boundary value problem. Analytical solution of this problem is given for a specific case. Numerical solution of the problem is presented by several examples. The results obtained in these examples are verified by simulations. The expected exit time computed by this method holds for any given 2D domain and any given starting position inside.

Keywords: Mobility Model, Correlated Diffusion Process, Expected Exit Time

ÖZ

HÜCRESEL ŞEBEKELERDE HAREKETLİLİK MODELİ İÇİN YENİ BİR 2 BOYUTLU BAĞIL YAYINIM SÜRECİ

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Bu çalışma, hücresel şebekelerde kullanılmak üzere yeni bir hareketlilik modeli sunmaktadır. Bu model, 2 boyutlu bağıl yayınım süreci tekniğine dayalıdır. x ve y eksenleri üzerindeki hareket bileşenleri birbirine istatistiki olarak bağımlıdır. Sunulan sürecin ortak yoğunluk fonksiyonu türetilmektedir. Bir sınır değeri problemi olarak ortaya konan herhangi bir alandan ortalama çıkış süresi, özel bir durum için analitik olarak çözülmektedir. Bu çıkış süresi, ayrıca bazı örnekler üzerinde sayısal olarak hesaplanarak elde edilen değerler simülasyon sonuçları ile desteklenmektedir. Problemin bir sınır değeri problemi olarak ortaya konması, çözümün isteğe göre seçilmiş bir 2 boyutlu alana ve bu alan içerisinde yine isteğe göre seçilmiş bir hareket başlangıç noktasına göre yapılmasına olanak tanımaktadır.

Anahtar Kelimeler: Hareketlilik Modeli, Yayınım Süreci, Ortalama Çıkış Süresi

To My Grandfather

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CHAPTER 1

INTRODUCTION

A mobile network must provide its subscribers a certain degree of freedom to move without any service interruption. To deliver an uninterrupted service, it must manage mobility of its users. The procedure of handling the entire group of tasks that a wireless network must perform to provide mobility to its users is called “mobility management”. Mobility management can be divided into two categories: location management and handover management. Location management tasks are handled when mobile station (MS) is in idle mode i.e., when it does not have an active connection. Location update, paging, routing area update, cell selection, cell reselection, and PLMN selection are some of the idle mode tasks in a cellular network. Handover management tasks are handled when MS is in active (or dedicated) mode. The aim of handover management tasks is to provide continuity of an active connection.

The research in mobility management is subject to rapid developments. These developments rely on hardware improvements as well as new algorithms or strategies. Performance analysis of a proposed strategy for performing a specific task of mobility management requires a mobility model. A mobility model is a description and characterization of movements of a mobile unit.

Mobility modeling is accomplished in various ways in literature. Some of the works attempting to model mobility are studied in the next chapter in detail. Motion has time-varying nature, hence characterization of MS movements is commonly accomplished by stochastic modeling. Stochastic processes are powerful tools in this subject. They can be gathered into four groups.

The first group has discrete index set and discrete state space. Markov chain is the most important example of this group. Random walk processes are based on regular Markov chains. They are used extensively for mobility modeling. They became very popular due to their simplicity and ease of use.

The second group has continuous index set and discrete state space. The most common processes of this group are continuous-time Markov chains and Poisson processes. Poisson processes are mainly used for two purposes: modeling connection arrival rates and direction change times.

The third group has discrete index set and continuous state space. An example of this sort of stochastic processes is Halris chains. It is a special type of Markov chains. This kind of processes are hardly used for mobility modeling purposes.

The last group has continuous index set and continuous state space. Diffusion processes are of this group. Fluid-flow mobility models rely on diffusion processes. Brownian motion process (sometimes called Wiener process in literature) is the most common diffusion process.

Brownian motion process is a suitable tool for mobility modeling. It has two parameters: C, and D. These parameters govern the behavior of movements. C is the mean parameter. Every time unit, the process moves C units on average. This parameter is sometimes called drift velocity or drift parameter. It determines average velocity of a moving body. D is the variance parameter. Variability of motion depends on this parameter. It is called location uncertainty in mobility modeling studies.

A common way of using Brownian motion as a mobility model is to consider the trajectory of two independent Brownian motions over cartesian coordinate system. The pattern resulting from this method can be of two forms. If the independent motion components are constructed by zero-mean processes, then the pattern will look like a drunk man wandering around. If one of the motion components or both are constructed with nonzero means, then the pattern will be untied. It will tend to go in a direction continuously and forever. The mobility patterns generated by this technique cannot go beyond these two forms.

In this thesis, a novel mobility model is introduced. This model is based on a so called "2D correlated diffusion process". Considering this process over cartesian coordinate system, x and y components are dependent on each other.

The first-order density function of the process is a joint normal density. The x and y components are jointly normal stochastic processes.

The main objective of presenting 2D correlated diffusion process as a mobility modeling technique is to provide possibility of controlling the amount of correlation between the motion components. This model avoids the limitation of two possible forms of mobility patterns in independent Brownian motions case.

Analytical description of 2D correlated diffusion process is provided by deriving its joint density function in the thesis. By this description, probability density of the location of an MS at a given time is available in a single function of two variables and time. Such a description has not been provided for the technique of independent Brownian motions so far.

The derivation of the density is provided in the thesis by first constructing a 2D correlated random walk. Then, limiting behavior of the random walk is analyzed. Taking the limit as time index goes to zero, the probability distribution function of the random walk generates a partial differential equation in two state variables and time. The solution of this differential equation is found to be a joint normal function satisfying all requirements of a first-order density function of a stochastic process. This solution is the first-order density function of the process and given by

$$f = A \exp\left\{ -\frac{1}{2(1-r^2)} \left[\frac{(x-x_0-C_xt)^2}{D_x^2 t} - \frac{2r(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt} + \frac{(y-y_0-C_yt)^2}{D_y^2 t} \right] \right\}$$

, where

$$A = \frac{1}{2\pi D_x D_y t\sqrt{1-r^2}}, \qquad |r| \leq 1$$

The proposed model allows to generate sample mobility characteristics by altering five parameters C_x, C_y, D_x, D_y, and r. These parameters appear in the joint density function. Many MS mobility profiles can be generated by adjusting these parameters.

In analyzing performances of mobility management strategies, residence times (or dwell times) in a registration area, a location area, or a cell are of great interest. After completely presenting the 2D correlated diffusion process, this thesis solves exit time problem from an arbitrary closed domain. The process can start at any given point inside any 2-dimensional domain. The expected duration of time the process stays inside that domain is computed. This duration of time corresponds to residence times in registration areas or cells.

Another motivation of the thesis was to provide flexibility in choosing the shape of registration area or cell from which the expected exit time is wanted. This flexibility exists also in choosing the initial position of the motion inside that cell. The expected exit time provided in the thesis can be from any arbitrary shaped region starting at any given point inside that region.

To solve the exit time problem, a difference equation with boundary conditions is constructed for the corresponding random walk such that the solution is the expected number of transitions starting at a given point until reaching the boundary. After taking the limit as time index goes to zero, a PDE with boundary conditions is obtained. The solution of this boundary value problem is the expected exit time of the 2D correlated diffusion process from the given domain.

The BVP obtained this way is a nonhomogeneous BVP with homogeneous boundary conditions. That is to say, the PDE of the problem is nonhomogeneous and the boundary condition is homogeneous. The PDE is a linear, second-order, and constant-coefficient equation. The BVP can be solved numerically for any given domain. In the thesis, two example numerical solutions are given. Analytical solution of the BVP is provided for a circular domain. This solution involves an assumption that the variance parameters of the process are equal.

CHAPTER 2

RELATED WORK

There are numerous mobility models in literature. They can be divided into several classes. One important family of mobility models are based on random walk processes. Another group are called fluid-flow models which use Brownian motion processes. Some models try to characterize cell or registration area residence times directly without considering a motion pattern. There are other mobility models which primarily concern direction changes of mobile unit. Some works attempt to model motion patterns in 3D environments.

A well-known strategy in random walk models is to let an MS move from one cell to its neighbors with the equal probability at every time unit. However, such an assumption does not allow setting, tuning, or altering mobility characteristics of the MS which is modeled by this method. Random walk models assume that MSs have the same mobility pattern. They move at a constant speed and do not retain location uncertainty.

[1], [2] and [3] are typical examples of the random walk model. In these works, every step of an MS is from one square cell to its neighbors with probability *1/4*, or from one hexagonal cell to its neighbors with probability *1/6*. These models are based on a regular Markov chain.

In [4], an enhancement is introduced to random walk models by considering each mobile unit to move roughly a straight line (with occasional backtracking) for a significant period of time before changing direction. It is also noted that in roughly an orthogonal road system, the new direction after changing direction tends to be roughly perpendicular to the previous direction.

In [5], a fractional Brownian motion (FBM) mobility model is presented. In this model, four independent Brownian motion processes are adopted to suit the motion in four bounds (west, east, north, and south). This paper includes the following statement which is important in expressing the motivation behind evolution from random walk to fluid-flow models. "Empirical transportation engineering studies show that at a given point, the mobile speed is normally distributed with the street speed limit as the mean and some standard deviation." According to this statement, it is concluded in the paper that the location distribution at a specific time follows a gaussian distribution. Therefore, Brownian motion is a suitable way of describing MS movements.

Some works directly attempt to characterize cell residence times. [6] and [8] propose a new mobility model, called hyper-Erlang distribution model. This model is used to characterize the cell residence time and obtain analytical results for the channel holding time. [7] classifies the mobility behavior of the subscribers concerning the covered travel length into three categories: type A - completely aimless motion, type B - preferred direction with a certain deviation and type C - completely directed motion. Given the random variable X of the travel length or the moving distance of a subscriber within a radio cell, and V of the velocity, the random variable T of the dwell time can be calculated as $T = X/V$. [35] allows for multiple platform types each having distinct mobility characteristics. These are characterized by the statistical properties of the dwell time of a platform of the given type. The dwell time of a platform is a random variable with three different characterizations. These are sum of negative exponentials, hyperexponential, and sum of hyperexponentials.

A great deal of mobility models in literature solve mobility problems in 1-dimensional or 2-dimensional environments. Other than these, [10], [19], [24], and [25] consider 3-dimensional case in indoor building environments. They assume proper boundary conditions on each floor and analytically model the mobility in multi-story buildings. Users move on the square-shaped floors of a building. A staircase region consists of staircases and passages and is located in the center. The remaining region is called the floor region. Horizontal and vertical speeds are uniformly distributed. Users' moving behaviors are statistically the same on each floor. They move straight until changing directions which occur according to Poisson process. When a user arrive at the outer wall, they go back to the incoming direction without delay.

Another group of works treat direction of users as a separate variable. [12] takes a squared region with a 5 meters grid pattern serving as a base for the model. This grid can be changed and accommodated to special conditions. Within this region several areas with different mobility parameters are defined. These are the approximate preferred direction of movement, the divergence from the preferred direction, the deviation of the velocity, and the probability for staying in this area at initialization. [21] presents a new model describing the mobility of vehicle-borne terminals under realistic urban traffic conditions. The model accounts for arbitrary urban street patterns and realistic terminal movements by a limited number of parameters that can be easily measured or derived from a city map. It introduces distribution functions of street length, direction changes at crossroads, and terminal velocity to find an analytical formulation. In [31] some assumptions are made to model MS mobility. A LA has the shape of a square. Mobile users move at a constant speed straightly until the change of moving directions. Moving direction changes occur according to a Poisson process and with the corresponding probabilities to left, right, up, and down.

[9] proposes a meta-model that is an integration of three key elements determining the mobility of users: the spatial environment, the user trip sequences, and the user movement dynamics (eg. speed). The framework is available for download as a stand-alone trace generator and may be used together with any simulation or emulation tool for mobile networks to evaluate a specific scenario.

CHAPTER 3

A NOVEL MOBILITY MODEL: 2D CORRELATED DIFFUSION PROCESS

2D correlated diffusion process is constructed in this chapter. Considering the process over cartesian coordinate system, x and y components are dependent on each other. The proposed process allows to change the amount of correlation between the components. The overall procedure of characterizing the process is summarized in figure 3.1. This procedure is followed in the following sections in detail. The procedure starts with description of 2D correlated random walk. The motion of this walk is constructed such that x and y components move together. This motion strategy creates dependency between the components.

Then, selection of the parameters such as, step sizes, transition probabilities, and time index of the random walk is performed. This selection step is of great importance since when transition from discrete-time to continuous-time is performed, these parameters will determine the characteristics of the resulting 2D correlated diffusion process.

The derivation procedure continues with analysis of limiting behavior of the random walk. Transition probability distribution of the random walk is shown to yield a differential equation when expanded according to Taylor's theorem and taking the limit as time index goes to zero. The solution of this equation is shown to be a joint normal function. This function is the first-order density function of the 2D correlated diffusion process.

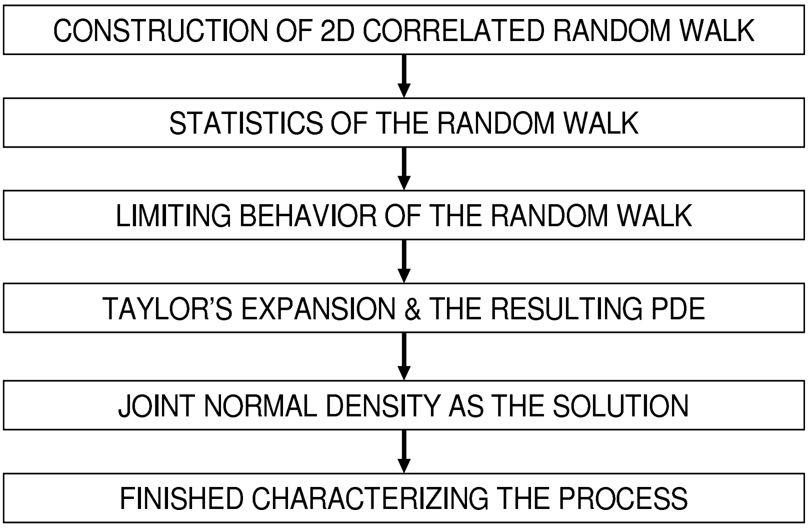

Figure 3.1: Derivation steps of the 2D correlated diffusion process.

3.1 Construction of 2D Correlated Random Walk

Let $\{X_n, Y_n; \quad n = 0, 1, , \ldots\}$ be a 2D correlated random walk. Moves of each component depends on each other according to the following transition probabilities.

$$p_1 = P\{X_{n+1} = X_n + 1, Y_{n+1} = Y_n + 1\}$$

$$p_2 = P\{X_{n+1} = X_n + 1, Y_{n+1} = Y_n - 1\}$$

$$p_3 = P\{X_{n+1} = X_n - 1, Y_{n+1} = Y_n + 1\}$$

$$p_4 = P\{X_{n+1} = X_n - 1, Y_{n+1} = Y_n - 1\}$$

$$p_1 + p_2 + p_3 + p_4 = 1$$

The probability of being in state (i, j), after n steps starting at (i_0, j_0) is computed recursively as

$$P_{(i_0,j_0)(i,j)}(n) = P\{X_n = i, Y_n = j \mid X_0 = 0, Y_0 = 0\}$$

$$= p_1 P_{(i_0,j_0)(i-1,j-1)}(n-1) + p_2 P_{(i_0,j_0)(i-1,j+1)}(n-1)$$

$$+ p_3 P_{(i_0,j_0)(i+1,j-1)}(n-1) + p_4 P_{(i_0,j_0)(i+1,j+1)}(n-1) \tag{3.1}$$

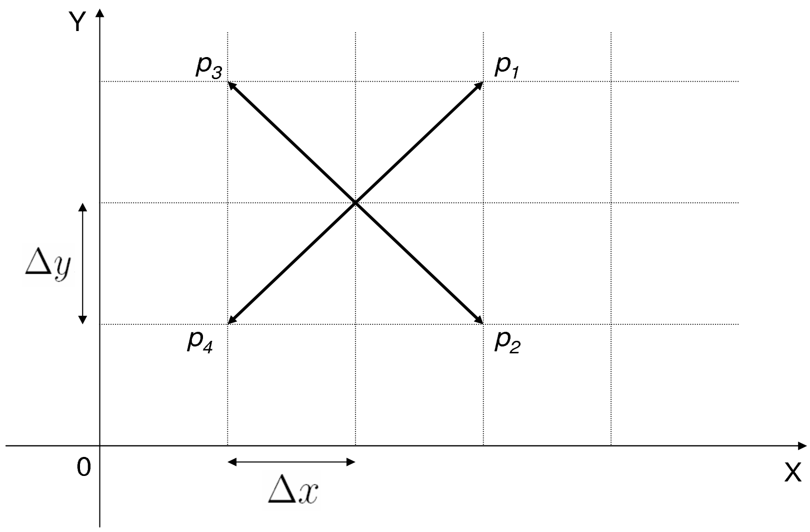

Figure 3.2: Moving diagram of 2D correlated random walk.

Suppose in this random walk, state changes have magnitudes $\Delta x, \Delta y$ for X_n, Y_n, respectively at time instants $\Delta t, 2\Delta t, \ldots$. Let x and x_0 be multiples of Δx, y and y_0 be multiples of Δy, and t be a multiple of Δt. Moving diagram of the random walk is given in figure 3.2. Let

$$f(t, x, y, x_0, y_0) \triangleq P\{X_t = x, Y_t = y \mid X_0 = x_0, Y_0 = y_0\}$$

$$= p_1 f(t - \Delta t, x - \Delta x, y - \Delta y, x_0, y_0) + p_2 f(t - \Delta t, x - \Delta x, y + \Delta y, x_0, y_0)$$

$$+ p_3 f(t - \Delta t, x + \Delta x, y - \Delta y, x_0, y_0) + p_4 f(t - \Delta t, x + \Delta x, y + \Delta y, x_0, y_0) \quad (3.2)$$

Let $n(t)$ be the number of transitions by time t.

$$n(t) \triangleq \frac{t}{\Delta t}$$

(x_0, y_0) is the initial position of the random walk and will be taken as $(x_0, y_0) = (0, 0)$ in the derivations of the following sections.

3.2 Statistics of the Random Walk

The expected value of $\{X_{n(t)}, n(t) = 0, 1, \ldots\}$ is

$$E\big[X_{n(t)}\big] = \Big[(p_1 + p_2)\Delta x - (p_3 + p_4)\Delta x\Big] \frac{t}{\Delta t}$$

$$= (p_1 + p_2 - p_3 - p_4)\frac{\Delta x}{\Delta t}t \tag{3.3}$$

and the expected value of $\{Y_{n(t)}, n(t) = 0, 1, \ldots\}$ is

$$E\big[Y_{n(t)}\big] = \Big[(p_1 + p_3)\Delta y - (p_2 + p_4)\Delta y\Big]\frac{t}{\Delta t}$$

$$= (p_1 + p_3 - p_2 - p_4)\frac{\Delta y}{\Delta t}t \tag{3.4}$$

Each component of the process can be expressed as a sum of $n(t)$ independent random variables as

$$X_{n(t)} = \sum_{i=1}^{n(t)} X_i$$

$$Y_{n(t)} = \sum_{i=1}^{n(t)} Y_i$$

, where (X_i, Y_i) pairs are generated according to the probability distribution

$$P\{X_i = \Delta x, Y_i = \Delta y\} = p_1$$

$$P\{X_i = \Delta x, Y_i = -\Delta y\} = p_2$$

$$P\{X_i = -\Delta x, Y_i = \Delta y\} = p_3$$

$$P\{X_i = -\Delta x, Y_i = -\Delta y\} = p_4$$

The variance of $X_{n(t)}$ is computed as follows.

$$V\big(X_{n(t)}\big) = n(t)V\big(X_i\big)$$

$$= n(t)\Big(E\big[X_i^2\big] - E\big[X_i\big]^2\Big)$$

$$= n(t)\Big[(p_1 + p_2)\Delta x^2 + (p_3 + p_4)(-\Delta x)^2 - \big((p_1 + p_2)\Delta x - (p_3 + p_4)\Delta x\big)^2\Big]$$

$$= n(t)\big[\Delta x^2 \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1} - \Delta x^2(p_1 + p_2 - p_3 - p_4)^2\big]$$

$$= n(t)\Delta x^2\big[1 - (p_1 + p_2 - p_3 - p_4)^2\big]$$

$$= \frac{\Delta x^2}{\Delta t}\big[1 - (p_1 + p_2 - p_3 - p_4)^2\big]t \tag{3.5}$$

Similarly, the variance of $Y_{n(t)}$ is

$$V\big(Y_{n(t)}\big) = n(t)V\big(Y_i\big)$$

$$= n(t)\Big(E\big[Y_i^2\big] - E\big[Y_i\big]^2\Big)$$

$$= n(t)\Big[(p_1+p_3)\Delta y^2 + (p_2+p_4)(-\Delta y)^2 - \big((p_1+p_3)\Delta y - (p_2+p_4)\Delta y\big)^2\Big]$$

$$= n(t)\big[\Delta y^2 \underbrace{(p_1+p_2+p_3+p_4)}_{1} - \Delta y^2 (p_1+p_3-p_2-p_4)^2\big]$$

$$= n(t)\Delta y^2\big[1-(p_1+p_3-p_2-p_4)^2\big]$$

$$= \frac{\Delta y^2}{\Delta t}\big[1-(p_1+p_3-p_2-p_4)^2\big]t \tag{3.6}$$

For the generic random variables X_i and Y_i, the covariance is

$$C = E\Big[\big(X_i - \eta_{X_i}\big)\big(Y_i - \eta_{Y_i}\big)\Big]$$

, where

$$E\big[X_i\big] = \eta_{X_i} = \Delta x(p_1+p_2-p_3-p_4) \tag{3.7}$$

$$E\big[Y_i\big] = \eta_{Y_i} = \Delta y(p_1+p_3-p_2-p_4) \tag{3.8}$$

The covariance can be rewritten as

$$C = E\big[X_iY_i\big] - E\big[X_i\big]E\big[Y_i\big]$$

The first term in the covariance is

$$E\big[X_iY_i\big] = p_1(\Delta x\Delta y) + p_2(\Delta x(-\Delta y)) + p_3(-\Delta x\Delta y) + p_4(-\Delta x(-\Delta y))$$

$$= \Delta x\Delta y(p_1-p_2-p_3+p_4) \tag{3.9}$$

Thus

$$C = \Delta x\Delta y(p_1-p_2-p_3+p_4) - \Delta x\Delta y(p_1+p_2-p_3-p_4)(p_1+p_3-p_2-p_4) \tag{3.10}$$

The correlation coefficient is

$$r_{xy} = \frac{C}{\sigma_x\sigma_y}$$

, where the variances are

$$V\big(X_i\big) = \sigma_x^2 = E\big[X_i^2\big] - E\big[X_i\big]^2$$

$$V\big(Y_i\big) = \sigma_y^2 = E\big[Y_i^2\big] - E\big[Y_i\big]^2$$

$$\sigma_x^2 = (p_1+p_2)\Delta x^2 + (p_3+p_4)(-\Delta x)^2$$

$$= \Delta x^2 \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1}$$
$$= \Delta x^2$$
$$\sigma_y^2 = (p_1 + p_3)\Delta y^2 + (p_2 + p_4)(-\Delta y)^2$$
$$= \Delta y^2 \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1}$$
$$= \Delta y^2$$

Then

$$r_{xy} = \frac{\Delta x \Delta y \big[(p_1 - p_2 - p_3 + p_4) - (p_1 + p_2 - p_3 - p_4)(p_1 + p_3 - p_2 - p_4)\big]}{\Delta x \Delta y}$$
$$= (p_1 - p_2 - p_3 + p_4) - (p_1 + p_2 - p_3 - p_4)(p_1 + p_3 - p_2 - p_4) \qquad (3.11)$$

Having computed the expected values and the variances of the X and Y components and the correlation coefficient of the random walk, the limiting values of these statistics are studied in the next section.

3.3 Limiting Behavior of the Random Walk

The question is how to choose the values p_1, p_2, p_3, p_4, Δx, and Δy so that as $\Delta t \to 0$

$$E\big[X_{n(t)}\big] \longrightarrow C_x t$$
$$E\big[Y_{n(t)}\big] \longrightarrow C_y t$$
$$V\big(X_{n(t)}\big) \longrightarrow D_x^2 t$$
$$V\big(Y_{n(t)}\big) \longrightarrow D_y^2 t$$
$$r_{xy} \longrightarrow r$$

, where C_x, C_y, D_x, D_y, and r are constants. The following constraints answer this question.

$$p_1 + p_2 = \frac{1}{2}\left(1 + \frac{C_x\sqrt{\Delta t}}{D_x}\right) \qquad (3.12)$$

$$p_3 + p_4 = \frac{1}{2}\left(1 - \frac{C_x\sqrt{\Delta t}}{D_x}\right) \qquad (3.13)$$

$$\Delta x = D_x\sqrt{\Delta t} \tag{3.14}$$

$$p_1 + p_3 = \frac{1}{2}\left(1 + \frac{C_y\sqrt{\Delta t}}{D_y}\right) \tag{3.15}$$

$$p_2 + p_4 = \frac{1}{2}\left(1 - \frac{C_y\sqrt{\Delta t}}{D_y}\right) \tag{3.16}$$

$$\Delta y = D_y\sqrt{\Delta t} \tag{3.17}$$

As $\Delta t \to 0$

$$\begin{aligned}
E\big[X_{n(t)}\big] &= (p_1 + p_2 - p_3 - p_4)\frac{\Delta x}{\Delta t}t \\
&= \frac{C_x\sqrt{\Delta t}}{D_x}\frac{D_x\sqrt{\Delta t}}{\Delta t}t \\
&= C_x t \longrightarrow C_x t \\
E\big[Y_{n(t)}\big] &= (p_1 + p_3 - p_2 - p_4)\frac{\Delta y}{\Delta t}t \\
&= \frac{C_y\sqrt{\Delta t}}{D_y}\frac{D_y\sqrt{\Delta t}}{\Delta t}t \\
&= C_y t \longrightarrow C_y t \\
V\big(X_{n(t)}\big) &= \frac{\Delta x^2}{\Delta t}\big[1 - (p_1 + p_2 - p_3 - p_4)^2\big]t \\
&= \frac{D_x^2\Delta t}{\Delta t}\left[1 - \left(\frac{C_x\sqrt{\Delta t}}{D_x}\right)^2\right]t \\
&\longrightarrow D_x^2 t \\
V\big(Y_{n(t)}\big) &= \frac{\Delta y^2}{\Delta t}\big[1 - (p_1 + p_3 - p_2 - p_4)^2\big]t \\
&= \frac{D_y^2\Delta t}{\Delta t}\left[1 - \left(\frac{C_y\sqrt{\Delta t}}{D_y}\right)^2\right]t \\
&\longrightarrow D_y^2 t
\end{aligned}$$

To satisfy the requirement $r_{xy} \to r$ as $\Delta t \to 0$. Let

$$p_1 - p_2 - p_3 + p_4 = r \tag{3.18}$$

Solving (3.18) together with

$$p_1 + p_2 + p_3 + p_4 = 1$$

the following equation is obtained.

$$p_1 + p_4 = \frac{1}{2}(1 + r) \tag{3.19}$$

Subtracting (3.16) from (3.12)

$$p_1 - p_4 = \frac{1}{2}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{2}\frac{C_y\sqrt{\Delta t}}{D_y} \tag{3.20}$$

Solving (3.19) together with (3.20)

$$p_1 = \frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 + r) \tag{3.21}$$

$$p_4 = \frac{1}{2}(1 + r) - p_1$$

$$= -\frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} - \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 + r) \tag{3.22}$$

$$p_2 = \frac{1}{2} + \frac{1}{2}\frac{C_x\sqrt{\Delta t}}{D_x} - p_1$$

$$= \frac{1}{2} + \frac{1}{2}\frac{C_x\sqrt{\Delta t}}{D_x} - \frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} - \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} - \frac{1}{4}(1 + r)$$

$$= \frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} - \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 - r) \tag{3.23}$$

$$p_3 = \frac{1}{2} - \frac{1}{2}\frac{C_x\sqrt{\Delta t}}{D_x} - p_4$$

$$= \frac{1}{2} - \frac{1}{2}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} - \frac{1}{4}(1 + r)$$

$$= -\frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 - r) \tag{3.24}$$

As a result the following set of choices of the values p_1, p_2, p_3, p_4, Δx, and Δy

$$p_1 = \frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 + r)$$

$$p_2 = \frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} - \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 - r)$$

$$p_3 = -\frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} + \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1 - r)$$

$$p_4 = -\frac{1}{4}\frac{C_x\sqrt{\Delta t}}{D_x} - \frac{1}{4}\frac{C_y\sqrt{\Delta t}}{D_y} + \frac{1}{4}(1+r)$$

$$\Delta x = D_x\sqrt{\Delta t}$$

$$\Delta y = D_y\sqrt{\Delta t}$$

will lead to the following limiting behavior of the correlated random walk process as $\Delta t \to 0$

$$E\big[X_{n(t)}\big] \longrightarrow C_x t$$

$$E\big[Y_{n(t)}\big] \longrightarrow C_y t$$

$$V\big(X_{n(t)}\big) \longrightarrow D_x^2 t$$

$$V\big(Y_{n(t)}\big) \longrightarrow D_y^2 t$$

$$r_{xy} \longrightarrow r$$

The limiting process $\{X(t), Y(t), t \geq 0\}$ constructed in this manner is the 2D correlated diffusion process such that its expected values and variances for the X and Y components together with the correlation coefficient are the limiting values of those of the 2D correlated random walk.

3.4 Taylor's Expansion, the PDE, and the Joint Normal Solution

To characterize the first-order density function of the 2D correlated diffusion process, t, x, y, x_0, and y_0 should be treated as continuous. Remember the function (3.2) defined in section 3.1. As $\Delta t \to 0$, this function becomes the first-order joint density function regarding the variables t, x, y, x_0, and y_0 as continuous. The corresponding first-order distribution function is

$$F(t, x, y, x_0, y_0) = P\{X(t) \leq x, Y(t) \leq y \mid X(0) = x_0, Y(0) = y_0\} \tag{3.25}$$

with

$$f(t, x, y, x_0, y_0) = \frac{\partial^2}{\partial x \partial y} F(t, x, y, x_0, y_0)$$

This thesis proposes that the density is joint normal $N(\eta_1, \eta_2; \sigma_1, \sigma_2; r_{xy})$, where

$$\eta_1 = x_0 + C_x t$$

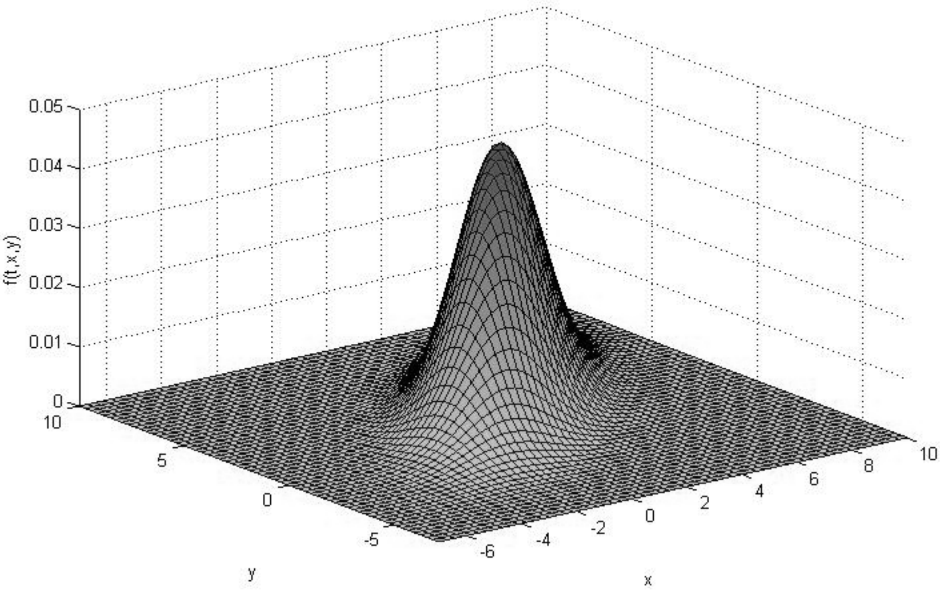

Figure 3.3: A sample plot of the joint density with Cx=Cy=0.5, Dx=Dy=1, r=0.7 for the process starting at the origin.

$$\eta_2 = y_0 + C_y t$$
$$\sigma_1 = D_x \sqrt{t}$$
$$\sigma_2 = D_y \sqrt{t}$$
$$r_{xy} = r$$

ie.

$$f = A \exp \left\{ -\frac{1}{2(1-r^2)} \left[\frac{(x - x_0 - C_x t)^2}{D_x^2 t} - \frac{2r(x - x_0 - C_x t)(y - y_0 - C_y t)}{D_x D_y t} + \frac{(y - y_0 - C_y t)^2}{D_y^2 t} \right] \right\} \quad (3.26)$$

This function is positive and its integral equals 1 if

$$A = \frac{1}{2\pi D_x D_y t \sqrt{1-r^2}}, \qquad |r| \leq 1$$

A sample plot of the joint density function is given in figure 3.3 for specific selections of the main parameters and time.

To prove this argument, components of (3.2) is approximated using Taylor's polynomial of degree two for functions of five variables. Taylor's theorem is given in appendix A.1 in detail. Approximations of the components are available in appendix A.2. Then, the approximated components are put in (3.2). Both sides of the resulting equation are divided by Δt. Taking the limit as $\Delta t \rightarrow 0$, the following PDE is obtained.

$$f_t = -C_x f_x - C_y f_y + D_x D_y r f_{xy} + \frac{1}{2} D_x^2 f_{xx} + \frac{1}{2} D_y^2 f_{yy} \tag{3.27}$$

This procedure of obtaining (3.27) is provided in appendix A.2 in detail. The solution of this equation is (3.26). Proof of this statement is given in appendix A.3. Hence, the 2D correlated diffusion process is fully characterized by explicitly determining its first-order density function.

The density has five parameters C_x, C_y, D_x, D_y, and r which determine the characteristics of motion. By adjusting these parameters, many mobility profiles can be generated. That is to say, these are the main controlling parameters of the proposed model.

C_x and C_y are drift parameters over the x and y axes, respectively. They determine the average velocities of the motion components. That is to say, the process moves C_x and C_y units on average over the corresponding axis at every time unit. Increasing C_x of positive value causes the motion to tend more to go in the positive x direction. Decreasing C_x of negative value causes the motion to tend more to go in the negative x direction. The same thing happens for the motion component over the y axis. If both C_x and C_y are adjusted together, their cummulative effect is observed.

D_x and D_y are the variance parameters regarding the x and y axes, respectively. They determine variability of the corresponding motion component. These parameters are positive valued and can be called location uncertainty parameters from MS point of view. By increasing them, larger variations during the same duration of time towards the corresponding axis are observed. Increase of both yields larger variation in both axes.

r is the correlation coefficient with $|r| < 1$. This parameter determines the amount of correlation between the motion components. This parameter provides the most important feature of the model. It allows to generate a great deal of mobility patterns formed using this model when altered within its range. Some sample realizations of the process with respect to various combinations of the main parameters are available in appendix C.

The proposed process has been fully described by giving its first-order density function explicitly. First, 2D correlated random walk was constructed, in which the motion components move together in four diagonal directions. This strategy resulted in a correlated motion pattern. Then, statistics of the walk were computed and taking the limit as time index goes to zero, they converged to those of the 2D correlated diffusion process. 2D correlated random walk became 2D correlated diffusion process, when transition from discrete-time to continuous-time was made. Finally, the probability distribution of the random walk yielded a PDE whose solution is the joint density of 2D correlated diffusion process when expanded using Taylor's theorem. And, it has been proven that the solution follows a joint normal density.

Diffusion processes are strong tools of mobility modeling since they allow time varying speeds of mobile units which they model. In addition to this powerful nature of diffusion processes, the proposed model goes one step further and provides to change the amount of correlation between the motion components. The proposed model is based on 2D correlated diffusion process. Motion of a moving body can be modeled by this process such that at a given time t, the location of him has a joint normal density as given in (3.26).

The closed form of the density function of the process can be used in many computations related to performance analysis of mobility management strategies. One famous computation is the expected duration of time an MS stays in a registration area or cell. The residence times (or dwell times) are frequently used in location management and handover management studies. The next chapter focusses on this topic.

CHAPTER 4

STATISTICAL DESCRIPTION OF RESIDENCE TIME

The residence time of an MS in a registration area or cell is an important metric in performance analysis of mobility management strategies. After finishing the complete description of the model in the previous chapter, the expected duration of time an MS stays in a registration area or cell is analyzed in this chapter. This duration of time corresponds to the exit time of the 2D correlated diffusion process from a given domain starting at an interior point.

The computation procedure of the exit time is followed through several steps. These steps are summarized in figure 4.1. Firstly, a difference equation with boundary condition is constructed for the corresponding 2D correlated random walk such that the solution of this equation is the expected number of periods until the process reaches the boundary on which it equals zero.

The difference equation constructed by this manner results in a PDE with boundary condition when the limit as time index goes to zero. The resulting equation is a linear, second-order, nonhomogeneous, and constant-coefficient PDE of elliptic type in two variables. And, the obtained boundary condition is homogeneous.

Then, this BVP is analyzed extensively. It is brought to its standard form through several steps. These steps which are rotation of axes, change of a dependent variable, and scaling are followed according to the methods given in [39]. Each step is described clearly in appendix B.3 in detail.

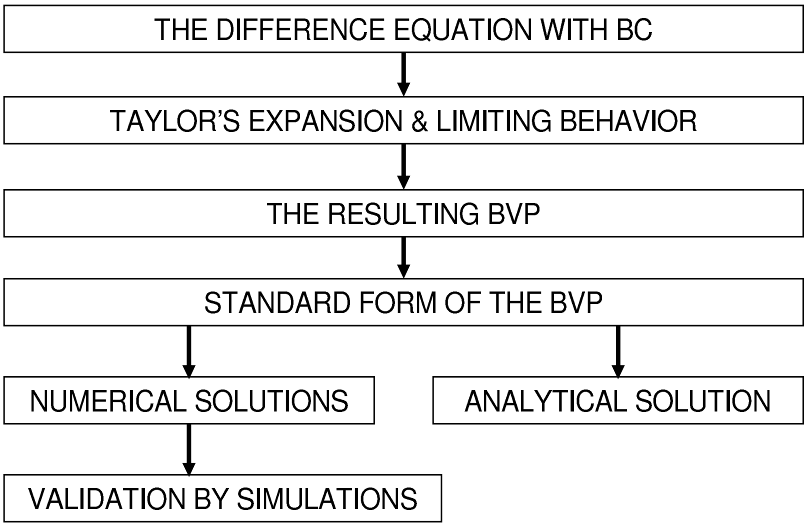

Figure 4.1: Procedure of exit time computation.

Transformation procedure of the BVP into its standard form and its numerical solution are demonstrated by an example which assumes hexagonal domain. The solution is verified by simulation. Furthermore, numerical solution and verification are repeated for another example with square domain. Finally, the BVP is solved analytically for circular-shaped domain with an assumption that variance parameters D_x and D_y are equal.

4.1 The Difference Equation, Taylor's Expansion, and the BVP

The 2D correlated random walk was described in chapter 3. In this random walk, state changes have magnitudes Δx and Δy at time instants Δt, $2\Delta t$,. . . . As $\Delta t \to 0$, the resulting process becomes a 2D correlated diffusion process with parameters C_x, C_y, D_x, D_y, and r if p_1, p_2, p_3, p_4, Δx, and Δy are selected as in section 3.3.

Let $T(x, y)$ be the expected number of steps for the walk to reach the boundary of the domain given in figure 4.2 starting at an arbitrary point (x, y) inside. $T(x, y)$ equals 0 over the boundary.

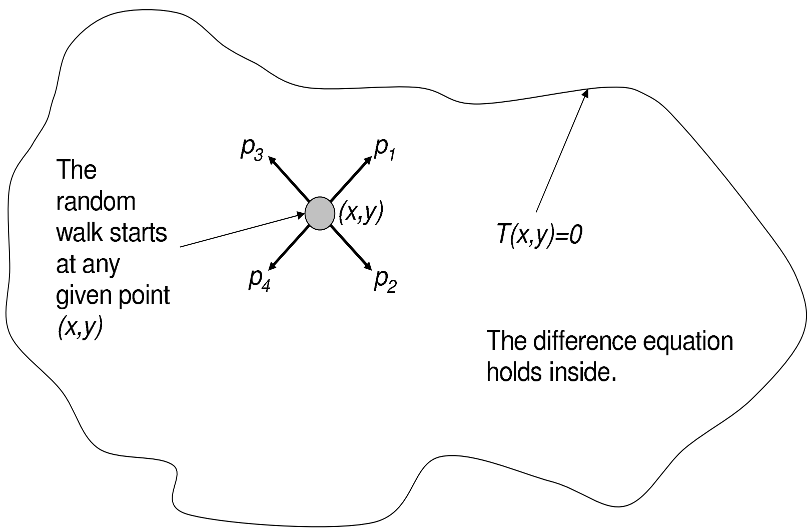

Figure 4.2: The difference equation with boundary condition.

Then, the following difference equation holds inside the domain.

$$\begin{aligned}
T(x,y) &= p_1\big[T(x+\Delta x, y+\Delta y)+\Delta t\big] + p_2\big[T(x+\Delta x, y-\Delta y)+\Delta t\big] \\
&\quad + p_3\big[T(x-\Delta x, y+\Delta y)+\Delta t\big] + p_4\big[T(x-\Delta x, y-\Delta y)+\Delta t\big] \\
&= p_1T(x+\Delta x, y+\Delta y) + p_2T(x+\Delta x, y-\Delta y) + p_3T(x-\Delta x, y+\Delta y) \\
&\quad + p_4T(x-\Delta x, y-\Delta y) + \underbrace{(p_1+p_2+p_3+p_4)}_{1} \\
&= p_1T(x+\Delta x, y+\Delta y) + p_2T(x+\Delta x, y-\Delta y) + p_3T(x-\Delta x, y+\Delta y) \\
&\quad + p_4T(x-\Delta x, y-\Delta y) + 1 \qquad (4.1)
\end{aligned}$$

Taylor's theorem for functions of two variables is applied to the components of $T(x,y)$. The approximated terms are plugged in (4.1). Terms are combined and arranged. Then, both sides of the resulting equation are divided by Δt. Taking the limit as $\Delta t \to 0$ and simplifying the equation further, the following PDE is obtained.

$$Lf = -1 \qquad (4.2)$$

, where L denotes the second-order differential operator

$$L = C_x\frac{\partial}{\partial x} + C_y\frac{\partial}{\partial y} + D_xD_yr\frac{\partial^2}{\partial x\partial y} + \frac{1}{2}D_x^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}D_y^2\frac{\partial^2}{\partial y^2} \qquad (4.3)$$

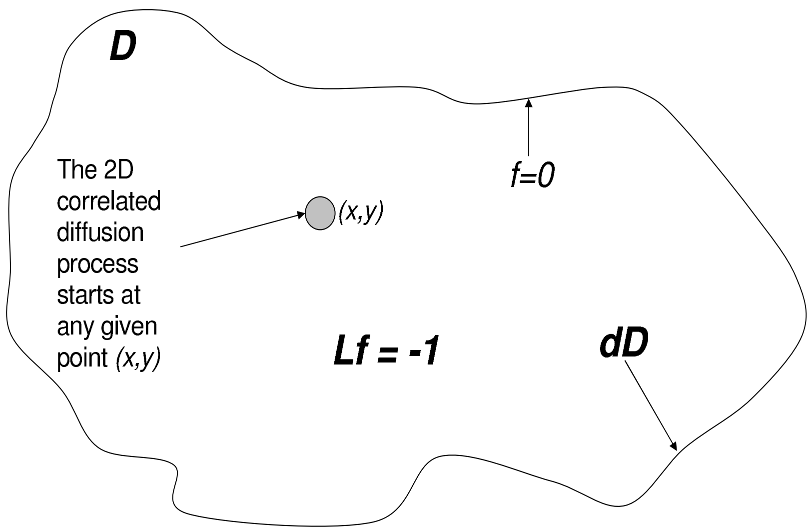

Figure 4.3: The resulting BVP.

, and

$$f \triangleq T(x, y)$$

The complete derivation of the resulting PDE is given in appendix B.1 in detail. The resulting BVP is shown in figure 4.3. The PDE holds inside the domain D and vanishes over the boundary dD. The solution of this BVP is the expected exit time of the 2D correlated diffusion process from D.

4.2 Standard Form of the BVP and Numerical Solutions

(4.2) is a second-order, linear, constant-coefficient, and nonhomogeneous PDE in two variables. In order to solve the resulting BVP involving (4.2), it should be brought into its standard form. Standard forms of second-order, linear, and constant-coefficient PDEs are explained in appendix B.2. The transformation procedure of the BVP into its standard form is summarized in figure 4.4. Each step of this procedure is explained in appendix B.3 in detail.

The transformation steps of the BVP into its standard form and numerical solution of the problem are demonstrated by example 1. The steps that are ex-

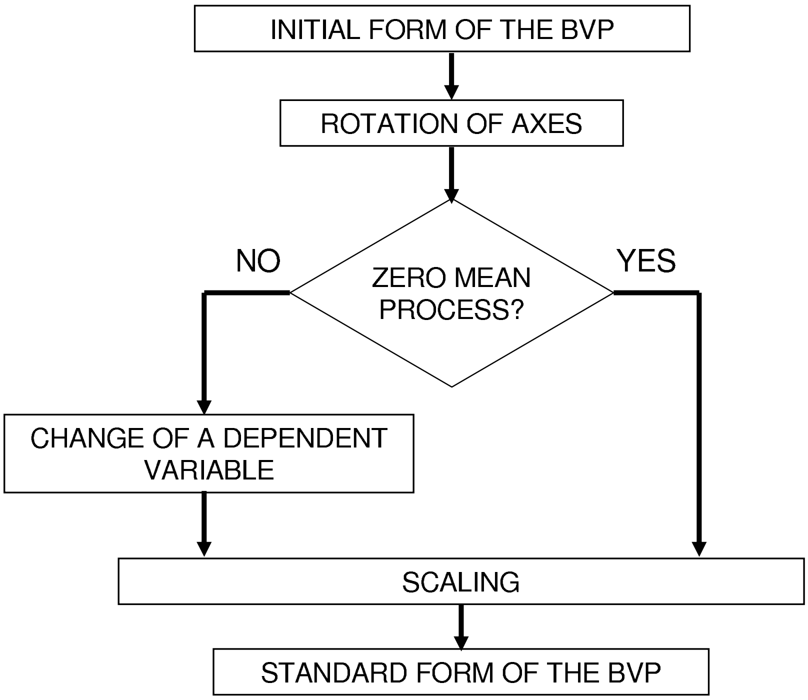

Figure 4.4: Bringing the BVP into its standard form.

plained in detail in appendix B.3 are applied to this example. Then, numerical solution is computed and verified by simulation.

Example 1. Consider an hexagon centered at the origin as given in figure 4.5. A motion starts inside with the setting

$$C_x = 1.5$$

$$C_y = 1.2$$

$$D_x = 2$$

$$D_y = 3$$

$$r = 0.5$$

Before the transformations, the PDE inside the hexagon is

$$Lu = -1$$

, where

$$L = C_x \frac{\partial}{\partial x} + C_y \frac{\partial}{\partial y} + D_x D_y r \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} D_x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2} D_y^2 \frac{\partial^2}{\partial y^2}$$

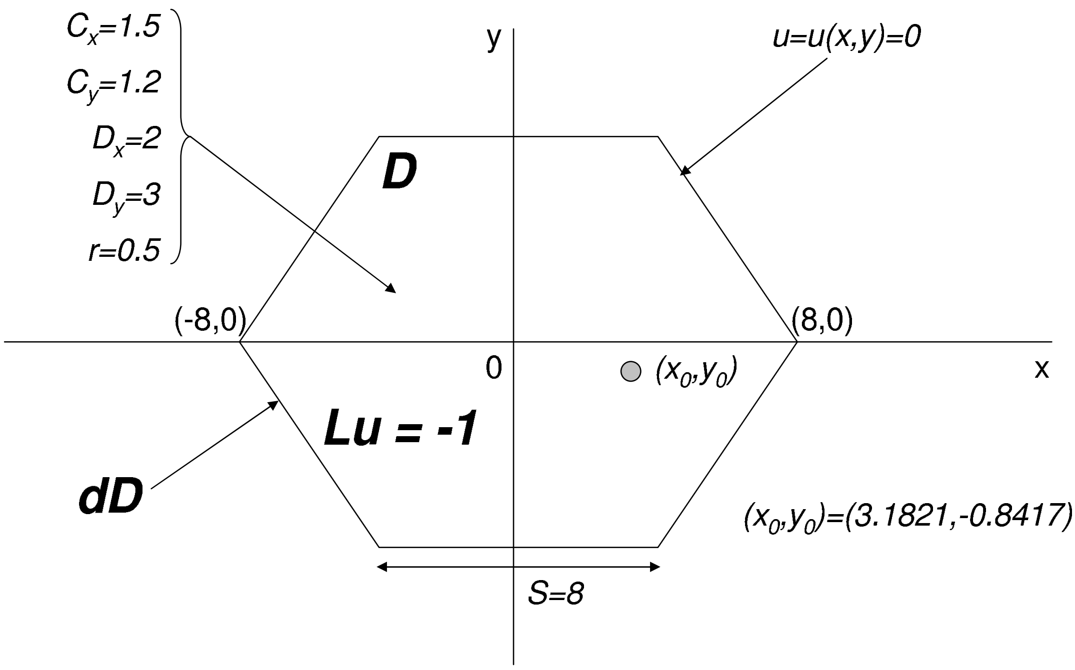

Figure 4.5: Description of the example 1.

, and

$$u = u(x, y)$$

$u(x, y)$ becomes 0 at the boundary of the hexagon. In the following steps, transformation procedure is described step-by-step.

(i) Rotation of axes. This step is described in figure 4.6. According to the settings given at the beginning of the example, the angle of rotation A defined in appendix B.3.1 is computed as

$$A = -25.0972°$$

The parameters a, b, c, and d which are given in appendix B.3.1 become

$$a = 1.2974$$

$$b = 5.2026$$

$$c = 0.8494$$

$$d = 1.7229$$

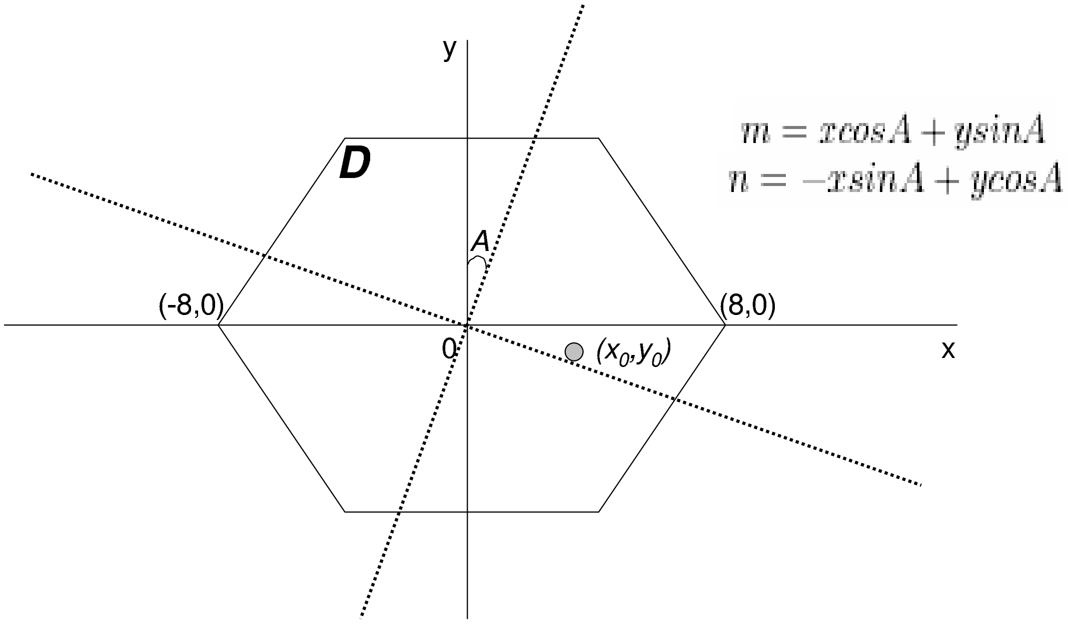

Figure 4.6: Rotation of axes for the example 1.

The resulting PDE from this transformation becomes

$$\tilde{L}w = -1$$

, where

$$\tilde{L} = a\frac{\partial^2}{\partial m^2} + b\frac{\partial^2}{\partial n^2} + c\frac{\partial}{\partial m} + d\frac{\partial}{\partial n}$$

, and

$$w = w(m, n)$$

The resulting domain is obtained by rotating the initial domain through an angle $A = 25.0972°$ counterclockwise or by rotating the axes through an angle $A = 25.0972°$ clockwise both are identical in effect. The resulting BVP from this step is sketched in figure 4.7.

(ii) Change of a dependent variable. The parameter K defined in appendix B.3.2 is computed as

$$K = 0.28167$$

The resulting PDE from this process becomes

$$au_{mm} + bu_{nn} - Ku = -exp\left\{\frac{c}{2a}m + \frac{d}{2b}n\right\}$$

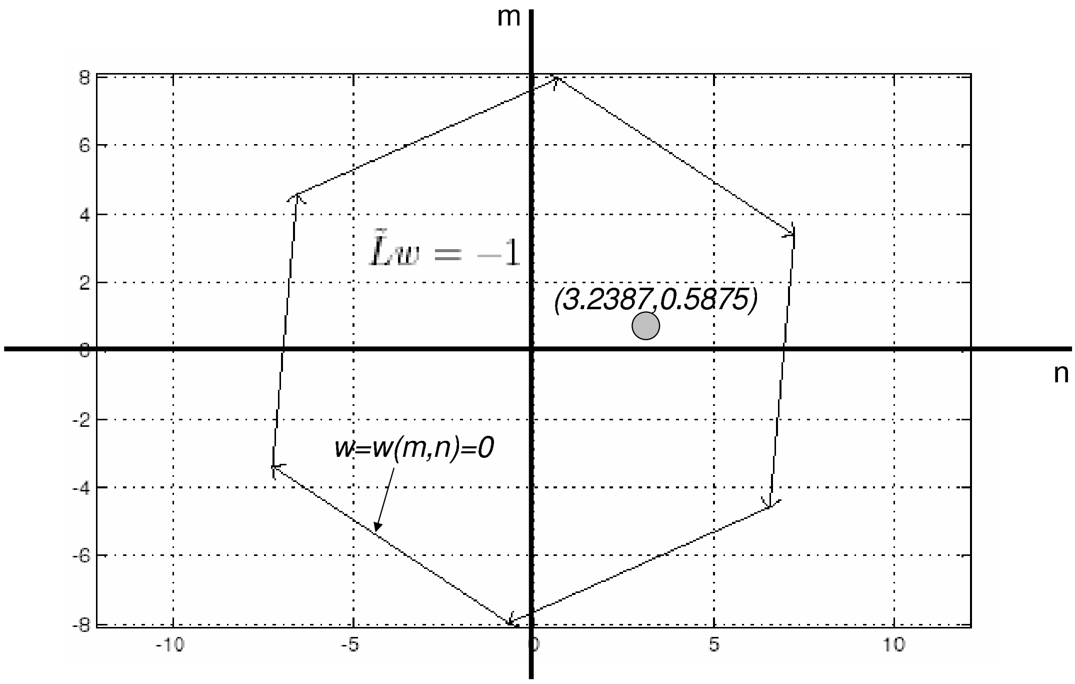

Figure 4.7: The resulting BVP from rotation of axes step for example 1.

with the values of a, b, c, d, and K given above. Note that the domain remains unchanged after this step.

(iii) Scaling. The related parameters defined in appendix B.3.3 are computed as

$$\xi = 0.70254$$

$$\eta = 0.71165$$

The resulting PDE from this transformation becomes

$$\tilde{\tilde{L}}w = f(x, y)$$

, where

$$\tilde{\tilde{L}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1 \qquad w = w(x, y)$$

, and

$$f(x, y) = -\frac{1}{K} exp\{\xi x + \eta y\}$$

with the values of K, ξ, and η given above. The scaling procedure is also applied to the domain. The resulting BVP from this step is sketched in figure 4.8.

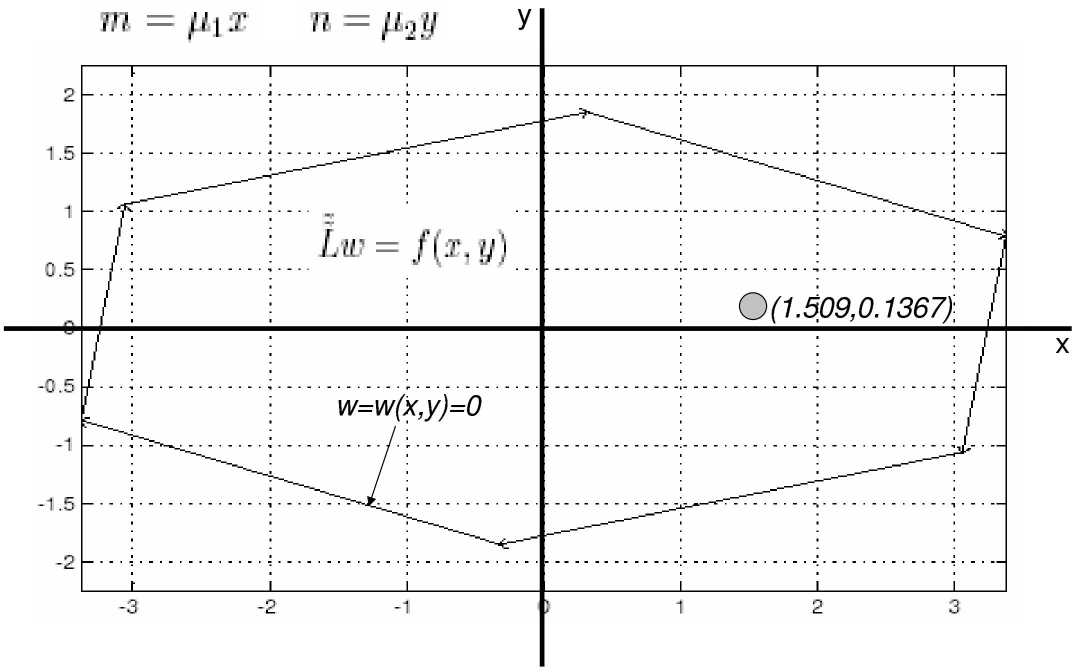

Figure 4.8: The resulting BVP from scaling step for example 1.

After completing the three steps, the PDE is now in its standard form. The solution of the BVP is then numerically computed and its plot over the transformed domain is given in figure 4.9. Top view of this solution is also available in figure 4.10. Selecting an arbitrary point $\left(\tilde{\tilde{x}}_0, \tilde{\tilde{y}}_0\right)$ as

$$\left(\tilde{\tilde{x}}_0, \tilde{\tilde{y}}_0\right) = (1.509, 0.1367)$$

, then the solution $\tilde{\tilde{z}}$ at his point is computed as

$$\tilde{\tilde{z}} = 5.636$$

$\tilde{\tilde{z}}$ is the solution of the transformed equation. In order to obtain the expected exit time from the initial domain, it should be transformed through the three steps in the reverse order. This back transformation procedure is described in figure 4.11. After the inverse scaling

$$(\tilde{x_0}, \tilde{y_0}) = (3.2387, 0.5875)$$

, and

$$\tilde{z} = 5.636$$

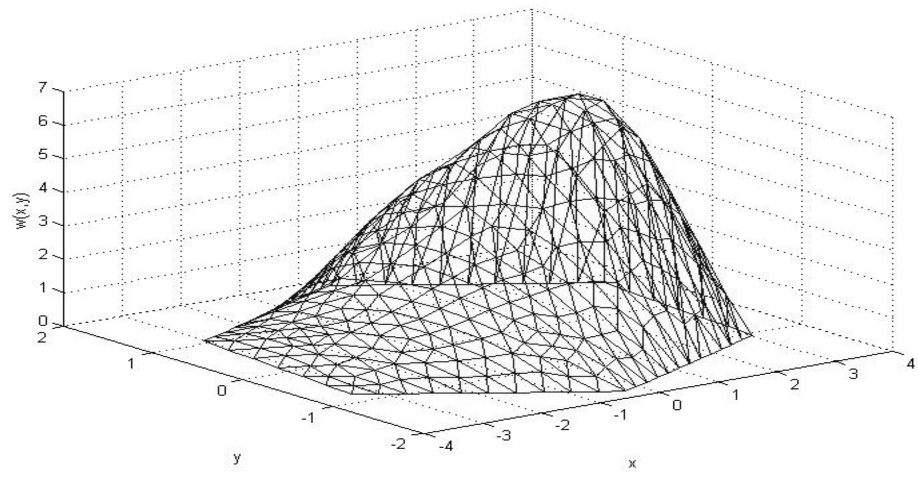

Figure 4.9: Solution plot for the transformed BVP in example 1.

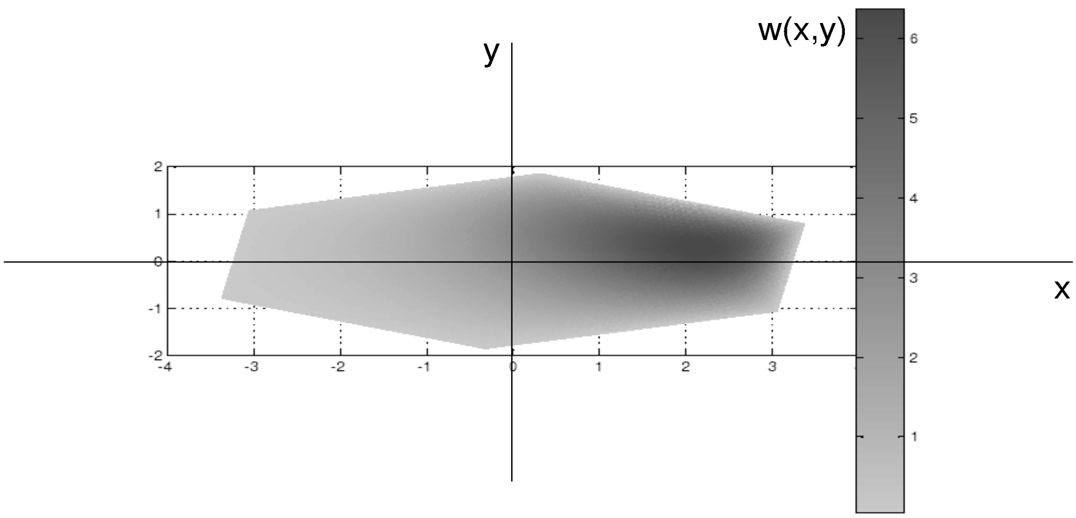

Figure 4.10: Top view of the solution for the transformed BVP in example 1.

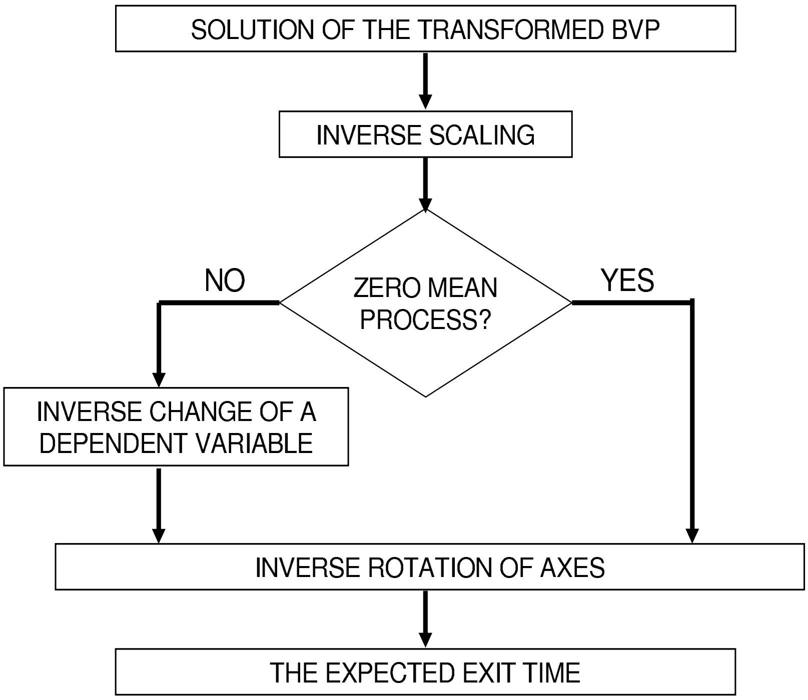

Figure 4.11: Back transformation of the solution into the initial case.

After the inverse change of a dependent variable

$$(\tilde{x_0}, \tilde{y_0}) = (3.2387, 0.5875)$$

, and

$$z = 1.7714$$

After the inverse rotation

$$(x_0, y_0) = (3.1821, -0.8417)$$

, and

$$z = 1.7714$$

The result obtained eventually is z that is the expected exit time from the hexagon described initially. To validate this result a simulation was performed with the setting

$$N = 100,000$$

, and

$$\Delta t = 0.0001$$

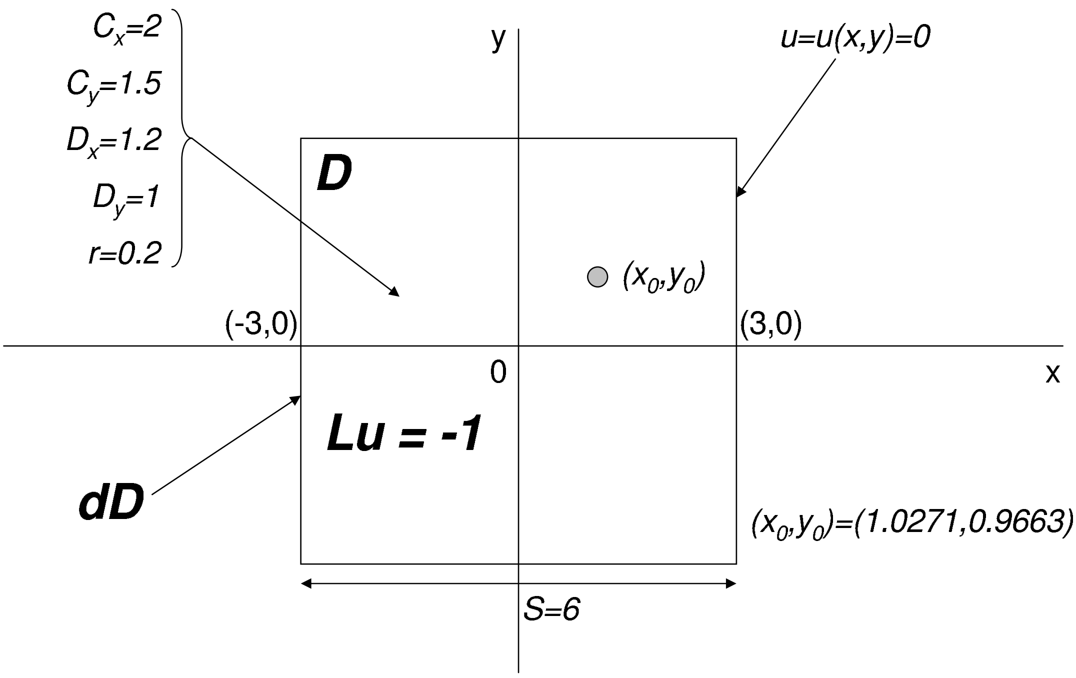

Figure 4.12: Description of the example 2.

, where N is the number of times the simulation is performed and Δt is the time step size described in section 3.1. In each simulation, motion was started at (x_0, y_0) and time was measured until the process reaches the boundary of the hexagon. Then, average of all samples was computed. The process was simulated using the techniques in chapter 3. Specifically, for the given setting (C_x, C_y, D_x, D_y, r) at the beginning of the example and the Δt value, corresponding transition probabilities p_1, p_2, p_3, p_4, and step size Δx were computed. Based on these probabilities and the step size, the process was simulated. Δt value used in this simulation is selected close to 0. This small selection of Δt results in a close realization to 2D correlated diffusion process.

The simulation resulted in

$$E[\tau]_{sim} = 1.7782$$

as the expected exit time from the domain. The difference between the numerical solution and the simulation result is computed as 0.3824%.

Example 2. In this example, the transformation steps are skipped. Only the mobility setting and the results are given. Consider a square centered at the

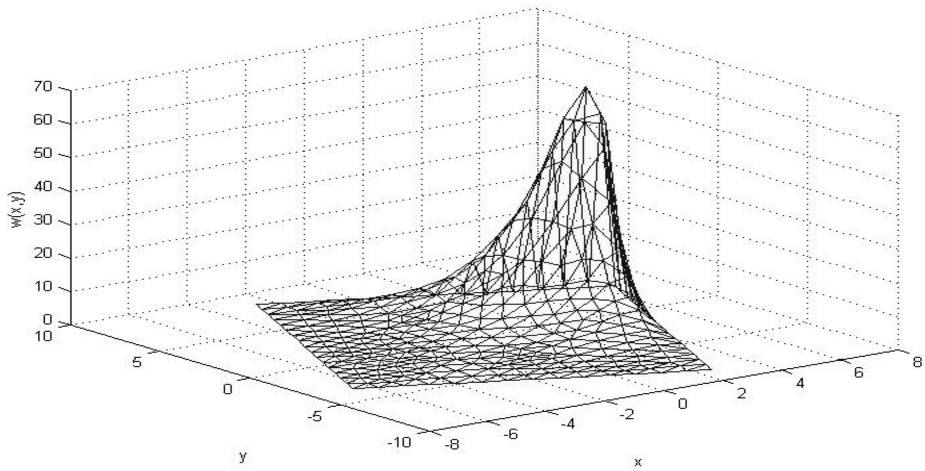

Figure 4.13: Solution plot for the transformed BVP in example 2.

origin as given in figure 4.12, and a motion inside with the setting

$$C_x = 2$$

$$C_y = 1.5$$

$$D_x = 1.2$$

$$D_y = 1$$

$$r = 0.2$$

The numerical solution of the BVP was found as

$$E[\tau]_{num} = 0.8212$$

for the motion started at the point

$$(x_0, y_0) = (1.0271, 0.9663)$$

The simulation resulted in

$$E[\tau]_{sim} = 0.82124$$

The difference between the results is 0.00487%. The numerical solution plot for the standard form of the equation over the transformed domain is available in figure 4.13. Top view of the solution is also given in figure 4.14.

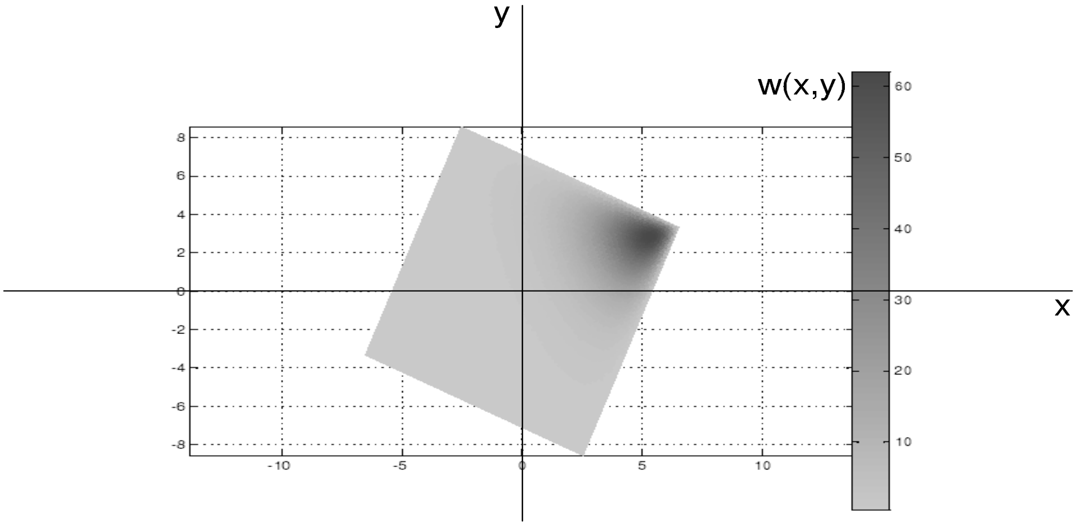

Figure 4.14: Top view of the solution for the transformed BVP in example 2.

As obvious from the results, the numerical solutions match the simulation results with small errors. Therefore, the expected exit time of a motion modeled by 2D correlated diffusion process can be computed by solving the BVP numerically for any domain shape and starting position inside. This technique provides important flexibility in selection of cell shape from which the exit time is desired.

4.3 Analytical Solution of the BVP

Before the transformations, the initial PDE was

$$Lu = -1$$

, where

$$L = C_x \frac{\partial}{\partial x} + C_y \frac{\partial}{\partial y} + D_x D_y r \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} D_x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2} D_y^2 \frac{\partial^2}{\partial y^2}$$

, and

$$u = u(x, y)$$

This PDE holds inside a general closed domain D with the boundary condition

$$u(x, y) = 0, \qquad \{x, y\} \in dD$$

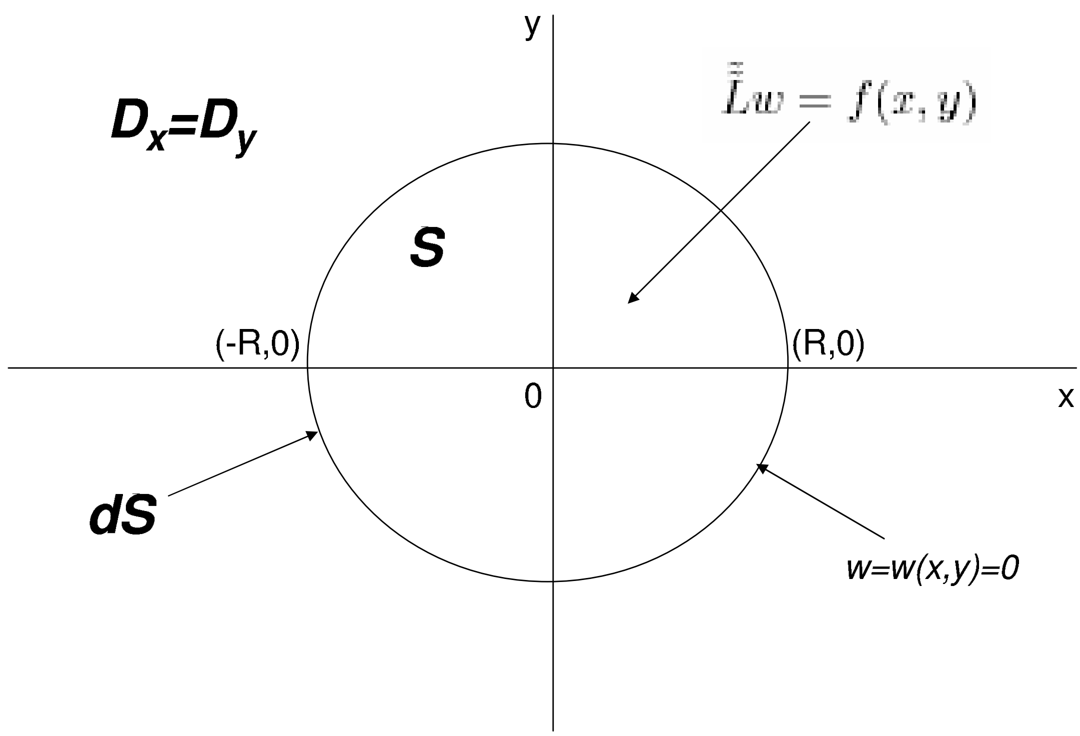

Figure 4.15: BVP with circular domain.

After the transformations performed in three steps, the resulting BVP became

$$\tilde{\tilde{L}}w = f(x, y) \tag{4.4}$$

, where

$$\tilde{\tilde{L}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1 \qquad w = w(x, y)$$

, and

$$f(x, y) = -\frac{1}{K} exp\{\xi x + \eta y\}$$

with the boundary condition

$$w(x, y) = 0, \qquad \{x, y\} \in d\tilde{\tilde{D}}$$

, where $d\tilde{\tilde{D}}$ is the boundary of the resulting domain $\tilde{\tilde{D}}$ from the transformation processes. The parameters K, ξ, and η were defined in appendix B.3.

The resulting PDE from the transformations is a Helmholtz equation whose basics are given in [40]. Brief description of this equation and the solution of BVP involving Helmholtz equation for a circular domain are given in appendix B.4.

For the circular domain the solution is given as

$$w = \sum_{n=1}^{\infty} \sum_{j=1}^{p_n} \frac{A_n^{(j)}}{\lambda_n - \lambda} w_n^{(j)} \tag{4.5}$$

, where p_n, $A_n^{(j)}$, λ_n, λ, and $w_n^{(j)}$ are defined in appendix B.4. This solution of the BVP corresponds to the expected exit time of the 2D correlated diffusion process from the circle centered at the origin. The Helmholtz equation (B.51) described in appendix B.4 has a constant λ and a function Φ which should be specified. For the selections of $\lambda = -1$ and $\Phi = -f$ the equations (B.51) and (4.4) become identical.

The solution (4.5) is given in polar coordinates. It should be transformed into cartesian coordinate system. Then, it is the solution of the transformed BVP involving (4.4). This solution must be transformed back into the initial case following the chart given in figure 4.11 to obtain the expected exit time from the circle given in figure 4.15.

The selection of a circle as the domain is for a purpose. A circle centered at the origin is not affected by the rotation of axes. That is to say, if the initial domain is a circle centered at the origin before the rotation of axes, then it is the same circle after the transformation. Hence, the effect of rotation of axes on the domain disappears.

The scaling step has an influence on the domain, too. It changes the shape of the domain. To avoid this, a selection of $D_x = D_y$ is sufficient. Such a selection makes the coefficients of $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ equal in (4.3). Hence, the circular domain before and after the scaling process are the same.

In this chapter, the exit time of a moving body modeled by 2D correlated diffusion process from an arbitrary-shaped cell has been computed. For the 2D correlated random walk, a difference equation with a boundary condition was written such that its solution is the expected number of periods until the process reaches the boundary of the domain. Then, each term in the difference equation was expanded using Taylor's theorem and taking the limit as time index goes to zero, the equation became a PDE. Taking into account the boundary condition which was set together with the difference equation, the problem turned into a BVP whose solution is the expected exit time. The resulting BVP was brought into its standard form and solved numerically in two examples. The results obtained in these examples were verified by simulations. Finally, analytical solution of the BVP was provided for a specific case.

The importance of representing the exit time problem in this way is that it allows to compute the solution of the BVP numerically for any cell shape. This flexibility is very useful and has not been provided in any works yet. Another flexibility is that the solution can be computed for a motion starting at any position inside a cell. It does not have to be center of mass. At a given time, no matter where the MS is, the expected duration of time until it reaches the cell boundary can be computed by this method.

For the analytical solution of the problem, some assumptions were made. The cell shape was this time restricted to a circle. And, the variance paremeters of the motion characterized by 2D correlated diffusion process were taken equal. These assumptions were due to the hardness of bringing the BVP into its standard form. Circular domains remain unchanged after the rotation of axes process. And, by the equal variances, the scaling step was avoided.

CHAPTER 5

CONCLUSIONS

This thesis proposed a novel mobility model based on 2D correlated diffusion process. In addition to the powerful nature of diffusion processes in mobility modeling, this process provides an important facility to alter the amount of correlation between motion components over x and y axes. As well as the correlation coefficient, there are four more parameters controlling behavior of the mobility pattern that is generated using this model. By adjusting these five parameters, many mobility patterns can be generated. Diffusion processes allow time varying speeds if used in mobility modeling. Therefore, the proposed model has also this property.

The proposed process was fully described by computing its joint density function. The computation procedure was provided in detail. First, a 2D correlated random walk was specified. Then, its statistics such as, mean, variance, and correlation coefficient were computed. Taking the limit as time index goes to zero, these statistics converged to a constant value multiplied by time and the resulting process became a 2D correlated diffusion process.

To characterize the joint density function of the process, the terms in the probability distribution of the random walk were expanded using Taylor's theorem. The expanded terms were put in the expression for the probability distribution. Then, both sides of the equation were divided by time index. Taking the limit as time index goes to zero, the expression resulted in a PDE whose solution is the joint density of the 2D correlated diffusion process. It was shown that the density has a joint normal form and it was specified explicitly.

After the complete presentation of the model, the works in this thesis involved computation of the expected residence time of an MS inside a general registration area or cell. The shapes of these entities were not restricted to basic shapes such as, square, circle, or hexagon. The method presented to compute the expected residence time is valid for any cell shape and for any motion starting at any given point inside the cell. These freedoms provide important flexibility to apply the model for a range of cell shapes.

The expected residence time computation was performed through several steps. First, a difference equation with boundary condition was set such that its solution is the expected number of periods for the 2D correlated random walk to reach a given closed boundary. Then, the terms in the difference equation were expanded using Taylor's theorem. The expanded terms were put in the difference equation and both sides were divided by time index. Taking the limit as the time index goes to zero, a PDE with boundary condition was obtained such that its solution is the expected exit time of the motion modeled by 2D correlated diffusion process from the given domain.

The resulting BVP was solved numerically by two examples. The results were supported by simulations for these examples. The errors between the simulation results and the numerical solutions were found to be 0.3824% and 0.00487% for the first and the second example, respectively. The analytical solution of the BVP was provided for a circular domain and with a constraint that the variance parameters of the motion modeled by the 2D correlated diffusion process are equal.

APPENDIX A

SUPPLEMENTARY DERIVATIONS FOR THE JOINT DENSITY

This appendix includes supplementary derivations for chapter 3 discussing the characterization of the 2D correlated diffusion process. The derivation procedure of the joint density function of the 2D correlated diffusion process was summarized in figure 3.1. It starts with construction of the 2D correlated random walk.

Then, expected value, variance, covariance, and correlation coefficient of the random walk are computed. Taking the limit as time index goes to zero, the random walk becomes 2D correlated diffusion process. In the fourth step, Taylor's theorem is applied to (3.2) and a PDE is obtained. Finally, this PDE is solved.

In the first section, Taylor's theorem for functions of n variables is discussed. The theorem is given in the vector form. This theorem is used to approximate the components of (3.2). Since the components are functions of five variables, the theorem is also provided for this specific case.

In the second section, Taylor's theorem is applied to the components of (3.2). The expressions resulting from these approximations are put into the equation. Then, both sides of the equation are divided by Δt. Taking the limit as $\Delta t \to 0$, a PDE whose solution is the joint density function of the 2D correlated diffusion process is obtained.

In the last section, the statement that (3.26) is the desired solution is verified by directly putting (3.26) in the obtained PDE. This solution is a joint normal function and it satisfies all requirements resulting from the limiting behavior of the random walk described in section 3.3.

A.1 Taylor's Theorem

In [38], Taylor's theorem for functions of n variables is given in the vector form. This section is a summary of Taylor's theorem quoted from [38].

Fix an integer $n > 0$ and let

$$J = (j_1, \ldots, j_n)$$

be a sequence of nonnegative integers. The norm of J is defined as

$$|J| = j_1 + \ldots + j_n$$

and $J!$ as

$$J! = j_1! \ldots j_n!$$

If $|J| = k$, the differential operator D_J is defined as

$$D_J = \frac{\partial^k}{\partial {x_1}^{j_1} \cdots \partial {x_n}^{j_n}}$$

Thus, if f is a real valued function of n variables

$$D_J f = \frac{\partial^k f}{\partial {x_1}^{j_1} \cdots \partial {x_n}^{j_n}}$$

If $\bar{v} = (x_1, \ldots x_n)$, $\bar{a} = (a_1, \ldots a_n) \in \mathbf{R}$, $(\bar{v} - \bar{a})^J$ is defined as

$$(\bar{v} - \bar{a})^J = (x_1 - a_1)^{j_1} \cdots (x_n - a_n)^{j_n}$$

In particular, if $n = 2$ and $J = (1, 3)$, then

$$|J| = 4$$

$$J! = 1!.3! = 6$$

$$D_J f = \frac{\partial^4 f}{\partial x_1 \partial {x_2}^3}$$

and

$$(\bar{v} - \bar{a})^J = (x_1 - a_1)(x_2 - a_2)^3$$

Let U be an open subset of $\mathbf{R^n}$, let $a \in U$, and let $f : U \to \mathbf{R}$ be a function whose partial derivatives of order $\leq (m+1)$ exist on U. The Taylor polynomial of f of degree m about a is defined as

$$P_m(\bar{v}) = \sum_{|J| \leq m} \frac{1}{J!} (D_J f)(\bar{a})(\bar{v} - \bar{a})^J \tag{A.1}$$

, where the expression on the right is the sum over all sequences $J = (j_1, \ldots, j_n)$ of nonnegative integers with $|J| = j_1 + \cdots + j_n \leq m$.

What follows is a couple of examples. For, $m = 2$ and $n = 1$

$$\bar{v} = x$$

$$\bar{a} = a$$

$$J = j$$

$$|J| = j$$

$$P_2(x) = f(a) + \frac{df}{dx}(a)(x - a) + \frac{1}{2}\frac{d^2 f}{dx^2}(a)(x - a)^2$$

For $m = 2$ and $n = 2$

$$\bar{v} = (x_1 + x_2)$$

$$\bar{a} = (a_1 + a_2)$$

$$J = (j_1, j_2)$$

$$|J| = j_1 + j_2$$

$$P_2(\bar{v}) = f(\bar{a}) + \frac{\partial}{\partial x_1} f(\bar{a})(x_1 - a_1) + \frac{\partial}{\partial x_2} f(\bar{a})(x_2 - a_2)$$
$$+ \frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})(x_1 - a_1)(x_2 - a_2) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})(x_1 - a_1)^2$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})(x_2 - a_2)^2$$

To characterize the joint density function, a Taylor series approximation for functions of five variables is needed, since the density will be of five variables (ie. t, x, y, x_0, and y_0). Approximation of degree 2 is sufficient. This case corresponds to $m = 2$ and $n = 5$.

$$\bar{v} = (x_1, x_2, x_3, x_4, x_5) \quad \bar{a} = (a_1, a_2, a_3, a_4, a_5)$$

$$J = (j_1, j_2, j_3, j_4, j_5) \quad |J| = j_1 + j_2 + j_3 + j_4 + j_5$$

$$P_2(\bar{v}) = f(\bar{a}) + \frac{\partial}{\partial x_1} f(\bar{a})(x_1 - a_1) + \frac{\partial}{\partial x_2} f(\bar{a})(x_2 - a_2)$$
$$+ \frac{\partial}{\partial x_3} f(\bar{a})(x_3 - a_3) + \frac{\partial}{\partial x_4} f(\bar{a})(x_4 - a_4) + \frac{\partial}{\partial x_5} f(\bar{a})(x_5 - a_5)$$
$$+ \frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})(x_1 - a_1)(x_2 - a_2) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})(x_1 - a_1)(x_3 - a_3)$$
$$+ \frac{\partial^2}{\partial x_1 \partial x_4} f(\bar{a})(x_1 - a_1)(x_4 - a_4) + \frac{\partial^2}{\partial x_1 \partial x_5} f(\bar{a})(x_1 - a_1)(x_5 - a_5)$$
$$+ \frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})(x_2 - a_2)(x_3 - a_3) + \frac{\partial^2}{\partial x_2 \partial x_4} f(\bar{a})(x_2 - a_2)(x_4 - a_4)$$
$$+ \frac{\partial^2}{\partial x_2 \partial x_5} f(\bar{a})(x_2 - a_2)(x_5 - a_5) + \frac{\partial^2}{\partial x_3 \partial x_4} f(\bar{a})(x_3 - a_3)(x_4 - a_4)$$
$$+ \frac{\partial^2}{\partial x_3 \partial x_5} f(\bar{a})(x_3 - a_3)(x_5 - a_5) + \frac{\partial^2}{\partial x_4 \partial x_5} f(\bar{a})(x_4 - a_4)(x_5 - a_5)$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial x_1^2} f(\bar{a})(x_1 - a_1)^2 + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} f(\bar{a})(x_2 - a_2)^2 + \frac{1}{2} \frac{\partial^2}{\partial x_3^2} f(\bar{a})(x_3 - a_3)^2$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial x_4^2} f(\bar{a})(x_4 - a_4)^2 + \frac{1}{2} \frac{\partial^2}{\partial x_5^2} f(\bar{a})(x_5 - a_5)^2$$

A.2 Approximation of the Components

Remember the function (3.2) defined in section 3.1. Taylor's approximation of order 2 for functions of 5 variables is applied to the components of this function as follows.

For $f(t - \Delta t, x - \Delta x, y - \Delta y, x_0, y_0)$

$$\bar{x} = (t - \Delta t, x - \Delta x, y - \Delta y, x_0, y_0)$$

$$\bar{v} = (x_1, x_2, x_3, x_4, x_5)$$

$$\bar{a} = (t, x, y, x_0, y_0)$$

Note that $(x_4 - a_4) = (x_5 - a_5) = 0$. The approximation simplifies to

$$f(t-\Delta t, x-\Delta x, y-\Delta y, x_0, y_0) \cong f(\bar{a}) + \frac{\partial}{\partial x_1} f(\bar{a})(-\Delta t) + \frac{\partial}{\partial x_2} f(\bar{a})(-\Delta x)$$
$$+\frac{\partial}{\partial x_3} f(\bar{a})(-\Delta y) + \frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})(-\Delta t)(-\Delta x) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})(-\Delta t)(-\Delta y)$$
$$+\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})(-\Delta x)(-\Delta y) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})(-\Delta t)^2 + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})(-\Delta x)^2$$
$$\frac{1}{2}\frac{\partial^2}{\partial x_3^2} f(\bar{a})(-\Delta y)^2 \quad \text{(A.2)}$$

Similarly, for $f(t-\Delta t, x-\Delta x, y+\Delta y, x_0, y_0)$

$$\bar{x} = (t-\Delta t, x-\Delta x, y+\Delta y, x_0, y_0)$$

$$\bar{v} = (x_1, x_2, x_3, x_4, x_5)$$

$$\bar{a} = (t, x, y, x_0, y_0)$$

$$f(t-\Delta t, x-\Delta x, y+\Delta y, x_0, y_0) \cong f(\bar{a}) + \frac{\partial}{\partial x_1} f(\bar{a})(-\Delta t) + \frac{\partial}{\partial x_2} f(\bar{a})(-\Delta x)$$
$$+\frac{\partial}{\partial x_3} f(\bar{a})(\Delta y) + \frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})(-\Delta t)(-\Delta x) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})(-\Delta t)(\Delta y)$$
$$+\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})(-\Delta x)(\Delta y) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})(-\Delta t)^2 + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})(-\Delta x)^2$$
$$\frac{1}{2}\frac{\partial^2}{\partial x_3^2} f(\bar{a})(\Delta y)^2 \quad \text{(A.3)}$$

For $f(t-\Delta t, x+\Delta x, y-\Delta y, x_0, y_0)$

$$\bar{x} = (t-\Delta t, x+\Delta x, y-\Delta y, x_0, y_0)$$

$$\bar{v} = (x_1, x_2, x_3, x_4, x_5)$$

$$\bar{a} = (t, x, y, x_0, y_0)$$

$$f(t-\Delta t, x+\Delta x, y-\Delta y, x_0, y_0) \cong f(\bar{a}) + \frac{\partial}{\partial x_1} f(\bar{a})(-\Delta t) + \frac{\partial}{\partial x_2} f(\bar{a})(\Delta x)$$
$$+\frac{\partial}{\partial x_3} f(\bar{a})(-\Delta y) + \frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})(-\Delta t)(\Delta x) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})(-\Delta t)(-\Delta y)$$
$$+\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})(\Delta x)(-\Delta y) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})(-\Delta t)^2 + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})(\Delta x)^2$$
$$\frac{1}{2}\frac{\partial^2}{\partial x_3^2} f(\bar{a})(-\Delta y)^2 \quad \text{(A.4)}$$

Lastly, for $f(t-\Delta t, x+\Delta x, y+\Delta y, x_0, y_0)$

$$\bar{x} = (t-\Delta t, x+\Delta x, y+\Delta y, x_0, y_0)$$

$$\bar{v} = (x_1, x_2, x_3, x_4, x_5)$$

$$\bar{a} = (t, x, y, x_0, y_0)$$

$$f(t-\Delta t, x+\Delta x, y+\Delta y, x_0, y_0) \cong f(\bar{a}) + \frac{\partial}{\partial x_1} f(\bar{a})(-\Delta t) + \frac{\partial}{\partial x_2} f(\bar{a})(\Delta x)$$
$$+\frac{\partial}{\partial x_3} f(\bar{a})(\Delta y) + \frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})(-\Delta t)(\Delta x) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})(-\Delta t)(\Delta y)$$
$$+\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})(\Delta x)(\Delta y) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})(-\Delta t)^2 + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})(\Delta x)^2$$
$$\frac{1}{2}\frac{\partial^2}{\partial x_3^2} f(\bar{a})(\Delta y)^2 \tag{A.5}$$

Putting the expressions (A.2), (A.3), (A.4), and (A.5) in (3.2) and rearranging the terms

$$f(t, x, y, x_0, y_0) = f(\bar{a}) = f(\bar{a}) \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1} + \frac{\partial}{\partial x_1} f(\bar{a})(-\Delta t) \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1}$$
$$+\frac{\partial}{\partial x_2} f(\bar{a})\Delta x(-p_1 - p_2 + p_3 + p_4) + \frac{\partial}{\partial x_3} f(\bar{a})\Delta y(-p_1 + p_2 - p_3 + p_4)$$
$$+\frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})\Delta t \Delta x(p_1 + p_2 - p_3 - p_4) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})\Delta t \Delta y(p_1 - p_2 + p_3 - p_4)$$
$$+\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})\Delta x \Delta y(p_1 - p_2 - p_3 + p_4) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})\Delta t^2 \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1}$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})\Delta x^2 \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1} + \frac{1}{2}\frac{\partial^2}{\partial x_3^2} f(\bar{a})\Delta y^2 \underbrace{(p_1 + p_2 + p_3 + p_4)}_{1} \tag{A.6}$$

Dividing both sides of (A.6) by Δt after necessary simplifications

$$0 = -\frac{\partial}{\partial x_1} f(\bar{a}) - \frac{\partial}{\partial x_2} f(\bar{a})\frac{\Delta x}{\Delta t}(p_1 + p_2 - p_3 - p_4) - \frac{\partial}{\partial x_3} f(\bar{a})\frac{\Delta y}{\Delta t}(p_1 + p_3 - p_2 - p_4)$$
$$+\frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a})\Delta x(p_1 + p_2 - p_3 - p_4) + \frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a})\Delta y(p_1 + p_3 - p_2 - p_4)$$
$$+\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a})\frac{\Delta x \Delta y}{\Delta t}(p_1 - p_2 - p_3 + p_4) + \frac{1}{2}\frac{\partial^2}{\partial x_1^2} f(\bar{a})\Delta t + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} f(\bar{a})\frac{\Delta x^2}{\Delta t}$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x_3^2} f(\bar{a})\frac{\Delta y^2}{\Delta t} \tag{A.7}$$

Taking the limit as $\Delta t \to 0$

$$0 = -\frac{\partial}{\partial x_1} f(\bar{a}) - \frac{\partial}{\partial x_2} f(\bar{a}) \underbrace{\frac{\Delta x}{\Delta t}(p_1 + p_2 - p_3 - p_4)}_{\to C_x} - \frac{\partial}{\partial x_3} f(\bar{a}) \underbrace{\frac{\Delta y}{\Delta t}(p_1 + p_3 - p_2 - p_4)}_{\to C_y}$$

$$+ \underbrace{\frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a}) \Delta x (p_1 + p_2 - p_3 - p_4)}_{\to 0} + \underbrace{\frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a}) \Delta y (p_1 + p_3 - p_2 - p_4)}_{\to 0}$$

$$+ \frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a}) \underbrace{\frac{\Delta x \Delta y}{\Delta t}}_{= D_x D_y} \underbrace{(p_1 - p_2 - p_3 + p_4)}_{= r} + \underbrace{\frac{1}{2} \frac{\partial^2}{\partial x_1^2} f(\bar{a}) \Delta t}_{\to 0} + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} f(\bar{a}) \underbrace{\frac{\Delta x^2}{\Delta t}}_{= D_x^2}$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial x_3^2} f(\bar{a}) \underbrace{\frac{\Delta y^2}{\Delta t}}_{= D_y^2} \tag{A.8}$$

Note that the terms $\frac{\partial^2}{\partial x_1 \partial x_2} f(\bar{a}) \Delta x (p_1 + p_2 - p_3 - p_4)$ and $\frac{\partial^2}{\partial x_1 \partial x_3} f(\bar{a}) \Delta y (p_1 + p_3 - p_2 - p_4)$ go to 0 since $\Delta x = D_x \sqrt{\Delta t}$ and $\Delta y = D_y \sqrt{\Delta t}$. (A.8) can be rewritten as

$$\frac{\partial}{\partial x_1} f(\bar{a}) = -C_x \frac{\partial}{\partial x_2} f(\bar{a}) - C_y \frac{\partial}{\partial x_3} f(\bar{a}) + D_x D_y r \frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a}) + \frac{1}{2} D_x^2 \frac{\partial^2}{\partial x_2^2} f(\bar{a})$$

$$+ \frac{1}{2} D_y^2 \frac{\partial^2}{\partial x_3^2} f(\bar{a}) \tag{A.9}$$

$\bar{v} = (x_1, x_2, x_3, x_4, x_5)$ is the transition variable. Since there is no risk of confusion, the following set of notational conversions are possible.

$$f(\bar{a}) = f(t, x, y, x_0, y_0) \triangleq f$$

$$\frac{\partial}{\partial x_1} f(\bar{a}) = \frac{\partial}{\partial x_1} f(\bar{v}) \bigg|_{\bar{v} = \bar{a}} = \frac{\partial}{\partial t} f(\bar{a}) \triangleq f_t$$

$$\frac{\partial}{\partial x_2} f(\bar{a}) = \frac{\partial}{\partial x_2} f(\bar{v}) \bigg|_{\bar{v} = \bar{a}} = \frac{\partial}{\partial x} f(\bar{a}) \triangleq f_x$$

$$\frac{\partial}{\partial x_3} f(\bar{a}) = \frac{\partial}{\partial x_3} f(\bar{v}) \bigg|_{\bar{v} = \bar{a}} = \frac{\partial}{\partial y} f(\bar{a}) \triangleq f_y$$

$$\frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{a}) = \frac{\partial^2}{\partial x_2 \partial x_3} f(\bar{v}) \bigg|_{\bar{v} = \bar{a}} = \frac{\partial^2}{\partial x \partial y} f(\bar{a}) \triangleq f_{xy}$$

$$\frac{\partial^2}{\partial x_2^2} f(\bar{a}) = \frac{\partial^2}{\partial x_2^2} f(\bar{v}) \bigg|_{\bar{v} = \bar{a}} = \frac{\partial^2}{\partial x^2} f(\bar{a}) \triangleq f_{xx}$$

$$\frac{\partial^2}{\partial x_3^2} f(\bar{a}) = \frac{\partial^2}{\partial x_3^2} f(\bar{v})\Big|_{\bar{v}=\bar{a}} = \frac{\partial^2}{\partial y^2} f(\bar{a}) \triangleq f_{yy}$$

According to these conversions, (A.9) can be rewritten as

$$f_t = -C_x f_x - C_y f_y + D_x D_y r f_{xy} + \frac{1}{2} D_x^2 f_{xx} + \frac{1}{2} D_y^2 f_{yy} \tag{A.10}$$

(A.10) is the final form of the PDE whose solution is

$$f = f(t, x, y, x_0, y_0)$$

, which is the joint density function of the correlated diffusion process.

A.3 Proof for the Solution

It suffices to plug (3.26) in (3.27). Derivatives of f are computed as follows.

$$f_t = \frac{-f}{t} - \frac{f}{2(1-r^2)} \Bigg[-\frac{2C_x(x - x_0 - C_x t)}{D_x^2 t} - \frac{(x - x_0 - C_x t)^2}{D_x^2 t^2}$$

$$+ \frac{2rC_x(y - y_0 - C_y t)}{D_x D_y t} + \frac{2rC_y(x - x_0 - C_x t)}{D_x D_y t} + \frac{2r(x - x_0 - C_x t)(y - y_0 - C_y t)}{D_x D_y t^2}$$

$$- \frac{2C_y(y - y_0 - C_y t)}{D_y^2 t} - \frac{(y - y_0 - C_y t)^2}{D_y^2 t^2} \Bigg] \tag{A.11}$$

$$f_x = \frac{-f}{2(1-r^2)} \left[\frac{2(x - x_0 - C_x t)}{D_x^2 t} - \frac{2r(y - y_0 - C_y t)}{D_x D_y t} \right] \tag{A.12}$$

$$f_y = \frac{-f}{2(1-r^2)} \left[\frac{2(y - y_0 - C_y t)}{D_y^2 t} - \frac{2r(x - x_0 - C_x t)}{D_x D_y t} \right] \tag{A.13}$$

$$f_{xy} = \frac{fr}{(1-r^2) D_x D_y t} + \frac{f}{4(1-r^2)^2} \left[\frac{2(x - x_0 - C_x t)}{D_x^2 t} - \frac{2r(y - y_0 - C_y t)}{D_x D_y t} \right]$$

$$\left[\frac{2(y - y_0 - C_y t)}{D_y^2 t} - \frac{2r(x - x_0 - C_x t)}{D_x D_y t} \right]$$

$$= \frac{fr}{(1-r^2) D_x D_y t} + \frac{f}{4(1-r^2)^2} \Bigg[-\frac{4r(x - x_0 - C_x t)^2}{D_x^3 D_y t^2}$$

$$+ \frac{4(x - x_0 - C_x t)(y - y_0 - C_y t)}{D_x^2 D_y^2 t^2} + \frac{4r^2(x - x_0 - C_x t)(y - y_0 - C_y t)}{D_x^2 D_y^2 t^2}$$

$$-\frac{4r(y-y_0-C_yt)^2}{D_xD_y^3t^2}\Bigg] \tag{A.14}$$

$$f_{xx} = \frac{-f}{(1-r^2)D_x^2t} + \frac{f}{4(1-r^2)^2}\left[\frac{2(x-x_0-C_xt)}{D_x^2t} - \frac{2r(y-y_0-C_yt)}{D_xD_yt}\right]^2$$

$$= \frac{-f}{(1-r^2)D_x^2t} + \frac{f}{4(1-r^2)^2}\Bigg[\frac{4(x-x_0-C_xt)^2}{D_x^4t^2} + \frac{4r^2(y-y_0-C_yt)^2}{D_x^2D_y^2t^2}$$

$$-\frac{8r(x-x_0-C_xt)(y-y_0-C_yt)}{D_x^3D_yt^2}\Bigg] \tag{A.15}$$

$$f_{yy} = \frac{-f}{(1-r^2)D_y^2t} + \frac{f}{4(1-r^2)^2}\left[\frac{2(y-y_0-C_yt)}{D_y^2t} - \frac{2r(x-x_0-C_xt)}{D_xD_yt}\right]^2$$

$$= \frac{-f}{(1-r^2)D_y^2t} + \frac{f}{4(1-r^2)^2}\Bigg[\frac{4(y-y_0-C_yt)^2}{D_y^4t^2} + \frac{4r^2(x-x_0-C_xt)^2}{D_x^2D_y^2t^2}$$

$$-\frac{8r(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_y^3t^2}\Bigg] \tag{A.16}$$

The terms $\frac{f}{(1-r^2)}$ immediately cancel in each of f_t, f_x, f_y, f_{xy}, f_{xx}, and f_{yy}.

$$\underbrace{f_t + C_xf_x + C_yf_y}_{(i)} = \underbrace{D_xD_yrf_{xy} + \frac{1}{2}D_x^2f_{xx} + \frac{1}{2}D_y^2f_{yy}}_{(ii)}$$

After the apparent simplifications

$$(i) = \frac{-(1-r^2)}{t} + \frac{1}{2}\frac{(x-x_0-C_xt)^2}{D_x^2t^2} - \frac{r(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2}$$

$$+\frac{1}{2}\frac{(y-y_0-C_yt)^2}{D_y^2t^2}$$

$$(ii) = \frac{r^2}{t} - \frac{r^2}{1-r^2}\frac{(x-x_0-C_xt)^2}{D_x^2t^2} + \frac{r}{1-r^2}\frac{(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2}$$

$$+\frac{r^3}{1-r^2}\frac{(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2} - \frac{r^2}{1-r^2}\frac{(y-y_0-C_yt)^2}{D_y^2t^2}$$

$$-\frac{1}{2t} + \frac{1}{2(1-r^2)}\frac{(x-x_0-C_xt)^2}{D_x^2t^2} + \frac{r^2}{2(1-r^2)}\frac{(y-y_0-C_yt)^2}{D_y^2t^2}$$

$$-\frac{r}{1-r^2}\frac{(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2} - \frac{1}{2t} + \frac{1}{2(1-r^2)}\frac{(y-y_0-C_yt)^2}{D_y^2t^2}$$

$$
+\frac{r^2}{2(1-r^2)}\frac{(x-x_0-C_xt)^2}{D_x^2t^2}-\frac{r}{1-r^2}\frac{(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2}
$$

$$
=\frac{r^2-1}{t}+\frac{(x-x_0-C_xt)^2}{D_x^2t^2}\underbrace{\left[\frac{-r^2}{1-r^2}+\frac{1}{2(1-r^2)}+\frac{r^2}{2(1-r^2)}\right]}_{\frac{1}{2}}
$$

$$
+\frac{(y-y_0-C_yt)^2}{D_y^2t^2}\underbrace{\left[\frac{-r^2}{1-r^2}+\frac{r^2}{2(1-r^2)}+\frac{1}{2(1-r^2)}\right]}_{\frac{1}{2}}
$$

$$
+\frac{(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2}\underbrace{\left[\frac{r}{1-r^2}+\frac{r^3}{1-r^2}-\frac{r}{1-r^2}-\frac{r}{1-r^2}\right]}_{-r}
$$

$$
=\frac{r^2-1}{t}+\frac{(x-x_0-C_xt)^2}{D_x^2t^2}\left(\frac{1}{2}\right)+\frac{(y-y_0-C_yt)^2}{D_y^2t^2}\left(\frac{1}{2}\right)
$$

$$
+\frac{(x-x_0-C_xt)(y-y_0-C_yt)}{D_xD_yt^2}(-r)
$$

$\Rightarrow$

$$
(i)=(ii)
$$

APPENDIX B

AUXILIARY INFORMATION FOR THE EXIT TIME PROBLEM

The procedure of exit time computation was summarized in figure 4.1. This procedure starts with setting a difference equation with boundary condition whose solution is the expected number of steps of the random walk to reach the boundary starting inside the given domain.

Then, the components of the difference equation are approximated using Taylor's theorem for functions of two variables given in appendix A.1. The approximated terms are put in the difference equation. Both sides of the equation is divided by Δt. Taking the limit as $\Delta t \rightarrow 0$, a PDE is obtained. The boundary condition for the difference equation is transformed to continuous time. Hence, a BVP whose solution is the expected exit time of a motion modeled by 2D correlated diffusion process from the given domain is obtained.

The PDE of the BVP is a second-order, linear, constant-coefficient, nonhomogeneous, and elliptic PDE in two variables. Basics of second-order PDEs and related properties are provided in the second section. Furthermore, standard forms of PDEs are discussed in this section.

In order to solve the BVP, it must be transformed into its standard form first. The transformation procedure is described in figure 4.4 and explained in the third section in detail. It involves the steps of rotation of axes, change of a dependent variable, and scaling.

The standard form of the PDE of the BVP can be written in the form of Helmholtz equation whose basics are provided in the last section. Solution of the BVP involving Helmholtz equation is given in this section for a circular domain.

B.1 Taylor's Approximation for the Difference Equation

The difference equation given by (4.1) is

$$T(x,y) = p_1 T(x+\Delta x, y+\Delta y) + p_2 T(x+\Delta x, y-\Delta y) + p_3 T(x-\Delta x, y+\Delta y)$$
$$+ p_4 T(x-\Delta x, y-\Delta y) + 1 \tag{B.1}$$

Applying Taylor's theorem for functions of two variables to the components of $T(x,y)$, the following results are obtained.

(i) $\quad T(x+\Delta x, y+\Delta y)$:

$$\bar{a} = (x,y)$$
$$\bar{v} = (x+\Delta x, y+\Delta y)$$
$$(\bar{v}-\bar{a}) = (\Delta x, \Delta y)$$

$\Rightarrow$

$$T(x+\Delta x, y+\Delta y) \cong T(x,y) + \frac{\partial}{\partial x}T(x,y)\Delta x + \frac{\partial}{\partial y}T(x,y)\Delta y$$
$$+\frac{\partial^2}{\partial x \partial y}T(x,y)\Delta x\Delta y + \frac{1}{2}\frac{\partial^2}{\partial x^2}T(x,y)\Delta x^2 + \frac{1}{2}\frac{\partial^2}{\partial y^2}T(x,y)\Delta y^2 \tag{B.2}$$

(ii) $\quad T(x+\Delta x, y-\Delta y)$:

$$\bar{a} = (x,y)$$
$$\bar{v} = (x+\Delta x, y-\Delta y)$$
$$(\bar{v}-\bar{a}) = (\Delta x, -\Delta y)$$

$\Rightarrow$

$$T(x+\Delta x, y-\Delta y) \cong T(x,y) + \frac{\partial}{\partial x}T(x,y)\Delta x + \frac{\partial}{\partial y}T(x,y)(-\Delta y)$$

$$+\frac{\partial^2}{\partial x \partial y}T(x,y)\Delta x(-\Delta y) + \frac{1}{2}\frac{\partial^2}{\partial x^2}T(x,y)\Delta x^2 + \frac{1}{2}\frac{\partial^2}{\partial y^2}T(x,y)(-\Delta y)^2 \tag{B.3}$$

(iii) $\quad T(x - \Delta x, y + \Delta y)$:

$$\bar{a} = (x, y)$$

$$\bar{v} = (x - \Delta x, y + \Delta y)$$

$$(\bar{v} - \bar{a}) = (-\Delta x, \Delta y)$$

$\Rightarrow$

$$T(x - \Delta x, y + \Delta y) \cong T(x,y) + \frac{\partial}{\partial x}T(x,y)(-\Delta x) + \frac{\partial}{\partial y}T(x,y)\Delta y$$

$$+\frac{\partial^2}{\partial x \partial y}T(x,y)(-\Delta x)\Delta y + \frac{1}{2}\frac{\partial^2}{\partial x^2}T(x,y)(-\Delta x)^2 + \frac{1}{2}\frac{\partial^2}{\partial y^2}T(x,y)\Delta y^2 \tag{B.4}$$

(iv) $\quad T(x - \Delta x, y - \Delta y)$:

$$\bar{a} = (x, y)$$

$$\bar{v} = (x - \Delta x, y - \Delta y)$$

$$(\bar{v} - \bar{a}) = (-\Delta x, -\Delta y)$$

$\Rightarrow$

$$T(x - \Delta x, y - \Delta y) \cong T(x,y) + \frac{\partial}{\partial x}T(x,y)(-\Delta x) + \frac{\partial}{\partial y}T(x,y)(-\Delta y)$$

$$+\frac{\partial^2}{\partial x \partial y}T(x,y)(-\Delta x)(-\Delta y) + \frac{1}{2}\frac{\partial^2}{\partial x^2}T(x,y)(-\Delta x)^2$$

$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}T(x,y)(-\Delta y)^2 \tag{B.5}$$

Plugging these expressions in (B.1), combining, and arranging the terms, (B.1) becomes

$$T(x,y) = T(x,y)\underbrace{(p_1 + p_2 + p_3 + p_4)}_{1} + \Delta x(p_1 + p_2 - p_3 - p_4)\frac{\partial}{\partial x}T(x,y)$$

$$+\Delta y(p_1 - p_2 + p_3 - p_4)\frac{\partial}{\partial y}T(x,y) + \Delta x \Delta y(p_1 - p_2 - p_3 + p_4)\frac{\partial^2}{\partial x \partial y}T(x,y)$$

$$+(\Delta x)^2\underbrace{(p_1 + p_2 + p_3 + p_4)}_{1}\frac{1}{2}\frac{\partial^2}{\partial x^2}T(x,y)$$

$$+(\Delta y)^2\underbrace{(p_1 + p_2 + p_3 + p_4)}_{1}\frac{1}{2}\frac{\partial^2}{\partial y^2}T(x,y) + \Delta t \tag{B.6}$$

Dividing both sides of (B.6) by Δt and after taking the limit as $\Delta t \to 0$, the resulting process becomes a 2D correlated diffusion process with setting (C_x, C_y, D_x, D_y, r). Simplifying (B.6) further, the following PDE is obtained.

$$\begin{aligned} 0 = C_x \frac{\partial}{\partial x} T(x,y) + C_y \frac{\partial}{\partial y} T(x,y) + D_x D_y r \frac{\partial^2}{\partial x \partial y} T(x,y) \\ + \frac{1}{2} D_x^2 \frac{\partial^2}{\partial x^2} T(x,y) + \frac{1}{2} D_y^2 \frac{\partial^2}{\partial y^2} T(x,y) + 1 \end{aligned} \tag{B.7}$$

Let

$$\begin{aligned} f &\triangleq T(x,y) \\ f_x &\triangleq \frac{\partial}{\partial x} T(x,y) \\ f_y &\triangleq \frac{\partial}{\partial y} T(x,y) \\ f_{xy} &\triangleq \frac{\partial^2}{\partial x \partial y} T(x,y) \\ f_{xx} &\triangleq \frac{\partial^2}{\partial x^2} T(x,y) \\ f_{yy} &\triangleq \frac{\partial^2}{\partial y^2} T(x,y) \end{aligned}$$

(B.7) can be rewritten as

$$C_x f_x + C_y f_y + D_x D_y r f_{xy} + \frac{1}{2} D_x^2 f_{xx} + \frac{1}{2} D_y^2 f_{yy} = -1 \tag{B.8}$$

It is convenient to write the equation in the form

$$Lf = -1 \tag{B.9}$$

, where L denotes the second-order differential operator

$$L = C_x \frac{\partial}{\partial x} + C_y \frac{\partial}{\partial y} + D_x D_y r \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} D_x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2} D_y^2 \frac{\partial^2}{\partial y^2} \tag{B.10}$$

B.2 Second-order PDEs and Related Properties

This section briefly describes the classification of second-order, linear, and constant-coefficient PDEs and their standard forms. The rest of the section is quoted from [39].

By the introduction of new variables m, n, and w, every second-order equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + hu_x + ku_y + eu = g(x,y) \tag{B.11}$$

, where a, b, c, d, e, and f are constants, can be transformed into one and only one of the following standard forms.

$$w_{mm} + w_{nn} + \gamma w = \varphi(m,n) \tag{B.12}$$

$$w_{mm} - w_{nn} + \gamma w = \varphi(m,n) \tag{B.13}$$

$$w_{mm} - w_n = \varphi(m,n) \tag{B.14}$$

$$w_{mm} + \gamma w = \varphi(m,n) \tag{B.15}$$

The equation (B.11) is called *elliptic* if it can be reduced to (B.12), and this case occurs if $ac - b^2 > 0$. It is called *hyperbolic* if it can be reduced to (B.13), and this case occurs if $ac - b^2 < 0$. If $ac = b^2$, then the equation can be reduced to (B.14) or (B.15) and is called *parabolic* or degenerate, respectively.

It should also be stated that γ is a constant with one of the values -1, 0, or 1. In the above list of standard forms, there are three elliptic equations, corresponding to the three choices $\gamma = -1, 0, 1$.

The determination of the general solution of the equation (B.11) is not possible except in special cases (constant-coefficient), and even when possible is not widely useful.

The importance of the result stated on the transformation of second-order equations into standard form is that it indicates that except for simple changes of variable, there are only *six* different nondegenerate second-order equations, collected into three classes, so that the study can be limited to these equations.

Further study of equations reveals that there are essential similarities among equations of the same class, profound differences between the equations of one class and those of another class.

B.3 Transformation of the BVP into Its Standard Form

The transformation process studied in this section is accomplished following the methods in [39]. This process is fully described in the following subsections.

B.3.1 Rotation of Axes

If the (m, n)-axes are obtained from the (x, y)-axes by rotation thorough angle A, then (m, n) and (x, y) are related by either of the pair of equations

$$m = xcosA + ysinA \tag{B.16}$$

$$n = -xsinA + ycosA \tag{B.17}$$

$$x = mcosA - nsinA \tag{B.18}$$

$$y = msinA + ncosA \tag{B.19}$$

$u(x, y)$ is a solution of

$$Lu = -1 \tag{B.20}$$

$$u = u(x, y) = f = f(x, y) = T(x, y)$$

$$u(x, y) = u(mcosA - nsinA, msinA + ncosA) = w(m, n)$$

With these change of variables, the differentials $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$, and $\frac{\partial^2}{\partial x \partial y}$ become

$$\frac{\partial}{\partial x} = \frac{\partial m}{\partial x}\frac{\partial}{\partial m} + \frac{\partial n}{\partial x}\frac{\partial}{\partial n} = cosA\frac{\partial}{\partial m} - sinA\frac{\partial}{\partial n} \tag{B.21}$$

$$\frac{\partial}{\partial y} = \frac{\partial m}{\partial y}\frac{\partial}{\partial m} + \frac{\partial n}{\partial y}\frac{\partial}{\partial n} = sinA\frac{\partial}{\partial m} + cosA\frac{\partial}{\partial n} \tag{B.22}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} &= \left(cosA\frac{\partial}{\partial m} - sinA\frac{\partial}{\partial n}\right)\left(cosA\frac{\partial}{\partial m} - sinA\frac{\partial}{\partial n}\right) \\ &= cos^2A\frac{\partial^2}{\partial m^2} - 2sinAcosA\frac{\partial^2}{\partial m \partial n} + sin^2A\frac{\partial^2}{\partial n^2}\end{aligned} \tag{B.23}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} = \left(sinA\frac{\partial}{\partial m} + cosA\frac{\partial}{\partial n}\right)\left(sinA\frac{\partial}{\partial m} + cosA\frac{\partial}{\partial n}\right)$$

$$= sin^2A\frac{\partial^2}{\partial m^2} + 2sinAcosA\frac{\partial^2}{\partial m\partial n} + cos^2A\frac{\partial^2}{\partial n^2} \tag{B.24}$$

$$\frac{\partial^2}{\partial xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} = \left(cosA\frac{\partial}{\partial m} - sinA\frac{\partial}{\partial n}\right)\left(sinA\frac{\partial}{\partial m} + cosA\frac{\partial}{\partial n}\right)$$

$$= sinAcosA\frac{\partial^2}{\partial m^2} + \left(cos^2A - sin^2A\right)\frac{\partial^2}{\partial m\partial n} - sinAcosA\frac{\partial^2}{\partial n^2} \tag{B.25}$$

Substituting (B.21), (B.22), (B.23), (B.24), and (B.25) in (B.20), (B.20) becomes

$$\tilde{L}w = -1 \tag{B.26}$$

, where

$$w = w(m, n)$$

, and

$$\tilde{L} = \frac{1}{2}D_x^2\left[cos^2A\frac{\partial^2}{\partial m^2} - 2sinAcosA\frac{\partial^2}{\partial m\partial n} + sin^2A\frac{\partial^2}{\partial n^2}\right]$$

$$+\frac{1}{2}D_y^2\left[sin^2A\frac{\partial^2}{\partial m^2} + 2sinAcosA\frac{\partial^2}{\partial m\partial n} + cos^2A\frac{\partial^2}{\partial n^2}\right]$$

$$+D_xD_yr\left[sinAcosA\frac{\partial^2}{\partial m^2} + \left(cos^2A - sin^2A\right)\frac{\partial^2}{\partial m\partial n} - sinAcosA\frac{\partial^2}{\partial n^2}\right]$$

$$+C_x\left[cosA\frac{\partial}{\partial m} - sinA\frac{\partial}{\partial n}\right] + C_y\left[sinA\frac{\partial}{\partial m} + cosA\frac{\partial}{\partial n}\right]$$

$$= \frac{\partial^2}{\partial m^2}\left[\frac{1}{2}D_x^2cos^2A + \frac{1}{2}D_y^2sin^2A + D_xD_yrsinAcosA\right]$$

$$+\frac{\partial^2}{\partial n^2}\left[\frac{1}{2}D_x^2sin^2A + \frac{1}{2}D_y^2cos^2A - D_xD_yrsinAcosA\right]$$

$$+\frac{\partial^2}{\partial m\partial n}\left[\frac{1}{2}D_x^2\big(-2sinAcosA\big) + \frac{1}{2}D_y^2\big(2sinAcosA\big) + D_xD_yr\big(cos^2A - sin^2A\big)\right]$$

$$+\frac{\partial}{\partial m}\big(C_xcosA + C_ysinA\big) + \frac{\partial}{\partial n}\big(-C_xsinA + C_ycosA\big) \tag{B.27}$$

After these transformations, in order that the mixed second partial derivative does not appear, the coefficient of $\frac{\partial^2}{\partial m\partial n}$ should be set to zero.

$$\frac{1}{2}D_x^2\big(-2sinAcosA\big) + \frac{1}{2}D_y^2\big(2sinAcosA\big) + D_xD_yr\big(cos^2A - sin^2A\big) = 0$$

$\Rightarrow$

$$2sinAcosA\left(-\frac{1}{2}D_x^2 + \frac{1}{2}D_y^2\right) + \big(cos^2A - sin^2A\big)D_xD_yr = 0 \tag{B.28}$$

Using the half-angle formulas

$$sin2A = 2sinAcosA$$

$$cos2A = cos^2A - sin^2A$$

(B.28) becomes

$$sin2A\left(-\frac{1}{2}D_x^2 + \frac{1}{2}D_y^2\right) + cos2A\big(D_xD_yr\big) = 0$$

⇒

$$tan2A = \frac{2D_xD_yr}{D_x^2 - D_y^2}$$

Selection of A

$$A = \frac{1}{2}tan^{-1}\left(\frac{2D_xD_yr}{D_x^2 - D_y^2}\right) \tag{B.29}$$

causes the term $\frac{\partial^2}{\partial m \partial n}$ to vanish. If the following set of parametric definitions are made

$$a \triangleq \frac{1}{2}D_x^2cos^2A + \frac{1}{2}D_y^2sin^2A + D_xD_yrsinAcosA$$

$$b \triangleq \frac{1}{2}D_x^2sin^2A + \frac{1}{2}D_y^2cos^2A - D_xD_yrsinAcosA$$

$$c \triangleq C_xcosA + C_ysinA$$

$$d \triangleq -C_xsinA + C_ycosA$$

, the differential operator $\tilde{L}$ can be rewritten as

$$\tilde{L} = a\frac{\partial^2}{\partial m^2} + b\frac{\partial^2}{\partial n^2} + c\frac{\partial}{\partial m} + d\frac{\partial}{\partial n} \tag{B.30}$$

It should also be noted that rotation of axes must also be applied to the domain where the PDE holds. The boundary condition

$$u(x,y) = 0, \qquad \{x,y\} \in dD \tag{B.31}$$

becomes

$$w(m,n) = 0, \qquad \{m,n\} \in d\tilde{D} \tag{B.32}$$

, where $d\tilde{D}$ is the boundary of the resulting domain $\tilde{D}$ from the transformation $(x,y) \to (m,n)$ (ie. rotation of axes).

B.3.2 Change of a Dependent Variable

The yielding equation from the rotation of axes was

$$\tilde{L}w = -1$$

, where

$$w = w(m, n)$$

, and

$$\tilde{L} = a\frac{\partial^2}{\partial m^2} + b\frac{\partial^2}{\partial n^2} + c\frac{\partial}{\partial m} + d\frac{\partial}{\partial n}$$

The aim of the change of a dependent variable is to remove the single partial derivatives $\frac{\partial}{\partial m}$ and $\frac{\partial}{\partial n}$ from $\tilde{L}$. To accomplish this let

$$w = exp\Big\{B_1 m + B_2 n\Big\}u \tag{B.33}$$

, where

$$u = u(m, n)$$

Let

$$e \triangleq exp\Big\{B_1 m + B_2 n\Big\}$$

Differentiating (B.33)

$$w_m = B_1 eu + u_m e \tag{B.34}$$

$$w_n = B_2 eu + u_n e \tag{B.35}$$

$$w_{mm} = B_1^2 eu + u_m B_1 e + u_{mm} e + B_1 e u_m \tag{B.36}$$

$$w_{nn} = B_2^2 eu + u_n B_2 e + u_{nn} e + B_2 e u_n \tag{B.37}$$

Substituting (B.34), (B.35), (B.36), and (B.37) in (B.26)

$$aB_1^2 eu + au_m B_1 e + au_{mm} e + aB_1 e u_m + bB_2^2 eu + bu_n B_2 e$$

$$+bu_{nn} e + bB_2 e u_n + cB_1 eu + cu_m e + dB_2 eu + du_n e = -1$$

Collecting terms together

$$aeu_{mm} + beu_{nn} + \big(aB_1 e + aB_1 e + ce\big)u_m + \big(bB_2 e + bB_2 e + de\big)u_n$$

$$+\big(aB_1^2e + bB_2^2e + cB_1e + dB_2e\big)u = -1$$

Dividing both sides by e

$$au_{mm} + bu_{nn} + \big(2aB_1 + c\big)u_m + \big(2bB_2 + d\big)u_n$$

$$+\big(aB_1^2 + bB_2^2 + cB_1 + dB_2\big)u = -\frac{1}{e} \tag{B.38}$$

In order to have the first partial derivatives vanish

$$2aB_1 + c = 0$$

$$2bB_2 + d = 0$$

$\Rightarrow$

$$B_1 = -\frac{c}{2a} \tag{B.39}$$

$$B_2 = -\frac{d}{2b} \tag{B.40}$$

(B.38) becomes

$$au_{mm} + bu_{nn} + \left[a\frac{c^2}{4a^2} + b\frac{d^2}{4b^2} + c\left(\frac{-c}{2a}\right) + d\left(\frac{-d}{2b}\right)\right]u$$

$$= -exp\left\{-\left(-\frac{c}{2a}m - \frac{d}{2b}n\right)\right\}$$

Rearranging the terms and making the necessary simplifications

$$au_{mm} + bu_{nn} + \left[-\frac{c^2}{4a} - \frac{d^2}{4b}\right]u = -exp\left\{\frac{c}{2a}m + \frac{d}{2b}n\right\} \tag{B.41}$$

Let

$$K \triangleq \frac{c^2}{4a} + \frac{d^2}{4b}$$

, then

$$au_{mm} + bu_{nn} - Ku = -exp\left\{\frac{c}{2a}m + \frac{d}{2b}n\right\} \tag{B.42}$$

The domain resulting from the change of a dependent variable step remains unchanged. That is to say, this process does not have any influence on the domain of the problem.

B.3.3 Scaling

The final step is a "change of scale"

$$m = \mu_1 x \tag{B.43}$$

$$n = \mu_2 y \tag{B.44}$$

, where μ_1 and μ_2 are chosen so that in the transformed equation the coefficients of w_{xx} and w_{yy} are equal in absolute value. The differentials $\frac{\partial^2}{\partial m^2}$ and $\frac{\partial^2}{\partial n^2}$ become

$$\frac{\partial^2}{\partial m^2} = \frac{1}{\mu_1^2}\frac{\partial^2}{\partial x^2}$$

$$\frac{\partial^2}{\partial n^2} = \frac{1}{\mu_2^2}\frac{\partial^2}{\partial y^2}$$

The condition

$$\frac{a}{\mu_1^2} = \frac{b}{\mu_2^2} = K$$

is satisfied if

$$\mu_1 = \sqrt{\frac{a}{K}} \tag{B.45}$$

$$\mu_2 = \sqrt{\frac{b}{K}} \tag{B.46}$$

Hence, after the transformation (B.42) becomes

$$Kw_{xx} + Kw_{yy} - Kw = -exp\left\{\frac{c}{2a}\sqrt{\frac{a}{K}}x + \frac{d}{2b}\sqrt{\frac{b}{K}}y\right\}$$

Dividing both sides by K

$$w_{xx} + w_{yy} - w = -\frac{1}{K}exp\left\{\frac{c}{2a}\sqrt{\frac{a}{K}}x + \frac{d}{2b}\sqrt{\frac{b}{K}}y\right\} \tag{B.47}$$

Let

$$\xi \triangleq \frac{c}{2a}\sqrt{\frac{a}{K}}$$

, and

$$\eta \triangleq= \frac{d}{2b}\sqrt{\frac{b}{K}}$$

Then, (B.47) can be rewritten as

$$\tilde{\tilde{L}}w = f(x, y) \tag{B.48}$$

, where

$$\tilde{\tilde{L}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1$$

, and

$$w = w(x, y) = u(\mu_1 x, \mu_2 y)$$

$f(x, y)$ constitutes the nonhomogeneous part of the equation.

$$f(x, y) = -\frac{1}{K} exp\{\xi x + \eta y\}$$

It should also be noted that scaling must also be applied to the domain where the PDE holds. The boundary condition

$$u(m, n) = 0, \qquad \{m, n\} \in d\tilde{D} \tag{B.49}$$

becomes

$$w(x, y) = 0, \qquad \{x, y\} \in d\tilde{\tilde{D}} \tag{B.50}$$

, where $d\tilde{\tilde{D}}$ is the boundary of the resulting domain $\tilde{\tilde{D}}$ from the transformation $(m, n) \rightarrow (x, y)$ (ie. scaling).

B.4 Notes on the Analytical Solution of the BVP

This section is quoted from [40]. Any elliptic equation with constant coefficients can be reduced to the Helmholtz equation

$$\Delta_2 w + \lambda w = -\Phi(x, y) \tag{B.51}$$

, where Δ_2 is a two-dimensional Laplace operator

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

, and λ is a constant.

The Helmholtz equation is called homogeneous is $\Phi = 0$ and and nonhomogeneous if $\Phi \neq 0$. A homogeneous BVP is a BVP for the homogeneous Helmholtz equation with homogeneous boundary conditions; a particular solution of a homogeneous BVP is $w = 0$.

The values λ_n of the parameter λ for which there are nontrivial solutions (solutions other than identical zero) of the homogeneous BVP are called eigenvalues and the corresponding solutions , $w = w_n$, are called eigenfunctions of the BVP.

There are infinitely many eigenvalues $\{\lambda_n\}$; the set of eigenvalues forms a discrete spectrum for the given BVP. All eigenvalues are positive, except for the eigenvalue $\lambda_0 = 0$ (the corresponding eigenfunction $w_0 = const$. The eigenvalues are numbered in order of increasing magnitudes, $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$.

B.4.1 Nonhomogeneous Helmholtz Equation with Homogeneous Boundary Conditions

Three cases are possible.

1. If the equation parameter λ is not equal to any one of the eigenvalues, then there exists the series solution.

$$w = \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n - \lambda} \tag{B.52}$$

, where

$$A_n = \frac{1}{\|w_n\|^2} \int_S \Phi w_n dS$$

, and

$$\|w_n\|^2 = \int_S w_n^2 dS$$

2. If λ is equal to some eigenvalue, $\lambda = \lambda_m$, then the solution of the nonhomogeneous problem exists only if the function Φ is orthogonal to w_m, i.e.

$$\int_S \Phi w_n dS = 0$$

In this case the system is expressed as

$$w = \sum_{n=1}^{m-1} \frac{A_n}{\lambda_n - \lambda_m} w_n + \sum_{n=m+1}^{\infty} \frac{A_n}{\lambda_n - \lambda_m} w_n + C w_m \tag{B.53}$$

, where C is an arbitrary constant.

3. If $\lambda = \lambda_m$ and $\int_S \Phi w_n dS \neq 0$, then the BVP for the nonhomogeneous equation does not have solutions.

Remark If p_n mutually orthogonal eigenfunctions $w_n^{(j)} (j = 1, 2, \ldots, p_n)$ correspond to each eigenvalue λ_n, then, for $\lambda \neq \lambda_n$, the solution is written as

$$w = \sum_{n=1}^{\infty} \sum_{j=1}^{p_n} \frac{A_n^{(j)}}{\lambda_n - \lambda} w_n^{(j)} \tag{B.54}$$

, where

$$A_n^{(j)} = \frac{1}{\left\| w_n^{(j)} \right\|^2} \int_S \Phi w_n^{(j)} dS$$

, and

$$\left\| w_n^{(j)} \right\|^2 = \int_S \left[w_n^{(j)} \right]^2 dS$$

B.4.2 Solution of the Homogeneous BVP for a Circular Domain

A two-dimensional nonhomogeneous Helmholtz equation in the polar coordinate system is written as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \lambda w = -\Phi(r, \varphi) \tag{B.55}$$

$$r = \sqrt{x^2 + y^2}$$

In what follows, the eigenvalues and eigenfunctions of homogeneous BVP for the homogeneous Helmholtz equation are given. The solutions of the corresponding nonhomogeneous problems can be constructed using the formulas presented in appendix B.4.1.

Considering a circle of radius R centered at the origin as the domain of the problem as given in figure 4.15 with the boundary condition

$$w = 0 \qquad \text{at} \qquad r = R \tag{B.56}$$

Eigenvalues

$$\lambda_{nm} = \frac{\mu_{nm}^2}{R^2}; \qquad n = 0, 1, 2, \ldots; m = 1, 2, 3, \ldots \tag{B.57}$$

Here, the μ_{nm} are positive zeros of the Bessel functions, $J_n(\mu) = 0$.

Eigenfunctions

$$w_{nm}^{(1)} = J_n\left(r\sqrt{\lambda_{nm}}\right) cosn\varphi \tag{B.58}$$

$$w_{nm}^{(2)} = J_n\left(r\sqrt{\lambda_{nm}}\right) sinn\varphi \tag{B.59}$$

The eigenfunctions possess the axial symetry property

$$w_{0m}^{(1)} = J_0\left(r\sqrt{\lambda_{0m}}\right)$$

The square of the norm of an eigenfunction is given by

$$\left\|w_{nm}^{(k)}\right\|^2 = \frac{1}{2}\pi R^2(1+\delta_{n0})\left[J_n'(\mu_{nm})\right]^2, \qquad k = 1, 2 \tag{B.60}$$

, where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j \end{cases}$$

APPENDIX C

SAMPLE REALIZATIONS OF THE PROCESS

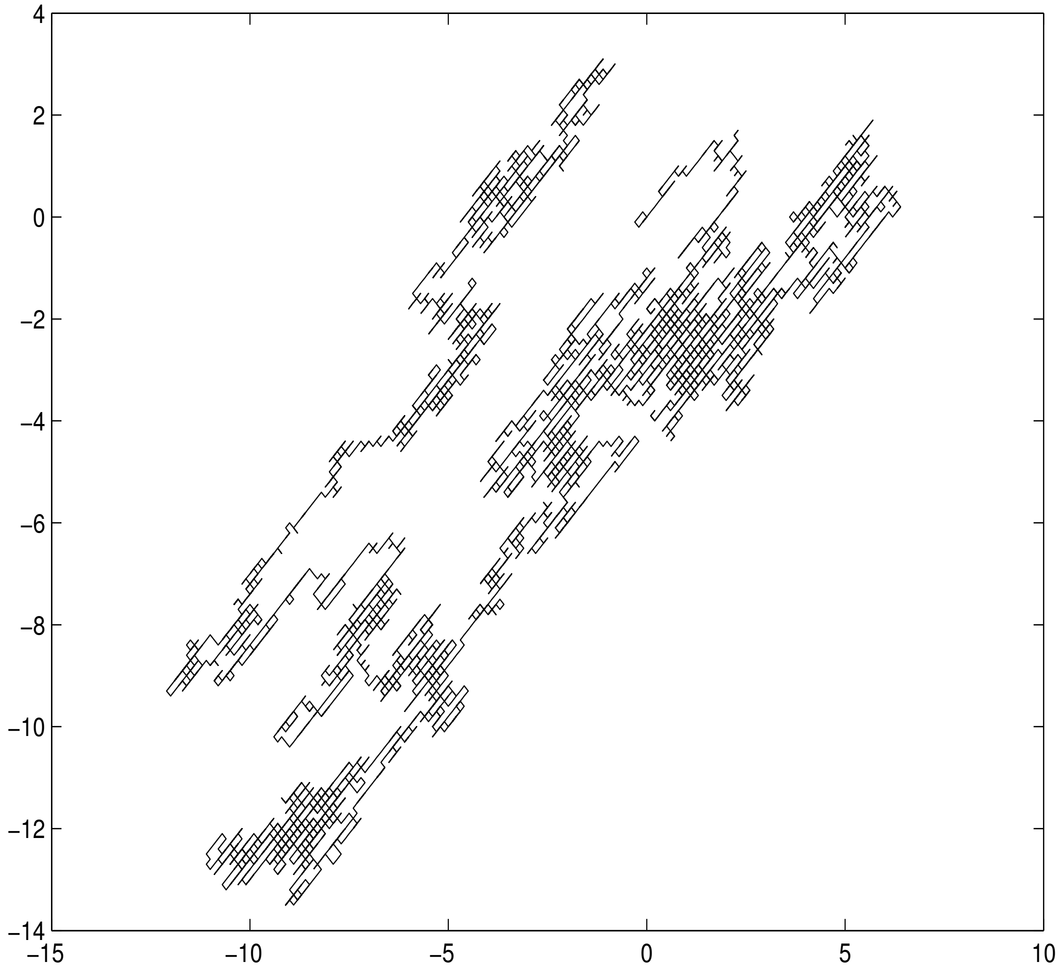

Figure A.1: Sample realization of the process with C_x=0, C_y=0, D_x=1, D_y=1, r=0.7, N=10,000, and Δt=0.01.

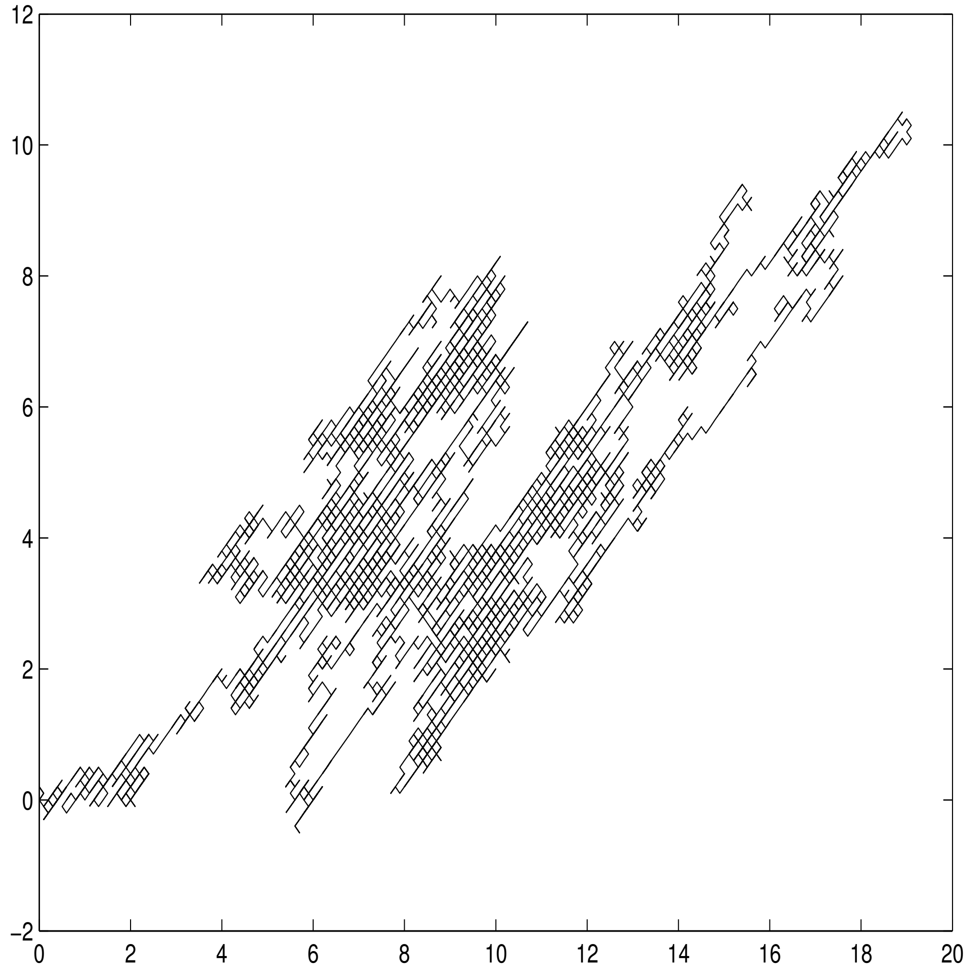

Figure A.2: Sample realization of the process with C_x=0.1, C_y=0.1, D_x=1, D_y=1, r=0.7, N=10,000, and Δt=0.01.

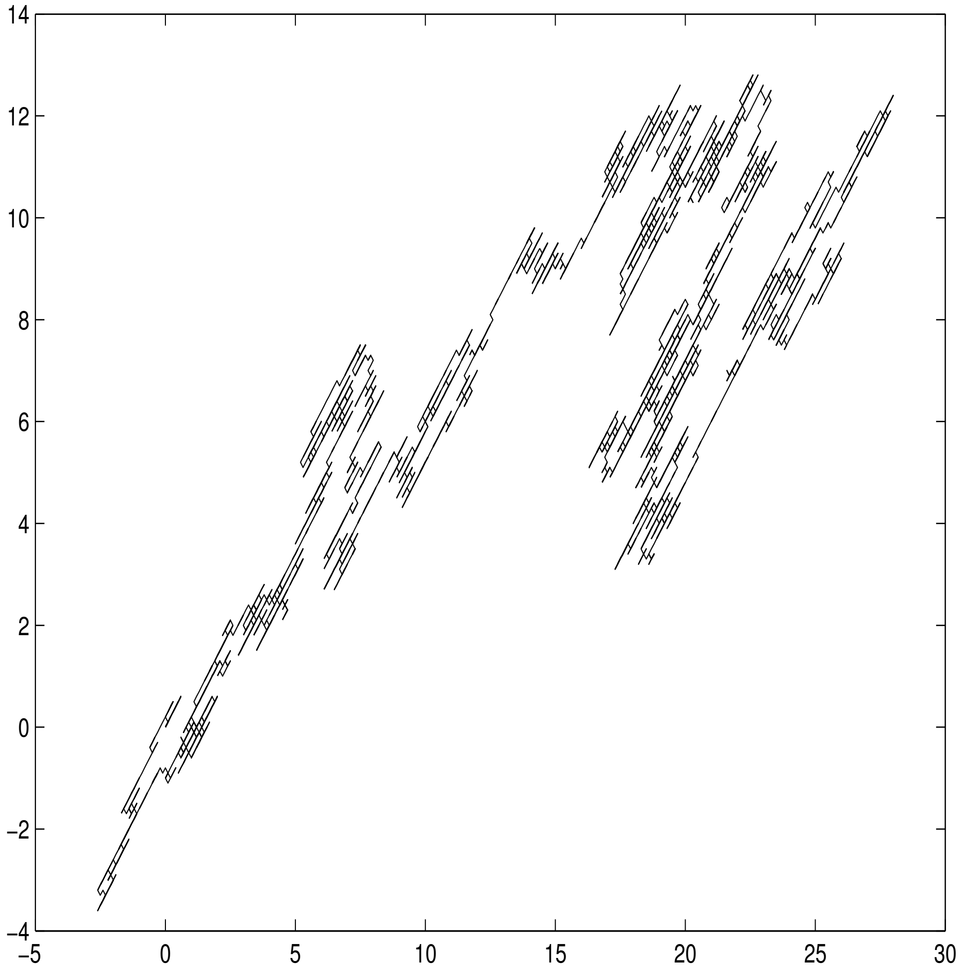

Figure A.3: Sample realization of the process with C_x=0.3, C_y=0.1, D_x=1, D_y=1, r=0.9, N=10,000, and Δt=0.01.

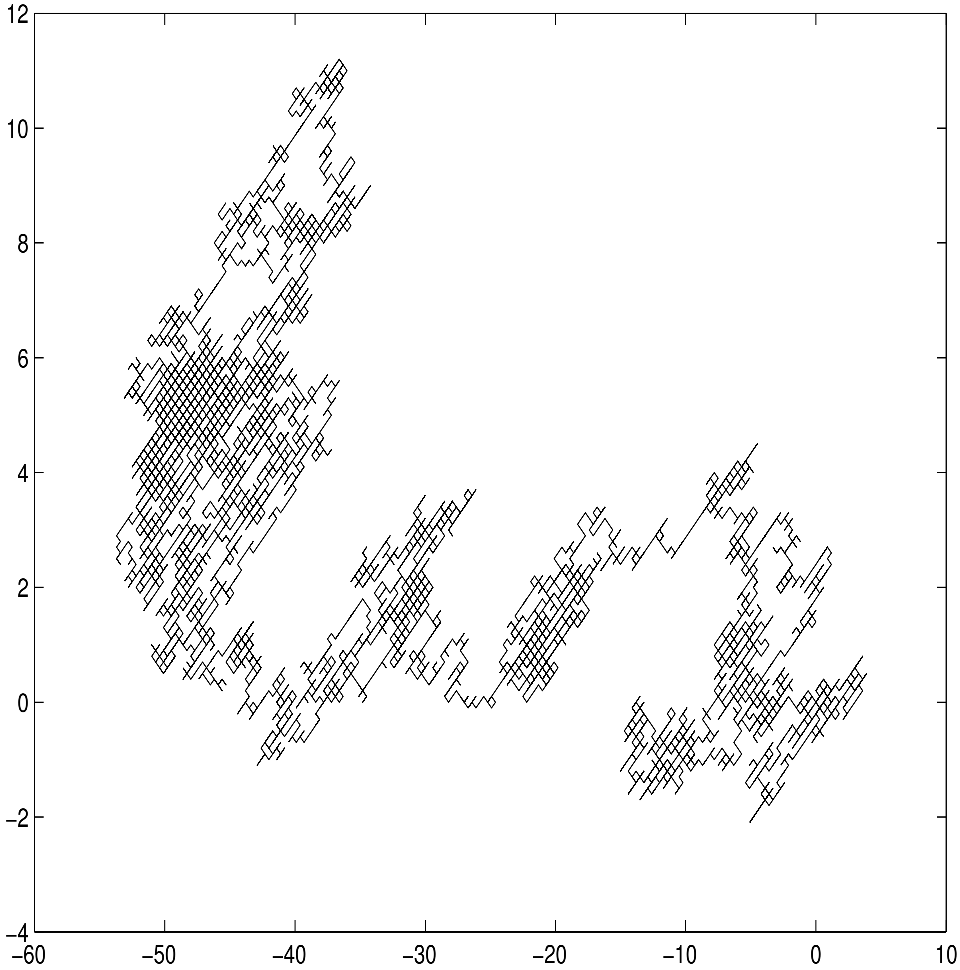

Figure A.4: Sample realization of the process with C_x=0.1, C_y=0.1, D_x=3, D_y=1, r=0.5, N=10,000, and Δt=0.01.

REFERENCES

[1] I. F. Akyildiz, Y.-B. Lin, W.-R. Lai, and R.-J. Chen, "A New Random Walk Model for PCS Networks," *IEEE Journal on Selected Areas in Communications,* vol. 18, issue 7, pp. 1254-1260, Jul. 2000.

[2] K.-H. Chiang and N. Shenoy, "A Random Walk Mobility Model for Location Management in Wireless Networks," *IEEE International Symposium on Personal, Indoor, and Mobile Radio Communications,* vol. 2, pp. E-43 – E-48, 30 Sep.-3 Oct. 2001.

[3] K.-H. Chiang and N. Shenoy, "A 2-D Random-Walk Mobility Model for Location-Management Studies in Wireless Networks," *IEEE Transactions on Vehicular Technology,* vol. 53, issue 2, pp. 413-424, Mar. 2004.

[4] B. Jabbari, Y. Zhou, and F. Hillier, "Random Walk Modeling of Mobility in Wireless Networks," *IEEE Vehicular Technology Conference,* vol. 1, pp. 639-643, 18-21 May 1998.

[5] E. Aleman-Llanes, D. Munoz-Rodriguez, and C. Molina, "FBM Mobility Modeling For Nomadic Subscribers," *Proceedings of IEEE Symposium on Computers and Communications,* pp. 315-319, 30 Jun.-2 Jul. 1998.

[6] Y. Fang and I. Chlamtac, "A New Mobility Model and its Application in the Channel Holding Time Characterization in PCS Networks," *Proceedings of Annual Joint Conference of the IEEE Computer and Communications Societies,* vol. 1, pp. 20-27, 21-25 Mar. 1999.

[7] E. Jugl and H. Boche, "Dwell Time Models for Wireless Communication Systems," *IEEE Vehicular Technology Conference,* vol. 5, pp. 2984-2988, 19-22 Sep. 1999.

[8] Y. Fang and I. Chlamtac, "Teletraffic Analysis and Mobility Modeling of PCS Networks," *IEEE Transactions on Communications,* vol. 47, issue 7, pp. 1062-1072, Jul. 1999.

[9] I. Stepanov, J. Hahner, C. Becker, J. Tian, and K. Rothermel, "A Meta-Model and Framework for User Mobility in Mobile Networks," *IEEE International Conference on Networks,* pp. 231-238, 28 Sep.-1 Oct. 2003.

[10] T. S. Kim, M. Y. Chung, D. K. Sung, and M. Sengoku, "A Mobility Analysis in 3-Dimensional PCS Environments," *IEEE Vehicular Technology Conference,* vol. 1, pp. 237-241, 28 Apr.-1 May 1996.

[11] J. Karnmann, M. Angermann, and B. Larni, "A New Mobility Model Based on Maps," *IEEE Vehicular Technology Conference,* vol. 5, pp. 3045-3049, 6-9 Oct. 2003.

[12] E. Jugl and H. Boche, "A New Mobility Model for Performance Evaluation of Future Mobile Communication Systems," *IEEE International Conference on Communications,* vol. 3, pp. 1751-1755, 6-10 Jun. 1999.

[13] D. C. Engelhart, A. Sivasubramaniam, C. L. Barrett, M. V. Marathe, J. P. Smith, and M. Morin, "A Spatial Analysis of Mobility Models: Application to Wireless Ad Hoc Network Simulation," *Proceedings of Simulation Symposium,* pp. 35-42, 18-22 Apr. 2004.

[14] M.-H. Yang, L.-W. Chen, Y.-C. Tseng, and J.-P. Sheu, "Traveling Salesman Mobility Model and Its Location Tracking in PCS Networks," *International Conference on Distributed Computing Systems,* pp. 517-523, 16-19 Apr. 2001.

[15] J. Scourias and T. Kuntz, "Activity-Based Mobility Modeling: Realistic Evaluation of Location Management Schemes for Cellular Networks," *IEEE Wireless Communications and Networking Conference,* vol. 1, pp. 296-300, 21-24 Sep. 1999.

[16] P. Camarda, G. Schiraldi, F. Talucci, and R. Valla, "Cellular Communication Mobility Model for a Finite Multiclass Population of Users," *IEEE Vehicular Technology Conference,* vol. 2, pp. 602-606, 4-7 May 1997.

[17] E.-S. Cho, G.-W. Jin, and C.-H. Cho, "Comparisons of Mobility Models in Cellular Systems," *IEEE Vehicular Technology Conference,* vol. 1, pp. 564-568, 19-22 Sep. 1999.

[18] J. Tian, J. Hahner, C. Becker, I. Stepanov, and K. Rothermel, "Graph-Based Mobility Model for Mobile Ad Hoc Network Simulation," *Proceedings of Simulation Symposium,* pp. 337-344, 14-18 Apr. 2002.

[19] T. S. Kim, M. Y. Chung, and D. K. Sung, "Mobility and Traffic Analyses in Three-Dimensional PCS Environments," *IEEE Transactions on Vehicular Technology,* vol. 47, issue 2, pp. 537-545, May 1998.

[20] I. F. Akyildiz, J. McNair, J. S. M. Ho, H. Uzunalioglu, and W. Wang, "Mobility Management in Next-Generation Wireless Systems," *Proceedings of the IEEE,* vol. 87, issue 8, pp. 1347-1384, Aug. 1999.

[21] P. I. Bratanov and E. Bonek, "Mobility Model of Vehicle-Borne Terminals in Urban Cellular Systems," *IEEE Transactions on Vehicular Technology,* vol. 52, issue 4, pp. 947-952, Jul. 2003.

[22] F. Ashtiani, J. A. Salehi, and M. R. Aref, "Mobility Modeling and Analytical Solution for Spatial Traffic Distribution in Wireless Multimedia Networks," *IEEE Journal on Selected Areas in Communications,* vol. 21, issue 10, pp. 1699-1709, Dec. 2003.

[23] M. M. Zonoozi, P. Dassanayake, and M. Faulkner, "Mobility Modelling and Channel Holding Time Distribution in Cellular Mobile Communication Systems," *IEEE Global Telecommunications Conference,* vol. 1, pp. 12-16, 13-17 Nov. 1995.

[24] T. S. Kim, J. K. Kwon, and D. K. Sung, "Mobility Modeling and Traffic Analysis in Three-Dimensional High-Rise Building Environments," *IEEE Transactions on Vehicular Technology,* vol. 49, issue 5, pp. 1633-1640, Sep. 2000.

[25] T. S. Kim, M. Y. Chung, D. K. Sung, and M. Sengoku, "Mobility Modeling and Traffic Analysis in Three-Dimensional Indoor Environments," *IEEE Transactions on Vehicular Technology,* vol. 47, issue 2, pp. 546-557, May 1998.

[26] P. Camarda, G. Schiraldi, and F. Talucci, "Mobility Modeling in Cellular Communication Networks," *Proceedings of IEEE Conference on Local Computer Networks,* pp. 518-525, 13-16 Oct. 1996.

[27] J. G. Markoulidakis, G. L. Lyberopoulos, D. F. Tsirkas, and E. D. Sykas, "Mobility Modeling in Third-Generation Mobile Telecommunications Systems," *IEEE Personal Communications,* vol. 4, issue 4, pp. 41-56, Aug. 1997.

[28] T. Liu, P. Bahl, and I. Chlamtac, "Mobility Modeling, Location Tracking, and Trajectory Prediction in Wireless ATM Networks," *IEEE Journal on Selected Areas in Communications,* vol. 16, issue 6, pp. 922-936, Aug. 1998.

[29] H. Xie and D. J. Goodman, "Mobility Models and Biased Sampling Problem," *International Conference on Personal Communications,* vol. 2, pp. 803-807, 12-15 Oct. 1993.

[30] D. Bansal, A. Chandra, R. Shorey, A. Kulshreshtha, and M. Gupta, "Mobility Models for Cellular Systems: Cell Topography and Handoff Probability," *IEEE Vehicular Technology Conference,* vol. 3, pp. 1794-1798, 16-20 May 1999.

[31] M. Y. Chung, T. S. Kim, H. S. Cho, and D. K. Sung, "Modeling of Terminal Mobility to Evaluate the Number of Location Updates," *IEEE International Conference on Communications,* vol. 3, pp. 1266-1270, 8-12 Jun. 1997.

[32] D. R. Basgeet, J. Irvine, A. Munro, P. Dugenie, and D. Kaleshi, "SMMT - Scalable Mobility Modeling Tool," *Proceedings of IEEE Vehicular Technology Conference,* vol. 1, pp. 102-106, 24-28 Sep. 2002.

[33] W. Navidi and T. Camp, "Stationary Distributions for the Random Waypoint Mobility Model," *IEEE Transactions on Mobile Computing,* vol. 3, issue 1, pp. 99-108, Jan.-Feb. 2004.

[34] C. Bettstetter, G. Resta, and P. Santi, "The Node Distribution of the Random Waypoint Mobility Model for Wireless Ad Hoc Networks," *IEEE Transactions on Mobile Computing,* vol. 2, issue 3, pp. 257-269, Jul.-Sep. 2003.

[35] P. V. Orlik and S. S. Rappaport, "Traffic Performance and Mobility Modeling of Cellular Communications with Mixed Platforms and Highly Variable Mobilities," *Proceedings of the IEEE,* vol. 86, issue 7, pp. 1464-1479, Jul. 1998.

[36] H. Heffes and K. M. Ryan, "User Mobility and Channel Holding Time in Mobile Communications," *IEEE Vehicular Technology Conference,* vol. 2, pp. 577-581, 4-7 May 1997.

[37] M. M. Zonoozi and P. Dassanayake, "User Mobility Modeling and Characterization of Mobility Patterns," *IEEE Journal on Selected Areas in Communications,* vol. 15, issue 7, pp. 1239-1252, Sep. 1997.

[38] Lawrence J. Corvin, Robert H. Szczarba, *Calculus in Vector Spaces.* New York: M. Dekker, 1995.

[39] Paul W. Berg, James L. McGregor, *Elementary Partial Differential Equations.* San Francisco: Holden-Day, 1966.

[40] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists.* Boca Raton: Chapman & Hall / CRC, 2002.

[41] A. K. Basu, *Introduction to Stochastic Processes.* Pangbourne: Alpha Science International, 2003.

[42] Edward P. C. Kao, *An Introduction to Stochastic Processes.* California: Duxbury, 1997.

[43] Sidney I. Resnick, *Adventures in Stochastic Processes.* Boston: Birkhauser, 1992.