INVERSE PROBLEMS FOR A SEMILINEAR HEAT EQUATION WITH MEMORY

MÜJDAT KAYA

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INVERSE PROBLEMS FOR A SEMILINEAR HEAT EQUATION WITH MEMORY

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Prof. Dr. Canan ÖZGEN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Şafak ALPAY
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Varga KALANTAROV
Co-Supervisor

Prof. Dr. A. Okay ÇELEBİ
Supervisor

Examining Committee Members

Prof. Dr. Zafer NURLU (METU)

Prof. Dr. A. Okay ÇELEBİ (METU)

Prof. Dr. Marat AKHMET (METU)

Prof. Dr. İ. Ethem ANAR (GAZİ Ü.)

Prof. Dr. Hasan TAŞELİ (METU)

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Name, Last name: Müjdat, Kaya

Signature:

ABSTRACT

INVERSE PROBLEMS FOR A SEMILINEAR HEAT EQUATIONS WITH MEMORY

Kaya, Müjdat

Ph.D, Department of Mathematics

Supervisor: Prof. Dr. A. Okay Çelebi

Co-Supervisor: Prof. Dr. Varga Kalantarov

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In this thesis, we study the existence and uniqueness of the solutions of the inverse problems to identify the memory kernel k and the source term h, derived from

$$\begin{aligned} \theta_t - k_0\Delta\theta &+ \int_{-\infty}^{t} k\Delta\theta ds + pg(\theta) = h, \qquad \Omega\times\mathbb{R}^+, \quad \Omega\subset\mathbb{R}^n, \\ \theta &= 0, \qquad x\in\partial\Omega, \quad t>0 \\ \theta(.,0) &= \theta_0, \quad x\in\Omega. \end{aligned}$$

First, we obtain the structural stability for k, when $p=1$ and the coefficient p, when $g(\theta)=\theta$.

To identify the memory kernel, we find an operator equation after employing the half Fourier transformation. For the source term identification, we make use of the direct application of the final overdetermination conditions.

Keywords: Structural stability, inverse problem, final overdetermination condition, memory kernel, source term, Paley-Wiener representation.

ÖZ

HAFIZALI YARI DOĞRUSAL ISI DENKLEMİ İÇİN TERS PROBLEMLER

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Bu çalışmada

$$\begin{aligned} \theta_t - k_0\Delta\theta &+ \int_{-\infty}^{t} k\Delta\theta ds + pg(\theta) = h, \qquad \Omega\times\mathbb{R}^+, \quad \Omega\subset\mathbb{R}^n \\ \theta &= 0 \qquad x\in\partial\Omega, \quad t>0 \\ \theta(.,0) &= \theta_0, \quad x\in\Omega \end{aligned}$$

denkleminden hafıza çekirdeği k ve kaynak terim h nin belirlenmesi için elde edilen ters problemlerin çözümlerinin varlığı ve tekliği gösterilmiştir. Öncelikle, yukarıdaki direk problemin çözümünün $p = 1$ için k' ya ve $g(\theta) = \theta$ için p' ye sürekli bağımlılığı incelenmiştir.

Hafıza çekirdeği k' yı belirleme problemi yarı Fourier dönüşümü kullanılarak bir operatör denkleme çevirilmiştir. Kaynak terim h yi belirmek için son karar verme şartları kullanılmıştır.

Anahtar Kelimeler: Yapısal kararlılık, ters problem, son karar verme şartı, hafıza çekirdeği, kaynak terimi, Paley- Wiener gösterimi.

To my family

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Two problems are called inverses of one another if the formulation of each needs all or partial solution of the other [26]. For some historical reasons, one of these problems may have been studied deeply while the other is newer. The former problem is called the direct and the latter the inverse problem. As primitive illustrative examples, we give the following two problems.

1. Find a polynomial $P(x)$ of degree n with zeros $x_1, \cdots, x_n$.

This problem is the inverse of the direct problem of finding zeros $x_1, \cdots, x_n$ of a given polynomial $P(x)$ of degree n. In this example, the solution of the inverse problem is trivial, i.e., $P(x) = C(x - x_1) \cdots (x - x_n)$, $C \neq 0$. Since C is arbitrary, solution is not unique.

2. Find a rule of the sequence if some terms $a_1, \cdots, a_k$ are given.

This is the inverse to the direct problem of finding the terms of the sequence with the given rule.

The origin of theory of inverse problems may be found in $19^{th}$and 20^{th} centuries. They include the problems of equilibrium figures for the rotating fluid, the kinematic problems in seismology, the inverse Sturm-Liuville problem, etc. Newton's problem of discovering forces making planets move in accordence with Kepler's laws was one of the first inverse problems in dynamics of mechanical sytems which was solved [32].

The theory of inverse problems for differential equations is being developed to solve problems of mathematical physics. In the study of direct problems, the solution of a given differential equation is derived by means of supplementary conditions. Whereas, in the inverse problems, the form of the equation is known but the equation is not known exactly. To determine the corresponding equation and its solution, some additional conditions, which are not given for the direct problem, must be imposed. The following example [32] will be helpful to understand the nature of inverse problems more clearly. Let us consider the non-homogeneous heat equation, where $u(x,t)$

$$u_t(x,t) = u_{xx}(x,t) + f(x), \quad 0 < x < \pi, \quad 0 < t < T \tag{1.1}$$

$$u(x,0) = 0, \qquad 0 < x < \pi, \tag{1.2}$$

$$u(0,t) = u(\pi,t) = 0 \qquad 0 \le t \le T \tag{1.3}$$

is to be determined. This is a well known example of an initial-boundary value problem. If $f(x)$ in (1.1) is specified as a square integrable function in $L^2(0,\pi)$, it is a direct problem whose solution $u(x,t) \in W_{2,0}^{2,1}((0,\pi)\times(0,T))$ may be obtained uniquely, for every T, $\quad T > 0$.

Assume that we want to find $u(x,t)$ and $f(x)$ satisfying (1.1)-(1.3). Now, the new problem is closely related to the direct problem, but it is clearly different. So the problem of finding $u(x,t)$ and $f(x)$ is an inverse problem of (1.1)-(1.3). We may observe here that, we have one more unknown, namely $f(x)$ in the case of inverse problem. To find a unique solution we need additional condition. To determine $u(x,t)$ and $f(x)$ satisfying (1.1)-(1.3) in which $f(x)$ appears as another unknown function. We impose the additional information

$$u(x,T) = \varphi(x), \qquad \varphi \in W_2^2(0,\pi). \tag{1.4}$$

to find a unique solution [32]. First, consider the solution

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \int_0^t f_k exp\{-k^2(t-\tau)\} d\tau \sin kx & (1.5) \\ &= \sum_{k=1}^{\infty} f_k k^{-2}[1 - exp\{-k^2 t\}] \sin kx, & (1.6) \end{aligned}$$

of the direct problem, which can be obtained by Fourier method, where $f_k = \frac{2}{\pi}\int_0^\pi f(x)\, \sin kx dx$ are the Fourier coefficients of the known function $f(x) \in L^2(0,\pi)$. But in the case of the inverse problem, $f(x)$ is not known and therefore, to determine the Fourier coefficients f_k becomes an important task. For this purpose, replacing t by T in (1.6) and using (1.4), we get

$$\varphi(x) = \sum_{k=1}^{\infty} f_k k^{-2}(1 - exp\{-k^2 T\}) \sin kx. \tag{1.7}$$

Then, (1.7) gives the Fourier-sine expansion of φ, which implies that

$$\varphi_k = \frac{2}{\pi}\int_0^\pi \varphi(x) \sin(kx) dx = f_k k^{-2}(1 - exp\{-k^2 T\})$$

and that

$$f_k = k^2(1 - exp\{-k^2 T\})^{-1} \varphi_k. \tag{1.8}$$

Hence, $f(x)$ is expressible as

$$f(x) = \sum_{k=0}^{\infty} k^2(1 - exp\{-k^2 T\})^{-1} \varphi_k \sin kx \tag{1.9}$$

which leads formally to

$$u(x,t) = \sum_{k=1}^{\infty}(1 - exp\{-k^2T\}^{-1}(1 - exp\{-k^2t\})\varphi_k \sin kx \tag{1.10}$$

on the substitution of (1.8) into (1.6). The series expansion (1.9) of $f(x)$ needs a convergence analysis. Since the system $\{\sin kx\}_{k=1}^{\infty}$ is complete in $L^2(0,\pi)$, it is known from Parseval's identity that

$$||f||_{2,\Omega}^2 = \sum_{k=1}^{\infty} \frac{2}{\pi} k^4 (1 - exp\{-k^2T\})^{-2}\varphi_k^2$$

implying the inequality

$$\sum_{k=1}^{\infty} \frac{2}{\pi} k^4 (1 - exp\{-k^2T\})^{-2}\varphi_k^2 \leq \frac{2}{\pi}(1 - exp\{-T\})^{-2} \sum_{k=1}^{\infty} k^4\varphi_k^2. \tag{1.11}$$

Obviously, the boundedness of the sum

$$\sum_{k=1}^{\infty} k^4\varphi_k^2 < \infty.$$

is required for the existence of $f(x)$. Thus, we have just deduced that the existence of a solution of this inverse problem depends on the proper choice of φ in (1.4).

A generalization of the problem in (1.1) - (1.4) may be given as follows: Find the functions $u \in W_{2,0}^{2,1}(Q_T)$ and $f \in L^2(\Omega)$ satisfying

$$u_t(x,t) - (Lu)(x,t) = f(x)h(x,t), \qquad (x,t) \in Q_T \tag{1.12}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{1.13}$$

$$u(x,t) = 0, \qquad (x,t) \in S_T, \tag{1.14}$$

subject to the final overdetermination condition

$$u(x,T) = \varphi(x), \qquad x \in \Omega, \tag{1.15}$$

where L is a given uniformly elliptic operator; h, φ are given functions, $\Omega \subset \mathbb{R}^n$ and $Q_T = \Omega \times (0,T)$. If $f(x) \in L^2(\Omega)$ is known, then (1.12) - (1.14) will be a direct problem and it has a unique solution $u \in W_{2,0}^{2,1}(Q_T)$. If $f(x)$ is also unknown, it is an inverse problem and we may solve it by using the condition (1.15). To show the existence and uniqueness of the solution of (1.12)-(1.15), we convert the problem to an operator equation. If we substitute t by T in (1.12) and use (1.15), we get

$$u_t(x,T) - (L\varphi)(x) = f(x)h(x,T). \tag{1.16}$$

Solving (1.16) for f we obtain

$$f(x) = \frac{1}{h(x,T)}[u_t(x,T) - (L\varphi)(x)]. \tag{1.17}$$

This may be written in the form

$$f = Af + \Psi \tag{1.18}$$

by defining

$$(Af)(x) := \frac{1}{h(x,T)}[u_t(x,T)], \quad x \in \Omega \tag{1.19}$$

and

$$\Psi(x) := \frac{-1}{h(x,T)}(L\varphi)(x), \quad x \in \Omega. \tag{1.20}$$

In this particular case, if h is choosen so that $|h(x,T)| \geq \delta > 0$ for $x \in \overline{\Omega}$, it is easy to see that

$$A : L_2(\Omega) \to L_2(\Omega).$$

Thus, we have corresponded the operator equation (1.18) to the problem(1.12)-(1.15). That is if we can solve (1.18) for f, then we may substitute this function in (1.12)-(1.14) to get a direct problem which has a unique solution. But let us note that f should be chosen so that u satisfies the final overdetermination condition (1.15). The uniqueness of (u, f) is carried out in the usual way, for example see [32].

Inverse problems in partial differential equations may be classfied according to the underlying partial differential equations, namely, inverse elliptic problems, inverse hyperbolic problems and inverse parabolic problems [32].

Inverse elliptic problems are investigated for the coefficient identification [6, 10] and the boundary identification [27].

Inverse hyperbolic problems are studied to determine the coefficients [2, 11, 24], the source terms [28, 41] and memory kernels appearing in the equations [24]. Likewise , there are many studies on inverse parabolic problems to identify the coefficients [1, 4, 20, 30, 38], the source terms [3, 12, 29], the memory kernels [5, 7, 8] and the boundary of the domain [19, 37].

Past history of most models leading to parabolic differential equations containing memory term is represented by some integrals. So, solving inverse problems in order to determine the kernel of these integrals are meaningful jobs. Since , we have studied an inverse problem with memory in our thesis, we mention some of the articles in detail up to an extend. The first study we will summarize is due to Colombo and Lorenzi [7] on the memory kernel depending on time and space variables.

They have dealt with identification problems related to open bounded sets of cylindrical domains $\quad \Omega = \Omega_1 \times \Omega_2, \quad \Omega_1$ and Ω_2 being, some intervals in $\mathbb{R}$

and a smooth enough domain in $\mathbb{R}^n$. They studied the existence and uniqueness of

$$u : [0,T] \times \Omega \to \mathbb{R} \ \text{ and } \ h : [0,T] \times \Omega_2 \to \mathbb{R}$$

satisfying

$$\begin{aligned} D_t u(t,x,y) &= B_1(x,D_x)u(t,x,y) + B_2(y,D_y)u(t,x,y) \\ &+ \int_0^t h(t-s,x)[B_1(x,D_x)u(s,x,y) \\ &+ B_2(y,D_y)u(s,x,y)]ds + f(t,x,y) \quad (1.21) \end{aligned}$$

for $\quad (t,x,y) \in [0,T] \times \Omega_1 \times \Omega_2,$

subject to

$$u(0,x,y) = u_0(x,y), \qquad (x,y) \in \Omega_1 \times \Omega_2 \qquad (1.22)$$

$$u(t,x,y) = 0, \qquad t \in [0,T], \qquad (x,y) \in (\partial\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \partial\Omega_2) \qquad (1.23)$$

and the final overdetermination condition

$$\int_{\Omega_2} \phi(y)u(t,x,y)dy = g(t,x), \quad (t,x) \in [0,T] \times \Omega_1, \qquad (1.24)$$

where

$$f : [0,T] \times \Omega \to \mathbb{R}, \quad u_0 : \Omega \to \mathbb{R}, \quad g : [0,T] \times \Omega_1 \to \mathbb{R}$$

and $\phi : \Omega_2 \to \mathbb{R}$ are known functions.

The corresponding abstract version of the above problem studied by Colombo and Lorenzi is determining $u : [0,T] \to X$ and $H : [0;T] \to \mathcal{L}(X)$ satisfying

$$u'(t) = (B_1 + B_2)u(t) + \int_0^t H(t-s)(B_1 + B_2)u(s)ds + f(t), \quad t \in [0,T] \quad (1.25)$$

$$u(0) = u_0, \qquad (1.26)$$

and the final overdetermination condition

$$\Phi(u(t)) = G(t), \qquad t \in [0,T]. \tag{1.27}$$

They have assumed that $f : [0,T] \to X$ is a known function, $u_0 \in X$ is a given element, $B_1 : \mathcal{D}(B_1) \subset X \to X$, $B_2 : \mathcal{D}(B_2) \subset X \to X$ are closed linear operators, Φ is a known bounded linear operator in $\mathcal{L}(X;\mathcal{L}(X))$ and $G : [0,T] \to \mathcal{L}(X)$ is a given operator valued function. They proved that (1.25) - (1.27) has a unique solution $(u,H) \in U^{2+\sigma,p}(B_2, B_1+B_2) \times W^{\sigma,p}((0,T);K)$, where $U^{s,p}(B_2, B_1+B_2) = [W^{s,p}((0,T);\mathcal{D}(B_2)) \cap W^{s-1,p}(0,T);\mathcal{D}((B_1+B_2-\mu I)(B_2-\mu I))]$ and K is a subalgebra of $\mathcal{L}(X) \cap \mathcal{L}(\mathcal{D}(B_1) \cap \mathcal{D}(B_2))$.

We will mention the article by Favini and Lorenzi [15] as the second example for the inverse problem to determine memory kernel k depending on time variable and the underlying equation is a singular integro-differential equation. The memory kernel k satisfies the following integro-differential equation in the complex Banach space X

$$Mu'(t,x) + Lu(t,x) = \int_0^t k(t-s)L_1 u(s,x)ds + f(t), \quad 0 \le t \le \tau \tag{1.28}$$

$$u(0) = u_0 \tag{1.29}$$

and the final overdetermination condition

$$\Phi[Mu(t,x)] = g(t), \quad 0 \le t \le \tau \tag{1.30}$$

where Φ is a given linear continuous functional and

$$f \in C^1([0,\tau];X), \quad g \in C^2([0,\tau];\mathbb{R}), \quad u_0 \in D(L) \tag{1.31}$$

are known functions. Favini and Lorenzi proved, under suitable conditions that the problem (1.28)-(1.30) has a unique solution (u, k) in $C^1([0,\tau]; D(L)) \times C([0,\tau])$, if L, L_1, M are closed linear operators from X into itself (M is not necessarily invertible) and the domains of L, L_1, M satisfy $D(L) \subseteq D(L_1) \cap D(M)$,

The inverse problem we will study in this thesis is based on the direct problem given by Giorgi, Pata and Marzocchi [16]. The differential equations appearing in our inverse problem is different from the problems given in [7, 15]. They have employed the semigroup theory and smooth functions in their investigations, but we discuss our problem in Sobolev spaces. Naturally, we need some other techniques to derive our results. In [16], the authors studied the problem satisfying

$$\theta_t(x,t) - k_0\Delta\theta(x,t) + \int_{-\infty}^{t} k(t-s)\Delta\theta(x,s)ds + g(\theta) = h(x,t), \Omega \times \mathbb{R}^+ \quad (1.32)$$

$$\theta(x,t) = 0 \qquad x \in \partial\Omega, \quad t > 0 \quad (1.33)$$

$$\theta(x,0) = \theta_0(x) \quad x \in \Omega, \quad (1.34)$$

where $\Omega \subset \mathbb{R}^n$, $\theta : \Omega \times \mathbb{R} \to \mathbb{R}$ is the temperature variation field relative to equilibrium reference value, $k : \mathbb{R}^+ \to \mathbb{R}$ is heat flux memory kernel and k_0 is instanteneous conductivity. (1.32) may be written as

$$\theta_t(x,t) - k_0\Delta\theta(x,t) + \int_0^t k(t-s)\Delta\theta(x,s)ds + g(\theta) = h(x,t) - \int_{-\infty}^0 k(t-s)\Delta\theta(x,s)ds.$$

The presence of the term $h(x,t) - \int_{-\infty}^0 k(t-s)\Delta\theta(x,s)ds$ in the above equation causes that the system (1.32)-(1.34) is non-autonomous. Since the system is non-autonomous, the family of operators mapping the initial value θ_0 into the solution $\theta(x,t)$ of (1.32)-(1.34) does not match the usual semigroup properties. For this reason, a different formulation is given. In order to get this new form, they define new variables

$$\theta^t(x,s) = \theta(x,t-s), \qquad s \geq 0$$

and

$$\eta^t(x,s) = \int_0^s \theta^t(x,\tau)d\tau = \int_{t-s}^t \theta(x,\tau)d\tau, \quad s \geq 0.$$

Assuming $k(\infty) = 0$, change of variables and integration by parts yield

$$\int_{-\infty}^t k(t-s)\Delta\theta(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds. \tag{1.35}$$

Defining $\mu(s) = -k'(s)$ and using (1.35), it is possible to write (1.32)-(1.34) in the following form

$$\begin{aligned}
\theta_t(x,t) - k_0\Delta\theta(x,t) &- \int_0^\infty \mu(s)\Delta\eta^t(x,s)ds + g(\theta) = h(x,t), \quad \Omega\times\mathbb{R}^+ & (1.36)\\
\eta_t^t(x,s) &= \theta(x,t) - \frac{\partial}{\partial s}\eta^t(x,s) \text{ on } \Omega\times\mathbb{R}^+\times\mathbb{R}^+, & (1.37)\\
\theta(x,t) &= \eta^t(x,s) = 0, \quad x\in\partial\Omega \quad t,s>0, & (1.38)\\
\theta(x,0) &= \theta_0(x), \quad x\in\Omega, & (1.39)\\
\eta^0(x,s) &= \eta_0(x,s) \quad x\in\Omega,\ s>0, & (1.40)
\end{aligned}$$

where, the term $\eta^0(x,s) = \int_{-s}^0 \theta(x,\tau)d\tau$ is the initial integrated past history of θ and assumed to vanish on $\partial\Omega$.

When h is independent of time, the new system (1.36)-(1.40) is autonomous dynamical system with respect to the unknown pair $(\theta(x,t),\eta^t(x,s))$. Hence, the asymptotic behaviour can be studied by the methods in the framework of the semigroup theory. Therefore, Giorgi, Pata and Marzocchi restricted themselves to (1.36)-(1.40). For simplicity, they assumed that the nonlinear part of the heat supply $g : \mathbb{R} \to \mathbb{R}$ as a polynomial of odd degree with positive leading coefficient, i.e., g is of the form

$$g(\theta) = \sum_{k=1}^{2p} g_{2p-k}\theta^{k-1}, \quad g_0 > 0, \quad p \in \mathbb{N}. \tag{1.41}$$

The authors also supposed that, the constant k_0 and the term μ in (1.36) satisfy the following hypothesis:

(C_1) $\quad k_0 > 0,$

(C_2) $\quad \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu(s) > 0,\ \mu'(s) \leq 0, \qquad \forall s \in \mathbb{R}^+,$

(C_3) $\quad \mu'(s) + \delta\mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+$ and some $\delta > 0$.

By(C_2) and (C_3), it is easy to see that $\mu(s)$ is allowed to have the form

$$\mu(s) = \frac{e^{-\delta s}}{s^\gamma}, \qquad 0 \leq \gamma < 1.$$

It is possible to write (1.36)-(1.40) in a compact form. First we denote

$$z(x,t) = (\theta(x,t), \eta^t(x,s)),$$

$$z_0 = (\theta_0, \eta_0)$$

and set

$$\mathcal{L}z = (k_0\Delta\theta + \int_0^\infty \mu(s)\Delta\eta^t(s)ds, \quad \theta - \eta') \tag{1.42}$$

and

$$\mathcal{G}(z) = (h - g(\theta), 0), \tag{1.43}$$

then the problem (1.36) - (1.40) takes the form

$$z_t = \mathcal{L}z + \mathcal{G}(z), \tag{1.44}$$

$$z(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{1.45}$$

$$z(x,0) = z_0. \tag{1.46}$$

In this article, the authors proved that if $h(x,t) \in L^2(\mathbb{R}^+, L^2(\Omega)), (1.41)$ and $(C_1)-(C_2)$ are satisfied $\quad z_0 = (\theta_0, \eta_0) \in \mathcal{H}$, where $\mathcal{H} := L^2(\Omega) \times L^2_\mu(\mathbb{R}^+, H_0^+)$, then there exists a unique function $z = (\theta, \eta^t)$ with

$$\theta \in L^\infty([0,T], L^2) \cap L^2([0,T], H_0^1) \cap L^{2p}([0,T], L^{2p}), \quad \forall T > 0 \tag{1.47}$$

$$\eta \in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1)) \quad \forall T > 0 \tag{1.48}$$

such that

$$z_t = \mathcal{L}z + \mathcal{G}(z) \tag{1.49}$$

in the weak sense and

$$z(x,t)|_{t=0} = z_0.$$

They also showed that

$$z \in C([0,T], \mathcal{H}), \qquad \forall T > 0$$

and the mapping

$$z_0 \longmapsto z(t) \in C(\mathcal{H}, \mathcal{H}), \qquad \forall t \in [0,T].$$

Furthermore, if $z_0 \in \mathcal{V} := H_0^1(\Omega) \times L^2_\mu(\mathbb{R}^+, H^2 \cap H_0^1)$, then it is proved that

$$\theta \in L^\infty([0,T],H_0^1)\cap L^2([0,T],H^2\cap H_0^1)\cap L^{2p}([0,T],L^{2p}), \quad \forall T>0 \quad (1.50)$$

$$\eta \in L^\infty([0,T],L^2_\mu(\mathbb{R}^+,H^2\cap H_0^1)) \quad \forall T>0 \qquad (1.51)$$

and

$$z\in C([0,T],\mathcal{V}), \qquad \forall T>0.$$

In the third chapter of this article, they have shown that the solution of the problem (1.43)-(1.45) has absorbing sets in $\mathcal{H}$ and $\mathcal{V}$, i.e.

$$\lim_{t\to\infty}||z(t)||^2_{\mathcal{H}}=\lim_{t\to\infty}(||\theta(t)||^2+\int_0^\infty \mu(s)||\nabla\eta^t(s)||^2ds)\le \rho^2_{\mathcal{H}}.$$

and

$$\lim_{t\to\infty}||z(t)||^2_{\mathcal{V}}=\lim_{t\to\infty}(||\nabla\theta(t)||^2+\int_0^\infty \mu(s)||\Delta\eta^t(s)||^2ds)\le \rho^2_{\mathcal{V}}.$$

The outline of this thesis is the following.
In Chapter 1, some definitions, basic facts and the functional inequalities which are used during this thesis are given.

In Chapter 2, we prove that the problem (1.44)-(1.46) when $g(\theta)$ is replaced by $p(x)\theta$ has a unique solution $z=(\theta,\eta)$ such that

$$\begin{aligned}\theta &\in L^\infty([0,T],L^2(\Omega))\cap L^2([0,T],L^2(\Omega)\cap H_0^1(\Omega)), \qquad \forall T>0\\ \eta &\in L^\infty([0,T],L^2_\mu(\mathbb{R}^+,H_0^1(\Omega))), \qquad \forall T>0.\end{aligned}$$

In Section 2.3, we show that the solution of the above problem continuously depends on the fuction p and the memory kernel μ.

Chapter 3 is devoted to define an inverse problem to identify the memory kernel μ appearing in the equation (1.44)-(1.46) when $g(\theta)$ is replaced by $p(x)\theta$. We prove that the corresponding inverse problem has a unique solution (z, μ).

Chapter 4 includes two inverse problems for coefficient identification of the above problem when $p(x) = 1$. In section 4.2, we show the inverse problem of recovering the evolution oh the source term of the form $h = H(t)M_(x, t)$ has a unique solution (z, H). In sections 4.3 and 4.4, we prove that the inverse problem of recovering a source term of the form $h = K(x)M_(x, t)$with two different final overdetermination conditions has a unique solution (z, K).

1.2 Some definitions and basic facts

In this section, we will give some definitions and auxiliary facts that will be used.

Definition 1.1. *Let Ω be a domain in $\mathbb{R}^n$ and $p \geq 1$ be a real number. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω for which*

$$\int_\Omega |u(x)|^p dx < \infty \qquad \textit{if} \qquad 1 \leq p < \infty$$

and

$$\sup_\Omega |u(x)| < \infty, \qquad a.e. \quad x \in \Omega \qquad \textit{if} \qquad p = \infty.$$

$L^p(\Omega)$ is a Banach space with the norm

$$||u||_p := \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty \qquad \text{if} \qquad 1 \leq p < \infty$$

and

$$||u||_\infty := \operatorname{ess\,sup}_\Omega \quad |u(x)| \quad \text{if} \quad p = \infty,$$

where

$$\operatorname{ess\,sup}_\Omega |u(x)| := \inf\{M : |u(x)| \leq M, a.e. \quad x \in \Omega\}.$$

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) := \int_\Omega u(x)v(x)dx, \qquad u, v \in L^2(\Omega).$$

Definition 1.2. *A sequence $\{x_n\}$ in a normed space X is said to be strongly convergent (or convergent in the norm) if there exists an $x \in X$ such that*

$$\lim_{n\to\infty} ||x_n - x||_X = 0.$$

Definition 1.3. *Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with the norm defined by*

$$||f||_X = \sup_{\substack{x \in X \\ ||x||=1}} |f(x)|$$

which is called the dual space of X and is denoted by X^*.

Definition 1.4. *A sequence $\{x_n\}$ in a normed space X is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X^*$,*

$$\lim_{n \to \infty} f(x_n) = f(x).$$

Definition 1.5. *Let $\{f_n\}$ be a sequence of bounded linear functionals in a normed space X^*. Then weak* convergence of $\{f_n\}$ means that there is an $f \in X^*$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$.*

For simplicity, we indicate $x_n \to x$ for convergence in norm, $x_n \rightharpoonup x$ for weak convergence and $x_n \overset{*}{\rightharpoonup}$ for weak* convergence.

Definition 1.6. *Let X be a Banach Space and X^* be its dual space. If $u \in X$ and $u^* \in X^*$, then we write $\langle u^*, u \rangle$ to define the real number $u^*(u)$. The symbol $\langle \cdot, \cdot \rangle$ denotes the pairing of X^* and X.*

Definition 1.7. *Let X denote a real Banach space with the norm $|| \cdot ||_X$. The space $L^p((0,T), X)$ consists of all measurable functions $u : (0,T) \to X$ with*

$$||u||_{L^p((0,T),X)} \equiv \left(\int_0^T ||u(\cdot, t)||_X^p dt \right)^{1/p} < \infty \quad if \quad 1 \le p < \infty$$

and

$$||u||_{L^\infty((0,T),X)} \equiv ess \sup_{0 \le t \le T} ||u(\cdot, t)||_X < \infty \quad if \quad p = \infty.$$

In this thesis, $||.||$ denotes the L^2 norm of the given function, on the given set.

Definition 1.8. *The space $C([0,T];X)$ consists of all continuous functions $u:[0,T]\to X$ such that*

$$||u||_{C([0,T];X)} \equiv \max_{0\le t\le T} ||u(\cdot,t)||_X < \infty.$$

Definition 1.9. *The support of a function u defined on Ω is the closure of the set of points where $u(x)$ is nonzero.*

Definition 1.10. *Let Ω be a non-empty open set in $\mathbb{R}^n$. A function f defined on Ω is called a test function if $f\in C^\infty(\Omega)$ and there is a compact set $K\subset\Omega$ such that the support of f lies in K. The set of all test functions on Ω is denoted by $\mathcal{D}(\Omega)=C_0^\infty(\Omega)$.*

Let α be an n-tuple of non-negative integers α_i, we set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The n-tuple α is called a multi-index.

Definition 1.11. *A distribution F is a linear mapping $F: C_0^\infty(\Omega)\to\mathbf{R}$ such that $F(v_j)\to 0$ for every sequence $\{v_j\}\subset C_0^\infty(\Omega)$ with support in a fixed compact set $K\subset\Omega$ and whose derivatives $D^\alpha v_j \to 0$ uniformly in K, as $j\to\infty$. If F and F_j are distribution in Ω, then $F_j\to F$ as distributions provided $F_j(v)\to F(v)$ for every $v\in C_0^\infty(\Omega)$. The support of a distribution F in Ω is the smallest (relatively) closed set $K\subset\Omega$ such that $F(v)=0$ whenever*

$v \in C_0^\infty(\Omega \setminus K)$.

The set of all distributions on $\mathcal{D}(\Omega)$ is denoted by $\mathcal{D}'(\Omega)$.

Definition 1.12. *Given a real (measurable) function $u \in \Omega$, we will write*

$$u \in L^p_{loc}(\Omega)$$

to mean $u \in L^p(\Omega')$ for any bounded domain Ω' with $(\overline{\Omega'}) \subset \Omega$.

Definition 1.13. *Suppose that $u, v \in L^1_{loc}(\Omega)$ and α is a multi-index. We say that v is the α^{th} weak or distributional partial derivative of u and write*

$$D^\alpha u = v,$$

provided that

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v \varphi dx$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

Definition 1.14. *Let $u \in L^1((0,T), X)$. We say $v \in L^1((0,T), X)$ is the weak derivative of u and write*

$$u' = v$$

provided that

$$\int_0^T u(t)\phi'(t) = -\int_0^T v(t)\phi(t)dt$$

for all scalar functions $\phi \in C_0^\infty(0,T)$.

Definition 1.15. *Given $u \in L^p((0,T), X)$, the function $v(t)$ with values in $X_0 \supset X$ is called the derivative of $u(t)$ in the distributional sense and is denoted by*

$$v(t) = \partial_t u(t),$$

if

$$\int_0^T \langle v(t), \phi(t) \rangle dt = -\int_0^T \langle u(t), \partial_t \phi(t) \rangle dt$$

for all $\phi \in C_0^\infty([0,T], X_0^*)$ *where* X_0^* *is the dual space of* X_0 *and* $\langle \cdot, \cdot \rangle$ *is the duality pairing between* X_0 *and* X_0^*. $C_0^\infty((0,T), X_0)$ *is the space of functions from* $C^\infty((0,T), X)$ *with compact support.*

Definition 1.16. *(The Sobolev Space) Let* k *be a non-negative integer and let* $1 \leq p \leq \infty$. *The Sobolev space* $W^{k,p}(\Omega)$ *is defined by*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) | \mathrm{D}^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}.$$

In $W^{k,p}(\Omega)$, we define a norm by

$$||u||_{k,p} := \left(\sum_{|\alpha| \leq k} ||D^\alpha u||_p^p \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty$$

and

$$||u||_{k,\infty} := \max_{0 \leq |\alpha| \leq k} ||D^\alpha u||_\infty \quad \text{if} \quad p = \infty.$$

For $p = 2$, we define an inner product by

$$(u, v)_k := \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

We also use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and $L^2(\Omega)$ for $W^{0,2}(\Omega)$.

Definition 1.17. *By* $W_0^{k,p}(\Omega)$, *we denote the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$.

Because of the definition 1.17, $u \in W_0^{k,p}(\Omega)$ if and only if there is a sequence of functions $u_m \in C_0^\infty(\Omega)$ such that $u_m \to u$ in $W^{k,p}(\Omega)$.

Definition 1.18. *Let* $1 < p, q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the dual space of* $W_0^{k,p}(\Omega)$ *is denoted by* $W^{-k,q}(\Omega)$.

$W^{-k,q}(\Omega)$ is the Banach space with the norm

$$||u||_{-k,q} := \sup\{|\langle u, v\rangle : v \in W_0^{k,p}(\Omega),\ ||v||_{k,p} \leq 1\}$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between $W^{-k,q}(\Omega)$ and $W_0^{k,p}(\Omega)$.

Definition 1.19. *Let X and Y be normed spaces such that $X \subset Y$. Then we say that, X is continuously imbedded into Y if there exists a positive constant $C > 0$ such that $||u||_Y \leq C||u||_X$, for all $u \in X$. Furthermore, we say that X is compactly imbedded into Y if it is continuously imbedded and each bounded sequence in X is precompact in Y.*

Theorem 1.20. *(Sobolev's inequality) [13] Assume that Ω is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$. Then we have the estimate*

$$||u||_q \leq C||Du||_p$$

for $1 \leq q \leq \frac{np}{n-p}$, the constant C depends on p, q, n and Ω.

Theorem 1.21. *(Rellich-Kondrasov) [25] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1. Then the following embeddings are compact.*

(i) if $p < n$, $\quad W^{1,p}(\Omega) \to L^q(\Omega), \quad 1 \leq q < \frac{np}{n-p}$,

(ii) if $p = n$, $\quad W^{1,n}(\Omega) \to L^q(\Omega), \quad 1 \leq q < \infty$,

(iii) if $p > n$, $\quad W^{1,p}(\Omega) \to C(\overline{\Omega})$.

If Ω is any bounded domain, the above mentioned theorem is valid for $W_0^{1,p}(\Omega)$.

Definition 1.22. [33] *For $0 < p < \infty$, $\quad H^p(\Pi)$ is the class of holomorphic functions F on Π such that*

$$||F||_p = \sup_{y>0} \left(\int_{-\infty}^{\infty} (F(x+iy)|^p dx \right)^{1/p} < \infty,$$

where $\Pi := \{z : Imz > 0\}$, *where* $z = x + iy, \quad x, y \in \mathbb{R}^n$. *Here* H *denotes the Hardy class.*

If $p \geq 1$, then $H^p(\Pi)$ is a Banach space with the above norm. Moreover, $H^2(\Pi)$ is an Hilbert space. The inner product in $H^2(\Pi)$ is given by

$$\langle F, G\rangle_2 = \int_{-\infty}^{\infty} F(x)\overline{G}(x)dx.$$

$F(z)$ and $G(z)$ are any functions in $H^2(\Pi)$ and $F(x)$ and $G(x)$ are their boundary functions.

The functions

$$\left\{\frac{\pi^{-1/2}}{z+i}\left(\frac{z-i}{z+i}\right)^n\right\}_0^\infty$$

form an orthonormal basis for $H^2(\Pi)$.

Definition 1.23. (Paley-Wiener Representation)[33] *For any given* $f \in L^2(0, \infty)$, *define*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz} f(t)dt, \quad z \in \Pi.$$

Then the mapping $U : f \to F$ is an isometry from $L^2(0, \infty)$ onto $H^2(\Pi)$. Its inverse $U^{-1} : F \to f$ can be calculated in this way. Let $f \in L^2(0, \infty)$ and $F \in H^2(\Pi)$, and assume that $F = Uf$. If $F(t)$ is the boundary function of $F(z)$, then

$$l.i.m._{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{ixt} F(t)dt = \begin{cases} f(x) & , \quad x > 0, \\ 0 & , \quad x < 0. \end{cases}$$

Moreover, for any $y > 0$,

$$l.i.m._{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{ixt} F(t + iy)dt = \begin{cases} e^{-yx} f(x) & , \quad x > 0, \\ 0 & , \quad x < 0, \end{cases}$$

where l.i.m. stands for limit in the mean and indicates that convergence is in the metric of $L^2(0,\infty)$.

In this work, $\langle\cdot,\cdot\rangle_{2,m}$ and $||\cdot||_{2,m},\ m=1,2$ denote the inner product and norm of H_0^1 and $H^2\cap H_0^1$, respectively.

We use the notation $\langle\cdot,\cdot\rangle_{2,1}=\langle\nabla\cdot,\nabla\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{2,2}=\langle\Delta\cdot,\Delta\cdot\rangle$.

For $\mu\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+),\quad \mu(s)\geq 0,\quad \mu'(s)\leq 0,\quad \forall s\in\mathbb{R}^+,\quad L^2_\mu(\mathbb{R}^+,L^2)$ denotes the Hilbert Space of funtcions $\varphi:\mathbb{R}^+\to L^2(\Omega)$ endowed with the inner product

$$\langle\varphi_1,\varphi_2\rangle_\mu=\int_0^\infty \mu(s)\langle\varphi_1(s),\varphi_2(s)\rangle ds.$$

and $||\varphi||_\mu$ denotes the corresponding weighted norm. We introduce the following inner products $\langle\cdot,\cdot\rangle_{m,\mu}$ and corresponding norms $\langle\cdot,\cdot\rangle_{m,\mu}(m=1,2)$ on $L^2_\mu(\mathbb{R}^+,H_0^1)$ and $L^2_\mu(\mathbb{R}^+,H^2\cap H_0^1)$ as

$$\langle\cdot,\cdot\rangle_{1,\mu}=\langle\nabla\cdot,\nabla\cdot\rangle_\mu \quad\text{and}\quad \langle\cdot,\cdot\rangle_{2,\mu}=\langle\Delta\cdot,\Delta\cdot\rangle_\mu.$$

As in [16], we introduce the Hilbert Spaces

$$\mathcal{H}=L^2(\Omega)\times L^2_\mu(\mathbb{R}^+,H_0^1)$$

and

$$\mathcal{V}=H_0^1(\Omega)\times L^2_\mu(\mathbb{R}^+,H^2\cap H_0^1),$$

respectively endowed with the inner produets

$$\langle w_1,w_2\rangle_{\mathcal{H}}=\langle\psi_1,\psi_2\rangle+\langle\varphi_1,\varphi_2\rangle_{1,\mu}$$

and

$$\langle w_1,w_2\rangle_{\mathcal{V}}=\langle\psi_1,\psi_2\rangle_{2,1}+\langle\varphi_1,\varphi_2\rangle_{2,\mu}$$

where $w_i = (\psi_i, \varphi_i) \in \mathcal{H}$ or $\mathcal{V}$ for $i = 1, 2$.

The norm on $\mathcal{H}$ is

$$||(\psi, \varphi)||^2_{\mathcal{H}} = ||\psi||^2 + \int_0^\infty \mu(s)||\nabla\varphi(s)||^2 ds.$$

1.3 Inequalities

We will also make use of the following functional and algebraic inequalities in this thesis.

1) Cauchy's inequality with ϵ.

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \qquad \forall a, b > 0, \epsilon > 0.$$

2) Cauchy - Schwarz inequality. Let H be a Hilbert space associated with the inner product $(.,.)$ and norm $||u|| = (u, u)^{1/2}$. Then

$$|(u, v)| \leq ||u||.||v||, \qquad \forall u, v \in H.$$

3) Poincaré inequality

$$\lambda_0(\Omega)||v||^2 \leq ||\nabla v||^2 \qquad \lambda_0 > 0, \forall v \in H^1_0$$

and

$$\gamma_0(\Omega)||\nabla v||^2 \leq ||\Delta v||^2 \qquad \lambda_0 > 0, \forall v \in H^2 \cap H^1_0$$

4) Young's inequality with ϵ.

$$ab \leq \epsilon a^p + C(\epsilon)b^q \qquad (a, b > 0, \quad \epsilon > 0) \text{ for } C(\epsilon) = \frac{(\epsilon p)^{-q/p}}{q}.$$

CHAPTER 2

RELATED DIRECT PROBLEM

2.1 Introduction

In this chapter, we will introduce the direct problem whose related inverse problem will be studied in Chapter 3. We will assume a particular linear part $p(x)\theta(x,t), \quad p(x) \geq 0$, instead of the nonlinear part of heat supply $g : \mathbb{R} \to \mathbb{R}$ in (1.36). This form of the problem is slightly different from the problem in (1.36)-(1.40). For the completeness of the discussions, we need the existence and uniqueness of solutions the new problem defined by

$$\theta_t(x,t) - k_0\Delta\theta(x,t) - \int_0^\infty \mu(s)\Delta\eta^t(x,s)ds + p(x)\theta(x,t) = h(x,t), \Omega \times \mathbb{R}^+ \tag{2.1}$$

$$\eta_t^t(x,s) = \theta(x,t) - \frac{\partial}{\partial s}\eta^t(x,s) \qquad \text{on } \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \tag{2.2}$$

$$\theta(x,t) = \eta^t(x,s) = 0, \qquad x \in \partial\Omega, \qquad t, s > 0, \tag{2.3}$$

$$\theta(x,0) = \theta_0(x), \qquad x \in \Omega, \tag{2.4}$$

$$\eta^0(x,s) = \eta_0(x,s), \qquad x \in \Omega, \quad s > 0. \tag{2.5}$$

The term $\eta^0(x,s) = \int_{-s}^0 \theta(x,\tau)d\tau$ is the initial integrated past history of θ and assumed to vanish on $\partial\Omega$. We denote

$$z(x,t) = (\theta(x,t), \eta^t(x,s))$$

$$z_0 = (\theta_0, \eta_0)$$

and set

$$\begin{aligned}\mathcal{L}z &= (k_0\Delta\theta(x,t) + \int_0^\infty \mu(s)\Delta\eta^t(s)ds, \quad \theta(x,t) - \eta^{'}(x,t-s)), &(2.6)\\ \mathcal{G}(z) &= (h(x,t) - p(x)\theta(x,t), \quad 0). &(2.7)\end{aligned}$$

Thus, (2.1)-(2.5) takes the compact form

$$\begin{aligned} z_t(x,t) &= \mathcal{L}z + \mathcal{G}(z), &(2.8)\\ z(x,t) &= 0, \qquad x \in \partial\Omega, \qquad t > 0, &(2.9)\\ z(x,0) &= z_0, \qquad x \in \Omega. &(2.10)\end{aligned}$$

2.2 Existence and Uniqueness of Solution

Theorem 2.1. *Suppose that* $h \in L^2(\mathbb{R}^+, L^2(\Omega))$, $\quad p(x) \geq 0, \quad p \in L^\infty(\Omega)$ *and* $z_0 = (\theta_0, \eta_0) \in \mathcal{H}$. *Then there exists a unique function* $z = (\theta, \eta)$ *with*

$$\begin{aligned} \theta &\in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega)), \qquad \forall T > 0 &(2.11)\\ \eta &\in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1(\Omega))), \qquad \forall T > 0 &(2.12)\end{aligned}$$

such that

$$z_t = \mathcal{L}z + \mathcal{G}(z) \tag{2.13}$$

is satisfied in the weak sense, and $z|_{t=0} = z_0$.

If we assume that $z_0 \in \mathcal{V}$, *then*

$$\begin{aligned} \theta &\in L^\infty([0,T], H_0^1(\Omega)) \cap L^2([0,T)], H^2(\Omega) \cap H_0^1(\Omega)), \qquad \forall T > 0, &(2.14)\\ \eta &\in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega))), \qquad \forall T > 0. &(2.15)\end{aligned}$$

Proof. We will give the proof by the standart Faedo-Galerkin Method. To use this method, we need orthonormal bases of $L^2(\Omega)$ and $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$.

We recall that there exists a smooth orthonormal basis $\{w_i\}_{j=1}^\infty$ of $L^2(\Omega)$ which is orthogonal in $H^1_0(\Omega)$. We will take a complete set of normalized eigenfunctions ω_j for $-\Delta$ in $H^1_0(\Omega)$, that is $-\Delta\omega_j = \nu_j\omega_j$, where ν_j are the eigenvalues corresponding to the eigenfunctions ω_j. We will select an orthonormal basis $\{\zeta_j\}_{j=1}^\infty$ of $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ which also belongs to $\mathcal{D}(\mathbb{R}^+, H^1_0(\Omega))$. Here, we recall that $\mathcal{D}(I, X)$ is the space of infinitely many differentiable functions with compact support in $I \subset \mathbb{R}$. We will complete the proof in five steps.

Step 1 (Faedo-Galerkin Scheme). We fix a finite time interval $(0, T)$, $T > 0$. Given an integer n, denote by P_n, Q_n the projections of $H^1_0(\Omega)$ and $L^2_\mu(\mathbb{R}^+, H^1_0)$ on the subspaces
Span $\{w_1, \cdots, w_n\} \subset H^1_0(\Omega)$ and Span $\{\zeta_1, \cdots, \zeta_n\} \subset L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$
respectively. We want to find a function $z_n = (\theta_n, \eta_n)$ of the form

$$\theta_n(t) = \sum_{j=1}^n a_j^n(t)\omega_j \qquad \text{and} \qquad \eta_n^t(s) = \sum_{j=1}^n b_j^n(t)\zeta_j(s)$$

satisfying

$$\langle \partial_t z_n, (\omega_k, \zeta_i)\rangle_{\mathcal{H}} = \langle \mathcal{L} z_n, (\omega_k, \zeta_j)\rangle_{\mathcal{H}} + \langle \mathcal{G}(z_n), (\omega_k, \zeta_i)\rangle_{\mathcal{H}}.$$

subject to $z_n \mid_{t=0} = (P_n\theta_0, Q_n\eta_0)$, where $\theta_0 = \theta(0, x)$ and $\eta_0 = \eta^0(x, s)$. By (2.6) and (2.7), this equation may be written as

$$\begin{aligned} \langle \partial_t z_n, (\omega_k, \zeta_i)\rangle_{\mathcal{H}} &= \langle (k_0\Delta\theta_n + \int_0^\infty \mu(s)\Delta\eta^t(s)ds, \theta_n - \eta_n^{'}), (\omega_k, \zeta_i)\rangle_{\mathcal{H}} \\ &+ \langle (h - p(x)\theta_n, 0), (\omega_k, \zeta_i)\rangle \qquad (2.16) \\ z_n \mid_{t=0} &= (P_n\theta_0, Q_n\eta_0) \end{aligned}$$

for a.e. $t \le T$, for every $k, j = 0, \cdots, n,$. We denote the zero vectors in Span $\{w_1, \cdots, w_n\}$ and Span$\{\zeta_1, \cdots, \zeta_n\}$by ω_0 and ζ_0, respectively.

Taking (ω_k, ζ_0) and (ω_0, ζ_k) for (ω_k, ζ_i) in (2.16) we get

$$\frac{d}{dt} a_k^n = -k_0 \nu_k a_k^n + \langle \int_0^\infty \mu(s) \Delta \eta_n^t(s) ds, \omega_k \rangle + \langle h, \omega_k \rangle - \langle p(x)\theta_n, \omega_k \rangle \tag{2.17}$$

$$\frac{d}{dt} b_k^n = \sum_{j=1}^n a_j^n \langle \omega_j, \zeta_k \rangle_{1,\mu} - \sum_{j=1}^n b_j^n \langle \zeta_j', \zeta_k \rangle_{1,\mu}. \tag{2.18}$$

If we use the divergence theorem to the second term in the right hand side of (2.17), we get

$$\langle \int_0^\infty \mu(s) \Delta \eta_n^t(s) ds, \omega_k \rangle = \sum_{j=1}^n b_j^n \langle \zeta_j, \omega_k \rangle_{1,\mu}.$$

Employing this identity in (2.17), we end up with a system of ordinary differential equations with respect to the unknown functions $a_k^n(t)$ and $b_k^n(t)$, of the form

$$\frac{d}{dt} a_k^n = -k_0 \nu_k a_k^n + \langle h, \omega_k \rangle - \langle p(x)\theta_n, \omega_k \rangle - \sum_{j=1}^n b_j^n \langle \zeta_j, \omega_k \rangle_{1,\mu} \tag{2.19}$$

$$\frac{d}{dt} b_k^k = \sum_{j=1}^n a_j^n \langle \omega_j, \zeta_k \rangle_{1,\mu} - \sum_{j=1}^n b_j^n \langle \zeta_j', \zeta_k \rangle_{1,\mu} \tag{2.20}$$

subject to initial condition

$$a_k^n(0) = \langle \theta_0, \omega_k \rangle \tag{2.21}$$

$$b_k^n(0) = \langle \eta_0, \zeta_k \rangle_{1,\mu}. \tag{2.22}$$

According to standart existence theory for ordinary differential equations, we have a continuous solution of (2.19)-(2.22) on some interval $(0, T_n)$. we may observe here that in fact $T_n = +\infty$.

Step 2 (Energy estimates). Now, we will look for energy estimate for the sequence (θ_n, η_n).

We multiply the equation (2.19) by a_k and (2.20) by b_k we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(a_k^n)^2 &= -k_0\nu_k(a_k^n)^2 - \sum_{j=1}^{n} b_j^n\langle\zeta_j, \omega_k a_k^n\rangle_{1,\mu} + \langle h, \omega_k a_k^n\rangle \\ &- \langle p(x)\theta_n, \omega_k a_k\rangle \qquad (2.23)\\ \frac{1}{2}\frac{d}{dt}(b_k^n)^2 &= \sum_{j=1}^{n} a_j^n\langle\omega_j, \zeta_k b_k^n\rangle_{1,\mu} - \sum_{j=1}^{n} b_j^n\langle\zeta_j', \zeta_k b_k^n\rangle_{1,\mu}. \qquad (2.24)\end{aligned}$$

Since $\mathcal{H}$ is a Hilbert space, the paralellogram law holds. Thus, $||z_n||_{\mathcal{H}}^2 = \sum_{k=1}^{n}(a_k^n)^2 + \sum_{k=1}^{n}(b_k^n)^2$. Summing over k the equations in (2.23) and (2.24), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||z_n||_{\mathcal{H}}^2 &= \langle k_0\Delta\theta_n, \theta_n\rangle + \langle h, \theta_n\rangle - \langle p(x)_n, \theta_n\rangle - \langle\eta_n^t, \theta_n\rangle_{1,\mu} \\ &+ \langle\eta_n^t, \theta_n\rangle_{1,\mu} - \langle\eta_n'^t, \theta_n\rangle_{1,\mu}.\end{aligned}$$

This equation can be written as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||z_n||_{\mathcal{H}}^2 &= \langle(k_0\Delta\theta_n + \int_0^\infty \mu(s)\Delta\eta^t(s)ds,\ \ \theta_n - \eta_n'), (\theta_n, \eta_n)\rangle_{\mathcal{H}} \\ &+ \langle(h - p(x)\theta_n, 0), (\theta_n, \eta_n)\rangle_{\mathcal{H}} \qquad (2.25)\\ &= -k_0||\nabla\theta_n||_{L^2(\Omega)}^2 - \langle\eta_n', \eta_n\rangle_{1,\mu} + \langle h, \theta_n\rangle - \langle p(x)\theta_n, \theta_n\rangle. \qquad (2.26)\end{aligned}$$

The application of the Schwarz inequality in the third and the fourth term in the right hand side of (2.26) and the definition of the inner product in the sense of $L^2(\Omega)$ yields the estimate

$$\langle h, \theta_n\rangle - \langle p(x)\theta_n, \theta_n\rangle \leq ||h||||\theta_n|| - \int_\Omega p(x)|\theta_n|^2 dx. \qquad (2.27)$$

Using (2.27) in (2.26) we get the bound

$$\frac{1}{2}\frac{d}{dt}||z_n||^2_{\mathcal{H}} \le -k_0||\nabla\theta_n||^2 - \langle \eta_n', \eta_n\rangle_{1,\mu} + ||h||||\theta_n|| - \int_\Omega p(x)|\theta|^2 dx. \quad (2.28)$$

Integration by parts and the condition (C_2) on $\mu(s)$ given in Section 1.1 imply that, the second term in the right hand side of (2.28) satisfies that

$$\langle \eta_n', \eta_n\rangle_{1,\mu} = -\int_0^\infty \mu'(s)||\nabla\eta_n||^2 ds \ge 0. \quad (2.29)$$

Employing the inequality (2.29) in (2.28) and recalling that $p(x) \ge 0$, we get

$$\frac{1}{2}\frac{d}{dt}||z_n||^2_{\mathcal{H}} + k_0||\nabla\theta_n||^2 \le ||h||||\theta_n||. \quad (2.30)$$

The application of the Young's inequality with ϵ to the term at the right hand side of (2.30) yields the estimate

$$||h||||\theta_n|| \le \frac{1}{4\epsilon}||h||^2 + \epsilon||\theta_n||^2. \quad (2.31)$$

Using Poincaré inequality for $||\theta_n||^2$ in (2.31), we end up with

$$||h||||\theta_n|| \le \frac{1}{4\epsilon}||h||^2 + \epsilon\lambda_0||\nabla\theta_n||^2. \quad (2.32)$$

Substituting (2.32) in (2.30) we get

$$\frac{d}{dt}||z_n||^2_{\mathcal{H}} + 2(k_0 - \epsilon\lambda_0)||\nabla\theta_n||^2 \le \frac{1}{2\epsilon}||h||^2. \quad (2.33)$$

The integration of (2.33) over $(0,t)$ for $t \in (0,T)$, leads the estimate

$$||z_n||^2_{\mathcal{H}} + 2(k_0 - \epsilon\lambda_0)\int_0^t ||\nabla\theta_n(\tau)||^2 d_\tau \le ||z_0||^2_{\mathcal{H}} + \frac{1}{2\epsilon}||h||^2 T,$$

or

$$||\theta_n||^2 + \int_0^\infty \mu(s)||\nabla\eta_n||^2 ds \leq ||z_0||^2_{\mathcal{H}} + \frac{1}{2\epsilon}||h||^2 T.$$

The above inequality for gives that

$$\begin{aligned} \theta_n \quad &\text{is bounded in} \qquad L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega)) \qquad (2.34)\\ \eta_n \quad &\text{is bounded in} \qquad L^\infty([0,T], L^2_\mu((\mathbb{R}^+, H_0^1(\Omega))). \qquad (2.35)\end{aligned}$$

We observe that the bounds are independent of n as we have expected.

Step 3 (Passage to limit)(2.34) and (2.35) shows that there exists a subsequence of (θ_n, η_n), which will be denoted by the same indices, satisfying

$$\begin{aligned} \theta_n \rightharpoonup \theta \qquad &\text{weakly-star in} \qquad L^\infty([0,T], L^2(\Omega)) \qquad (2.36)\\ \theta_n \rightharpoonup \theta \qquad &\text{weakly in} \qquad L^2([0,T], H_0^1(\Omega)) \qquad (2.37)\\ \eta_n \rightharpoonup \eta \qquad &\text{weakly-star in} \qquad L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1(\Omega))). \qquad (2.38)\end{aligned}$$

For a fixed integer m, choose a function
$u = (\sigma, \xi) \in \mathcal{D}((0,T), H_0^1(\Omega)) \times \mathcal{D}((0,T), \mathcal{D}(\mathbb{R}^+, H_0^1(\Omega)))$ of the form

$$\sigma(t) = \sum_{j=1}^m \tilde{a}_j(t)\omega_j \qquad \text{and} \qquad \xi^t(s) = \sum_{j=1}^m \tilde{b}_j(t)\zeta_j(s)$$

where $\{\tilde{a}_j\}_{i=j}^m$ and $\{\tilde{b}_j\}_{i=j}^m$ are given functions in $\mathcal{D}((0,T))$. Then (2.16) holds with $(\sigma(t), \xi^t)$ replaced by (ω_k, ζ_j), i.e,

$$\begin{aligned} \langle \partial_t z_n, (\sigma(t), \xi^t)\rangle_{\mathcal{H}} \quad &= \quad \langle (k_0\Delta\theta_n + \int_0^\infty \mu(s)\Delta\eta^t(s)ds, \theta_n - \eta_n^{'})(\sigma(t), \xi^t)\rangle_{\mathcal{H}}\\ &+ \quad \langle (h - p(x)\theta_n, 0)(\sigma(t), \xi^t)\rangle_{\mathcal{H}}. \qquad (2.39)\end{aligned}$$

By the definition of the inner product in the sense of $\mathcal{H}$, by (2.39) we get

$$\begin{aligned} \langle \partial_t z_n, (\sigma(t), \xi^t) \rangle_{\mathcal{H}} &= \langle (k_0 \Delta\theta_n, \sigma(t) \rangle_{L^2(\Omega)} + \langle \int_0^\infty \mu(s) \Delta\eta_n^t(s) ds, \sigma(t) \rangle_{L^2(\Omega)} \\ &+ \langle \theta_n, \xi^t \rangle_{1,\mu} - \langle \eta_n^{'}, \xi^t \rangle_{1,\mu} + \langle h - p(x)\theta_n, \sigma(t) \rangle_{L^2(\Omega)}. \end{aligned} \tag{2.40}$$

If we use (2.36)-(2.38), we may pass to the limit as n tends to ∞ except for the term

$$\langle \eta_n^{'}, \xi^t \rangle_{1,\mu} = \int_0^\infty \mu(s) \langle \nabla\eta^{'}, \nabla\xi^t \rangle_{L^2(\Omega)} ds.$$

We denote $\langle\langle \cdot, \cdot \rangle\rangle$ the duality mapping, between $H_\mu^1(\mathbb{R}^+, H_0^1(\Omega))$ and its dual space. For $f \in H_\mu^1(\mathbb{R}^+, H_0^1(\Omega))$ and g in its dual space

$$\langle\langle f, g \rangle\rangle = \int_0^\infty \mu(s) \langle \nabla f, \nabla g \rangle_{L^2(\Omega)} ds.$$

Using this definition of the duality map, we observe that

$$\langle\langle \varphi^{'}, \xi \rangle\rangle = - \int_0^\infty \mu^{'}(s) \langle \nabla\varphi, \nabla\xi^t(s) \rangle_{L^2(\Omega)} ds - \int_0^\infty \mu(s) \langle \nabla\varphi, \nabla\xi^{'}(s) \rangle_{L^2(\Omega)} ds \tag{2.41}$$

for $\varphi \in H_\mu^1(\mathbb{R}^+, H_0^1(\Omega))$ and ξ is in its dual space. Indeed, the identity (2.41) may be adopted for every $\varphi \in L_\mu^2(\mathbb{R}^+, H_0^1(\Omega))$, if $\xi \in H_\mu^1(\mathbb{R}^+, H_0^1(\Omega)) \cap L^2(\frac{\mu'}{\mu})^2(\mathbb{R}^+, H_0^1(\Omega))$. Thus, we may compute the limit of $\langle \eta_n^{'}, \xi^t \rangle_{1,\mu}$ as $n \to \infty$, and we get

$$\partial_t z_n \to z_t \ \text{ in } \ \mathcal{D}'((0,T), H_0^1(\Omega)) \times \mathcal{D}'((0,T), \mathcal{D}(\mathbb{R}^+, H_0^1(\Omega))).$$

Integrating (2.39) over $(0,T)$ and passing to limits we get

$$\begin{aligned}\int_0^T \langle z, u_t\rangle_{\mathcal{H}} dt &= \int_0^T [k_0\langle \nabla\theta, \nabla\sigma\rangle + \langle \eta, \sigma\rangle_{1,\mu} - \langle \theta, \xi\rangle_{1,\mu} + \langle\langle \eta', \xi\rangle\rangle \\ &- \langle h, \sigma\rangle]dt + \int_0^T p(x)\theta\sigma dx dt. \end{aligned} \tag{2.42}$$

So the limit of the sequnce $z_n = (\theta_n, \eta_n)$ is the solution of our problem.

Step 4 (Uniqueness of the solution). Suppose that there are two functions $z^1 = (\theta_1, \eta_1)$ and $z^2 = (\theta_2, \eta_2)$ satisfying (2.2), i.e,

$$z_t^1 = (\theta_1, \eta_1)_t = (k_0\Delta\theta_1 + \int_0^\infty \mu(s)\Delta\eta_1^t(s)ds + h - p(x)\theta_1, \ \ \theta_1 - \eta_1') \tag{2.43}$$

and

$$z_t^2 = (\theta_2, \eta_2)_t = (k_0\Delta\theta_2 + \int_0^\infty \mu(s)\Delta\eta_2^t(s)ds + h - p(x)\theta_2, \ \ \theta_2 - \eta_2') \tag{2.44}$$

with the initial conditions $z_0^1 = z_0^2 = z_0$. If we set $\tilde{\theta} := \theta_1 - \theta_2$, $\tilde{\eta} := \eta_1 - \eta_2$ and $\tilde{z} := z^1 - z^2$, subtraction of (2.44) from (2.43) gives

$$\tilde{z}_t = (k_0\Delta\tilde{\theta} + \int_0^\infty \mu(s)\Delta\tilde{\eta} - p(x)\tilde{\theta}, \tilde{\theta} - \tilde{\eta}'). \tag{2.45}$$

The inner product of $\tilde{z}_t$ and $\tilde{z}$ in the sense of $\mathcal{H}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\tilde{z}||_{\mathcal{H}}^2 &= -k_0||\nabla\tilde{\theta}||^2 - \int_0^\infty \mu(s)\langle \nabla\tilde{\eta}, \nabla\tilde{\theta}\rangle ds - \langle p(x)\tilde{\theta}, \tilde{\theta}\rangle + \langle \nabla\tilde{\theta}, \nabla\tilde{\eta}\rangle_\mu \\ &- \int_0^\infty \mu(s)\frac{d}{ds}||\nabla\tilde{\eta}||^2 ds. \end{aligned} \tag{2.46}$$

We know that $\int_0^\infty \mu(s)\langle \nabla\tilde{\eta}, \nabla\tilde{\theta}\rangle ds = \langle \nabla\tilde{\theta}, \nabla\tilde{\eta}\rangle_\mu$. The cancellations of these terms in (2.46) gives

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}||_{\mathcal{H}}^2 = -k_0||\nabla\tilde{\theta}||^2 - \langle p(x)\tilde{\theta}, \tilde{\theta}\rangle - \int_0^\infty \mu(s)\frac{d}{ds}||\nabla\tilde{\eta}||^2 ds.$$

Keeping in mind that

$$-k_0||\nabla\tilde{\theta}||^2 \leq 0, \qquad -\langle p(x)\tilde{\theta}, \tilde{\theta}\rangle \leq 0$$

and

$$-\int_0^\infty \mu(s)\frac{d}{ds}||\nabla\tilde{\eta}||^2 ds = \int_0^\infty \mu^{'}(s)||\nabla\tilde{\eta}||^2 \leq 0,$$

we end up with the following inequality

$$\frac{d}{dt}||\tilde{z}||^2 \leq 0. \tag{2.47}$$

This differential inequality yields that

$$||\tilde{z}(t)||^2 \leq ||\tilde{z}(0)||^2. \tag{2.48}$$

Since $\tilde{z}(0) = z_0^1 - z_0^2 = 0$, (2.48) implies that $\tilde{z}(t)$ is identically zero, which implies uniqueness of the solution.

Step 5 (Further regularity). At this step, we investigate that, under some appropriate assumptions, the solution of the problem (2.1)-(2.5) is in a more smooth space. we write the equation (2.8) in the form

$$\theta_t(x,t) - k_0\Delta\theta(x,t) - \int_0^\infty \mu(s)\Delta\mu^t(s)ds + p(x)\theta(x,t) = h(x,t) \tag{2.49}$$

$$\eta_t^t(x,s) = \theta(x,t) - \eta_s^t(x,s). \tag{2.50}$$

Our aim is to find an upper bound for $||z||_{\mathcal{V}}^2$. For this end, we multiply (2.49) by $-\Delta\theta$ in $L^2(\Omega)$ and the Laplacian of (2.50) by $\Delta\eta$ in $L^2_\mu(\Omega)$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla\theta||^2 &+ k_0||\Delta\theta||^2 + \int_0^\infty \mu(s)\langle\Delta\eta^t, \Delta\theta\rangle_{L^2(\mu)} \\ &= \langle p(x)\theta - h, \Delta\theta\rangle_{L^2(\mu)} \qquad (2.51)\\ \frac{1}{2}\frac{d}{dt}||\Delta\eta^t||^2_\mu &= \langle\Delta\theta, \Delta\eta^t\rangle_\mu - \langle\Delta\eta^t_s, \Delta\eta\rangle_\mu. \qquad (2.52)\end{aligned}$$

Addition of (2.51) and (2.52) results in

$$\frac{d}{dt}||z||^2_\nu + 2k_0||\Delta\theta||^2 + 2\langle\eta', \eta\rangle_{2,\mu} = 2\langle p(x)\theta - h, \Delta\theta\rangle. \qquad (2.53)$$

We observe that $\langle\eta', \eta\rangle_{2,\mu} = -\int_0^\infty \mu'(s)||\Delta\eta(s)||^2 ds$ is positive. Using this fact and using the triangle inequality for the term in the right hand side of (2.53) gives the estimate

$$\frac{d}{dt}||z||^2_\nu + 2k_0||\Delta\theta||^2 \le 2\int_\Omega |p(x)||\theta\Delta\theta|dx + 2\int_\Omega |h||\Delta\theta|dx. \qquad (2.54)$$

The application of Young's inequality with ϵ for the terms in the right hand side of the inequality (2.54) yields

$$\begin{aligned}\frac{d}{dt}||z||^2_{\mathcal{V}} &+ 2k_0||\Delta\theta||^2 \le \frac{1}{2\epsilon}||h||^2 + 2\epsilon||\nabla\theta||^2 \\ &+ 2\epsilon_1||p||_{L^\infty(\Omega)}||\nabla\theta||^2 + \frac{||p||_{L^\infty(\Omega)}}{2\epsilon_1}||\theta||^2. \qquad (2.55)\end{aligned}$$

If we use Poincaré inequality for the last term in the right hand side of (2.55), we get the estimate,

$$\begin{aligned}\frac{d}{dt}||z||^2_{\mathcal{V}} &+ (2k_0 - 2\epsilon - 2\epsilon_1||p||_{L^\infty(\Omega)} - \frac{||p||_{L^\infty(\Omega)}}{2\epsilon_1}\lambda_0\gamma_0)||\nabla\theta||^2 \\ &\le \frac{1}{2\epsilon}||h||^2. \qquad (2.56)\end{aligned}$$

The integration of (2.56) over $(0,t)$, $\quad 0 \leq t \leq T$, gives the following inequality

$$\begin{aligned} ||z(t)||^2_{\mathcal{V}} &+ \int_0^t (2k_0 - 2\epsilon - 2\epsilon_1 ||p||_{L^\infty(\Omega)} - \frac{||p||_{L^\infty(\Omega)}}{2\epsilon_1}\lambda_0\gamma_0)||\nabla\theta||^2 \\ &\leq \frac{1}{2\epsilon}\int_0^T ||h||^2 dt + ||z_0||^2_{\nu}. \end{aligned} \tag{2.57}$$

If we choose ϵ and ϵ_1 so that the coefficient of $||\nabla\theta||^2$ in (2.57) is positive, the bound we get in (2.57)shows that

$$\begin{aligned} \theta &\in L^\infty([0,T], H^1_0(\Omega)) \cap L^2([0,T], H^2(\Omega) \cap H^1_0(\Omega)) \\ \eta &\in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H^2(\Omega) \cap H^1_0(\Omega))). \end{aligned}$$

□

2.3 Some properties for the structural stability of the equation

In this section, we will investigate the continuous dependence of the solutions of (2.8)-(2.10) on the coefficient p and the memory kernel μ. We will start with the continous dependence on $p(x)$.

Theorem 2.2. *The solution of the problem (2.8)-(2.10) depends continuously on $p(x)$.*

Proof. To show continuous dependence on $p(x)$, we will consider two solutions z_1 and z_2 of (2.8)-(2.10) for two different choices for $p(x)$, i.e., $p_1(x)$ and $p_2(x)$

with the respective initial conditions z_0^1 andz_0^2 , where $z_0^1 = z_0^2$:

$$\begin{aligned} z_t^1(x,t) = (k_0\nabla\theta_1(x,t) &+ \int_0^\infty \mu(s)\nabla\eta_1^t(x,s)ds \\ &+ h(x,t) - p_1(x)\theta_1(x,t), \ \ \theta_1(x,t) - \eta_1^{t\prime}(x,s)) \quad (2.58) \\ z_t^2(x,t) = (k_0\nabla\theta_2(x,t) &+ \int_0^\infty \mu(s)\nabla\eta_2^t(x,s)ds \\ &+ h(x,t) - p_2(x)\theta_2(x,t), \ \ \theta_2(x,t) - \eta_2^{t\prime}(x,s)) \quad (2.59) \end{aligned}$$

Subtracting (2.59) from (2.58), and setting $\tilde{z} := z^1 - z^2$, $\tilde{\theta} := \theta_1 - \theta_2$, $\tilde{\eta} := \eta_1 - \eta_2$, $\tilde{p} := p_1 - p_2$, we get

$$\begin{aligned} \tilde{z}_t(x,t) &= (k_0\Delta\tilde{\theta}(x,t) + \int_0^\infty \mu(s)\Delta\tilde{\eta}^t(x,s)ds - p_1(x)\tilde{\theta}(x,t) \\ &- \tilde{p}(x)\theta_2(x,t), \tilde{\theta}(x,t) - \tilde{\eta}^{t\prime}(x,s)). \end{aligned} \quad (2.60)$$

Taking the inner product of $\tilde{z}_t$ with $\tilde{z}$ in the sense of $\mathcal{H}$, (2.60) gives

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}||_{\mathcal{H}}^2 + k_0||\nabla\tilde{\theta}||^2 + \langle p_1\tilde{\theta}, \tilde{\theta}\rangle + \langle \tilde{\eta}', \tilde{\eta}\rangle_{1,\mu} = -\langle \tilde{p}\theta_2, \tilde{\theta}\rangle. \quad (2.61)$$

Since $\langle p_1\tilde{\theta}, \tilde{\theta}\rangle$,$k_0||\nabla\tilde{\theta}||^2$ and $\langle \tilde{\eta}', \tilde{\eta}\rangle_{1,\mu}$ are positive , (2.61) becomes

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}||_{\mathcal{H}}^2 \leq |\langle \tilde{p}\theta_2, \tilde{\theta}\rangle|. \quad (2.62)$$

We need an estimate for

$$|\langle \tilde{p}\theta_2, \tilde{\theta}\rangle| = |\int_\Omega \tilde{p}(x)\theta_2(x)\tilde{\theta}(x)dx|. \quad (2.63)$$

The application of Hölder's inequality in (2.63) gives

$$\begin{aligned} |\langle \tilde{p}(x)\theta_2(x), \tilde{\theta}(x)\rangle| &\leq \int_\Omega |\tilde{p}(x)\theta_2(x)|\tilde{\theta}(x)|dx \\ &\leq \left(\int_\Omega (\tilde{p}(x)\theta_2(x))^2 dx\right)^{1/2} \left(\int_\Omega (\tilde{\theta}(x))^2 dx\right)^{1/2} \\ &\leq ||\tilde{p}||^{1/2}_{L^\infty(\Omega)}.||\theta_2||.||\tilde{\theta}||. \end{aligned} \tag{2.64}$$

Using Young's inequality for the right hand side of (2.64) we get

$$|\langle \tilde{p}(x)\theta_2(x), \tilde{\theta}(x)\rangle| \leq ||\tilde{p}||^{1/2}_{L^\infty(\Omega)}.||\theta_2||.||\tilde{\theta}|| \leq \frac{1}{2}||\tilde{\theta}||^2 + \frac{1}{2}||\tilde{p}||_{L^\infty(\Omega)}.||\theta_2||^2. \tag{2.65}$$

Substitution of (2.65) in (2.62) yields

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}||^2_{\mathcal{H}} \leq \frac{1}{2}||\tilde{\theta}||^2 + \frac{1}{2}||\tilde{p}||_{L^\infty(\Omega)}.||\theta_2||^2. \tag{2.66}$$

Since $||\tilde{\theta}||^2 \leq ||\tilde{z}||^2_{\mathcal{H}}$, we may write (2.66) as

$$\frac{d}{dt}||\tilde{z}||^2_{\mathcal{H}} - ||\tilde{z}||^2_{\mathcal{H}} \leq ||\tilde{p}||_{L^\infty(\Omega)}.||\theta_2||^2. \tag{2.67}$$

Using Gronwall's inequality, we get

$$||\tilde{z}(t)||^2_{\mathcal{H}} \leq e^T ||\tilde{p}||_{L^\infty(\Omega)} \int_0^T ||\theta_2(\tau)||^2 d\tau, \quad 0 \leq t \leq T. \tag{2.68}$$

From (2.68), we deduce that $||\tilde{z}(t)||_{\mathcal{H}} \to 0$ as $||\tilde{p}||_{L^\infty(\Omega)} \to 0$. This gives the continuous dependendce on $p(x)$. □

We will show the continous dependence of the solution of the problem (2.8)-(2.10) on μ for a general nonlinearity $g(\theta(x,t))$ instead of $p(x)\theta(x,t)$. We write $g(\theta(x,t))$ in the place of $p(x)\theta(x,t)$ in (2.8)-(2.10) and we get the problem (1.44)-(1.46). Now, let us show the continuous dependence of the solution of the equation (1.44)-(1.46) on μ.

Theorem 2.3. *The solution of the direct problem (1.44) - (1.46) continously depends on the memory μ if $\mu \in C^1(0,\infty)\cap L^\infty[0,\infty)\cap L^1(0,\infty)$ and $g(\theta(x,t))$ as in (1.41).*

Proof. Let z_1 and z_2 be two solutions corresponding to μ_1 and μ_2 respectively;

$$\begin{aligned} z_{1,t}(x,t) &= (k_0\Delta\theta_1(x,t) + \int_0^\infty \mu_1(s)\Delta\eta_1^t(x,s)ds \\ &+ h(x,t) - g(\theta_1(x,t)), \theta_1(x,t) - \eta_1^{t\prime}(x,s)), \end{aligned} \tag{2.69}$$

$$\begin{aligned} z_{2,t}(x,t) &= (k_0\Delta\theta_2(x,t) + \int_0^\infty \mu_2(s)\Delta\eta_2^t(x,s)ds \\ &+ h(x,t) - g(\theta_2(x,t)), \theta_2(x,t) - \eta_2^{t\prime}(x,s)). \end{aligned} \tag{2.70}$$

Let us form the difference of (2.69) and (2.70). Defining the new variables $\tilde{z} := z_1 - z_2$, $\quad \tilde{\theta} := \theta_1 - \theta_2$, $\quad \tilde{\eta} := \eta_1 - \eta_2 \quad$ and $\quad \tilde{\mu} := \mu_1 - \mu_2$, we have

$$\begin{aligned} \tilde{z}_t(x,t) &= (k_0\Delta\tilde{\theta}(x,t) + \int_0^\infty \tilde{\mu}(s)\Delta\eta_2^t(x,s)ds + \int_0^\infty \mu_1(s)\Delta\tilde{\eta}^t(x,s)ds \\ &- (g(\theta_1(x,t)) - g(\theta_2(x,t))), \tilde{\theta}(x,t) - \tilde{\eta}^{t\prime}(x,s)). \end{aligned} \tag{2.71}$$

Multiplying both sides of the above equation by $\tilde{z} = (\tilde{\theta}, \tilde{\eta})$ in the sense of $\mathcal{H}$ with the weight function $\mu_1(s)$, we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\tilde{z}||^2_{\mathcal{H}} &= -k_0||\nabla\tilde{\theta}(.,t)||^2 - \langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}} \\ &- \langle g(\theta_1(x,t)) - g(\theta_2(x,t)), \tilde{\theta}(x,t)\rangle - \langle \tilde{\eta^{t\prime}}(x,s), \tilde{\eta}^t(x,s)\rangle_{1,\mu_1}. \end{aligned} \tag{2.72}$$

Our aim is to find an upper bound for $||\tilde{z}||^2_{\mathcal{H}}$ depending on $\tilde{\mu}$. To this end, we will estimate the right hand side of (2.72). From [16], we know that

$$-\langle g(\theta_1(x,t)) - g(\theta_2(x,t)), \tilde{\theta}(x,t)\rangle \leq C \mid\mid \tilde{\theta}(.,t) \mid\mid^2 \quad ,$$

C depending on g and $\langle \tilde{\eta}^{t\prime}(x,s), \tilde{\eta}\rangle_{1,\mu_1} \geq 0$. If we use these facts in (2.72), we get

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}(.,t)||^2_{\mathcal{H}} + k_0||\nabla\tilde{\theta}(.,t)||^2 \leq -\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}} + C||\tilde{\theta}(.,t)||^2. \quad (2.73)$$

By the Poincaré's inequality for the second term at the left hand side of (2.73) we get

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}(.,t)||^2_{\mathcal{H}} + \lambda_0 k_0||\tilde{\theta}(.,t)||^2 - C||\tilde{\theta}(.,t)||^2 \leq -\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}} \quad (2.74)$$

or

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}(.,t)||^2_{\mathcal{H}} + (\lambda_0 k_0 - C)||\tilde{\theta}(.,t)||^2 \leq -\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}}. \quad (2.75)$$

If the function g is chosen so that C satisfies $\lambda_0 k_0 - C \geq 0$, we find

$$\frac{1}{2}\frac{d}{dt}||\tilde{z}(.,t)||^2_{\mathcal{H}} \leq -\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}}. \quad (2.76)$$

Now, let us study the right hand side of the equation (2.76). By the definition of $\langle .,.\rangle_{1,\tilde{\mu}}$ we get

$$\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}} = \int_0^\infty \tilde{\mu}(s) \int_\Omega \nabla\eta_2^t(x,s)\nabla\tilde{\theta}(x,t)dxds \quad (2.77)$$

so

$$|\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}}| \leq \int_0^\infty |\tilde{\mu}(s)|||\nabla\eta_2^t(.,s)||||\nabla\tilde{\theta}(.,t)||ds. \quad (2.78)$$

Since an absorbing ball in $\mathcal{V} := H_0^1 \times L^2_\mu(\mathbb{R}^+, H^2 \cap H_0^1)$ exists [16], there is an R for every $t \in [0, \infty)$ depending on h satisfying

$$||\nabla\tilde{\theta}(.,t)|| \leq R. \tag{2.79}$$

Employing (2.79) in (2.78), we get

$$|\langle \eta_2^t(x,s), \tilde{\theta}(x,t)\rangle_{1,\tilde{\mu}} \leq R\int_0^\infty |\tilde{\mu}(s)|||\nabla\eta_2^t(.,s)||ds. \tag{2.80}$$

The right hand side of (2.80) may be written as

$$\begin{aligned} R & \int_0^\infty |\tilde{\mu}(s)|||\nabla\eta_2^t(.,s)||ds = R\int_0^\infty |\tilde{\mu}(s)|^{1/2}|\tilde{\mu}(s)|^{1/2}||\nabla\eta_2^t(.,s)||ds \\ \leq & R \left(\int_0^\infty |\tilde{\mu}(s)|ds\right)^{1/2}\left(\int_0^\infty |\tilde{\mu}(s)|||\nabla\eta_2^t(.,s)||^2 ds\right)^{1/2}. \end{aligned} \tag{2.81}$$

Substituting (2.81) in (2.76), we end up with

$$\frac{d}{dt}||\tilde{z}(.,t)||_{\mathcal{H}} \leq R\left(\int_0^\infty |\tilde{\mu}(s)|ds\right)^{1/2}\left(\int_0^\infty |\tilde{\mu}(s)|||\nabla\eta_2^t(.,s)||^2 ds\right)^{1/2}. \tag{2.82}$$

Solving the differential inequality, we get the following bound for $||\tilde{z}(.,t)||_{\mathcal{H}}$:

$$||\tilde{z}(.,t)||_{\mathcal{H}} \leq R\int_0^\infty |\tilde{\mu}(s)|ds \int_0^t \left(\int_0^\infty |\tilde{\mu}(s)|||\nabla\eta_2^\tau(.,s)||^2 ds\right)^{1/2} d\tau. \tag{2.83}$$

It is trivial that $||\tilde{z}(.,t)||_{\mathcal{H}}$ tends to zero as $|\tilde{\mu}(s)|$ approaches to 0. This proves the continuous dependence of the solution z on μ.

□

CHAPTER 3

THE INVERSE PROBLEM FOR MEMORY KERNEL IDENTIFICATION

3.1 Introduction

In this chapter, firstly we will define an inverse problem to identify the memory kernel μ. Secondly, we will prove that this inverse problem has a unique solution.

3.2 The identification of memory kernel

In Chapter 2, we have proved that if $h \in L^2(\mathbb{R}^+; L^2(\Omega))$, $p \in L^\infty(\Omega)$, $z_0 = (\theta_0, \eta_0) \in \mathcal{H}$ are given and the conditions $(C_1), (C_2), (C_3)$ on μ are satisfied then the problem

$$\theta_t(x,t) = k_0 \Delta\theta(x,t) + \int_0^\infty \mu(s)\Delta\eta^t(x,s)ds + h(x,t) - \theta(x,t) \text{ on } \Omega \times \mathbb{R}^+ \quad (3.1)$$

$$\eta_t^t(x,s) = \theta(x,t) - \eta'(x,t-s) \text{ on } \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \quad (3.2)$$

$$z(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \quad (3.3)$$

$$z(x,0) = z_0, \quad x \in \Omega, \quad s > 0 \quad (3.4)$$

has a unique solution $z(x,t) = (\theta(x,t), \eta(x,t-s))$ where

$$\begin{aligned} \theta &\in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T)], H_0^1(\Omega)), \qquad \forall T > 0 \\ \eta &\in L^\infty([0,T], L^2_\mu(\mathbb{R}^+, H_0^1(\Omega))), \qquad \forall T > 0. \end{aligned}$$

We can easily check that this problem is well-posed for the unknown functions θ, η. In this section, we want to identify $\mu(s)$ besides $\theta(x,t)$ and $\eta^t(x,s)$ appearing in the problem (3.1)-(3.4). It means that, we have one more unknown function. Thus, the new problem we are interested in is ill-posed and is called as an inverse problem for μ. It is clear that in this inverse problem $\theta(x,t)$ and $\eta^t(x,s)$ can not be determined uniquely with the conditions (3.3) and (3.4). To solve this ill-posed problem uniquely, we must impose additional appropriate constraints on $\theta(x,t)$ and $\eta(x,t-s)$ which are called as final overdetermination conditions. We propose

$$\int_\Omega \theta(x,t)\varphi(x)dx = A(t) \tag{3.5}$$

and

$$\int_\Omega \eta(x,t-s)\varphi(x)dx = B(t-s), \tag{3.6}$$

as the final overdetermination conditions, where $A(t)$, $B(t-s)$ and $\varphi \in \mathcal{D}(\Omega)$ will be determined later.

In order to identify $\mu(s), \theta(x,t)$ and $\eta^t(x,s)$ in (3.1)-(3.6), we will follow the technique given earlier in the study of the inverse problem (1.12)-(1.15). That is, we will convert the inverse problem (3.1)-(3.6) to an operator equation for μ and we will show that this operator equation has a fixed point. Using this value of μ we find a direct problem for θ and η. In order to get the operator equation, we will use Paley-Wiener representation (half Fourier transformation) given in Section 1.2. Let us recall that for any $f \in L^2(0,\infty)$, the half Fourier transformation is defined by

$$\widehat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz} f(t) dt, \quad z \in \Pi \tag{3.7}$$

where $\Pi = \{(x, y) : y \geq 0\}$. The mapping given by (3.7) is an isometry from $L^2(0, \infty)$ onto $H^2(\Pi)$.

(i) The operator Equation.

Since $h(x, t)$ in (3.1) has the property that $h(x, t) \in L^2(\mathbb{R}^+, L^2(\Omega))$, we can compute the half Fourier transformation of (3.1):

$$\widehat{\theta_t(x,t)} = k_0 \Delta \widehat{\theta(x,t)} + [\int_0^\infty \mu(s) \widehat{\Delta\eta(x, t-s)} ds] + \widehat{h(x,t)} - \widehat{\theta(x,t)}. \tag{3.8}$$

If we multiply each term in (3.8) by $\varphi(x)$ and integrate over Ω, we find

$$\begin{aligned} \widehat{A'(t)} &= \frac{k_0}{\sqrt{2\pi}} \int_0^\infty \int_\Omega e^{itz} \Delta\theta(x,t) \varphi(x) dx dt \\ &+ \int_\Omega \left[\int_0^\infty \mu(s) \widehat{\Delta\eta(x, t-s)} ds \right] \varphi(x) dx \\ &+ \widehat{\beta(t)} - \widehat{A(t)}, \end{aligned} \tag{3.9}$$

using (3.5) and (3.6) where, $\widehat{\beta(t)} := \int_\Omega \widehat{h(x,t)} \varphi(x) dx$. We need to study the integrant of the second term in the right hand side of (3.9):

$$\left[\int_0^\infty \mu(s) \widehat{\Delta\eta(x, t-s)} ds \right] = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty e^{itz} \mu(s) \Delta\eta(x, t-s) dt\, ds. \tag{3.10}$$

Let us use the substitution $\xi := t - s$ in (3.10) ; then

$$\left[\int_0^\infty \mu(s)\widehat{\Delta\eta(x,t-s)}ds\right] = \frac{1}{\sqrt{2\pi}}\int_0^\infty\int_{-s}^\infty e^{(\xi+s)iz}\mu(s)\Delta\eta(x,\xi)d\xi ds \tag{3.11}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty\int_0^\infty e^{isz}e^{i\xi z}\mu(s)\Delta\eta(x,\xi)d\xi ds$$

$$+ \frac{1}{\sqrt{2\pi}}\int_0^\infty\int_{-s}^0 e^{isz}e^{i\xi z}\mu(s)\Delta\eta(x,\xi)d\xi ds \tag{3.12}$$

Using (3.7), the first term in (3.12) is

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty\int_0^\infty e^{isz}e^{i\xi z}\mu(s)\Delta\eta(x,\xi)d\xi ds = \sqrt{2\pi}\widehat{\mu(s)}\Delta\widehat{\eta(x,\xi)} \tag{3.13}$$

and the second term is

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty\int_{-s}^0 e^{isz}e^{i\xi z}\mu(s)\Delta\eta(x,\xi)d\xi ds = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{isz}\mu(s)\int_{-s}^0 e^{i\xi z}\Delta\eta(x,\xi)d\xi ds. \tag{3.14}$$

If we use (3.13) and (3.14) in (3.9) we get

$$\widehat{A'(t)} = -\widehat{A(t)} + \widehat{\beta(t)} + k_0\int_\Omega \Delta\widehat{\theta(x,t)}\varphi(x)dx + \sqrt{2\pi}\widehat{\mu(s)}\int_\Omega \Delta\widehat{\eta(x,\xi)}\varphi(x)dx$$

$$+ \frac{1}{\sqrt{2\pi}}\int_\Omega \varphi(x)\int_0^\infty e^{isz}\mu(s)\int_{-s}^0 e^{i\xi z}\Delta\eta(x,\xi)d\xi ds\, dx. \tag{3.15}$$

Solving (3.15) for $\widehat{\mu(s)}$, we get

$$\widehat{\mu(s)} = \frac{\widehat{A'(t)} + \widehat{A(t)} - \widehat{\beta(t)} - k_0\int_\Omega \Delta\widehat{\theta(x,t)}\varphi(x)dx}{\sqrt{2\pi}\int_\Omega \Delta\widehat{\eta(x,\xi)}\varphi(x)dx}$$

$$- \frac{\frac{1}{\sqrt{2\pi}}\int_\Omega \varphi(x)\int_0^\infty e^{isz}\mu(s)\int_{-s}^0 e^{i\xi z}\Delta\eta(x,\xi)d\xi ds dx}{\sqrt{2\pi}\int_\Omega \Delta\widehat{\eta(x,\xi)}\varphi(x)dx}. \tag{3.16}$$

On the other hand, the half Fourier transform of (3.2) is

$$\widehat{\eta_t(x, t-s)} = \widehat{\theta(x,t)} - \widehat{\eta'(x, t-s)}. \tag{3.17}$$

By the definition , the term on the left hand side of (3.17) is

$$\widehat{\eta_t(x, t-s)} = \int_0^\infty e^{-itz} \eta_t(x, t-s) dt. \tag{3.18}$$

Integration by parts in (3.18) yields

$$\widehat{\eta_t(x, t-s)} = \frac{1}{\sqrt{2\pi}} [-\eta_0(x,s) - iz \int_0^\infty e^{itz} \eta(x, t-s) dt]. \tag{3.19}$$

The second term in the right hand side of (3.17) gives

$$\begin{aligned} \widehat{\eta'(x, t-s)} &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz} \frac{\partial}{\partial s} (\eta(x, t-s)) dt \\ &= \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial s} \left(\int_0^\infty e^{itz} \eta(x, t-s) dt \right). \end{aligned} \tag{3.20}$$

Substituting (3.19) and (3.20) in (3.17); and then multiplying the resulting equation with $\varphi(x)$ in the sense of L^2, we get

$$\begin{aligned} \frac{-1}{\sqrt{2\pi}} \int_\Omega \eta_0(x,s) \varphi(x) dx &- \frac{iz}{\sqrt{2\pi}} \int_0^\infty e^{itz} B(t-s) dt \\ &= \widehat{A(t)} - \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial s} (\int_0^\infty e^{itz} B(t-s) dt). \end{aligned} \tag{3.21}$$

This equality gives a relatipn between $A(t)$ and $B(t-s)$.

(ii) The equivalence of The Inverse Problem and The Operator Equations

First, we assume that the inverse problem (3.1)-(3.6) has a solution $(\theta(x,t), \eta(x,t-s), \mu(s))$. By following the steps for getting $\widehat{\mu(s)}$, we reach the operator equations (3.16) and (3.21).

Secondly, we assume that (3.16) and (3.21) are satisfied by a function $\mu(s)$. If we substitute this $\mu(s)$ in (3.1), we get the direct problem (3.1)-(3.4), which has a unique solution $(\theta(x,t), \eta(x,t-s))$ by Theorem 2.1. Now we have to show that the solution $(\theta(x,t), \eta(x,t-s))$ satisfies the final overdetermination conditions (3.5) and(3.6). To show that (3.5) is satisfied we transform (3.1) to get

$$\widehat{\theta_t(x,t)} = k_0 \widehat{\Delta\theta(x,t)} + [\widehat{\int_0^\infty \mu(s)\Delta\eta(x}, t-s)ds] + \widehat{h(x,t)} - \widehat{\theta(x,t)}. \quad (3.22)$$

Multipliying (3.22) by $\varphi(x)$ and integrating over Ω we get

$$\begin{aligned}
\int_\Omega \widehat{\theta_t(x,t)}\varphi(x)dx &= k_0 \int_\Omega \widehat{\Delta\theta(x,t)}\varphi(x)dx + \sqrt{2\pi}\int_\Omega \widehat{\mu(s)}\widehat{\Delta\eta(x,\xi)}\varphi(x)dx \\
&+ \frac{1}{\sqrt{2\pi}} \int_\Omega \varphi(x) \int_0^\infty e^{isz}\mu(s) \int_{-s}^0 e^{i\xi z}\Delta\eta(x,\xi)d\xi ds dx \\
&+ \widehat{\beta(t)} - \int_\Omega \widehat{\theta(x,t)}\varphi(x)dx, \quad (3.23)
\end{aligned}$$

where $\widehat{\beta(t)} = \int_\Omega \widehat{h(x,t)}dx$. Since $\mu(s)$ satisfies (3.16), we have the identity

$$\begin{aligned}
\widehat{A'(t)} &+ \widehat{A(t)} - \widehat{\beta(t)} - k_0 \int_\Omega \widehat{\Delta\theta(x,t)}\varphi(x)dx \\
&- \frac{1}{\sqrt{2\pi}} \int_\Omega \varphi(x) \int_0^\infty e^{isz}\mu(s) \int_{-s}^0 e^{i\xi z}\Delta\eta(x,\xi)d\xi ds\, dx \\
&= \sqrt{2\pi}\int_\Omega \widehat{\mu(s)}\widehat{\Delta\eta(x,\xi)}\varphi(x)dx. \quad (3.24)
\end{aligned}$$

Subtracting (3.24) from (3.23) we find

$$\int_\Omega \widehat{\theta_t(x,t)}\varphi(x)dx - \widehat{A'(t)} = -\int_\Omega \widehat{\theta(x,t)}\varphi(x)dx + \widehat{A(t)}, \quad (3.25)$$

or

$$\int_\Omega \left[\widehat{\theta_t(x,t)} + \widehat{\theta(x,t)}\right] \varphi(x)dx = \widehat{A'(t)} + \widehat{A(t)}. \tag{3.26}$$

Using the definition of half Fourier transformation we can write (3.26) as

$$\int_\Omega \frac{1}{\sqrt{2\pi}}\varphi(x) \int_0^\infty e^{itz}[\theta_t(x,t) + \theta(x,t)]dtdx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz}[A'(t) + A(t)]dt, \tag{3.27}$$

or

$$\int_0^\infty e^{itz} \left(\left(\int_\Omega \varphi(x) \left[\theta_t(x,t) + \theta(x,t)\right] dx \right) - \left[A'(t) + A(t)\right] \right) dt = 0. \tag{3.28}$$

From (3.28) we deduce that

$$\int_\Omega \varphi(x) \left[\theta_t(x,t) + \theta(x,t)\right] dx - \left[A'(t) + A(t)\right] = 0, \quad \text{a.e } t \in [0, \infty) \tag{3.29}$$

If we define

$$u(t) := \int_\Omega \varphi(x)\theta(x,t)dx - A(t), \tag{3.30}$$

(3.29) becomes

$$u_t(t) + u(t) = 0, \text{ for a.e.} t \in [0, \infty) \tag{3.31}$$

which has the solution $u(t) = u(0)e^{-t}$. If we impose

$$A(0) = \int_\Omega \theta_0(x)\varphi(x)dx$$

as a compatibility condition on $A(t)$, then $u(0) = 0$ by (3.30). Hence, $u(t)$ will be the trivial solution of (3.31), which gives

$$A(t) = \int_\Omega \theta(x,t)\varphi(x)dx, \tag{3.32}$$

showing that the final overdetermination condition (3.5) is satisfied.

Secondly, we will show that $(\theta(x,t), \eta(x,t-s))$ satisfies the condition (3.6). We recall that, using the substitution $\eta(x,t-s) = \int_{t-s}^{t} \theta(x,\tau)d\tau$, the problem (1.32)-(1.34) takes the form (1.36)-(1.40). By the definition of $B(t-s)$ and $A(t)$, we get that

$$\begin{aligned} B(t-s) &= \int_\Omega \varphi(x)\eta(x,t-s)dx = \int_\Omega \varphi(x) \int_{t-s}^{t} \theta(x,\tau)d\tau dx \\ &= \int_{t-s}^{t} \int_\Omega \varphi(x)\theta(x,\tau)dxd\tau = \int_{t-s}^{t} A(x,\tau)d\tau. \end{aligned}$$

This argument shows that $B(t-s)$ is determined uniquely by $A(t)$. Hence, the second final overdetermination condition (3.6)is satisfied.

Since the inverse problem (3.1)-(3.6) and the couple of the operator equations (3.16),(3.21) are equivalent, we need to show the existence of a $\widehat{\mu(s)}$ satisfying (3.16) and (3.21).

(iii) The Existence of The Solution of The Operator Equations

We will show that the operator equation (3.16)have a solution $\widehat{\mu(s)}$. Before starting to study on the existence, we will try to write the equation (3.16) in a compact form. From now on, the function $\varphi(x)$ in (3.5) will be specified as a solution of the eigenvalue problem

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega, \tag{3.33}$$

$$\varphi(x) = 0, \quad x \in \partial\Omega. \tag{3.34}$$

for the smallest eigenvalue λ. With this choice of $\varphi(x)$,
the term $-k_0 \int_\Omega \widehat{\Delta\theta(x,t)}\varphi(x)dx$ in (3.16) becomes

$$-k_0 \int_\Omega \widehat{\Delta\theta(x,t)}\varphi(x)dx = k_0\lambda \int_\Omega \widehat{\theta(x,t)}\varphi(x) = k_0\lambda\widehat{A(t)}$$

and the term $\int_\Omega \widehat{\Delta\eta(x,\xi)}\varphi(x)dx$ becomes

$$\int_\Omega \widehat{\Delta\eta(x,\xi)}\varphi(x)dx = -\lambda \int_\Omega \widehat{\eta(x,\xi)}\varphi(x)dx = -\lambda\widehat{B(\xi)}.$$

Utilizing these identities, the operator equation (3.16) takes the form

$$\widehat{\mu(s)} = \frac{\widehat{\Gamma(t)} - \frac{\lambda}{\sqrt{2\pi}} \int_0^\infty e^{isz}\mu(s) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds}{-\lambda\sqrt{2\pi}\widehat{B(\xi)}} \tag{3.35}$$

where

$$\widehat{\Gamma(t)} := \widehat{A'(t)} + \widehat{A(t)} - \widehat{\beta(t)} + k_0\lambda\widehat{A(t)}. \tag{3.36}$$

We may represent $\widehat{\mu(s)}$ as

$$\widehat{\mu(s)} = N(z) - \frac{1}{2\pi\widehat{B(\xi)}} \left[\int_0^\infty e^{isz}\mu(s) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds\right] \tag{3.37}$$

or

$$\mu(s) := (N(z))^{-1} + (T\mu)(s), \tag{3.38}$$

where

$$N(z) := \frac{\widehat{\Gamma(t)}}{-\lambda\sqrt{2\pi}\widehat{B(\xi)}}, \tag{3.39}$$

$(N(z))^{-1}$ is the inverse of $N(z)$ with respect to the half Fourier transformation and

$$\widehat{(T\mu)(s)} := \left[\frac{-1}{2\pi\widehat{B(\xi)}} \int_0^\infty e^{isz}\mu(s) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds\right]. \tag{3.40}$$

The Fixed point Argument

Now, we will show that the operator equation (3.40) has a fixed point in a certain set. Since the operator T is linear, we will use the following fixed point theorem.

Theorem 3.1. *Suppose that*

(i) We are given an operator $T : M \subseteq X \to M$, i.e., M is mapped into itself by T;

(ii) M is a closed nonempty set in a complete metric space (X, d);

(iii) T is contractive, i.e.,

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in M$ and for a fixed k, $\quad 0 \leq k < 1$.

Then, T has exactly one fixed point on M.

Proof. See, [40]. □

Now, we have to construct a set M satisfying the conditions of Theorem 3.1.

We observe that, the conditions (C_2) and (C_3) given in Chapter 1.1 are satisfied for the value of $\mu(s) = e^{-\alpha s}, \quad (\alpha > 0)$, and the direct problem (3.1)-(3.4) is uniquely soluble. Using this fact, we will start with the set

$$S := \{e^{-\alpha s} : s \in [0, \infty), \quad \text{for all } \alpha > \epsilon > 0\} \cup \{0\}. \tag{3.41}$$

We define the set M as

$$M := \left\{ \mu : \mu = \sum_{i=1}^{n} a_i \rho_i, \quad \rho_i \in S, \quad n \text{ is finite or infinite}, \quad a_i > 0 \right\}, \tag{3.42}$$

which is a closed subset of $L^2(0, \infty)$.

The Operator T is from M into M

Now, we will show that the operator T is from M into M. For this end, we will fix μ in M and try to show that the image of $\mu(s)$ under T will be an element of M. Any element μ of M is written as $\mu = \sum_{j=1}^{n} a_j \rho_j, \quad \rho_j \in S, \quad \forall j,.$ We can easily show that $\int_0^\infty (e^{-\alpha_j s})^2 ds \leq \infty$, for all $\alpha_j > 0$. Hence , $e^{-\alpha_j s} \in$

$L^2(0,\infty)$. Since $L^2(0,\infty)$ is a vector space, any linear combinations of $e^{-\alpha_j s}$ is an element of $L^2(0,\infty)$. In particular $\mu(s) \in L^2(0,\infty)$. Therefore, $\mu(s)$ is transformable into $H^2(\Pi)$. The half Fourier transformation of $e^{-\alpha_j s}$ is

$$\widehat{e^{-\alpha_j s}} = \int_0^\infty e^{-izs} e^{-\alpha_j s} ds = \frac{1}{\alpha_j - iz}.$$

Since the half Fourier transformation is linear, the transformation of $\mu(s)$ will be of the form

$$\widehat{\mu(s)} = \sum_{j=1}^{l} a_j \frac{1}{\alpha_j - iz}. \tag{3.43}$$

From (3.40), we know that

$$\widehat{T(\mu)} = \frac{-1}{2\pi \widehat{B(\xi)}} \left[\int_0^\infty e^{isz} \mu(s) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds \right]$$

and

$$\widehat{B(\xi)}\widehat{T(\mu)} := \frac{-1}{2\pi} \left[\int_0^\infty e^{isz} \mu(s) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds \right]. \tag{3.44}$$

Since $\mu(s) \in M$, $\quad \mu(s) = \sum_{j=1}^{\ell} a_j e^{-\alpha_j s}$. We want to show that $T(\mu) \in M$. We have proved that, the transformation of any element in M is of the form (3.43), so if $T(\mu) \in M$, then

$$\widehat{T(\mu)} = \sum_{j=1}^{n} b_j \frac{1}{\beta_j - iz}. \tag{3.45}$$

Writing (3.45) in (3.44), we get

$$\frac{-1}{2\pi} \int_0^\infty e^{isz} \left(\sum_{j=1}^{\ell} a_j e^{-\alpha_j s} \right) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds = \frac{1}{\sqrt{2\pi}} \left(\widehat{B(\xi)} \sum_{j=1}^{m} \frac{b_j}{\beta_{j-iz}} \right). \tag{3.46}$$

The left hand side of (3.46) may be written in the form

$$-\frac{1}{2\pi}\sum_{j=1}^{\ell} a_j \int_0^\infty e^{isz} t_j(s,z)ds,$$

where $t_j(s,z) = e^{-\alpha_j s}\int_{-s}^0 e^{i\xi z}B(\xi)d\xi$. Using the basis of $H^2(\Pi)$, we can write the left hand side of (3.46) as

$$\frac{-1}{2(\pi)^{3/2}}\sum_{j=1}^{\ell} a_j \sum_{n=0}^{\infty} k_{jn}\frac{(z-i)^n}{(z+i)^{n+1}} = \frac{-1}{2(\pi)^{3/2}}\sum_{n=0}^{\infty}\left(\sum_{j=1}^{\ell} a_j k_{jn}\right)\frac{(z-i)^n}{(z+i)^{n+1}}$$
$$= \sum_{n=0}^{\infty} K_n(l)\frac{(z-i)^n}{(z+i)^{n+1}},$$

where $K_n(l) = \frac{-1}{2(\pi)^{3/2}}\sum_{j=1}^{\ell} a_j k_{jn}$.

By the same way, the right hand side of (3.46), may be written also

$$\frac{1}{\sqrt{2}\pi^{3/2}}\sum_{n=0}^{\infty} g_n \frac{(z-i)^n}{(z+i)^{n+1}}\sum_{n=0}^{\infty}\left(\sum_{j=1}^{m} b_j h_{jn}\right)\frac{(z-i)^n}{(z+i)^{n+1}}, \tag{3.47}$$

where g_n's are Fourier coefficients of $\widehat{B(\xi)}$ and h_{jn}'s are Fourier coefficients of $\frac{b_j}{\beta_{j-iz}}$ with respect to the basis of $H^2(\Pi)$. Substituting $\frac{1}{\sqrt{2}\pi^{3/2}}\sum_{j=1}^{m} b_j h_{jn} = F_n(m)$, the equation (3.47) becomes

$$\frac{1}{\sqrt{2}\pi^{3/2}}\sum_{n=0}^{\infty} g_n \frac{(z-i)^n}{(z+i)^{n+1}}\sum_{n=0}^{\infty}\left(\sum_{j=1}^{m} b_j h_{jn}\right)\frac{(z-i)^n}{(z+i)^{n+1}}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} g_k F_{n-k}(m)\frac{(z-i)^n}{(z+i)^{n+1}}.$$

Sustituting these in (3.46), we have

$$\sum_{n=0}^{\infty} K_n(l)\frac{(z-i)^n}{(z+i)^{n+1}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} g_k F_{n-k}(m)\frac{(z-i)^n}{(z+i)^{n+1}},$$

or

$$\sum_{k=0}^{n} g_k F_{n-k}(m) = K_n(l).$$

This equality shows that, we can find proper Fourier coeffcients of the function in (3.45) for suitably chosen $B(\xi)$. So, we deduce that the operator T is from M into M.

The operator T is a Contraction Mapping

Now, we will show that, T is a contraction mapping. For this end, we need to estimate $L^2(0,\infty)$ norm of T. Since, the half Fourier transform is an isometry from $L^2(0,\infty)$ onto $H^2(\Pi)$, we will estimate $H^2(\Pi)$ norm of $\widehat{T(\mu)}$. By the definition of $H^2(\Pi)$ norm, we have

$$||\widehat{T(\mu_1)} - \widehat{T(\mu_2)}||^2_{H^2(\Pi)} = \frac{1}{4(\pi)^2} \sup_{y>0} \int_{-\infty}^{\infty} \left| \frac{\int_0^\infty e^{isz}(\mu_1(s) - \mu_2(s)) \int_{-s}^0 e^{i\xi z} B(\xi) d\xi ds}{\widehat{B(\xi)}} \right|^2 dx. \tag{3.48}$$

Substituting $g(z,s) := \int_{-s}^0 e^{i\xi z} B(\xi) d\xi$ in (3.48), it becomes

$$||\widehat{T(\mu_1)} - \widehat{T(\mu_2)}||^2_{H^2(\Pi)} = \frac{1}{4(\pi)^2} \sup_{y>0} \int_{-\infty}^{\infty} \left| \frac{\int_0^\infty e^{isz}(\mu_1(s) - \mu_2(s)) g(z,s) ds}{\widehat{B(\xi)}} \right|^2 dx. \tag{3.49}$$

The term in the right hand side of (3.49) satisfies the inequality

$$\begin{aligned}
&\frac{1}{4(\pi)^2} \sup_{y>0} \int_{-\infty}^{\infty} \left| \frac{\int_0^\infty e^{isz} \mu(s) g(z,s) ds}{\widehat{B(\xi)}} \right|^2 dx \\
&\le \frac{1}{4(\pi)^2} \sup_{y>0} \int_{-\infty}^{\infty} \frac{\sup_{s\in(0,\infty)} |g(z,s)|^2}{|\widehat{B(\xi)}|^2} |\widehat{\mu(s)}|^2 dx \\
&\le \frac{1}{4(\pi)^2} \left(\sup_{z\in\Pi} \frac{\sup_{s\in(0,\infty)} |g(z,s)|^2}{|\widehat{B(\xi)}|^2} \right) \sup_{y>0} \int_{-\infty}^{\infty} |\widehat{\mu(s)}|^2 dx \\
&= \frac{1}{4(\pi)^2} \left(\sup_{z\in\Pi} \frac{\sup_{s\in(0,\infty)} |g(z,s)|^2}{|\widehat{B(\xi)}|^2} \right) ||\widehat{\mu(s)}||^2_{H^2(\Pi)}.
\end{aligned} \tag{3.50}$$

Using (3.50) in (3.49), we find that

$$||\widehat{T(\mu_1)} - \widehat{T(\mu_2)}||^2 \leq \frac{1}{4(\pi)^2}\left(\sup_{z\in\Pi}\frac{\sup_{s\in(0,\infty)}|g(z,s)|^2}{|\widehat{B(\xi)}|^2}\right)||\widehat{\mu(s)}||^2_{H^2(\Pi)}. \quad (3.51)$$

The equation (3.51) shows that if $B(\xi)$ satisfies the property that

$$\frac{1}{4(\pi)^2}\left(\sup_{z\in\Pi}\frac{\sup_{s\in(0,\infty)}|g(z,s)|^2}{|\widehat{B(\xi)}|^2}\right) < 1,$$

then the operator T is a contraction mapping.

The operator T and the set M satisfy the conditions given in the Theorem 3.1. So, Using this theorem, we prove that there exist a unique μ satisfying (3.37). Hence the inverse problem (3.1)-(3.4) with the final overdetermination conditions (3.5),(3.6) has a unique solution (z, μ).

CHAPTER 4

TWO INVERSE PROBLEMS FOR THE SOURCE TERM IDENTIFICATION

4.1 Introduction

In this chapter, recovering of the evolution of a source term and recovering a source term depending on x of the problem (3.1)-(3.4) with zero boundary and initial conditions will be studied.

4.2 Recovering the Evolution of The Source Term

We will consider the problem

$$\theta_t(x,t) - k_0\Delta\theta(x,t) - \int_0^\infty \mu(s)\Delta\eta(x,t-s)ds + \theta(x,t) = H(t)M_0(x,t) \quad (4.1)$$

$$\eta_t(x,t-s) = \theta(x,t) - \eta_s(x,t-s) \quad (4.2)$$

$$z(x,0) = 0 \quad x \in \Omega \quad (4.3)$$

$$z(x,t) = 0 \quad x \in \partial\Omega \quad t,s > 0, \quad (4.4)$$

where $z(x,t) = (\theta(x,t), \eta(x,t-s))$ and the final overdetermination conditions are

$$\int_\Omega \theta(x,t)\varphi_1(x)dx = A(t), \tag{4.5}$$

$$\int_\Omega \eta(x,t-s)\varphi_1(x)dx = B(t-s), \tag{4.6}$$

where $\varphi_1(x) \in \mathcal{D}(\Omega)$.

Definition 4.1. A pair of functions $(z(x,t), H(t))$ is said to be a generalized solution of the inverse problem (4.1)-(4.6) if $z \in \mathcal{H}, \quad H \in L^2(0,\infty)$ and all of the relations (4.1) - (4.6) are satisfied.

To show the existence of the solution defined above, we will construct a linear operator equation under the assumptions

$$A \in L_\infty[0,\infty), \tag{4.7}$$

$$B \in L_\infty[0,\infty), \quad \text{with respect to } t \text{ and}$$

$$B \in L^2_\mu(\mathbb{R}^+) \quad \text{with respect to } s, \tag{4.8}$$

$$\varphi_1 \in \mathcal{D}(\Omega) \tag{4.9}$$

and

$$M_0 \in L_\infty(\Omega \times [0,\infty)). \tag{4.10}$$

For this construction, first, we multiply (4.1) by $\varphi_1(x)$, integrate over Ω to get

$$\begin{aligned} \int_\Omega \theta_t(x,t)\varphi_1(x)dx &- k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx - \int_0^\infty \mu(s) \int_\Omega \eta(x,t-s)\Delta\varphi_1(x)dxds \\ &+ \int_\Omega \theta(x,t)\sigma(x)dx = H(t) \int_\Omega M_0(x,t)\varphi_1(x)dx. \end{aligned} \tag{4.11}$$

We employ (4.5) in (4.11) to find

$$A'(t) + A(t) - k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx \;-\; \int_0^\infty \mu(s) \int_\Omega \eta(t-s)\Delta\varphi_1(x)dxds$$
$$= H(t) \int_\Omega M_0(x,t)\varphi_1(x)dx. \quad (4.12)$$

Solving (4.12) for $H(t)$, we find

$$H(t) = \frac{A'(t)+A(t)}{\int_\Omega M_0(x,t)\varphi_1(x)dx} - \frac{1}{\int_\Omega M_0(x,t)\varphi_1(x)dx}$$
$$\times \quad \left[k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx + \int_0^\infty \mu(s) \int_\Omega \eta(t-s)\Delta\varphi_1(x)dxds\right]. \quad (4.13)$$

We set

$$G(t) := -\int_\Omega M_0(x,t)\varphi_1(x)dx, \quad (4.14)$$

$$\psi(t) := -\frac{A'(t)+A(t)}{G(t)} \quad (4.15)$$

and

$$(TH)(t) = \frac{1}{G(t)}\left[k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx + \int_0^\infty \mu(s) \int_\Omega \eta(x,t-s)\Delta\varphi_1(x)dxds\right]. \quad (4.16)$$

Thus, one of the relation between H and $z = (\theta, \eta)$ may be specified as a linear operator

$$T : L^2(0,\infty) \to L^2(0,\infty)$$

with values

$$H(t) = (TH)(t) + \psi(t). \quad (4.17)$$

If we multiply (4.2) by $\varphi_1(x)$, integrate over Ω and employ (4.5),(4.6), we get

$$B_t(t-s) = A(t) - B_s(t-s) \quad (4.18)$$

Theorem 4.2. *The inverse problem (4.1)-(4.6)is soluble if and only if the operator equations (4.17),(4.18) are soluble.*

Proof. If the inverse problem (4.1)-(4.6) has a solution then all of the relations (4.1) - (4.6) hold. We follow the steps to get (4.17),(4.18). So, (4.17), (4.18) are soluble.

To prove the " only if " part, we assume that (4.17) has a solution $H(t)$. We substitute this $H(t)$ in the equation (4.1), then (4.1)-(4.4) becomes a direct problem for $(\theta(x,t), \eta^t(x,s))$ which has a unique solution by Theorem 2.1. So we need only to show that the solution of the direct problem (4.1) - (4.4) satisfies the final overdetermination conditions (4.5)-(4.6). To this end, let

$$\int_\Omega \theta(x,t)\varphi_1(x)dx = A_1(t) \tag{4.19}$$

and

$$\int_\Omega \eta(x,t-s)\varphi_1(x)dx = B_1(t-s). \tag{4.20}$$

Multiplying (4.1) by $\sigma(x)$, integrating over Ω and employing (4.19), we get

$$\begin{aligned} A_1'(t) &- k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx - \int_0^\infty \mu(s) \int_\Omega \eta(x,t-s)\Delta\varphi_1(x) + A_1(t) \\ &= H(t) \int_\Omega M_0(x,t)\varphi_1(x)dx. \end{aligned} \tag{4.21}$$

Since (4.17) is satisfied, we have also

$$\begin{aligned} A'(t) &- k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx - \int_0^\infty \mu(s) \int_\Omega \eta(x,t-s)\Delta\varphi_1(x) + A(t) \\ &= H(t). \int_\Omega M_0(x,t)\varphi_1(x)dx. \end{aligned} \tag{4.22}$$

If we subtract (4.22) from (4.21), we get

$$(A_1 - A)'(t) + (A_1 - A)(t) = 0. \tag{4.23}$$

The differential equation (4.23) implies that $(A_1 - A)(t) = (A_1 - A)(0)e^{-t}$. Since $(A_1 - A)(0) = 0$, we conclude that $A_1 = A$.

For the second final overdetemination conditions, we must remember that

$$\eta(x, t-s) = \int_{(t-s)}^{t} \theta(x,\tau)d\tau. \tag{4.24}$$

By(4.20), we have $B_1(t-s) = \int_\Omega \eta(x,t-s)\varphi_1(x)dx$. Using (4.24) in this equality we have

$$\begin{aligned} B_1(t-s) &= \int_\Omega \int_{(t-s)}^{t} \theta(x,\tau)d\tau\varphi_1(x)dx \\ &= \int_{(t-s)}^{t} A_1(x,\tau)d\tau. \end{aligned}$$

Since $A_1 = A$, we have then,

$$B_1(t-s) = \int_{(t-s)}^{t} A(x,\tau)d\tau = B(t-s).$$

□

We have shown that, to solve the inverse problem (4.1)-(4.6), is equaivalet to find a fixed point of (4.17)-(4.18). Now, we will show that the operator equation (4.17) has a fixed point.

Theorem 4.3. *If $G \in L_\infty(0,\infty)$ and $|G(t)| > \delta > 0, \quad \forall t \in [0,\infty)$ and if $M_0 \in L_\infty(\Omega \times \mathbb{R}^+)$ satisfying*

$$\frac{1}{\delta}\frac{|\Omega|}{2\sqrt{\epsilon\alpha}}\left(||M_0||_{L^\infty(\Omega\times\mathbb{R}^+)}\right)\left(k_0||\Delta\varphi_1||_{L^\infty(\Omega)} + ||\nabla\varphi_1||_{L^\infty(\Omega)}(\tilde{\mu})^{1/2}\right) \le 1,$$

where $\tilde{\mu} = \int_0^\infty \mu(s)ds$*, then (4.17) has a unique solution* $H(t)$*.*

Proof. Our aim is to show the existence of the solution of (4.17) by using the contraction mapping principle since T is linear.

Now, we will estimate the $L^2(0,\infty)$ norm of T.

Using (4.16) and keeping in mind that $|G(t)| > \delta, \quad \forall t \in [0,\infty)$, we get

$$\begin{aligned}
&||T(H)||_{L^2(0,\infty)} = \\
&||\frac{1}{G(t)}\left[k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx + \int_0^\infty \mu(s)\int_\Omega \eta(t-s)\Delta\varphi_1(x)dxds\right]||_{L^2(0,\infty)} \\
&\le \delta^{-1}||k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx + \int_0^\infty \mu(s)\int_\Omega \eta(t-s)\Delta\varphi_1(x)dxds||_{L^2(0,\infty)} \\
&\le \delta^{-1}\Bigg(||k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx||_{L^2(0,\infty)} \\
&+||\int_0^\infty \mu(s)\int_\Omega \eta(x,t-s)\Delta\varphi_1(x)dxds||_{L^2(0,\infty)}\Bigg). \qquad (4.25)
\end{aligned}$$

First, we estimate

$$||k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx||^2_{L^2(0,\infty)}.$$

By definition, we have

$$||\int_\Omega \theta(x,t)\Delta\varphi_1(x)dx||^2_{L^2(0,\infty)} = \int_0^\infty \left(\int_\Omega \theta(x,t)\Delta\varphi_1(x)dx\right)^2 dt. \qquad (4.26)$$

We will study only the integrand $\left(\int_\Omega \theta(x,t)\Delta\varphi_1(x)dx\right)^2$.

$$\begin{aligned}
\left(\int_\Omega \theta(x,t)\Delta\varphi_1(x)dx\right)^2 &\le \left(||\Delta\varphi_1(x)||_{L^\infty(\Omega)}\int_\Omega \theta(x,t)dx\right)^2 \\
&= ||\Delta\varphi_1||^2_{L^\infty(\Omega)}(\int_\Omega \theta(x,t)dx)^2. \qquad (4.27)
\end{aligned}$$

By Hölder's inequality, the second term in the right hand side of (4.27) satisfy the following inequality

$$(\int_\Omega \theta(x,t)dx)^2 \le (\int_\Omega (\theta(x,t))^2 dx)(\int_\Omega dx) = ||\theta(.,t)||^2_{L^2(\Omega)}|\Omega|. \tag{4.28}$$

Using (4.28) and (4.27) in (4.26), we end up with

$$||\int_\Omega \theta(x,t)\Delta\varphi_1(x)dx||^2_{L^2(0,\infty)} \le ||\Delta\varphi_1||^2_{L^\infty(\Omega)}|\Omega| \int_0^\infty ||\theta(.,t)||^2_{L^2(\Omega)}dt. \tag{4.29}$$

We may estimate $||\theta||_{L^2((0,\infty),L^2(\Omega))}$, using

$$||\theta(.,t)||_{L^2(\Omega)} \le ||z(t)||_{\mathcal{H}} \le \frac{1}{2\sqrt{\epsilon\alpha}}||h(t)||_{L^2(\Omega)} = \frac{1}{2\sqrt{\epsilon\alpha}}||H(t)M_0(.,t)||_{L^2(\Omega)}, \tag{4.30}$$

where $\alpha := min\{(2k_0\lambda_0 + 2 - \epsilon), \delta\}$, λ_0 is Poincaré inequality multiple, ϵ is Young's inequality multiple and δ as in C_2.

We start with the following identity

$$||\theta||^2_{L^2((0,\infty),L^2(\Omega))} = \int_0^\infty ||\theta(.,t)||^2_{L^2(\Omega)}dt \tag{4.31}$$

Using (4.30) in (4.31), we get

$$\begin{aligned} ||\theta||_{L^2((0,\infty),L^2(\Omega))} &\le ||z||_{L^2((0,\infty),\mathcal{H})} \le \frac{1}{2\sqrt{\epsilon\alpha}}||HM_0||_{L^2((0,\infty),L^2(\Omega))} \\ &= \frac{1}{2\sqrt{\epsilon\alpha}}\left(\int_0^\infty \int_\Omega (H(t)M_0(x,t))^2 dxdt\right)^{1/2} \\ &\le \left(\sup_{x\in\Omega, t\in(0,\infty)} (M_0(x,t))^2|\Omega|\frac{1}{4\epsilon\alpha}||H||^2_{L^2(0,\infty)}\right)^{1/2} \\ &= ||M_0||_{L^\infty(\Omega\times(0,))}|\Omega|^{1/2}\frac{1}{2\sqrt{\epsilon\alpha}}||H||_{L^2(0,\infty)}. \end{aligned} \tag{4.32}$$

If we employ (4.27) and (4.32) in (4.26), we get

$$
\begin{aligned}
&||k_0 \int_\Omega \theta(x,t)\Delta\varphi_1(x)dx||_{L^2(0,\infty)} \leq \\
&k_0||\Delta\varphi_1(x)||_{L^\infty(\Omega)} \frac{|\Omega|}{2\sqrt{\epsilon\alpha}}(||M_0||_{L^\infty(\Omega\times(0,\infty))})^2||H||_{L^2(0,\infty)}. \qquad (4.33)
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
&|| \int_0^\infty \mu(s) \int_\Omega \eta(x,t-s)\Delta\varphi_1(x)dxds||^2_{L^2(0,\infty)} \\
&= || \int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)\nabla\varphi_1(x)dxds||^2_{L^2(0,\infty)} \qquad (4.34)\\
&= \int_0^\infty \left(\int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)\nabla\varphi_1(x)dxds \right)^2 dt. \qquad (4.35)
\end{aligned}
$$

We will study the integrand of (4.35)

$$
\begin{aligned}
&\left(\int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)\nabla\varphi_1(x)dxds \right)^2 \leq \\
&||\nabla\varphi_1||^2_{L^\infty(\Omega)} \left(\int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)dxds \right)^2. \qquad (4.36)
\end{aligned}
$$

The term $\left(\int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)dxds \right)^2$ in (4.36) may be estimated as

$$
\begin{aligned}
&\left(\int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)dxds \right)^2 \\
&=\left(\int_0^\infty (\mu(s)^{1/2})(\mu(s)^{1/2}) \int_\Omega \nabla\eta(x,t-s)dxds \right)^2 \\
&\leq\left[(\int_0^\infty \mu(s)ds)^{1/2} (\int_0^\infty \mu(s)(\int_\Omega \nabla\eta(x,t-s)dx)^2 ds)^{1/2} \right]^2 \\
&= \tilde{\mu}\left(\int_0^\infty \mu(s)(\int_\Omega \nabla\eta(x,t-s)dx)^2 ds \right) \\
&\leq \tilde{\mu}|\Omega| \int_0^\infty \mu(s)||\nabla\eta(.,t-s)||^2_{L^2(\Omega)}ds \leq \tilde{\mu}|\Omega|||z(t)||^2_{\mathcal{H}}, \qquad (4.37)
\end{aligned}
$$

where $\tilde{\mu} = \int_0^\infty \mu(s)ds$. Using (4.37) in (4.35), we conclude that

$$\begin{aligned} &|| \int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)\nabla\varphi_1(x)dxds||_{L^2(0,\infty)} \\ &\leq ||\nabla\varphi_1||_{L^\infty(\Omega)}(\tilde{\mu})^{1/2}|\Omega|^{1/2}||z(t)||_{\mathcal{H}}. \end{aligned} \tag{4.38}$$

We recall that

$$\begin{aligned} ||HM_0||_{L^2((0,\infty),L^2(\Omega)} &= \left(\int_0^\infty \int_\Omega (H(t)M_0(x,t))^2 dxdt \right)^{1/2} \\ \leq \quad & \left((||M_0||_{L^\infty(\Omega\times(0,\infty))})^2 |\Omega| \frac{1}{4\epsilon\alpha} \right)^{1/2} ||H||_{L^2(0,\infty)}. \end{aligned}$$

Using this inequality in (4.30), we find

$$||z||_{L^2(0,\infty),\mathcal{H}} \leq \frac{|\Omega|^{1/2}}{2\sqrt{\epsilon\alpha}} \left((||M_0||_{L^\infty(\Omega\times(0,\infty))}) \right) ||H||_{L^2(0,\infty)} \tag{4.39}$$

If we employ (4.39) in (4.38), we get that

$$\begin{aligned} &|| \int_0^\infty \mu(s) \int_\Omega \nabla\eta(x,t-s)\nabla\varphi_1(x)dxds||_{L^2(0,\infty)} \\ &\leq ||\nabla\varphi_1||_{L^\infty(\Omega)}(\tilde{\mu})^{1/2}|\Omega|^{1/2}\frac{|\Omega|^{1/2}}{2\sqrt{\epsilon\alpha}} \left(||M_0||_{L^\infty(\Omega\times(0,\infty))} \right) ||H||_{L^2(0,\infty)} \end{aligned} \tag{4.40}$$

We use (4.33) and (4.40) in (4.25) to end up with

$$\begin{aligned} ||(TH)(t)|| \leq \frac{1}{\delta} &\left(k_0||\Delta\varphi_1||_{L^\infty(\Omega)} \frac{|\Omega|}{2\sqrt{\epsilon\alpha}} \left(||M_0||_{L^\infty(\Omega\times(0,\infty))} \right) \right. \\ + \quad & \left. ||\nabla\varphi_1||_{L^\infty(\Omega)}(\tilde{\mu})^{1/2}\frac{|\Omega|}{2\sqrt{\epsilon\alpha}} \left(||M_0||_{L^\infty(\Omega\times(0,\infty))} \right) ||H||_{L^\infty(0,\infty)}. \right. \end{aligned} \tag{4.41}$$

Since, we assumed that, M_0 satisfies

$$\frac{1}{\delta}\frac{|\Omega|}{2\sqrt{\epsilon\alpha}} \left(||M_0||_{L^\infty(\Omega\times(0,\infty))} \right) \left(k_0||\Delta\varphi_1||_{L^\infty(\Omega)} + ||\nabla\varphi_1||_{L^\infty(\Omega)}(\tilde{\mu})^{1/2} \right) < 1,$$

then the operator T will be contraction mapping, therefore it will have a unique fixed point which is H. □

4.3 Recovering a Source Term of The Form $K(x)M_0(x,t)$.

We will consider the inverse problem of recovering $K(x)$ satisfying

$$\theta_t(x,t) - k_0\Delta\theta(x,t) - \int_0^\infty \mu(s)\Delta\eta(t-s)ds + \theta(x,t) = K(x)M_0(x,t) \tag{4.42}$$

$$\eta_t(x,t-s) = \theta(x,t) - \eta_s(x,t-s), \quad x \in \Omega, \quad t,s > 0 \tag{4.43}$$

$$z(x,0) = 0 \qquad x \in \Omega \tag{4.44}$$

$$z(x,t) = 0 \qquad x \in \partial\Omega, \quad t,s > 0 \tag{4.45}$$

where $z(x,t) = (\theta(x,t), \eta(x,t-s))$ and the final over determination conditions

$$\theta(x,T) = \gamma(x) \tag{4.46}$$

$$\eta(x,T-s) = \beta(x,s). \tag{4.47}$$

Definition 4.4. A pair of functions (z,K) is said to be generalized solution of the inverse problem (4.42)-(4.47) if $z \in \mathcal{H}, \quad K \in L^2(\Omega)$ and all of the relations (4.42)-(4.47) are satisfied.

We will show the existence and uniqueness of the solution defined above by reducing the problem (4.42)-(4.47) in to an operator equation for $K(x)$. To this end, we assume that

$$\gamma \in H_0^1(\Omega), \qquad \beta \in L_\mu^2((0,\infty), H_0^1), \tag{4.48}$$

and

$$M_0 \in L^\infty(\Omega \times (0,\infty)). \tag{4.49}$$

First, we replace t by T in (4.42) and get

$$\theta_t(x,T) - k_0 \Delta\theta(x,T) - \int_0^\infty \mu(s)\Delta\eta(T-s)ds + \theta(x,T) = K(x)M_0(x,T). \tag{4.50}$$

Employing (4.46) and (4.47) in (4.50), we end up with

$$\theta_t(x,T) - k_0\Delta\gamma(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x) = K(x)M_0(x,T). \tag{4.51}$$

If we solve (4.51) for $K(x)$, we find

$$K(x) = \frac{-k_0\Delta\gamma(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x)}{M_0(x,T)} + \frac{\theta_t(x,T)}{M_0(x,T)}. \tag{4.52}$$

Setting

$$\Phi(x) = \frac{-k_0\Delta\gamma(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x)}{M_0(x,T)}, \tag{4.53}$$

and

$$(AK)(x) := \frac{\theta_t(x,T)}{M_0(x,T)}, \tag{4.54}$$

the correspondence between $z = (\theta, \eta)$ and $K(x)$ may be given as

$$K(x) = \Phi(x) + (AK)(x). \tag{4.55}$$

Second, we again replace t by T in (4.43) and get

$$\eta_t(x, T-s) = \gamma(x) - \eta_s(x, T-s), \quad x \in \Omega, \quad s > 0 \tag{4.56}$$

A is a linear operator from $L^2(\Omega)$ to $L^2(\Omega)$ if $M_0(x,T) > \delta > 0, \quad \forall x \in \Omega$.

Theorem 4.5. *Assume that the constant k_0 in (4.42) is such that $-\frac{1}{k_0}$ is not* *an eigenvalue for Δ in Ω. Then, the inverse problem (4.42)-(4.47) is soluble if and only if the operator equations(4.55),(4.56) are soluble.*

Proof. If the inverse problem (4.42)-(4.47) is soluble, we follow the steps to get (4.55) and (4.56). Hence, we conclude that (4.55),(4.56) are soluble.

Now, we assume that (4.55) and (4.56) have a solution $K(x)$. If we substitute this value of $K(x)$ in (4.42), we get a direct problem (4.42)-(4.45) which has a unique solution $(\theta(x,t), \eta^t(x,s))$ by Theorem 2.1. The problem is whether this solution $(\theta(x,t), \eta^t(x,s))$ satisfies (4.46),(4.47) or not.

By the discussions in Chapter 1.1, we know that, the problem (4.42)-(4.45) is equivalent to

$$\theta_t(x,t) - k_0\Delta\theta(x,t) - \int_{-\infty}^{t} k(t-s)\Delta\theta(s)ds + \theta(x,t) = K(x)M_0(x,t) \quad (4.57)$$

subject to the conditions

$$\begin{aligned} \theta(x,t) &= 0 \qquad x \in \partial\Omega, \quad t > 0 &(4.58)\\ \theta(x,0) &= 0 \qquad x \in \Omega. &(4.59) \end{aligned}$$

Then, (4.57) gives

$$\theta_t(x,T) - k_0\Delta\theta(x,T) - \int_{-\infty}^{T} k(T-s)\Delta\theta(s)ds + \theta(x,T) = K(x).M_0(x,T). \quad (4.60)$$

Since (4.55) holds, (4.51) must hold. Because of the equivalence of the problems given above (4.51) takes the form

$$\theta_t(x,T) - k_0\Delta\gamma(x) - \int_{-\infty}^{T} k(T-s)\Delta\theta(s)ds + \gamma(x) = K(x).M_0(x,T). \quad (4.61)$$

Subtracting (4.61) from (4.60), we find

$$k_0\Delta(\gamma(x)-\theta(x,T))-(\gamma(x)-\theta(x,T))=0. \tag{4.62}$$

Since $\gamma(x)|_{\partial\Omega}=\theta(x)|_{\partial\Omega}$, (4.62) has homogeneous boundary conditions. We assumed that $-\frac{1}{k_0}$ is not an eigenvalue for Δ in Ω then $(\varphi(x)-\theta(x,T))=0$ showing that $\theta(x,t)$ satisfies (4.46). □

Theorem 4.6. *Assume that $M_0(x,t)\in L_\infty(\Omega\times(0,\infty)), |M_{0t}|\in L^1((0,\infty)L_\infty(\Omega))$ and $|M_0(x,T)|>\delta>0, \forall x\in\Omega$. If M_{0t} satisfies the inequality*

$$\frac{1}{2\sqrt{\epsilon\alpha}}\frac{1}{\delta}||K||\left(\int_0^\infty \sup_{x\in\Omega}|M_{0t}(x,t)|dt\right)^{1/2}<1,$$

then $K(x)=\Phi(x)+AK$ has a unique solution in $L^2(\Omega)$.

Proof. Since the operator A is linear, we use contraction mapping principle for the proof. We estimate $L^2(\Omega)$ norm of A as follows: First, we differentiate (4.42) and (4.43) to get

$$\theta_{tt}(x,t)-k_0\Delta\theta_t(x,t)-\int_0^\infty \mu(s)\Delta\eta_t(t-s)ds+\theta_t(x,t)=h_t(x,t) \tag{4.63}$$

$$\eta_{tt}(x,t-s)=\theta_t(x,t)-\eta_{st}(x,t-s), \quad x\in\Omega, \quad t,s>0 \tag{4.64}$$

We multiply (4.63) by $\theta_t(x,t)$ in the sense of $L^2(\Omega)$ and get

$$\frac{1}{2}\frac{d}{dt}||\theta_t(t)||^2+k_0||\nabla\theta_t(t)||^2+\langle\nabla\eta_t,\nabla\theta_t\rangle_\mu+||\theta_t(t)||^2=\langle h_t,\theta_t\rangle. \tag{4.65}$$

And we multiply (4.64) by $-\Delta\eta(x,t-s)$ in the sense of L^2_μ to get

$$\frac{1}{2}\int_0^\infty \mu(s)\frac{d}{dt}||\nabla\eta_t||^2ds-\langle\nabla\eta_t,\nabla\theta_t\rangle_\mu-\frac{1}{2}\int_0^\infty \mu^{'}(s)||\nabla\eta_t||^2ds=0 \tag{4.66}$$

We add (4.65) and (4.66) to find

$$\frac{1}{2}\frac{d}{dt}||z_t(t)||^2_{\mathcal{H}}+k_0||\nabla\theta_t(t)||^2+||\theta_t(t)||^2-\frac{1}{2}\int_0^\infty \mu^{'}(s)||\nabla\eta_t||^2 ds = \langle h_t, \theta_t\rangle. \quad (4.67)$$

Using the fact $\mu^{'} + \delta\mu \leq 0, \quad \delta > 0$ in (4.67), multiplying the resulting inequality by 2 and using Schwarz inequality for the term in the right hand side of (4.67), we end up with

$$\frac{d}{dt}||z_t(t)||^2_{\mathcal{H}} + 2k_0||\nabla\theta_t(t)||^2 + 2||\theta_t(t)||^2 + \delta\int_0^\infty \mu(s)||\nabla\eta_t||^2 ds \leq ||h_t|||\theta_t||| \quad (4.68)$$

If we use Poincaré inequality for the second term in the left hand side of (4.68), we find

$$\frac{d}{dt}||z_t(t)||^2_{\mathcal{H}} + 2k_0\lambda_0||\theta_t(t)||^2 + 2||\theta_t(t)||^2 + \delta\int_0^\infty \mu(s)||\nabla\eta_t||^2 ds \leq ||h_t||||\theta_t|| \quad (4.69)$$

Using Young's inequality with ϵ in the right hand side of (4.69), we may write it in the following form

$$\frac{d}{dt}||z_t(t)||^2_{\mathcal{H}} + (2k_0\lambda_0 + 2 - \epsilon)||\theta_t(t)||^2 + \delta\int_0^\infty \mu(s)||\nabla\eta_t||^2 ds \leq \frac{1}{4\epsilon}||h_t||^2. \quad (4.70)$$

We set $\alpha := min\{(2k_0\lambda_0 + 2 - \epsilon), \delta\}$ to write (4.70) as

$$\frac{d}{dt}||z_t(t)||^2_{\mathcal{H}} + \alpha||z_t(t)||^2_{\mathcal{H}} \leq \frac{1}{4\epsilon}||h_t||^2. \quad (4.71)$$

Solving the differential inequality (4.71), we find

$$||z_t(t)||^2_{\mathcal{H}} \leq \frac{1}{4\epsilon\alpha}||h_t||^2_{L^2(\mathbb{R}^+, L^2(\Omega))}. \quad (4.72)$$

We know that

$$||h_t||^2_{L^2(\mathbb{R}^+,L^2(\Omega))} = \int_0^\infty ||h_t(t)||^2 dt = \int_0^\infty ||K(.)M_{0t}(.,t)||^2 dt. \tag{4.73}$$

Using (4.73) in (4.72), we get the the following estimate

$$||z_t(t)||^2_{\mathcal{H}} \leq \frac{1}{4\epsilon\alpha}||K||^2 \int_0^\infty \sup_{x\in\Omega} |M_{0t}(x,t)|dt. \tag{4.74}$$

By (4.54), we deduce that

$$||(AK)(x)|| \leq \frac{1}{2\sqrt{\epsilon\alpha}}\frac{1}{\delta}||K||\left(\int_0^\infty \sup_{x\in\Omega} |M_{0t}(x,t)|dt\right)^{1/2}. \tag{4.75}$$

Since, we assumed that M_{0t} satisfies

$$\frac{1}{2\sqrt{\epsilon\alpha}}\frac{1}{\delta}\left(\int_0^\infty \sup_{x\in\Omega} |M_{0t}(x,t)|dt\right)^{1/2} < 1,$$

we get that, the operator A is a contraction mapping and it will have a unique fixed point K. □

4.4 Recovering a Source Term of The Form $K(x).M_0(x,t)$ using integral final overdetermination condition

We will consider the problem (4.42)-(4.45) with the final overdetermination conditions of the type

$$\int_0^T \theta(x,t)\varphi_2(t)dt = \gamma(x) \tag{4.76}$$

and

$$\int_0^T \eta(x,t-s)\varphi_2(t)dt = \beta(x,s), \tag{4.77}$$

where $\varphi_2(t) \in \mathcal{D}(0,T)$. We will covert the inverse problem (4.42)-(4.45) with (4.76)-(4.77) to an operator equation. To this end, first we multiply (4.42) with $\varphi_2(t)$, integrate over $(0,T)$ and employ (4.76)-(4.77) to get

$$\int_0^T \theta_t(x,t)\varphi_2(t)dt - k_0\Delta\gamma(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x) = K(x)M(x), \tag{4.78}$$

where

$$M(x) = \int_0^T M_0(x,t)\varphi_2(t)dt, \tag{4.79}$$

assuming $M(x) > \delta > 0$. We solve (4.78) for $K(x)$ and find

$$K(x) = \frac{-k_0\Delta\gamma(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x)}{M(x)} + \frac{\int_0^T \theta_t(x,t)\varphi_2(t)dt}{M(x)}. \tag{4.80}$$

We set

$$\Phi(x) := \frac{k_0\Delta\gamma(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x)}{M(x)} \tag{4.81}$$

$$(PK)(x) := \frac{\int_0^T \theta_t(x,t)\varphi_2(t)dt}{M(x)} \tag{4.82}$$

The correspondence between $z = (\theta, \eta)$ and $K(x)$ may be given as follows;

$$K(x) = \Phi(x) + (PK)(x), \tag{4.83}$$

where P is a linear operator from $L^2(\Omega)$ to $L^2(\Omega)$ if $M(x) > \delta > 0, \forall x \in \Omega$. Second, we multiply (4.43) by $\varphi_2(t)$ and integrate over $(0,T)$, employ (4.76) and get identity

$$\int_0^T \eta_t(x,t-s)\varphi_2(t)dt = \gamma(x) - \int_0^T \eta_s(x,t-s)\varphi_2(t)dt \tag{4.84}$$

Theorem 4.7. *Assume that the constant k_0 in (4.42) is such that $-\frac{1}{k_0}$ is not an eigenvalue for Δ in Ω. Then, the inverse problem (4.42)-(4.45) with (4.76),(4.77) is soluble if and only if the operator equations(4.83),(4.84)are soluble.*

Proof. If the inverse problem (4.42)-(4.45) with (4.76),(4.77) is soluble, then we follow the steps to get the operator equations (4.83),(4.84). Hence , we deduce that the operator equations are soluble.

For the "only if" part, we assume that (4.83), (4.84) has a solution $K(x)$, if we substitute this value of $K(x)$ in (4.42), we get the direct problem (4.42)-(4.47) which has a unique solution $(\theta(x,t), \eta^t(x,s))$ by Theorem 2.1.

We have the some problem as in the proof of Theorem 4.5. The problem is whether this solution $(\theta(x,t), \eta^t(x,s))$ satisfies (4.76), (4.77) or not. By the discussions in Chapter 1.1, the problem (4.42)-(4.45) is equivalent to (4.57)-(4.59). Let us use this equivalence, if we multiply (4.57) by $\varphi_2(t)$ and integrate over $(0,T)$, we get

$$\begin{aligned}\int_0^T \theta_t(x,t)\varphi_2(t)dt \quad &- \quad k_0\int_0^T \Delta\theta(x,t)\varphi_2(t)dt - \int_0^T \varphi_2(t)\int_{-\infty}^t k(t,s)\Delta\theta(x,s)ds \\ &+ \quad \int_0^T \theta(x,t)\varphi_2(t)dt = K(x)\int_0^T M_0(x,t)\varphi_2(t)dt \end{aligned} \tag{4.85}$$

Since (4.83) has a solution, it satisfies

$$\int_0^T \theta_t(x,t)\varphi_2(t) - k_0\Delta\varphi(x) - \int_0^\infty \mu(s)\Delta\beta(x,s)ds + \gamma(x) = K(x).\int_0^T M_0(x,t)\varphi_2(t)dt \tag{4.86}$$

Because of the equivalence of (4.42)-(4.45) and (4.57)-(4.59), (4.86) takes the form

$$\int_0^T \theta_t(x,t)\varphi_2(t)dt \;-\; k_0\Delta\gamma(x) - \int_0^T \varphi_2(t)\int_{-\infty}^t k(t-s)\Delta\theta(x,s)ds + \varphi(x)$$
$$= K(x).\int_0^T M_0(x,t)\varphi_2(t)ds \tag{4.87}$$

Subtracting (4.87) from (4.85), we get

$$-k_0[\Delta(\int_0^T \theta(x,t)\varphi_2(t)dt - \gamma(x))] + [\int_0^T \theta(x,t)\varphi_2(t)dt - \gamma(x)] = 0. \tag{4.88}$$

Since $\int_0^T \theta(x,t)\varphi_2(t)dt|_{\partial\Omega} = \gamma(x)|_{\partial\Omega}$, the equation (4.88) has homogeneous boundary condition. We assumed that $-\frac{1}{k_0}$ is not an eigenvalue for Δ in Ω, then the equation (4.88) has only the trivial solution. Hence, $\gamma(x) = \int_0^T \theta(x,t)\varphi_2(t)dt$, which shows that $\theta(x,t)$ satisfies (4.76). □

Theorem 4.8. *Assume that* $|M(x)| > \delta > 0$, $M_{0t} \in L^1((0,\infty), L_\infty(\Omega))$.
If M_{0t} *satisfies the inequality*

$$\frac{1}{2\sqrt{\epsilon\alpha}}\frac{1}{\delta}\tilde{\varphi_2}\left(\int_0^\infty \sup_{x\in\Omega}|M_t(x,t)|dt\right)^{1/2} < 1,$$

where $\tilde{\varphi_2}^2 = \int_0^T \varphi_2(\tau)^2 d\tau$, *then the operator equation (4.83) has a unique solution* $K(x)$ *in* $L^2(\Omega)$.

Proof. For the proof, we use contracting mapping principle. Thus, we need to estimate $L^2(\Omega)$ norm of $(PK)(x)$. By the definition of $(PK)(x)$, we have

$$||(PK)||^2 = ||\frac{\int_0^T \theta_t(x,t)\varphi_2(t)dt}{M(x)}||^2. \tag{4.89}$$

Since $M(x) > \delta > 0$, (4.89) satisfies the following inequality ;

$$||(PK)||^2 \leq \frac{1}{\delta^2}||\int_0^T \theta_t(x,t)\varphi_2(t)dt||^2 = \frac{1}{\delta^2}\int_\Omega\Big(\int_0^T \theta_t(x,\tau)\varphi_2(\tau)d\tau\Big)^2 dx. \tag{4.90}$$

If we apply Hölder's inequality for the integrand of the last term of (4.90)

$$||(PK)(x)||^2 \leq \frac{1}{\delta^2}\tilde{\varphi_2}^2\int_0^T ||\theta_t(t)||^2 dt \leq \frac{1}{\delta^2}\tilde{\varphi_2}^2||z(t)||_{\mathcal{H}}, \tag{4.91}$$

where $\tilde{\varphi_2}^2 = \int_0^T \varphi_2(\tau)^2 d\tau$. Using (4.72) in (4.91), we get

$$||(PK)(x)|| \leq \frac{1}{2\sqrt{\epsilon\alpha}}\frac{1}{\delta}\tilde{\varphi_2}||K||\Big(\int_0^\infty \sup_{x\in\Omega}|M_{0t}(x,t)|dt\Big)^{1/2}. \tag{4.92}$$

Since, M_{0t} satisfies that

$$\frac{1}{2\sqrt{\epsilon\alpha}}\frac{1}{\delta}\tilde{\varphi_2}\Big(\int_0^\infty \sup_{x\in\Omega}|M_{0t}(x,t)|dt\Big)^{1/2} < 1,$$

then the operator P is a contraction mapping and it will have a unique fixed point K. □

Remark: The results of Chapter 4 are also valid, if we take the governing differential equation as

$$\theta_t(x,t) - k_0\Delta\theta(x,t) - \int_0^\infty \mu(s)\Delta\eta(x,t-s)ds + g(\theta(x,t)) = h(x,t) \tag{4.93}$$

$$\eta_t(x,t-s) = \theta(x,t) - \eta_s(x,t-s) \tag{4.94}$$

$$z(x,0) = 0 \quad x \in \Omega \tag{4.95}$$

$$z(x,t) = 0 \quad x \in \partial\Omega \quad t,s > 0, \tag{4.96}$$

where $z(x,t) = (\theta(x,t), \eta(x,t-s))$ and $g(\theta)$ as in (1.41).

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VITA

Müjdat Kaya was born in Ankara, on 15th May, 1973. He has completed his high school education in Ankara Gazi Lisesi. He started his undergraduate studies at the Department of Mathematics Education, M.E.T.U., in 1990. He took his B.Sc. degree in Mathematics Education and in Mathematics as a double major in 1995. He became a research assistant at the Department of Mathematics, M.E.T.U., in the same year and studied his graduate studies. He received his M.Sc. degree in Mathematics in December 1998, with the thesis entitled as " Existence of Continuous Linear Right Inverse for Linear Partial Differential Operators with Constant Coefficients ". He is now an instructor in the Department of Mechanical Engineering, Başkent University.