INVERSE PROBLEMS FOR PARABOLIC EQUATIONS

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# INVERSE PROBLEMS FOR PARABOLIC EQUATIONS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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# ABSTRACT

## INVERSE PROBLEMS FOR PARABOLIC EQUATIONS

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In this thesis, we study inverse problems of restoration of the unknown function in a boundary condition, where on the boundary of the domain there is a convective heat exchange with the environment. Besides the temperature of the domain, we seek either the temperature of the environment in Problem I and II, or the coefficient of external boundary heat emission in Problem III and IV. An additional information is given, which is the overdetermination condition, either on the boundary of the domain (in Problem III and IV) or on a time interval (in Problem I and II). If solution of inverse problem exists, then the temperature can be defined everywhere on the domain at all instants.

The thesis consists of six chapters. In the first chapter, there is the introduction where the definition and applications of inverse problems are given and definition of the four inverse problems, that we will analyze in this thesis, are stated. In the second chapter, some definitions and theorems which we will use to obtain some conclusions about the corresponding direct problem of our four inverse problems are stated, and the conclusions about direct problem are obtained. In the third, fourth, fifth and sixth chapters we have the analysis of inverse problems I, II, III and IV, respectively.

Keywords: inverse problem, partial differential equation of parabolic type, integral equation

# ÖZ

## PARABOLİK DENKLEMLERDE TERS PROBLEMLER

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Bu tezde, bölgenin sınırında çevreyle konvektif ısı alışverişi olan ters problemin sınır koşulundaki bilinmeyen fonksiyonun bulunmasını inceliyoruz. Bölgenin sıcaklığıyla beraber, çevrenin sıcaklığını (Problem I ve III) veya sınırın dış emisyon katsayısını (Problem II ve IV) bulmaya çalışıyoruz. Ek bir bilgi I. ve II. Problemlerde zaman aralığı üzerinde, III. ve IV. Problemlerde bölgenin sınırı üzerinde veriliyor. Eğer ters problemin çözümü varsa, bölgenin her yerinde ve her zaman için sıcaklık bulunabilir.

Tez altı bölümden oluşmaktadır.

Giriş bölümünde ters problemlerin tanımı ve uygulama alanları ve- riliyor ve tezde inceleyeceğimiz dört ters problemin tanımları belirtiliyor.

İkinci bölümde bu dört ters probleme ait olan direkt problem hakkında sonuçlara ulaşabilmek için kullanacağımız bazı tanım ve teoremleri belirtiyor ve direkt problemle ilgili sonuçları elde ediyoruz.

Üçüncü, dördüncü, beşinci ve altıncı bölümlerde sırasıyla Problem I, II ,III ve IV' ü inceliyoruz.

Anahtar Kelimeler: ters problem, parabolik kısmi diferansiyel denklem, integral denklemleri

To my family

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# CHAPTER 1

# INTRODUCTION

In the study of differential equations, the usual procedure is to find the solution to a given differential equation with some initial and/or boundary conditions. In inverse problems, the situation is reversed; one is given the solution or some information about the solution to an unknown differential equation and the differential equation must be determined.

We can give another definition by recalling Professor J.B. Keller's paper [20]. We call two problems "inverses" of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time while the other is newer and not so well understood. In such cases the former is called the "direct problem" while the other is called the "inverse problem".

However there is often another, more important difference between these two problems. Hadamard [12] introduced the concept of a well posed problem, originating from the philosophy that the mathematical model of a physical problem has to have the properties of existence, uniqueness and stability of the solution. If one of the properties fails to hold then the problem is called ill-posed. It turns out that many interesting and important inverse problems in science lead to ill-posed problems, while the corresponding direct problems are well-posed.

In direct problems, the solution of a given differential equation or system of equations is the unknown part, while in inverse problems the equation itself is unknown. Of course inverse problems may be classified arbitrarily, but in our thesis classification is by the type of information that is being sought in the solution procedure. We classify inverse problems as:

Backward or Retrospective Problem: The initial conditions are to be found.

Coefficient Inverse Problem: This is a classical parameter problem where a constant multiplier in the governing (main) equation is to be found.

Boundary Inverse Problem: Some missing information at the boundary of the domain is to be found.

In the field of inverse problems, in the last two decades there has been a rapid development. The enormous increase in computing power and the development of powerful numerical methods made it possible to simulate real world direct problems of growing complexity. Since in many applications in science and engineering, the inverse question of determining causes for desired or observed effects is really the final question, this lead to a growing appetite in applications for posing and solving inverse problems, which in turn stimulated mathematical research. This began mainly for linear problems, but more recently it has also been done for nonlinear problems.

One of the important researchers who studied in the field of inverse boundary problems is V.Isakov. Some of his important studies are on the subjects which are inverse problems of gravimetry and related problems of imaging [15], inverse problems of conductivity and its applications to medical imaging, inverse scattering problems , finding constitutional laws from experimental data and uniqueness of the continuation for hyperbolic equations and systems of mathematical physics [16, 17, 18, 19]. Also other researchers as M. Lassas, Y. Kurylev, A. Katchalov, A. Katsuda [25, 26, 27, 28, 29], R. Kress [23, 24], M. Yamamoto [34], and R. Chapko [5, 6, 7] have studied inverse boundary problems.

In the other classes of inverse problems, there are also countless recent studies in the last decade. Some of the important researchers are A.K. Alekseev (retrospective and boundary inverse problems) [1, 2], T. Shores (inverse coefficient problem)[33], A. Hasanov (inverse coefficient problem) [13], and A. Denisov (inverse coefficient problem) [8].

Applications of inverse problems are growing very rapidly as well and now they include physics, geophysics, chemistry, medicine and engineering, and have a great deal of attention by mathematicians, statisticians, physicists and engineers.

For many centuries people are searching for hiding places by tapping walls and analyzing the echo. Generally inverse problems are those of finding some characteristics of a medium from knowledge of some fields interacting with the medium. These fields are usually measured outside the medium, for instance on its boundary.

A specific example is in the field of nondestructive testing or medical imaging, in which one seeks to image the interior of an object without damaging the object. To do this, some kind of energy, electromagnetic, thermal or mechanical, is applied to the boundary of the object. The energy flows through the object in a quantifiable way that depends on the interior structure. By taking boundary measurements of the object's physical response, one tries to deduce the interior structure of the object. The flow of energy is generally governed by a partial differential equation and the interior structure of the object is manifest as unknown coefficients or boundary coefficients of the partial differential equation.

Some of the applications are:

Geophysical prospecting, general acoustics problems, inverse scattering theory: Given a signal and an unknown obstacle, then determining what does the obstacle look like.

Industrial prospecting: Finding fluid flows from only the boundary measurements.It is used when the direct measurement is difficult, particularly in the case of fluid flows.

Medical diagnosis, scanners, tomography, electrical imaging: Finding out something about the inside of a body from measurements taken only from the outside.

Medical imaging using electricity (EIT), for example to diagnose water in the lung.

There are many other applications like industrial process monitoring using electrical measurement, detection of abandoned plastic anti-personnel mines electromagnetically, photoelasticity-visualisation of the stress inside a transparent object, electromagnetic monitoring of molten metal flow, inverse quantum scattering, radar imaging, imaging at NASA, seismography and petroleum exploration, acoustic pyrometry, neutron reflectometry and analysis of thin films, radiometry and radiocarbon dating, neutron radiography, solutions of fiber optics.

In this thesis we will analyze inverse problems of the third type, in which the unknown function is in the boundary condition, that is boundary inverse problem.

In this thesis, we have four inverse boundary problems. In the $1^{st}$ and $3^{rd}$ problems, the scalar function in the boundary condition (that is the temperature of the environment) is unknown while in the $2^{nd}$ and $4^{th}$ problems the coefficient function (that is the coefficient of external boundary heat emission) is unknown. In the problems we need extra information about solution. This extra information is point or integral overdetermination condition, which is a given knowledge either on the boundary of the domain (problem III and IV) or on a time interval (problem I and II).

Let $\Omega \subset R^m$ be a bounded simply connected domain with boundary $\Gamma \in C^{(1+\alpha)}$, $\alpha \in (0,1)$. In the cylinder $Q = \Omega \times (0,T]$, where $T > 0$, with lateral area $S = \Gamma \times (0,T]$, consider the following inverse problems.

**Problem I:** Find a pair of functions $\{u(x,t), f(x)\}$ satisfying

$$u_t - \Delta u = g(x,t) \text{ on } Q \tag{1.1}$$

$$u(x,0) = a(x) \text{ on } \bar{\Omega} \tag{1.2}$$

$$\partial_n u + \sigma u = h(x,t)f(x) + b(x,t) \text{ on } S \tag{1.3}$$

$$\ell(u) = \chi(x) \text{ on } \Gamma, \tag{1.4}$$

where $g(x,t)$, $a(x)$, $\sigma(x)$, $h(x,t)$, $b(x,t)$, $\chi(x)$ are given and $n$ is the outward pointing normal to $\Gamma$. The expression $\ell(u)$ has one of the forms

$$\ell(u) = u(x,t_1), 0 < t_1 < T$$

or

$$\ell(u) = \int_0^T u(x,\tau)\omega(\tau)d\tau,$$

where $t_1$ is chosen and $\omega \in L_1(0,T)$ is given. These conditions are called terminal and integral boundary observations respectively.

We have $\partial u/\partial n_x = lim_{y\to x}[\partial u(y,t)/\partial n_x]$, $y \in K$ , where $K$ is a closed cone with the vertex at point $x$ that is entirely contained in $\Omega \cup \{x\}$.

**Problem II:** Find a pair of functions $\{u(x,t), \sigma(x)\}$ satisfying (1.1), (1.2), (1.4) and

$$\frac{\partial u}{\partial n} + \sigma u = b(x,t) \text{ on } S, \tag{1.5}$$

where $g(x,t)$, $a(x)$, $b(x,t)$ and $\chi(x)$ are given functions.

**Problem III:** Find a pair $\{u(x,t), f(t)\}$ satisfying

$$\begin{aligned} u_t - \Delta u &= g \text{ on } Q & (1.6) \\ u(x,0) &= a \text{ on } \bar{\Omega} & (1.7) \\ \frac{\partial u}{\partial n} + \sigma u &= hf + b \text{ on } S & (1.8) \\ \Psi(u) &= \bar{\chi} \text{ on } [0,T], & (1.9) \end{aligned}$$

where $g(x,t)$, $a(x)$, $\sigma(x,t)$, $h(x,t)$, $b(x,t)$, $\bar{\chi}(t)$ are given functions and

$$\Psi(u) = u(x_0, t)$$

or

$$\Psi(u) = \int_{\Gamma} u(\xi, \tau)\nu(\tau) dS_\xi,$$

where $x_0 \in \Gamma$ is fixed and $\nu \in L_1(\Gamma)$ is given. These conditions are called a point boundary observation and integral boundary observation respectively.

**Problem IV:** Find a pair of functions $\{u(x,t), \sigma(t)\}$ satisfying (1.6), (1.7), (1.9) and

$$\frac{\partial u}{\partial n} + \sigma u = b \text{ on } S, \tag{1.10}$$

where $g(x,t)$, $a(x)$, $b(x,t)$, $\bar{\chi}(t)$ are given functions.

# CHAPTER 2

# PRELIMINARIES

## 2.1 Maximum Principles

In the proof of uniqueness theorem for Problem I and in the proof of existence and uniqueness theorem for Problem III, maximum principles for parabolic and elliptic equations, and some results on the the derivative of solution $u$ of the corresponding direct problem with respect to the outward pointing normal at extreme points will be needed. So, let us state these theorems first.

Let

$$L(u) = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u \tag{2.1}$$

for $x \in R^m$. The operator $L$ is called elliptic at a point $x^*$ if and only if

$$\sum_{i,j=1}^{m} a_{ij}(x^*)\xi_i\xi_j \geq \mu(x^*) \sum_{i=1}^{m} \xi_i^2$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in R^m$. If $L$ is elliptic at each point of the domain $Q$, then $L$ is called elliptic in $Q$. If $L$ is elliptic in $Q$ and there exists a constant $\mu_0 > 0$ such that $\mu(x) \geq \mu_0$ then $L$ is called uniformly elliptic in $Q$.

THEOREM 2.1.1. *[32] (p. 61)The maximum Principle of E.Hopf: Let $L$ be uniformly elliptic and $L(u) \geq 0$ in a domain $Q$. Suppose the coefficients $a_{ij}$ and $b_i$ are uniformly bounded. If $u$ attains a maximum $M$ at an interior point of $Q$ then $u \equiv M$ in $Q$.*

THEOREM 2.1.2. *[32] (p. 65) Let $L(u) \geq 0$ in $Q$, in which $L$ is uniformly elliptic. Suppose $u \leq M$ in $Q$ and $u = M$ at a boundary point $P$. Assume $P$ lies on the* *boundary of a ball $K_1$ in $Q$. If $u$ is continuous in $Q \cup P$ and an outward directional derivative $\frac{\partial u}{\partial n}$ exists at $P$, then $\frac{\partial u}{\partial n} > 0$ at $P$ unless $u \equiv M$.*

Let $\Omega$ be a bounded, simply connected domain in $R^m$, $Q = \Omega \times (0,T]$, $\Gamma = \partial\Omega$ and $S = \Gamma \times (0,T]$. Now define the differential operator $L$ for parabolic equations in the domain $Q$ as

$$L(u) = \sum_{i,j=1}^{m} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t}. \tag{2.2}$$

The operator $L$ is parabolic at a point $(x^*, t^*)$ if and only if there exists $\mu(x,t) > 0$ such that

$$\sum_{i,j=1}^{m} a_{ij}(x^*, t^*)\xi_i \xi_j \geq \mu(x^*, t^*) \sum_{i=1}^{m} \xi_i^2$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in R^m$. The operator $L$ is called parabolic in a domain $Q$ if and only if it is parabolic at each point of that domain. The operator $L$ is uniformly parabolic in a domain $Q$ if and only if there exists a constant $\mu_0 > 0$ such that $L$ is parabolic in $Q$ and $\mu(x,t) > \mu_0$ in $Q$.

Theorem 2.1.3. *[32](pp173) Let $L$ be uniformly parabolic, $L(u) \geq 0$ in a domain $Q$ and suppose the coefficients of $L$ are bounded. Suppose that the maximum of $u$ in $Q$ is $M$ and that it is attained at some interior point $P = (x^*, t^*)$. Denote by $Q(t^*)$ the connected component of the intersection of the hyperplane $t = t^*$ with $Q$, which contains $P$. Then $u \equiv M$ on $Q(t^*)$. If $K = (x,t)$ is a point of $Q$ which can be connected to $P$ by a path in $Q$ consisting only of horizontal segments and upward vertical segments, then $u = M$ at $K$.*

*Remark* 2.1.1. *The above theorem is valid if the point $P = (x^*, t^*)$ is on a horizontal component $\partial Q(t^*)$ of the boundary $\partial Q := \{(x,t) : (x,t) \in \Omega \times t = 0$ or $\in \partial\Omega \times [0,T]\}$ of $Q$, provided that $u$ and the derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $\frac{\partial u}{\partial x_i}$ and $\frac{\partial u}{\partial t}$ are all continuous on $Q \cup \partial Q(t^*)$. For example, if $P$ is on the top side $\Omega \times \{t = T\}$ of a cylinder domain $Q = \Omega \times (0,T)$, where $\Omega$ is a domain of $R^m$.*

Theorem 2.1.4. *[32] (p. 174) Let $L$ be uniformly parabolic, $L(u) \geq 0$ and $L$ has bounded coefficients in $Q = \Omega \times (0,T]$, where $\Omega \subset R^m$. Suppose* $\max_Q u = M$

*is attained at a point $P$ on $\partial Q$, where $\partial Q = \{(x,t) : (x,t) \in \Omega \times t = 0$ or $\in \partial\Omega \times [0,T]\}$. Assume a sphere $K_1$ through $P$ can be constructed whose interior lies entirely in $Q$ and in which $u < M$. Also assume that the radial direction from the center of sphere $K_1$ to $P$ is not parallel to t-axis. Then, if $\frac{\partial}{\partial n}$ denotes any directional derivative in an outward direction, then $\frac{\partial u}{\partial n} > 0$ at $P$ unless $u \equiv M$.*

## 2.2 About Direct Problem

In the proof of existence and uniqueness theorem for the direct problem, continuity of the volume potential $V(x,t) = \int_{T_0}^{t} \int_{\Omega} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau$, continuity of the integral $W(x,t) = \int_{T_0}^{t} \int_{\Omega} G(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau$ and continuity of partial derivatives of these integrals with respect to $x$ and $t$ will be needed. So we have the following definition, lemma and theorems.

Let $L$ be the parabolic in $Q = \bar{\Omega} \times [T_0, T_1]$.

Definition 2.2.1. *[9] (p. 3) A fundamental solution of $Lu = 0$ in $Q$ is a function $G(x,t,\xi,\tau)$ defined for all $(x,t), (\xi,\tau) \in Q$, $t > \tau$ which satisfies the conditions*

*i) for fixed $(\xi,\tau) \in Q$ it satisfies, as a function of $(x,t), x \in \Omega, \tau < t < T_1$, the equation $Lu = 0$.*

*ii) for every continuous function $f(x)$ in $\bar{\Omega}$, if $x \in \Omega$, then*

$$\lim_{t \to \tau} \int_{\Omega} G(x,t,\xi,\tau) f(\xi) d\xi = f(x). \tag{2.3}$$

Lemma 2.2.1. *[9] (p. 7) Let $f(x,y)$ be a continuous function of $(x,y)$, $x,y$ are in a compact subset $S$ of $R^m$ and $x \neq y$. Let $\int_{S(x,\epsilon)} |f(x,y)| dy \to 0$ as $\epsilon \to 0$ uniformly with respect to $x \in S$, where $S(x,\epsilon) = S \cap B(x,\epsilon)$. Then for any bounded measurable function $g(y)$ in $S$, the improper integral*

$$\varphi(x) = \int_{S} f(x,y) g(y) dy$$

*is a continuous function in $S$.*

Let us state the function $G$, the fundamental solution of the equation $Lu = 0$

[9] (p. 4).

$$G(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + Z_0(x,t,\xi,\tau).$$

Here

$$Z(x,t,\xi,\tau) = C(\xi,\tau)w^{\xi,\tau}(x,t,\xi,\tau),$$

where

$$C(\xi,\tau) = (2\sqrt{\pi})^{-m}[\det(a_{ij}(\xi,\tau))]^{1/2}$$

$$w^{\xi,\tau}(x,t,\xi,\tau) = (t-\tau)^{-m/2}\exp[-\frac{\upsilon^{\xi,\tau}(x,\xi)}{4(t-\tau)}]$$

$$\upsilon^{\xi,\tau}(x,\xi) = \sum_{i,j=1}^{m} a_{ij}(\xi,\tau)(x_i-\xi_i)(x_j-\xi_j)$$

and

$$Z_0(x,t,\xi,\tau) = \int_\tau^t \int_\Omega Z(x,t,\nu,\sigma)\Phi(\nu,\sigma,\xi,\tau)d\nu d\sigma,$$

where $\Phi$ is the solution of the integral equation

$$\Phi(x,t,\xi,\tau) = LZ(x,t,\xi,\tau) + \int_\tau^t \int_\Omega LZ(x,t,\nu,\sigma)\Phi(\nu,\sigma,\xi,\tau)d\nu d\sigma.$$

Here

$$\begin{aligned} LZ(x,t,\xi,\tau) &= \sum_{i,j=1}^{m} a_{ij}(x,t) - a_{ij}(\nu,\sigma)\frac{\partial^2 Z(x,t,\nu,\sigma)}{\partial x_i \partial x_j} + \\ &+ \sum_{i=1}^{m} b_i(x,t)\frac{\partial Z(x,t,\nu,\sigma)}{\partial x_i} + c(x,t)Z(x,t,\nu,\sigma). \end{aligned}$$

The solution $\Phi$ is of the form

$$\Phi(x,t,\xi,\tau) = \sum_{k=1}^{m}(LZ)_k(x,t,\xi,\tau),$$

where

$$(LZ)_1 = LZ$$

$$(LZ)_{k+1} = \int_{\tau}^{t} \int_{\Omega} [LZ(x,t,\nu,\sigma)][(LZ)_k(\nu,\sigma,\xi,\tau)] d\nu d\sigma.$$

To find $\Phi$ in this form, method of successive approximations is used, [14] (p. 259). Similar calculations will later be done to find the function $\varphi$ in the proof of existence and uniqueness theorem for the direct problem.

THEOREM 2.2.1. *[9] (p. 4) Let $f(x,t)$ be a continuous function in $Q = \bar{\Omega} \times [T_0, T_1]$, where $\Omega$ is a bounded domain in $R^m$. Then*

$$J(x,t,\tau) = \int_{\Omega} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi \tag{2.4}$$

*is a continuous function in $(x,t,\tau)$, $x \in \bar{\Omega}$, $T_0 \leq \tau < t \leq T_1$ and*

$$\lim_{\tau \to t} J(x,t,\tau) = f(x,t) \tag{2.5}$$

*uniformly with respect to $(x,t)$, $x \in S$, $T_0 < t \leq T_1$, where $S$ is a closed subset of $\Omega$.*

THEOREM 2.2.2. *[9] (p. 8) If $f(x,t)$ is a bounded measurable function in $Q$ then the volume potential*

$$V(x,t) = \int_{T_0}^{t} \int_{\Omega} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau \tag{2.6}$$

*is a continuous function in $Q$.*

THEOREM 2.2.3. *[9] (p. 8) If $f(x,t)$ is a continuous function in $Q$ then the volume potential (2.6) has first continuous partial derivatives with respect to $x$ for $x \in \Omega$, $T_0 < t \leq T_1$ and*

$$\frac{\partial V}{\partial x_i} = \int_{T_0}^{t} \int_{\Omega} \frac{\partial}{\partial x_i} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau. \tag{2.7}$$

THEOREM 2.2.4. *[9] (p. 9) If $f(x,t)$ is a continuous function in $Q$ and locally Hölder continuous function with exponent $\beta$ in $x \in \Omega$ uniformly with respect to $t$, then the volume potential (2.6) has second continuous partial derivatives with* *respect to $x$ for $x \in \Omega$, $T_0 < t \leq T_1$, and*

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \int_{T_0}^{t} \int_{\Omega} \frac{\partial^2}{\partial x_i \partial x_j} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau. \tag{2.8}$$

THEOREM 2.2.5. *[9] (p. 12) If $f(x,t)$ is a continuous function in $Q$ and locally Hölder continuous function with exponent $\beta$ in $x \in \Omega$ uniformly with respect to $t$, then the volume potential (2.6) has first continuous partial derivative with respect to $t$ for $x \in \Omega$, $T_0 < t \leq T_1$, and*

$$\frac{\partial V}{\partial t} = f(x,t) + \int_{T_0}^{t} \int_{\Omega} \sum_{i,j=1}^{m} a_{ij}(\xi,\tau) \frac{\partial^2}{\partial x_i \partial x_j} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau. \tag{2.9}$$

THEOREM 2.2.6. *[9] (p. 13) If $f(x,t)$ is a continuous function in $Q$ and locally Hölder continuous function with exponent $\beta$ in $x \in \Omega$ uniformly with respect to $t$, then the volume potential (2.6) satisfies the equation*

$$\sum_{i,j=1}^{m} a_{ij}(x,t) \frac{\partial^2 V(x,t)}{\partial x_i \partial x_j} - \frac{\partial V(x,t)}{\partial t} = -f(x,t) +$$
$$+ \int_{T_0}^{t} \int_{\Omega} \sum_{i,j=1}^{m} [a_{ij}(x,t) - a_{ij}(\xi,\tau)] \frac{\partial^2}{\partial x_i \partial x_j} Z(x,t,\xi,\tau) d\xi d\tau$$

*for $x \in \Omega$, $T_0 < t \leq T_1$.*

THEOREM 2.2.7. *[9] (p. 21) If $f(x,t)$ is a continuous function in $Q$ then the function*

$$W(x,t) = \int_{T_0}^{t} \int_{\Omega} G(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau \tag{2.10}$$

*is a continuous function in $Q$ and $\frac{\partial W}{\partial x_i}$ are continuous functions for $x \in \Omega$, $T_0 < t \leq T_1$. If $f(x,t)$ is also locally Hölder continuous in $x \in \Omega$, uniformly with respect to $t$, then $\frac{\partial^2 W(x,t)}{\partial x_i \partial x_j}$ and $\frac{\partial W}{\partial t}$ are continuous functions for $x \in \Omega$, $T_0 < t \leq T_1$, and*

$$LW(x,t) = -f(x,t).$$

Now we will state another theorem that will be used in the proof of existence and uniqueness theorem for the solution of direct problem.

THEOREM 2.2.8. *[9] (p. 137) Let the operator L in (2.2) be parabolic in* $Q = \bar{\Omega} \times [0,T]$*, that is for all* $\xi = (\xi_1, \ldots, \xi_m) \neq 0, \sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j > 0$*, the coefficients of L satisfy the following Hölder conditions*

$$
\begin{aligned}
|a_{ij}(x,t) - a_{ij}(x_0,t_0)| &\leq A\left(|x - x_0|^{\alpha} + |t - t_0|^{\alpha/2}\right) && (2.11)\\
|b_i(x,t) - b_i(x_0,t)| &\leq A|x - x_0|^{\alpha} && (2.12)\\
|c(x,t) - c(x_0,t)| &\leq A|x - x_0|^{\alpha}. && (2.13)
\end{aligned}
$$

*Furthermore let* $\Gamma$ *belong to* $C^{1+\alpha}$ *for some* $\alpha \in (0,1)$*, and let* $\varphi$ *be a continuous function on* $\Gamma \times [0,T]$*. Then, for any* $x_0 \in \Gamma$*,* $0 < t \leq T$*, the function* $U(x,t) = \int_0^T \int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau$ *satisfies*

$$
\lim_{x \to x_0, x \in K} \frac{\partial U(x,t)}{\partial n(x_0,t)} = -\frac{1}{2}\varphi(x_0,t) + \int_0^T \int_\Gamma \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)} \varphi(\xi,\tau) d_\xi d\tau. \qquad (2.14)
$$

Now we will state the existence and uniqueness theorem for the direct problem.

THEOREM 2.2.9. *[9] (p. 144) Let the operator L in (2.2) be parabolic in* $Q = \bar{\Omega} \times [0,T]$*, that is for all* $\xi = (\xi_1, \ldots, \xi_m) \neq 0, \sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j > 0$*, the coefficients of L satisfy the Hölder conditions (2.11)-(2.13) and let* $\Gamma$ *belong to* $C^{1+\alpha}$*. If* $a(x)$ *is continuous in* $\bar{\Omega}$ *and vanishes in some Q-neighborhood of* $\partial Q$*, and if* $f$ *is continuous on* $\Gamma \times [0,T]$*, then there exists a unique solution of the direct problem*

$$
\begin{aligned}
Lu(x,t) &= g(x,t) \text{ on } \Omega \times (0,T] && (2.15)\\
u(x,0) &= a(x) \text{ on } \bar{\Omega} && (2.16)\\
\frac{\partial u(x,t)}{\partial n(x,t)} + \sigma(x,t)u(x,t) &= f(x,t) \text{ on } \Gamma \times (0,T]. && (2.17)
\end{aligned}
$$

*Proof.* We first prove the existence. We will try to find the solution in the form

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau + \int_\Omega G(x,t,\xi,0)a(\xi)d\xi\\
&- \int_0^t \int_\Omega G(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau,
\end{aligned}
$$

where $\varphi$ is to be determined. Here, $G$ is the fundamental solution of the equation $Lu = 0$. Consider the function

$$\begin{aligned} F(x,t) &= \int_\Omega \frac{\partial G(x,t,\xi,0)}{\partial n(x,t)} a(\xi) d\xi - \int_0^t \int_\Omega \frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)} g(\xi,\tau) d\xi d\tau \\ &+ \sigma(x,t) \int_\Omega G(x,t,\xi,0) a(\xi) d\xi - \sigma(x,t) \int_0^t \int_\Omega G(x,t,\xi,\tau) g(\xi,\tau) d\xi d\tau \\ &- f(x,t) \text{ on } \Gamma \times (0,T]. \end{aligned}$$

We have, for $0 < \alpha < 1$, $F$ is a continuous function satisfying

$$|F(x,t)| \leq \text{constant}.$$

Let us prove this. First prove continuity of $F$ and then boundedness. We will first show that $G$ is a continuous function.

$$G(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + \int_\tau^t \int_\Omega Z(x,t,\nu,\sigma) \Phi(\nu,\sigma,\xi,\tau) d\nu d\sigma,$$

where $Z(x,t,\xi,\tau)$ is a continuous function of $(x,t,\xi,\tau)$, by definition of $Z$. Moreover $\Phi(x,t,\xi,\tau)$ is Hölder continuous in $x$, more precisely, for any $0 < \beta < \alpha < 1$, we have

$$\begin{aligned} |\Phi(x,t,\xi,\tau) - \Phi(y,t,\xi,\tau)| \leq & \frac{c|x-y|^\beta}{(t-\tau)^{(m+2-\gamma)/2}} \\ & \{\exp(-\frac{\lambda^*|x-\xi|^2}{t-\tau}) + \exp(-\frac{\lambda^*|y-\xi|^2}{t-\tau})\}, \end{aligned}$$

where $\gamma = \alpha - \beta$ and $\lambda^*$ is a positive constant by [9] (p. 17). We have $\Phi(x,t,\xi,\tau)$ is a bounded measurable function since it is Hölder continuous on a bounded domain. Then by using Lemma 2.2.1, $G$ is a continuous function of $(x,t,\xi,\tau)$ for $x, \xi \in \bar{\Omega}$, $T_0 \leq \tau < t \leq T_1$. By Theorem 2.2.3 and Theorem 2.2.5, the derivatives $\frac{\partial G}{\partial x_i}$ and $\frac{\partial G}{\partial t}$ are continuous, that is $\frac{\partial G}{\partial n}$ is continuous. We know the functions $a$, $f$, $g$, $\sigma$ are all continuous. So by the definition of $F$ and by using Lemma 2.2.1, $F$ is a continuous function. Now let us prove boundedness of $F$. If $\mu = (\mu_1, \mu_2, \ldots, \mu_m, 0)$ is inward normal to $S$ at some point $(x,t)$, then inward conormal to $S$ at $(x,t)$ is

$$n = (n_1, n_2, \ldots, n_m, 0) \text{ with } n_i = \sum_{j=1}^{m} a_{ij}\mu_j$$

and

$$\begin{aligned}\frac{\partial}{\partial n} &= (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m}, 0).\vec{n} \\ &= (\sum_{j=1}^{m} a_{1j}\mu_j)\frac{\partial}{\partial x_1} + \ldots + (\sum_{j=1}^{m} a_{nj}\mu_j)\frac{\partial}{\partial x_m}.\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial G}{\partial n} &= \frac{\partial Z}{\partial n} + \frac{\partial Z_0}{\partial n} \\ &= (\frac{\partial Z}{\partial x_1}, \frac{\partial Z}{\partial x_2}, \ldots, \frac{\partial Z}{\partial x_m}, 0).\vec{n} + (\frac{\partial Z_0}{\partial x_1}, \frac{\partial Z_0}{\partial x_2}, \ldots, \frac{\partial Z_0}{\partial x_m}, 0).\vec{n} \\ &= (\sum_{j=1}^{m} a_{1j}\mu_j)(\frac{\partial Z}{\partial x_1} + \frac{\partial Z_0}{\partial x_1}) + \ldots + (\sum_{j=1}^{m} a_{nj}\mu_j)(\frac{\partial Z}{\partial x_m} + \frac{\partial Z_0}{\partial x_m}),\end{aligned}$$

where $a_{ij}$ are Hölder continuous in $x$ and $t$, in $\bar{\Omega} \times [0,T]$. So $a_{ij}$ are bounded, hence $\sum_{j=1}^{m} a_{ij}\mu_j$ are also bounded. Then, using the equations [9] (p. 134)

$$|G(x,t,\xi,\tau)| \leq c/(t-\tau)^{\mu}|x-\xi|^{m-2\mu}, 0 < \mu < 1 \quad (2.18)$$

$$|D_x Z(x,t,\xi,\tau)| \leq c/(t-\tau)^{\mu}|x-\xi|^{m+1-2\mu}, \frac{1}{2} < \mu < 1 \quad (2.19)$$

$$|D_x Z_0(x,t,\xi,\tau)| \leq c/(t-\tau)^{\mu}|x-\xi|^{m+1-2\mu-\alpha}, 1-\frac{\alpha}{2} < \mu < 1 \quad (2.20)$$

we have,

$$|\frac{\partial G}{\partial n}| \leq c.|D_x Z + D_x Z_0| \leq \frac{c}{(t-\tau)^{\mu}|x-\xi|^{m+1-2\mu-\alpha}}$$

Now,

$$
\begin{aligned}
|F(x,t)| &= |\int_\Omega \frac{\partial G(x,t,\xi,0)}{\partial n(x,t)} a(\xi) d\xi - \int_0^t \int_\Omega \frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)} g(\xi,\tau) d\xi d\tau \\
&+ \sigma(x,t) \int_\Omega G(x,t,\xi,0) a(\xi) d\xi - \sigma(x,t) \int_0^t \int_\Omega G(x,t,\xi,\tau) g(\xi,\tau) d\xi d\tau \\
&- f(x,t)| \\
&\leq \int_\Omega |\frac{\partial G(x,t,\xi,0)}{\partial n(x,t)}||a(\xi)| d\xi + \int_0^t \int_\Omega |\frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)}||g(\xi,\tau)| d\xi d\tau \\
&+ |\sigma(x,t)| \int_\Omega |G(x,t,\xi,0)||a(\xi)| d\xi + |\sigma(x,t)| \int_0^t \int_\Omega |G(x,t,\xi,\tau)| \\
& \quad |g(\xi,\tau)| d\xi d\tau \\
&+ |f(x,t)| \text{ on } \Gamma \times (0,T].
\end{aligned}
$$

Since $a$, $g$, $f$ and $\sigma$ are continuous functions they are bounded on $\Gamma \times [0,T]$. So,

$$
\begin{aligned}
|F(x,t)| &\leq \int_\Omega c_1 |\frac{1}{(t-0)^\mu |x-\xi|^{m+1-2\mu-\alpha}}| d\xi + \\
&+ \int_0^t \int_\Omega c_2 |\frac{1}{(t-\tau)^\mu |x-\xi|^{m+1-2\mu-\alpha}}| d\xi d\tau \\
&+ c_3 \int_\Omega |\frac{1}{(t-0)^\mu |x-\xi|^{m-2\mu}}| d\xi + \\
&+ \int_0^t \int_\Omega |\frac{1}{(t-\tau)^\mu |x-\xi|^{m-2\mu}}| d\xi d\tau + c_5 \\
&= c_1 (\frac{1}{t^\mu}) (\frac{-|x-\xi|^{-m-1+2\mu+\alpha+m}}{-1+2\mu+\alpha} |_\Omega) + \\
&+ c_2 (\frac{|t-\tau|^{-\mu+1}}{-\mu+1} |_0^t) (\frac{-|x-\xi|^{-m-1+2\mu+\alpha+m}}{-1+2\mu+\alpha} |_\Omega) \\
&+ c_3 (\frac{1}{t^\mu}) (\frac{-|x-\xi|^{-m+2\mu+m}}{2\mu} |_\Omega) + \\
&+ c_4 (\frac{|t-\tau|^{-\mu+1}}{-\mu+1}) (\frac{-|x-\xi|^{-m+2\mu+m}}{2\mu}) + c_5 \\
&\leq \text{const.}
\end{aligned}
$$

If $\varphi(x,t)$ is a continuous function on $\Gamma \times [0,T]$, then by using Theorem 2.2.8 and definition of $F$ we find an equation to which the boundary condition (2.17) reduces. By Theorem 2.2.8 the function $U(x,t) = \int_0^t \int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau$ satisfies

$$\lim_{x\to x_0} \frac{\partial U(x,t)}{\partial n(x_0,t)} = -\frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}\varphi(\xi,\tau)dS_\xi d\tau.$$

Then our solution $u$ satisfies the equation

$$\begin{aligned}\lim_{x\to x_0} \frac{\partial u(x,t)}{\partial n(x_0,t)} &= -\frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}\varphi(\xi,\tau)dS_\xi d\tau + \\ &+ \int_\Omega \frac{\partial G(x_0,t,\xi,0)}{\partial n(x_0,t)} a(\xi)d\xi \\ &- \int_0^t \int_\Omega \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)} g(\xi,\tau)d\xi d\tau. \end{aligned} \quad (2.21)$$

The condition (2.17) reduces to

$$\lim_{x\to x_0, x\in K} \frac{\partial u(x,t)}{\partial n(x_0,t)} = -\sigma(x_0,t)u(x_0,t) + f(x_0,t). \quad (2.22)$$

From (2.21) and (2.22) we have

$$\begin{aligned}-\sigma(x_0,t)u(x_0,t) + f(x_0,t) &= -\frac{1}{2}\varphi(x_0,t) \\ &+ \int_0^t \int_\Gamma \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}\varphi(\xi,\tau)dS_\xi d\tau \\ &+ \int_\Omega G(x_0,t,\xi,0)a(\xi)d\xi \\ &- \int_0^t \int_\Omega G(x_0,t,\xi,\tau)g(\xi,\tau)d\xi d\tau. \end{aligned}$$

Then,

$$
\begin{aligned}
\varphi(x_0,t) &= -2\{-\sigma(x_0,t)u(x_0,t)+f(x_0,t) \\
&\quad -\int_0^t\int_\Gamma \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}\varphi(\xi,\tau)dS_\xi d\tau \\
&\quad -\int_\Omega G(x_0,t,\xi,0)a(\xi)d\xi+\int_0^t\int_\Omega G(x_0,t,\xi,\tau)g(\xi,\tau)d\xi d\tau\} \\
&= 2\sigma(x_0,t)\int_0^t\int_\Gamma G(x_0,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau \\
&\quad +2\sigma(x_0,t)\int_\Omega G(x_0,t,\xi,0)a(\xi)d\xi \\
&\quad -2\sigma(x_0,t)\int_0^t\int_\Omega G(x_0,t,\xi,\tau)g(\xi,\tau)d\xi d\tau \\
&\quad +2\int_0^t\int_\Gamma \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}\varphi(\xi,\tau)dS_\xi d\tau \\
&\quad +2\int_\Omega \frac{\partial G(x_0,t,\xi,0)}{\partial n(x_0,t)}a(\xi)d\xi \\
&\quad -2\int_0^t\int_\Omega \frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}g(\xi,\tau)d\xi d\tau-2f(x,t) \\
&= 2\int_0^t\int_\Gamma[\sigma(x_0,t)G(x_0,t,\xi,\tau)+\frac{\partial G(x_0,t,\xi,\tau)}{\partial n(x_0,t)}]\varphi(\xi,\tau)dS_\xi d\tau \\
&\quad +2F(x_0,t).
\end{aligned}
$$

So we have obtained that

$$
\begin{aligned}
\varphi(x,t) &= 2\int_0^t\int_\Gamma[\frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)}+\sigma(x,t)G(x,t,\xi,\tau)]\varphi(\xi,\tau)dS_\xi d\tau \\
&\quad +2F(x,t). \qquad (2.23)
\end{aligned}
$$

Setting

$$M(x,t,\xi,\tau)=2\frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)}+2\sigma(x,t)G(x,t,\xi,\tau)$$

and using (2.18) and the inequality

$$|\frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)}|\le c/(t-\tau)^\mu|x-\xi|^{m+a-2\mu-\gamma},$$

where $1-\frac{\gamma}{2}<\mu<1$, $0<\gamma<1$, we obtain

$$\begin{aligned}|M(x,t,\xi,\tau)| &\leq 2|\frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)}|+2|\sigma(x,t)||G(x,t,\xi,\tau)| \\ &\leq 2c(t-\tau)^{\mu}|x-\xi|^{m+1-2\mu-\gamma}+2\frac{|\max_{\Gamma\times[0,T]}\sigma(x,t)|.c}{(t-\tau)^{\mu}|x-\xi|^{m-2\mu}} \\ &\leq c/(t-\tau)^{\mu}|x-\xi|^{m+1-2\mu-\gamma}.\end{aligned}$$

Since $\frac{\partial G}{\partial n}$, $G$ and $\sigma$ are continuous functions, we conclude $M$ is a continuous function. Now let us try to find a series solution $\varphi$ to the integral equation (2.23), [14] (p. 259).

$$\varphi(x,t)=2\int_0^t\int_\Gamma[\frac{\partial G(x,t,\xi,\tau)}{\partial n(x,t)}+\sigma(x,t)G(x,t,\xi,\tau)]\varphi(\xi,\tau)dS_\xi d\tau+2F(x,t). \tag{2.24}$$

We know that $F$ is a continuous and bounded function. $M$ is a continuous function and $|M|\leq c/(t-\tau)^{\mu}|x-\xi|^{m-2\mu}$ for $0<\mu<1$. Let,

$$\varphi_1(x,t) = 2F(x,t)+2\int_0^t\int_\Gamma M(x,t,\xi,\tau)\varphi_0(\xi,\tau)dS_\xi d\tau \tag{2.25}$$

$$\vdots$$

$$\varphi_n(x,t) = 2F(x,t)+2\int_0^t\int_\Gamma M(x,t,\xi,\tau)\varphi_{n-1}(\xi,\tau)dS_\xi d\tau. \tag{2.26}$$

Then we have

$$\varphi_1(\xi,\tau)=2F(\xi,\tau)+2\int_0^\tau\int_\Gamma M(\xi,\tau,\xi_1,\tau_1)\varphi_0(\xi_1,\tau_1)dS_{\xi_1}d\tau_1$$

$$
\begin{aligned}
\varphi_2(x,t) &= 2F(x,t) + 2\int_0^t \int_\Gamma M(x,t,\xi,\tau)[2F(\xi,\tau) + 2\int_0^\tau \int_\Gamma M(\xi,\tau,\xi_1,\tau_1) \\
& \quad \varphi_0(\xi_1,\tau_1) dS_{\xi_1} d\tau_1] dS_\xi d\tau \\
&= 2F(x,t) + 4\int_0^t \int_\Gamma M(x,t,\xi,\tau) F(\xi,\tau) dS_\xi d\tau \\
&+ 4\int_0^t \int_\Gamma \int_0^\tau \int_\Gamma M(x,t,\xi,\tau) M(\xi,\tau,\xi_1,\tau_1) \varphi_0(\xi_1,\tau_1) dS_{\xi_1} d\tau_1 dS_\xi d\tau
\end{aligned} \tag{2.27}
$$

and

$$
\begin{aligned}
\varphi_3(x,t) &= 2F(x,t) + 4\int_0^t \int_\Gamma M(x,t,\xi,\tau) F(\xi,\tau) dS_\xi d\tau \\
&+ 8\int_0^t \int_\Gamma \int_0^\tau \int_\Gamma M(x,t,\xi,\tau) M(\xi,\tau,\xi_1,\tau_1) F(\xi_1,\tau_1) dS_{\xi_1} d\tau_1 dS_\xi d\tau \\
&+ 8\int_0^t \int_\Gamma \int_0^\tau \int_\Gamma \int_0^{\tau_1} \int_\Gamma M(x,t,\xi,\tau) M(\xi,\tau,\xi_1,\tau_1) M(\xi_1,\tau_1,\xi_2,\tau_2) \\
& \quad \varphi_0(\xi_2,\tau_2) dS_{\xi_2} d\tau_2 dS_{\xi_1} d\tau_1 dS_\xi d\tau.
\end{aligned} \tag{2.28}
$$

We introduce the integral operator $K_t$ defined by

$$
K_t f(x,t) = \int_0^t \int_\Gamma M(x,t,\xi,\tau) f(\xi,\tau) dS_\xi d\tau. \tag{2.29}
$$

Then (2.25) becomes

$$
\varphi_1(x,t) = 2F(x,t) + 2K_t \varphi_0(x,t) \tag{2.30}
$$

and (2.26) becomes

$$
\varphi_n(x,t) = 2F(x,t) + 2K_t \varphi_{n-1}(x,t). \tag{2.31}
$$

Further (2.25), (2.27), (2.28) and (2.31) takes the form

$$\begin{aligned}
\varphi_1(x,t) &= 2F(x,t) + 2K_t\varphi_0(x,t) \\
\varphi_2(x,t) &= 2F(x,t) + 4K_tF(x,t) + 4K_t^2\varphi_0(x,t) \\
\varphi_3(x,t) &= 2F(x,t) + 4K_tF(x,t) + 8K_t^2F(x,t) + 8K_t^3\varphi_0(x,t) \\
&\vdots \\
\varphi_n(x,t) &= 2F(x,t) + 4K_tF(x,t) + \ldots + 2^{n-1}K_t^{n-2}F(x,t) + 2^nK_t^{n-1}F(x,t) \\
&+ 2^nK_t^n\varphi_0(x,t). \qquad (2.32)
\end{aligned}$$

Define

$$R_n(x,t) := 2^nK_t^n\varphi_0(x,t). \qquad (2.33)$$

Hence, as $n \to \infty$, (2.24) can be expressed as

$$\varphi(x,t) = 2F(x,t) + \sum_{n=1}^{\infty} 2^{n+1}K_t^nF(x,t). \qquad (2.34)$$

It remains to show that $R_n(x,t) \to 0$ as $n \to \infty$, and (2.34) converges and represents a solution for to (2.24). Since $M(x,t,\xi,\tau)$ and $F(x,t)$ are continuous functions we have

$$|M(x,t,\xi,\tau)| \leq N \qquad |F(x,t)| \leq m. \qquad (2.35)$$

We assume initial approximation $\varphi_0(x,t)$ is also bounded, that is

$$|\varphi_0(x,t)| \leq c. \qquad (2.36)$$

Then

$$\begin{aligned}
|K_t\varphi_0(x,t)| &= \left|\int_0^t\int_\Gamma M(x,t,\xi,\tau)\varphi_0(\xi,\tau)dS_\xi d\tau\right| \\
&\leq Ncc_1.|t-0| \\
|K_t^2\varphi_0(x,t)| &= |\int_0^t\int_\Gamma\int_0^\tau\int_\Gamma M(x,t,\xi,\tau)M(\xi,\tau,\xi_1,\tau_1)\varphi_0(\xi_1,\tau_1)dS_{\xi_1}d\tau_1 \\
&\quad dS_\xi d\tau| \\
&\leq N^2cc_1\int_0^t \tau d\tau \\
&= N^2cc_1|\frac{t^2}{2}-0| \\
&\vdots \\
|K_t^n\varphi_0(x,t)| &\leq N^ncc_1^n|\frac{t^n}{n!}| \qquad (2.37)
\end{aligned}$$

and

$$|K_t^nF(x,t)| \leq N^nmc_1^n|\frac{t^n}{n!}|. \qquad (2.38)$$

Hence,

$$|R_n(x,t)| \leq \frac{2^nN^nmc_1^n|t^n|}{n!}$$

and $R_n(x,t)\to 0$ as $n\to\infty$, for all values of $(x,t)$. From (2.38), it follows that

$$\begin{aligned}
|\varphi(x,t)| &= |2F(x,t)+\sum_{n=1}^{\infty}2^{n+1}K_t^nF(x,t)| \\
&\leq 2|F(x,t)|+\sum_{n=1}^{\infty}2^{n+1}|K_t^nF(x,t)|
\end{aligned}$$

is dominated by

$$2m+\sum_{n=1}^{\infty}2^{n+1}N^nmc_1^n\frac{t^n}{n!},$$

which converges absolutely for all $t$. Since (2.34) satisfies equation (2.24) and converges for all $t$, it is a solution of (2.24). That is there exists a continuous bounded solution $\varphi$ to the integral equation (2.23) expressed in the form

$$\varphi(x,t) = 2F(x,t) + 2\sum_{\nu=1}^{\infty}\int_0^t\int_\Gamma M_\nu(x,t,\xi,\tau)F(\xi,\tau)dS_\xi d\tau, \tag{2.39}$$

where

$$\begin{aligned} M_1(x,t,\xi,\tau) &= M(x,t,\xi,\tau) \\ M_{\nu+1}(x,t,\xi,\tau) &= \int_0^t\int_\Gamma M(x,t,\eta,\sigma)M_\nu(\eta,\sigma,\xi,\tau)dS_\eta d\sigma. \end{aligned}$$

Having proved that $u$ satisfies (2.17), we will now prove (2.15) and (2.16) are also satisfied. The function $u$ satisfies (2.15) by Theorem 2.2.7 and by the equation that $L\Gamma = 0$. We have

$$\begin{aligned} Lu &= L(\int_0^t\int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau + \int_\Omega G(x,t,\xi,0)a(\xi)d\xi \\ &\quad -\int_0^t\int_\Omega G(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau) \\ &= L\int_0^t\int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau + L\int_\Omega G(x,t,\xi,0)a(\xi)d\xi \\ &\quad -L\int_0^t\int_\Omega G(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau \\ &= \int_0^t\int_\Gamma LG(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau + \int_\Omega LG(x,t,\xi,0)a(\xi)d\xi - (-f(x,t)) \\ &= f(x,t). \end{aligned}$$

Moreover, $u$ satisfies (2.16) by (2.18), and since the $1^{st}$ and $3^{rd}$ integrals approach to 0 as $t$ approach 0, and $2^{nd}$ integral tends to $a(x)$.

$$\begin{aligned} \lim_{t\to 0} u(x,t) &= \lim_{t\to 0}[\int_0^t\int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau + \int_\Omega G(x,t,\xi,0)a(\xi)d\xi \\ &\quad - \int_0^t\int_\Omega G(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau] \\ &= \lim_{t\to 0}\int_\Omega G(x,t,\xi,0)a(\xi)d\xi \\ &= a(x). \end{aligned} \tag{2.40}$$

For uniqueness we will use the following lemma.

LEMMA 2.2.2. *[9] (p. 146) If $u$ is a solution of (2.15)-(2.17), if $L$ is parabolic in $Q$ and the coefficients of $L$ satisfy (2.11)-(2.13), and if $S$ is of class $C^{1+\alpha}$ for $\alpha \in (0,1)$, then for all $(x,t) \in Q$,*

$$|u(x,t)| \leq K(l.u.b_Q|g| + l.u.b_{\Gamma\times[0,T]}|f| + l.u.b._\Omega|a|),$$

*where $K$ is a constant depending only on $L$, $\sigma$, $\Omega$.*

By this lemma, let us prove the uniqueness of solution $u$ of the direct problem. Let $u_1$ and $u_2$ be solutions of the direct problem. Then $\bar{u} = u_1 - u_2$ satisfy

$$\begin{aligned} L\bar{u} &= 0 \text{ on } \Omega \times (0,T] \\ \bar{u}(x,0) &= 0 \text{ on } \Omega \\ \frac{\partial \bar{u}}{\partial n} + \sigma\bar{u} &= 0 \text{ on } \Gamma \times (0,T]. \end{aligned}$$

So

$$|\bar{u}| \leq l.u.b._Q 0 + l.u.b._{\Gamma\times(0,T]} 0 + l.u.b._\Omega 0 = 0,$$

which means

$$\bar{u} = 0.$$

Then, $u_1 = u_2$, which means the solution is unique.

□

## 2.3 Auxiliary Statements I

We present some results for the direct problem (1.1)-(1.3) with given functions $f(x) \in C(\Gamma)$ and $\sigma(x,t) \in C(\bar{S})$ in condition (1.3).

THEOREM 2.3.1. *There exists a unique $u \in C^{2,1}(Q) \cap C(\bar{Q})$ that satisfies equations (1.1)-(1.3) in the classical sense. (Moreover the functions $\partial u/\partial x_i$ are uniformly continuous on $\bar{\Omega} \times [\epsilon, T]$ for any $\epsilon \in (0,T)$, and the function $\partial u/\partial n$ is bounded* *on $\bar{S}$.) The solution to problem (1.1)-(1.3) can be found in the form*

$$\begin{aligned} u(x,t) &= \int_0^t \int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau + \int_\Omega G(x,t,\xi,0)a(\xi)d\xi \\ &+ \int_0^t \int_\Omega G(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau, \end{aligned} \quad (2.41)$$

*where $G$ is the fundamental solution to the heat operator $(\Delta - \partial t)$ in a wider domain $\bar{\Omega} \subset \Omega_0$, and $\varphi(x,t)$ is found as a solution of the integral equation*

$$\varphi(x,t) = -2\int_0^t \int_\Gamma [\frac{\partial G(x,t,\xi,\tau)}{\partial n} + \sigma(x,t)G(x,t,\xi,\tau)]\varphi(\xi,\tau)dS_\xi d\tau + 2F(x,t), \quad (2.42)$$

*where*

$$\begin{aligned} F(x,t) &= -\int_\Omega \frac{\partial G(x,t,\xi,0)}{\partial n}a(\xi)d\xi + \int_0^t \int_\Omega \frac{\partial G(x,t,\xi,\tau)}{\partial n}g(\xi,\tau)d\xi d\tau \\ &- \sigma(x,t)\int_\Omega G(x,t,\xi,0)a(\xi)d\xi + \sigma(x,t)\int_0^t \int_\Omega G(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau \\ &+ b(x,t) + h(x,t)f(x). \end{aligned} \quad (2.43)$$

*For brevity we write equation (2.42) in the form $(I - \bar{B})\varphi = \bar{F}$, where $\bar{F} = 2F$ and $\bar{B}$ is the integral operator with the following kernel*

$$M(x,t,\xi,\tau) = -2\frac{\partial G(x,t,\xi,\tau)}{\partial n} - 2\sigma(x,t)G(x,t,\xi,\tau) \quad (2.44)$$

*and this kernel satisfies*

$$|M(x,t,\xi,\tau)| \le \frac{c}{(t-\tau)^\mu |x-\xi|^{m+1-2\mu-\alpha}}, \quad (2.45)$$

*where $1 - \alpha < \mu < 1$. The solution of the integral equation (2.42) is of the form*

$$2F(x,t) + 2\sum_{k=1}^{\infty} \int_0^t \int_\Gamma M_k(x,t,\xi,\tau)F(\xi,\tau)dS_\xi d\tau, \quad (2.46)$$

*where $M_1 = M$ and $M_{k+1}(x,t,\xi,\tau) = \int_0^t \int_\Gamma M(x,t,y,\eta)M_k(y,\eta,\xi,\tau)dS_y d\eta$. The* *series in (2.46) is convergent absolutely and uniformly on $\bar{S}$.*

*Proof.* We have $Lu = u_t - \Delta u$ is parabolic in $Q$. Also we have the coefficients of $L$, that are $a_{ij} = 1$ for all $i, j = 1 \ldots m$, $b_i = 0$ for all $i = 1 \ldots m$ and $c = 0$, satisfy the Hölder conditions (2.11)-(2.13), $\Gamma$ belong to $C^{1+\alpha}$, $a(x)$ is continuous in $\bar{\Omega}$ and vanishes in some $Q$-neighborhood of $\partial Q$ and $f$ is continuous on $\Gamma \times [0, T]$. Then by Theorem 2.2.9 there exists a unique solution of the direct problem $(1.1) - (1.3)$. □

THEOREM 2.3.2. *If $\varphi \in C(\bar{S})$, then for any $(x, t) \in \Gamma \times (0, T]$, the function*

$$V(x,t) = \int_0^t \int_\Gamma G(x,t,\xi,\tau)\varphi(\xi,\tau)dS_\xi d\tau$$

*satisfies the following relation*

$$\frac{\partial V}{\partial n} = \frac{1}{2}\varphi(x,t) + \int_0^t \int_\Gamma \frac{\partial G(x,t,\xi,\tau)}{\partial n}\varphi(\xi,\tau)dS_\xi d\tau, \tag{2.47}$$

*where $n$ is the outward pointing normal at $x \in \Gamma$.*

By this property, for the problem (1.1)-(1.3) we have

$$\begin{aligned} \frac{\partial u}{\partial n} &= \frac{1}{2}\varphi(x,t) + \int_0^t \int_\Gamma \frac{\partial G(x,t,\xi,\tau)}{\partial n}\varphi(\xi,\tau)dS_\xi d\tau \\ &+ \int_\Omega \frac{\partial G(x,t,\xi,0)}{\partial n}a(\xi)d\xi - \int_0^t \int_\Omega \frac{\partial G(x,t,\xi,\tau)}{\partial n}g(\xi,\tau)d\xi d\tau. \end{aligned} \tag{2.48}$$

To prove the uniqueness theorem for Problem I, we will need two lemmas.

LEMMA 2.3.1. *[30] Let $Lu = \sum_{i,j=1}^n a_{ij}u_{x_ix_j} + \sum_{i=1}^n b_iu_{x_i}$ and $u \in C^{2,1}(Q) \cap C(\bar{Q})$ be such that*

$$u_t - Lu + cu \geq 0 \text{ on } Q \tag{2.49}$$

$$u(x,0) \geq 0 \text{ on } \bar{\Omega} \tag{2.50}$$

$$\alpha_0\frac{\partial u}{\partial n} + \beta_0 u \geq 0 \text{ on } S \tag{2.51}$$

*where* $\alpha_0 \geq 0$, $\beta_0 \geq 0$, $\alpha_0 + \beta_0 > 0$ *on* $S$ *and* $c = c(x,t)$ *is bounded in* $Q$. *Then* $u(x,t) \geq 0$ *on* $\bar{Q}$. *Moreover* $u(x,t) > 0$ *on* $Q$ *unless it is identically zero. That means either* $u(x,t) > 0$ *or* $u(x,t) \equiv 0$ *on* $Q$.

*Proof.* Consider the case $c \geq 0$ in $Q$. If $u < 0$ then there exists $(x_0, t_0) \in \bar{Q}$ such that $u(x_0, t_0)$ is a negative minimum in $Q$. Since $u(x_0, 0) \geq 0$ on $\bar{\Omega}$ by assumption of lemma,$(x_0, t_0)$ is either in $Q$ or on $S$ by maximum principle for parabolic equations. Since u is not a negative constant, by strict maximum principle $(x_0, t_0) \in S$, and so by assumption of lemma we have

$$\alpha(x_0, t_0)\frac{\partial u(x_0, t_0)}{\partial n} + \beta_0(x_0, t_0)u(x_0, t_0) \geq 0. \tag{2.52}$$

If $\alpha(x_0, t_0) = 0$, then equation (2.51) becomes $\beta_0(x_0, t_0)u(x_0, t_0) \geq 0$ but this is a contradiction since $\beta_0 \geq 0$ by assumption of lemma and $u(x_0, t_0) \leq 0$ by the assumption of proof.

If $\alpha(x_0, t_0) \geq 0$ then by equation (2.51), $\frac{\partial u(x_0,t_0)}{\partial n} \geq 0$ since $\beta_0 \geq 0$ by assumption of lemma and $u(x_0, t_0) \leq 0$ by the assumption of proof, and this is a contradiction by maximum principle, since by max principle when u is assumed to be a negative min,$\frac{\partial u(x_0,t_0)}{\partial n} \leq 0$ must be satisfied. So $u(x,t) \geq 0$ in $\bar{Q}$. Moreover, if $u(x,t) = 0$ at some $(x_1, t_1)$ in Q, then $u(x_1, t_1)$ is a minimum and by strict minimum principle $u(x,t) \equiv 0$ in Q. Therefore either $u(x,t) \geq 0$ or $u(x,t) \equiv 0$ in $Q$. For $c(x,t) \geq 0$, $u(x,t) \geq 0$ is proved. For arbitrary bounded $c$, choose a constant $\gamma \geq -c$ and define $v(x,t) = \exp^{-\gamma t} u(x,t)$. Then,

$$v_t - Lv + (\gamma + c)v \geq 0 = \exp^{-\gamma t}(u_t - Lu + cu) \geq 0 \text{ on } Q,$$

since $(u_t - Lu + cu) \geq 0$ on $Q$ by assumption of lemma.

$$v(x,0) = \exp^{-\gamma . 0} u(x,0) \geq 0 \text{ on } \Omega,$$

by assumption of lemma.

$$\alpha_0 \frac{\partial v}{\partial n} + \beta_0 v = \exp^{-\gamma t}(\alpha_0 \frac{\partial u}{\partial n} + \beta_0 u) \geq 0 \text{ on } S,$$

since $(\alpha_0 \frac{\partial u}{\partial n} + \beta_0 u) \geq 0$ on $S$. That means $v$ satisfies the equation

$$\begin{aligned} v_t - Lv + (\gamma + c)v &\geq 0 \text{ on } Q \\ v(x,0) &\geq 0 \text{ on } \Omega \\ \alpha_0 \frac{\partial v}{\partial n} + \beta_0 v &\geq 0 \text{ on } S. \end{aligned}$$

Since $\gamma + c \geq 0$, the above conclusion for $c \geq 0$ implies $v \geq 0$, and either $v > 0$ or $v \equiv 0$ in $Q$. $u = \exp^{\gamma t} u(x,t)$, so $u \geq 0$, and either $u > 0$ or $u \equiv 0$ in $Q$, for any $c$. By this lemma, we have proved that $u(x,t) \geq 0$ on $Q$. □

LEMMA 2.3.2. *Assume $\sigma$, $b \in C(\bar{S})$, $\omega \in L_1(0,T)$, $\sigma b \geq 0$ on $S$, $\omega \geq 0$ on $[0,T]$ and $u \in C^{2,1}(Q)$ is a solution to the problem*

$$\begin{aligned} u_t - \Delta u = 0 \text{ on } Q \\ u(x,0) = 0 \text{ on } \bar{\Omega} \\ \partial_n u + \sigma u = b \text{ on } S. \end{aligned} \tag{2.53}$$

*Then $u(x,t)$ has the properties*

1. *If $b(x,t) \not\equiv 0$ on $S$, then $u(x,t) \geq 0$ on $Q$ and $u(x,T) > 0$ on $\bar{\Omega}$.*
2. *If $\int_0^T b(x,\tau)\omega(\tau)d\tau \not\equiv 0$ on $\Gamma$, then $\int_0^T u(x,\tau)\omega(\tau)d\tau > 0$ on $\bar{\Omega}$.*
3. *If $\int_0^T u(x,\tau)\omega(\tau)d\tau \not\equiv 0$ on $\Gamma$, then $\int_0^T u(x,\tau)\omega(\tau)d\tau > 0$ on $\bar{\Omega}$.*

*Proof.* By Theorem 2.3.1, we know $u \in C(\bar{Q})$. It follows by Lemma 2.3.1 that $u(x,t) \geq 0$ on $Q$.

1) If there exists $x_0 \in \Omega$ at which $u(x_0,T) = 0$, then by strict maximum principle $u(x,t) \equiv 0$ on $Q$. In this case we have $\alpha_0 \frac{\partial u}{\partial n} + \beta_0 u = b \equiv 0$. This is a contradiction to the assumption of lemma that $b(x,t) \not\equiv 0$.

If there exists $x_0 \in \Gamma$ such that $u(x_0,T) = 0$, then by maximum principle for parabolic equations, either $\frac{\partial u(x_0,T)}{\partial n} < 0$ or $u(x,t) \equiv u(x_0,T)$ on $Q$. In the first case since $b(x_0,T) = \frac{\partial u(x_0,T)}{\partial n} + \alpha(x_0)u(x_0,T)$, $b \geq 0$ and $u(x_0,T) = 0$ we have $\frac{\partial u(x_0,T)}{\partial n} = b(x_0,T) \geq 0$, which is a contradiction. In the second case, $u(x,t) \equiv$

$u(x_0,T)\equiv 0$, so $b(x_0,T)=\frac{\partial u(x_0,T)}{\partial n}+\alpha(x_0)u(x_0,T)\equiv 0$, and this is a contradiction to $b\not\equiv 0$. So there is no $x_0\in\bar\Omega$ such that $u(x_0,T)=0$. So $u(x_0,T)>0$ for all $x\in\bar\Omega$.

2) By the assumption $\int_0^T b(x,\tau)\omega(\tau)d\tau\not\equiv 0$ on $\Gamma$, there exists an open set $\Gamma^+\subset\Gamma$ such that $\int_0^T b(x,\tau)\omega(\tau)d\tau>0$ on $\Gamma^+$. Introduce

$$S^+=\Gamma^+\times(0,T]$$

and

$$B^+=\left\{(x,t)\in S^+ : b(x,t)\omega(t)>0\right\}.$$

The set $B^+$ is a measurable subset of $S$ and $mes_m B^+>0$. If it were not, $\int_0^T b(x,\tau)\omega(\tau)d\tau=0$ on $B^+$ and this would be a contradiction to the assumption of lemma. Let

$$t_1=inf\left\{t\in(0,T] : B^+\cap(\Gamma\times\{t\})\neq\emptyset\right\}$$

and

$$t_2=sup\left\{t\in(0,T] : B^+\cap(\Gamma\times\{t\})\neq\emptyset\right\}.$$

We have $t_2>t_1$ since $mes_m B^+>0$. Let

$$U=\{t\in[t_1,t_2] : \omega(t)>0\}.$$

The set $U$ is measurable and $mes_1 U>0$. If $U$ were not measurable, then $\int_0^T b(x,\tau)\omega(\tau)d\tau\equiv 0$, and this is a contradiction to the assumption of the lemma. Write $U=U_1+U_2$, where $U_1=U\cap[t_1,\frac{t_1+t_2}{2}]$ and $U_2=U\cap[\frac{t_1+t_2}{2},t_2]$. Both $U_1$ and $U_2$ are measurable, since $U$ is measurable. Without loss of generality we assume $mes_1 U_1>0$ and $mes_1 U_2>0$. (If not divide the interval $[t_1,t_2]$ into two parts and redefine $U_1,U_2$ such that both have positive measure. This is possible since $U$ has positive measure.)

For all $t\in[\frac{t_1+t_2}{2},T]$ we have $u(x,t)>0$ on $\bar\Omega$. To see this let $t_3=\inf t\in U$ and $t_4=\sup t\in U$. Consider $u(x,t)$ as a solution of problem (2.53) on the cylinder $\Omega\times(t_3,\alpha]$ for any $\alpha\in U$. Since $b(x,t)\not\equiv 0$ on $\Gamma\times(t_3,\alpha]$ ( since $b\omega>0$ on $B_+$, $\omega>0$ on $\bar\Omega\subset B_+$ we have $b>0$ on $U$), then by the previous part of the lemma we have $u(x,\alpha) > 0$ on $\bar{\Omega}$ for all $\alpha \in U$. Hence $u(x,t) > 0$ on $\bar{\Omega}$ for all $t \in U$. Then

$$\begin{aligned}\int_0^T u(x,\tau)\omega(\tau)d\tau &\geq \int_U u(x,\tau)\omega(\tau)d\tau \text{ , since } u\omega \geq 0 \text{ on } U \text{ and } U \subset [0,T] \\ &> 0 \text{ on } \bar{\Omega} \text{ , since } u > 0 \text{ and } \omega > 0 \text{ on } U \text{ .}\end{aligned}$$

3) Take $\Gamma^+ = \{x \in \Gamma : \int_0^T u(x,\tau)\omega(\tau)d\tau > 0\}$ open subset of $\Gamma$. We know $u \geq 0$ on $Q$. Then $\chi(x) = \int_0^T u(x,\tau)\omega(\tau)d\tau \geq 0$ on $\bar{\Omega}$. Set

$$B^+ = \{(x,t) \in S : u(x,t)\omega(t) > 0\}.$$

We have $mes_m B^+ > 0$, since otherwise $\chi(x) \equiv 0$ on $\Gamma$ and this is a contradiction to the assumption. Let

$$t_1 = \inf\{t \in [0,T] : (x,t) \in B^+\}$$

and

$$t_2 = \sup\{t \in [0,T] : (x,t) \in B^+\}.$$

Here $t_2 > t_1$ since $mes_m B^+ > 0$. Set

$$U = \{t \in [t_1,t_2] : \omega(t) > 0\}.$$

We have $mes_1 U > 0$, since otherwise we would have a contradiction to the assumption of lemma. Let $U = U_1 \cup U_2$. We have $mes_1 U_1 > 0$ and $mes_1 U_2 > 0$, as in previous part of lemma. By strict minimum principle, $u(x,t) > 0$ on $\bar{\Omega}$ for almost all $t \in U$. We also know $\omega > 0$ for $t \in U$, then we have

$$\int_0^T u(x,\tau)\omega(\tau)d\tau \geq \int_U u(x,\tau)\omega(\tau)d\tau > 0 \text{ on } \bar{\Omega}.$$

That ends the proof of the lemma.

□

## 2.4 Auxiliary Statements II

THEOREM 2.4.1. *Suppose $f \in C([0,T])$, $\sigma \in C(\bar{S})$ and condition (3.5) is satisfied. Then there exists a unique function $u \in C^{2,1}(Q) \cap C^{\alpha,\frac{\alpha}{2}}(\bar{Q})$ satisfying equations (1.6)-(1.8).*

To study Problem III, we need some results on the solvability of the Abel equation

$$K[\varphi] = \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau = \chi(t) \text{ on } [0,T].$$

**Proposition 2.4.2.** *If the condition*

$$\chi \in C^{1/2}([0,T]), \qquad \hat{F}(t) = \frac{d}{dt}\int_0^t \frac{\chi(\tau)}{\sqrt{t-\tau}} d\tau \in C([0,T]) \tag{2.54}$$

*is satisfied, then there exists a unique solution $\varphi \in C([0,T])$ of the Abel equation.*

*Proof.* We will use

$$\int_\xi^t \frac{dx}{(t-x)^{1/2}(x-\xi)^{1/2}} = \pi.$$

Let us prove this first.

$$\begin{aligned}
\int_\xi^t \frac{dx}{(t-x)^{1/2}(x-\xi)^{1/2}} &= \int_0^{t-\xi} \frac{du}{u^{1/2}(t-\xi-u)^{1/2}} \quad \text{, for } u = t-\xi \\
&= \int_0^{\pi/2} \frac{(t-\xi)2\sin\theta\cos\theta d\theta}{\sqrt{t-\xi}\sin\theta\sqrt{t-\xi}\cos\theta} \\
&\quad \text{, for } u = (t-\xi)\sin^2\theta \\
&= 2\theta \ \Big|_0^{\pi/2} \\
&= \pi.
\end{aligned}$$

We will also use Dirichlet Formula and Dirichlet's Extended Formulawhen changing the order of integration.

Dirichlet's Formula [3] (p. 4): If $\phi(x,y)$ is finite in $T$ and if its discontinuities, if any, are regularly distributed, then

$$\int_a^b \int_a^x \phi(x,y) dy dx = \int_a^b \int_y^b \phi(x,y) dx dy.$$

Dirichlet's Extended Formula [3] (p. 4): If $\phi(x,y)$ is finite in $T$ and if its discontinuities, if any, are regularly distributed and if $\mu, \lambda, \nu$ are constants such that $0 \le \mu, \lambda, \nu \le 1$, then

$$\int_a^b \int_a^x \frac{\phi(x,y)dydx}{(x-y)^\lambda (b-x)^\mu (y-a)^\nu} = \int_a^b \int_y^b \frac{\phi(x,y)dxdy}{(x-y)^\lambda (b-x)^\mu (y-a)^\nu}.$$

Now let us prove Proposition 2.4.2. We have the equation

$$\int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau = \chi(t).$$

Multiplying both sides with $\frac{1}{\sqrt{z-t}}$ and integrating with respect to $t \in [0,z]$ we have

$$\int_0^z \frac{1}{\sqrt{z-t}} \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau dt = \int_0^z \frac{\chi(t)}{\sqrt{z-t}} dt.$$

Changing the order of integration by Dirichlet's Extended Formula leads to

$$\int_0^z \int_\tau^z \frac{\varphi(\tau)}{\sqrt{z-t}\sqrt{t-\tau}} dt d\tau = \int_0^z \frac{\chi(t)}{\sqrt{z-t}} dt.$$

Then

$$\int_0^z \varphi(\tau) \int_\tau^z \frac{1}{\sqrt{z-t}\sqrt{t-\tau}} dt d\tau = \int_0^z \frac{\chi(t)}{\sqrt{z-t}} dt$$

$$\pi \int_0^z \varphi(\tau) d\tau = \int_0^z \frac{\chi(t)}{\sqrt{z-t}} dt.$$

Taking derivative of both sides with respect to $z$, we obtain

$$\varphi(z) = \frac{1}{\pi} \frac{d}{dz} \int_0^z \frac{\chi(t)}{\sqrt{z-t}} dt$$

is the solution of Abel's equation. Let us prove the uniqueness. Let $\varphi_1$, $\varphi_2$ be two solutions of the Abel's equation. That is

$$\int_0^t \frac{\varphi_i(\tau)}{\sqrt{t-\tau}} d\tau \text{ on } [0,T],$$

for $i = 1, 2$. Then,

$$\int_0^t \frac{\varphi_1(\tau)}{\sqrt{t-\tau}} d\tau - \int_0^t \frac{\varphi_2(\tau)}{\sqrt{t-\tau}} d\tau = \chi(t) - \chi(t) = 0 \text{ on } [0, T],$$

so

$$\int_0^t \frac{(\varphi_1 - \varphi_2)(\tau)}{\sqrt{t-\tau}} d\tau = 0 \text{ on } [0, T].$$

Thus,

$$(\varphi_1 - \varphi_2)(\tau) = 0, \text{ a.e. on } [0, T]$$

and

$$\varphi_1 = \varphi_2, \text{ a.e. on } [0, T].$$

So the solution of the Abel's equation is unique. □

**Proposition 2.4.3.** *If*

$$\chi \in W^{1,1}(0, T), \qquad \chi(0) = 0, \qquad \hat{F}(t) = \int_0^t \frac{\chi'(\tau)}{\sqrt{t-\tau}} d\tau \in C([0, T]) \tag{2.55}$$

*is satisfied, then there exists a solution of the Abel's equation.*

*Proof.* Let $\varphi(t) = \frac{1}{\pi} \int_0^t \frac{\chi'(t)}{\sqrt{t-\tau}} d\tau$. Multiplying both sides with $\frac{1}{\sqrt{t-\tau}}$ and integrating with respect to $\tau \in [0, t]$, we obtain

$$\begin{aligned}
\int_0^t \frac{1}{\sqrt{t-\tau}} \varphi(\tau) d\tau &= \int_0^t \frac{1}{\pi} \frac{d\tau}{\sqrt{t-\tau}} \int_0^\tau \frac{\chi'(u)}{\sqrt{\tau - u}} du \\
&= \frac{1}{\pi} \int_0^t \int_0^\tau \frac{\chi'(u)}{\sqrt{t-\tau}\sqrt{\tau-u}} du d\tau \\
&= \frac{1}{\pi} \int_0^t \int_u^t \frac{\chi'(u)}{\sqrt{t-\tau}\sqrt{\tau-u}} d\tau du \\
&\quad \text{, by Dirichlet's Extended Formula} \\
&= \frac{1}{\pi} \int_0^t \chi'(u) \int_u^t \frac{d\tau}{\sqrt{t-\tau}\sqrt{\tau-u}} du \\
&= \frac{1}{\pi} \int_0^t \chi'(u) \pi du \\
&= \chi(t) - \chi(0) \\
&= \chi(t) \text{ , since } \chi(0) = 0 \text{ by (2.55).}
\end{aligned}$$

So $\varphi(t) = \frac{1}{\pi}\int_0^t \frac{\chi^{'}(u)}{\sqrt{t-\tau}}d\tau$ is the solution of Abel's equation. Uniqueness follows from Proposition 2.4.2. □

*Remark* 2.4.1. *The function* $\hat{F}(t) = \int_0^t \frac{\chi^{'}(\tau)}{\sqrt{t-\tau}}d\tau$ *is continuous on* $[0,T]$ *if the estimate* $|\chi^{'}(\tau)| \leq \frac{c}{\tau^{\mu}}$, $0 \leq \mu \leq \frac{1}{2}$ *is valid.*

*Proof.* First let us prove

$$\int_0^t \frac{\chi^{'}(\tau)}{\sqrt{t-\tau}}d\tau = \frac{d}{dt}\int_0^t \frac{\chi(\tau)}{\sqrt{t-\tau}}d\tau,$$

if the conditions (2.55) are satisfied. By using integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_0^t \frac{\chi(\tau)}{\sqrt{t-\tau}}d\tau &= \frac{d}{dt}[\chi(\tau)(-2)\sqrt{t-\tau}|_0^t - \int_0^t \chi^{'}(\tau)(-2)\sqrt{t-\tau}d\tau] \\
&= \frac{d}{dt}[-2\chi(t)0 + \chi(0)2\sqrt{t} + 2\int_0^t \chi^{'}(\tau)\sqrt{t-\tau}d\tau] \\
&= \frac{d}{dt}[2\int_0^t \chi^{'}(\tau)\sqrt{t-\tau}d\tau] \\
&= 2\chi'(t)\sqrt{t-t} + 2\int_0^t \frac{d}{dt}[\chi^{'}(\tau)\sqrt{t-\tau}]d\tau, \\
&\quad \text{by Leibnitz's Rule} \\
&= \int_0^t \frac{\chi^{'}(\tau)}{\sqrt{t-\tau}}d\tau.
\end{aligned}$$

The function $\hat{F}(t) = \int_0^t \frac{\chi^{'}(\tau)}{\sqrt{t-\tau}}d\tau = \frac{d}{dt}\int_0^t \frac{\chi(\tau)}{\sqrt{t-\tau}}d\tau$ is defined only for $t > 0$; The continuity on $[0,T]$ is understood in the sense that it's continuous for $t \in (0,T)$, left continuous at $t = T$ and can be defined for continuity at $t = 0$. Let us show that $\hat{F}$ is continuous if we set $\hat{F}(0) = 0$.

$$\begin{aligned}
0 \leq |\hat{F}(t)| &\leq c\int_0^t \frac{d\tau}{\tau^{\mu}\sqrt{t-\tau}} \\
&= c\int_0^t \frac{\tau^{1/2-\mu}}{\sqrt{t-\tau}\sqrt{\tau-0}}d\tau \\
&\leq c\max_{\tau\in[0,t]}(\tau^{1/2-\mu})\int_0^t \frac{1}{\sqrt{t-\tau}\sqrt{\tau-0}}d\tau \\
&= c\pi t^{1/2-\mu}\text{ , for } t > 0.
\end{aligned}$$

So $\hat{F}(t) \to 0$ as $t \to 0^+$. This means $\hat{F}$ is right continuous at $t = 0$ if we set $\hat{F}(0) = 0$. Now let $\Delta > 0$, then we have

$$0 \leq |\Delta \hat{F}(t)| \tag{2.56}$$

and

$$\begin{aligned}
|\Delta \hat{F}(t)| &\leq \left| \int_0^{t+\Delta} \frac{\chi'(\tau)}{\sqrt{t+\Delta-\tau}} d\tau - \int_0^t \frac{\chi'(\tau)}{\sqrt{t-\tau}} d\tau \right| \\
&\leq \left| \int_0^t \left( \frac{\chi'(\tau)}{\sqrt{t+\Delta-\tau}} - \frac{\chi'(\tau)}{\sqrt{t-\tau}} \right) d\tau \right| + \left| \int_t^{t+\Delta} \frac{\chi'(\tau)}{\sqrt{t+\Delta-\tau}} d\tau \right| \\
&\leq \left| \int_0^t \frac{c}{\tau^\mu} \left( \frac{1}{\sqrt{t+\Delta-\tau}} - \frac{1}{\sqrt{t-\tau}} \right) d\tau \right| + \left| \int_t^{t+\Delta} \frac{\chi'(\tau)}{\sqrt{t+\Delta-\tau}} d\tau \right| \\
&= \left| \int_0^t \frac{c}{\tau^\mu} \left( \frac{\sqrt{\sqrt{t-\tau}} - \sqrt{t+\Delta-\tau}}{\sqrt{t+\Delta-\tau}\sqrt{\sqrt{t-\tau}}} \right) d\tau \right| + \left| \int_t^{t+\Delta} \frac{\chi'(\tau)}{\sqrt{t+\Delta-\tau}} d\tau \right| \\
&\leq \left| \int_0^t \frac{c}{\tau^\mu} \left( \frac{\sqrt{\sqrt{t-\tau}} - \sqrt{t+\Delta-\tau}}{\sqrt{t+\Delta-\tau}\sqrt{\sqrt{t-\tau}}} \right) d\tau \right| + \\
&\quad \left| \int_t^{t+\Delta} \frac{c}{\tau^\mu \sqrt{t+\Delta-\tau}} d\tau \right| \\
&\leq |c\sqrt{\Delta} I| + \left| \int_t^{t+\Delta} \frac{c}{\tau^\mu \sqrt{t+\Delta-\tau}} d\tau \right| \\
&\leq |c\sqrt{\Delta} I|.
\end{aligned}$$

We represent $I = I_1 + I_2$, where

$$I_1 = \int_0^{t/2} \frac{d\tau}{\tau^\mu \sqrt{t+\Delta-\tau}\sqrt{t-\tau}}$$

and

$$I_2 = \int_{t/2}^t \frac{d\tau}{\tau^\mu \sqrt{t+\Delta-\tau}\sqrt{t-\tau}}.$$

Clearly,

$$
\begin{aligned}
0 \le I_1 &= \int_0^{t/2} \frac{d\tau}{\tau^\mu \sqrt{t+\Delta-\tau}\sqrt{t-\tau}} \\
&\le \max_{\tau\in[0,t/2]} \left( \frac{1}{\sqrt{t+\Delta-\tau}\sqrt{t-\tau}} \right) \int_0^{t/2} \frac{d\tau}{\tau^\mu} \\
&\le \frac{1}{\sqrt{t+\Delta-\frac{t}{2}}\sqrt{t-\frac{t}{2}}} \int_0^{t/2} \frac{d\tau}{\tau^\mu} \\
&= \frac{(t/2)^{1/2-\mu}}{(1-\mu)\sqrt{\frac{t}{2}+\Delta}},
\end{aligned}
$$

and

$$
\begin{aligned}
0 \le I_2 &= \int_{t/2}^{t} \frac{d\tau}{\tau^\mu \sqrt{t+\Delta-\tau}\sqrt{t-\tau}} \\
&\le \max_{\tau\in[t/2,t]} \left( \frac{1}{\tau^\mu} \right) \int_{t/2}^{t} \frac{d\tau}{\sqrt{t+\Delta-\tau}\sqrt{t-\tau}} \\
&= \left(\frac{2}{t}\right)^\mu \int_{t/2}^{0} \frac{-dz}{\sqrt{z+\Delta}\sqrt{z}} \\
&= \left(\frac{2}{t}\right)^\mu \int_0^{\arctan\sqrt{\frac{t}{2\Delta}}} \frac{\Delta 2\tan\theta\sec^2\theta}{\sqrt{\Delta}\tan\theta\sqrt{\Delta}\sec\theta} d\theta \\
&= \left(\frac{2}{t}\right)^\mu \int_0^{\arctan\sqrt{\frac{t}{2\Delta}}} 2\sec\theta d\theta \\
&= \left(\frac{2}{t}\right)^\mu 2\ln\left|\frac{\sqrt{t+2\Delta}+\sqrt{t}}{\sqrt{2\Delta}}\right|.
\end{aligned}
$$

So we have

$$
I = I_1 + I_2 \le \frac{(t/2)^{1/2-\mu}}{(1-\mu)\sqrt{\frac{t}{2}+\Delta}} + \left(\frac{2}{t}\right)^\mu 2\ln\left|\frac{\sqrt{t+2\Delta}+\sqrt{t}}{\sqrt{2\Delta}}\right|.
$$

Let us find the limit of $\sqrt{\Delta}I$ as $\sqrt{\Delta}\to 0^+$. Since

$$
0 \le I \le \frac{(t/2)^{1/2-\mu}}{(1-\mu)\sqrt{\frac{t}{2}+\Delta}} + \left(\frac{2}{t}\right)^\mu 2\ln\left|\frac{\sqrt{t+2\Delta}+\sqrt{t}}{\sqrt{2\Delta}}\right|,
$$

we have

$$\begin{aligned} 0 &\leq \lim_{\sqrt{\Delta}\to 0^+} \sqrt{\Delta} I \\ &\leq \lim_{\sqrt{\Delta}\to 0^+} \sqrt{\Delta}\left( \frac{(t/2)^{1/2-\mu}}{(1-\mu)\sqrt{\frac{t}{2}+\Delta}} + \left(\frac{2}{t}\right)^{\mu} 2\ln\left|\frac{\sqrt{t+2\Delta}+\sqrt{t}}{\sqrt{2\Delta}}\right| \right) \\ &= \lim_{u\to 0^+} \left( \frac{(t/2)^{1/2-\mu}}{(1-\mu)\sqrt{\frac{t}{2}+u^2}} + 2\left(\frac{2}{t}\right)^{\mu} u\ln\left|\frac{\sqrt{t+2u^2}+\sqrt{t}}{\sqrt{2}u}\right| \right). \end{aligned}$$

So $0 \leq \lim_{\sqrt{\Delta}\to 0^+} \sqrt{\Delta} I \leq 0$, that's $\lim_{\sqrt{\Delta}\to 0^+} \sqrt{\Delta} I = 0$. We know that

$$0 \leq |\Delta \hat{F}| \leq c\sqrt{\Delta} I + 2\sqrt{\Delta}\left(\frac{c}{t^{\mu}}\right),$$

so we have $|\Delta \hat{F}| \to 0$ as $\sqrt{\Delta} \to 0^+$. Similarly for the case $\sqrt{\Delta} \to 0^-$ we find $|\Delta \hat{F}| \to 0$. That means $\lim_{\sqrt{\Delta}\to 0} |\Delta \hat{F}| = 0$, and then we can conclude $\hat{F}(t)$ is continuous on $[0, T]$. □

The following lemma shows the necessity of the condition $\chi \in C^{1/2}([0, T])$ for the solvability of the Abel's equation in $C([0, T])$.

LEMMA 2.4.1. $K \in L\left(C([0, T]), C^{1/2}([0, T])\right)$

*Proof.* $\varphi \in C([0, T])$ by assumption. $K[\varphi] = \chi \in C([0, T])$ by assumption again. So obviously $K : C([0, T]) \to C([0, T])$. $K$ is a linear operator since

$$\begin{aligned} K[c_1\varphi_1 + c_2\varphi_2] &= \int_0^t \frac{c_1\varphi_1 + c_2\varphi_2}{\sqrt{t-\tau}} d\tau \\ &= \int_0^t \frac{c_1\varphi_1}{\sqrt{t-\tau}} d\tau + \int_0^t \frac{c_2\varphi_2}{\sqrt{t-\tau}} d\tau \\ &= c_1 K[\varphi_1] + c_2 K[\varphi_2]. \end{aligned}$$

Also $K$ is a bounded function since $K[\varphi]$ is in $C([0, T])$ on bounded domain. So

$K \in L(C([0,T]))$. Now set $\Phi = \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau$ and $\Phi(0) = \lim_{t\to 0^+} \Phi(t) = 0$. Then,

$$\begin{aligned}
0 \leq |\Delta\Phi(t)| &\leq \left| \int_0^{t+\Delta} \frac{\varphi(\tau)}{\sqrt{t+\Delta-\tau}} d\tau - \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau \right| \\
&\leq \left| \int_0^t \varphi(\tau) \left( \frac{1}{\sqrt{t+\Delta-\tau}} - \frac{1}{\sqrt{t-\tau}} \right) d\tau \right| + \\
&\quad \left| \int_t^{t+\Delta} \frac{\varphi(\tau)}{\sqrt{t+\Delta-\tau}} d\tau \right| \\
&\leq \|\varphi(\tau)\|.[\left| \int_0^t \left( \frac{1}{\sqrt{t+\Delta-\tau}} - \frac{1}{\sqrt{t-\tau}} \right) d\tau \right| + \\
&\quad \left| \int_t^{t+\Delta} \frac{1}{\sqrt{t+\Delta-\tau}} d\tau \right|] \\
&= \|\varphi(\tau)\|.[| -2\sqrt{t+\Delta-\tau} + 2\sqrt{t-\tau}||_0^t + | - \\
&\quad 2\sqrt{t+\Delta-\tau}||_t^{t+\Delta}] \\
&= \|\varphi(\tau)\|.[| -2\sqrt{\Delta} + 2\sqrt{t+\Delta} - 2\sqrt{t}| + 2\sqrt{\Delta}] \\
&\leq \|\varphi(\tau)\|.2\sqrt{\Delta}.
\end{aligned}$$

This implies that $\Phi \in C^{1/2}([0,T])$.

□

# CHAPTER 3

# ANALYSIS OF PROBLEM I

**Problem I:** Find a pair of functions $\{u(x,t), f(x)\}$ satisfying

$$u_t - \Delta u = g(x,t) \text{ on } Q \tag{3.1}$$

$$u(x,0) = a(x) \text{ on } \bar{\Omega} \tag{3.2}$$

$$\partial_n u + \sigma u = h(x,t)f(x) + b(x,t) \text{ on } S \tag{3.3}$$

$$\ell(u) = \chi(x) \text{ on } \Gamma, \tag{3.4}$$

where $g(x,t)$, $a(x)$, $\sigma(x)$, $h(x,t)$, $b(x,t)$, $\chi(x)$ are given and $n$ is the outward pointing normal to $\Gamma$. The expression $\ell(u)$ has one of the forms

$$\ell(u) = u(x, t_1), 0 < t_1 < T$$

or

$$\ell(u) = \int_0^T u(x,\tau)\omega(\tau)d\tau,$$

where $t_1$ is chosen and $\omega \in L_1(0,T)$ is given. These conditions are called terminal and integral boundary observations respectively.

Definition 3.0.1. *A pair $\{u(x,t), f(x)\}$ is called a solution to Problem I if $u \in C^{2,1}(Q)$, $f \in C(\Gamma)$ and these functions satisfy equations (3.1)-(3.4) in the classical sense.*

## 3.1 Uniqueness Theorem for Problem I

Theorem 3.1.1. *Assume that smoothness conditions*

$$g \in C^{\alpha,0}(\bar{Q}), \qquad a \in C^1(\bar{\Omega}), \qquad h, b \in C(\bar{S}) \tag{3.5}$$

*hold,* $\sigma(x) \in C(\Gamma)$*,* $\sigma(x) \geq 0$ *on* $\Gamma$*,* $\omega(t) \geq 0$ *on* $[0,T]$*,* $\ell(h) > 0$ *almost everywhere on* $\Gamma$*, the function* $h(x,t) > 0$ *on* $S$ *and it is monotone nondecreasing with respect to* $t$*. Then a solution to inverse Problem I is unique.*

Let us define the equations

$$
\begin{aligned}
u_t - \Delta u &= 0 \text{ on } Q && (3.6)\\
u(x,0) &= 0 \text{ on } \bar{\Omega} && (3.7)\\
\frac{\partial u}{\partial n} + \sigma u &= hf \text{ on } S && (3.8)\\
\ell(u) &= 0 \text{ on } \Gamma. && (3.9)
\end{aligned}
$$

Assume Problem I has two pair of solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$. Then

$$(u_1 - u_2)_t - \Delta(u_1 - u_2) = 0 \text{ on } Q$$

$$(u_1 - u_2)(x,0) = 0 \text{ on } \bar{\Omega}$$

$$\frac{\partial(u_1 - u_2)}{\partial n} + \sigma(u_1 - u_2) = h(f_1 - f_2) = h\bar{f} \text{ on } S$$

$$\ell(u_1 - u_2) = \chi - \chi = 0 \text{ on } \Gamma.$$

So $u = u_1 - u_2$, $f = f_1 - f_2$ satisfies the equations (3.6)-(3.9). If we can show that solution of (3.6)-(3.9) is $u(x,t) \equiv 0$, $f(x) \equiv 0$, this will mean $u_1 \equiv u_2$ and $f_1 \equiv f_2$, and that means the solution of Problem I is unique. So to prove Problem I, it is enough to prove that if $\{u(x,t), f(x)\}$ satisfies (3.6)-(3.9), then $u(x,t) = 0$ and $f(x) = 0$.

Assume that there exists a pair $\{u, f\} \neq \{0, 0\}$ that satisfies equations (3.6)-(3.9). Introduce $f^+ = max(0, f(x))$ and $f^- = max(0, -f(x))$. Then we have $f^\mp \geq 0$ on $\Gamma$, $f(x) = f^+(x) - f^-(x)$, $f^\mp \in C(\Gamma)$ and $f^+(x)f^-(x) = 0$ on $\Gamma$. Define $u^\mp(x,t)$ be solutions to problem

$$
\begin{aligned}
u_t^\mp - \Delta u &= 0 \text{ on } Q && (3.10)\\
u_t^\mp(x,0) &= 0 \text{ on } \bar{\Omega} && (3.11)\\
\frac{\partial u^\mp}{\partial n} + \sigma u &= hf^\mp \text{ on } S. && (3.12)
\end{aligned}
$$

By Theorem 2.3.1, $u^{\mp} \in C^{1,2}(Q) \cap C(\bar{Q})$ and $u^{\mp}$ satisfies equations (3.10),(3.11) and (3.12) in the classical sense. Since $f = f^+ - f^-$ and (3.10),(3.11) and (3.12) are linear and a solution to problem (3.6)-(3.8) is unique by Theorem 2.3.1, we obtain $u = u^+ - u^-$ and by (3.9) we have $\ell(u) = \ell(u^+) - \ell(u^-) = 0$ on $\Gamma$, that is $\ell(u^+) = \ell(u^-) = 0$ on $\Gamma$. Set $\ell(u^{\mp}) = \bar{\chi}(x)$ on $\Gamma$, where $\bar{\chi}(x)$ is a given function. If $f^{\mp} \equiv 0$, then $u^{\mp} \equiv 0$ on $Q$. If $f^{\mp} \not\equiv 0$ then since $h > 0$ by assumption of theorem and $f^{\mp} \geq 0$ by definition $h(x,t)f^{\mp}(x) \geq 0$ on S, and by maximum principle $u^{\mp}(x,t) \geq 0$ on Q (was proved in Lemma 2.3.2). We know that $\bar{\chi}(x) = u^{\mp}(x,t_0)$ or $\bar{\chi}(x) = \int_0^T u^{\mp}(x.\tau)\omega(\tau)d\tau$. Therefore in both cases we have $\bar{\chi}(x) \geq 0$ on $\Gamma$. Set $\ell(u^{\mp}) = \bar{\chi}^{\mp}(x)$ on $\bar{\Omega}$, where $\bar{\chi}^{\mp}(x) \in C(\bar{\Omega})$, $\bar{\chi}^{\mp}(x) \geq 0$ on $\bar{\Omega}$ and $\bar{\chi}^+(x) = \bar{\chi}^-(x) = \bar{\chi}(x)$ on $\Gamma$.

Further we consider two cases.

Case 1: $f^- = 0$ and $f = f^+ \neq 0$. Then $u = u^+$ and therefore $\bar{\chi} = 0$ on $\Gamma$, since $u^- = 0$.

For $\ell(u) = u(x,t_1)$, we have $u^+(x,t) = \bar{\chi}(x) = 0$ on $\Gamma$, and it is the minimum value of $u$. By the lemma on the normal derivative for parabolic equations, either $\frac{\partial u^+(x,t_1)}{\partial n_x} < 0$ on $\Gamma$ or $u^+(x,t) = 0$ in $\Omega \times [0,t_1]$. In the case $\frac{\partial u^+(x,t_1)}{\partial n_x} < 0$, substituting $t = t_1$ into equation (3.8) we obtain $\frac{\partial u^+(x,t_1)}{\partial n_x} + \sigma(x)u^+(x,t_1) = h(x,t_1)f^+(x)$. So, $f^+(x) < 0$, and this is a contradiction to the definition of $f^+$ that $f^+ \geq 0$ on $\Gamma$. In the case $u^+(x,t) = 0$ in $\Omega \times [0,t_1]$, we have $f^+ \equiv 0$ and that is a contradiction to the assumption that $f^+ \neq 0$.

For $\ell(u) = \int_0^T u(x,\tau)\omega(\tau)d\tau$, since $\ell(hf^+) = \ell(h)f^+ \not\equiv$ on $\Gamma$, it follows by Lemma 2.3.2 part (ii) that $\ell(u^+) > 0$ on $\bar{\Omega}$, and this is a contradiction to $\ell(u^+) = \ell(u^-) = 0$ on $\Gamma$. Thus the case $f^- = 0$ , $f = f^+ \neq 0$ is excluded. Furthermore $f^+ = 0$, $f = f^- \neq 0$ on $\Gamma$ can be reduced to the previous case since problem (3.6)-(3.9) is linear and homogeneous. So the case $f^+ = 0$, $f = f^- \neq 0$ on $\Gamma$ is also excluded.

Case 2: Let us consider $f^+ \not\equiv 0, f^- \not\equiv 0$. Then by Lemma 2.3.2 part(2), since $\ell(hf^+) \not\equiv 0$ and $\ell(hf^-) \not\equiv 0$, we have $\bar{\chi}^+(x) > 0$ and $\bar{\chi}^-(x) > 0$ on $\bar{\Omega}$ and therefore $\bar{\chi}(x) > 0$ on $\Gamma$. We know $h$ is a monotone nondecreasing function of t, that is $h_t > 0$, and $f^{\mp}$ are functions of x. Then we have $(hf^{\mp})_t = h_t f^{\mp} + hf_t^{\mp} = h_t f^{\mp} + 0 > 0$. So the functions $hf^{\mp}$ are monotone nondecreasing functions with respect to t on

$Q$. Then $u^{\mp}$ are also monotone nondecreasing functions with respect to $t$ on $Q$. To prove this let $w^{\mp} = u^{\mp}(x, t+\delta) - u^{\mp}(x,t)$ for fixed $\delta > 0$. We see that

$$\begin{aligned}
w_t^{\mp} - \Delta w_t^{\mp} &= u_2^{\mp}(x, t+\delta)\frac{\partial(t+\delta)}{\partial t} - u_2^{\mp}(x,t) - \\
&\quad - \Delta u^{\mp}(x, t+\delta) + \Delta u^{\mp}(x,t) \\
&= [u_2^{\mp}(x, t+\delta) - \Delta u^{\mp}(x, t+\delta)] - \\
&\quad - [u_2^{\mp}(x,t) - \Delta u^{\mp}(x,t)] \\
&= 0 \text{ on } Q \\
w^{\mp}(x,0) &= u^{\mp}(x,\delta) - 0 \text{ on } \bar{\Omega} \\
\frac{\partial w^{\mp}(x,t)}{\partial n} + \sigma w^{\mp}(x,t) &= \frac{\partial u^{\mp}(x,t+\delta)}{\partial n} - \frac{\partial u^{\mp}(x,t)}{\partial n} + \sigma u^{\mp}(x,t+\delta) \\
&- \sigma u^{\mp}(x,t) \\
&= h(x,t+\delta) f^{\mp}(x) - h(x,t) f^{\mp}(x) \text{ on } S.
\end{aligned}$$

So $w^{\mp}(x,0)$ satisfies

$$w_t^{\mp} - \Delta w_t^{\mp} = 0 \text{ on } Q$$

$$w^{\mp}(x,0) = u^{\mp}(x,\delta) \text{ on } \bar{\Omega}$$

$$\frac{\partial w^{\mp}}{\partial n} + \sigma w^{\mp} = f^{\mp}(x)[h(x,t+\delta) - h(x,t)] \text{ on } S.$$

Since $f^{\mp} \not\equiv 0$, by assumption, and $h(x,t+\delta) - h(x,t) \geq 0$, since h is monotone nondecreasing with respect to t, we have $[h(x,t+\delta) - h(x,t)]f^{\mp} \not\equiv 0$ on $S$, so by Lemma 2.3.2 part(1), $w^{\mp} \geq 0$ on $Q$. So $u^{\mp}(x,t+\delta) - u^{\mp}(x,t) \geq 0$, so $u^{\mp}$ are monotone nondecreasing with respect to $t$. By this property, $u_t^{\mp} \geq 0$ on $Q$ and therefore $\ell(\Delta u^{\mp}) \geq 0$ on $\Omega$, for both choice of $\ell(u)$. $u_t^{\mp} - \Delta u^{\mp} = 0 \Rightarrow u_t^{\mp} = \Delta u^{\mp} \Rightarrow \ell(u^{\mp}) = \ell(\Delta u^{\mp})$ on $\Omega$. By the relations $\ell(u_t^{\mp}) = \ell(\Delta u^{\mp})$ on $\Omega$, we see that $\Delta\bar{\chi}^{\mp}(x) \geq 0$ on $\Omega$, since $\Delta\bar{\chi}^{\mp} \geq 0 = \Delta\ell(u^{\mp}) = \ell(\Delta u^{\mp}) = \ell(u_t^{\mp}) \geq 0$. This means $\bar{\chi}^{\mp}(x)$ are subharmonic functions on $\Omega$. By the maximum principle for elliptic equations we have

$$\max_{x\in\bar{\Omega}} \bar{\chi}^{+}(x) = \max_{x\in\Gamma} \bar{\chi}^{+}(x) = \max_{x\in\Gamma} \bar{\chi}(x) = \max_{x\in\Gamma} \bar{\chi}^{-}(x) = \max_{x\in\bar{\Omega}} \bar{\chi}^{-}(x).$$

This means, $\max_{x\in\bar{\Omega}} \bar{\chi}^{+}(x) = \max_{x\in\bar{\Omega}} \bar{\chi}^{-}(x)$ on $\bar{\Omega}$. Assume that this positive maximum is attained at some $x_0 \in \Gamma$. Apply the operator $\ell$ to (3.12). Then by Theorem 2.3.1 and by the theorem on transposing an integral and a limit we obtain

$$\ell(\frac{\partial u^{\mp}}{\partial n} + \sigma u^{\mp}) = \ell(hf^{\mp}) \text{ on } \Gamma.$$

So

$$\ell(\frac{\partial u^{\mp}}{\partial n}) + \sigma \ell(u^{\mp}) = \ell(h) f^{\mp} \text{ on } \Gamma,$$

since $\sigma$ and $f$ are functions of $x$. Then

$$\frac{\bar{\chi}^{\mp}}{\partial n} + \sigma \bar{\chi}^{\mp}) = \ell(h) f^{\mp} \text{ on } \Gamma.$$

By substituting here the maximum point $x_0$ and the functions $\bar{\chi}^{\mp}(x)$ on $\bar{\Omega}$ and taking into account that $\partial_n \bar{\chi}^{\mp}(x_0) \geq 0$, either $\partial_n \bar{\chi}^{\mp}(x_0) > 0$ or $\partial_n \bar{\chi}^{\mp}(x_0) = 0$, that is $\bar{\chi}^{\mp}(x_0)$ is constant, so we have the inequalities $\ell(h) f^{\mp}(x_0) \geq \sigma(x_0)\bar{\chi}(x_0)$ on $\Gamma$.

If $\sigma(x_0) \neq 0$, then $f^{\mp}(x_0) > 0$ on $\Gamma$, and this is a contradiction to $f^{+}f^{-} = 0$ on $\Gamma$.

If $\sigma(x_0) = 0$, then $\partial_n \bar{\chi}^{\mp}(x_0) = \ell(h(x_0,t)) f^{\mp}(x_0)$. In the case of $\partial_n \bar{\chi}^{\mp}(x_0) > 0$ we obtain $\partial_n \bar{\chi}^{\mp}(x_0) > 0 \Rightarrow \ell(h) f^{\mp}(x_0) > 0 \Rightarrow f^{+}(x_0) f^{-}(x_0) > 0$ on $\Gamma$, and this is a contradiction to $f^{+}f^{-} = 0$ on $\Gamma$. In the case of $\partial_n \bar{\chi}^{\mp}(x_0) = 0$, by maximum principle and by Hopf strict maximum principle, we have $\bar{\chi} = c > 0$ on $\bar{\Omega}$ and $\partial_n \bar{\chi}^{+}(x) = 0$ on $\Gamma$, i.e., $\ell(h) f^{+}(x) = \sigma(x).c$ and $\ell(h) f^{-}(x) \geq \sigma(x).c$ on $\Gamma$, since $\partial_n \bar{\chi}^{-}(x) > 0$ on $\Gamma$. Then, there exists $y \in \Gamma$ such that $\sigma(y) \neq 0 \Rightarrow f^{+}(y) f^{-}(y) > 0$ and this is a contradiction. If $\sigma(x) \equiv 0$ on $\Gamma$, then $f^{+}(x) \equiv 0$ on $\Gamma$, and this is a contradiction to the assumption $f^{+} \not\equiv 0$. So the case $f^{+} \not\equiv 0$ and $f^{-} \not\equiv 0$ is also excluded. So $f^{\mp} \equiv 0$ then $f \equiv 0$. For this $f \equiv 0$ we have $u \equiv 0$ also and this proves the theorem.

## 3.2 An Operator Equation of the First Kind

We can readily show that the Inverse Problem I is equivalent to an operator equation of the first kind. Denote by $u_0(x,t)$ a solution of direct problem (3.1)-(3.3) with $f(x) = 0$ on $\Gamma$. Since $f$ is a function of $x$ and $f(x) = 0$ on $\Gamma$, we can say that $f(x) = 0$ on $S$ and we can say $u_0(x,t)$ is a solution of the problem

$$u_t - \Delta u = g(x,t) \text{ on } Q$$

$$u(x,0) = a(x) \text{ on } \bar{\Omega}$$

$$\partial_n u + \sigma u = b(x,t) \text{ on } S.$$

By Theorem 2.3.1, $u_0(x,t)$ exists, it is in $C^{2,1}(Q) \cap C(\bar{Q})$ and is unique. Then $\{u, f\}$ is a solution to Problem I if and only if $\{u - u_0, f\}$ is a solution to Problem $I_0$ where Problem $\mathrm{I}_0$ is

$$\begin{aligned} u_t - \Delta u &= 0 \text{ on } Q \\ u(x,0) &= 0 \text{ on } \bar{\Omega} \\ \frac{\partial u}{\partial n} + \sigma u &= hf \text{ on } S \\ \ell(u) &= \chi(x) - \ell(u_0) \text{ on } \Gamma \end{aligned}$$

To prove this let $\{u, f\}$ be a solution of Problem I. Clearly

$$(u - u_0)_t - \Delta(u - u_0) = u_t - \Delta u - (u_{0_t} - \Delta u_0) = g(x,t) - g(x,t) = 0 \text{ on } Q$$

$$(u - u_0)(x,0) = u(x,0) - u_0(x,0) = a(x) - a(x) = 0 \text{ on } \bar{\Omega}$$

$$\frac{\partial(u - u_0)}{\partial n} + \sigma(u - u_0) = (\frac{\partial u}{\partial n} + \sigma u) - (\frac{\partial u_0}{\partial n} + \sigma u_0) = hf + b - b = hf \text{ on } S$$

$$\ell(u - u_0) = \ell(u) - \ell(u_0) = \chi(x) - \ell(u_0) \text{ on } \Gamma.$$

So $\{u-u_0, f\}$ is a solution to Problem $I_0$. Conversely assume $\{u-u_0, f\}$ is a solution of Problem $I_0$. Then,

$$(u-u_0)_t-\Delta(u-u_0)=0, \text{ then } u_t-\Delta u-g(x,t)=0, \text{ then } u_t-\Delta u=g(x,t) \text{ on } Q$$

and

$$(u-u_0)(x,0)=0, \text{ then } u(x,0)-a(x)=0, \text{ then } u(x,0)=a(x) \text{ on } \bar{\Omega}$$

$$\frac{\partial(u-u_0)}{\partial n}+\sigma(u-u_0)=hf, \text{ then } \frac{\partial u}{\partial n}+\sigma u-b=hf \text{ on } S$$

$$\ell(u-u_0)=\chi(x)-\ell(u_0), \text{ then } \ell(u)=\chi(x) \text{ on } \Gamma.$$

So $\{u, f\}$ is a solution of Problem I. We have proved that $\{u, f\}$ is a solution of Problem I if and only if $\{u-u_0, f\}$ is a solution to Problem $I_0$.

Choose $f \in C(\Gamma)$ and consider a solution of problem

$$\begin{aligned} v_t-\Delta v&=0 \text{ on } Q \\ v(x,0)&=0 \text{ on } \bar{\Omega} \\ \frac{\partial v}{\partial n}+\sigma v&=hf \text{ on } S. \end{aligned} \tag{3.13}$$

The solution can be found in the simple layer potential form as

$$v(x,t)=\int_0^t\int_\Gamma G(x,t,\xi,\tau)(I-\bar{B})^{-1}hf dS_\xi d\tau \tag{3.14}$$

by substituting $g(x,t)=0$, $a(x)=0$, $b(x,t)=0$ in the unique solution of the direct problem in Theorem 2.3.1. We see that $\ell(v)=\chi(x)$ is equivalent to

$$\ell(\int_0^t\int_\Gamma G(x,t,\xi,\tau)(I-\bar{B})^{-1}hf dS_\xi d\tau)=\chi(x) \text{ on } \Gamma. \tag{3.15}$$

Introduce the linear operator

$$K: C(\Gamma)\to C(\Gamma)$$

$$K[f] = \ell(\int_0^t \int_\Gamma G(x,t,\xi,\tau)(I-\bar{B})^{-1} hf dS_\xi d\tau) = \chi(x). \tag{3.16}$$

It follows from estimate (2.45) that $K \in L(C(\Gamma))$ is a compact operator. Thus if $\{u, f\}$ is a solution to Problem $I_0$, then f is a solution to the following integral equation of the first kind

$$K[f] = \chi(x). \tag{3.17}$$

Conversely, if f is a solution to the integral equation (3.17), consider a solution to direct problem (3.13) with given $f$. This solution can be expressed in the form of a simple layer potential. Let us apply the operator $\ell$ to (3.14). Then,

$$\ell(v) = \ell(\int_0^t \int_\Gamma G(x,t,\xi,\tau)(I-\bar{B})^{-1} hf dS_\xi d\tau) = K[f] = \chi(x).$$

So $\{u, f\}$ is a solution to Inverse Problem $I_0$. This proves the following proposition.

**Proposition 3.2.1.** *Inverse Problem I is equivalent to equation (3.17) of the first kind with the compact operator $K$.*

*Remark* 3.2.1. *Solving an operator of the first kind is a classical example of an ill-posed problem. So Inverse Problem I is an ill-posed problem.*

As we know, a problem is well-posed if a solution exists, unique and depends continuously on the initial data. It is ill-posed if it fails to satisfy at least one of these conditions. Theorem 3.1.1 gives sufficient conditions for the uniqueness of a solution of Inverse Problem I, so it also gives sufficient conditions for the uniqueness of a solution to integral equation (3.17).

## 3.3 An Operator Equation of the Second Kind

Now we will derive an operator equation of the second kind and we will show that solvability of inverse Problem I is equivalent to solvability of this operator equation of the second kind.

Let $u_0(x,t)$ be solution of direct problem (3.1)-(3.3) with $f(x) = 0$ on $\Gamma$. That is

$$u_t - \Delta u = g(x,t) \text{ on } Q$$

$$u(x,0) = a(x) \text{ on } \bar{\Omega}$$

$$\partial_n u + \sigma u = b(x,t) \text{ on } S.$$

Suppose $h(x,t)$ is such that $|\ell(h)| \geq \delta > 0$ on $\Gamma$. By virtue of this condition, the multiplication by the function $\ell(h)(x)$ is a continuous one to one operator from $C(\Gamma)$ to $C(\Gamma)$. Set $\hat{h} = h[\ell(h)]^{-1}$. Choose some $\varphi \in C(\Gamma)$ and consider a solution $v(x,t)$ of the problem

$$\begin{aligned} v_t - \Delta v &= 0 \text{ on } Q \\ v(x,0) &= 0 \text{ on } \bar{\Omega} \\ \frac{\partial v}{\partial n} + \sigma v &= \hat{h}\varphi \text{ on } S. \end{aligned} \tag{3.18}$$

By Theorem 2.3.1, this solution $v(x,t)$ exists and is unique. Define the operator $B : C(\Gamma) \to C(\Gamma)$ as

$$B\varphi = \ell(\partial_n v), \tag{3.19}$$

where $v$ is a solution of (3.18) with given $\varphi$. Obviously $B$ is a linear operator, since $B(c_1\varphi_1 + c_2\varphi_2) = \ell(\partial_n(c_1\varphi_1 + c_2\varphi_2) = c_1\ell(\varphi_1) + c_2\ell(\varphi_2) = c_1B\varphi_1 + c_2B\varphi_2)$. By Theorem 2.3.1, $DomB = C(\Gamma)$, $RangeB \subset C(\Gamma)$.

**Proposition 3.3.1.** *Let $|\ell(h)| > 0$ and $|\sigma(x)| > 0$ on $\Gamma$, let $h \in C(\bar{S})$, and $\chi, \sigma \in C(\Gamma)$. Then Problem $I_0$ is equivalent to the operator equation*

$$(I - B)\varphi = \psi, \tag{3.20}$$

*in $C(\Gamma)$ where $\psi = \sigma\chi \in C(\Gamma)$.*

*Proof.* Equivalence of the inverse problem and operator equation means that two implications hold. First, if $\{u, f\}$ is a solution to Problem $\mathrm{I}_0$ then $\varphi = \ell(h)f$ is a solution to (3.20) with $\psi = \sigma\chi$ on $\Gamma$. Second, conversely, if $\varphi \in C(\Gamma)$ is a solution to equation (3.20) with some $\psi \in C(\Gamma)$ then $u(x,t)$ is a solution to (3.18) with the same satisfying $\ell(u) = \sigma^{-1}\psi$. Now for the first implication, let $\{u, f\}$ be solution of Problem $\mathrm{I}_0$. That is

$$u_t - \Delta u = 0 \text{ on } Q \tag{3.21}$$

$$u(x,0) = 0 \text{ on } \bar{\Omega} \tag{3.22}$$

$$\frac{\partial u}{\partial n} + \sigma u = hf \text{ on } S \tag{3.23}$$

$$\ell(u) = \chi(x) \text{ on } \Gamma. \tag{3.24}$$

By equation (3.23), $\frac{\partial u}{\partial n} + \sigma u = hf = \hat{h}\varphi$, for $\varphi = \ell(h)f$. Applying the operator $\ell$ to (3.23), which is possible by Theorem 2.3.1, and taking account of (3.24), we have

$$\ell(\frac{\partial u}{\partial n}) + \sigma\chi = \ell(h)f = \varphi. \tag{3.25}$$

On the other hand, $u(x,t)$ is a solution to (3.18) with $\psi = \ell(h)f$. By (3.25), $\varphi \in C(\Gamma)$ is a solution to (3.20) with $\psi \in C(\Gamma)$, since

$$(I - B)\varphi = \varphi - B\varphi = \ell(h)f - \ell(\partial_n u) = \ell(h)f - (\ell(h)f - \sigma\chi) = \sigma\chi = \psi.$$

Conversely let $\varphi \in C(\Gamma)$ be a solution to (3.20) with $\psi(\Gamma)$. Consider $u(x,t)$ which is a solution to (3.21)-(3.23) with $f = (\ell(h))^{-1}\varphi \in C(\Gamma)$. That is

$$\begin{aligned} u_t - \Delta u &= 0 \text{ on } Q \\ u(x,0) &= 0 \text{ on } \bar{\Omega} \\ \frac{\partial u}{\partial n} + \sigma u &= h(\ell(h))^{-1}\varphi \text{ on } S. \end{aligned}$$

On one hand, $\varphi$ as a solution of (3.20) satisfies

$$(I - B)\varphi = \psi,$$

or

$$\varphi - B\varphi = \psi.$$

Then,

$$\varphi - \ell(\partial_n v) = \psi,$$

and

$$\ell(\partial_n v) + \psi = \varphi.$$

That is

$$\ell(\partial_n v) + \psi = \varphi, \tag{3.26}$$

where $v$ is a solution of (3.18); and on the other hand, by the uniqueness of a solution to direct problem (3.21)-(3.23), we have $u = v$; hence applying $\ell$ to (3.23) we have

$$\ell(\partial_n u) + \sigma\ell(u) = \varphi. \tag{3.27}$$

Subtracting (3.27) from (3.26) we have

$$[\ell(\partial_n u) + \sigma\ell(u)] - [\ell(\partial_n v) + \psi] = \varphi - \varphi.$$

That is,

$$-\psi + \sigma\ell(u) = 0,$$

or we may write

$$\sigma[\ell(u) - \sigma^{-1}\psi] = 0 \text{ on } \Gamma.$$

So

$$\ell(u) = \sigma^{-1}\psi \text{ on } \Gamma, \text{ since} |\sigma(x)| > 0,$$

and

$$\ell(u) = \chi \text{ on } \Gamma.$$

This proves $\{u, f\}$ is a solution of Problem $\mathrm{I}_0$. The proposition is proved.

□

*Remark* 3.3.1. *By the given definition of equivalence of Inverse Problem $I_0$ to operator equation (3.20), we see that a solution of Problem $I_0$ is unique only if a solution of operator equation is unique.*

## 3.4 Kernel of Inverse Problem I

If in Theorem 3.1.1 we omit the condition $\ell(h) > 0$ a.e. on $\Gamma$, then a solution to Inverse Problem I is not unique. However if the remaining conditions are satisfied, then we can completely describe the kernel of the problem. Let us classify the statement.

DEFINITION 3.4.1. *A pair of functions $\{u, f\}$ belongs to the kernel of Inverse Problem I if $u \in C^{2,1}(Q)$, $f \in C(\Gamma)$ and these functions satisfy (3.6)-(3.9). In this case we write $(u, f) \in Ker$.*

Consider the operator multiplication by the function

$$\ell(h) : C(\Gamma) \to C(\Gamma)$$

$$(\ell(h)\varphi)(x) = \ell(h)(x)\varphi(x).$$

Denote the kernel of the operator by

$$N = \{\varphi \in C(\Gamma) : \ell(h)\varphi = 0 \text{ on } \Gamma\}.$$

Moreover, $N = \{0\}$ under conditions of Theorem 3.1.1.

**Proposition 3.4.1.** *Let $h \in C(\bar{S})$, $\sigma \in C(\Gamma)$, $\sigma(x) \geq 0$ on $\Gamma$, $\omega(t) \geq 0$ on $[0, T]$, $h(x,t) \geq 0$ on $S$ and $h$ be monotone nondecreasing with respect to $t$. Then, $\{u, f\} \in Ker$ if and only if $f \in N$ and $u$ is a solution to problem (3.6)-(3.8). Moreover $\ell(u) = 0$ on $\Omega$.*

*Proof.* ($\Rightarrow$)Let $\{u, f\} \in Ker$. Introduce $f^{\mp}(x)$, $u^{\mp}(x)$ as in Theorem 1, satisfying $f = f^{+} - f^{-}$, $u = u^{+} - u^{-}$, $\ell(u^{+}) = \ell(u^{-}) = \bar{\chi}(x)$ on $\Gamma$. Moreover we have $\ell(u_t^{\mp}) \geq 0$, $\Delta\ell(u^{\mp} \geq 0$ on $\Omega$. Set $\ell(u^{\mp}) = \bar{\chi}^{\mp} \in C(\bar{\Omega})$. We know by Lemma 2.3.2 that since $hf^{\mp} \not\equiv 0$ then $u^{\mp} \geq 0$ on $\bar{Q}$; if $hf^{\mp} \equiv 0$ then $u^{\mp} \equiv 0$ that is $\bar{\chi}^{\mp} \equiv 0$. So we have $\bar{\chi}^{\mp}(x) = \ell(u^{\mp}(x,t)) \geq 0$ on $\Omega$, since $u^{\mp}(x,t) \geq 0$ on $\bar{Q}$. $\bar{\chi}^{\mp}$ attains its maximum on $\Gamma$, since by maximum principle $\ell(u^{\mp}) = \Delta\bar{\chi}^{\mp} \geq 0$ on $\Omega$, then $\max\{\bar{\chi}^{\mp}(x) : x \in \bar{\Omega}\} = \max\{\bar{\chi}^{\mp}(x) : x \in \Gamma\}$. If this maximum value vanishes, since 0 is the minimum value of $\bar{\chi}^{\mp}$, then $\bar{\chi}^{\mp} \equiv 0$ on $\bar{\Omega}$. That is by

(3.8) we have $\ell(h)f^{\mp} \equiv 0$, and this means $f^{\mp} \in N$. Suppose that the maximum is positive and attained at some $x_0 \in \Gamma$. Since $\bar{\chi}^+ = \bar{\chi}^- = \bar{\chi}$ on $\Gamma$, we have $\bar{\chi}(x_0) = \max\{\bar{\chi}^+(x) : x \in \bar{\Omega}\} = \max\{\bar{\chi}^-(x) : x \in \bar{\Omega}\}$. Then, by applying $\ell$ operator to (3.8) we have

$$\partial_n \bar{\chi}^{\mp}(x) + \sigma(x)\bar{\chi}(x) = \ell(h)f^{\mp} \text{ on } \Gamma. \tag{3.28}$$

We have either $\partial_n \bar{\chi}^+(x_0) = 0$ or $\partial_n \bar{\chi}^-(x_0) = 0$. Otherwise, if both are not 0, we obtain $\ell(h)f^{\mp}(x_0) > 0$ but this contradicts $f^+f^- \equiv 0$ on $\Gamma$. Let $\partial_n \bar{\chi}^-(x_0) = 0$. We know in this case $\partial_n \bar{\chi}^+(x_0) \geq 0$. By strict maximum principle, we have $\bar{\chi}^-(x) = \text{const} > 0$, on $\bar{\Omega}$. Then $\partial_n \bar{\chi}^-(x) = 0$, $\bar{\chi} = \bar{\chi}^-(x) = \text{const}$ on $\Gamma$, so any point of $\Gamma$ is a maximum point for $\bar{\chi}^{\mp}(x)$. By (3.28), $\ell(h)f^-(x) = \sigma(x)$.const and $\ell(h)f^+(x) \geq \sigma(x)$.const on $\Gamma$, which is possible only if $\sigma(x) = 0$ on $\Gamma$ (otherwise, if $\sigma > 0$, then $f^{\mp} > 0$ on $\Gamma$ and that is a contradiction), i.e. $\ell(h)f^- = 0$, then we have $f^- \in N$. Since $\bar{\chi}^+ = \text{const}$ on $\Gamma$, by strict max principle either $\bar{\chi}^+ = \text{const}$ on $\bar{\Omega}$ or $\partial_n \bar{\chi}^+ > 0$ on $\Gamma$. If $\bar{\chi}^+ = \text{const}$ on $\bar{\Omega}$ then $\ell(h)f^+ = 0$, so $f^+ \in N$. If $\partial_n \bar{\chi}^+ > 0$ on $\Gamma$, then $\ell(h)f^+ > 0$ on $\Gamma$, this is possible if $f^+ > 0$ on $\Gamma$, i.e. $f^- = 0$, $f$=$f^+$. Then $u(x,t) = u^+(x,t)$ on $Q$, but since $h(x,t)f^+(x) \not\equiv 0$ on $S$, by Lemma 2.3.2 $\ell(u^+) > 0$ on $\bar{\Omega}$, and this is a contradiction to $\ell(u^+) = \ell(u) = 0$ on $\Gamma$ (to the assumption that $\{u, f\} \in Ker$).

($\Leftarrow$) Now let us assume $f$ be in $N$, $u$ be solution of (3.6)-(3.8) with this $f$. We will prove in this case $\ell(u) = 0$ on $\bar{\Omega}$. We have $\ell(h)f^{\mp}(x) = \ell(h)(x)f^{\mp}(x)$, therefore $f \in N$ if and only if $f^{\mp} \in N$. Hence it is sufficient to prove for $f \in N$ and $f \geq 0$ on $\Gamma$. Choose such a function $f$ and consider solution of (3.6)-(3.8). This solution satisfies $\ell(u) \geq 0$, $\Delta\ell(u) \geq 0$ on $\Omega$ (that was proved in Theorem 3.1.1), $\partial_n \ell(u) + \sigma\ell(u) = \ell(h)f = 0$ on $\Gamma$. We know $\ell(h)f = 0$, since f is assumed to be in $N$, the function $\ell(u)$ cannot have positive maximum $\bar{\Omega}$. Otherwise, that is if $\ell(u)$ has a positive maximum on $\bar{\Omega}$, since $\Delta\ell(u) \geq 0$ on $\Omega$, then the maximum occurs on $\Gamma$ by maximum principle, say at a point $x_0 \in \Gamma$. Then $\partial_n \ell(u)(x_0) > 0$ but this is a contradiction to $\partial_n \ell(u) + \sigma\ell(u) = \ell(h)f = 0$. So $\ell(u) \leq 0$ on $\bar{\Omega}$. We also know $\ell(u) \geq 0$ on $\bar{\Omega}$, so $\ell(u) = 0$ on $\bar{\Omega}$, except for the case $\ell(u) = \text{const} \geq 0$ and $\sigma(x) = 0$ on $\Gamma$. So continue with this case. Now, if $\ell(u) = u(x, t_1) = \text{const} \geq 0$,

$0 \le u(x,t) \le u(x,t_1)$, $x \in \bar{\Omega}$, $t \in [0,t_1]$ since $u$ is nondecreasing with respect to $t$, which was proved in Theorem 3.1.1. Hence by strict maximum principle $u(x,t) = \text{const} = 0$ on $\bar{\Omega} \times [0,t_1]$. $u(x,t)$ takes its maximum in $\Omega$ for $t = t_1$, so by maximum principle it is constant. But we know $u = 0$ in $\Omega$ for $t = 0$ so this constant is 0. So $\ell(u) = 0$ on $\bar{\Omega}$. If $\ell(u) = \int_0^T u(x,\tau)\omega(\tau)d\tau$, where $u$ satisfies $u_t - \Delta u = 0$ on $Q$, $u(x,0) = 0$ on $\bar{\Omega}$, $\frac{\partial u}{\partial n} = hf$ on $S$. Since $\ell(h)f = 0$ on $\Gamma$ by the assumption $f \in N$, i.e. $\ell(h)f = \int_0^T h(x,\tau)f(x)\omega(\tau)d\tau = 0$, on $\Gamma$. So $h(x,t)f(x)\omega(t) = 0$, on $\Gamma$ for almost all $t \in [0,T]$. Then $\int_\tau^t h(x,\eta)\omega(\eta)d\eta$, on $\Gamma$ for any $\tau, t \in [0,T]$. If we integrate (3.6) with respect to $t$ from 0 to $\tau$, then we have

$$\int_0^\tau u_2(x,\eta)d\eta = \int_0^\tau \Delta u(x,\eta)d\eta,$$

so

$$u(x,\tau) - u(x,0) = \Delta \int_0^\tau u(x,\eta)d\eta,$$

then

$$u(x,\tau) = \Delta \int_0^\tau u(x,\eta)d\eta.$$

Multiply this with $\omega(\tau)$ and integrate with respect to $\tau$ from 0 to $t$, then

$$0 \le \int_0^t u(x,\tau)\omega(\tau)d\tau = \Delta[\int_0^t \omega(\tau)\int_0^\tau u(x,\eta)d\eta d\tau] = \Delta W,$$

if we define

$$W := \int_0^t \omega(\tau)\int_0^\tau u(x,\eta)d\eta d\tau.$$

Applying the same operations to (3.8) leads to

$$\begin{aligned}
\partial_n W &= \int_0^t \omega(\tau) \int_0^\tau \partial_n u(x,\eta) d\eta d\tau \\
&= \int_0^t \omega(\tau) \int_0^\tau h(x,\eta) f(x) d\eta d\tau \\
&\geq \int_0^t \omega(\tau) f(x) \min_{\eta \in (0,\tau)} h(x,\eta)[\tau - 0] d\tau \\
&= \int_0^t \omega(\tau) f(x) h(x,0) \tau d\tau, \\
&\quad \text{since } h \text{ is monotone nondecreasing with respect to t} \\
&= f(x) h(x,0) \int_0^t \omega(\tau) \tau d\tau \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_n W &= \int_0^t \omega(\tau) \int_0^\tau h(x,\eta) f(x) d\eta d\tau \\
&\leq \int_0^t \omega(\tau) \int_0^\tau h(x,\tau) f(x) d\eta d\tau,
\end{aligned}$$

since $\max_{\eta \in (0,\tau)} h(x,\eta) = h(x,\tau)$ because $h$ is monotone nondecreasing with respect to $t$. Changing order of the integration we have

$$\begin{aligned}
\partial_n W &= \int_0^t \int_\eta^t \omega(\tau) h(x,\tau) f(x) d\tau d\eta \\
&= \int_0^t 0 d\eta \text{ , was proved before} \\
&= 0.
\end{aligned}$$

So we have obtained

$$0 \leq \partial_n W \leq 0,$$

which means

$$\partial_n W = 0 \text{ on } \Gamma.$$

Therefore we have

$$\Delta W(x,t) \geq 0 \text{ on } \Omega$$

$$\partial_n W(x,t) = 0 \text{ on } \Gamma \text{for} t \in [0,T].$$

If $W(x_0,t)$ is a maximum point of $W$ with $x_0 \in \Gamma$, then $W$ must satisfy $\partial_n W(x_0,t) > 0$ but this is a contradiction since we know $\partial_n W(x,t) = 0$ on $\Gamma$. So there is no maximum of W on $\Gamma$. So $W$ takes its maximum in $\Omega$, and in this case by maximum principle for elliptic equations $W$ is a constant with respect to $x$. So we have,

$$W(x,t) = W(t),$$

which means

$$\Delta W = 0 \text{ on } \Omega$$

and

$$\int_0^t u(x,\tau)\omega(\tau)d\tau = \Delta W = 0 \text{ on } \Omega \text{ for } t \in [0,T],$$

then

$$u(x,t)\omega(t) = 0 \text{ on } \bar{\Omega} \text{ a.e. } t \in [0,T],$$

and

$$\ell(u) = 0 \text{ on } \bar{\Omega}.$$

□

*Remark* 3.4.1. *In Proposition (3.4.1), we proved in addition that if $f \in N$ then $u(x,t) = 0$ on $\bar{\Omega} \times [0,t_1]$ in the case of a terminal boundary observation, and $\omega(t)u(x,t) = 0$ on $\bar{\Omega}$ for almost all $t \in [0,T]$.*

# CHAPTER 4

# ANALYSIS OF PROBLEM II

**Problem II:** Find a pair of functions $\{u(x,t), \sigma(x)\}$ satisfying (3.1), (3.2), (3.4) and

$$\partial_n u + \sigma u = b(x,t) \text{ on } S, \tag{4.1}$$

where $g(x,t)$, $a(x)$, $b(x,t)$ and $\chi(x)$ are given functions.

DEFINITION 4.0.2. *A pair $\{u(x,t), \sigma(x)\}$ is called a solution to Problem II if $u \in C^{2,1}(Q)$, $\sigma \in C(\Gamma)$, $\sigma(x) > 0$ on $\Gamma$ and these functions satisfy (3.1), (3.2), (3.4) and (4.1) in the classical sense.*

## 4.1 Uniqueness Theorem for Problem II

THEOREM 4.1.1. *Assume that smoothness conditions (3.5) hold, $a(x) = 0$ on $\Omega$, $g(x,t) \geq 0$ in $Q$, $b(x,t) \geq 0$ on $S$, $\chi(x) > 0$ on $\Gamma$, $g$ and $b$ are monotone nondecreasing with respect to $t$. Then the solution to the inverse Problem II is unique.*

Assume there exists two pairs of functions $\{u^1, \sigma^1\}$ and $\{u^2, \sigma^2\}$ that satisfy Problem II. Then,

$$\begin{aligned} u_t^i - \Delta u^i &= g(x,t) \text{ on } Q \\ u^i(x,0) &= a(x) \text{ on } \bar{\Omega} \\ \frac{\partial u^i}{\partial n} + \sigma u^i &= b \text{ on } S \\ \ell(u^i) &= \chi \text{ on } \Gamma, \end{aligned}$$

for $i = 1, 2$. So the functions $\bar{u} = u^2 - u^1$, $\bar{\sigma} = \sigma^2 - \sigma^1$ satisfy

$$\bar{u}_t - \Delta \bar{u} = g(x,t) - g(x,t) = 0 \text{ on } Q$$

$$\bar{u}(x,0) = a(x) - a(x) = 0 \text{ on } \bar{\Omega}$$
$$\frac{\partial \bar{u}}{\partial n} + \sigma \bar{u} = b(x,t) - b(x,t) = 0 \text{ on } S$$
$$\ell(\bar{u}) = \chi(x) - \chi(x) = 0 \text{ on } \Gamma,$$

and this problem is equivalent to

$$\begin{aligned} \bar{u}_t - \Delta \bar{u} &= 0 \text{ on } Q \\ \bar{u}(x,0) &= 0 \text{ on } \bar{\Omega} \\ \frac{\partial \bar{u}}{\partial n} + \sigma^2 \bar{u} &= u^1 \bar{\sigma} \text{ on } S \\ \ell(\bar{u}) &= 0 \text{ on } \Gamma. \end{aligned} \tag{4.2}$$

This problem is in the form of Problem I for $u = \bar{u}$, $f = \sigma$. Since $g \in C^{\alpha,0}(\bar{Q})$, $b \in C(\bar{S})$, we have $u^1 \in C^{2,1}(Q) \cap C(\bar{Q})$ for given $\sigma^1$. By the assumption of Theorem 4.1.1, $g$ and $b$ are nonnegative and non-decreasing with respect to $t$. Hence $u^1(x,t)$ is nonnegative on $\bar{Q}$, by Lemma 2.3.2 part(i), and $u^1(x,t)$ is nondecreasing with respect to $t$i which was proved in Theorem 3.1.1. Since $\sigma^2(x) \geq 0$ on $\Gamma$ by definition of solution of Problem II, and $\ell(u^1) > 0$ on $\Gamma$ by the assumption of the theorem, all conditions of Theorem 3.1.1 hold. So solution $(\bar{u}, \bar{\sigma})$ of the problem (4.2). So $\bar{u} = 0$, $\bar{\sigma} = 0$ is the unique solution of problem (4.2). Then $\bar{u} = u^2 - u^1 = 0$ and $\bar{\sigma} = \sigma^2 - \sigma^1 = 0$, so $u^2 = u^1$ and $\sigma^2 = \sigma^1$, and that means solution of Inverse Problem II is unique.

## 4.2 Derivation of Operator Equation

Suppose $|\chi(x)| > 0$ on $\Gamma$. Choose a $\sigma \in C(\Gamma)$ and consider a solution $u(x,t;\sigma)$ to the problem

$$u_t - \Delta u = g(x,t) \text{ on } Q \tag{4.3}$$

$$u(x,0) = a(x) \text{ on } \bar{\Omega} \tag{4.4}$$

$$\frac{\partial u}{\partial n} + \sigma u = b(x,t) \text{ on } S. \tag{4.5}$$

Since $g \in C^{\alpha,0}(\bar{Q})$, $a \in C^1(\bar{\Omega})$ and $b \in C(\bar{S})$, this solution $u$ exists and is unique and $u \in C^{2,1}(Q) \cap C(\bar{Q})$. Introduce the nonlinear operator

$$U : C(\Gamma) \to C(\Gamma)$$

$$U\sigma = [\ell(b) - \ell(\partial_n u)]\chi^{-1}.$$

We have $DomU = C(\Gamma)$, since it can be applied to any function in $C(\Gamma)$. Consider the equation

$$U\sigma = \sigma. \tag{4.6}$$

We will show that if (4.6) has a solution with $\sigma(x) > 0$ a.e. on $\Gamma$, then Inverse Problem II is solvable. Let $\sigma \in C(\Gamma)$ be a solution to equation (4.6). Consider a solution $u(x,t;\sigma)$ to direct problem (4.3)-(4.5). Apply operator $\ell$ to equation (4.5), which is possible by Theorem 2.3.1, and obtain the relation

$$\ell(\partial_n u) + \sigma\ell(u) = \ell(b), \tag{4.7}$$

in $C(\Gamma)$. Multiplying (4.6) by $\chi$ and adding to (4.7) we obtain

$$\begin{aligned} \ell(\partial_n u) + \sigma\ell(u) &= \ell(b) \\ U\sigma\chi &= \sigma\chi. \end{aligned}$$

Then,

$$\begin{aligned} U\sigma\chi + \ell(\partial_n u) + \sigma\ell(u) &= \ell(b) + \sigma\chi \\ (\ell(b) - \ell(\partial_n u))\chi^{-1}\chi + \ell(\partial_n u) + \sigma\ell(u) &= \sigma\chi + \ell(b), \end{aligned}$$

which implies

$$\begin{aligned} \sigma\ell(u) &= \sigma\chi \\ \sigma[\ell(u)-\chi] &= 0 \text{ on } \Gamma. \end{aligned}$$

Since $\sigma > 0$ we have

$$\ell(u) = \chi \text{ on } \Gamma.$$

So $\{u, \sigma\}$ is a solution to Problem II. So we have proved the following proposition.

**Proposition 4.2.1.** *Let $g \in C^{\alpha,0}(\bar{Q})$, $a \in C^1(\bar{\Omega})$, $b \in C(\bar{S})$, $\chi \in C(\Gamma)$ and $|\chi(x)| > 0$ on $\Gamma$. Then, if (4.6) has a solution $\sigma \in C(\Gamma)$ with $\sigma(x) > 0$ a.e. on $\Gamma$, then Problem II is solvable.*

# CHAPTER 5

# ANALYSIS OF PROBLEM III

**Problem III:** Find a pair $\{u(x,t), f(t)\}$ satisfying

$$u_t - \Delta u = g \text{ on } Q \tag{5.1}$$

$$u(x,0) = a \text{ on } \bar{\Omega} \tag{5.2}$$

$$\frac{\partial u}{\partial n} + \sigma u = hf + b \text{ on } S \tag{5.3}$$

$$\Psi(u) = \bar{\chi} \text{ on } [0,T], \tag{5.4}$$

where $g(x,t)$, $a(x)$, $\sigma(x,t)$, $h(x,t)$, $b(x,t)$, $\bar{\chi}(t)$ are given functions and

$$\Psi(u) = u(x_0, t)$$

or

$$\Psi(u) = \int_\Gamma u(\xi,\tau)\nu(\tau)dS_\xi,$$

where $x_0 \in \Gamma$ is fixed and $\nu \in L_1(\Gamma)$ is given. These conditions are called a point boundary observation and integral boundary observation respectively.

Definition 5.0.1. *By a solution of Problem III, we understand a classical solution $u \in C^{2,1}(Q)$, $f \in C([0,T])$.*

We denote the solution of direct problem (5.1)-(5.3) with $f = 0$ by $u_0(x,t)$, and introduce the function

$$\chi(t) = \bar{\chi}(t) - \Psi(u_0)(t).$$

THEOREM 5.0.2. *Suppose the smoothness conditions (3.5) and*

$$\chi \in C^{1/2}([0,T]) \ , \ \chi(0) = 0 \ , \ F(t) = \frac{d}{dt}\int_0^t \frac{\chi(\tau)}{\sqrt{t-\tau}} d\tau \in C([0,T]) \tag{5.5}$$

*are satisfied, $h \in C^{\alpha,0}(\bar{S})$, $\sigma \in C(\bar{S})$, $\Psi(u_0)(0) = \bar{\chi}(0)$ and $|\Psi(h) > 0|$ on $[0,T]$. Then there exists unique solution of Problem III.*

## 5.1 An Operator Equation of the First Kind

We denoted $u_0(x,t)$ as solution of direct problem (5.1)-(5.3) with $f = 0$. By Theorem 5.0.2, $u_0(x,t)$ exists, unique and $u_0 \in C^{2,1}(Q) \cap C(\bar{Q})$. We find that $\{u, f\}$ is a solution of Problem III if and only if $\{u - u_0, f\}$ is a solution of Problem $\mathrm{III}_0$, which is

$$\begin{aligned} u_t - \Delta u &= 0 \text{ on } Q \\ u(x,0) &= 0 \text{ on } \bar{\Omega} \\ \frac{\partial u}{\partial n} + \sigma u &= hf \text{ on } S \\ \Psi(u) &= \chi = \bar{\chi} - \Psi(u_0) \text{ on } [0,T], \end{aligned}$$

provided that $\chi(0) = \bar{\chi}(0) - \Psi(u_0)(0) = 0$. That means, Problem III is equivalent to Problem $\mathrm{III}_0$ with a given $\chi(t)$ such that $\chi(0) = 0$. Let us prove this.

($\Rightarrow$)Let $\{u, f\}$ be solution of Problem III. Then,

$$\begin{aligned} u_t - \Delta u &= g \text{ on } Q \\ u(x,0) &= a \text{ on } \bar{\Omega} \\ \partial_n u + \sigma u &= hf + b \text{ on } S \\ \Psi(u) &= \bar{\chi}(t) \text{ on } [0,T], \end{aligned}$$

and we know there exists unique solution $u_0$ of

$$\begin{aligned}(u_{0_t}) - \Delta u_0 &= 0 \text{ on } Q\\ u_0(x,0) &= 0 \text{ on } \bar{\Omega}\\ \partial_n u_0 + \sigma u_0 &= hf \text{ on } S.\end{aligned}$$

So $\{u - u_0, f\}$ satisfies the equation

$$(u-u_0)_t - \Delta(u-u_0) = g - 0 = g \text{ on } Q$$

$$(u-u_0)(x,0) = a - 0 = a \text{ on } \bar{\Omega}$$

$$\partial_n(u-u_0) + \sigma(u-u_0) = hf + b - hf = b \text{ on } S$$

$$\Psi(u-u_0) = \Psi(u) - \Psi(u_0) = \bar{\chi} - \Psi(u_0) = \chi \text{ on } [0,T].$$

So $\{u - u_0, f\}$ is a solution of Problem III$_0$.

($\Leftarrow$)Now let $\{u - u_0, f\}$be a solution of Problem III$_0$. Then we have

$$\begin{aligned}(u-u_0)_t - \Delta(u-u_0) &= g \text{ on } Q\\ (u-u_0)(x,0) &= a \text{ on } \bar{\Omega}\\ \partial_n(u-u_0) + \sigma(u-u_0) &= b \text{ on } S\\ \Psi(u-u_0) &= \chi \text{ on } [0,T].\end{aligned}$$

Then,

$$\begin{aligned}u_t - \Delta u - 0 &= g \text{ on } Q\\ u(x,0) - 0 &= a \text{ on } \bar{\Omega}\\ \partial_n u + \sigma u - hf &= b \text{ on } S\\ \Psi(u) - \Psi(u_0) &= \chi \text{ on } [0,T].\end{aligned}$$

That is $\{u, f\}$ is a solution of Problem III. Everywhere in the following, we assume that $\Psi(h) > 0$ on $[0,T]$. We know $\Psi(h) \in C([0,T])$ since $h \in C([0,T])$ was assumed. Then we have $\varphi := \Psi(h)f \in C([0,T])$ if and only if $f \in C([0,T])$. We have $\hat{h}(x,t) = [\Psi(h)]^{-1}h(x,t) \in C(S)$ and $\Psi(\hat{h}) = \Psi(h[\Psi(h)]^{-1}) = \Psi(h)[\Psi(h)]^{-1} = 1$. Since $hf = \hat{h}\Psi(h)f = \hat{h}\varphi$, it follows that $\{u,f\}$ is a solution of Problem $\text{III}_0$ if and only if $\{u,\varphi\}$ is a solution of the problem

$$\begin{aligned} u_t - \Delta u &= 0 \text{ on } Q && (5.6)\\ u(x,0) &= 0 \text{ on } \bar{\Omega} && (5.7)\\ \frac{\partial u}{\partial n} + \sigma u &= \hat{h}\varphi \text{ on } S && (5.8)\\ \Psi(u) &= \chi(t) \text{ on } [0,T]. && (5.9) \end{aligned}$$

Let us first show that Problem $\text{III}_0$ is equivalent to an operator equation of the first kind.

LEMMA 5.1.1. *Suppose that $\chi \in C^{1/2}([0,T])$, $\chi(0) = 0$, $h \in C(\bar{S})$ and $|\Psi(h)| > 0$ on $[0,T]$. Then Problem $III_0$ is equivalent to an operator equation $\hat{K}[\varphi] = \chi$, where $\hat{K}$ is an integral operator of the Volterra type whose kernel has a weak singularity.*

*Proof.* ($\Rightarrow$)Let $\{u,\varphi\}$ be solution of the Problem $\text{III}_0$. Then $\{u,\varphi\} \in C^{2,1}([0,T]) \times C([0,T])$ satisfies (5.6)-(5.9). We have $\Psi(u)(0) = \chi(0) = 0$ by assumption of the lemma. $\Psi(u) \equiv \chi$ and $\chi \in C^{1/2}([0,T])$, so $\Psi(u) \in C^{1/2}([0,T])$. Choose $\varphi \in C([0,T])$. Then solution of direct problem is of the form

$$u(x,t) = \int_0^t \int_\Gamma G(x,t,\xi,\tau)(I - \bar{B})^{-1}(\hat{h}\varphi)(\xi,\tau)dS_\xi d\tau$$

by Theorem 5.0.2. Apply the operator $\Psi$ to both sides, then

$$\Psi(u) = \int_0^t \Psi\left(\int_\Gamma G(x,t,\xi,\tau)(I - \bar{B})^{-1}(\hat{h}\varphi)(\xi,\tau)dS_\xi\right) d\tau := \hat{K}[\varphi].$$

Since $u$ is solution of Problem $\text{III}_0$, we have

$$\Psi(u) = \chi \text{ on } [0,T].$$

So,

$$\Psi(u) = \hat{K}[\varphi] = \chi \text{ on } [0,T].$$

Then $\varphi \in C([0,T])$ satisfies the operator equation of Volterra type of first kind.

($\Leftarrow$)Assume $\varphi \in C([0,T])$ satisfies the operator equation with some $\chi \in C^{1/2}([0,T]), \chi(0) = 0$. Let $u$ be the solution of direct problem (5.6)-(5.8) with this $\chi$. $u$ can be found in the simple layer potential form

$$u(x,t) = \int_0^t \int_\Gamma G(x,t,\xi,\tau)(I-\bar{B})^{-1}(\hat{h}\varphi)(\xi,\tau)dS_\xi d\tau.$$

We have

$$\Psi(u) = \hat{K}[\varphi] = \chi \text{ on } [0,T].$$

So $\{u,\varphi\}$ is solution of problem $\text{III}_0$. □

Lemma 5.1.2. *Suppose $h \in C^{\alpha,0}(\bar{S})$, $|\Psi(h)| > 0$ on $[0,T]$. Then the operator*

$$\hat{K}[\varphi] = \int_0^t \Psi\left(\int_\Gamma G(x,t,\xi,\tau)(I-\bar{B})^{-1}(\hat{h}\varphi)(\xi,\tau)dS_\xi\right)d\tau$$

*can be represented in the form $\hat{K} = K + \bar{K}$, where $K[\varphi] = \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}}d\tau$ and the kernel of $\bar{K}$ has a weaker singularity than $K$, more precisely*

$$\bar{K}[\varphi] = \int_0^t \frac{k(t,\tau)}{(t-\tau)^\mu}\varphi(\tau)d\tau,$$

*with $\mu < 1/2$ and $k \in C_t^1$.*

*Proof.* We consider two cases depending on the form of the overdetermination.

Case 1: The case $\Psi(u) = u(x_0,t)$, a point overdetermination. The solution of problem (5.6)-(5.8) can be found in the simple layer potential form

$$u(x,t) = \int_0^t \int_\Gamma G(x,t,\xi,\tau)\bar{\varphi}(\xi,\tau)dS_\xi d\tau,$$

where

$$G(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + Z_0(x,t,\xi,\tau).$$

Then using formulas (2.41)-(2.47) we have

$$
\begin{aligned}
u(x_0,t) &= 2\int_0^t \int_\Gamma Z(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)\varphi(\tau)dS_\xi d\tau \\
&+ 2\int_0^t \int_\Gamma Z(x_0,t,\xi,\tau) \\
&\left(\sum_{k=1}^{\infty}\int_0^\tau \int_\Gamma M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)\varphi(\sigma)dS_\eta d\sigma\right) dS_\xi d\tau \\
&+ 2\int_0^t \int_\Gamma Z_0(x_0,t,\xi,\tau) \\
&\left(\hat{h}(\xi,\tau)\varphi(\tau) + \sum_{k=1}^{\infty}\int_0^\tau \int_\Gamma M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)\varphi(\sigma)dS_\eta d\sigma\right) dS_\xi d\tau \\
&= K_1[\varphi] + K_2[\varphi] + K_3[\varphi].
\end{aligned}
$$

Here $K_3[\varphi]$ is an operator of Volterra type with bounded kernel. Let us explain.

Changing the order of integration by Dirichlet's Theorem, we obtain

$$\begin{aligned}
K_3[\varphi] &= 2\int_0^t \int_\Gamma Z_0(x_0,t,\xi,\tau) \\
&\quad \left(\hat{h}(\xi,\tau)\varphi(\tau) + \sum_{\nu=1}^{\infty}\int_0^\tau \int_\Gamma M_\nu(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)\varphi(\sigma)dS_\eta d\sigma\right) dS_\xi d\tau \\
&= \int_0^t \left[2\int_\Gamma Z_0(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi\right]\varphi(\tau)d\tau + \\
&+ 2\int_0^t \int_\Gamma Z_0(x_0,t,\xi,\tau) \\
&\quad \left[\int_0^\tau \int_\Gamma \sum_{k=1}^{\infty} M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)\varphi(\sigma)dS_\eta d\sigma\right] dS_\xi d\tau \\
&= \int_0^t \left[2\int_\Gamma Z_0(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi\right]\varphi(\tau)d\tau + \\
&+ \int_0^t \int_0^z \left[2\int_\Gamma\int_\Gamma Z_0(x_0,t,\xi,\tau)\sum_{k=1}^{\infty} M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta dS_\xi\right] \\
&\quad \varphi(\sigma)d\sigma d\tau \\
&= \int_0^t \left[2\int_\Gamma Z_0(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi\right]\varphi(\sigma)d\sigma + \\
&+ \int_0^t \left[\int_\sigma^t 2\int_\Gamma\int_\Gamma Z_0(x_0,t,\xi,\tau)\sum_{k=1}^{\infty} M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta dS_\xi d\tau\right] \\
&\quad \varphi(\sigma)d\sigma \\
&= \int_0^t \{2\int_\Gamma Z_0(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi + \\
&+ 2\int_\sigma^t \int_\Gamma\int_\Gamma Z_0(x_0,t,\xi,\tau)\sum_{k=1}^{\infty} M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta dS_\xi d\tau\}\varphi(\sigma)d\sigma.
\end{aligned}$$

So the kernel of $K_3$ is

$$\begin{aligned}
KerK_3 &= 2\int_\Gamma Z_0(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi \\
&+ 2\int_\sigma^t \int_\Gamma\int_\Gamma Z_0(x_0,t,\xi,\tau)\sum_{k=1}^{\infty} M_k(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta dS_\xi d\tau.
\end{aligned}$$

We know

$$Z_0(x,t,\xi,\tau) = \int_\tau^t \int_D Z(x,t,\eta,\sigma)\hat{\Phi}(\eta,\sigma,\xi,\tau)d\eta d\sigma$$

is a continuous function for $(x,t,\xi,\tau) \in (\bar{\Omega}\times[0,T])^2$, $\bar{\Omega}\subset D$ where $\hat{\Phi}$ is the solution of the integral equation

$$\hat{\Phi}(x,t,\xi,\tau) = -MZ - \int_\tau^t \int_D MZ(x,t,\eta,\sigma)\hat{\Phi}(\eta,\sigma,\xi,\tau)d\eta d\sigma,$$

where $MZ = \partial_t Z - \Delta Z$, by parametrix method, [9](p. 4). $M_1(x,t,\xi,\tau) = M(x,t,\xi,\tau)$ is bounded by Theorem 2.3.1. So

$$M_k(x,t,\xi,\tau) = \int_\tau^t \int_D M_{k-1}(x,t,\eta,\sigma)M(\eta,\sigma,\xi,\tau)d\eta d\sigma$$

are also bounded. Furthermore $\hat{h} = h(\Psi(h))^{-1}$ is continuous since $h$ is continuous, so it is bounded on bounded domain. So kernel of $K_3[\varphi]$ is continuous on bounded domain $\bar{\Omega}$, that means it is bounded. Let us consider $K_2[\varphi]$.

$$\begin{aligned} K_2[\varphi] &= \int_0^t \int_\Gamma Z(x_0,t,\xi,\tau)\left(\int_0^\tau \int_\Gamma M(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)\varphi(\sigma)dS_\eta dS_\sigma\right)dS_\xi d\tau \\ &= \int_0^t \int_0^\tau \left(\int_\Gamma \int_\Gamma Z(x_0,t,\xi,\tau)M(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta dS_\xi\right)\varphi(\sigma)d\sigma d\tau. \end{aligned}$$

Assuming $1-\frac{\alpha}{2}<\mu<1$, we have

$$\begin{aligned} &\left|\int_\Gamma \int_\Gamma Z(x_0,t,\xi,\tau)M(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta dS_\xi\right| \\ \leq\ & \int_\Gamma Z(x,t,\xi,\tau)\left|\int_\Gamma M(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta\right| dS_\xi. \end{aligned}$$

We know

$$|M(\xi,\tau,\eta,\sigma)| \leq \frac{c}{(\tau-\sigma)^\mu|\xi-\eta|^{m+1-2\mu-\alpha}}$$

and

$$\begin{aligned} |Z(x,t,\xi,\tau)| &= |c(t-\tau)^{-m/2}\exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right)| \\ &\leq |c(t-\tau)^{-m/2}|. \end{aligned}$$

Since $\min m = 1$, we get

$$\begin{aligned} \int_\Gamma Z(x,t,\xi,\tau)\left|\int_\Gamma M(\xi,\tau,\eta,\sigma)\hat{h}(\eta,\sigma)dS_\eta\right|dS_\xi &\leq \frac{c}{(\tau-\sigma)^\mu}\int_\Gamma Z(x,t,\xi,\tau)dS_\xi \\ &\leq \frac{c}{(\tau-\sigma)^\mu(t-\tau)^{-m/2}} \\ &\leq \frac{c}{(\tau-\sigma)^\mu(t-\tau)^{-1/2}}. \end{aligned}$$

So,

$$\begin{aligned} K_2[\varphi] &\leq \int_0^t\int_0^\tau \frac{\varphi(\sigma)}{(t-\tau)^{1/2}(\tau-\sigma)^\mu}d\sigma d\tau \\ &= \int_0^t\left[\int_\sigma^t \frac{d\tau}{(t-\tau)^{1/2}(\tau-\sigma)^\mu}\right]\varphi(\sigma)d\sigma. \end{aligned}$$

Remembering the Beta function defined by

$$\int_0^1 (1-\rho)^{a-1}\rho^{b-1}d\rho = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

we find the kernel of $K_2[\varphi]$ as

$$\begin{aligned}
k_2(t,\tau) &= \int_\sigma^t \frac{d\tau}{(t-\tau)^{1/2}(\tau-\sigma)^\mu} \\
&= \int_0^1 \frac{(t-\sigma)d\rho}{(1-\rho)^{1/2}(t-\sigma)^{1/2}(\tau-\sigma)^\mu}\ , \text{ for } \rho = \frac{\tau-\sigma}{t-\sigma} \\
&= \int_0^1 \frac{(t-\sigma)d\rho}{(1-\rho)^{1/2}(t-\sigma)^{1/2}\frac{(\tau-\sigma)^\mu}{(t-\sigma)^\mu}(t-\sigma)^\mu} \\
&= \int_0^1 \frac{(t-\sigma)^{\frac{1}{2}-\mu}}{(1-\rho)^{1/2}\rho^\mu}d\rho \\
&= (t-\sigma)^{\frac{1}{2}-\mu}\int_0^1 (1-\rho)^{-1/2}\rho^{-\mu}d\rho \\
&= (t-\sigma)^{\frac{1}{2}-\mu}\frac{\Gamma(\frac{1}{2})\Gamma(1-\mu)}{\Gamma(\frac{3}{2}-\mu)}.
\end{aligned}$$

Since $1-\frac{\alpha}{2} < \mu < 1$, we have $\frac{1}{2}-\mu \in (-\frac{1}{2}, 0)$, and we see that singularity of $k_2(t,\tau)$ is weaker than that of $(t-\tau)^{1/2}$. Therefore the singularity of the operator $K_2$ is weaker than that of $(t-\tau)^{1/2}$. Now consider

$$\begin{aligned}
K_1[\varphi] &= 2\int_0^t\int_\Gamma Z(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)\varphi(\tau)dS_\xi d\tau \\
&= \int_0^t [2\int_\Gamma Z(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi]\varphi(\tau)d\tau \\
&:= \int_0^t \frac{k_1(t,\tau)}{(t-\tau)^{1/2}}\varphi(\tau)d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
k_1(t,\tau) &= 2(t-\tau)^{1/2}\int_\Gamma Z(x_0,t,\xi,\tau)\hat{h}(\xi,\tau)dS_\xi \\
&= 2(2\sqrt{\pi})^{-m}(t-\tau)^{\frac{1-m}{2}}\int_\Gamma \exp\left(\frac{-|x-\xi|^2}{4(t-\tau)}\right)\hat{h}(\xi,\tau)dS_\xi.
\end{aligned}$$

We know $\Psi(\hat{h}) = \hat{h}(x_0,t) = \Psi(h(\Psi(h))^{-1}) = \Psi(h)(\Psi(h))^{-1} = 1$ for $t \in [0,T]$. Let $U(x_0,d) = \{x \in R^m : |x-x_0| < d\}$. Suppose, in addition to the assumptions of lemma, that $\hat{h}(x,t) = \hat{h}(x_0,t)$ is valid for all $t \in [0,T]$ and $x \in U(x_0,d)\cap\Gamma := \Gamma_d$

with some $d > 0$. We first represent the kernel $k_1(t,\tau)$ in the form

$$\begin{aligned} k_1(t,\tau) &= 2(2\sqrt{\pi})^{-m}(t-\tau)^{\frac{1-m}{2}}\{\int_{\Gamma_d} \exp\left(\frac{-|x-\xi|^2}{4(t-\tau)}\right)\hat{h}(\xi,\tau)dS_\xi + \\ &+ \int_{\Gamma/\Gamma_d} \exp\left(\frac{-|x-\xi|^2}{4(t-\tau)}\right)\hat{h}(\xi,\tau)dS_\xi\} \\ &:= J_1 + J_2, \end{aligned}$$

where

$$J_2 \leq 2(2\sqrt{\pi})^{-m}(t-\tau)^{\frac{1-m}{2}} \exp\left(\frac{-d^2}{4(t-\tau)}\right) \int_{\Gamma/\Gamma_d} |\hat{h}(\xi,\tau)|dS_\xi.$$

Hence, $J_2(t,\tau)$ is a continuous function differentiable with respect to $t$ on $0 \leq \tau \leq t \leq T$ if we continuously define it as $\tau \to (t-0)$. We assume that $d > 0$ is small enough to ensure that $\Gamma_d$ is uniquely projected on the tangent plane $\pi(x_0)$. Let us translate the origin to $x_0$ and let $z_m = \psi(z_1, z_2, ..., z_{m-1}) = \psi(z')$, $z' \in \nu_d$ be the equation of $\Gamma_d$ in the local coordinate system where $\nu_d$ is the orthogonal projection of $\Gamma_d$ onto $\pi(x_0) : \{z \in R^m : z_m = 0\}$.($z_m = \psi(z'), z' \in \nu_d$ is the equation of $\Gamma_d$ in local coordinate system, $\nu_d$ is orthogonal projection of $\Gamma_d$ on the tangent plane $\pi(x_0)$, $z_m = 0$ is the equation of the tangent plane $\pi(x_0)$.) Remembering that $\Psi(\hat{h}) = \hat{h}(x_0,t) = 1$ on $\Gamma_d$, consider

$$\begin{aligned} J_1(t,\tau) &= 2(2\sqrt{\pi})^{-m}(t-\tau)^{\frac{1-m}{2}} \int_{\Gamma_d} \exp\left(\frac{-|x-\xi|^2}{4(t-\tau)}\right)dS_\xi \\ &= 2(2\sqrt{\pi})^{-m}(t-\tau)^{\frac{1-m}{2}} \int_{\nu_d} \exp\left(-\frac{z_1^2+z_2^2+\ldots+z_{m-1}^2+\psi^2}{4(t-\tau)}\right)dS_z. \end{aligned}$$

We can find $dS_z$ as

$$\begin{aligned} dS_z &= \sqrt{(\frac{\partial z_m}{\partial z_1})^2 + (\frac{\partial z_m}{\partial z_2})^2 + \ldots + (\frac{\partial z_m}{\partial z_m})^2} dz' \\ &= \sqrt{(\psi_{z_1})^2 + (\psi_{z_2})^2 + \ldots + (\psi_{z_{m-1}})^2 + 1} dz' \\ &= \sqrt{1 + \sum_{i=1}^{m-1} (\psi_i)^2} dz' \text{ , where } \psi_i := \frac{\partial \psi}{\partial z_i} \\ &:= J(z')dz'. \end{aligned}$$

There exists numbers $a, b$ such that

$$K_a = \{z \in \pi(x_0) : |z_k| \leq a, k = 1, 2, \ldots, m-1\}$$

and $K_a \subset \nu_d \subset K_b$ for $0 < a < d < b$. We know

$$0 \leq J_1 \leq c(t-\tau)^{\frac{1-m}{2}} \int_{K_b} \exp\left(-\frac{z_1^2 + z_2^2 + \ldots + z_{m-1}^2 + \psi^2}{4(t-\tau)}\right) J(z')dz'.$$

The first part of inequality is because all terms of $J_1$ are positive, and the second inequality is because of definition of $K_b$. Also we know

$$J(z') = \sqrt{1 + \sum_{i=1}^{m-1} (\psi_i)^2}.$$

Since $d$ is small enough such that $\nu_d$ has the properties of $\Gamma_d$, so since $\Gamma \in C^{1+\alpha}(\bar{S})$ we have $\psi \in C^{1+\alpha}(\bar{S})$. Then $\psi_i \in C^{\alpha}(\bar{S})$. That means $\psi_i$ are continuous so bounded on $S$. Then the jacobian $J(z')$ is also bounded, that is

$$J(z') \leq c.$$

We have $J_1 \leq c$, for $0 \leq \tau < t \leq T$. Indeed we have

$$J_1(t,\tau) \leq$$
$$2(2\sqrt{\pi})^{-m} \int_{-b}^{b} \cdots \int_{-b}^{b} (t-\tau)^{\frac{1-m}{2}} \exp\left(-\frac{z_1^2 + z_2^2 + \ldots + z_{m-1}^2 + \psi^2}{4(t-\tau)}\right) J(z')dz'.$$

Let $z_k = 2\sqrt{t-\tau}y_k$ for $k = 1, 2, \ldots, m-1$ and let $h = \frac{b}{2\sqrt{t-\tau}}$. Then,

$$dz_k = 2\sqrt{t-\tau}dy_k$$

$$dz' = (2\sqrt{t-\tau})^{m-1}dy' = 2^{m-1}(t-\tau)^{\frac{m-1}{2}}dy'.$$

So we have

$$\begin{aligned}
& J_1(t,\tau) \\
\leq\ & \pi^{-m/2} \int_{-h}^{h} \cdots \int_{-h}^{h} (t-\tau)^{\frac{1-m}{2}} \\
& \exp\left(-|y'|^2 - \frac{\psi^2(2\sqrt{t-\tau}y_1, \ldots, 2\sqrt{t-\tau}y_{m-1})}{4(t-\tau)}\right) 2^{m-1}(t-\tau)^{\frac{m-1}{2}}dy' \\
=\ & \pi^{-m/2} \int_{-h}^{h} \cdots \int_{-h}^{h} \exp\left(-|y'|^2 - \frac{\psi^2(2\sqrt{t-\tau}y_1, \ldots, 2\sqrt{t-\tau}y_{m-1})}{4(t-\tau)}\right) dy' \\
\leq\ & \text{const},
\end{aligned}$$

since $\exp\left(-|y'|^2 - \frac{\psi^2}{4(t-\tau)}\right)$ is bounded on $[-h, h]$. Let us calculate $\lim_{\tau\to t} J_1(t,\tau)$, $t > 0$. Since $\psi \in C^{1+\alpha}(\bar{S})$, and $\psi(0) = 0$ (which holds because $z' = 0$ on $\pi(x_0)$, $z_m = 0$ is the equation of $\pi(x_0)$, so $z_m = \psi(z') = 0$ for $z' = 0$). Then we have

$|\psi(z')| \leq c|z'|^{1+\alpha}$. In addition $J(z') \to J(0) = 1$ as $z' \to 0$.Therefore

$$\begin{aligned}
\lim_{\tau \to t} J_1(t,\tau) &= \pi^{-m/2} \int_{-h}^{h} \dots \int_{-h}^{h} \exp\left(-|y'|^2 - \frac{\psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)}\right) dy' \\
&= \pi^{-m/2} \left(\int_{-\infty}^{\infty} \exp(-y^2) dy\right)^{m-1} \\
&= \pi^{-m/2} \left(\sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-x^2 - y^2) dx dy}\right)^{m-1} \\
&= \pi^{-m/2} \left(\int_{0}^{2\pi} \int_{0}^{\infty} \exp(-r^2) r dr d\theta\right)^{\frac{m-1}{2}} \\
&= \pi^{-m/2} \pi^{\frac{m-1}{2}} \\
&= \frac{1}{\sqrt{\pi}}.
\end{aligned}$$

The second equation above is because $\lim_{\tau \to t} \frac{\psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)} = 0$ by L'Hospital Rule. Hence

$$J_1(t,\tau) = \frac{1}{\sqrt{\pi}} + \bar{J}_1(t,\tau),$$

where

$$\begin{aligned}
\bar{J}_1(t,\tau) &\leq \pi^{-m/2} \{ \int_{-h}^{h} \dots \int_{-h}^{h} \exp(-|y'|^2 - \frac{\psi^2}{4(t-\tau)}) J dy' \\
&\quad - \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp(-|y|^2) dy' \} \\
&= \pi^{-m/2} \{ \int_{-h}^{h} \dots \int_{-h}^{h} \exp(-|y|^2)[\exp(-\frac{\psi^2}{4(t-\tau)}) J - 1] dy' + \\
&\quad +2 \int_{h}^{\infty} \dots \int_{h}^{\infty} \exp(-|y|^2) dy' \} \\
&:= \pi^{-m/2} (I_1 + I_2^{m-1}).
\end{aligned}$$

Here

$$I_1 = \int_{-h}^{h} \dots \int_{-h}^{h} \exp(-|y|^2)[\exp(-\frac{\psi^2}{4(t-\tau)}) J - 1] dy'$$

and

$$I_2^{m-1} = \int_{h}^{\infty} \dots \int_{h}^{\infty} \exp(-|y|^2) dy'.$$

We have

$$
\begin{aligned}
I_2^2 &= \int_h^\infty \int_h^\infty \exp(-y_1^2 - y_2^2) dy_1 dy_2 \\
&= \int_0^{\pi/2} \int_h^\infty \exp(-r^2) r dr d\theta \\
&\leq \pi(\frac{-1}{2} \exp(-r^2)|_h^\infty) \\
&= \pi \exp(-h^2) \\
&= \pi \exp(\frac{-b^2}{4(t-\tau)}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2^{m-1} &= \int_h^\infty \ldots \int_h^\infty \exp(-y_1^2 - y_2^2 - \ldots - y_{m-1}^2) dy_1 dy_2 \ldots dy_{m-1} \\
&= \left( \int_h^\infty \int_h^\infty \exp(-y_1^2 - y_2^2) dy_1 dy_2 \right)^{\frac{m-1}{2}} \\
&\leq \left( 2\pi \int_h^\infty \int_h^\infty \exp(-r^2 r dr) \right)^{\frac{m-1}{2}} \\
&= 2\pi(-\frac{1}{2} \exp(-r^2)|_h^\infty)^{\frac{m-1}{2}} \\
&= (\pi \exp(-h^2))^{\frac{m-1}{2}} \\
&= (\pi)^{\frac{m-1}{2}} \exp^{\frac{-(m-1)b^2}{8(t-\tau)}}.
\end{aligned}
$$

Let $\frac{(m-1)b^2}{8} = c$. We know $c > 0$. Let $(t-\tau) = x$. As $x \to 0^+$ we have $\exp(cx) > x^\lambda$ for any $\lambda > 0$. Then $\exp(\frac{c}{x}) > (\frac{1}{x})^\lambda \Rightarrow \exp(-\frac{c}{x}) < (\frac{1}{x})^{-\lambda} \Rightarrow \exp(-\frac{c}{x}) < x^\lambda$. That means $\exp(\frac{-c}{(t-\tau)}) < (t-\tau)^\lambda$. So

$$
I_2^{m-1} \leq (\pi)^{\frac{m-1}{2}} \exp^{\frac{-(m-1)b^2}{8(t-\tau)}} = o((t-\tau)^\lambda),
$$

as $\tau \to t - 0$, for any $\lambda > 0$. A similar estimate is also valid for $\partial_t I_2^{m-1}$. Let us now estimate $I_1(t,\tau)$ as

$$
I_1(t,\tau) \leq \pi^{\frac{-m}{2}} \int_{-h}^h \ldots \int_{-h}^h \exp(-|y'|^2) \left| \exp(\frac{-\psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)}) J - 1 \right| dy'.
$$

We have $|\exp(\frac{-\Psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)})| \leq 1$, $J(0)=1$ so we have

$$\begin{aligned} & \left|\exp(\frac{-\Psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)})J(2\sqrt{t-\tau}y')-1\right| \\ \leq\ & c|J(2\sqrt{t-\tau}y')-J(0)|+\frac{\Psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)} \\ :=\ & R. \end{aligned}$$

Since $J=1+\sum\Psi_i^2$ and $\Psi_i \in C^\alpha$, we have $|J(z')-J(0)| = |\sum\Psi_i^2(2\sqrt{t-\tau}y')| \leq c(|2\sqrt{t-\tau}y'|^\alpha)^2$, which was proved before. That is equal to $c(t-\tau)^\alpha|y'|^{2\alpha}$. So,

$$\begin{aligned} \frac{-\Psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)} &= |\frac{-\Psi^2(2\sqrt{t-\tau}y')}{4(t-\tau)}| \\ &\leq \frac{c((2\sqrt{t-\tau}y')^{1+\alpha})^2}{4(t-\tau)} \\ &= c(t-\tau)^{\alpha+1-1}|y'|^{2\alpha+2}. \end{aligned}$$

Then,

$$R \leq c[(t-\tau)^\alpha|y'|^{2\alpha}+(t-\tau)^\alpha|y'|^{2\alpha}+2].$$

So as $\tau \to t$ for $y \in \nu_d$, we have

$$|I_1| \leq c(t-\tau)^\alpha,$$

i.e.

$$J_1(t,\tau)=\frac{1}{\sqrt{\pi}}+\bar{J}_1(t,\tau),$$

where $|\bar{J}_1| \leq \pi^{-m/2}(I_1+2I_2) \leq c(t-\tau)^\alpha$. A similar estimate is valid for $\partial_t\bar{J}_1$. Moreover the singularity of $\partial_t\bar{J}_1$ is integrable for $\alpha > 0$. Therefore we have obtained the desired representation in the case under consideration.

Let us consider the general case of point overdetermination, that is without the condition $\hat{h}=1$ in $U(x_0,d)$. By assumption of lemma, the function $\hat{h}$ satisfies the Hölder condition with respect to $x$ at $x_0$ uniformly with respect to $t$, i.e. there exists $c > 0$ and $d > 0$ such that for all $x \in U(x_0, d)$, for all $t \in [0, T]$, we have

$$|\hat{h}(x,t) - \hat{h}(x_0,t)| = |\hat{h}(x,t) - 1| \leq c|x - x_0|^\alpha.$$

Let us estimate the kernel $K_1$ in this case by representing it in the form $K_1 = J_1 + J_2$. As before, $J_2(t,\tau)$ is continuous and differentiable with respect to $t$. Let us estimate the new $J_1$ as

$$\begin{aligned} J_1(t,\tau) &= 2(2\sqrt{\pi})^{-m}(t-\tau)^{(1-m)/2}[\int_{\nu_d} \exp(\frac{-|z'|^2 - \Psi^2}{4(t-\tau)})\hat{h}(0,\tau)J(0)dz' + \\ &+ \int_{\nu_d} \exp(\frac{-|z'|^2 - \Psi^2}{4(t-\tau)})[\hat{h}(z',z)J(z') - \hat{h}(0,z)J(0)]dz'], \end{aligned}$$

we have

$$|\hat{h}(z',z)J(z') - \hat{h}(0,z)J(0)| \leq c|z'|^\alpha,$$

for $z' \in \nu_d$, $\tau \in [0,T]$, since $\hat{h} \in C^{\alpha,0}(\bar{S})$ and $J \leq c$. Using similar conditions we can find

$$\lim_{\tau \to t} J_1(t,\tau) = \frac{1}{\pi},$$

and

$$J_1(t,\tau) = \frac{1}{\pi} + O((t-\tau)^{\alpha/2}).$$

Case 2: The case $\Psi(u) = \int_\Gamma u(x,t)\nu(x)dS_x$. □

## 5.2 An Operator Equation of the Second Kind

Under certain conditions Problem III is equivalent to an integral equation of the Volterra type of the second kind with an integrable kernel. (Volterra integral equation of second type is of the form $\phi(x) = \lambda \int K(x,\tau)\phi(\tau)d\tau + f(x)$)

LEMMA 5.2.1. *Suppose the condition (2.55) is satisfied, $h \in C^{\alpha,0}(\bar{S})$ and $|\Psi(h)| > 0$ on $[0,T]$. Then Problem $III_0$ is equivalent to an operator equation of the second kind.*

*Proof.* It follows from Lemma 5.1.1 and Lemma 5.1.2 that Problem $\text{III}_0$ is equivalent to integral equation

$$\chi(z) = \frac{1}{\sqrt{\pi}}\int_0^z \frac{\varphi(\tau)}{\sqrt{z-\tau}}d\tau + \int_0^z \frac{\hat{k}(z,\tau)}{(z-\tau)^{\mu}}\varphi(\tau)d\tau$$

with $\mu < \frac{1}{2}$. Multiplying both sides with $\frac{1}{\sqrt{t-\tau}}$, then integrating with respect to $z$ from 0 to $t$ and then taking derivative with respect to $t$ we have

$$\frac{d}{dt}\int_0^t \frac{\chi(z)}{\sqrt{t-z}}dz = \frac{d}{dt}\int_0^t \frac{1}{\sqrt{t-z}}\left[\frac{1}{\sqrt{\pi}}\int_0^z \frac{\varphi(\tau)}{\sqrt{z-\tau}}d\tau + \int_0^z \frac{\hat{k}(z,\tau)}{(z-\tau)^{\mu}}\varphi(\tau)d\tau\right]dz.$$

The left hand side is continuous on $[0,T]$, since $\frac{d}{dt}\int_0^t \frac{\chi(z)}{\sqrt{t-z}}dz = \int_0^t \frac{\chi'(z)}{\sqrt{t-z}}dz \in C([0,T])$,by assumption of lemma. It is not difficult to see that

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_0^t \frac{dz}{\sqrt{t-z}}\int_0^z \frac{\varphi(\tau)}{\sqrt{z-\tau}}d\tau &= \frac{1}{\sqrt{\pi}}\int_0^t \varphi(\tau)\int_\tau^t \frac{dz}{\sqrt{t-z}\sqrt{z-\tau}} \\ &= \frac{1}{\sqrt{\pi}}\int_0^t \varphi(\tau)d\tau\pi \\ &= \sqrt{\pi}\int_0^t \varphi(\tau)d\tau\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \frac{dz}{\sqrt{t-z}}\int_0^z \frac{\hat{k}(z,\tau)}{(z-\tau)^{(1/2-\alpha/2)}}\varphi(\tau)d\tau \\ = &\int_0^t \varphi(\tau)\left[\int_\tau^t \frac{\hat{k}(z,\tau)}{\sqrt{t-z}(z-\tau)^{(1/2-\alpha/2)}}\right]d\tau \\ = &\int_0^t \varphi(\tau)(t-\tau)^{\alpha/2}\left[\int_0^1 \frac{\hat{k}((t-\tau)\rho+\tau,\tau)}{(1-\rho)^{1/2}\rho^{(1/2-\alpha/2)}}d\rho\right]d\tau \\ := &\int_0^t K(t,\tau)\varphi(\tau)d\tau,\end{aligned}$$

where $K(t,\tau) = (t-\tau)^{\alpha/2}\int_0^1 \frac{\hat{k}((t-\tau)\rho+\tau,\tau)}{(1-\rho)^{1/2}\rho^{(1/2-\alpha/2)}}d\rho$. Clearly $K(t,t) = 0$. Also we have $0 \le (t-\tau)^{\alpha/2} \le c$, for some $c$ for $0 \le \tau \le t \le T$, $\alpha \in (0,1)$. Furthermore

$\hat{k} \in C_t^1$, so $\hat{k}$ is bounded on $[0,T]$, which implies $K \leq c$ on $[0,T]$. Then we have

$$\begin{aligned} |K(t,\tau)| &\leq (t-\tau)^{\alpha/2} c \int_0^1 (1-\rho)^{1/2-1} \rho^{(1/2-\alpha/2)-1} \\ &\leq c \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} \\ &\leq c. \end{aligned}$$

□

So $K(t,\tau) \in L_1(G)$, where $G = \{0 \leq \tau \leq t \leq T\}$. Similarly $K_t(t,\tau) \in L_1(G)$. Then, by Leibnitz's Rule we have

$$\begin{aligned} F(t) &= \sqrt{\pi}\frac{d}{dt}\int_0^t \varphi(\tau)d\tau + \frac{d}{dt}\int_0^t K(t,\tau)\varphi(\tau)d\tau \\ &= \sqrt{\pi}\varphi(t) + K(t,t)\varphi(t)\frac{d}{dt}t - K(t,0)\varphi(0)\frac{d}{dt}0 + \int_0^t K_t(t,\tau)\varphi(\tau)d\tau \\ &= \sqrt{\pi}\varphi(t) + \int_0^t K_t(t,\tau)\varphi(\tau)d\tau. \end{aligned}$$

That is we obtain the integral equation

$$F(t) = \sqrt{\pi}\varphi(t) + \int_0^t \partial_t K(t,\tau)\varphi(\tau)d\tau, \tag{5.10}$$

and this is an equation of Volterra type whose Kernel has a weak singularity.

## 5.3 Existence and Uniqueness Theorem for Problem III

*Proof.* It was shown that Problem III is equivalent to Problem $\text{III}_0$. In Lemmas 5.1.1, 5.1.2 and 5.2.1 we have shown that Problem $\text{III}_0$ is equivalent to integral equation (5.10), and this integral equation is solvable [4] (p. 93). □

# CHAPTER 6

# ANALYSIS OF PROBLEM IV

**Problem IV:** Find a pair of functions $\{u(x,t), \sigma(t)\}$ satisfying (5.1), (5.2), (5.4) and

$$\frac{\partial u}{\partial n} + \sigma u = b \text{ on } S, \tag{6.1}$$

where $g(x,t)$, $a(x)$, $b(x,t)$, $\bar{\chi}(t)$ are given functions.

DEFINITION 6.0.1. *By a solution to Problem IV, we understand a classical solution $u \in C^{2,1}(Q)$, $\sigma \in C([0,T])$.*

## 6.1 Uniqueness Theorem for Problem IV

THEOREM 6.1.1. *Suppose (3.5) is satisfied, $|\chi(x)| > 0$ on $[0,T]$. Then the solution of Problem IV is unique.*

Assume there exists two pair of functions $\{u_1(x,t), \sigma_1(t)\}$, $\{u_2(x,t), \sigma_2(t)\}$ such that $\{u_i, \sigma_i\}$ satisfy Problem IV for $i = 1, 2$. Then $\bar{u} = u_2 - u_1$, $\bar{\sigma} = \sigma_2 - \sigma_1$ satisfies the equation

$$\begin{aligned} \bar{u}_t - \Delta \bar{u} &= 0 \text{ on } Q \\ \bar{u}(x,0) &= 0 \text{ on } \bar{\Omega} \\ \partial_n \bar{u} + \sigma_2 \bar{u} &= \bar{\sigma} u_1 \text{ on } S \\ \Psi(\bar{u}) &= 0 \text{ on } [0,T]. \end{aligned} \tag{6.2}$$

Consider this relation as Problem III for $u = \bar{u}$, $f = \bar{\sigma}$. We have all conditions of Theorem 5.0.2 are satisfied, that are, $g = 0 \in C^{\alpha,0}(\bar{Q})$, $a = 0 \in C^1(\bar{\Omega})$, $f = 0 \in C(\bar{S})$, $h = u_1 \in C^{2,1}(Q) \cap C^{\alpha,\alpha/2}(\bar{Q}) \in C^{\alpha,0}(\bar{S})$, $\chi(t) = 0$ satisfies conditions (5.5), $\sigma = \sigma_2 \in C(\bar{S})$, $\Psi(u_0)(0) = \bar{\chi}(0)$, $|\Psi(h)| = |\Psi(u_1)| > 0$ on $[0, T]$ since $u_1 > 0$ by Lemma 2.3.2. Then by Theorem 5.0.2, the solution of the problem (6.2) is unique and that unique solution is $(\bar{u}, \bar{\sigma}) = 0$. So we have $u_2 = u_1$ and $\sigma_2 = \sigma_1$, which means the solution of Problem IV is unique.

## 6.2 Derivation of Operator Equation

Let us derive an operator equation for Problem IV, then the solvability of this equation will imply the solvability of the inverse problem IV.

Suppose $|\chi(t)| > 0$ on $[0, T]$, choose $\sigma \in C([0, T])$ and consider the solution $U(x, t; \sigma)$ of

$$U_t - \Delta U = g \text{ on } Q \tag{6.3}$$

$$U(x, 0) = a(x) \text{ on } \bar{\Omega} \tag{6.4}$$

$$\frac{\partial U}{\partial n} + \sigma U = b \text{ on } S. \tag{6.5}$$

For $g \in C^{\alpha,0}(\bar{Q})$, $a \in C^1(\bar{\Omega})$ and $b \in C(\bar{S})$, the solution $U$ exists and is unique and has the desired differential properties by Theorem 2.3.1. Let us introduce the non-linear operator $\mathcal{D}$ as

$$D : C([0, T]) \to C([0, T])$$

$$D\sigma = [\Psi(b) - \Psi(\partial_n U)]\chi^{-1}$$

with $Dom(D) = C([0, T])$. Consider the equation

$$D\sigma = \sigma. \tag{6.6}$$

If equation (6.6) has a solution $\sigma(t)$ with the property $|\sigma(t)| > 0$ almost everywhere on $[0, T]$, then the Inverse Problem IV is solvable. Let $\sigma \in C([0, T])$ be a solution of equation (6.6). Let us consider the solution $U(x, t; \sigma)$ of the direct problem (6.3)-(6.4). By applying the overdetermination operator $\Psi$ to equation

(6.5) we obtain

$$\Psi(\partial_n U) + \sigma \Psi(U) = \Psi(b),$$

in $C([0,T])$. Now multiplying equation (6.6) by $\chi$ and adding to the equation above, we obtain

$$D\sigma\chi + \Psi(\partial_n U) + \sigma \Psi(U) = \chi\sigma + \Psi(b).$$

Since $D\sigma = [\Psi(b) - \Psi(\partial_n U)]\chi^{-1}$, we have

$$\sigma(t)[\Psi(U) - \chi](t) = 0,$$

on $[0,T]$. By the assumption that $\sigma(t) > 0$ a.e on $[0,T]$, we have $\Psi(U)(t) = \chi(t)$ on $[0,T]$, and this means the pair $\{U, \sigma\}$ satisfies Problem IV. We have proved the following proposition

**Proposition 6.2.1.** *Suppose that $g \in C^{\alpha,0}(\bar{Q})$, $a \in C^1(\bar{\Omega})$, $b \in C(\bar{S})$, $\chi \in C([0,T])$ and $|\chi(t)| > 0$ on $[0,T]$. If the operator equation (6.6) has a solution $\sigma \in C([0,T])$ satisfying $|\sigma(t)| > 0$ almost everywhere on $[0,T]$, then the Inverse Problem IV is solvable.*

# CHAPTER 7

# CONCLUSION

In this thesis, we studied four inverse boundary problems, Problems I, II, III and IV, with overdetermination condition on the boundary of the domain or on the time interval. We showed that, under certain conditions, solution of each problem is unique, if it exists. After showing uniqueness of the solution, we showed that each problem is equivalent to an integral equation of first or second kind or both. That means if the operator equation is solvable, then the inverse problem is also solvable. Showing equivalence of the inverse problem to an integral equation simplifies our study to show that an inverse problem is solvable.

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