CONSERVED CHARGES IN ASYMPTOTICALLY (ANTI)-DE SITTER SPACETIME

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ABSTRACT

CONSERVED CHARGES IN ASYMPTOTICALLY (ANTI)-DE SITTER SPACETIME

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In this master's thesis, the Killing vectors are introduced and the Killing equation is derived. Also, some information is given about the cosmological constant. Then, the Abbott-Deser (AD) energy is reformulated by linearizing the Einstein equation with cosmological constant. From the linearized Einstein equation, Killing charges are derived by using the properties of Killing vectors. Using this formulation, energy is calculated for some specific cases by using the Schwarzschild-de Sitter metric. Last, the Einstein-Gauss-Bonnet model is studied. The equations of motion are calculated by solving the generic action at quadratic order. Following this, all energy calculations are renewed for this model.

Some useful relations and calculations are shown in Appendix (A-B) parts.

Keywords: de-Sitter spacetime, Killing vector, Conserved charges.

ÖZ

ASİMPTOTİK (ANTİ)-DE SITTER UZAYZAMANINDA KORUNAN YÜKLER

GÜLLÜ, İBRAHİM
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Bu master çalışmasında, Killing vektörler tanımlandı ve Killing denklemi çıkarıldı. Ayrıca evrenbilimsel sabit, de-Sitter ve Anti-de Sitter uzayları hakkında bilgi verildi. Sonra, Abbott-Deser (AD) enerjisi, evrenbilimsel sabitli Einstein denklemi doğrusallaştırılarak yeniden formüle edildi. Doğrusallaştırılmış Einstein denkleminden, Killing vektörlerin özellikleri kullanılarak Killing yükleri (Deser-Tekin denklemi) çıkarıldı. Schwarzschild-de Sitter metriği kullanılarak özel durumlar için enerji hesaplandı. Son olarak Einstein-Gauss-Bonnet (GB) modeli çalışıldı. İkinci dereceden genel eylem çözülerek hareket denklemleri hesaplandı. Bundan sonra, tüm enerji hesaplamaları bu model için tekrarlandı.

Bazı faydalı hesaplamalar ek (A-B) kısımlarında gösterilmiştir.

Anahtar Kelimeler: de-Sitter uzayzamanı, Killing vektör, Korunan yükler.

to my lovely sister, *Emel Egger*

and

admirable brother, *İsmail Güllü*

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CHAPTER 1

INTRODUCTION

Conserved quantities, such as energy-momentum, electric charge, angular momentum, baryon number etc., are important in the description of physical phenomena. In the presence of gravity, definition of certain conserved charges (especially the energy) become rather tricky. Our task in this thesis is to give a review of the techniques of defining conserved charges in asymptotically (Anti)de-Sitter spaces developed by Abbott-Deser (AD) [3] and Deser-Tekin (DT) [1]. We will carry out the computations in great detail. These methods are an extension of the Arnowitt-Deser-Misner (ADM) [4] methods which work for asymptotically flat geometries.

We define the global charges primarily in D dimensional quadratic theories. We first present a reformulation of the original definition of conserved charges in cosmological Einstein theory; then we derive the generic form of the energy for quadratic gravity theories in D dimensions and specifically study the ghost-free low energy string-inspired model: Gauss-Bonnet (GB) plus Einstein terms [1].

A definition of gauge invariant conserved (global) charges in a diffeomorphism-invariant theory rests on the "Gauss law" and the presence of asymptotic Killing symmetries. More explicitly, in any diffeomorphism-invariant gravity theory, a vacuum satisfying the classical equations of motion is chosen as the background relative to which excitations and any background gauge-invariant properties (such as energy) are defined. Two important model-independent features of these charges are: First, the vacuum itself has zero charge; secondly, they are expressible as surface integrals [1, 2, 3, 4].

In the first chapter, we will define the Killing vector field and give some useful properties of it. Then we will see how Einstein equations can be linearized to reformulate the AD energy. Using that reformulation, we will calculate the energy in (A)dS spaces. Following that we will look at the Einstein-GB model. We will then state our conclusion and describe some open questions. In the two Appendices, we will give some useful calculations that will help us in this journey.

CHAPTER 2

KILLING VECTORS

Tensor calculus is largely concerned with how quantities change under coordinate transformations. It is of particular interest when a quantity does not change, *i.e.* remains invariant, under coordinate transformations. For example, coordinate transformations which leave a metric invariant are of importance since they contain information about the symmetries of a Riemannian manifold. In an ordinary Euclidean space, there are two sorts of transformations: descrete ones, like reflections, and continuous ones, like translations and rotations. In most applications, these latter types are the more important ones and they can in principle be obtained systematically by obtaining the so-called Killing vectors of a metric, which we now discuss below.

A metric g_{ab} is invariant under the transformation $x^a \to x'^a$ if $g'_{ab}(y) = g_{ab}(y)$ for all coordinates y^c. An infinitesimal coordinate transformation is $x^a \to x'^a = x^a + \epsilon X^a(x)$ where X^a denotes the vector field and x^a denotes a vector in that field.

Differentiating x'^a gives

$$\frac{\partial x'^a}{\partial x^b} = \frac{\partial x^a}{\partial x^b} + \epsilon \frac{\partial X^a}{\partial x^b} = \delta^a_b + \epsilon \partial_b X^a,$$

and the metric transforms

$$g_{ab}(x) = \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g'_{cd}(x').$$

Then $x^a \to x'^a$ will be an isometry if

$$\begin{aligned} g_{ab}(x) &= \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g_{cd}(x') \\ &= \left(\frac{\partial}{\partial x^a}(x^c + \epsilon X^c) \right) \left(\frac{\partial}{\partial x^b}(x^d + \epsilon X^d) \right) g_{cd}(x^e + \epsilon X^e) \end{aligned}$$

We expand $g_{cd}(x^e + \epsilon X^e)$ using the Taylor's theorem,

$$g_{ab}(x) = (\delta^c_a + \epsilon \partial_a X^c)(\delta^d_b + \epsilon \partial_b X^d)\{g_{cd}(x) + \epsilon X^e \partial_e g_{cd}(x) + ...\}$$

$$\begin{aligned} g_{ab}(x) &= g_{cd}(\delta^c_a + \epsilon \partial_a X^c)(\delta^d_b + \epsilon \partial_b X^d) + \epsilon X^e \partial_e g_{cd}(x)(\delta^c_a + \epsilon \partial_a X^c)(\delta^d_b + \epsilon \partial_b X^d), \\ &= g_{ab}(x) + \epsilon\{g_{ad} \partial_b X^d + g_{bd} \partial_a X^d + X^e \partial_e g_{ab}\} + O(\epsilon^2), \end{aligned}$$

$$\Rightarrow g_{ad} \partial_b X^d + g_{bd} \partial_a X^d + X^e \partial_e g_{ab} = 0.$$

Where we use Einstein sum convention. Now define:

$$\mathcal{L}_x g_{ab} = X^c \partial_c g_{ab} + g_{ca} \partial_b X^c + g_{cb} \partial_a X^c.$$

In the first term we make $c \to e$ transformation, in the second and third term $c \to d$ that gives us

$$\mathcal{L}_x g_{ab} = X^e \partial_e g_{ab} + g_{ad} \partial_b X^d + g_{bd} \partial_a X^d = 0.$$

This is called *Lie derivative* and since in the expression for a Lie derivative of a tensor, all occurrences of the partial derivatives may be replaced by covariant derivatives. Therefore we can make the $\partial_a \rightarrow \nabla_a$ substitution, that is

$$X^e \nabla_e g_{ab} + g_{ad} \nabla_b X^d + g_{bd} \nabla_a X^d = 0.$$

Since we work in a metric compatible system $\nabla_b g_{ab} = 0$, we have

$$g_{ad} \nabla_b X^d + g_{bd} \nabla_a X^d = 0.$$

Finally we have

$$\nabla_b X_a + \nabla_a X_b = 0, \tag{2.1}$$

where X_a is a vector field that leaves the metric invariant and such a vector field is called a Killing vector [5, 6].

The importance of Killing vectors comes from the symmetry considerations. Translational or rotational symmetries will give us conserved quantities. With the first, we can get energy and momentum; and with the latter, angular momentum.

CHAPTER 3

MODELS WITH A COSMOLOGICAL CONSTANT

A characteristic feature of general relativity is that the source for the gravitational field is the entire energy-momentum tensor. In non-gravitational physics, only changes in energy from one state to another are measurable; the normalization of the energy is arbitrary. For example, the motion of a particle with potential energy $V(x)$ is precisely the same as that with a potential energy $V(x) + V_0$, for any constant V_0. In gravitation, however, the actual value of the energy matters, not just the differences between states.

This behavior opens up the possibility of vacuum energy: an energy density characteristic of empty space. One feature that we might want the vacuum to exhibit is that it must not pick out a preferred direction; it will still be possible to have a nonzero energy density if the associated energy-momentum tensor is Lorentz invariant in locally inertial coordinates. Lorentz invariance implies that the corresponding energy-momentum tensor should be proportional to the metric,

$$T^{(vac)}_{\hat{\mu}\hat{\nu}} = -\rho_{vac}\eta_{\hat{\mu}\hat{\nu}} \, , \tag{3.1}$$

(where "$\hat{\mu}\hat{\nu}$" denote locally inertial coordinates and ρ_{vac} is the vacuum energy density), since $\eta_{\hat{\mu}\hat{\nu}}$ is the only Lorentz invariant (0,2) tensor. This generalizes straightforwardly from inertial coordinates to arbitrary coordinates as

$$T_{\mu\nu}^{(vac)} = -\rho_{vac} g_{\mu\nu} \,. \tag{3.2}$$

Comparing with the perfect-fluid energy-momentum tensor $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$, we find that the vacuum looks like a perfect fluid with an isotropic pressure opposite in sign to the energy density,

$$p_{vac} = -\rho_{vac}. \tag{3.3}$$

The energy density should be constant throughout spacetime, since a gradient would not be Lorentz invariant.

If we decompose the energy-momentum tensor into a matter piece $T_{\mu\nu}^{(M)}$ and a vacuum piece $T_{\mu\nu}^{(vac)} = -\rho_{vac} g_{\mu\nu}$, the Einstein's equation is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G (T_{\mu\nu}^{(M)} - \rho_{vac} g_{\mu\nu}) \,, \tag{3.4}$$

where $R_{\mu\nu}$ and R are Ricci tensor and scalar, G is Newton's constant. Soon after inventing general relativity, Einstein tried to find a static cosmological model, since that was what astronomical observations of the time seemed to imply. The result was the Einstein static universe. In order for this static cosmology[1] to solve the field equation with an ordinary matter source, it was necessary to add a new term called the cosmological constant, Λ, which enters as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.5}$$

[1] We now know that even with a cosmological constant, static universe is not possible.

The cosmological constant is precisely equivalent to introducing a vacuum energy density

$$\rho_{vac} = \frac{\Lambda}{8\pi G}. \tag{3.6}$$

The terms “cosmological constant” and “vacuum energy” are essentially interchangeable [7].

Maximally symmetric solutions of the cosmological Einstein equation with $T_{\mu\nu}$, are de Sitter and Anti-de Sitter spaces which we now briefly describe.

3.1 DE-SITTER SPACE

De Sitter space corresponds to a four-dimensional surface in a flat five-dimensional space with metric $(-,+,+,+,+)$ described by

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = \frac{3}{\Lambda} \quad , \Lambda > 0\,. \tag{3.7}$$

The symmetries of this space are then the ten rotations and boosts of this five-dimensional embedding space. Rotations among the $z_1 - z_4$ clearly result in spacelike Killing vectors. However, boosts which mix $z_1 - z_4$ with z_0 can lead to Killing vectors which are timelike. The Killing vector which corresponds to a mixing of z_4 and z_0 is

$$\bar{\xi}_a = (-z_4, 0, 0, 0, z_0). \tag{3.8}$$

Now

$$\bar{\xi}^2 = -z_4^2 + z_0^2, \tag{3.9}$$

so

$$\bar{\xi}^2 < 0, \tag{3.10}$$

if and only if

$$|z_4| > |z_0|. \tag{3.11}$$

This is a distinctive feature of de Sitter space which implies the existence of a cosmological horizon.

To be more specific, consider the de Sitter space metric in the form

$$d\tau^2 = -dt^2 + f^2(t)[dx^2 + dy^2 + dz^2]\,, \tag{3.12}$$

where

$$f(t) = exp\sqrt{\frac{1}{3}\Lambda t}\,. \tag{3.13}$$

The Killing vector for this symmetry is

$$\bar{\xi}^\mu = (-1, \sqrt{\frac{1}{3}\Lambda x})\,. \tag{3.14}$$

Now

$$\bar{\xi}^2 = -1 + \frac{1}{3}\Lambda f^2 |x|^2 \quad , \tag{3.15}$$

which is timelike in the region

$$\frac{1}{3}\Lambda f^2 |x|^2 < 1\,. \tag{3.16}$$

This $\bar{\xi}_\mu$ generates a Killing energy. However the restriction (3.16) limits the region of applicability of this quantity. In order for $E(\bar{\xi})$ to act like an energy, the surface of integration must lie inside the event horizon defined by

$$\frac{1}{3}\Lambda f^2 |x|^2 = 1 \quad [3,\ 8]\,. \tag{3.17}$$

3.2 ANTI DE-SITTER SPACE

Anti-de Sitter space is the covering space for the four-dimensional surface, in a flat five-dimensional space with metric (-,+,+,+,-), described by

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 - z_4^2 = \frac{3}{\Lambda}, \Lambda < 0. \tag{3.18}$$

Once again the symmetries of this space are just the rotations and boosts in the five-dimensional embedding space. Here, however, there is a global timelike Killing vector corresponding to the rotation mixing z_0 and z_4,

$$\bar{\xi}_a = (-z_4, 0, 0, 0, z_0), \tag{3.19}$$

and

$$\bar{\xi}^2 = -z_4^2 - z_0^2 < 0. \tag{3.20}$$

Thus, $\bar{\xi}_a$ is timelike everywhere [note that the condition (3.17) excludes the point $z_4 = z_0 = 0$], and there is no cosmological horizon [3].

CHAPTER 4

REFORMULATION OF AD ENERGY

4.1 CONSERVED CHARGES

We first look at how conserved charges arise in a generic gravity theory coupled to a covariantly conserved bounded matter source $\tau_{\mu\nu}$. Consider the following equations of motion that come from a Lagrangian

$$\Phi_{\mu\nu}(g, R, \nabla R, R^2, ...) = \kappa\tau_{\mu\nu}, \tag{4.1}$$

where $\Phi_{\mu\nu}$ is the "Einstein tensor" of a local, invariant, but otherwise arbitrary, gravity action and κ is an effective coupling constant. We work in generic D dimensions.

Now we will decompose our metric into the sum of two parts:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{4.2}$$

where $\bar{g}_{\mu\nu}$ solves (4.1) for $\tau_{\mu\nu} = 0$ and a deviation part $h_{\mu\nu}$ that vanishes sufficiently rapidly at infinity and it is not necessarily small everywhere.

Separating the field equations (4.1) into a part linear in $h_{\mu\nu}$ and collecting all other non-linear terms and the matter source $\tau_{\mu\nu}$ in $T_{\mu\nu}$ that constitute the total source, one obtains

$$\mathcal{O}(\bar{g})_{\mu\nu\alpha\beta}h^{\alpha\beta} = \kappa T_{\mu\nu}, \tag{4.3}$$

$\Phi_{\mu\nu}(\bar{g}, \bar{R}, \bar{\nabla}\bar{R}, \bar{R}^2, ...) = 0$, by assumption; the operator $\mathcal{O}(\bar{g})$ depends only on the background metric $\bar{g}_{\mu\nu}$.

The linearization of the (source free) Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \tag{4.4}$$

follows as

$$R^L_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R} - \frac{1}{2}\bar{g}_{\mu\nu}R^L + \Lambda h_{\mu\nu} + O(h^2) + ... = 0\,,$$

the background terms will be zero because $\bar{g}_{\mu\nu}$ itself satisfies the field equation (4.4). Here, $[\nabla_\mu, \nabla_\nu]V_\lambda = R_{\mu\nu\lambda}{}^\sigma V_\sigma$ and $R_{\mu\nu} \equiv R_{\mu\lambda\nu}{}^\lambda$ and the constant curvature vacuum $\bar{g}_{\mu\nu}$ has Riemann, Ricci and scalar curvatures that are

$$\bar{R}_{\mu\lambda\nu\beta} = \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\nu}\bar{g}_{\lambda\beta} - \bar{g}_{\mu\beta}\bar{g}_{\lambda\nu})\,; \tag{4.5}$$

$$\begin{aligned}
\bar{R}_{\mu\nu} &= \bar{R}_{\mu\lambda\nu\beta}\bar{g}^{\beta\lambda}\,,\\
&= \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\nu}\bar{g}_{\lambda\beta} - \bar{g}_{\mu\beta}\bar{g}_{\lambda\nu})\bar{g}^{\beta\lambda}\,,\\
&= \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\nu}D - \delta^\lambda_\mu\bar{g}_{\lambda\nu})\,,\\
&= \frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}(D-1) = \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\,;
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\bar{R} &= \bar{R}_{\mu\nu}\bar{g}^{\mu\nu}\,,\\
&= \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\bar{g}^{\mu\nu} = \frac{2\Lambda}{(D-2)}D\,.
\end{aligned} \tag{4.7}$$

Using (4.7), we get:

$$R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{1}{2}h_{\mu\nu}\frac{2D\Lambda}{(D-2)} + \Lambda h_{\mu\nu} + O(h^2) + ... = 0\,.$$

Collecting the terms we get

$$R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{1}{2}h_{\mu\nu}R^L - \frac{2\Lambda h_{\mu\nu}}{(D-2)} + O(h^2) + ... = 0\,.$$

We define all terms of second and higher order in $h_{\mu\nu}$ and the matter source $\tau_{\mu\nu}$ to be the gravitational energy-momentum tensor and write $\mathcal{G}^L_{\mu\nu}$ as

$$\mathcal{G}^L_{\mu\nu} \equiv R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{(D-2)}h_{\mu\nu} \equiv \kappa T_{\mu\nu}\,. \tag{4.8}$$

The left hand side of (4.8) obeys the background Bianchi identity

$$\bar{\nabla}_\mu(R^{\mu\nu}_L - \frac{1}{2}\bar{g}^{\mu\nu}R_L - \frac{2\Lambda}{(D-2)}h^{\mu\nu}) = 0 \tag{4.9}$$

or

$$\bar{\nabla}_\mu T^{\mu\nu} = 0\,. \tag{4.10}$$

Therefore, we have a background conserved energy-momentum tensor. However, the derivative in (4.9) is a background covariant, not an ordinary derivative; furthermore, only integrals over divergences of contravariant vector densities have invariant content, so (4.8) and (4.10) cannot be used directly to construct conserved quantities. To overcome this problem we contract $T^{\mu\nu}$ with a Killing vector $\bar{\xi}_\mu$ that is:

$$\bar{\nabla}_\mu(T^{\mu\nu}\bar{\xi}_\nu) = (\bar{\nabla}_\mu T^{\mu\nu})\bar{\xi}_\nu + \frac{1}{2}T^{\mu\nu}(\bar{\nabla}_\mu\bar{\xi}_\nu + \bar{\nabla}_\nu\bar{\xi}_\mu) = 0\,,$$

since $T^{\mu\nu} = T^{\nu\mu}$ and with the help of the Killing equation, we have the above equation. Now the quantity $(\bar{\xi}_\nu T^{\mu\nu})$ is a vector density whose covariant divergence becomes an ordinary one, and gives the desired conservation law,

$$\bar{\nabla}_\mu(\bar{\xi}_\nu T^{\mu\nu}) = \partial_\mu(\bar{\xi}_\nu T^{\mu\nu}) = 0\,,$$

or

$$\bar{\nabla}_\mu(\sqrt{-\bar{g}}\; T^{\mu\nu}\bar{\xi}_\nu) \equiv \partial_\mu(\sqrt{-\bar{g}}\; T^{\mu\nu}\bar{\xi}_\nu) = 0\,. \tag{4.11}$$

Therefore the conserved Killing charges are expressed as

$$Q^\mu(\bar{\xi}) = \int_M d^{D-1}x\, \sqrt{-\bar{g}}\; T^{\mu\nu}\bar{\xi}_\nu = \oint_\Sigma dS_i F^{\mu i}\,. \tag{4.12}$$

Here M is a spatial $(D-1)$ hypersurface and Σ is its $(D-2)$ dimensional boundary and i ranges over $(1, 2, ..., d-2)$ and $F^{\mu i}$ is an antisymmetric tensor whose explicit form will be written below [1, 4].

4.2 LINEAR FORM OF THE EINSTEIN EQUATION

In order to write the spatial volume integrals as surface integrals, we need to carry out the linearization of the relevant tensors. In this part that is what we shall do.

4.2.1 THE METRIC $g_{\mu\nu}$

We will take the signature to be $(-,+,+,+,...)$. We know that any metric must satisfy $g_{\mu\nu}g^{\nu\alpha}=\delta_\mu{}^\alpha$ where $\delta g_{\mu\nu}=h_{\mu\nu}$,

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+\delta g_{\mu\nu}\,,$$

$$(\bar{g}_{\mu\nu}+\delta g_{\mu\nu})g^{\nu\alpha}=\delta_\mu{}^\alpha \Rightarrow g^{\mu\nu}=\bar{g}^{\mu\nu}-\delta g^{\mu\nu}\,.$$

4.2.2 LINEARIZATION OF THE CHRISTOFFEL SYMBOL

We linearize $\Gamma^\mu_{\alpha\beta}=\frac{1}{2}g^{\mu\nu}(\partial_\alpha g_{\beta\nu}+\partial_\beta g_{\alpha\nu}-\partial_\nu g_{\alpha\beta})$ with the use of $g^{\mu\nu}\longrightarrow\bar{g}^{\mu\nu}-\delta g^{\mu\nu}$ and $g_{\beta\nu}\longrightarrow\bar{g}_{\beta\nu}+\delta g_{\beta\nu}$. That is ,

$$\delta\Gamma^\mu_{\alpha\beta}=\frac{1}{2}\delta g^{\mu\nu}(\partial_\alpha\bar{g}_{\beta\nu}+\partial_\beta\bar{g}_{\alpha\nu}-\partial_\nu\bar{g}_{\alpha\beta})+\frac{1}{2}\bar{g}^{\mu\nu}(\delta\partial_\alpha g_{\beta\nu}+\delta\partial_\beta g_{\alpha\nu}-\delta\partial_\nu g_{\alpha\beta})\,,$$

$$\delta\Gamma^\mu_{\alpha\beta}=\frac{1}{2}\bar{g}^{\mu\nu}(\bar{\nabla}_\alpha\delta g_{\beta\nu}+\bar{\nabla}_\beta\delta g_{\alpha\nu}-\bar{\nabla}_\nu\delta g_{\alpha\beta})\,. \tag{4.13}$$

4.2.3 LINEAR FORMS OF RIEMANN, RICCI TENSORS AND RICCI SCALAR

The Riemann tensor is

$$R^\mu{}_{\alpha\beta\nu}=\partial_\beta\Gamma^\mu_{\alpha\nu}-\partial_\nu\Gamma^\mu_{\alpha\beta}+\Gamma^\sigma_{\alpha\nu}\Gamma^\mu_{\sigma\beta}-\Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\nu}\,, \tag{4.14}$$

which can be linearized as

$$\delta R^\mu{}_{\alpha\beta\nu}=\partial_\beta\delta\Gamma^\mu_{\alpha\nu}-\partial_\nu\delta\Gamma^\mu_{\alpha\beta}+(\delta\Gamma^\sigma_{\alpha\nu})\Gamma^\mu_{\sigma\beta}+\Gamma^\sigma_{\alpha\nu}\delta\Gamma^\mu_{\sigma\beta}-(\delta\Gamma^\sigma_{\alpha\beta})\Gamma^\mu_{\sigma\nu}-\Gamma^\sigma_{\alpha\beta}\delta\Gamma^\mu_{\sigma\nu}\,.$$

To find the partial derivatives of the Christoffel symbol we can use covariant derivative,

$$\bar{\nabla}_\beta(\delta\Gamma^\mu_{\alpha\nu}) = \partial_\beta\delta\Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\beta\sigma}\delta\Gamma^\sigma_{\alpha\nu} - \Gamma^\sigma_{\beta\alpha}\delta\Gamma^\mu_{\sigma\nu} - \Gamma^\sigma_{\beta\nu}\delta\Gamma^\mu_{\sigma\alpha}\,,$$

$$\Rightarrow \partial_\beta\delta\Gamma^\mu_{\alpha\nu} = \bar{\nabla}_\beta(\delta\Gamma^\mu_{\alpha\nu}) - \Gamma^\mu_{\beta\sigma}\delta\Gamma^\sigma_{\alpha\nu} + \Gamma^\sigma_{\beta\alpha}\delta\Gamma^\mu_{\sigma\nu} + \Gamma^\sigma_{\beta\nu}\delta\Gamma^\mu_{\sigma\alpha}\,,$$

and

$$\bar{\nabla}_\nu(\delta\Gamma^\mu_{\alpha\beta}) = \partial_\nu\delta\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\nu\sigma}\delta\Gamma^\sigma_{\alpha\beta} - \Gamma^\sigma_{\nu\alpha}\delta\Gamma^\mu_{\sigma\beta} - \Gamma^\sigma_{\nu\beta}\delta\Gamma^\mu_{\sigma\alpha}\,,$$

$$\Rightarrow \partial_\nu\delta\Gamma^\mu_{\alpha\beta} = \bar{\nabla}_\nu(\delta\Gamma^\mu_{\alpha\beta}) - \Gamma^\mu_{\nu\sigma}\delta\Gamma^\sigma_{\alpha\beta} + \Gamma^\sigma_{\nu\alpha}\delta\Gamma^\mu_{\sigma\beta} + \Gamma^\sigma_{\nu\beta}\delta\Gamma^\mu_{\sigma\alpha}\,.$$

If we insert these into our linearized Riemann tensor, we get

$$\begin{aligned}\delta R^\mu{}_{\alpha\beta\nu} &= \bar{\nabla}_\beta(\delta\Gamma^\mu_{\alpha\nu}) - \Gamma^\mu_{\beta\sigma}\delta\Gamma^\sigma_{\alpha\nu} + \Gamma^\sigma_{\beta\alpha}\delta\Gamma^\mu_{\sigma\nu} + \Gamma^\sigma_{\beta\nu}\delta\Gamma^\mu_{\sigma\alpha} - \bar{\nabla}_\nu(\delta\Gamma^\mu_{\alpha\beta}) \\ &\quad + \Gamma^\mu_{\nu\sigma}\delta\Gamma^\sigma_{\alpha\beta} - \Gamma^\sigma_{\nu\alpha}\delta\Gamma^\mu_{\sigma\beta} - \Gamma^\sigma_{\nu\beta}\delta\Gamma^\mu_{\sigma\alpha} + (\delta\Gamma^\sigma_{\alpha\nu})\Gamma^\mu_{\sigma\beta} + \Gamma^\sigma_{\alpha\nu}\delta\Gamma^\mu_{\sigma\beta} \\ &\quad - (\delta\Gamma^\sigma_{\alpha\beta})\Gamma^\mu_{\sigma\nu} - \Gamma^\sigma_{\alpha\beta}\delta\Gamma^\mu_{\sigma\nu}\end{aligned}$$

yielding;

$$\delta R^\mu{}_{\alpha\beta\nu} = \bar{\nabla}_\beta(\delta\Gamma^\mu_{\alpha\nu}) - \bar{\nabla}_\nu(\delta\Gamma^\mu_{\alpha\beta})\,. \tag{4.15}$$

$\mu \leftrightarrow \beta$ contraction gives the linear Ricci tensor

$$\delta R^\mu{}_{\alpha\mu\nu} = \bar{\nabla}_\mu(\delta\Gamma^\mu_{\alpha\nu}) - \bar{\nabla}_\nu(\delta\Gamma^\mu_{\alpha\mu}) = \delta R_{\alpha\nu}\,,$$

or by $\alpha \leftrightarrow \mu$

$$\delta R_{\mu\nu} = \bar{\nabla}_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \bar{\nabla}_\mu(\delta\Gamma^\alpha_{\alpha\nu})\,.$$

Now let us write this in terms of $\delta g_{\alpha\beta}$

$$
\begin{aligned}
\delta R_{\mu\nu} &= \bar{\nabla}_\alpha \left[\frac{1}{2}\bar{g}^{\alpha\sigma}(\bar{\nabla}_\mu \delta g_{\nu\sigma} + \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\nabla}_\sigma \delta g_{\mu\nu})\right] \\
&\quad -\bar{\nabla}_\mu \left[\frac{1}{2}\bar{g}^{\alpha\sigma}(\bar{\nabla}_\alpha \delta g_{\sigma\nu} + \bar{\nabla}_\nu \delta g_{\alpha\sigma} - \bar{\nabla}_\sigma \delta g_{\alpha\nu})\right] .
\end{aligned}
$$

In the last term if we make the $\sigma \leftrightarrow \alpha$ exchange, it will cancel the fourth term

$$
\delta R_{\mu\nu} = \frac{1}{2}\{\bar{\nabla}^\sigma \bar{\nabla}_\mu \delta g_{\nu\sigma} + \bar{\nabla}^\sigma \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\nabla}^\sigma \bar{\nabla}_\sigma \delta g_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{g}^{\alpha\sigma} \delta g_{\alpha\sigma}\}
$$

and we are left with

$$
\delta R_{\mu\nu} = \frac{1}{2}\{\bar{\nabla}^\sigma \bar{\nabla}_\mu \delta g_{\nu\sigma} + \bar{\nabla}^\sigma \bar{\nabla}_\nu \delta g_{\mu\sigma} - \bar{\Box} \delta g_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu \delta g\} , \tag{4.16}
$$

where $\delta g = h = \bar{g}^{\alpha\sigma} h_{\alpha\sigma}$ and $\bar{\nabla}^\sigma \bar{\nabla}_\sigma \equiv \bar{\Box}$.

The linear part of the Ricci scalar reads

$$
\begin{aligned}
\delta R &= \delta(g^{\mu\nu} R_{\mu\nu}) = (\delta R_{\mu\nu})\bar{g}^{\mu\nu} - \bar{R}_{\mu\nu} \delta g^{\mu\nu} , \\
&= \frac{1}{2}(-\bar{\Box} h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\sigma\mu} + \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\sigma\nu})\bar{g}^{\mu\nu} - \frac{2}{(D-2)} \Lambda \bar{g}_{\mu\nu} h^{\mu\nu} , \\
&= \frac{1}{2}(-\bar{\Box} h - \bar{\Box} h + \bar{\nabla}^\sigma \bar{\nabla}^\mu h_{\sigma\mu} + \bar{\nabla}^\sigma \bar{\nabla}^\nu h_{\sigma\nu}) - \frac{2}{(D-2)} \Lambda h ,
\end{aligned}
$$

or by taking $\nu \to \mu$

$$
\begin{aligned}
\delta R &= \frac{1}{2}(-2\bar{\Box} h + 2\bar{\nabla}^\sigma \bar{\nabla}^\mu h_{\sigma\mu}) - \frac{2}{(D-2)} \Lambda h , \\
&= -\bar{\Box} h + \bar{\nabla}^\sigma \bar{\nabla}^\mu h_{\sigma\mu} - \frac{2}{(D-2)} \Lambda h .
\end{aligned} \tag{4.17}
$$

4.2.4 LINEAR FORM OF THE EINSTEIN EQUATION

Now, we are ready to write (4.8) in terms of the deviation part of the metric. That is,

$$\begin{aligned}\mathcal{G}_L^{\mu\nu} &= R_L^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}R_L - \frac{2}{(D-2)}\Lambda h^{\mu\nu} \\ &= \frac{1}{2}(-\bar{\Box}h^{\mu\nu} - \bar{\nabla}^{\mu}\bar{\nabla}^{\nu}h + \bar{\nabla}_{\sigma}\bar{\nabla}^{\nu}h^{\sigma\mu} + \bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h^{\sigma\nu}) \\ &\quad -\frac{1}{2}\bar{g}^{\mu\nu}(-\bar{\Box}h + \bar{\nabla}_{\sigma}\bar{\nabla}_{\alpha}h^{\sigma\alpha} - \frac{2}{(D-2)}\Lambda h) - \frac{2}{(D-2)}\Lambda h^{\mu\nu}. \quad (4.18)\end{aligned}$$

We can also show that this equation is background covariantly constant:

$$\begin{aligned}\bar{\nabla}_{\mu}\mathcal{G}_L^{\mu\nu} &= \frac{1}{2}(-\bar{\nabla}_{\mu}\bar{\Box}h^{\mu\nu} - \bar{\Box}\bar{\nabla}^{\nu}h + \bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\bar{\nabla}^{\nu}h^{\sigma\mu} + \bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h^{\sigma\nu}) \\ &\quad -\frac{1}{2}(-\bar{\nabla}^{\nu}\bar{\Box}h + \bar{\nabla}^{\nu}\bar{\nabla}_{\sigma}\bar{\nabla}_{\alpha}h^{\sigma\alpha}) + \frac{1}{(D-2)}\Lambda\bar{\nabla}^{\nu}h - \frac{2}{(D-2)}\Lambda\bar{\nabla}_{\mu}h^{\mu\nu}.\end{aligned}$$

In the fourth term change the places of the covariant derivatives to get the first term with an opposite sign:

$$[\bar{\nabla}_{\mu}, \bar{\nabla}_{\sigma}]\bar{\nabla}^{\mu}h^{\sigma\nu} = \bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h^{\sigma\nu} - \bar{\nabla}_{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}^{\mu}h^{\sigma\nu},$$

$$\Rightarrow \bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h^{\sigma\nu} = [\bar{\nabla}_{\mu}, \bar{\nabla}_{\sigma}]\bar{\nabla}^{\mu}h^{\sigma\nu} + \bar{\nabla}_{\sigma}\bar{\Box}h^{\sigma\nu}.$$

In the last term, we make the $\sigma \rightarrow \mu$ transformation

$$\bar{\nabla}_{\mu}\bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h^{\sigma\nu} = [\bar{\nabla}_{\mu}, \bar{\nabla}_{\sigma}]\bar{\nabla}^{\mu}h^{\sigma\nu} + \bar{\nabla}_{\mu}\bar{\Box}h^{\mu\nu}.$$

Moreover

$$\begin{aligned}
[\bar{\nabla}_\mu, \bar{\nabla}_\sigma]\bar{\nabla}^\mu h^{\sigma\nu} &= \bar{R}_{\mu\sigma}{}^{\mu}{}_{\lambda}\bar{\nabla}^\lambda h^{\sigma\nu} + \bar{R}_{\mu\sigma}{}^{\sigma}{}_{\lambda}\bar{\nabla}^\mu h^{\lambda\nu} + \bar{R}_{\mu\sigma}{}^{\nu}{}_{\lambda}\bar{\nabla}^\mu h^{\sigma\lambda} \\
&= \bar{R}_{\sigma\lambda}\bar{\nabla}^\lambda h^{\sigma\nu} - \bar{R}_{\mu\lambda}\bar{\nabla}^\mu h^{\lambda\nu} \\
&\quad + \frac{2\Lambda}{(D-1)(D-2)}(\delta^\nu_\mu \bar{g}_{\sigma\lambda} - \bar{g}_{\mu\lambda}\delta^\nu_\sigma)\bar{\nabla}^\mu h^{\sigma\lambda} \\
&= \frac{2\Lambda}{(D-2)}(\bar{g}_{\sigma\lambda}\bar{\nabla}^\lambda h^{\sigma\nu} - \bar{g}_{\mu\lambda}\bar{\nabla}^\mu h^{\lambda\nu}) \\
&\quad + \frac{2\Lambda}{(D-1)(D-2)}(\bar{\nabla}^\nu h - \bar{\nabla}_\lambda h^{\nu\lambda}) \\
&= \frac{2\Lambda}{(D-2)}(\bar{\nabla}_\sigma h^{\sigma\nu} - \bar{\nabla}_\lambda h^{\lambda\nu}) \\
&\quad + \frac{2\Lambda}{(D-1)(D-2)}(\bar{\nabla}^\nu h - \bar{\nabla}_\lambda h^{\nu\lambda})\,.
\end{aligned}$$

If we make the $\lambda \to \sigma$ substitution the term in the first parenthesis will vanish and we will get

$$\bar{\nabla}_\mu \bar{\nabla}_\sigma \bar{\nabla}^\mu h^{\sigma\nu} = \frac{2\Lambda}{(D-1)(D-2)}(\bar{\nabla}^\nu h - \bar{\nabla}_\lambda h^{\nu\lambda}) + \bar{\nabla}_\mu \bar{\Box} h^{\mu\nu}\,.$$

In the fifth and sixth terms of $\bar{\nabla}_\mu \mathcal{G}^{\mu\nu}$, we use the same procedure. Making suitable index substitutions we get

$$\bar{\nabla}^\nu \bar{\Box} h = \frac{-2\Lambda}{(D-2)}\bar{\nabla}^\nu h + \bar{\Box}\bar{\nabla}^\nu h\,,$$

$$\begin{aligned}
\bar{\nabla}^\nu \bar{\nabla}_\sigma \bar{\nabla}_\mu h^{\sigma\mu} &= \frac{2\Lambda}{(D-1)(D-2)}\bar{\nabla}^\nu h - \frac{2\Lambda}{(D-1)(D-2)}\bar{\nabla}_\mu h^{\mu\nu}(2D-1) \\
&\quad + \bar{\nabla}_\mu \bar{\nabla}_\sigma \bar{\nabla}^\nu h^{\sigma\mu}\,.
\end{aligned}$$

When these results are inserted into (4.18), one will have

$$\bar{\nabla}_\mu \mathcal{G}_L^{\mu\nu} = \frac{\Lambda}{(D-2)(D-1)}(\bar{\nabla}^\nu h - \bar{\nabla}_\lambda h^{\nu\lambda}) - \frac{\Lambda}{(D-2)}\bar{\nabla}^\nu h + \frac{(2D-1)\Lambda}{(D-2)(D-1)}\bar{\nabla}_\mu h^{\mu\nu}$$

$$-\frac{\Lambda}{(D-2)(D-1)}\bar{\nabla}^{\nu}h+\frac{\Lambda}{(D-2)}\bar{\nabla}^{\nu}h-\frac{2\Lambda}{(D-2)}\bar{\nabla}_{\mu}h^{\mu\nu}=0\,.$$

In the first term the covariant derivative of h will cancel with the fourth term and we change the indices of the remaining part of the first term: $\lambda \to \mu$. Thus we conclude that the energy-momentum tensor is background covariantly constant, that is

$$\bar{\nabla}_{\mu}\mathcal{G}_{L}^{\mu\nu}=0\,,\qquad (\text{or } \bar{\nabla}_{\mu}T^{\mu\nu}=0)\,. \tag{4.19}$$

4.3 KILLING CHARGES

Let us recall that there are two facets of a proper energy definition: First, identification of the "Gauss law", whose existence is guaranteed by gauge invariance; second, choice of the proper vacuum, possessing sufficient Killing symmetries with respect to which global, background gauge-invariant, generators can be defined; these will always appear as surface integrals in the asymptotic vacuum [2].

In converting the volume integrals to surface integrals, let us follow a route, which will be convenient in the higher curvature cases. We take the energy-momentum tensor in a Killing vector field and collect terms in the covariant derivative to get surface terms:

$$\begin{aligned}
2\bar{\xi}_{\nu}\mathcal{G}_{L}^{\mu\nu} &= 2\bar{\xi}_{\nu}R_{L}^{\mu\nu}-\bar{\xi}_{\nu}\bar{g}^{\mu\nu}R_{L}-\frac{4\Lambda}{(D-2)}\bar{\xi}_{\nu}h^{\mu\nu}\\
&= \bar{\xi}_{\nu}\{-\bar{\nabla}_{\rho}\bar{\nabla}^{\rho}h^{\mu\nu}-\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}h+\bar{\nabla}_{\sigma}\bar{\nabla}^{\nu}h^{\sigma\mu}+\bar{\nabla}_{\sigma}\bar{\nabla}^{\mu}h^{\sigma\nu}\}\\
&\quad -\bar{\xi}^{\mu}\{-\bar{\nabla}_{\rho}\bar{\nabla}^{\rho}h+\bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}h^{\sigma\nu}-\frac{2\Lambda}{(D-2)}h\}-\frac{4\Lambda}{(D-2)}\bar{\xi}_{\nu}h^{\mu\nu}
\end{aligned}$$

where we used (4.16) and (4.17). We rename the indices for convenience;

second term: $\nu \to \rho$

third term: $\sigma \to \nu, \nu \to \rho$

fourth term: $\sigma \to \rho$

sixth term: $\sigma \to \rho$. Thus our equation becomes

$$
\begin{aligned}
2\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} &= -\bar{\xi}_\nu \bar{\nabla}_\rho \bar{\nabla}^\rho h^{\mu\nu} - \bar{\xi}_\rho \bar{\nabla}^\mu \bar{\nabla}^\rho h + \bar{\xi}_\rho \bar{\nabla}_\nu \bar{\nabla}^\rho h^{\nu\mu} + \bar{\xi}_\nu \bar{\nabla}_\rho \bar{\nabla}^\mu h^{\rho\nu} \\
&\quad + \bar{\xi}^\mu \bar{\nabla}_\rho \bar{\nabla}^\rho h - \bar{\xi}^\mu \bar{\nabla}_\rho \bar{\nabla}_\nu h^{\rho\nu} + \frac{2\Lambda}{(D-2)} \bar{\xi}^\mu h - \frac{4\Lambda}{(D-2)} \bar{\xi}_\nu h^{\mu\nu} \,.
\end{aligned}
$$

To collect all terms, we use the commutator relation of a vector that gives us the Riemann tensor. In the first, fourth, fifth and sixth terms the Killing vectors are taken inside the covariant derivative with extra terms that will come from the derivative of the Killing vectors. In the second and third terms, places of derivatives must change, after that the Killing vectors can be taken inside the derivative with two additional terms, the second comes from exchange of derivatives. After these calculations we are left with

$$
\begin{aligned}
2\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} &= -\bar{\nabla}_\rho (\bar{\xi}_\nu \bar{\nabla}^\rho h^{\mu\nu}) + (\bar{\nabla}_\rho \bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) - \bar{\nabla}_\rho (\bar{\xi}^\rho \bar{\nabla}^\mu h) + (\bar{\nabla}_\rho \bar{\xi}^\rho)(\bar{\nabla}^\mu h) \\
&\quad + \bar{\nabla}_\rho (\bar{\xi}^\rho \bar{\nabla}_\nu h^{\mu\nu}) - (\bar{\nabla}_\rho \bar{\xi}^\rho)(\bar{\nabla}_\nu h^{\mu\nu}) + \bar{\nabla}_\rho (\bar{\xi}_\nu \bar{\nabla}^\mu h^{\rho\nu}) - (\bar{\nabla}_\rho \bar{\xi}_\nu)(\bar{\nabla}^\mu h^{\rho\nu}) \\
&\quad + \bar{\nabla}_\rho (\bar{\xi}^\mu \bar{\nabla}^\rho h) - (\bar{\nabla}_\rho \bar{\xi}^\mu)(\bar{\nabla}^\rho h) - \bar{\nabla}_\rho (\bar{\xi}^\mu \bar{\nabla}_\nu h^{\rho\nu}) + (\bar{\nabla}_\rho \bar{\xi}^\mu)(\bar{\nabla}_\nu h^{\rho\nu}) \\
&\quad + \frac{2\Lambda}{(D-2)} h^{\nu\mu} \bar{\xi}_\nu + \frac{2\Lambda}{(D-2)} h \bar{\xi}^\mu - \frac{4\Lambda}{(D-2)} h^{\nu\mu} \bar{\xi}_\nu \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)} (h^{\nu\mu} \bar{\xi}_\nu - h \bar{\xi}^\mu)
\end{aligned}
$$

The fourth, sixth and the eighth terms vanish because of the Killing equation

$$\begin{aligned}
2\bar{\xi}_\nu G_L^{\mu\nu} &= \bar{\nabla}_\rho\{-\bar{\xi}_\nu\bar{\nabla}^\rho h^{\mu\nu} - \bar{\xi}^\rho\bar{\nabla}^\mu h + \bar{\xi}^\rho\bar{\nabla}_\nu h^{\mu\nu} + \bar{\xi}_\nu\bar{\nabla}^\mu h^{\rho\nu} + \bar{\xi}^\mu\bar{\nabla}^\rho h - \bar{\xi}^\mu\bar{\nabla}_\nu h^{\rho\nu}\} \\
&\quad + \frac{2\Lambda}{(D-2)}h\bar{\xi}^\mu - \frac{2\Lambda}{(D-2)}h^{\nu\mu}\bar{\xi}_\nu + \frac{2\Lambda}{(D-2)(D-1)}(h^{\nu\mu}\bar{\xi}_\nu - h\bar{\xi}^\mu) \\
&\quad + (\bar{\nabla}_\rho\bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) - (\bar{\nabla}_\rho\bar{\xi}^\mu)(\bar{\nabla}^\rho h) + (\bar{\nabla}_\rho\bar{\xi}^\mu)(\bar{\nabla}_\nu h^{\rho\nu})\,.
\end{aligned}$$

We will look at the last three terms closely:

$$(\bar{\nabla}_\rho\bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) = \bar{\nabla}_\rho(h^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\nu) - h^{\mu\nu}(\bar{\nabla}_\rho\bar{\nabla}^\rho\bar{\xi}_\nu)\,.$$

Operating on (2.1) with $\bar{\nabla}^\mu$, one gets

$$\bar{\nabla}^\mu\bar{\nabla}_\mu\bar{\xi}_\nu + \bar{\nabla}^\mu\bar{\nabla}_\nu\bar{\xi}_\mu = 0\,.$$

The second term can be written in the commutator form that is

$$\bar{\Box}\bar{\xi}_\nu + [\bar{\nabla}^\mu, \bar{\nabla}_\nu]\bar{\xi}_\mu = 0\,,$$

or simply

$$\bar{\Box}\bar{\xi}_\nu = -\frac{2\Lambda}{(D-2)}\bar{g}_{\nu\lambda}\bar{\xi}^\lambda\,.$$

Using this relation, we have

$$(\bar{\nabla}_\rho\bar{\xi}_\nu)(\bar{\nabla}^\rho h^{\mu\nu}) = \bar{\nabla}_\rho(h^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\nu) + \frac{2\Lambda}{(D-2)}h^{\mu\nu}\bar{g}_{\nu\lambda}\bar{\xi}^\lambda\,,$$

and

$$(\bar{\nabla}_\rho\bar{\xi}^\mu)(\bar{\nabla}^\rho h) = -\bar{\nabla}_\rho(h\bar{\nabla}^\mu\bar{\xi}^\rho) + \frac{2\Lambda}{(D-2)}h\bar{\xi}^\mu\,.$$

Using the property of a Killing vector, shown in the appendix (see (9.3))

$$(\bar{\nabla}_\rho \bar{\xi}^\mu)(\bar{\nabla}_\nu h^{\rho\nu}) = -\bar{\nabla}_\rho (h^{\rho\nu} \bar{\nabla}^\mu \bar{\xi}_\nu) - \frac{2\Lambda}{(D-2)(D-1)} (\bar{\xi}_\nu h^{\mu\nu} - \bar{\xi}^\mu h) \, .$$

Finally collecting these results, we have

$$2\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} = \bar{\nabla}_\rho \{ -\bar{\xi}_\nu \bar{\nabla}^\rho h^{\mu\nu} - \bar{\xi}^\rho \bar{\nabla}^\mu h + \bar{\xi}^\rho \bar{\nabla}_\nu h^{\mu\nu} + \bar{\xi}_\nu \bar{\nabla}^\mu h^{\rho\nu} + \bar{\xi}^\mu \bar{\nabla}^\rho h$$

$$- \bar{\xi}^\mu \bar{\nabla}_\nu h^{\rho\nu} + h^{\mu\nu} \bar{\nabla}^\rho \bar{\xi}_\nu + h \bar{\nabla}^\mu \bar{\xi}^\rho - h^{\rho\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \} \, . \tag{4.20}$$

Since the charge densities are surface terms, the Killing charges become

$$Q^\mu(\bar{\xi}) = \frac{1}{4\Omega_{(D-2)} G_D} \oint_\Sigma dS_i \{ -\bar{\xi}_\nu \bar{\nabla}^i h^{\mu\nu} - \bar{\xi}^i \bar{\nabla}^\mu h + \bar{\xi}^i \bar{\nabla}_\nu h^{\mu\nu} + \bar{\xi}_\nu \bar{\nabla}^\mu h^{i\nu}$$

$$+ \bar{\xi}^\mu \bar{\nabla}^i h - \bar{\xi}^\mu \bar{\nabla}_\nu h^{i\nu} + h^{\mu\nu} \bar{\nabla}^i \bar{\xi}_\nu + h \bar{\nabla}^\mu \bar{\xi}^i - h^{i\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \} \, . \tag{4.21}$$

Here $dS_i \equiv \sqrt{-det\bar{g}} \;\, d\Omega_i$ where i ranges over $(1, 2, ..., D-2)$; the charge is normalized by dividing with the (D-dimensional) Newton's constant G_D and the solid angle $d\Omega_{D-2}$.

Before calculating the conserved charges Q^0, we check our expression to see whether it is gauge invariant or not, and we will look if it goes to the ADM charges in the limit of an asymptotically flat background ($\bar{\nabla}_j \to \partial_j$) in which case our timelike Killing vector is $\bar{\xi}_\mu = (1, \mathbf{0})$ [1].

First we will look at the gauge-invariance. Under an infinitesimal diffeomorphism, generated by a vector δ_ζ, the deviation part of the metric transforms as

$$\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu \quad [1]. \tag{4.22}$$

First we will look at the linear Ricci scalar:

$$R_L = (g^{\mu\nu} R_{\mu\nu})_L \,,$$

$$\begin{aligned} \Rightarrow \delta_\zeta R_L &= \delta_\zeta(\bar{g}^{\mu\nu} R^L_{\mu\nu} - \bar{R}_{\mu\nu} h^{\mu\nu}) \\ &= \bar{g}^{\mu\nu} \delta_\zeta R^L_{\mu\nu} - \frac{2\Lambda}{(D-2)} \bar{g}^{\mu\nu} \delta_\zeta h_{\mu\nu} \,. \end{aligned}$$

Now using (4.16), we have

$$\begin{aligned} \delta_\zeta R_L &= \bar{g}^{\mu\nu} \frac{1}{2} \delta_\zeta(-\bar{\Box} h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\sigma\mu} + \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\sigma\nu}) \\ &\quad - \frac{2\Lambda}{(D-2)} \bar{g}^{\mu\nu} (\bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu) \\ &= -\bar{\Box} \delta_\zeta h + \bar{\nabla}^\sigma \bar{\nabla}^\mu \delta_\zeta h_{\sigma\mu} - \frac{4\Lambda}{(D-2)} \bar{\nabla}^\mu \zeta_\mu \,. \end{aligned}$$

With the help of (4.22), we can write

$$\begin{aligned} \delta_\zeta R_L &= -\bar{g}^{\mu\nu} \bar{\Box} (\bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu) + \bar{\nabla}^\sigma \bar{\nabla}^\mu (\bar{\nabla}_\sigma \zeta_\mu + \bar{\nabla}_\mu \zeta_\sigma) - \frac{4\Lambda}{(D-2)} \bar{\nabla}^\mu \zeta_\mu \\ &= -2 \bar{\Box} \bar{\nabla}^\mu \zeta_\mu + \bar{\nabla}^\sigma \bar{\nabla}^\mu \bar{\nabla}_\sigma \zeta_\mu + \bar{\nabla}^\sigma \bar{\Box} \zeta_\sigma - \frac{4\Lambda}{(D-2)} \bar{\nabla}^\mu \zeta_\mu \,. \end{aligned}$$

We look examine the second and third terms carefully:

$$[\bar{\nabla}^\mu, \bar{\nabla}_\sigma] \zeta_\mu = \bar{\nabla}^\mu \bar{\nabla}_\sigma \zeta_\mu - \bar{\nabla}_\sigma \bar{\nabla}^\mu \zeta_\mu \,,$$

$$\bar{\nabla}^\mu \bar{\nabla}_\sigma \zeta_\mu = \frac{2\Lambda}{(D-2)} \zeta_\sigma + \bar{\nabla}_\sigma \bar{\nabla}^\mu \zeta_\mu \,.$$

Therefore, we have

$$\bar{\nabla}^\sigma \bar{\nabla}^\mu \bar{\nabla}_\sigma \zeta_\mu = \frac{2\Lambda}{(D-2)} \bar{\nabla}^\sigma \zeta_\sigma + \bar{\Box} \bar{\nabla}^\mu \zeta_\mu \,.$$

In the third term the same calculations can be done to get

$$\bar{\nabla}^\sigma \bar{\Box} \zeta_\sigma = \frac{2\Lambda}{(D-2)} \bar{\nabla}^\beta \zeta_\beta + \bar{\Box} \bar{\nabla}^\sigma \zeta_\sigma$$

We have the background gauge invariance of the linear Ricci scalar

$$\delta_\zeta R_L = 0\,.$$

Therefore

$$\delta_\zeta \mathcal{G}^L_{\mu\nu} = \delta_\zeta R^L_{\mu\nu} - \frac{2\Lambda}{(D-2)} \delta_\zeta h_{\mu\nu}.$$

Using (4.16) and (4.22), we have

$$\begin{aligned}
\delta_\zeta \mathcal{G}^L_{\mu\nu} &= \frac{1}{2}(-\bar{\Box}\bar{\nabla}_\mu \zeta_\nu - \bar{\Box}\bar{\nabla}_\nu \zeta_\mu - \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\nabla}^\beta \zeta_\beta - \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\nabla}^\alpha \zeta_\alpha \\
&\quad + \bar{\nabla}^\sigma \bar{\nabla}_\nu \bar{\nabla}_\sigma \zeta_\mu + \bar{\nabla}^\sigma \bar{\nabla}_\nu \bar{\nabla}_\mu \zeta_\sigma + \bar{\nabla}^\sigma \bar{\nabla}_\mu \bar{\nabla}_\sigma \zeta_\nu + \bar{\nabla}^\sigma \bar{\nabla}_\mu \bar{\nabla}_\nu \zeta_\sigma) \\
&\quad - \frac{2\Lambda}{(D-2)} (\bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu)\,.
\end{aligned}$$

Just sa before, let us look at the terms that are in the second line: The fifth term is:

$$\bar{\nabla}^\sigma \bar{\nabla}_\nu \bar{\nabla}_\sigma \zeta_\mu = \frac{2\Lambda}{(D-2)(D-1)} (\bar{g}_{\nu\mu} \bar{\nabla}^\sigma \zeta_\sigma - \bar{g}_{\sigma\mu} \bar{\nabla}^\sigma \zeta_\nu) + \bar{\nabla}^\sigma \bar{\nabla}_\sigma \bar{\nabla}_\nu \zeta_\mu\,.$$

The sixth term is:

$$\bar{\nabla}^\sigma \bar{\nabla}_\nu \bar{\nabla}_\mu \zeta_\sigma = \frac{2\Lambda}{(D-2)(D-1)} (\bar{\nabla}_\nu \zeta_\mu - \bar{\nabla}_\mu \zeta_\nu) + \bar{\nabla}^\sigma \bar{\nabla}_\mu \bar{\nabla}_\nu \zeta_\sigma\,.$$

The third term is:

$$\bar{\nabla}^\sigma \bar{\nabla}_\mu \bar{\nabla}_\sigma \zeta_\nu = \frac{2\Lambda}{(D-2)(D-1)} (\bar{g}_{\mu\nu} \bar{\nabla}^\sigma \zeta_\sigma - \bar{g}_{\sigma\nu} \bar{\nabla}^\sigma \zeta_\mu) + \bar{\nabla}^\sigma \bar{\nabla}_\sigma \bar{\nabla}_\mu \zeta_\nu\,.$$

The fourth term is:

$$\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\zeta_{\sigma} = \frac{2\Lambda}{(D-2)}\bar{\nabla}_{\mu}\zeta_{\nu} + \frac{2\Lambda}{(D-2)(D-1)}(\bar{\nabla}_{\mu}\zeta_{\nu} - \bar{g}_{\mu\nu}\bar{\nabla}^{\sigma}\zeta_{\sigma}) + \frac{2\Lambda}{(D-2)}\bar{\nabla}_{\nu}\zeta_{\mu} + \bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\nabla}^{\sigma}\zeta_{\sigma}\,.$$

Collecting all these, terms we end up with

$$\delta_{\zeta}\mathcal{G}^{L}_{\mu\nu} = 0\,,$$

which means that $\mathcal{G}^{L}_{\mu\nu}$ is gauge-invariant. Therefore, we have $\delta_{\zeta}R^{L}_{\mu\nu} = \frac{2\Lambda}{(D-2)}\delta_{\zeta}h_{\mu\nu}$.

Hence $\delta_{\zeta}Q^{\mu} = 0$; that is, the Killing charge is indeed background gauge-invariant.

Now we will examine (4.21) in the limit of an asymptotically flat background, which should yield the ADM charge. Let us just look at the mass to begin with. With $\bar{\xi}_{\mu} = (1, \mathbf{0})$, we have $\bar{\xi}_{i} = 0$, $\bar{\xi}_{0} = 1$ and $\bar{\xi}^{0} = -1$ in flat space with the metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. We have $h = -h_{00} + h_{ii}$. Being in Cartesian coordinates, we can take the covariant derivatives to partial derivatives ($\bar{\nabla}_{j} \to \partial_{j}$). Hence,

$$M = \frac{1}{4\Omega_{(D-2)}G_{D}}\oint_{\Sigma} dS_{i}\{\bar{\xi}_{0}\partial^{0}h^{i0} - \bar{\xi}_{0}\partial^{i}h^{00} - \bar{\xi}_{0}\partial^{i}h - \bar{\xi}_{0}\partial^{0}h^{i0} + \partial_{j}h^{ij}\}\,.$$

Expanding the last term we have

$$\begin{aligned} M &= \frac{1}{4\Omega_{(D-2)}G_{D}}\oint_{\Sigma} dS_{i}\{-\partial^{i}h^{00} - \partial^{i}h_{jj} + \partial^{i}h_{00} + \partial_{j}h^{ij}\} \\ &= \frac{1}{4\Omega_{(D-2)}G_{D}}\oint_{\Sigma} dS_{i}\{\partial_{j}h^{ij} - \partial^{i}h_{jj}\} \end{aligned}$$

which is the usual ADM mass [6].

CHAPTER 5

THE ENERGY OF SDS SOLUTIONS

Having established the energy formula for asymptotically (A)dS spaces, we can now evaluate the energy of Schwarzschild-de Sitter (SdS) solutions.

In static coordinates, the line element of D-dimensional SdS reads

$$\begin{aligned} ds^2 &= -\left(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2}\right)dt^2 \\ &\quad +\left(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2}\right)^{-1}dr^2+r^2d\Omega^2_{D-2}\,, \end{aligned} \tag{5.1}$$

where $l^2 \equiv (D-2)(D-1)/2\Lambda > 0$. The background ($r_0 = 0$) Killing vector is $\bar{\xi}^\mu = (-1, \mathbf{0})$, which is timelike everywhere for AdS ($l^2 < 0$), but remains timelike for dS ($l^2 > 0$) only inside the cosmological horizon: $\bar{g}_{\mu\nu}\bar{\xi}^\mu\bar{\xi}^\nu = -(1-\frac{r^2}{l^2})$ [1].

5.1 THE $D = 4$ CASE

Let us concentrate on $D = 4$ first and calculate the surface integral (4.21) not at $r \to \infty$, but at some finite distance r from the origin; this will not be gauge-invariant, since energy is to be measured only at infinity. Nevertheless, for dS space (which has a horizon that keeps us from going smoothly to infinity), let us first keep r finite as an intermediate step. The integral becomes

$$E(r) = \frac{r_0}{2G} \frac{(1 - \frac{r^2}{l^2})}{(1 - \frac{r_0}{r} - \frac{r^2}{l^2})} \,. \tag{5.2}$$

For AdS, take $r \to \infty$ and we get the usual mass $\frac{r_0}{2G}$. For dS, we can only consider small r_0 limit, which do not change the location of the background horizon, that also yields $\frac{r_0}{2G}$ [1].

5.2 ENERGY FOR D DIMENSIONS

In this case $h \approx 0$. From the line element (5.1) $g_{00}, g_{rr}, \bar{g}_{00}, \bar{g}_{rr}$ and the corresponding h terms can be calculated. These are

$$g_{00} = -\left(1 - (\frac{r_0}{r})^{D-3} - \frac{r^2}{l^2}\right) \,,\; g_{rr} = \left(1 - (\frac{r_0}{r})^{D-3} - \frac{r^2}{l^2}\right)^{-1} \,, \tag{5.3}$$

and the background metric components will be

$$\bar{g}_{00} = -\left(1 - \frac{r^2}{l^2}\right) \,,\; \bar{g}_{rr} = -\left(1 - \frac{r^2}{l^2}\right)^{-1} \,, \tag{5.4}$$

since $r_0 = 0$ for the background. From (4.2)

$$h_{00} = (\frac{r_0}{r})^{D-3} \,, \quad \bar{g}^{00}\bar{g}^{00}h_{00} = h^{00} = \frac{(\frac{r_0}{r})^{D-3}}{(1 - \frac{r^2}{l^2})^2} \,. \tag{5.5}$$

$$h_{rr} = \frac{(\frac{r_0}{r})^{D-3}}{(1 - (\frac{r_0}{r})^{D-3} - \frac{r^2}{l^2})(1 - \frac{r^2}{l^2})} \,, \quad h^{rr} = \frac{(\frac{r_0}{r})^{D-3}(1 - \frac{r^2}{l^2})}{(1 - (\frac{r_0}{r})^{D-3} - \frac{r^2}{l^2})} \,. \tag{5.6}$$

Using (4.21), we have

$$Q^0 = \frac{4\pi}{16\pi G_D} \lim_{r\to\infty} r^{D-2}\{\bar{\xi}_0 \bar{\nabla}^0 h^{r0} - \bar{\xi}_0 \bar{\nabla}^r h^{00} + h^{00}\bar{\nabla}^r \bar{\xi}_0 - h^{rr}\bar{\nabla}^0 \bar{\xi}_r + \bar{\nabla}_\nu h^{r\nu}\}$$

where the constant factors come from the normalization constant and the integration element dS_i.

$$Q^0 = \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{\bar{\xi}^0\bar{\nabla}_0 h^{r0} - \bar{\xi}_0\bar{\nabla}^r h^{00} + h^{00}\bar{\nabla}^r\bar{\xi}_0 - h^{rr}\bar{\nabla}^0\bar{\xi}_r + \bar{\nabla}_0 h^{r0} + \bar{\nabla}_i h^{ri}\}$$

and

$$Q^0 = \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{-\bar{\xi}_0\bar{\nabla}^r h^{00} + h^{00}\bar{\nabla}^r\bar{\xi}_0 - h^{rr}\bar{\nabla}^0\bar{\xi}_r + \bar{\nabla}_i h^{ri}\}\,,$$

playing with indices

$$Q^0 = \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{\bar{g}_{00}\bar{g}^{rr}\bar{\nabla}_r h^{00} + h^{00}\bar{g}_{00}\bar{\nabla}_0\bar{\xi}_r + h^{rr}\bar{g}_{rr}\bar{\nabla}_0\bar{\xi}_r + \bar{\nabla}_i h^{ri}\}\,,$$

and the terms in the middle cancel with each other since $h = 0$. Therefore we are left with

$$\begin{aligned}
Q^0 &= \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{\bar{g}_{00}\bar{g}^{rr}\bar{\nabla}_r h^{00} + \bar{\nabla}_i h^{ri}\} \\
&= \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{\bar{g}^{00}\bar{g}^{00}\bar{g}_{00}\bar{g}^{rr}\bar{\nabla}_r h_{00} + \bar{\nabla}_i h^{ri}\} \\
&= \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{-\bar{\nabla}_r h_{00} + \bar{\nabla}_i h^{ri}\} \\
&= \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{-\partial_r h_{00} + 2\Gamma^0_{r0}h_{00} + \partial_r h^{rr} + \Gamma^r_{ij}h^{ij} + \Gamma^i_{ir}h^{rr}\} \\
&= \frac{4\pi}{16\pi G_D}\lim_{r\to\infty} r^{D-2}\{-\partial_r h_{00} + h_{00}\bar{g}^{00}\partial_r\bar{g}_{00} + \partial_r h^{rr} + \frac{1}{2}h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr} \\
&\quad + \frac{1}{2}h^{rr}\bar{g}^{ij}\partial_r\bar{g}_{ij}\}\,.
\end{aligned}$$

The last term can be expanded

$$Q^0 = \frac{4\pi}{16\pi G_D} \lim_{r\to\infty} r^{D-2}\{-\partial_r h_{00} + h_{00}\bar{g}^{00}\partial_r\bar{g}_{00} + \partial_r h^{rr} + \frac{1}{2}h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr}$$

$$+\frac{1}{2}h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr} + \frac{1}{2}h^{rr}\bar{g}^{ij}\partial_r\bar{g}_{ij}\}\,,$$

where $\bar{g}^{ij}\partial_r\bar{g}_{ij} = \frac{1}{r^2}$ when $i = j \neq r$, therefore we have

$$Q^0 = \frac{4\pi}{16\pi G_D} \lim_{r\to\infty} r^{D-2}\{-\partial_r h_{00} + h_{00}\bar{g}^{00}\partial_r\bar{g}_{00} + \partial_r h^{rr} + h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr} + \frac{1}{2}h^{rr}\bar{g}^{ij}\partial_r\bar{g}_{ij}\}\,.$$

Lets look at our expression term by term in the limit of $r \to \infty$:

$$-\partial_r h_{00} = (D-3)\frac{r_0^{D-3}}{r^{D-2}} \qquad -r^{D-2}\partial_r h_{00} = (D-3)r_0^{D-3}.$$

The second term

$$r^{D-2}h_{00}\bar{g}^{00}\partial_r\bar{g}_{00} = -r^{D-2}(\frac{r_0}{r})^{D-3}\frac{1}{(1-\frac{r^2}{l^2})}\frac{2r}{l^2} = r_0^{D-3}\frac{1}{(\frac{1}{r^2}-\frac{1}{l^2})}\frac{2}{l^2} = 2r_0^{D-3}\,;$$

the third term

$$\begin{aligned}
\partial_r h^{rr} &= (D-3)(\frac{r_0}{r})^{D-4}(-\frac{r_0}{r^2})(1-\frac{r^2}{l^2})\frac{1}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})} \\
&\quad +(\frac{r_0}{r})^{D-3}(-\frac{2r}{l^2})\frac{1}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})} \\
&\quad +(\frac{r_0}{r})^{D-3}(1-\frac{r^2}{l^2})\frac{\{(D-3)(\frac{r_0}{r})^{D-4}(-\frac{r_0}{r^2})+\frac{2r}{l^2}\}}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})^2}.
\end{aligned}$$

We will operate on this with r^{D-2},

$$\begin{aligned}
r^{D-2}\partial_r h^{rr} &= -(D-3)r_0^{D-3} + 2r_0^{D-3} - 2r_0^{D-3}\,, \\
&= -(D-3)r_0^{D-3}.
\end{aligned}$$

The fourth term is

$$h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr} = \frac{(\frac{r_0}{r})^{D-3}(\frac{2r}{l^2})}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})}\,,$$

and

$$r^{D-2}h^{rr}\bar{g}^{rr}\partial_r\bar{g}_{rr} = -2r_0^{D-3}$$

$$\frac{1}{2}h^{rr}\bar{g}^{ij}\partial_r\bar{g}_{ij} = \frac{(D-2)}{r}\frac{(\frac{r_0}{r})^{D-3}(1-\frac{r^2}{l^2})}{(1-(\frac{r_0}{r})^{D-3}-\frac{r^2}{l^2})}\,,$$

$$r^{D-2}\frac{1}{2}h^{rr}\bar{g}^{ij}\partial_r\bar{g}_{ij} = (D-2)r_0^{D-3}.$$

Hence, adding all the above we get the energy in D-dimensions

$$E = \frac{(D-2)}{4G_D}r_0^{D-3}\,. \tag{5.7}$$

Here r_0 can be arbitrarily large in the AdS case but must be small in dS [1].

5.3 THE $D = 3$ CASE

Let us note that analogous computations can also be carried out in $D = 3$; the proper solution to consider is

$$ds^2 = -(1-r_0-\frac{r^2}{l^2})dt^2 + (1-r_0-\frac{r^2}{l^2})^{-1}dr^2 + r^2d\phi^2\,, \tag{5.8}$$

for which the energy is $E = r_0/2G$ again, but now r_0 is a dimensionless constant and the Newton constant G has dimensions of $1/mass$ [1, 9].

CHAPTER 6

STRING-INSPIRED GRAVITY

In flat backgrounds, the ghost freedom of low energy string theory requires the quadratic corrections to Einstein's gravity to be of the Gauss-Bonnet (GB) form, an argument that should carry over to the AdS backgrounds. Below we construct and compute the energy of various asymptotically (A)dS spaces that solve the generic Einstein plus quadratic gravity theories, particularly the Einstein-GB model [1, 10].

At quadratic order, the generic action is

$$I = \int d^D x \sqrt{-g} \{ \frac{1}{\kappa} R + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma (R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2) \} \tag{6.1}$$

In $D = 4$, the GB part (γ terms) is a surface integral and plays no role in the equations of motion. In $D > 4$, on the contrary, GB is the only viable term, since non-zero α, β produce ghosts [11]. Here $\kappa = 2\Omega_{D-2} G_D$, where G_D is the D-dimensional Newton's constant [1].

After lengthy calculations, that are shown in the appendix B, we reach the equations of motion

$$
\begin{aligned}
& \frac{1}{\kappa}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\alpha R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) \\
+ & (2\alpha+\beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R \\
+ & 2\gamma[RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} + R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau} \\
- & 2R_{\mu\sigma}R_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\sigma\rho} - 4R^2_{\sigma\rho} + R^2)] \\
+ & \beta\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\beta(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} = \tau_{\mu\nu}\,. \qquad (6.2)
\end{aligned}
$$

In the absence of matter, flat space is a solution of these equations; but more important is that (A)dS is also a solution [1]. The cosmological constant can be found using Eq. (6.2):

$$
\begin{aligned}
& \frac{1}{\kappa}(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}) + 2\alpha\bar{R}(\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}) \\
+ & (2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu)\bar{R} \\
+ & 2\gamma[\bar{R}\bar{R}_{\mu\nu} - 2\bar{R}_{\mu\sigma\nu\rho}\bar{R}^{\sigma\rho} + \bar{R}_{\mu\sigma\rho\tau}\bar{R}_\nu{}^{\sigma\rho\tau} \\
- & 2\bar{R}_{\mu\sigma}\bar{R}_\nu{}^\sigma - \frac{1}{4}\bar{g}_{\mu\nu}(\bar{R}^2_{\tau\lambda\sigma\rho} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2)] \\
+ & \beta\bar{\Box}(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}) + 2\beta(\bar{R}_{\mu\sigma\nu\rho} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}_{\sigma\rho})\bar{R}^{\sigma\rho} = 0\,. \qquad (6.3)
\end{aligned}
$$

The terms that have covariant derivatives will be zero by using (4.5), (4.6) and (4.7). The other terms can be calculated one by one:

$$
\frac{1}{\kappa}(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}) = \frac{1}{\kappa}(\frac{2}{(D-2)}\Lambda\bar{g}_{\mu\nu} - \frac{1}{2}\frac{2D\Lambda}{(D-2)}) = -\frac{1}{\kappa}\Lambda\bar{g}_{\mu\nu}\,,
$$

and the second one is

$$
2\alpha\bar{R}(\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}) \;\; = \;\; 2\alpha\left(\frac{4D\Lambda^2}{(D-2)^2} - \frac{1}{4}\frac{4D^2\Lambda^2}{(D-2)^2}\right)\bar{g}_{\mu\nu}
$$

$$= \quad -2D\alpha\frac{\Lambda^2(D-4)}{(D-2)^2}\bar{g}_{\mu\nu}\,.$$

The next one is a bit longer than the others:

$$\begin{aligned}
I &= [\bar{R}\bar{R}_{\mu\nu} - 2\bar{R}_{\mu\sigma\nu\rho}\bar{R}^{\sigma\rho} + \bar{R}_{\mu\sigma\rho\tau}\bar{R}_{\nu}{}^{\sigma\rho\tau} \\
&\quad -2\bar{R}_{\mu\sigma}\bar{R}_{\nu}{}^{\sigma} - \frac{1}{4}\bar{g}_{\mu\nu}(\bar{R}^2_{\tau\lambda\sigma\rho} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2)] \\
&= \frac{4D\Lambda^2}{(D-2)^2}\bar{g}_{\mu\nu} - \frac{8\Lambda^2}{(D-2)^2(D-1)}(\bar{g}_{\mu\nu}\bar{g}_{\sigma\rho} - \bar{g}_{\mu\rho}\bar{g}_{\sigma\nu})\bar{g}^{\sigma\rho} \\
&\quad + \frac{4\Lambda^2}{(D-2)^2(D-1)^2}(\bar{g}_{\mu\rho}\bar{g}_{\sigma\tau} - \bar{g}_{\mu\tau}\bar{g}_{\sigma\rho})(\bar{g}^{\rho}_{\nu}\bar{g}^{\sigma\tau} - \bar{g}^{\tau}_{\nu}\bar{g}^{\sigma\rho}) \\
&\quad - \frac{8\Lambda^2}{(D-2)^2}\bar{g}_{\mu\sigma}\bar{g}^{\sigma}_{\nu} - \frac{1}{4}\frac{4D\Lambda^2(D-3)}{(D-2)(D-1)}\bar{g}_{\mu\nu}\,.
\end{aligned}$$

If we look carefully to the second and the fourth terms, we can see that they are of the same type and can be added together

$$\begin{aligned}
I &= \frac{4D\Lambda^2}{(D-2)^2}\bar{g}_{\mu\nu} - \frac{16\Lambda^2}{(D-2)^2}\bar{g}_{\mu\nu} \\
&\quad + \frac{8\Lambda^2}{(D-2)^2(D-1)}\bar{g}_{\mu\nu} - \frac{D\Lambda^2(D-3)}{(D-2)(D-1)}\bar{g}_{\mu\nu}\,,
\end{aligned}$$

and we get

$$\begin{aligned}
I &= \frac{\Lambda^2\bar{g}_{\mu\nu}}{(D-2)^2(D-1)}\{4D(D-1) - 16(D-1) + 8 - D(D-3)(D-2)\} \\
&= -\frac{\Lambda^2\bar{g}_{\mu\nu}}{(D-2)^2(D-1)}(D^3 - 9D^2 + 26D - 24)\,,
\end{aligned}$$

and finally we have

$$I = -\frac{(D-4)(D-3)}{(D-2)(D-1)}\Lambda^2\bar{g}_{\mu\nu}.$$

Similarly

$$2\beta(\bar{R}_{\mu\sigma\nu\rho} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}_{\sigma\rho})\bar{R}^{\sigma\rho} = 2\beta\frac{4\Lambda^2}{(D-2)^2(D-1)}(\bar{g}_{\mu\nu}\bar{g}_{\sigma\rho} - \bar{g}_{\mu\rho}\bar{g}_{\sigma\nu})\bar{g}^{\sigma\rho}$$

$$-\frac{1}{4}\frac{4D\Lambda^2}{(D-2)^2}\bar{g}_{\mu\nu} = -2\beta\frac{\Lambda^2(D-4)}{(D-2)^2}\bar{g}_{\mu\nu}\,.$$

Adding all these terms and equating the sum to zero, one can find

$$-\frac{1}{\kappa}\Lambda\bar{g}_{\mu\nu} - 2D\alpha\frac{\Lambda^2(D-4)}{(D-2)^2}\bar{g}_{\mu\nu} - 2\gamma\frac{(D-4)(D-3)}{(D-2)(D-1)}\Lambda^2\bar{g}_{\mu\nu} - 2\beta\frac{\Lambda^2(D-4)}{(D-2)^2}\bar{g}_{\mu\nu} = 0\,,$$

and

$$-\frac{1}{2\Lambda\kappa} = \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \gamma\frac{(D-4)(D-3)}{(D-2)(D-1)}\,, \tag{6.4}$$

where $\Lambda \neq 0$ [1, 12]. Several comments are in order here: In the string-inspired Einstein-GB model ($\alpha = \beta = 0$ and $\gamma > 0$), only AdS background ($\Lambda < 0$) is allowed (the Einstein constant κ is positive in our conventions). String theory is known to prefer AdS to dS [1, 13] and we can see why this is so in the uncompactified theory. Another interesting limit is the "traceless" theory ($D\alpha = -\beta$), which, in the absence of a γ term, does not allow constant curvature spaces unless the Einstein term is also dropped. For $D = 4$, the γ term drops out, and the pure quadratic theory allows (A)dS solutions with arbitrary Λ. For $D > 4$, (6.4) leaves a two-parameter set (say α, β) of allowed solutions [1].

Now we will linearize the total energy-momentum tensor $T_{\mu\nu}$ to first order in $h_{\mu\nu}$ and define the total energy-momentum tensor $T_{\mu\nu}$ as we did before. With the help of equations in appendix A, the total energy-momentum tensor can be calculated (shown in appendix B)

$$T_{\mu\nu}(h) \quad = \quad T_{\mu\nu}(\bar{g}) + \mathcal{G}^L_{\mu\nu}\left\{\frac{1}{\kappa} + \frac{4\Lambda D\alpha}{(D-2)} + \frac{4\Lambda\beta}{(D-1)} + \frac{4\Lambda\gamma(D-3)(D-4)}{(D-1)(D-2)}\right\}$$

$$
\begin{aligned}
&+(2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\right)R^L \\
&+\beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L\right) \\
&-2\Lambda^2 h_{\mu\nu}\left\{\frac{1}{2\Lambda\kappa}+\frac{(D-4)}{(D-2)^2}(D\alpha+\beta)+\frac{\gamma(D-4)(D-3)}{(D-2)(D-1)}\right\}\ . \qquad (6.5)
\end{aligned}
$$

Using (6.4), one has $T_{\mu\nu}(\bar{g})=0$ and the last term also vanishes, yielding

$$
\begin{aligned}
T_{\mu\nu} \;\; = \;\; & \mathcal{G}^L_{\mu\nu}\left\{-\frac{1}{\kappa}+\frac{4\Lambda D}{(D-2)^2}(2\alpha+\frac{\beta}{(D-1)})\right\} \\
&+(2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}\right)R^L \\
&+\beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L\right)\ . \qquad (6.6)
\end{aligned}
$$

This is a background conserved tensor ($\bar{\nabla}^\mu T_{\mu\nu}=0$) and it is checked explicitly in appendix B. An important aspect of (6.6) is the sign change of the $\frac{1}{\kappa}$ term relative to Einstein theory, due to the GB contributions. Hence in the Einstein-GB limit, we have $T_{\mu\nu} = -{\mathcal{G}_{\mu\nu}}^L/\kappa$, with the overall sign exactly opposite to that of the cosmological Einstein theory. However, this does not mean that E is negative there [1, 14].

There remains now to obtain a Killing energy expression from (6.6), namely, to write $\bar{\xi}_\nu T^{\mu\nu}$ as a surface integral. The first term is the usual AD piece (4.21), which we derived in chapter 4. The term in the middle (which has the coefficient $2\alpha+\beta$), is easy to handle. First we take the indices up and than operate on this equation with a Killing vector, say $\bar{\xi}_\nu$,

$$
\bar{\xi}^\mu\bar{\Box}R^L-\bar{\xi}^\nu\bar{\nabla}^\mu\bar{\nabla}_\nu R^L+\frac{2\Lambda}{(D-2)}\bar{\xi}^\mu R_L. \qquad (6.7)
$$

In the first term the covariant derivative must be taken outside to get surface terms:

$$\begin{aligned}
\bar{\xi}^{\mu}\bar{\nabla}_{\alpha}\bar{\nabla}^{\alpha}R_L &= \bar{\nabla}_{\alpha}(\bar{\xi}^{\mu}\bar{\nabla}^{\alpha}R_L) - (\bar{\nabla}_{\alpha}\bar{\xi}^{\mu})(\bar{\nabla}^{\alpha}R_L) \\
&= \bar{\nabla}_{\alpha}(\bar{\xi}^{\mu}\bar{\nabla}^{\alpha}R_L) - \bar{\nabla}_{\alpha}(R_L\bar{\nabla}^{\alpha}\bar{\xi}^{\mu}) + R_L(\bar{\Box}\bar{\xi}^{\mu}) \\
&= \bar{\nabla}_{\alpha}\{\bar{\xi}^{\mu}\bar{\nabla}^{\alpha}R_L - R_L\bar{\nabla}^{\alpha}\bar{\xi}^{\mu}\} - \frac{2\Lambda}{(D-2)}R_L\bar{\xi}^{\mu} \,.
\end{aligned}$$

In the second term of (6.7), we can easily change the places of covariant and contravariant derivatives because of the Ricci scalar R_L, that is

$$\bar{\xi}^{\nu}\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}R_L = \bar{\xi}^{\nu}\bar{\nabla}_{\nu}\bar{\nabla}^{\mu}R_L \,,$$

and making the $\nu \to \alpha$ substitution, we have

$$\begin{aligned}
\bar{\xi}^{\alpha}\bar{\nabla}_{\alpha}\bar{\nabla}^{\mu}R_L &= \bar{\nabla}_{\alpha}(\bar{\xi}^{\alpha}\bar{\nabla}^{\mu}R_L) - (\bar{\nabla}_{\alpha}\bar{\xi}^{\alpha})(\bar{\nabla}^{\mu}R_L) \\
&= \bar{\nabla}_{\alpha}(\bar{\xi}^{\alpha}\bar{\nabla}^{\mu}R_L) \,,
\end{aligned}$$

where $\bar{\nabla}_{\alpha}\bar{\xi}^{\alpha}$ is zero because of the Killing equation. Inserting these results into (6.7), the surface terms can be taken out

$$\begin{aligned}
& \bar{\xi}^{\mu}\bar{\Box}R^{L} - \bar{\xi}^{\nu}\bar{\nabla}^{\mu}\bar{\nabla}_{\nu}R^{L} + \frac{2\Lambda}{(D-2)}\bar{\xi}^{\mu}R_L \\
= \;& \bar{\nabla}_{\alpha}\{\bar{\xi}^{\mu}\bar{\nabla}^{\alpha}R_L - R_L\bar{\nabla}^{\alpha}\bar{\xi}^{\mu}\} - \frac{2\Lambda}{(D-2)}R_L\bar{\xi}^{\mu} + \frac{2\Lambda}{(D-2)}R_L\bar{\xi}^{\mu} - \bar{\nabla}_{\alpha}(\bar{\xi}^{\alpha}\bar{\nabla}^{\mu}R_L) \\
= \;& \bar{\nabla}_{\alpha}\{\bar{\xi}^{\mu}\bar{\nabla}^{\alpha}R_L - \bar{\xi}^{\alpha}\bar{\nabla}^{\mu}R_L + R_L\bar{\nabla}^{\mu}\bar{\xi}^{\alpha}\} \,.
\end{aligned}$$

The last term in (6.6) can be written as a surface term plus extra terms:

$$\begin{aligned}
\bar{\xi}_{\nu}\bar{\Box}\mathcal{G}_L^{\mu\nu} &= \bar{\xi}_{\nu}\bar{\nabla}_{\alpha}\bar{\nabla}^{\alpha}\bar{\mathcal{G}}_L^{\mu\nu} \\
&= \bar{\nabla}_{\alpha}\{\bar{\xi}_{\nu}\bar{\nabla}^{\alpha}\mathcal{G}_L^{\mu\nu}\} - (\bar{\nabla}_{\alpha}\bar{\xi}_{\nu})(\bar{\nabla}^{\alpha}\mathcal{G}_L^{\mu\nu}) \,,
\end{aligned}$$

where we have put the Killing vector inside the covariant derivative. In the second term we can freely move the α indices and afterwards, we can also take terms inside the covariant derivative with an extra term. Hence, we get

$$\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} = \bar{\nabla}_\alpha \{ \bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu \} + \mathcal{G}_L^{\mu\nu} \bar{\Box} \bar{\xi}_\nu \,.$$

Now we can add and subtract the terms $\bar{\nabla}_\alpha \{ \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} \}$ and $\bar{\nabla}_\alpha \{ \mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \}$

$$\begin{aligned} \bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} &= \bar{\nabla}_\alpha \{ \bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu + \mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \} \\ &\quad + \mathcal{G}_L^{\mu\nu} \bar{\Box} \bar{\xi}_\nu + \bar{\nabla}_\alpha \{ \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} \} - \bar{\nabla}_\alpha \{ \mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \}. \end{aligned}$$

If we expand the last two terms we can see that: (i) When the covariant derivative hits on the Killing vector $\bar{\xi}_\nu$, it will be zero in the first one with the use of (2.1), because α and ν are symmetric in $\mathcal{G}_L^{\alpha\nu}$. (ii) With the help of Bianchi identity (4.19), the term $(\bar{\nabla}_\alpha \mathcal{G}_L^{\alpha\nu})(\bar{\nabla}^\mu \bar{\xi}_\nu)$ is zero. Hence we are left with

$$\begin{aligned} \bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu} &= \bar{\nabla}_\alpha \{ \bar{\xi}_\nu \bar{\nabla}^\alpha \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^\alpha \bar{\xi}_\nu + \mathcal{G}_L^{\alpha\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \} \\ &\quad + \mathcal{G}_L^{\mu\nu} \bar{\Box} \bar{\xi}_\nu + \bar{\xi}_\nu \bar{\nabla}_\alpha \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\alpha\nu} \bar{\nabla}_\alpha \bar{\nabla}^\mu \bar{\xi}_\nu \,. \end{aligned} \tag{6.8}$$

Using (B.1), (B.2) and (B.3), we can write $\bar{\xi}_\nu \bar{\Box} \mathcal{G}_L^{\mu\nu}$ as a surface term. Collecting everything, the final form of the conserved charges for the generic quadratic theory reads

$$\begin{aligned} Q^\mu(\bar{\xi}) &= \{ -\frac{1}{\kappa} + \frac{8\Lambda}{(D-2)^2}(\alpha D + \beta) \} \int d^{D-1}x \sqrt{-\bar{g}}\, \bar{\xi}_\nu \, \mathcal{G}_L^{\mu\nu} \\ &\quad + (2\alpha + \beta) \int dS_i \sqrt{-\bar{g}} \{ \bar{\xi}^\mu \bar{\nabla}^i R_L - \bar{\xi}^i \bar{\nabla}^\mu R_L + R_L \bar{\nabla}^\mu \bar{\xi}^i \} \\ &\quad + \beta \int dS_i \sqrt{-\bar{g}} \{ \bar{\xi}_\nu \bar{\nabla}^i \mathcal{G}_L^{\mu\nu} - \bar{\xi}_\nu \bar{\nabla}^\mu \mathcal{G}_L^{i\nu} - \mathcal{G}_L^{\mu\nu} \bar{\nabla}^i \bar{\xi}_\nu + \mathcal{G}_L^{i\nu} \bar{\nabla}^\mu \bar{\xi}_\nu \}. \end{aligned} \tag{6.9}$$

Now let us compute the energy of an asymptotically SdS geometry that might be a solution to our generic model. Should such a solution exist, we only require its asymptotic behavior to be

$$h_{00} \approx + \left(\frac{r_0}{r}\right)^{D-3}, \quad h^{rr} \approx + \left(\frac{r_0}{r}\right)^{D-3} + O(r_0^2). \tag{6.10}$$

It is easy to see that for asymptotically SdS spaces the second and the third lines of (6.9) do not contribute, since for any Einstein space, to linear order

$$R^L_{\mu\nu} = \frac{2\Lambda}{D-2} h_{\mu\nu}, \tag{6.11}$$

which in turn yields $R_L = \bar{g}^{\mu\nu} R^L_{\mu\nu} - [2\Lambda/(D-2)]h = 0$ and thus $\mathcal{G}^L_{\mu\nu} = 0$ in the asymptotic region. Therefore the total energy of the full (α, β, γ) system, for geometries that are asymptotically SdS, is given only by the first term in (6.9),

$$E_D = \left\{ -1 + \frac{8\Lambda\kappa}{(D-2)^2}(\alpha D + \beta) \right\} \frac{(D-2)}{4G_D} r_0^{D-3}, \quad D > 4, \tag{6.12}$$

where γ is implicitly assumed not to vanish. For $D = 4$, equivalently from (6.5), it reads (for models with an explicit Λ)

$$E_4 = \{1 + 2\Lambda\kappa(4\alpha + \beta)\} \frac{r_0}{2G_4} \quad [1]. \tag{6.13}$$

In $D = 3$, the GB density vanishes identically and the energy expression has the same form of the $D = 4$ model, with the difference that r_0 comes from the metric (5.8) [1].

From (6.12), the asymptotically SdS solution seemingly has negative energy, in the Einstein-GB model:

$$E = -\frac{(D-2)}{4G_D} r_0^{D-3}. \tag{6.14}$$

While this is, of course, correct in terms of the usual SdS signs, their exact form is [14]

$$ds^2 = g_{00} dt^2 + g_{rr} dr^2 + r^2 d\Omega_{D-2}\,, \tag{6.15}$$

$$\begin{aligned} -g_{00} = g_{rr}^{-1} &= 1 + \frac{r^2}{4\kappa\gamma(D-3)(D-4)} \\ &\times \left(1 \pm \left[1 + 8\gamma(D-3)(D-4)\frac{r_0^{D-3}}{r^{D-1}}\right]^{\frac{1}{2}}\right) . \end{aligned} \tag{6.16}$$

Note that there is a branching here, with qualitatively different asymptotics: Schwarzschild and Schwarzschild-AdS,

$$\begin{aligned} -g_{00} &= 1 - \left(\frac{r_0}{r}\right)^{D-3}, \\ &= 1 + \left(\frac{r_0}{r}\right)^{D-3} + \frac{r^2}{\kappa\gamma(D-3)(D-4)} . \end{aligned} \tag{6.17}$$

[Here we restored γ, using $\kappa\gamma(D-3)(D-4) = -l^2$.] The first solution has the usual positive (for positive r_0 of course) ADM energy $E = +(D-2)r_0^{D-3}/4G_D$, since the GB term does not contribute when expanded around flat space. On the other hand, the second solution which is asymptotically SdS, has the wrong sign for the "mass term". However, to actually compute the energy here, one needs our energy expression (6.9), and not simply the AD formula which is valid only for cosmological Einstein theory. Now from (6.17), we have

$$h_{00} \approx -\left(\frac{r_0}{r}\right)^{D-3}, \quad h^{rr} \approx -\left(\frac{r_0}{r}\right)^{D-3} + O(r_0^2)\,, \tag{6.18}$$

whose sign is opposite to that of the usual SdS. This sign just compensates the flipped sign in the energy definition, so the energy (6.12) reads $E = (D-2)r_0^{D-3}/4G_D$ and the AdS branch, just like the flat branch, has positive energy, after the GB effects are taken into account also in the energy definition. Thus, for every Einstein-GB external solution, energy is positive and AdS vacuum is stable [1, 14].

CHAPTER 7

CONCLUSION

We have defined the energy of generic Einstein plus cosmological term plus quadratic gravity theories as well as pure quadratic models in all D, for both asymptotically flat and (A)dS spaces. The higher derivative terms do not change the form of the energy expression in flat backgrounds. On the other hand, for asymptotically (A)dS backgrounds (which are generically solutions to these equations, even in the absence of an explicit cosmological constant), the energy expressions (6.9) essentially reduce to the AD formula [up to higher order corrections that vanish for spacetimes that asymptotically approach (A)dS at least as fast as SdS spaces] [1].

Among quadratic theories, we have studied the string-inspired Einstein-GB model in more detail. This one, in the absence of an explicit cosmological constant, has both flat and AdS vacua. The AdS vacuum has specific cosmological constant and some of these are negative. These constants are determined by the Newton's constant and the GB coefficient. The constants that are negative being fixed from the string expansion to be positive. The explicit spherically symmetric black hole solutions in this theory consist of two branches: asymptotically Schwarzschild spaces with a positive mass parameter or asymptotically Schwarzschild-AdS spaces with a negative one. The asymptotically Schwarzschild branch has the usual positive ADM energy. Using the compensation of two minus signs in the solution and in the correct energy definition, we noted that the AdS branch has likewise positive energy and that the AdS vacuum was a stable zero energy state [1].

In this thesis, out of conserved charges our explicit examples were related to the energy. In fact, with this formalism, we can easily study the angular momenta of black holes in both asymptotically AdS and flat spacetimes.

Once a background Killing vector is given, our formalism provides us with a conserved charge. For the case of angular momentum, we just take (in four dimensions) the Killing vector to be $\bar{\xi}^{\mu} = (0, 0, 0, 1)$. For the details, we refer the reader to [15].

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APPENDIX A

LINEARIZATION EXPRESSIONS FOR PURE QUADRATIC TERMS

In this section, we will carry out the linearization of certain tensors that appear in the equation of motion (6.2).

$$
\begin{aligned}
\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma}) &= (\delta R_{\mu\rho\nu\sigma})\bar{R}^{\rho\sigma} + \bar{R}_{\mu\rho\nu\sigma}\delta(R^{\rho\sigma}) \\
&= (\delta R_{\mu\rho\nu\sigma})\bar{g}^{\alpha\rho}\bar{g}^{\theta\sigma}\bar{R}_{\alpha\theta} + \bar{R}_{\mu\rho\nu\sigma}\delta(g^{\alpha\rho}g^{\theta\sigma}R_{\alpha\theta}) \\
&= (\delta R_{\mu\rho\nu\sigma})\bar{g}^{\alpha\rho}\bar{g}^{\theta\sigma}\frac{2}{(D-2)}\Lambda\bar{g}_{\alpha\theta} \\
&\quad +\bar{R}_{\mu\rho\nu\sigma}\{-(\delta g^{\alpha\rho})\bar{g}^{\theta\sigma}\bar{R}_{\alpha\theta} - \bar{g}^{\alpha\rho}(\delta g^{\theta\sigma})\bar{R}_{\alpha\theta} + \bar{g}^{\alpha\rho}\bar{g}^{\theta\sigma}\delta R_{\alpha\theta}\}
\end{aligned}
$$

Using (4.6) and (4.7), we get

$$
\begin{aligned}
(\delta(R_{\mu\rho\nu\sigma})R^{\rho\sigma}) &= \delta R_{\mu\rho\nu\sigma}\bar{g}^{\alpha\rho}\delta^{\sigma}_{\alpha}\frac{2}{(D-2)}\Lambda + \bar{R}_{\mu\rho\nu\sigma}\left\{-(\delta g^{\alpha\rho})\bar{g}^{\theta\sigma}\frac{2}{(D-2)}\Lambda\bar{g}_{\alpha\theta}\right. \\
&\quad \left. -\,\bar{g}^{\alpha\rho}(\delta g^{\theta\sigma})\frac{2}{(D-2)}\Lambda\bar{g}_{\alpha\theta} + \bar{g}^{\alpha\rho}\bar{g}^{\theta\sigma}\delta R_{\alpha\theta}\right\} \\
&= (\delta R_{\mu\rho\nu\alpha})\bar{g}^{\alpha\rho}\frac{2}{(D-2)}\Lambda \\
&\quad +\bar{R}_{\mu\rho\nu\sigma}\left\{-(\delta g^{\sigma\rho})\frac{2}{(D-2)}\Lambda - (\delta g^{\rho\sigma})\frac{2}{(D-2)}\Lambda + \bar{g}^{\alpha\rho}\bar{g}^{\theta\sigma}\delta R_{\alpha\theta}\right\}.
\end{aligned}
$$

Inserting (4.5) in $\bar{R}_{\mu\rho\nu\sigma}$

$$\begin{aligned}\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma}) &= \frac{2}{(D-2)}\Lambda R^L_{\mu\nu} - \frac{4\Lambda^2}{(D-1)(D-2)^2}(h_{\mu\nu} - \bar{g}_{\mu\nu}h) \\ &+ \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\rho\nu}) \\ &\times \left\{\delta g^{\sigma\rho}\frac{4\Lambda}{(D-2)} + \bar{g}^{\alpha\rho}\bar{g}^{\theta\sigma}\delta R_{\alpha\theta}\right\},\end{aligned}$$

expanding the parentheses in the second line we get

$$\begin{aligned}\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma}) &= \frac{2}{(D-2)}\Lambda R^L_{\mu\nu} - \frac{4\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} + \frac{4\Lambda^2}{(D-1)(D-2)^2}\bar{g}_{\mu\nu}h \\ &- \frac{8\Lambda^2}{(D-1)(D-2)^2}\bar{g}_{\mu\nu}h + \frac{8\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} \\ &+ \frac{2\Lambda}{(D-1)(D-2)}R^L_{\alpha\theta}(\bar{g}_{\mu\nu}\bar{g}^{\theta\sigma}\delta^\alpha_\sigma - \delta^\theta_\mu\delta^\rho_\nu).\end{aligned}$$

By inserting the definition of $h = \bar{g}_{\alpha\theta}h^{\alpha\theta}$

$$\begin{aligned}\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma}) &= \frac{2}{(D-2)}\Lambda R^L_{\mu\nu} + \frac{4\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} \\ &- \frac{4\Lambda^2}{(D-1)(D-2)^2}\bar{g}_{\mu\nu}h^{\alpha\theta}\bar{g}_{\alpha\theta} + \frac{2\Lambda}{(D-1)(D-2)}R^L_{\alpha\theta}\bar{g}_{\mu\nu}\bar{g}^{\alpha\theta} \\ &- \frac{2\Lambda}{(D-1)(D-2)}R^L_{\mu\nu} \\ &= \frac{(R^L_{\mu\nu}2\Lambda(D-1) - 2\Lambda R^L_{\mu\nu})}{(D-1)(D-2)} + \frac{4\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} \\ &- \frac{2}{(D-2)}\Lambda\bar{g}_{\alpha\theta}\frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}h^{\alpha\theta} \\ &+ \frac{2\Lambda}{(D-1)(D-2)}R^L_{\alpha\theta}\bar{g}_{\mu\nu}\bar{g}^{\alpha\theta},\end{aligned}$$

where $\frac{2}{(D-2)}\Lambda\bar{g}_{\alpha\theta} = \bar{R}_{\alpha\theta}$,

$$\begin{aligned}\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma}) &= \frac{2\Lambda(D-2)}{(D-1)(D-2)}R^L_{\mu\nu} \\ &+ \frac{4\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} + \frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}(R^L_{\alpha\theta}\bar{g}^{\theta\alpha} - \bar{R}_{\alpha\theta}h^{\alpha\theta})\end{aligned}$$

and the last term in the parentheses is equal to R^L. We end up with

$$\delta(R_{\mu\rho\nu\sigma}R^{\rho\sigma}) = \frac{2\Lambda}{(D-1)}R^L_{\mu\nu} + \frac{4\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} + \frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}R^L\,. \tag{A.1}$$

Now we will linearize $R_{\mu\rho\sigma\alpha}R_\nu{}^{\rho\sigma\alpha}$:

$$\delta(R_{\mu\rho\sigma\alpha}R_\nu{}^{\rho\sigma\alpha}) = (\delta R_{\mu\rho\sigma\alpha})\bar{R}_\nu{}^{\rho\sigma\alpha} + \bar{R}_{\mu\rho\sigma\alpha}\delta R_\nu{}^{\rho\sigma\alpha}\,.$$

We follow the same path here

$$\begin{aligned}\delta(R_{\mu\rho\sigma\alpha}R_\nu{}^{\rho\sigma\alpha}) &= (\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\rho\beta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\eta}\bar{R}_{\nu\beta\theta\eta} + \bar{R}_{\mu\rho\sigma\alpha}\delta(g^{\rho\beta}g^{\sigma\theta}g^{\alpha\eta}R_{\nu\beta\theta\eta})\\ &= \frac{2\Lambda}{(D-1)(D-2)}(\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\rho\beta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\eta}(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta})\\ &\quad -\bar{R}_{\mu\rho\sigma\alpha}(\delta g^{\rho\beta})\bar{g}^{\sigma\theta}\bar{g}^{\alpha\eta}\bar{R}_{\nu\beta\theta\eta} - \bar{R}_{\mu\rho\sigma\alpha}\bar{g}^{\rho\beta}(\delta g^{\sigma\theta})\bar{g}^{\alpha\eta}\bar{R}_{\nu\beta\theta\eta}\\ &\quad -\bar{R}_{\mu\rho\sigma\alpha}\bar{g}^{\rho\beta}\bar{g}^{\sigma\theta}(\delta g^{\alpha\eta})\bar{R}_{\nu\beta\theta\eta} + \bar{R}_{\mu\rho\sigma\alpha}\bar{g}^{\rho\beta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\eta}\delta R_{\nu\beta\theta\eta}\,.\end{aligned}$$

Using (4.5), we have

$$\begin{aligned}\delta(R_{\mu\rho\sigma\alpha}R_\nu{}^{\rho\sigma\alpha}) &= \frac{2\Lambda}{(D-1)(D-2)}\{(\delta R_{\mu\rho\sigma\alpha})(\delta^\sigma_\nu\,\delta^\rho_\eta\,\bar{g}^{\alpha\eta} - \delta^\alpha_\nu\,\delta^\rho_\theta\,\bar{g}^{\sigma\theta})\\ &\quad +(\bar{g}_{\mu\sigma}\bar{g}_{\rho\alpha} - \bar{g}_{\mu\alpha}\bar{g}_{\rho\sigma})\bar{g}^{\rho\beta}(\delta g^{\sigma\theta})\bar{g}^{\alpha\eta}\bar{R}_{\nu\beta\theta\eta}\}\\ &\quad -\frac{4\Lambda^2}{(D-1)^2(D-2)^2}\\ &\qquad \times\{(\delta g^{\rho\beta})\bar{g}^{\sigma\theta}\bar{g}^{\alpha\eta}(\bar{g}_{\mu\sigma}\bar{g}_{\rho\alpha} - \bar{g}_{\mu\alpha}\bar{g}_{\rho\sigma})(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta})\\ &\quad +\bar{g}^{\rho\beta}(\delta g^{\sigma\theta})\bar{g}^{\alpha\eta}(\bar{g}_{\mu\sigma}\bar{g}_{\rho\alpha} - \bar{g}_{\mu\alpha}\bar{g}_{\rho\sigma})(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta})\\ &\quad +\bar{g}^{\rho\beta}\bar{g}^{\sigma\theta}(\delta g^{\alpha\eta})(\bar{g}_{\mu\sigma}\bar{g}_{\rho\alpha} - \bar{g}_{\mu\alpha}\bar{g}_{\rho\sigma})(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta})\}\,,\end{aligned}$$

expanding the parentheses we get

$$\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R_{\nu}{}^{\rho\sigma\alpha}) &= \frac{2\Lambda}{(D-1)(D-2)}\{(\delta R_{\mu\eta\nu\alpha})\bar{g}^{\alpha\eta} - (\delta R_{\mu\theta\sigma\nu})\bar{g}^{\theta\sigma} \\
&+\delta^{\beta}_{\alpha}\,\delta^{\theta}_{\mu}\,\bar{g}^{\alpha\eta}\,\delta R_{\nu\beta\theta\eta} - \delta^{\eta}_{\mu}\,\delta^{\beta}_{\sigma}\,\bar{g}^{\sigma\theta}\,\delta R_{\nu\beta\theta\eta}\} \\
&-\frac{4\Lambda^2}{(D-1)^2(D-2)^2}\{\delta g^{\rho\beta}(\delta^{\theta}_{\mu}\delta^{\eta}_{\rho} - \delta^{\eta}_{\mu}\delta^{\theta}_{\rho})(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta}) \\
&+\delta g^{\sigma\theta}(\bar{g}_{\rho\beta}\bar{g}_{\mu\sigma}\delta^{\eta}_{\rho} - \delta^{\beta}_{\sigma}\delta^{\eta}_{\mu})(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta}) \\
&+\delta g^{\alpha\eta}(\delta^{\theta}_{\mu}\delta^{\beta}_{\alpha} - \bar{g}^{\sigma\theta}\bar{g}_{\mu\alpha}\delta^{\beta}_{\sigma})(\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \bar{g}_{\nu\eta}\bar{g}_{\beta\theta})\}\,.
\end{aligned}$$

We rewrite the indices according to the Kronecker delta factors

$$\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R_{\nu}{}^{\rho\sigma\alpha}) &= \frac{2\Lambda}{(D-1)(D-2)}\{(\delta R_{\mu\eta\nu\alpha})\bar{g}^{\alpha\eta} + (\delta R_{\mu\theta\nu\sigma})\bar{g}^{\theta\sigma} \\
&+(\delta R_{\nu\alpha\mu\eta})\bar{g}^{\alpha\eta} - \bar{g}^{\sigma\theta}\delta R_{\nu\sigma\theta\mu}\} \\
&-\frac{4\Lambda^2}{(D-1)^2(D-2)^2}\{\delta g^{\rho\beta}(\delta^{\theta}_{\mu}\,\delta^{\eta}_{\rho}\,\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \delta^{\theta}_{\mu}\,\delta^{\eta}_{\rho}\,\bar{g}_{\nu\eta}\bar{g}_{\beta\theta} \\
&-\delta^{\eta}_{\mu}\,\delta^{\theta}_{\rho}\,\bar{g}_{\nu\eta}\bar{g}_{\beta\eta} + \delta^{\eta}_{\mu}\,\delta^{\theta}_{\rho}\,\bar{g}_{\nu\eta}\bar{g}_{\beta\theta}) \\
&+\delta g^{\sigma\theta}(\delta^{\eta}_{\rho}\,\delta^{\rho}_{\eta}\,\bar{g}_{\mu\sigma}\bar{g}_{\nu\theta} - \delta^{\eta}_{\rho}\,\delta^{\rho}_{\theta}\,\bar{g}_{\mu\sigma}\bar{g}_{\nu\eta} - \delta^{\beta}_{\sigma}\,\delta^{\eta}_{\mu}\,\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} + \delta^{\beta}_{\sigma}\,\delta^{\eta}_{\mu}\,\bar{g}_{\nu\eta}\bar{g}_{\beta\theta}) \\
&+\delta g^{\alpha\eta}(\delta^{\theta}_{\mu}\,\delta^{\beta}_{\alpha}\,\bar{g}_{\nu\theta}\bar{g}_{\beta\eta} - \delta^{\theta}_{\mu}\,\delta^{\beta}_{\alpha}\,\bar{g}_{\nu\eta}\bar{g}_{\beta\theta} - \delta^{\beta}_{\sigma}\,\delta^{\sigma}_{\nu}\,\bar{g}_{\mu\alpha}\bar{g}_{\beta\eta} + \delta^{\beta}_{\sigma}\,\delta^{\sigma}_{\beta}\,\bar{g}_{\mu\alpha}\bar{g}_{\nu\eta})\} \\
&= \frac{8\Lambda}{(D-1)(D-2)}\{R^{L}_{\mu\nu} - \frac{2\Lambda}{(D-1)(D-2)}(h_{\mu\nu} - \bar{g}_{\mu\nu}h\} \\
&-\frac{4\Lambda}{(D-1)(D-2)}(4h\bar{g}_{\mu\nu} - 6h_{\mu\nu} + 2Dh_{\mu\nu})
\end{aligned}$$

collecting terms, we end up with

$$\delta(R_{\mu\rho\sigma\alpha}R_{\nu}{}^{\rho\sigma\alpha}) = \frac{8\Lambda}{(D-1)(D-2)}R^{L}_{\mu\nu} - \frac{8\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu}\,. \tag{A.2}$$

The calculations for the Riemann tensor squared are similar but longer. We start as usual

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) &= (\delta R_{\mu\rho\sigma\alpha}\bar{R}^{\mu\rho\sigma\alpha} + \bar{R}_{\mu\rho\sigma\alpha}\delta R^{\mu\rho\sigma\alpha} \\
&= (\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\bar{R}_{\beta\eta\theta\nu} + \bar{R}_{\mu\rho\sigma\alpha}\delta(g^{\mu\beta}g^{\rho\eta}g^{\sigma\theta}g^{\alpha\nu}R_{\beta\eta\theta\nu}) \\
&= (\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\beta\theta}\bar{g}_{\eta\nu} - \bar{g}_{\beta\nu}\bar{g}_{\eta\theta}) \\
&\quad -\bar{R}_{\mu\rho\sigma\alpha}\{(\delta g^{\mu\beta})\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\bar{R}_{\beta\eta\theta\nu} + \bar{g}^{\mu\beta}(\delta g^{\rho\eta})\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\bar{R}_{\beta\eta\theta\nu} \\
&\quad +\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}(\delta g^{\sigma\theta})\bar{g}^{\alpha\nu}\bar{R}_{\beta\eta\theta\nu} + \bar{g}^{\mu\beta}\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}(\delta g^{\alpha\nu})\bar{R}_{\beta\eta\theta\nu}\} \\
&\quad +\bar{R}_{\mu\rho\sigma\alpha}\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\delta R_{\beta\eta\theta\nu}\,.
\end{aligned}
$$

Again we insert (4.5), except for the first one which will be done later,

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) &= \delta R_{\mu\rho\sigma\alpha}\frac{2\Lambda}{(D-2)(D-1)}(\delta^{\mu}_{\theta}\delta^{\rho}_{\nu}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu} - \delta^{\mu}_{\nu}\delta^{\rho}_{\theta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}) \\
&\quad -\bar{R}_{\mu\rho\sigma\alpha}\{(\delta g^{\mu\beta})\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\beta\theta}\bar{g}_{\eta\nu} - \bar{g}_{\beta\nu}\bar{g}_{\eta\theta}) \\
&\quad +\bar{g}^{\mu\beta}(\delta g^{\rho\eta})\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\beta\theta}\bar{g}_{\eta\nu} - \bar{g}_{\beta\nu}\bar{g}_{\eta\theta}) \\
&\quad +\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}(\delta g^{\sigma\theta})\bar{g}^{\alpha\nu}\frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\beta\theta}\bar{g}_{\eta\nu} - \bar{g}_{\beta\nu}\bar{g}_{\eta\theta}) \\
&\quad +\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}(\delta g^{\alpha\nu})\frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\beta\theta}\bar{g}_{\eta\nu} - \bar{g}_{\beta\nu}\bar{g}_{\eta\theta})\} \\
&\quad +\frac{2\Lambda}{(D-2)(D-1)}(\bar{g}_{\mu\sigma}\bar{g}_{\rho\alpha} - \bar{g}_{\mu\alpha}\bar{g}_{\rho\sigma})\bar{g}^{\mu\beta}\bar{g}^{\rho\eta}\bar{g}^{\sigma\theta}\bar{g}^{\alpha\nu}\delta R_{\beta\eta\theta\nu}\,.
\end{aligned}
$$

Expanding the parentheses and renaming the indices we get

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) &= \frac{2\Lambda}{(D-2)(D-1)}\{(\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\sigma\mu}\bar{g}^{\alpha\rho} - (\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\sigma\rho}\bar{g}^{\alpha\mu} \\
&\quad +(\delta R_{\beta\eta\theta\nu})\bar{g}^{\beta\theta}\bar{g}^{\eta\nu} - (\delta R_{\beta\eta\theta\nu})\bar{g}^{\eta\theta}\bar{g}^{\beta\nu}\} \\
&\quad -\bar{R}_{\mu\rho\sigma\alpha}\frac{2\Lambda}{(D-2)(D-1)}\{(\delta g^{\mu\sigma})\bar{g}^{\rho\alpha} - (\delta g^{\mu\alpha})\bar{g}^{\rho\sigma} + (\delta g^{\rho\alpha})\bar{g}^{\sigma\mu} \\
&\quad -(\delta g^{\rho\sigma})\bar{g}^{\alpha\mu} + (\delta g^{\sigma\mu})\bar{g}^{\alpha\rho} - (\delta g^{\sigma\rho})\bar{g}^{\alpha\mu} + (\delta g^{\alpha\rho})\bar{g}^{\mu\sigma} - (\delta g^{\alpha\mu})\bar{g}^{\sigma\rho}\}\,.
\end{aligned}
$$

Now inserting (4.5) in for the background Riemann tensor

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) \;=\;& \frac{2\Lambda}{(D-2)(D-1)}\{(\delta R_{\mu\rho\sigma\alpha})\bar{g}^{\sigma\mu}\bar{g}^{\alpha\rho} + (\delta R_{\mu\rho\alpha\sigma})\bar{g}^{\sigma\rho}\bar{g}^{\alpha\mu} \\
& +(\delta R_{\beta\eta\theta\nu})\bar{g}^{\beta\theta}\bar{g}^{\eta\nu} + (\delta R_{\beta\eta\nu\theta})\bar{g}^{\eta\theta}\bar{g}^{\beta\nu}\} \\
& -\frac{8\Lambda^2}{(D-2)^2(D-1)^2}(\bar{g}_{\mu\sigma}\bar{g}_{\rho\alpha} - \bar{g}_{\mu\alpha}\bar{g}_{\rho\sigma})\{(\delta g^{\mu\sigma})\bar{g}^{\rho\alpha} - (\delta g^{\mu\alpha})\bar{g}^{\rho\sigma} \\
& +(\delta g^{\rho\alpha})\bar{g}^{\sigma\mu} - (\delta g^{\rho\sigma})\bar{g}^{\alpha\mu}\}\,.
\end{aligned}
$$

Collecting terms we get

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) \;=\;& \frac{2\Lambda}{(D-2)(D-1)}\left\{\bar{g}^{\sigma\mu}\left(R^L_{\mu\sigma} - \frac{2\Lambda}{(D-2)(D-1)}(h_{\mu\sigma} - \bar{g}_{\mu\sigma})\right)\right. \\
& + \bar{g}^{\alpha\mu}\left(R^L_{\mu\alpha} - \frac{2\Lambda}{(D-2)(D-1)}(h_{\mu\alpha} - \bar{g}_{\mu\alpha})\right) \\
& + \bar{g}^{\beta\theta}\left(R^L_{\beta\theta} - \frac{2\Lambda}{(D-2)(D-1)}(h_{\beta\theta} - \bar{g}_{\beta\theta})\right) \\
& \left. + \bar{g}^{\beta\nu}\left(R^L_{\beta\nu} - \frac{2\Lambda}{(D-2)(D-1)}(h_{\beta\nu} - \bar{g}_{\beta\nu})\right)\right\} \\
& -\frac{8\Lambda^2}{(D-2)^2(D-1)^2} \\
& \quad\times\{hD - h - h + hD + hD - h - h + hD\}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) \;=\;& \frac{2\Lambda}{(D-2)(D-1)}\left\{\bar{g}^{\sigma\mu}R^L_{\mu\sigma} + \frac{2\Lambda}{(D-2)}h + \bar{g}^{\alpha\mu}R^L_{\alpha\mu}\right. \\
& \left. +\frac{2\Lambda}{(D-2)}h + \bar{g}^{\beta\theta}R^L_{\beta\theta} + \frac{2\Lambda}{(D-2)}h + \bar{g}^{\beta\nu}R^L_{\nu\beta} + \frac{2\Lambda}{(D-2)}h\right\} \\
& -\frac{32\Lambda^2}{(D-2)^2(D-1)}h\,.
\end{aligned}
$$

To get the linear form of the Ricci scalar, we remember that $h = \bar{g}^{\mu\nu}h_{\mu\nu}$ and write it in a suitable form

$$
\begin{aligned}
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) &= \frac{2\Lambda}{(D-2)(D-1)}\Big\{\bar{g}^{\sigma\mu}R^L_{\mu\sigma} + \bar{g}^{\alpha\mu}R^L_{\alpha\mu} + \bar{g}^{\beta\theta}R^L_{\beta\theta} \\
&\quad + \bar{g}^{\beta\nu}R^L_{\nu\beta} + \frac{8\Lambda}{(D-2)}h\Big\} - \frac{32\Lambda^2}{(D-2)^2(D-1)}h \\
&= \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\sigma\mu}R^L_{\mu\sigma} - \frac{4\Lambda^2}{(D-2)^2(D-1)}h \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\alpha\mu}R^L_{\mu\alpha} - \frac{4\Lambda^2}{(D-2)^2(D-1)}h \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\beta\theta}R^L_{\beta\theta} - \frac{4\Lambda^2}{(D-2)^2(D-1)}h \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\beta\nu}R^L_{\beta\nu} - \frac{4\Lambda^2}{(D-2)^2(D-1)}h \\
&= \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\sigma\mu}R^L_{\mu\sigma} - \frac{2\Lambda}{(D-2)(D-1)}\frac{2\Lambda}{(D-2)}\bar{g}_{\mu\sigma}h^{\mu\sigma} \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\alpha\mu}R^L_{\alpha\mu} - \frac{2\Lambda}{(D-2)(D-1)}\frac{2\Lambda}{(D-2)}\bar{g}_{\mu\alpha}h^{\mu\alpha} \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\beta\theta}R^L_{\beta\theta} - \frac{2\Lambda}{(D-2)(D-1)}\frac{2\Lambda}{(D-2)}\bar{g}_{\beta\theta}h^{\beta\theta} \\
&\quad + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}^{\beta\nu}R^L_{\beta\nu} - \frac{2\Lambda}{(D-2)(D-1)}\frac{2\Lambda}{(D-2)}\bar{g}_{\beta\nu}h^{\beta\nu} \\
&= \frac{2\Lambda}{(D-2)(D-1)}\Big\{(\bar{g}^{\sigma\mu}R^L_{\mu\sigma} - \bar{R}_{\mu\sigma}h^{\mu\sigma}) + (\bar{g}^{\alpha\mu}R^L_{\mu\alpha} - \bar{R}_{\mu\alpha}h^{\mu\alpha}) \\
&\quad + (\bar{g}^{\beta\theta}R^L_{\beta\theta} - \bar{R}_{\beta\theta}h^{\beta\theta}) + (\bar{g}^{\beta\nu}R^L_{\beta\nu} - \bar{R}_{\beta\nu}h^{\beta\nu})\Big\} \, .
\end{aligned}
$$

All the terms in parentheses are equal to the linear form of the Ricci scalar, finally we have

$$
\delta(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}) = \frac{8\Lambda}{(D-2)(D-1)}R^L \, . \tag{A.3}
$$

Let us calculate $\delta(RR_{\mu\nu})$, that is,

$$
\delta(RR_{\mu\nu}) = (\delta R)\bar{R}_{\mu\nu} + \bar{R}\delta R_{\mu\nu} \, .
$$

Using (4.6) and (4.7), we find

$$\delta(RR_{\mu\nu}) = \frac{2D\Lambda}{(D-2)} R^L_{\mu\nu} + \frac{2\Lambda}{(D-2)} \bar{g}_{\mu\nu} R_L \,. \tag{A.4}$$

The $R^\sigma{}_\mu R_{\nu\sigma}$ term can be linearized as follows:

$$\delta(R^\sigma{}_\mu R_{\nu\sigma}) = \delta(g^{\sigma\theta} R_{\theta\mu} R_{\nu\sigma}) = (\delta R_{\theta\mu}) \bar{R}_{\nu\sigma} \bar{g}^{\sigma\theta} + \bar{R}_{\theta\mu} (\delta R_{\nu\sigma}) \bar{g}^{\sigma\theta} - \bar{R}_{\theta\mu} \bar{R}_{\nu\sigma} \delta g^{\sigma\theta} \,,$$

using the same procedure we get it as

$$\begin{aligned}
\delta(R^\sigma{}_\mu R_{\nu\sigma}) &= (\delta R_{\theta\mu}) \bar{g}^{\sigma\theta} \frac{2\Lambda}{(D-2)} \bar{g}_{\nu\sigma} + \bar{g}_{\theta\mu} (\delta R_{\nu\sigma}) \bar{g}^{\sigma\theta} \frac{2\Lambda}{(D-2)} \\
&\quad - \bar{g}_{\theta\mu} \bar{g}_{\nu\sigma} \delta g^{\sigma\theta} \frac{4\Lambda^2}{(D-2)^2} \\
&= \frac{4\Lambda}{(D-2)} R^L_{\mu\nu} - \frac{4\Lambda^2}{(D-2)^2} h_{\mu\nu} \,.
\end{aligned} \tag{A.5}$$

For linearizing $R^2_{\mu\nu}$, the same procedure can be applied

$$\begin{aligned}
\delta R^2_{\mu\nu} &= \delta(R^{\mu\nu} R_{\mu\nu}) = \delta(g^{\mu\theta} g^{\nu\beta} R_{\theta\beta} R_{\mu\nu}) \\
&= (\delta R_{\theta\beta}) \bar{g}^{\mu\theta} \bar{g}^{\nu\beta} \bar{R}_{\mu\nu} + \bar{R}_{\theta\beta} \bar{g}^{\mu\theta} \bar{g}^{\nu\beta} (\delta R_{\mu\nu}) - \bar{R}_{\theta\beta} (\delta g^{\mu\theta}) \bar{g}^{\nu\beta} \bar{R}_{\mu\nu} \\
&\quad - \bar{R}_{\theta\beta} \bar{g}^{\mu\theta} (\delta g^{\nu\beta}) \bar{R}_{\mu\nu} \\
&= (\delta R_{\theta\beta}) \bar{g}^{\mu\theta} \bar{g}^{\nu\beta} \bar{g}_{\mu\nu} \frac{2\Lambda}{(D-2)} + \bar{g}_{\theta\beta} \bar{g}^{\mu\theta} \bar{g}^{\nu\beta} (\delta R_{\mu\nu}) \frac{2\Lambda}{(D-2)} \\
&\quad - \bar{g}_{\theta\beta} (\delta g^{\mu\theta}) \bar{g}^{\nu\beta} \bar{g}_{\mu\nu} \frac{4\Lambda^2}{(D-2)^2} - \bar{g}_{\theta\beta} (\delta g^{\nu\beta}) \bar{g}^{\mu\theta} \bar{g}_{\mu\nu} \frac{4\Lambda^2}{(D-2)^2} \\
&= \frac{2\Lambda}{(D-2)} \{ (\delta R_{\theta\beta}) \delta^\theta_\nu \bar{g}^{\nu\beta} + (\delta R_{\mu\nu}) \delta^\mu_\beta \bar{g}^{\nu\beta} \} \\
&\quad - \frac{4\Lambda^2}{(D-2)^2} \{ (\delta g^{\mu\theta}) \delta^\nu_\theta \bar{g}_{\mu\nu} - (\delta g^{\nu\beta}) \delta^\mu_\beta \bar{g}_{\mu\nu} \} \\
&= \frac{2\Lambda}{(D-2)} \{ (\delta R_{\nu\beta}) \bar{g}^{\nu\beta} - \bar{R}_{\nu\beta} \delta g^{\nu\beta} + (\delta R_{\mu\nu}) \bar{g}^{\nu\mu} - \bar{R}_{\mu\nu} \delta g^{\mu\nu} \} \\
&= \frac{4\Lambda}{(D-2)} R^L \,,
\end{aligned} \tag{A.6}$$

where we have used (4.6).

We know that

$$\delta(R_{\mu\sigma\nu\rho}g^{\sigma\rho}) = (\delta R_{\mu\sigma\nu\rho})\bar{g}^{\sigma\rho} - \bar{R}_{\mu\sigma\nu\rho}\delta g^{\sigma\rho} = R^L_{\mu\nu}\,,$$

and

$$\begin{aligned}(\delta R_{\mu\sigma\nu\rho})\bar{g}^{\sigma\rho} &= R^L_{\mu\nu} + \bar{R}_{\mu\sigma\nu\rho}\delta g^{\sigma\rho} \\ &= R^L_{\mu\nu} + \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\mu\nu}\bar{g}_{\sigma\rho} - \bar{g}_{\mu\rho}\bar{g}_{\sigma\nu})h^{\sigma\rho} \\ &= R^L_{\mu\nu} - \frac{2\Lambda}{(D-1)(D-2)}(h_{\mu\nu} - \bar{g}_{\mu\nu}h)\,. \qquad \text{(A.7)}\end{aligned}$$

With the help of (4.7) we get

$$\delta(R^2) = 2\bar{R}\delta R = 2\bar{R}R^L = \frac{4\Lambda D}{(D-2)}R^L. \qquad \text{(A.8)}$$

Using (8.3), (8.6) and (8.8) we can calculate the linearization of the GB combination as

$$\begin{aligned}\delta(R^2_{\tau\lambda\rho\sigma} - 4R^2_{\sigma\rho} + R^2) &= \frac{8\Lambda}{(D-1)(D-2)}R^L - \frac{16\Lambda}{(D-2)}R^L + \frac{4\Lambda D}{(D-2)}R^L \\ &= \frac{4\Lambda}{(D-1)(D-2)}(D^2-5D+6)R^L \\ &= \frac{4\Lambda}{(D-1)(D-2)}(D-3)(D-2)R^L \\ &= \frac{4\Lambda(D-3)}{(D-1)}R^L\,. \qquad \text{(A.9)}\end{aligned}$$

Let us calculate the GB density of a cosmological space: Using (4.5), the $\bar{R}^2_{\tau\lambda\rho\sigma}$ term can be calculated,

$$\bar{R}^2_{\tau\lambda\rho\sigma} = \bar{R}_{\tau\lambda\rho\sigma}\bar{R}^{\tau\lambda\rho\sigma} = \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\tau\rho}\bar{g}_{\lambda\sigma} - \bar{g}_{\tau\sigma}\bar{g}_{\lambda\rho})\bar{g}^{\tau\theta}\bar{g}^{\lambda\beta}\bar{g}^{\rho\gamma}\bar{g}^{\sigma\alpha}\bar{R}_{\theta\beta\gamma\alpha}$$

$$
\begin{aligned}
&= \frac{4\Lambda^2}{(D-1)^2(D-2)^2}(\delta^\theta_\rho\,\delta^\beta_\sigma\,\bar{g}^{\rho\gamma}\bar{g}^{\sigma\alpha} - \delta^\theta_\sigma\,\delta^\beta_\rho\,\bar{g}^{\rho\gamma}\bar{g}^{\sigma\alpha})(\bar{g}_{\theta\gamma}\bar{g}_{\beta\alpha} - \bar{g}_{\theta\alpha}\bar{g}_{\beta\gamma}) \\
&= \frac{4\Lambda^2}{(D-1)^2(D-2)^2}(D^2 - \delta^\gamma_\alpha\,\delta^\alpha_\gamma - \delta^\theta_\beta\,\delta^\beta_\theta + D^2) \\
&= \frac{4\Lambda^2}{(D-1)^2(D-2)^2}2D(D-1) = \frac{8\Lambda^2}{(D-1)(D-2)^2}D\,. \qquad \text{(A.10)}
\end{aligned}
$$

For calculating $\bar{R}^2_{\sigma\rho}$ we use (4.16)

$$
\begin{aligned}
\bar{R}^2_{\sigma\rho} = \bar{R}_{\sigma\rho}\bar{R}^{\sigma\rho} &= \bar{R}_{\sigma\rho}\bar{R}_{\beta\theta}\bar{g}^{\sigma\beta}\bar{g}^{\rho\theta} = \frac{4\Lambda^2}{(D-2)^2}\bar{g}_{\sigma\rho}\bar{g}_{\beta\theta}\bar{g}^{\sigma\beta}\bar{g}^{\rho\theta} \\
&= \frac{4\Lambda^2}{(D-2)^2}\delta^\beta_\rho\delta^\rho_\beta = \frac{4\Lambda^2}{(D-2)^2}D\,, \qquad \text{(A.11)}
\end{aligned}
$$

and for $\bar{R}^2$ we use (4.7)

$$
\bar{R}^2 = \frac{4\Lambda^2}{(D-2)^2}D^2. \qquad \text{(A.12)}
$$

Hence the GB density, which is $\bar{R}^2_{\tau\lambda\rho\sigma} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2$, can be calculated with the help of (8.10), (8.11), and (8.12) as

$$
\begin{aligned}
\bar{R}^2_{\tau\lambda\rho\sigma} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2 &= \frac{8\Lambda^2}{(D-1)(D-2)^2}D - \frac{16\Lambda^2}{(D-2)^2} + \frac{4\Lambda^2}{(D-2)^2}D^2 \\
&= \frac{4\Lambda^2}{(D-1)(D-2)}D(D-3)\,. \qquad \text{(A.13)}
\end{aligned}
$$

APPENDIX B

KILLING ENERGY EXPRESSION FOR GENERIC QUADRATIC THEORY

In this section we are going to derive equations of motion coming from the action (6.1). We will look at term by term and carry out the calculations, ending with collecting all terms together. We are going to take all terms in $\delta g^{\mu\nu}$ parenthesis for finding the variation.

$$
\begin{aligned}
I_1 = \int d^D x \sqrt{-g}\frac{1}{\kappa}R \Rightarrow \delta I_1 &= \int d^D x \sqrt{-g}\frac{1}{\kappa}\delta R + \int d^D x \delta(\sqrt{-g})\frac{1}{\kappa}R \\
\delta(\sqrt{-g}) &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \\
\delta I_1 &= \int d^D x \frac{1}{\kappa}(\sqrt{-g}\delta R - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}R) \\
\delta R &= \delta(g^{\mu\nu}R_{\mu\nu}) = (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta(R_{\mu\nu}) \\
\delta(R_{\mu\nu}) &= \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\mu \delta\Gamma^\alpha_{\alpha\nu} \\
\delta R &= (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}(\nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\mu \delta\Gamma^\alpha_{\alpha\nu})
\end{aligned}
$$

The second term is zero and we are left with

$$\delta R = (\delta g^{\mu\nu})R_{\mu\nu} \, ,$$

and

$$I_1 = \int d^D x \frac{1}{\kappa}\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} \,.$$

Next

$$\begin{aligned}
I_2 &= \int d^D x\sqrt{-g}\alpha R^2 \Rightarrow \delta I_2 = \int d^D x \left\{\delta(\sqrt{-g})\alpha R^2 + \sqrt{-g}\alpha 2R\delta R\right\} \\
\delta I_2 &= \int d^D x\sqrt{-g}\alpha\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2 + 2R\delta R\} \\
R(\delta R) &= (\delta g^{\mu\nu})R_{\mu\nu}R + Rg^{\mu\nu}(\delta R_{\mu\nu}) \\
Rg^{\mu\nu}(\delta R_{\mu\nu}) &= Rg^{\mu\nu}(\nabla_\alpha \delta\Gamma^\alpha_{\ \mu\nu} - \nabla_\mu \delta\Gamma^\alpha_{\ \alpha\nu}) \,,
\end{aligned}$$

(in the second term we make the change : $\mu \leftrightarrow \alpha$)

$$\begin{aligned}
Rg^{\mu\nu}(\delta R_{\mu\nu}) &= R(\nabla_\alpha g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu} - \nabla_\alpha g^{\alpha\nu}\delta\Gamma^\mu_{\ \mu\nu}) \\
&= \nabla_\alpha\{R(g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu} - g^{\alpha\nu}\delta\Gamma^\mu_{\ \mu\nu})\} - (g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu} - g^{\alpha\nu}\delta\Gamma^\mu_{\ \mu\nu})(\nabla_\alpha R)
\end{aligned}$$

The first term is zero.

$$Rg^{\mu\nu}\delta R_{\mu\nu} = (g^{\alpha\nu}\delta\Gamma^\mu_{\ \mu\nu} - g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu})\nabla_\alpha R \,,$$

using (4.13), we get

$$\begin{aligned}
Rg^{\mu\nu}(\delta R_{\mu\nu}) &= \Big\{\frac{1}{2}g^{\alpha\nu}g^{\mu\sigma}(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu}) \\
&\quad - \frac{1}{2}g^{\mu\nu}g^{\alpha\sigma}(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu})\Big\}\nabla_\alpha R \,,
\end{aligned}$$

(in the second term : $\sigma \leftrightarrow \nu$)

$$Rg^{\mu\nu}(\delta R_{\mu\nu}) = \Big\{\frac{1}{2}g^{\alpha\nu}g^{\mu\sigma}(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\mu\sigma} - \nabla_\sigma\delta g_{\mu\nu})$$

$$
\begin{aligned}
& - \frac{1}{2} g^{\mu\sigma} g^{\alpha\nu} (\nabla_\mu \delta g_{\nu\sigma} + \nabla_\sigma \delta g_{\nu\mu} - \nabla_\nu \delta g_{\mu\sigma}) \Big\} (\nabla_\alpha R) \\
= & \Big\{ \frac{1}{2} 2 g^{\mu\sigma} g^{\alpha\nu} (\nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu}) \Big\} (\nabla_\alpha R) \\
= & \; g^{\mu\sigma} g^{\alpha\nu} (\nabla_\nu \delta g_{\mu\sigma})(\nabla_\alpha R) - g^{\mu\sigma} g^{\alpha\nu} (\nabla_\sigma \delta g_{\mu\nu})(\nabla_\alpha R) \\
= & \; g^{\mu\sigma} g^{\alpha\nu} \{ \nabla_\nu ((\delta g_{\mu\sigma}) \nabla_\alpha R) - (\nabla_\nu \nabla_\alpha R) \delta g_{\mu\sigma} \} \\
& + g^{\mu\sigma} g^{\alpha\nu} \{ -\nabla_\sigma ((\delta g_{\mu\nu}) \nabla_\alpha R) + (\nabla_\sigma \nabla_\alpha R) \delta g_{\mu\nu} \} \\
= & \; -g^{\mu\sigma} g^{\alpha\nu} (\nabla_\nu \nabla_\alpha R) \delta g_{\mu\sigma} + g^{\mu\sigma} g^{\alpha\nu} (\nabla_\sigma \nabla_\alpha R) \delta g_{\mu\nu} \\
= & \; -g^{\mu\sigma} (\nabla^\alpha \nabla_\alpha R) \delta g_{\mu\sigma} + (\nabla^\mu \nabla^\nu R) \delta g_{\mu\nu} \\
= & \; g_{\mu\sigma} (\Box R) \delta g^{\mu\sigma} - (\nabla_\mu \nabla_\nu R) \delta g^{\mu\nu} \, ,
\end{aligned}
$$

and taking $\sigma \to \nu$ in the first term

$$
R g^{\mu\nu} (\delta R_{\mu\nu}) = g_{\mu\nu} (\Box R) \delta g^{\mu\nu} - (\nabla_\mu \nabla_\nu R) \delta g^{\mu\nu} \, .
$$

Therefore

$$
R(\delta R) = \delta g^{\mu\nu} R_{\mu\nu} R + g_{\mu\nu} (\Box R) \delta g^{\mu\nu} - (\nabla_\mu \nabla_\nu R) \delta g^{\mu\nu} \, ,
$$

and we end up with

$$
\begin{aligned}
\delta I_2 &= \int d^D x \sqrt{-g} \alpha \{ -\frac{1}{2} g_{\mu\nu} R^2 + 2 (R_{\mu\nu} R + g_{\mu\nu} \Box R - \nabla_\mu \nabla_\nu R) \} \delta g^{\mu\nu} \\
&= \int d^D x \sqrt{-g} \{ 2\alpha R (R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R) + 2\alpha (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) R \} \delta g^{\mu\nu}
\end{aligned}
$$

For the β part we do similar manipulations

$$
I_3 = \int d^D x \beta R_{\mu\nu}^2 \sqrt{-g}
$$

$$
\begin{aligned}
\delta I_3 &= \int d^D x\beta\{R_{\mu\nu}R^{\mu\nu}\delta(\sqrt{-g}) + (\delta R_{\mu\nu})R^{\mu\nu}\sqrt{-g} + R_{\mu\nu}(\delta R^{\mu\nu})\sqrt{-g}\} \\
&= \int d^D x\beta\{R_{\mu\nu}R^{\mu\nu}\delta(\sqrt{-g}) + (\delta R_{\mu\nu})R^{\mu\nu}\sqrt{-g} + R_{\mu\nu}\delta(g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta})\sqrt{-g}\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \delta I_3 &= \int d^D x\beta\{-\frac{1}{2}\sqrt{-g}g_{\sigma\rho}(\delta g^{\sigma\rho})R_{\mu\nu}R^{\mu\nu} + (\delta R_{\mu\nu})R^{\mu\nu}\sqrt{-g} \\
&\quad +R_{\mu\nu}[(\delta g^{\mu\alpha})g^{\nu\beta}R_{\alpha\beta} + g^{\mu\alpha}(\delta g^{\nu\beta})R_{\alpha\beta} + g^{\mu\alpha}g^{\nu\beta}(\delta R_{\alpha\beta})]\sqrt{-g}\} \\
&= \int d^D x\beta\sqrt{-g}\{-\frac{1}{2}g_{\sigma\rho}(\delta g^{\sigma\rho})R_{\mu\nu}R^{\mu\nu} + (\delta R_{\mu\nu})R^{\mu\nu} \\
&\quad +R_{\mu\nu}(\delta g^{\mu\alpha})g^{\nu\beta}R_{\alpha\beta} + R_{\mu\nu}g^{\mu\alpha}(\delta g^{\nu\beta})R_{\alpha\beta} + R_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}(\delta R_{\alpha\beta})\} \\
&= \int d^D x\beta\sqrt{-g}\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R_{\sigma\rho}R^{\sigma\rho} + (\delta R_{\mu\nu})R^{\mu\nu} \\
&\quad +R_{\mu\alpha}(\delta g^{\mu\nu})g^{\alpha\beta}R_{\nu\beta} + R_{\beta\nu}g^{\beta\alpha}(\delta g^{\nu\mu})R_{\alpha\mu} + R_{\mu\nu}g^{\mu\alpha}g^{\nu\beta}(\delta R_{\alpha\beta})\} \\
&= \int d^D x\beta\sqrt{-g}\{-\frac{1}{2}g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho} + R_{\mu}{}^{\beta}R_{\nu\beta} + R_{\beta}{}^{\mu}R_{\beta\nu}\}(\delta g^{\mu\nu}) \\
&\quad +\int d^D x\beta\sqrt{-g}\{(\delta R_{\mu\nu})R^{\mu\nu} + (\delta R_{\alpha\beta})R^{\alpha\beta}\} \\
&= \int d^D x\beta\sqrt{-g}\{(-\frac{1}{2}g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho} + 2R_{\mu}{}^{\beta}R_{\nu\beta})(\delta g^{\mu\nu}) + 2(\delta R_{\mu\nu})R^{\mu\nu}\}
\end{aligned}
$$

Let us look to the second term, that is

$$
\begin{aligned}
R^{\mu\nu}(\delta R_{\mu\nu}) &= R^{\mu\nu}(\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - \nabla_\mu\delta\Gamma^\alpha_{\alpha\nu}) \\
&= R^{\mu\nu}\nabla_\alpha\delta\Gamma^\alpha_{\mu\nu} - R^{\mu\nu}\nabla_\mu\delta\Gamma^\alpha_{\alpha\nu} \\
&= \nabla_\alpha(R^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - (\delta\Gamma^\alpha_{\mu\nu})\nabla_\alpha R^{\mu\nu} - \nabla_\alpha(R^{\alpha\nu}\delta\Gamma^\mu_{\mu\nu}) + (\delta\Gamma^\mu_{\mu\nu})\nabla_\alpha R^{\alpha\nu} \\
&= (\delta\Gamma^\mu_{\mu\nu})\nabla_\alpha R^{\alpha\nu} - (\delta\Gamma^\alpha_{\mu\nu})\nabla_\alpha R^{\mu\nu}\,.
\end{aligned}
$$

The first and the third terms are boundary terms. Using (4.13),

$$
R^{\mu\nu}(\delta R_{\mu\nu}) = \frac{1}{2}g^{\mu\beta}(\nabla_\mu\delta g_{\nu\beta} + \nabla_\nu\delta g_{\mu\beta} - \nabla_\beta\delta g_{\mu\nu})(\nabla_\alpha R^{\alpha\nu})
$$

$$
\begin{aligned}
& -\frac{1}{2}g^{\alpha\beta}(\nabla_\mu \delta g_{\nu\beta} + \nabla_\nu \delta g_{\mu\beta} - \nabla_\beta \delta g_{\mu\nu})(\nabla_\alpha R^{\mu\nu}) \\
= & \frac{1}{2}g^{\mu\beta}(\nabla_\nu \delta g_{\mu\beta})(\nabla_\alpha R^{\alpha\nu}) \\
& -\frac{1}{2}g^{\alpha\beta}(\nabla_\mu \delta g_{\nu\beta} + \nabla_\nu \delta g_{\mu\beta} - \nabla_\beta \delta g_{\mu\nu})(\nabla_\alpha R^{\mu\nu}) \\
= & -\frac{1}{2}g^{\mu\beta}\delta g_{\mu\beta}(\nabla_\nu \nabla_\alpha R^{\alpha\nu}) + \frac{1}{2}g^{\alpha\beta}\delta g_{\nu\beta}(\nabla_\mu \nabla_\alpha R^{\mu\nu}) \\
& +\frac{1}{2}g^{\alpha\beta}\delta g_{\mu\beta}(\nabla_\nu \nabla_\alpha R^{\mu\nu}) - \frac{1}{2}g^{\alpha\beta}\delta g_{\mu\nu}(\nabla_\beta \nabla_\alpha R^{\mu\nu}) \,.
\end{aligned}
$$

In the third term we change ν with μ

$$
\begin{aligned}
R^{\mu\nu}(\delta R_{\mu\nu}) &= \frac{1}{2}g_{\mu\beta}\delta g^{\mu\beta}(\nabla_\nu \nabla_\alpha R^{\alpha\nu}) + g^{\alpha\beta}\delta g_{\nu\beta}(\nabla_\mu \nabla_\alpha R^{\mu\nu}) \\
& +\frac{1}{2}\delta g^{\mu\nu}(\nabla^\alpha \nabla_\alpha R_{\mu\nu}) \,.
\end{aligned}
$$

In the first term β goes to ν

$$
\begin{aligned}
R^{\mu\nu}(\delta R_{\mu\nu}) &= \frac{1}{2}g_{\mu\nu}(\nabla_\beta \nabla_\alpha R^{\alpha\beta})\delta g^{\mu\nu} + g^{\alpha\beta}(\nabla_\mu \nabla_\alpha R^{\mu\nu})\delta g_{\nu\beta} \\
& +\frac{1}{2}(\Box R_{\mu\nu})\delta g^{\mu\nu} \,.
\end{aligned}
$$

The first term can be calculated by using the Bianchi identity

$$
\nabla_\alpha (R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R) = 0 \,,
$$

$$
\begin{aligned}
\nabla_\alpha R^{\alpha\beta} &= \frac{1}{2}g^{\alpha\beta}\nabla_\alpha R \\
\nabla_\beta \nabla_\alpha R^{\alpha\beta} &= \frac{1}{2}g^{\alpha\beta}\nabla_\beta \nabla_\alpha R \\
\nabla_\beta \nabla_\alpha R^{\alpha\beta} &= \frac{1}{2}\nabla^\alpha \nabla_\alpha R \,, \\
\nabla_\beta \nabla_\alpha R^{\alpha\beta} &= \frac{1}{2}\Box R \,.
\end{aligned}
$$

The second term can be calculated as follows:

$$\begin{aligned}
\nabla_\mu \nabla_\alpha R^{\mu\alpha} &= [\nabla_\mu, \nabla_\alpha] R^{\mu\nu} + \nabla_\alpha \nabla_\mu R^{\mu\alpha} \\
&= R_{\mu\alpha}{}^{\mu}{}_{\lambda} R^{\lambda\nu} + R_{\mu\alpha}{}^{\nu}{}_{\lambda} R^{\mu\lambda} + \nabla_\alpha \nabla_\mu R^{\mu\alpha} \\
&= R_{\alpha\lambda} R^{\lambda\nu} + R_{\mu\alpha}{}^{\nu}{}_{\lambda} R^{\mu\lambda} + \nabla_\alpha \nabla_\mu R^{\mu\alpha} \,,
\end{aligned}$$

from Bianchi identity the last term is $\frac{1}{2}\nabla_\alpha \nabla^\nu R$ and

$$\begin{aligned}
g^{\alpha\beta} \nabla_\mu \nabla_\alpha R^{\mu\nu} \delta g_{\nu\beta} &= R^{\beta}{}_{\lambda} R^{\lambda\nu} \delta g_{\nu\beta} + R_{\mu}{}^{\beta\nu}{}_{\lambda} R^{\mu\lambda} \delta g_{\nu\beta} + \frac{1}{2} \nabla^\beta \nabla^\nu R \delta g_{\nu\beta} \\
&= -R_{\beta\lambda} R^{\lambda}{}_{\nu} \delta g^{\nu\beta} - R_{\mu\beta\nu\lambda} R^{\mu\lambda} \delta g^{\nu\beta} - \frac{1}{2} \nabla_\beta \nabla_\nu R \delta g^{\nu\beta} \\
&= -R_{\mu\lambda} R^{\lambda}{}_{\nu} \delta g^{\mu\nu} - R_{\beta\mu\nu\lambda} R^{\beta\lambda} \delta g^{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R \delta g^{\mu\nu} \\
&= (-R_{\nu\beta} R^{\beta}{}_{\mu} + R_{\mu\sigma\nu\rho} R^{\sigma\rho} - \frac{1}{2} \nabla_\mu \nabla_\nu R) \delta g^{\mu\nu}
\end{aligned}$$

And the β part of our integral becomes

$$\begin{aligned}
\delta I_3 &= \int d^D x \beta \sqrt{-g} \Big\{ -\frac{1}{2} g_{\mu\nu} R_{\sigma\rho} R^{\sigma\rho} + 2 R^{\sigma\rho} R_{\mu\sigma\nu\rho} \\
&\quad + \frac{1}{2} g_{\mu\nu} (\Box R) - (\nabla_\mu \nabla_\nu R) + (\Box R_{\mu\nu}) \Big\} \delta g^{\mu\nu} \\
\delta I_3 &= \int d^D x \beta \sqrt{-g} \Big\{ -\frac{1}{2} g_{\mu\nu} R_{\sigma\rho} R^{\sigma\rho} + 2 R^{\sigma\rho} R_{\mu\sigma\nu\rho} \\
&\quad + g_{\mu\nu} (\Box R) - \frac{1}{2} g_{\mu\nu} (\Box R) - (\nabla_\mu \nabla_\nu R) + (\Box R_{\mu\nu}) \Big\} \delta g^{\mu\nu} \\
\delta I_3 &= \int d^D x \sqrt{-g} \Big\{ 2\beta (R_{\mu\sigma\nu\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho}) R^{\sigma\rho} \\
&\quad + \beta \Box (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \beta (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) R \Big\} \delta g^{\mu\nu} \,.
\end{aligned}$$

In the γ part, that is, $I_4 = \int d^D x \sqrt{-g} \gamma (R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2)$, we will only find the variation of the first term. Others have already been calculated.

Our starting point is as before,

$$\begin{aligned}
\delta I_4' &= \int d^Dx\gamma\left\{\delta(\sqrt{-g})R^2_{\mu\nu\rho\sigma}+\sqrt{-g}\delta(R^2_{\mu\nu\rho\sigma})\right\}\\
&-4\int d^Dx\sqrt{-g}\left\{2\gamma(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho}\right.\\
&\quad\left.+\gamma\Box(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R)+\gamma(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R\right\}\delta g^{\mu\nu}\\
&+\int d^Dx\sqrt{-g}\left\{2\gamma R(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R)+2\gamma(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R\right\}\delta g^{\mu\nu}\\
\delta I_4' &= \int d^Dx\sqrt{-g}\gamma\left\{-\frac{1}{2}g_{\tau\lambda}(\delta g^{\tau\lambda})R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\delta(R_{\mu\nu\rho\sigma})R^{\mu\nu\rho\sigma}\right.\\
&\qquad\qquad\left.+R_{\mu\nu\rho\sigma}\delta(R^{\mu\nu\rho\sigma})\right\},\\
\delta I_4' &= \int d^Dx\sqrt{-g}\gamma\left\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R_{\tau\lambda\rho\sigma}R^{\tau\lambda\rho\sigma}\right.\\
&\left.+\delta(g_{\mu\beta}R^{\beta}{}_{\nu\rho\sigma})R^{\mu\nu\rho\sigma}+R_{\mu\nu\rho\sigma}\delta(g^{\nu\beta}g^{\alpha\sigma}g^{\rho\theta}R^{\mu}{}_{\beta\theta\alpha})\right\},\\
\delta I_4' &= \int d^Dx\sqrt{-g}\gamma\left\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2_{\tau\lambda\rho\sigma}+(\delta g_{\mu\beta})R^{\beta}{}_{\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right.\\
&+R_{\mu\nu}{}^{\theta\alpha}(\delta g^{\nu\beta})R^{\mu}{}_{\beta\theta\alpha}+R_{\mu}{}^{\beta\theta}{}_{\sigma}(\delta g^{\alpha\sigma})R^{\mu}{}_{\beta\theta\alpha}\\
&\left.+R_{\mu}{}^{\beta}{}_{\rho}{}^{\alpha}(\delta g^{\rho\theta})R^{\mu}{}_{\beta\theta\alpha}+(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}+R_{\mu}{}^{\beta\theta\alpha}(\delta R^{\mu}{}_{\beta\theta\alpha})\right\},\\
\delta I_4' &= \int d^Dx\sqrt{-g}\gamma\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2_{\tau\lambda\rho\sigma}+(\delta g_{\mu\nu})\{R^{\nu}{}_{\beta\rho\sigma}R^{\mu\beta\rho\sigma}\\
&+R_{\beta\nu}{}^{\theta\alpha}(\delta g^{\nu\mu})R^{\beta}{}_{\mu\theta\alpha}+R_{\alpha}{}^{\beta\theta}{}_{\nu}(\delta g^{\mu\nu})R^{\alpha}{}_{\beta\theta\mu}\\
&+R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}(\delta g^{\mu\nu})R^{\rho}{}_{\beta\nu\alpha}+(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}+R_{\beta}{}^{\nu\rho\sigma}(\delta R^{\beta}{}_{\nu\rho\sigma})\}.
\end{aligned}$$

In the last term we make the $\mu\to\beta,\beta\to\nu,\theta\to\rho,\alpha\to\sigma$ transformations.

$$\begin{aligned}
\delta I_4' &= \int d^Dx\sqrt{-g}\gamma\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2_{\tau\lambda\rho\sigma}+(\delta g^{\mu\nu})\{-R_{\nu\beta\rho\sigma}R_{\mu}{}^{\beta\rho\sigma}\\
&+R_{\beta\nu}{}^{\theta\alpha}R^{\beta}{}_{\mu\theta\alpha}+R_{\alpha}{}^{\beta\theta}{}_{\nu}R^{\alpha}{}_{\beta\theta\mu}\\
&+R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}R^{\rho}{}_{\beta\nu\alpha}\}+2(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}\}.
\end{aligned}$$

In the term $R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}R^{\rho}{}_{\beta\nu\alpha}$, we make the changes $\alpha \to \rho, \theta \to \alpha, \mu \leftrightarrow \nu$ to get

$$\begin{aligned}
\delta I_4' &= \int d^D x\sqrt{-g}\gamma\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2_{\tau\lambda\rho\sigma} \\
&+(\delta g^{\mu\nu})\{-R_{\nu\beta\rho\sigma}R_{\mu}{}^{\beta\rho\sigma} + R_{\beta\nu}{}^{\theta\alpha}R^{\beta}{}_{\mu\theta\alpha} + 2R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}R^{\rho}{}_{\beta\nu\alpha}\} \\
&+2(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}\}\,, \\
\delta I_4' &= \int d^D x\sqrt{-g}\gamma\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2_{\tau\lambda\rho\sigma} \\
&+(\delta g^{\mu\nu})\{-R_{\nu\beta\rho\sigma}g_{\mu\theta}R^{\theta\beta\rho\sigma} + g_{\nu\sigma}R^{\beta\sigma\theta\alpha}R_{\beta\mu\theta\alpha} + 2R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}R^{\rho}{}_{\beta\nu\alpha}\} \\
&+2(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}\}\,.
\end{aligned}$$

In the term $g_{\nu\sigma}R^{\beta\sigma\theta\alpha}R_{\beta\mu\theta\alpha}$, the suitable transformations are $\nu \leftrightarrow \mu, \sigma \leftrightarrow \theta, \alpha \to \rho$:

$$\begin{aligned}
\delta I_4' &= \int d^D x\sqrt{-g}\gamma\{-\frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})R^2_{\tau\lambda\rho\sigma} \\
&+(\delta g^{\mu\nu})\{-R_{\nu\beta\rho\sigma}g_{\mu\theta}R^{\theta\beta\rho\sigma} + g_{\mu\theta}R^{\beta\theta\sigma\rho}R_{\beta\nu\sigma\rho} + 2R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}R^{\rho}{}_{\beta\nu\alpha}\} \\
&+2(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}\}\,, \\
\delta I_4' &= \int d^D x\sqrt{-g}\gamma\{(\delta g^{\mu\nu})(-\frac{1}{2}g_{\mu\nu}R^2_{\tau\lambda\rho\sigma} \\
&+2R_{\rho}{}^{\beta}{}_{\mu}{}^{\alpha}R^{\rho}{}_{\beta\nu\alpha}) + 2(\delta R^{\beta}{}_{\nu\rho\sigma})R_{\beta}{}^{\nu\rho\sigma}\}\,, \\
\delta I_4' &= \int d^D x\sqrt{-g}\gamma\{(\delta g^{\mu\nu})(-\frac{1}{2}g_{\mu\nu}R^2_{\tau\lambda\rho\sigma} + 2R^{\rho\beta}{}_{\mu}{}^{\alpha}R_{\rho\beta\nu\alpha}) \\
&+2(\delta R^{\beta}{}_{\nu\sigma\rho})R_{\beta}{}^{\nu\sigma\rho}\}\,, \\
\delta I_4' &= \int d^D x\sqrt{-g}\gamma\{(\delta g^{\mu\nu})(-\frac{1}{2}g_{\mu\nu}R^2_{\tau\lambda\rho\sigma} + 2g_{\mu\theta}R^{\rho\beta\theta\alpha}R_{\rho\beta\nu\alpha}) \\
&+2(\delta R^{\beta}{}_{\nu\sigma\rho})R_{\beta}{}^{\nu\sigma\rho}\}\,.
\end{aligned}$$

Setting $\theta \leftrightarrow \alpha, \rho \to \sigma$ in $g_{\mu\theta} R^{\rho\beta\theta\alpha} R_{\rho\beta\nu\alpha}$, we have

$$\begin{aligned} \delta I'_4 &= \int d^D x \sqrt{-g} \gamma \{ (\delta g^{\mu\nu}) (-\frac{1}{2} g_{\mu\nu} R^2_{\tau\lambda\sigma\rho} + 2 g_{\mu\alpha} R^{\sigma\beta\alpha\theta} R_{\sigma\beta\nu\theta}) \\ &+2(\delta R^{\beta}{}_{\nu\sigma\rho}) R_{\beta}{}^{\nu\sigma\rho} \} . \end{aligned}$$

$$\begin{aligned} g_{\mu\alpha} R^{\sigma\beta\alpha\theta} R_{\sigma\beta\nu\theta} &= g_{\mu\alpha} R^{\alpha\theta\sigma\beta} R_{\nu\theta\sigma\beta} \\ &= R_{\mu}{}^{\theta\sigma\beta} R_{\nu\theta\sigma\beta} \end{aligned}$$

and making the substitutions $\mu \leftrightarrow \nu, \theta \to \sigma, \sigma \to \rho, \beta \to \tau$ will give us $R_{\nu}{}^{\sigma\rho\tau} R_{\mu\sigma\rho\tau}$ and

$$\begin{aligned} \delta I'_4 &= \int d^D x \sqrt{-g} \gamma \{ (\delta g^{\mu\nu}) (-\frac{1}{2} g_{\mu\nu} R^2_{\tau\lambda\sigma\rho} + 2 R_{\nu}{}^{\sigma\rho\tau} R_{\mu\sigma\rho\tau}) \\ &+2(\delta R^{\beta}{}_{\nu\sigma\rho}) R_{\beta}{}^{\nu\sigma\rho} \} . \end{aligned}$$

The last term is

$$R_{\beta}{}^{\nu\sigma\rho} (\delta R^{\beta}{}_{\nu\sigma\rho}) = R_{\mu}{}^{\nu\sigma\rho} (\delta R^{\mu}{}_{\nu\sigma\rho}) ,$$

where we have changed β with μ. Using (4.15)

$$\begin{aligned} R_{\mu}{}^{\nu\sigma\rho} (\delta R^{\mu}{}_{\nu\sigma\rho}) &= R_{\mu}{}^{\nu\sigma\rho} (\nabla_{\sigma} \delta \Gamma^{\mu}_{\nu\rho} - \nabla_{\rho} \delta \Gamma^{\mu}_{\nu\sigma}) , \\ &= R_{\mu}{}^{\nu\sigma\rho} \nabla_{\sigma} \delta \Gamma^{\mu}_{\nu\rho} - R_{\mu}{}^{\nu\sigma\rho} \nabla_{\rho} \delta \Gamma^{\mu}_{\nu\sigma}) , \\ &= R_{\mu}{}^{\nu\sigma\rho} \nabla_{\sigma} \delta \Gamma^{\mu}_{\nu\rho} - R_{\mu}{}^{\nu\rho\sigma} \nabla_{\sigma} \delta \Gamma^{\mu}_{\nu\rho}) . \end{aligned}$$

Here we only change the names of indices (ρ with σ):

$$\begin{aligned} R_{\mu}{}^{\nu\sigma\rho} (\delta R^{\mu}{}_{\nu\sigma\rho}) &= 2 R_{\mu}{}^{\nu\sigma\rho} \nabla_{\sigma} \delta \Gamma^{\mu}_{\nu\rho} \\ &= 2 \nabla_{\sigma} (R_{\mu}{}^{\nu\sigma\rho} \delta \Gamma^{\mu}_{\nu\rho}) - 2 (\nabla_{\sigma} R_{\mu}{}^{\nu\sigma\rho}) \delta \Gamma^{\mu}_{\nu\rho} . \end{aligned}$$

Using (4.13)

$$\begin{aligned}
R_{\mu}{}^{\nu\sigma\rho}(\delta R^{\mu}{}_{\nu\sigma\rho}) &= -2(\nabla_{\sigma}R_{\mu}{}^{\nu\sigma\rho})\frac{1}{2}g^{\mu\beta}(\nabla_{\nu}\delta g_{\rho\beta}+\nabla_{\rho}\delta g_{\nu\beta}-\nabla_{\beta}\delta g_{\nu\rho})\,,\\
&= -\{\nabla_{\nu}(\nabla_{\sigma}R^{\beta\nu\sigma\rho}\delta g_{\rho\beta})-(\nabla_{\nu}\nabla_{\sigma}R^{\beta\nu\sigma\rho})\delta g_{\rho\beta}\\
&\quad +\nabla_{\rho}(\nabla_{\sigma}R^{\beta\nu\sigma\rho}\delta g_{\nu\beta})-(\nabla_{\rho}\nabla_{\sigma}R^{\beta\nu\sigma\rho})\delta g_{\nu\beta}\\
&\quad -\nabla_{\beta}(\nabla_{\sigma}R^{\beta\nu\sigma\rho}\delta g_{\nu\rho})+(\nabla_{\beta}\nabla_{\sigma}R^{\beta\nu\sigma\rho})\delta g_{\nu\rho}\}\,,\\
&= (\nabla_{\nu}\nabla_{\sigma}R^{\beta\nu\sigma\rho})\delta g_{\rho\beta}+(\nabla_{\rho}\nabla_{\sigma}R^{\beta\nu\sigma\rho})\delta g_{\nu\beta}-(\nabla_{\beta}\nabla_{\sigma}R^{\beta\nu\sigma\rho})\delta g_{\nu\rho}\,,\\
&= -(\nabla_{\nu}\nabla_{\sigma}R_{\beta}{}^{\nu\sigma}{}_{\rho})\delta g^{\rho\beta}+(\nabla_{\rho}\nabla_{\sigma}R_{\beta\nu}{}^{\sigma\rho})\delta g^{\nu\beta}\\
&\quad -(\nabla_{\beta}\nabla_{\sigma}R^{\beta}{}_{\nu}{}^{\sigma}{}_{\rho})\delta g^{\nu\rho}\,,
\end{aligned}$$

for obtaining $\delta g^{\mu\nu}$, we make suitable changes in the indices

$$\begin{aligned}
R_{\mu}{}^{\nu\sigma\rho}(\delta R^{\mu}{}_{\nu\sigma\rho}) &= -(\nabla_{\beta}\nabla_{\sigma}R_{\nu}{}^{\beta\sigma}{}_{\mu})\delta g^{\mu\nu}+(\nabla_{\rho}\nabla_{\sigma}R_{\nu\mu}{}^{\sigma\rho})\delta g^{\mu\nu}\\
&\quad -(\nabla_{\beta}\nabla_{\sigma}R^{\beta}{}_{\mu}{}^{\sigma}{}_{\nu})\delta g^{\mu\nu}\,.
\end{aligned}$$

The term in the middle will vanish because of symmetry and anti-symmetry of μ and ν indices.

$$R_{\mu}{}^{\nu\sigma\rho}(\delta R^{\mu}{}_{\nu\sigma\rho}) = 2(\nabla_{\beta}\nabla_{\sigma}R^{\beta}{}_{\mu}{}^{\sigma}{}_{\nu})\delta g^{\mu\nu}\,.$$

In the Bianchi identity we have

$$\nabla_{\sigma}R^{\alpha}{}_{\beta\mu\nu}+\nabla_{\mu}R^{\alpha}{}_{\beta\nu\sigma}+\nabla_{\nu}R^{\alpha}{}_{\beta\sigma\mu}=0\,,$$

when we replace the σ index with α

$$\nabla_{\alpha}R^{\alpha}{}_{\beta\mu\nu}+\nabla_{\mu}R^{\alpha}{}_{\beta\nu\alpha}+\nabla_{\nu}R^{\alpha}{}_{\beta\alpha\mu}=0\,,$$

changing indices: $\alpha \to \sigma$, $\beta \to \nu$, $\mu \to \beta$, $\nu \to \mu$ we get

$$\nabla_\sigma R^\sigma{}_{\nu\beta\mu} + \nabla_\beta R^\sigma{}_{\nu\mu\sigma} + \nabla_\mu R^\sigma{}_{\nu\sigma\beta} = 0$$

$$\Rightarrow \quad \nabla_\sigma R^\sigma{}_{\nu\beta\mu} = \nabla_\beta R_{\nu\mu} - \nabla_\mu R_{\nu\beta}\,,$$

$$\Rightarrow \quad \nabla^\beta \nabla_\sigma R^\sigma{}_{\nu\beta\mu} = \nabla^\beta \nabla_\beta R_{\nu\mu} - \nabla^\beta \nabla_\mu R_{\nu\beta}\,,$$

$$\Rightarrow \quad \nabla_\beta \nabla_\sigma R^\sigma{}_\nu{}^\beta{}_\mu = \Box R_{\nu\mu} - \nabla_\beta \nabla_\mu R_\nu{}^\beta\,.$$

The last term can be written

$$\begin{aligned}
\nabla_\beta \nabla_\mu R_\nu{}^\beta &= [\nabla_\beta, \nabla_\mu] R_\nu{}^\beta + \nabla_\mu \nabla_\beta R_\nu{}^\beta\,, \\
&= R_{\beta\mu\nu}{}^\lambda R_\lambda{}^\beta + R_{\beta\mu}{}^\beta{}_\lambda R_\nu{}^\lambda + \nabla_\mu \nabla_\beta R_\nu{}^\beta\,, \\
&= R_{\beta\mu\nu}{}^\lambda R_\lambda{}^\beta + R_{\mu\lambda} R_\nu{}^\lambda + \frac{1}{2} g_\nu{}^\beta \nabla_\mu \nabla_\beta R\,, \\
&= -R_{\mu\beta\nu}{}^\lambda R_\lambda{}^\beta + R_{\mu\lambda} R_\nu{}^\lambda + \frac{1}{2} \nabla_\mu \nabla_\nu R\,,
\end{aligned}$$

making suitable index transformations we have

$$\nabla_\beta \nabla_\mu R_\nu{}^\beta = -R_{\mu\sigma\nu\rho} R^{\sigma\rho} + R_{\mu\sigma} R_\nu{}^\sigma + \frac{1}{2} \nabla_\mu \nabla_\nu R\,.$$

Therefore we get

$$\begin{aligned}
\delta I_4' &= \int d^D x \sqrt{-g}\, \gamma (\delta g^{\mu\nu}) \{ (-\frac{1}{2} g_{\mu\nu} R^2_{\tau\lambda\sigma\rho} + 2 R_\nu{}^{\sigma\rho\tau} R_{\mu\sigma\rho\tau}) \\
&\quad + 2(2\Box R_{\mu\nu} + 2 R_{\mu\sigma\nu\rho} R^{\sigma\rho} - 2 R_{\mu\sigma} R_\nu{}^\sigma - \nabla_\mu \nabla_\nu R) \}
\end{aligned}$$

and

$$\delta I_4 = \int d^D x \sqrt{-g}\, \gamma (\delta g^{\mu\nu}) \{ (-\frac{1}{2} g_{\mu\nu} R^2_{\tau\lambda\sigma\rho} + 2 R_\nu{}^{\sigma\rho\tau} R_{\mu\sigma\rho\tau})$$

$$
\begin{aligned}
&+2(2\Box R_{\mu\nu} + 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} - 2R_{\mu\sigma}R_{\nu}{}^{\sigma} - \nabla_\mu\nabla_\nu R)\} \\
&-4\int d^Dx\sqrt{-g}\{2\gamma(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} + \gamma\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \\
&+\gamma(g_{\mu\nu}\,\Box - \nabla_\mu\nabla_\nu)R\}(\delta g^{\mu\nu}) \\
&+\int d^Dx\sqrt{-g}\{2\gamma R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + 2\gamma(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R\}(\delta g^{\mu\nu})\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta I_4 \;=\; &\int d^Dx\sqrt{-g}\gamma(\delta g^{\mu\nu})\{-\frac{1}{2}g_{\mu\nu}R^2_{\tau\lambda\sigma\rho} + 2R_\nu{}^{\sigma\rho\tau}R_{\mu\sigma\rho\tau} \\
&-(4R_{\mu\sigma\nu\rho} + 4R_{\mu\sigma}R_\nu{}^\sigma - 2g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} + 2R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R)\}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta I_4 \;=\; &\int d^Dx\sqrt{-g}\gamma(\delta g^{\mu\nu})\{-\frac{1}{2}g_{\mu\nu}R^2_{\tau\lambda\sigma\rho} + 2R_\nu{}^{\sigma\rho\tau}R_{\mu\sigma\rho\tau} \\
&-4R_{\mu\sigma}R_\nu{}^\sigma - 4R_{\mu\sigma\nu\rho}R^{\sigma\rho} + 2g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho} + 2R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R)\}\,.
\end{aligned}
$$

At last we will add these integrals and get the equations of motion:

$$
\begin{aligned}
I \;&=\; I_1 + I_2 + I_3 + I_4 \\
&= \int d^Dx\frac{1}{\kappa}\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(\delta g^{\mu\nu}) \\
&\quad+\int d^Dx\sqrt{-g}\{2\alpha R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + 2\alpha(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R\}(\delta g^{\mu\nu}) \\
&\quad+\int d^Dx\sqrt{-g}\{2\beta(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} + \beta\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \\
&\quad+\beta(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R\}(\delta g^{\mu\nu}) \\
&\quad+\int d^Dx\sqrt{-g}\gamma(\delta g^{\mu\nu})\{-\frac{1}{2}g_{\mu\nu}R^2_{\tau\lambda\sigma\rho} + 2R_\nu{}^{\sigma\rho\tau}R_{\mu\sigma\rho\tau} \\
&\quad-4R_{\mu\sigma}R_\nu{}^\sigma - 4R_{\mu\sigma\nu\rho}R^{\sigma\rho} + 2g_{\mu\nu}R_{\sigma\rho}R^{\sigma\rho} + 2R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R)\}\,, \\
&= \int d^Dx\sqrt{-g}\{\frac{1}{\kappa}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)
\end{aligned}
$$

$$
\begin{aligned}
&+2\alpha R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + (2\alpha+\beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R \\
&+2\gamma[RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} + R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau} \\
&-2R_{\mu\sigma}R_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\sigma\rho} - 4R^2_{\sigma\rho} + R^2)] \\
&+\beta\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\beta(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho}\}(\delta g^{\mu\nu})\,.
\end{aligned}
$$

Hence the equations of motion is

$$
\begin{aligned}
&\frac{1}{\kappa}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\alpha R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) \\
&+(2\alpha+\beta)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R \\
&+2\gamma[RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} + R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau} \\
&-2R_{\mu\sigma}R_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\sigma\rho} - 4R^2_{\sigma\rho} + R^2)] \\
&+\beta\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + 2\beta(R_{\mu\sigma\nu\rho} - \frac{1}{4}g_{\mu\nu}R_{\sigma\rho})R^{\sigma\rho} = \tau_{\mu\nu}\,.
\end{aligned}
$$

In this part we will find the linearized form of equations of motion that we derived in the previous section. Then we will check if it is a background conserved tensor or not. At last the generic form of Killing charges will be found.

Let us start with the $\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)$ term of the previous equation.

$$
\begin{aligned}
\Box(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) &= \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R \\
\delta(\Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R) &= (\Box R_{\mu\nu})^L - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}R^L - \frac{1}{2}h_{\mu\nu}\bar{\Box}\bar{R} - \frac{1}{2}\bar{g}_{\mu\nu}(\Box)^L\bar{R}\,.
\end{aligned}
$$

The last two terms vanishes and we get

$$
\begin{aligned}
(\Box R_{\mu\nu})^L - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}R^L &= \bar{\Box}R^L_{\mu\nu} + (\Box)_L\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}R^L \\
&= \bar{\Box}R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}R^L + \frac{2\Lambda}{(D-2)}(\Box)_L\bar{g}_{\mu\nu}\,.
\end{aligned}
$$

The linearization of Box operator is,

$$
\begin{aligned}
(\Box)_L\bar{g}_{\mu\nu} &= (\nabla^\alpha\nabla_\alpha)_L\bar{g}_{\mu\nu}\,, \\
\nabla_\alpha g_{\mu\nu} = 0 &\Rightarrow (\nabla_\alpha g_{\mu\nu})_L = 0\,, \\
&\Rightarrow (\nabla_\alpha)_L\bar{g}_{\mu\nu} + \bar{\nabla}_\alpha h_{\mu\nu} = 0 \\
&\Rightarrow (\nabla_\alpha)_L\bar{g}_{\mu\nu} = -\bar{\nabla}_\alpha h_{\mu\nu}\,, \\
\nabla_\alpha g_{\mu\nu} = 0 &\Rightarrow \nabla^\alpha\nabla_\alpha g_{\mu\nu} = 0 \\
&\Rightarrow (\nabla^\alpha\nabla_\alpha g_{\mu\nu})_L = 0\,, \\
&\Rightarrow (\nabla^\alpha)_L\bar{\nabla}_\alpha\bar{g}_{\mu\nu} + \bar{\nabla}^\alpha(\nabla_\alpha)_L\bar{g}_{\mu\nu} + \bar{\Box}h_{\mu\nu} = 0\,,
\end{aligned}
$$

and we know that

$$
(\nabla^\alpha)_L\bar{\nabla}_\alpha\bar{g}_{\mu\nu} = 0\,.
$$

Therefore,

$$
(\Box)_L\bar{g}_{\mu\nu} = (\nabla^\alpha\nabla_\alpha)_L\bar{g}_{\mu\nu} = (\nabla^\alpha)_L\bar{\nabla}_\alpha\bar{g}_{\mu\nu} + \bar{\nabla}^\alpha(\nabla_\alpha)_L\bar{g}_{\mu\nu} = -\bar{\Box}h_{\mu\nu}\,,
$$

$$
\begin{aligned}
\delta(\Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box R) &= \bar{\Box}R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{\Box}R^L + \frac{2\Lambda}{(D-2)}\bar{\Box}h_{\mu\nu} \\
&= \bar{\Box}(R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L + \frac{2\Lambda}{(D-2)}h_{\mu\nu}) \\
&= \bar{\Box}\mathcal{G}^L_{\mu\nu}\,.
\end{aligned}
$$

The other terms can be linearized as usual and we get

$$
\begin{aligned}
T_{\mu\nu} &= T_{\mu\nu}(\bar{g}) + \frac{1}{\kappa}(R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{1}{2}h_{\mu\nu}\bar{R}) \\
&\quad + 2\alpha R^L(\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}) + 2\alpha\bar{R}(R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R^L - \frac{1}{4}h_{\mu\nu}\bar{R}) \\
&\quad + (2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu)R^L \\
&\quad + 2\gamma[\delta(RR_{\mu\nu}) - 2\delta(R_{\mu\sigma\nu\rho}R^{\sigma\rho}) + \delta(R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}) \\
&\quad - 2\delta(R_{\mu\sigma}R_\nu{}^\sigma) - \frac{1}{4}\bar{g}_{\mu\nu}\delta(R^2_{\tau\lambda\sigma\rho} - 4R^2_{\sigma\rho} + R^2) \\
&\quad - \frac{1}{4}h_{\mu\nu}(\bar{R}^2_{\tau\lambda\sigma\rho} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2)] \\
&\quad + \beta\bar{\Box}\mathcal{G}^L_{\mu\nu} + 2\beta\{\delta(R_{\mu\sigma\nu\rho}R^{\sigma\rho}) - \frac{1}{4}\bar{g}_{\mu\nu}\delta(R^2_{\sigma\rho}) - \frac{1}{4}h_{\mu\nu}\bar{R}^2_{\sigma\rho}\}\,.
\end{aligned}
$$

Using (4.5), (4.6), (4.7), and equations of appendix A, we can reorganize the above equation: Lets start with adding and subtracting $\frac{2\Lambda}{(D-2)}$ to the first term.

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{\kappa}(R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{(D-2)}h_{\mu\nu} + \frac{2\Lambda}{(D-2)}h_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R}) \\
&= \frac{1}{\kappa}(\mathcal{G}^L_{\mu\nu} + \frac{(2-D)\Lambda}{(D-2)}h_{\mu\nu}) = \frac{1}{k}(\mathcal{G}^L_{\mu\nu} - \Lambda h_{\mu\nu})\,.
\end{aligned}
$$

We make the same things in the α terms

$$
\begin{aligned}
\mathcal{B} &= 2\alpha\left(R^L\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R^L\bar{R} + \bar{R}R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}R^L - \frac{1}{4}h_{\mu\nu}\bar{R}^2\right) \\
&= 2\alpha\left(\frac{2\Lambda}{(D-2)}R^L\bar{g}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R^L\frac{2D\Lambda}{(D-2)} + R^L_{\mu\nu}\frac{2D\Lambda}{(D-2)}\right.
\end{aligned}
$$

$$
- \frac{1}{4}\bar{g}_{\mu\nu}\frac{2D\Lambda}{(D-2)}R^L - \frac{1}{4}h_{\mu\nu}\frac{4D^2\Lambda^2}{(D-2)^2}\Bigg) \, ,
$$

adding and subtracting $\frac{2\Lambda}{(D-2)}h_{\mu\nu}$ we have

$$
\begin{aligned}
\mathcal{B} &= \frac{4\alpha D\Lambda}{(D-2)}\Bigg(R^L_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}R^L + \frac{1}{D}\bar{g}_{\mu\nu}R^L - \frac{1}{4}\bar{g}_{\mu\nu}R^L \\
&\quad - \frac{2\Lambda}{(D-2)}h_{\mu\nu} + \frac{2\Lambda}{(D-2)}h_{\mu\nu} - \frac{1}{4}h_{\mu\nu}\frac{2D\Lambda}{(D-2)}\Bigg) \, ,
\end{aligned}
$$

and

$$
\mathcal{B} = \frac{4\alpha D\Lambda}{(D-2)}\mathcal{G}^L_{\mu\nu} + \frac{4\alpha\Lambda}{(D-2)}\bar{g}_{\mu\nu}R^L - \frac{2\alpha(D-4)D\Lambda^2}{(D-2)^2}h_{\mu\nu} \, .
$$

For the γ term we do the same calculations and we end up with

$$
\begin{aligned}
\mathcal{C} &= 2\gamma\Bigg\{\frac{2\Lambda D}{(D-2)}R^L_{\mu\nu} + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}R^L \\
&\quad - 2\left(\frac{2\Lambda}{(D-1)}R^L_{\mu\nu} + \frac{2\Lambda}{(D-2)(D-1)}\bar{g}_{\mu\nu}R^L + \frac{4\Lambda^2}{(D-2)^2(D-1)}h_{\mu\nu}\right) \\
&\quad + \frac{8\Lambda}{(D-2)(D-1)}R^L_{\mu\nu} - \frac{8\Lambda^2}{(D-2)^2(D-1)}h_{\mu\nu} \\
&\quad - 2(\frac{4\Lambda}{(D-2)}R^L_{\mu\nu} - \frac{4\Lambda^2}{(D-2)^2}h_{\mu\nu}) \\
&\quad - \frac{1}{4}\bar{g}_{\mu\nu}\frac{4(D-3)\Lambda}{(D-1)}R^L - \frac{1}{4}\frac{4D(D-3)\Lambda^2}{(D-1)(D-2)}h_{\mu\nu}\Bigg\} \, , \\
&= 2\gamma\{\frac{\Lambda R^L_{\mu\nu}}{(D-2)(D-1)}[2D(D-1) - 4(D-2) + 8 - 8(D-1)] \\
&\quad + \frac{\Lambda\bar{g}_{\mu\nu}R^L}{(D-2)(D-1)}[2(D-1) - 4 - (D-3)(D-2)] \\
&\quad + \frac{\Lambda^2 h_{\mu\nu}}{(D-2)^2(D-1)}[-16 + 8(D-1) - D(D-3)(D-2)]\} \, , \\
&= 2\gamma\{\frac{2\Lambda R^L_{\mu\nu}}{(D-2)(D-1)}(D-3)(D-4) \\
&\quad - \frac{\Lambda\bar{g}_{\mu\nu}R^L}{(D-2)(D-1)}(D-3)(D-4) \\
&\quad - \frac{\Lambda^2 h_{\mu\nu}}{(D-2)^2(D-1)}(D^3 - 5D^2 - 2D + 24)\} \, ,
\end{aligned}
$$

again we add and subtract $\frac{2\Lambda}{(D-2)}h_{\mu\nu}$

$$\mathcal{C} = \frac{4\gamma\Lambda(D-3)(D-4)}{(D-2)(D-1)}\{R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{(D-2)}h_{\mu\nu} + \frac{2\Lambda}{(D-2)}h_{\mu\nu} - \frac{\Lambda(D^3-5D^2-2D+24)}{2(D-2)(D-3)(D-4)}h_{\mu\nu}\},$$

after some easy calculations we can get

$$\mathcal{C} = \frac{4\gamma\Lambda(D-3)(D-4)}{(D-2)(D-1)}\mathcal{G}^L_{\mu\nu} - \frac{2\gamma\Lambda^2(D-3)(D-4)}{(D-2)(D-1)}h_{\mu\nu}.$$

After the same calculations, the β terms can be written as

$$\begin{aligned}\mathcal{D} &= 2\beta\left\{\frac{2\Lambda}{(D-1)}R^L_{\mu\nu} + \frac{2\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}R^L + \frac{4\Lambda^2}{(D-1)(D-2)^2}h_{\mu\nu} - \frac{\Lambda}{(D-2)}\bar{g}_{\mu\nu}R^L - \frac{\Lambda^2 D}{(D-2)^2}h_{\mu\nu}\right\} \\ &= \frac{4\beta\Lambda}{(D-1)}G^L_{\mu\nu} + \frac{2\beta\Lambda}{(D-2)}\bar{g}_{\mu\nu}R^L - \frac{2\beta\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L - \frac{2\beta\Lambda^2(D-4)}{(D-2)^2}h_{\mu\nu}.\end{aligned}$$

We add all these to get the following result,

$$\begin{aligned}T_{\mu\nu}(h) &= T_{\mu\nu}(\bar{g}) + \mathcal{G}^L_{\mu\nu}\left\{\frac{1}{\kappa} + \frac{4\Lambda D\alpha}{(D-2)} + \frac{4\Lambda\beta}{(D-1)} + \frac{4\Lambda\gamma(D-3)(D-4)}{(D-1)(D-2)}\right\} \\ &\quad +(2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu})R^L \\ &\quad +\beta(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L) \\ &\quad -2\Lambda^2 h_{\mu\nu}\left\{\frac{1}{2\Lambda\kappa} + \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \frac{\gamma(D-4)(D-3)}{(D-2)(D-1)}\right\}.\end{aligned}$$

With the help of

$$\frac{-1}{2\Lambda\kappa} = \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \frac{\gamma(D-4)(D-3)}{(D-1)(D-2)}$$

we have

$$\frac{4\gamma(D-4)(D-3)}{(D-1)(D-2)} = -\frac{2}{\kappa} - \frac{4\Lambda(D-4)}{(D-2)^2}D\alpha - \frac{4\Lambda(D-4)}{(D-2)^2}\beta$$

and the constant terms of $\mathcal{G}^L_{\mu\nu}$ becomes

$$\begin{aligned} & \frac{1}{\kappa} + \frac{4\Lambda D\alpha}{(D-2)} + \frac{4\Lambda\beta}{(D-1)} + \frac{4\Lambda\gamma(D-3)(D-4)}{(D-1)(D-2)} \\ = & \frac{1}{\kappa} + \frac{4\Lambda D\alpha}{(D-2)} + \frac{4\Lambda\beta}{(D-1)} - \frac{2}{\kappa} - \frac{4\Lambda(D-4)}{(D-2)^2}D\alpha - \frac{4\Lambda(D-4)}{(D-2)^2}\beta \\ = & -\frac{1}{\kappa} + \frac{4\Lambda D}{(D-2)^2}(2\alpha + \frac{\beta}{(D-1)}), \end{aligned}$$

and, $T(\bar{g}) = 0$ that we show in our work, we left with

$$\begin{aligned} T_{\mu\nu} = & \mathcal{G}^L_{\mu\nu}\{-\frac{1}{\kappa} + \frac{4\Lambda D}{(D-2)^2}(2\alpha + \frac{\beta}{(D-1)})\} \\ & +(2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu})R^L \\ & +\beta(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L). \end{aligned}$$

Lets check if it is a background conserved tensor or not. We already know that $\mathcal{G}^L_{\mu\nu}$ is background conserved tensor. Therefore, we will only show that the derivatives of the last two terms are zero.

$$\bar{\nabla}^\mu(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu}R^L)$$

$$= \bar{\nabla}_\nu\bar{\Box}R^L - \bar{\Box}\bar{\nabla}_\nu R^L + \frac{2\Lambda}{(D-2)}\bar{\nabla}_\nu R^L$$

If we change the places of the derivatives of the first term we find

$$\bar{\nabla}_\nu\bar{\Box}R^L = \bar{\Box}\bar{\nabla}_\nu R^L - \frac{2\Lambda}{(D-2)}\bar{\nabla}_\nu R^L$$

and

$$
\begin{aligned}
&\bar{\nabla}_\nu \bar{\Box} R^L - \bar{\Box}\bar{\nabla}_\nu R^L + \frac{2\Lambda}{(D-2)}\bar{\nabla}_\nu R^L \\
= \ & \bar{\Box}\bar{\nabla}_\nu R^L - \frac{2\Lambda}{(D-2)}\bar{\nabla}_\nu R^L - \bar{\Box}\bar{\nabla}_\nu R^L + \frac{2\Lambda}{(D-2)}\bar{\nabla}_\nu R^L = 0\,.
\end{aligned}
$$

Let us look to the $\bar{\nabla}^\mu(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L)$ it is $\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{\nabla}_\nu R^L$. The $\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu}$ can be written as

$$
\begin{aligned}
\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu} &= [\bar{\nabla}^\mu, \bar{\nabla}^\beta\bar{\nabla}_\beta]\mathcal{G}^L_{\mu\nu} + \bar{\Box}\bar{\nabla}^\mu\mathcal{G}^L_{\mu\nu}\,, \\
&= \bar{\nabla}^\beta[\bar{\nabla}^\mu, \bar{\nabla}_\beta]\mathcal{G}^L_{\mu\nu} + [\bar{\nabla}^\mu, \bar{\nabla}^\beta]\bar{\nabla}_\beta\mathcal{G}^L_{\mu\nu}\,, \\
&= \bar{\nabla}^\beta\bar{g}^{\mu\sigma}[\bar{\nabla}_\sigma, \bar{\nabla}_\beta]\mathcal{G}^L_{\mu\nu} + \bar{g}^{\mu\sigma}[\bar{\nabla}_\sigma, \bar{\nabla}_\beta]\bar{\nabla}^\beta\mathcal{G}^L_{\mu\nu}\,, \\
&= \bar{\nabla}^\beta\bar{g}^{\mu\sigma}(\bar{R}_{\sigma\beta\mu}{}^\lambda\mathcal{G}^L_{\lambda\nu} + \bar{R}_{\sigma\beta\nu}{}^\lambda\mathcal{G}^L_{\mu\lambda}) \\
&\quad + \bar{g}^{\mu\sigma}(\bar{R}_{\sigma\beta}{}^\beta{}_\lambda\bar{\nabla}^\lambda\mathcal{G}^L_{\mu\nu} + \bar{R}_{\sigma\beta\mu}{}^\lambda\bar{\nabla}^\beta\mathcal{G}^L_{\lambda\nu} + \bar{R}_{\sigma\beta\mu}{}^\lambda\bar{\nabla}^\beta\mathcal{G}^L_{\mu\lambda})\,.
\end{aligned}
$$

Using (4.5) we get

$$
\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu} = -2\frac{2\Lambda}{(D-1)(D-2)}\bar{\nabla}_\nu\bar{g}^{\mu\sigma}\mathcal{G}^L_{\mu\sigma}\,.
$$

Using (4.8) the $\bar{g}^{\mu\sigma}\mathcal{G}^L_{\mu\sigma}$ term can be written as

$$
\begin{aligned}
\bar{g}^{\mu\sigma}\mathcal{G}^L_{\mu\sigma} &= \bar{g}^{\mu\sigma}(R^L_{\mu\sigma} - \frac{1}{2}\bar{g}_{\mu\sigma}R^L - \frac{2\Lambda}{(D-2)}h_{\mu\sigma}) \\
&= \bar{g}^{\mu\sigma}R^L_{\mu\sigma} - \frac{1}{2}DR^L - \frac{2\Lambda}{(D-2)}h\,.
\end{aligned}
$$

Here we use the linear Ricci tensor (4.16)

$$
\begin{aligned}
\bar{g}^{\mu\sigma}\mathcal{G}^L_{\mu\sigma} &= \frac{1}{2}\bar{g}^{\mu\sigma}(-\bar{\Box}h_{\mu\sigma} - \bar{\nabla}_\mu\bar{\nabla}_\sigma h + \bar{\nabla}^\beta\bar{\nabla}_\sigma h_{\beta\mu} + \bar{\nabla}^\beta\bar{\nabla}_\mu h_{\beta\sigma}) \\
&\quad -\frac{1}{2}DR^L - \frac{2\Lambda}{(D-2)}h
\end{aligned}
$$

and we end up with

$$\bar{g}^{\mu\sigma}\mathcal{G}^L_{\mu\sigma} = \frac{(2-D)}{2}R^L$$

where the $\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu}$ term will equal to

$$\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu} = \frac{2\Lambda}{(D-1)}\bar{\nabla}_\nu R^L\,.$$

Therefore,

$$\bar{\nabla}^\mu\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{\nabla}R^L = \frac{2\Lambda}{(D-1)}\bar{\nabla}R^L - \frac{2\Lambda}{(D-1)}\bar{\nabla}R^L = 0\,.$$

Hence, $T_{\mu\nu}$ is background conserved tensor.

There remains now to obtain a Killing energy expression from below equation, namely, to write $\bar{\xi}_\nu T^{\mu\nu}$ as a surface integral.

$$\begin{aligned}
\bar{g}^{\mu\sigma}\mathcal{G}^L_{\mu\sigma}T_{\mu\nu} \quad &= \quad \mathcal{G}^L_{\mu\nu}\{-\frac{1}{\kappa} + \frac{4\Lambda D}{(D-2)^2}(2\alpha + \frac{\beta}{(D-1)})\} \\
&\quad +(2\alpha+\beta)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{(D-2)}\bar{g}_{\mu\nu})R^L \\
&\quad +\beta(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{g}_{\mu\nu}R^L)\,.
\end{aligned}$$

Here is some useful equations that will help us in this purpose.

From (2.1) we have

$$\bar{\Box}\bar{\xi}_\nu = -\frac{2\Lambda}{(D-2)}\bar{g}_{\nu\lambda}\bar{\xi}^\lambda\,, \tag{B.1}$$

and

$$\begin{aligned}
\bar{\nabla}_\alpha\bar{\nabla}^\mu\mathcal{G}^{\alpha\nu}_L \quad &= \quad [\bar{\nabla}_\alpha, \bar{\nabla}^\mu]\mathcal{G}^{\alpha\nu}_L + \bar{\nabla}^\mu\bar{\nabla}_\alpha\mathcal{G}^{\alpha\nu}_L \\
&= \quad \bar{R}_\alpha{}^{\mu\alpha}{}_\lambda\mathcal{G}^{\lambda\nu}_L + \bar{R}_\alpha{}^{\mu\nu}{}_\lambda\mathcal{G}^{\alpha\lambda}_L
\end{aligned}$$

using (4.5) and (4.6) we find

$$\bar{\nabla}_\alpha \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} = \frac{2\Lambda D}{(D-2)(D-1)} \mathcal{G}_L^{\mu\nu} + \bar{g}^{\mu\nu} \frac{\Lambda}{(D-1)} R_L \,.$$

Therefore,

$$\bar{\xi}_\nu \bar{\nabla}_\alpha \bar{\nabla}^\mu \mathcal{G}_L^{\alpha\nu} = \frac{2\Lambda D}{(D-2)(D-1)} \bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} + \bar{\xi}^\mu \frac{\Lambda}{(D-1)} R_L. \tag{B.2}$$

From Bianchi identity we have

$$\bar{R}_{\mu\nu\sigma}{}^\beta + \bar{R}_{\sigma\mu\nu}{}^\beta + \bar{R}_{\nu\sigma\mu}{}^\beta = 0 \,.$$

Multiplying this equation with $\bar{\xi}_\beta$ we get

$$\bar{R}_{\mu\nu\sigma}{}^\beta \bar{\xi}_\beta + \bar{R}_{\sigma\mu\nu}{}^\beta \bar{\xi}_\beta + \bar{R}_{\nu\sigma\mu}{}^\beta \bar{\xi}_\beta = 0 \,,$$

and using the commutator property of Riemann tensor we have

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \bar{\xi}_\sigma + [\bar{\nabla}_\sigma, \bar{\nabla}_\mu] \bar{\xi}_\nu + [\bar{\nabla}_\nu, \bar{\nabla}_\sigma] \bar{\xi}_\mu = 0 \,.$$

Expanding these commutators and using the Killing vector equation we get

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\xi}_\sigma + \bar{\nabla}_\nu \bar{\nabla}_\sigma \bar{\xi}_\mu + \bar{\nabla}_\sigma \bar{\nabla}_\mu \bar{\xi}_\nu = 0 \,,$$

and in the second term we change the places of σ and μ with a minus sign where it will be a commutation with the first term,

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] \bar{\xi}_\sigma = -\bar{\nabla}_\sigma \bar{\nabla}_\mu \bar{\xi}_\nu$$

$$\bar{R}_{\mu\nu\sigma}{}^\beta \bar{\xi}_\beta = -\bar{\nabla}_\sigma \bar{\nabla}_\mu \bar{\xi}_\nu$$

changing indices $\sigma \to \alpha, \mu \leftrightarrow \beta$

$$-\bar{R}_{\beta\nu\alpha}{}^{\mu}\bar{\xi}_{\mu} = \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\xi}_{\nu}$$

$$\bar{R}^{\mu}{}_{\alpha\beta\nu}\bar{\xi}_{\mu} = \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\xi}_{\nu}$$

again using (4.5) we have

$$\begin{aligned} \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\xi}_{\nu} &= \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}^{\mu}_{\beta}\bar{g}_{\alpha\nu} - \bar{g}^{\mu}_{\nu}\bar{g}_{\alpha\beta})\bar{\xi}_{\mu}\,, \\ &= \frac{2\Lambda}{(D-1)(D-2)}(\bar{g}_{\alpha\nu}\bar{\xi}_{\beta} - \bar{g}_{\alpha\beta}\bar{\xi}_{\nu})\,. \end{aligned} \tag{B.3}$$

Therefore,

$$\begin{aligned} &(i) \quad \mathcal{G}_L^{\mu\nu}\bar{\Box}\bar{\xi}_{\nu} = -\frac{2\Lambda}{(D-2)}\mathcal{G}_L^{\mu\nu}\bar{\xi}_{\nu}\,, \\ &(ii) \quad \bar{\xi}_{\nu}\bar{\nabla}_{\alpha}\bar{\nabla}^{\mu}\mathcal{G}_L^{\alpha\nu} = \frac{2\Lambda D}{(D-2)(D-1)}\bar{\xi}_{\nu}\mathcal{G}_L^{\mu\nu} + \frac{\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L\,, \\ &(iii) \quad -\mathcal{G}_L^{\alpha\nu}\bar{\nabla}_{\alpha}\bar{\nabla}^{\mu}\bar{\xi}_{\nu} = -\mathcal{G}_L^{\alpha\nu}\bar{g}^{\mu\beta}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\xi}_{\nu} \\ &\qquad = \frac{\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L + \frac{2\Lambda}{(D-1)(D-2)}\bar{\xi}_{\nu}\mathcal{G}_L^{\mu\nu}\,. \end{aligned}$$

So that $\bar{\xi}_{\nu}\bar{\Box}\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L$ can be written as

$$\begin{aligned} \bar{\xi}_{\nu}\bar{\Box}\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L &= \bar{\nabla}_{\alpha}\{\bar{\xi}_{\nu}\bar{\nabla}^{\alpha}\mathcal{G}_L^{\mu\nu} - \bar{\xi}_{\nu}\bar{\nabla}^{\mu}\mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu}\bar{\nabla}^{\alpha}\bar{\xi}_{\nu} + \mathcal{G}_L^{\alpha\nu}\bar{\nabla}^{\mu}\bar{\xi}_{\nu}\} \\ &\quad -\frac{2\Lambda}{(D-2)}\mathcal{G}_L^{\mu\nu}\bar{\xi}_{\nu} + \frac{2\Lambda D}{(D-2)(D-1)}\bar{\xi}_{\nu}\mathcal{G}_L^{\mu\nu} \\ &\quad +\frac{\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L + \frac{\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L \\ &\quad +\frac{2\Lambda}{(D-1)(D-2)}\bar{\xi}_{\nu}\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L\,, \end{aligned}$$

and

$$\bar{\xi}_{\nu}\bar{\Box}\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{(D-1)}\bar{\xi}^{\mu}R_L = \bar{\nabla}_{\alpha}\{\bar{\xi}_{\nu}\bar{\nabla}^{\alpha}\mathcal{G}_L^{\mu\nu} - \bar{\xi}_{\nu}\bar{\nabla}^{\mu}\mathcal{G}_L^{\alpha\nu} - \mathcal{G}_L^{\mu\nu}\bar{\nabla}^{\alpha}\bar{\xi}_{\nu} + \mathcal{G}_L^{\alpha\nu}\bar{\nabla}^{\mu}\bar{\xi}_{\nu}\}$$

$$+\frac{4\Lambda}{(D-2)(D-1)}\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu}\,.$$

With the help of chapter 4, we can write

$$\begin{aligned}
\bar{\xi}_\nu T^{\mu\nu} &= \bar{\xi}_\nu \mathcal{G}_L^{\mu\nu}\{-\frac{1}{\kappa}+\frac{4\Lambda D}{(D-2)^2}(2\alpha+\frac{\beta}{(D-1)})\}+\frac{4\Lambda\beta}{(D-2)(D-1)}\bar{\xi}_\nu \mathcal{G}_L^{\mu\nu} \\
&+(2\alpha+\beta)\bar{\nabla}_\alpha\{\bar{\xi}^\mu\bar{\nabla}^\alpha R_L-\bar{\xi}^\alpha\bar{\nabla}^\mu R_L+R_L\bar{\nabla}^\mu\bar{\xi}^\alpha\} \\
&+\beta\bar{\nabla}_\alpha\{\bar{\xi}_\nu\bar{\nabla}^\alpha\mathcal{G}_L^{\mu\nu}-\bar{\xi}_\nu\bar{\nabla}^\mu\mathcal{G}_L^{\alpha\nu}-\mathcal{G}_L^{\mu\nu}\bar{\nabla}^\alpha\bar{\xi}_\nu+\mathcal{G}_L^{\alpha\nu}\bar{\nabla}^\mu\bar{\xi}_\nu\}\,.
\end{aligned}$$

The first two terms can be added to have

$$\begin{aligned}
\bar{\xi}_\nu T^{\mu\nu} &= \bar{\xi}_\nu \mathcal{G}_L^{\mu\nu}\{-\frac{1}{\kappa}+\frac{8\Lambda}{(D-2)^2}(\alpha D+\beta)\} \\
&+(2\alpha+\beta)\bar{\nabla}_\alpha\{\bar{\xi}^\mu\bar{\nabla}^\alpha R_L-\bar{\xi}^\alpha\bar{\nabla}^\mu R_L+R_L\bar{\nabla}^\mu\bar{\xi}^\alpha\} \\
&+\beta\bar{\nabla}_\alpha\{\bar{\xi}_\nu\bar{\nabla}^\alpha\mathcal{G}_L^{\mu\nu}-\bar{\xi}_\nu\bar{\nabla}^\mu\mathcal{G}_L^{\alpha\nu}-\mathcal{G}_L^{\mu\nu}\bar{\nabla}^\alpha\bar{\xi}_\nu+\mathcal{G}_L^{\alpha\nu}\bar{\nabla}^\mu\bar{\xi}_\nu\}\,,
\end{aligned}$$

and taking the surface terms where $\alpha \to i$

$$\begin{aligned}
Q^\mu(\bar{\xi}) &= \{-\frac{1}{\kappa}+\frac{8\Lambda}{(D-2)^2}(\alpha D+\beta)\}\int d^{D-1}x\sqrt{-g}\bar{\xi}_\nu\mathcal{G}_L^{\mu\nu} \\
&+(2\alpha+\beta)\int dS_i\sqrt{-g}\{\bar{\xi}^\mu\bar{\nabla}^i R_L-\bar{\xi}^i\bar{\nabla}^\mu R_L+R_L\bar{\nabla}^\mu\bar{\xi}^i\} \\
&+\beta\int dS_i\sqrt{-g}\{\bar{\xi}_\nu\bar{\nabla}^i\mathcal{G}_L^{\mu\nu}-\bar{\xi}_\nu\bar{\nabla}^\mu\mathcal{G}_L^{i\nu}-\mathcal{G}_L^{\mu\nu}\bar{\nabla}^i\bar{\xi}_\nu+\mathcal{G}_L^{i\nu}\bar{\nabla}^\mu\bar{\xi}_\nu\}\,.
\end{aligned}$$

This is the final form of the conserved charges for the generic quadratic theory.