COLLIDING GRAVITATIONAL PLANE WAVES: BELL-SZEKERES SOLUTION

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

EFSUN CAMBAZ

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
PHYSICS

AUGUST 2005

Approval of the Graduate School of Natural and Applied Sciences.

Prof. Dr. Canan ÖZGEN
Director

I cerify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science

Prof. Dr. Sinan BİLİKMEN
Head of Department

This is to cerify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Atalay KARASU
Supervisor

Examining Commitee Members

Assoc. Prof. Dr. Bayram TEKİN (METU, PHYS)

Prof. Dr. Atalay KARASU (METU, PHYS)

Prof. Dr. Ayşe KARASU (METU, PHYS)

Assoc. Prof. Dr. Yusuf İPEKOĞLU (METU, PHYS)

Asst. Prof. Dr. Refik TURHAN (ANKARA UNV.)

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Lastname: Efsun CAMBAZ

Signature :

ABSTRACT

COLLIDING GRAVITATIONAL PLANE WAVES:
BELL-SZEKERES SOLUTION

Cambaz, Efsun
M.Sc., Department of Physics
Supervisor: Prof. Dr. Atalay Karasu

August 2005, 41 pages

The collision of pure electromagnetic plane waves with collinear polarization in Einstein-Maxwell theory and the collision of gravitational plane waves in vacuum Einstein theory are studied. The singularity structure of the Bell-Szekeres and the Szekeres solutions is examined by using curvature invariants. As a final work, the collision of the plane waves in dilaton gravity theory is studied and also the singularity structure of the corresponding space-time is examined.

Keywords: Gravitation, Colliding Plane Waves, Bell-Szekeres metric

ÖZ

ÇARPIŞAN GRAVİTASYONEL DÜZLEM DALGALAR:
BELL-SZEKERES ÇÖZÜMÜ

Cambaz, Efsun
Yüksek Lisans, Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Atalay Karasu

Ağustos 2005, 41 sayfa

Einstein-Maxwell teorisinde eş-çizgisel kutuplanıma sahip elektromagnetik düzlem dalgaların çarpışması ve boşluk Einstein teorisinde gravitasyonel düzlem dalgaların çarpışması ele alındı. Bell-Szekeres ve Szekeres çözümleri için tanımlanan uzay-zamanın tekillik yapısı eğrilik değişmezleri kullanılarak incelendi. Son olarak, dilaton gravitasyon teorisinde düzlem dalgaların çarpışması ele alındı ve bu metriğin tanımladığı uzay-zamanın tekillik yapısı incelendi.

Anahtar Kelimeler: Gravitasyon, Çarpışan Düzlem Dalgalar, Bell-Szekeres Metriği

To My Parents…

ACKNOWLEDGMENTS

I would like to express my deepest thanks to my supervisor Prof. Dr. Atalay Karasu for his guidance, suggestions, encouragements and insight during the course of this thesis.

I am indebted to my family for their moral and financial support, encouragement and love throughout the time that has gone for getting my M.S. degree.

I am also grateful to my friends for their friendship and support during my life in Ankara.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The colliding plane wave solutions have been an important topic in classical general relativity. They are the exact solutions describing the collision of plane wave in a flat background. The work on this subject was proposed by Penrose [1] in 1965. Since then, many exact colliding plane wave solutions have been constructed [2]. The first results on exact solutions of the vacuum Einstein equations representing colliding plane gravitational waves with collinear polarizations were obtained by Szekeres [3], and by Khan and Penrose [4]. Later, Nutku and Halil [5] generalized this solution to the case of noncollinear polarizations. The first exact solution of the Einstein-Maxwell equations representing colliding plane shock electromagnetic waves with collinear polarizations was obtained by Bell and Szekeres [6]. Later, Halil [7], Gürses and Halilsoy [8], Griffiths [9] and Chandrasekhar and Xanthopoulos [10] studied exact solutions of the Einstein-Maxwell equations describing the collision of gravitational and electromagnetic waves. The main result of these exact solutions is that the future of the collision surface is bounded by a curvature singularity in future directions. This fact could be considered as an inevitable effect of the nonlinear gravitational focusing. It has been expected that the study of the colliding plane wave solutions may tell us about the nature of the space-time singularity. The singularity structure of the colliding plane wave geometries has been investigated by Sbytov [11], Tipler [12] and Bonnor and Vickers [13]. The structure of the governing field equations for colliding plane waves, their physical and geometrical interpretations, and various particular solutions and techniques have been described in [2].

Colliding plane wave solutions are not only important in classical general relativity but also in the higher dimensional gravity. Plane wave metrics in various dimensions provide exact solutions in the string theory [14]. It is well-known that in the low energy effective action of string theory, there are dilaton fields and various kinds of multi-form fields, coupled with each other in the supergravity action. The first exact solutions of the colliding plane waves in Einstein-Maxwell-Dilaton gravity theories were obtained by Gürses and Sermutlu [15]. This problem is formulated for the collinear polarization case and it was shown that when the dilaton coupling constant vanishes one of the solutions reduces to the well-known Bell-Szekeres solution in the Einstein-Maxwell theory and more recently Halilsoy and Sakalli [16] have obtained the extension of Bell-Szekeres solution in Einstein-Maxwell-Axion theory.

In [17-24], the colliding plane wave solutions in the dilatonic gravity, in the higher dimensional gravity, and in the higher dimensional Einstein-Maxwell theory were discussed.

In this thesis, we study the Bell-Szekeres solution, the Szekeres solution and the colliding gravitational plane waves in dilaton gravity.

In Chapter 2, we briefly review the known properties of the colliding plane waves in general relativity.

In Chapter 3, we discuss the solution of Bell-Szekeres and the solution of Szekeres.

In Chapter 4, we study the exact solutions of two colliding gravitational plane waves in dilaton field.

CHAPTER 2

COLLIDING PLANE WAVES IN GENERAL RELATIVITY

In this chapter we will discuss the collision of gravitational and electromagnetic plane waves in general relativity.

2.1 Plane Gravitational Waves

In the electromagnetic theory, Maxwell's equations are linear, so, electromagnetic waves pass through each other without any interaction. However, in general relativity Einstein's field equations are non-linear, so, interactions occur between gravitational waves while they pass through each other. This property attracted many authors to find a solution to the problem of head-on collisions of gravitational waves.

When searching for exact solutions it is appropriate that the approaching waves have plane symmetry, because for plane waves it is possible to formulate the problem in such a way that exact solutions can be found before and after the interactions.

Our consideration of gravitational waves starts from the pioneering work of Einstein and is based on the linearized form of field equations. In this approximation, we shall see that plane wave solutions lead to the result that gravitational waves are transverse and possess two polarization states. Also, in Einstein's theory, gravitational waves are considered as perturbations of space-time that propagate with the speed of light.

2.1.1 The Linearized Field Equations

We assume that the metric describing the space-time is slightly different from the Minkowski metric η_{ab} which describes the flat space-time [25]:

$$g_{ab} = \eta_{ab} + \varepsilon h_{ab} \tag{2.1}$$

where $a, b = 0,1,2,3$ and ε is a dimensionless parameter and, throughout, we will neglect terms of second order or higher in ε. We also impose that the space-time is asymptotically flat, that is, if r denotes a radial parameter, then

$$\lim_{r\to\infty} h_{ab} = 0\,. \tag{2.2}$$

Since $g_{ab}g^{bc} = \delta_a^c$, than the inverse of the metric is given by

$$g^{ab} = \eta^{ab} - \varepsilon h^{ab}\,. \tag{2.3}$$

The Christoffel symbols are defined by

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{dc,b} + g_{db,c} - g_{bc,d}). \tag{2.4}$$

Since η_{ab} is constant then by using (2.1) and (2.4), we can write the Christoffel symbols as

$$\Gamma^a_{bc} = \frac{1}{2}\varepsilon(h^a{}_{c,b} + h^a{}_{b,c} - h_{bc,}{}^a). \tag{2.5}$$

The Riemann tensor (or curvature tensor) is defined by

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bd,c} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \quad , \tag{2.6}$$

and with the equation (2.5) this becomes

$$R_{abcd} = g_{ae}R^e{}_{bcd} = \frac{1}{2}\varepsilon(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}) . \tag{2.7}$$

Then the Ricci tensor is

$$R_{ab} = g^{cd}R_{cadb} = \frac{1}{2}\varepsilon(h^c{}_{a,bc} + h^c{}_{b,ac} - \Box\, h_{ab} - h_{,ab}) , \tag{2.8}$$

where $h = \eta^{ab}h_{ab} = h^a{}_a$ and $\Box$ is the D'Alembertian operator defined as

$$\begin{aligned}
\Box &= \eta^{ab}\partial_a\partial_b \\
&= \partial^a\partial_a \\
&= \frac{\partial^2}{\partial t^2} - \nabla^2 \\
&= \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).
\end{aligned}$$

Contracting R_{ab} with g_{ab}, the Ricci Scalar is obtained as

$$R = g^{ab}R_{ab} = \varepsilon(h^{cd}{}_{,cd} - \Box\, h), \tag{2.9}$$

and finally the Einstein field tensor, G_{ab}, in the weak gravitational field is

$$G_{ab}=R_{ab}-\frac{1}{2}g_{ab}R=\frac{1}{2}\varepsilon(h^{c}{}_{a,bc}+h^{c}{}_{b,ac}-\Box h_{,ab}-\eta_{ab}h^{cd}{}_{,cd}+\eta_{ab}\Box h) \tag{2.10}$$

The linearized Einstein field equations are then

$$G_{ab}=\kappa T_{ab}, \tag{2.11}$$

where T_{ab} is the energy-momentum tensor and κ is the gravitational constant.

2.1.2 Gauge Transformations

Let us consider what happens to linearized equations under a coordinate transformation of the form

$$x^a \to x'^a = x^a + \varepsilon\xi^a. \tag{2.12}$$

Applying this to the transformation formula for g_{ab} given by

$$g_{ab}(x)=\frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}g'_{cd}(x'),$$

we find the transformation of h_{ab}, namely,

$$h_{ab}\to h'_{ab}=h_{ab}-2\xi_{(a,b)}, \tag{2.13}$$

where the bracket denotes symmetrization. This is called a gauge transformation. We can see that both the linearized curvature tensor (2.7) and its contractions are gauge invariant quantities, that are unchanged to first order in ε by transformations of the form (2.13). To fix the gauge, we go back to field equations and define new variables ψ_{ab} by

$$\psi_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h, \tag{2.14}$$

then (2.8), (2.9) and (2.10) become

$$R_{ab} = \frac{1}{2}\varepsilon(\psi^{c}{}_{a,bc} + \psi^{c}{}_{b,ac} - \Box\, h_{ab}), \tag{2.15}$$

$$R = \frac{1}{2}\varepsilon(2\psi^{cd}{}_{,cd} - \Box\, h_{ab}), \tag{2.16}$$

$$G_{ab} = \frac{1}{2}\varepsilon(\psi^{c}{}_{a,bc} + \psi^{c}{}_{b,ac} - \Box\,\psi_{ab} - \eta_{ab}\psi^{ab}{}_{,ab}). \tag{2.17}$$

This suggest that our field equations will reduce to wave equations if we impose the condition

$$\psi^{a}{}_{b,a} = h^{a}{}_{b,a} - \frac{1}{2}h_{,b} = 0, \tag{2.18}$$

which is called the Einstein, de Donder, Hilbert, or Fock gauge. Then by (2.17), Einstein's full field equations reduce to

$$\frac{1}{2}\varepsilon\,\Box\,\psi_{ab} = -\kappa T_{ab}. \tag{2.19}$$

Then, the vacuum field equations in the Einstein's gauge reduce to

$$\Box \psi_{ab} = 0. \tag{2.20}$$

Combining (2.20) and (2.14), we find that h_{ab} must satisfy the classical wave equation

$$\Box h_{ab} = 0. \tag{2.21}$$

Thus, we conclude that, in linearized theory, gravitational effects propagate as waves with the speed of light.

2.1.3 Linearized Plane Gravitational Waves

We look for a simple solution of the linearized vacuum field equations which represents an infinite plane wave propagating in the x-direction. We start by introducing the coordinates

$$(x^0, x^1, x^2, x^3) = (t, x, y, z)$$

and adopt the ansatz

$$h_{ab} = h_{ab}(t, x) \tag{2.22}$$

which requires

$$h_{ab,2} = h_{ab,3} = 0\,. \tag{2.23}$$

Then by using (2.7), we find 20 independent components of Riemann tensor [25]. From the linearized vacuum field equations in the form $R_{ab} = 0$, some of the components of Riemann tensor will vanish. We can see that, the non-zero components of Riemann tensor only involve the components h_{22}, h_{23} and h_{33}. Then h_{ab} can be written in this form

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & h_{33} \end{bmatrix} \quad . \tag{2.24}$$

We sharpen our ansatz (2.22) by requiring

$$h_{ab} = h_{ab}(t - x), \tag{2.25}$$

so that it clearly represents a solution propagating in the x direction with the speed of light. If we use the Einstein gauge condition (2.18) and the gauge freedom (2.13), where ξ_a also satisfies the wave equation, there might exist a coordinate system in which $h_{33} = -h_{22}$ and h_{ab} has only $h_{22}(t-x)$ and $h_{23}(t-x)$ components. Hence, h_{ab} will be in this canonical form

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{bmatrix} \quad . \tag{2.26}$$

Now, we consider the physical significance of these two independent functions in the next section.

2.1.4 Polarization States

Let us first define the line element in general form as

$$ds^2 = g_{ab} dx^a dx^b , \tag{2.27}$$

where $a, b = 0,1,2,3.$ In the case $h_{23} = 0,$ this line element becomes

$$ds^2 = dt^2 - dx^2 - [1 - \varepsilon h_{22}(t-x)]dy^2 - [1 + \varepsilon h_{22}(t-x)]dz^2 , \tag{2.28}$$

which is called an '$h_{22} -$ wave'. Let us suppose that h_{22} is some oscillatory function of $u\,(= t - x)$ so that there are values when $h_{22} > 0$ and values when $h_{22} < 0$. As seen from the metric (2.28), this wave causes oscillations only in the $yz -$ plane. This implies that an $h_{22} -$ wave has a transverse character and we refer to this state as a wave with + polarization.

On the other hand, in the case h_{22} =0, the line element (2.28) takes the form

$$ds^2 = dt^2 - dx^2 - [1 - \varepsilon h_{23}(t-x)]d\bar{y}^2 - [1 + \varepsilon h_{23}(t-x)]d\bar{z}^2 , \tag{2.29}$$

by performing a rotation through 45° in the $yz -$ plane given by

$$y \to \bar{y} = \frac{1}{\sqrt{2}}(y+z), \qquad z \to \bar{z} = \frac{1}{\sqrt{2}}(-y+z). \tag{2.30}$$

This is called '$h_{23} -$ wave'. This wave is also transverse and produces the same effect as an $h_{22} -$ wave but with the axes rotated 45°. We refer to this state as a wave with X polarization.

Clearly, a general wave is a superposition of these two polarization states. The fact that the two polarization states are at 45° to each other contrasts with the two polarization states of an electromagnetic wave, which are 90° to each other.

2.2 Exact Plane Gravitational Waves

If we introduce double null coordinates defined by

$$u = t - x, \qquad v = t + x \tag{2.32}$$

in (2.28), then an h_{22} – wave has a line element of the form

$$ds^2 = dudv - f^2(u)dy^2 - g^2(u)dz^2, \tag{2.33}$$

where

$$f^2(u) = 1 - \varepsilon h_{22}(u), \qquad g^2(u) = 1 + \varepsilon h_{22}(u). \tag{2.34}$$

From the line element (2.33), only one vacuum field equation can be found

$$f''/f + g''/g = 0, \tag{2.35}$$

where prime denotes derivative with respect to *u*. Let us denote the first term by the function $h(u)$,i.e.

$$f''/f = -g''/g = h\ . \tag{2.36}$$

Hence, any choice of arbitrary function $h(u)$ gives rise to a vacuum solution. Such exact solutions are called linearly polarized plane gravitational waves. They represent plane-fronted waves, abstracted from any sources, propagating in the x-direction.

If we carry out the coordinate transformation,

$$U = u, \quad V = v + y^2 ff' + z^2 gg', \quad Y = fy, \quad Z = gz,$$

then the line element is transformed into Brinkmann form

$$ds^2 = h(U)(Z^2 - Y^2)dU^2 + dUdV - dY^2 - dZ^2 \tag{2.37}$$

which shows the explicit dependence on function $h(u)$. This function represents the amplitude of the polarized wave.

Such solutions allow us to investigate the question of the scattering of gravitational waves. Unlike electromagnetic theory, where the linearity of the theory means that electromagnetic waves pass through each other unaltered and so one can superpose 2 solutions; there is, in general, no superposition principle in general relativity. Indeed we may expect the non-linearity of the theory to reveal itself in the interaction of two gravitational waves. However, (2.37) does reveal a limited superposition principle in that two plane waves moving in the same direction can be superposed simply by adding their corresponding $h(u)$ functions. Thus when moving in the same direction, two such gravitational waves do not scatter one another. To exhibit scattering, we need two waves moving in different directions. If we consider two linearly polarized waves colliding at an angle, we can always find a class of observers who consider the collision to be head on. Hence, it is sufficient to work in a coordinate system in which the waves appear to collide head on.

2.2.1 Collision of Plane Waves

In this section we consider the head on collision between plane gravitational and electromagnetic waves.

In collision problems, we may choose two null coordinates u and v such that the wave fronts of the waves are represented by u=0 and v=0 (see figure 2.1). The metric describing the plane-fronted gravitational waves is given by

$$ds^2 = 2dudr + H(u,Y,Z)du^2 - dY^2 - dZ^2, \tag{2.38}$$

where $H(u,Y,Z)$ characterizes the nature of the wave.

For an impulsive plane gravitational wave function H can be taken as

$$H = a\delta(u)(Z^2 - Y^2) \tag{2.39}$$

where a is the amplitude of the wave and $\delta(u)$ is the Dirac-delta function, which is defined by the requirements

$$\delta(u) = \begin{cases} 0 & \text{if } u \neq 0 \\ \infty & \text{if } u = 0 \end{cases}, \tag{2.40}$$

$$\int_{-\infty}^{\infty} f(u)\delta(u)du = f(0). \tag{2.41}$$

Alternatively, for a plane gravitational shock wave

$$H = a^2\theta(u)(Z^2 - Y^2), \tag{2.42}$$

where $\theta(u)$ is the Heaviside step function defined by

$$\theta(u) = \begin{cases} 0 & \text{if } u \le 0 \\ 1 & \text{if } u > 0 \end{cases}, \tag{2.43}$$

and for an electromagnetic plane wave

$$H = a^2\theta(u)(Z^2 + Y^2). \tag{2.44}$$

The form of metric (2.38), however, is unsuitable for a discussion of colliding plane waves since it contains only one null coordinate. It is therefore convenient to transform this metric to the Szekeres line element [25], which is

$$ds^2 = e^{-M} dudv + e^{-U}(e^V \cosh W dy^2 - 2\sinh W dydz + e^{-V} \cosh W dz^2) \tag{2.45}$$

where *M, U, V* and *W* are functions of u and v in general. In the study of collision problems, it is convenient to divide the space-time into four regions labeled I(u <0, v<0), II (u >0, v<0), III (u <0, v>0) and IV (u >0, v>0), as shown in figure 2.1. These regions are bounded by the two null hypersurfaces u =0 and v=0.

The metric functions *U, V, M* and *W* must take different forms in different regions. That is

Region I : U=V=W=M=0,

Region II : $U = U(u), V = V(u), W = W(u), M = M(u)$,

Region III : $U = U(v), V = V(v), W = W(v), M = M(v)$,

Region IV : $U = U(u,v), V = V(u,v), W = W(u,v), M = M(u,v)$.

Region I is the flat Minkowski space-time, it is assumed that the collision is taking place in the absence of any background field, and regions II and III contain the approaching waves from opposite directions. Region IV is the interaction region in which the metric is in the form (2.45) (see figure 2.1). The metric coefficients in region IV are uniquely determined by a characteristic initial value problem with data determined on the null hypersurfaces $u=0$ and $v=0$.

The function W determines the rotation of the wave polarization vectors. If we have constant and parallel polarization then one can put W=0.

The general recipe to construct the colliding plane wave solutions is to solve the field equations in region IV and then reduce the solutions to other regions, requiring the metric to be continuous and invertible in order to paste the solutions in different regions.

More importantly, as a physical solution one has to impose some kind of junctions on the metric to get an acceptable physical solution. The physical conditions can be translated into conditions on the metric are called the junction conditions.

In the collision problem, we generally use the Lichnerowicz or the O'Brien and Synge (OS) boundary conditions. The Lichnerowicz conditions require that there should exist a coordinate system in which the components of the metric and electromagnetic potential are at least of class C^1 on the null surfaces. However, OS require that

$$g_{ab}, \qquad g^{ab}g_{ab,0}, \qquad g^{a0}g_{ab,0}$$

be continuous across the null surface (note that 0 in the above formulae for $u=0$ or $v=0$). Moreover, OS condition means that the metric functions U, V, M and W need to be continuous and $U_u=0$ across the junction at $u=0$. The same happens at the junction at $v=0$. However, the above Lichnerowicz or OS junction condition on the metric is not enough. To be physically sensible, the curvature invariants R and $R_2=R_{ab}R^{ab}$ should not blow up at the null boundaries.

Usually, when discussing the colliding plane wave solutions, one does not put on any constraints on the Riemann tensor R_{abcd}, $R_4 = R_{abcd} R^{abcd}$ or other higher curvature invariants.

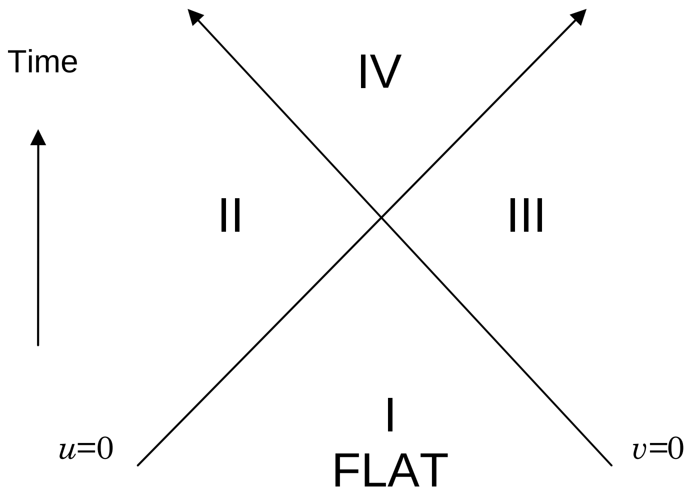

Figure 2.1: Space-time is divided into four regions. Two space-like coordinates have been suppressed. Region I is the flat background, regions II and III contain the approaching waves, and region IV is the interaction region following the collision at the point $u = 0, v = 0$.

CHAPTER 3

THE BELL-SZEKERES AND THE SZEKERES SOLUTIONS

In this chapter, we discuss the Bell-Szekeres solution, which describes colliding electromagnetic waves in Einstein-Maxwell gravity and the Szekeres solution which describes the collision of two step gravitational plane waves.

3.1 The Bell-Szekeres Solution

Here, we discuss the Bell-Szekeres solution, which is the first solution of the Einstein-Maxwell field equations and describes the collision of two step electromagnetic plane waves with collinear polarization. The line element for the Szekeres solution is given by

$$ds^2 = 2e^{-M}dudv + e^{-U}(e^V dx^2 + e^{-V}dy^2) \ . \tag{3.1}$$

The metric functions U, V and M depend on the null coordinates u and v. The non-zero Christoffel symbols of the metric (3.1) can be calculated to be

$$\Gamma^u{}_{uu} = -M_u ,$$

$$\Gamma^u{}_{yy} = \tfrac{1}{2}(U_v + V_v)e^{M-U-V} ,$$

$$\Gamma^u{}_{xx} = \tfrac{1}{2}(U_v - V_v)e^{M-U+V} ,$$

$$
\begin{aligned}
\Gamma^{v}{}_{vv} &= -M_v, \\
\Gamma^{v}{}_{yy} &= \tfrac{1}{2}(U_u + V_u)e^{M-U-V}, \\
\Gamma^{v}{}_{xx} &= \tfrac{1}{2}(U_u - V_u)e^{M-U+V}, \\
\Gamma^{y}{}_{yu} &= -\tfrac{1}{2}(U_u + V_u), \\
\Gamma^{y}{}_{yv} &= -\tfrac{1}{2}(U_v + V_v), \\
\Gamma^{x}{}_{xu} &= -\tfrac{1}{2}(U_u - V_u), \\
\Gamma^{x}{}_{xv} &= -\tfrac{1}{2}(U_v - V_v).
\end{aligned}
\tag{3.2}
$$

Here we have abbreviated the derivatives by a subscript, e.g. $M_u = \dfrac{\partial M}{\partial u}$. Using (2.6), the components of the Riemann tensor are calculated as

$$
\begin{aligned}
R_{uxux} &= \left(\frac{1}{2}U_{uu} + \frac{1}{2}M_u(U_u - V_u) - \frac{1}{4}(U_u)^2 - \frac{1}{4}(V_u)^2\right)e^{-U+V}, \\
R_{uyuy} &= \left(\frac{1}{2}U_{uu} + \frac{1}{2}M_u(U_u + V_u) - \frac{1}{4}(U_u)^2 - \frac{1}{4}(V_u)^2\right)e^{-U-V}, \\
R_{vxvx} &= \left(\frac{1}{2}U_{vv} + \frac{1}{2}M_v(U_v - V_v) - \frac{1}{4}(U_v)^2 - \frac{1}{4}(V_v)^2\right)e^{-U+V}, \\
R_{vyvy} &= \left(\frac{1}{2}U_{vv} + \frac{1}{2}M_v(U_v + V_v) - \frac{1}{4}(U_v)^2 - \frac{1}{4}(V_v)^2\right)e^{-U-V}, \\
R_{uxvx} &= \left(-\frac{1}{2}V_{uv} + \frac{1}{4}U_uV_v + \frac{1}{4}U_vV_u\right)e^{-U+V}, \\
R_{uyvy} &= \left(\frac{1}{2}V_{uv} - \frac{1}{4}U_uV_v - \frac{1}{4}U_vV_u\right)e^{-U-V}, \\
R_{uvuv} &= -e^{-M}M_{uv}.
\end{aligned}
\tag{3.3}
$$

Using (2.8), the components of Ricci tensors are then given by

$$
\begin{aligned}
R_{uu} &= U_{uu} + M_u U_u - \frac{1}{2}(U_u^2 + V_u^2), \\
R_{vv} &= U_{vv} + M_v U_v - \frac{1}{2}(U_v^2 + V_v^2), \\
R_{uv} &= U_{uv} + M_{uv} - \frac{1}{2}(U_u U_v + V_u V_v), \\
R_{yy} &= e^{M-U-V}\left(V_{uv} - \frac{1}{2}(U_v V_u + U_u V_v)\right), \\
R_{xx} &= -e^{M-U+V}\left(V_{uv} - \frac{1}{2}(U_v V_u + U_u V_v)\right).
\end{aligned}
\tag{3.4}
$$

The electromagnetic vector potential has a single non-zero component $A = (0,0,A,0)$, where A is a function of both u and v. The components of the electromagnetic field strength

$$F = \tfrac{1}{2} F_{ab} dx^a \Lambda dx^b = dA , \tag{3.5}$$

where $A = A_b dx^b$, are

$$F_{uy} = A_u \quad , \quad F_{vy} = A_v \ . \tag{3.6}$$

The energy-momentum tensor defined by

$$T_{ab} = \frac{1}{4\pi}(g^{cd} F_{ad} F_{bc} - \frac{1}{4} g_{ab} F_{cd} F^{cd}) , \tag{3.7}$$

has the following non-vanishing components

$$T_{uu} = \frac{1}{4\pi} e^{U+V} (A_u)^2 ,$$

$$T_{vv} = \frac{1}{4\pi} e^{U+V} (A_v)^2 ,$$

$$T_{uv} = \frac{1}{4\pi} e^{U+V} A_u A_v \ ,$$

$$T_{yy} = T_{xx} = \frac{1}{4\pi} e^{M} A_u A_v \ . \tag{3.8}$$

Using the Einstein field equation

$$R_{ab} = \kappa (T_{ab} - \tfrac{1}{2} g_{ab} T) , \tag{3.9}$$

where the trace of the energy momentum tensor $T = 0$ for the Bell-Szekeres metric, the Einstein-Maxwell field equations in region IV can be written as

$$U_{uv} - U_u U_v = 0, \tag{3.10}$$

$$-U_u V_v - U_v V_u + 2V_{uv} = \frac{\kappa}{2\pi} e^{U+V} A_u A_v , \tag{3.11}$$

$$-V_u A_v - V_v A_u = 2A_{uv} , \tag{3.12}$$

$$-(U_u)^2 - (V_u)^2 + 2U_{uu} + 2M_u U_u = \frac{\kappa}{2\pi} e^{U+V} (A_u)^2 , \tag{3.13}$$

$$-(U_v)^2 - (V_v)^2 + 2U_{vv} + 2M_v U_v = \frac{\kappa}{2\pi} e^{U+V} (A_v)^2 , \tag{3.14}$$

$$2M_{uv} + U_u U_v - V_u V_v = 0. \tag{3.15}$$

Here, the last equation can be derived from the other equations so it is not independent. The problem is to find the solution of the above equations. It is easy to see that the integration of the first equation gives

$$e^{-U} = f(u) + g(v), \tag{3.16}$$

where f and g are arbitrary decreasing functions in the interaction region. Here, we can find the initial data for the functions U, V, and M. In region II, (3.16) becomes

$$e^{-U} = f(u) + g(0), \tag{3.17}$$

while in region III

$$e^{-U} = f(0) + g(v), \tag{3.18}$$

where $f(0) + g(0) = 1$. We set without loss of generality $f(0) = g(0) = 1/2$. Hence, the initial data of e^{-U} determines the functions f and g. It can be shown that (3.11) and (3.12) are the integrability conditions for the other equations (3.13), (3.14) and (3.15). First we find the functions U, V and A from (3.10), (3.11) and (3.12), then the function M can be obtained by integrating (3.13) and (3.14). Therefore, we should solve the equations (3.11) and (3.12) first.

It is useful to change the variables (u, v) to (f, g), so that the field equations become

$$-U_f V_g - U_g V_f + 2V_{fg} = \frac{\kappa}{2\pi} e^{U+V} A_f A_g, \tag{3.19}$$

$$-V_f A_{,g} - V_g A_{,f} = 2A_{fg}, \tag{3.20}$$

$$M_u = -\frac{f_{uu}}{f_u} + \frac{1}{2}\frac{f_u}{f+g} - \frac{f+g}{2f_u}\left[(V_u)^2 + \frac{\kappa}{2\pi} e^{U+V} (A_u)^2\right], \tag{3.21}$$

$$M_v = -\frac{g_{vv}}{g_v} + \frac{1}{2}\frac{g_v}{f+g} - \frac{f+g}{2g_v}\left[(V_v)^2 + \frac{\kappa}{2\pi} e^{U+V} (A_v)^2\right], \tag{3.22}$$

by changing variables

$$A_u = A_f f_u, \quad A_v = A_g g_v, \quad U_u = U_f f_u, \quad U_v = U_g g_v, \quad V_u = V_f f_u, \quad V_v = V_g g_v.$$

Here, it is suitable to put the equations (3.21) and (3.22) in the form

$$e^{-M} = \frac{f_u g_v}{\sqrt{f+g}} e^{-S}, \tag{3.23}$$

where S satisfies

$$S_f = -\frac{f+g}{2}\left[(V_f)^2 + \frac{\kappa}{2\pi} e^{U+V} (A_f)^2\right], \tag{3.24}$$

$$S_g = -\frac{f+g}{2}\left[(V_g)^2 + \frac{\kappa}{2\pi} e^{U+V} (A_g)^2\right]. \tag{3.25}$$

An exact solution to the equation (3.19) and (3.20) is

$$\begin{gathered} V = \log(rw - pq) - \log(rw + pq), \\ A = \gamma(pw - rq), \end{gathered} \tag{3.26}$$

where $\gamma^2 = \dfrac{8\pi}{\kappa}$ and

$$r = (\frac{1}{2} + f)^{1/2}, \ p = (\frac{1}{2} - f)^{1/2}, \ w = (\frac{1}{2} + g)^{1/2}, \ q = (\frac{1}{2} - g)^{1/2}. \tag{3.27}$$

Since, we determine the functions $U, V,$ and A, now we can integrate the equations (3.24) and (3.25) to find the function M. The integration gives for S

$$\begin{gathered} S = -\frac{1}{2}\log(f+g) + \frac{1}{2}\log\left(\frac{1}{2} - f\right) + \log\left(\frac{1}{2} + f\right) \\ + \log\left(\frac{1}{2} - \mathrm{g}\right) + \log\left(\frac{1}{2} + \mathrm{g}\right) - \log\mathrm{c}, \end{gathered} \tag{3.28}$$

where c is the integration constant. Then the equation (3.26) becomes

$$e^{-M} = \frac{c f_u g_v}{\sqrt{\frac{1}{2} - f}\sqrt{\frac{1}{2} + f}\sqrt{\frac{1}{2} - g}\sqrt{\frac{1}{2} + g}}. \tag{3.29}$$

Here, e^{-M} is not continuous across the boundaries because of the terms $\sqrt{\frac{1}{2} - f}$ and $\sqrt{\frac{1}{2} - g}$. Bell-Szekeres have given definitions for f and g to be

$$f = \frac{1}{2} - (\sin P)^{n_1}, \qquad g = \frac{1}{2} - (\sin Q)^{n_2}, \tag{3.30}$$

where $P = au\theta(u)$ and $Q = bv\theta(v)$ with the arbitrary constants a and b . The parameters n_1 and n_2 are determined by the initial data. Physically, these parameters determine the character of the wavefront; mathematically, they determine the continuity of the metric functions on the wavefronts $u = 0, v = 0$. The terms $\sqrt{\frac{1}{2} - f}$ and $\sqrt{\frac{1}{2} - g}$ exactly cancel the effect of the term $c f_u g_v$ when $n_1 = n_2 = 2$. Then, e^{-M} becomes continuous across the boundaries. Here, it is appropriate to choose $c = \frac{1}{4ab}$ in order to put M zero in region I. Then,

$$M = 0 \tag{3.31}$$

in all regions for the Bell-Szekeres metric. Therefore, the metric is piecewise C^1 and metric functions satisfy the required O'Brien-Synge junction conditions. In terms of P and Q the solution of the complete set of equations (3.10)-(3.15) is

$$
\begin{aligned}
U &= -\log\cos(P-Q) - \log\cos(P+Q), \\
V &= \log\cos(P-Q) - \log\cos(P+Q), \\
W &= 0, \\
M &= 0.
\end{aligned}
\tag{3.32}
$$

The solution in the interaction region takes the very simple form given by

$$
ds^2 = 2dudv + \cos^2(P-Q)dx^2 + \cos^2(P+Q)dy^2 . \tag{3.33}
$$

To interpret this solution as a plane wave we have to transform the metric to Brinkmann form (2.38) by doing transformations:

$$
\begin{aligned}
\tilde{x} &= \cos(P-Q)x, \\
\tilde{y} &= \cos(P+Q)y, \\
r &= v + \tfrac{1}{2}\tan(P-Q)P_u\tilde{x}^2 + \tfrac{1}{2}\tan(P+Q)P_u\tilde{y}^2, \\
\tilde{u} &= u,
\end{aligned}
\tag{3.34}
$$

where the profile function is found to be (2.44), which means that the approaching waves are electromagnetic plane waves.

Now, we will look for the singularities in region IV for the Bell-Szekeres metric. The metric coefficient e^{-U}, according to (3.16), is given by $f(u)+g(v)$ where f and g are decreasing functions from the value 1/2. It is therefore inevitable that a singularity will develop as $f+g \to 0$.

A general analysis of space-time singularity requires all invariants. In 4-dimensinonal case there are fourteen curvature invariants [26]. Here, we give three of them which are R, $R_2\,(=R^{ab}R_{ab})$ and $R_4(=R^{abcd}R_{abcd})$. For the Bell-Szekeres metric the quadratic Riemann invariant R_4 is calculated as

$$
\begin{aligned}
R_4 &= 4R^{uxux}R_{uxux} + 4R^{uyuy}R_{uyuy} + 4R^{vxvx}R_{vxvx} \\
&+ 4R^{vyvy}R_{vyvy} + 8R^{uxvx}R_{uxvx} + 8R^{uyvy}R_{uyvy} \\
&= 4(U_u)^2(U_v)^2 + 4U_{uu}U_{vv} + 4(V_{uv})^2 \\
&+ 2((U_u)^2 - U_{uu})(V_v)^2 + 2((U_v)^2 - U_{vv})(V_u)^2 \\
&- 6(U_v)^2U_{uu} - 6(U_u)^2U_{vv}
\end{aligned}
\tag{3.35}
$$

$$
R_4 = 32a^2b^2 \tag{3.36}
$$

which is finite. The nature of the space-time singularity for the Bell-Szekeres metric has been considered by Matzner and Tipler [27], Clarke and Hayward [28] and more recently by Helliwell and Konkowski [29]. They have shown that the Bell-Szekeres solution is free of curvature singularities. It has been also shown that the hypersurface $f + g = 0$ on which the opposing waves focus each other is a Cauchy horizon rather than a curvature singularity.

For the solutions to be physical, the curvature invariants R and R_2 need to not blow up in the region IV. If we calculate them for Bell-Szekeres we see that $R = 0$ and

$$
\begin{aligned}
R_2 &= 2R^{uv}R_{uv} + R^{uu}R_{uu} + R^{vv}R_{vv} + R^{xx}R_{xx} + R^{yy}R_{yy} \\
&= 2U_{uu}U_{vv} - U_{uu}(U_v^2 + V_v^2) - U_{uv}(U_u^2 + V_u^2) + 2U_{uv}^2 \\
&+ 2V_{uv}^2 + U_u^2V_v^2 + U_v^2V_u^2 - 2V_{uv}(U_uV_v + U_vV_u)
\end{aligned}
\tag{3.37}
$$

$$
R_2 = 16a^2b^2 \tag{3.38}
$$

which is also finite. Thus we can conclude that, the Bell-Szekeres metric has no singularity [2].

We also examine whether the Bell-Szekeres metric is conformally flat or not. In general relativity, this is determined by the Weyl tensor which is the trace free part of the Riemann tensor and given by

$$\begin{aligned} C_{abcd} &= R_{abcd} - \frac{1}{2}(R_{bc}g_{ad} - R_{bd}g_{ac} - R_{ac}g_{bd} + R_{ad}g_{bc}) \\ &\quad + \frac{1}{6}R(g_{bc}g_{ad} - g_{bd}g_{ac}). \end{aligned} \tag{3.39}$$

A metric is said to be conformally flat if its Weyl tensor vanishes everywhere. Using (3.3) and (3.4), the non-zero components of the Weyl tensor can be calculated as

$$\begin{aligned} C_{uxux} &= -\frac{1}{2}(V_{uu} - U_u V_u)e^{-U+V}, \\ C_{uyuy} &= \frac{1}{2}(V_{uu} - U_u V_u)e^{-U-V}, \\ C_{vxvx} &= -\frac{1}{2}(V_{vv} - U_v V_v)e^{-U+V}, \\ C_{vyvy} &= \frac{1}{2}(V_{vv} - U_v V_v)e^{-U-V}. \end{aligned} \tag{3.40}$$

Here, there are two linearly independent components. We can choose them to be C_{uyuy} and C_{vyvy}. If we substitute the solution (3.26) into these tensors, we see that all components (3.40) are zero in all four regions. So, we can say that the interior of each region is conformally flat. However, if we calculate the Weyl tensor on the boundaries of the interaction region, on $u = 0$ and $v = 0$, it is found to be proportional to Dirac- delta function (2.43). That is,

$$C_{uyuy} = -a \sin bv \cos bv \delta(u)\theta(v), \qquad C_{vyvy} = -b \sin au \cos au \delta(v)\theta(u). \tag{3.41}$$

Thus we can conclude that, in the Bell-Szekeres solution, the collision of two step electromagnetic plane waves always generates impulsive gravitational waves along the null boundaries. This is an interesting feature of the Bell-Szekeres solution.

3.2 The Szekeres Solution

The first exact solution (that was published) which describes a collision between plane waves was in fact that of Szekeres (1970) [3]. It describes the collision of two gravitational plane waves. In the Szekeres solution, the approaching waves have constant and parallel polarizations. Since, the approaching waves are gravitational waves, the electromagnetic vector potential becomes zero, and then we can rewrite the field equations (3.10)-(3.15) as

$$U_{uv} - U_u U_v = 0, \tag{3.42}$$

$$2V_{uv} = U_u V_v + U_v V_u, \tag{3.43}$$

$$2M_u U_u = (U_u)^2 + (V_u)^2 - 2U_{uu}, \tag{3.44}$$

$$2M_v U_v = (U_v)^2 + (V_v)^2 - 2U_{vv}, \tag{3.45}$$

$$2M_{uv} = -U_u U_v + V_u V_v\ . \tag{3.46}$$

From (3.42), we have the same solutions (3.16), (3.17) and (3.18). The equation (3.43), in terms of f and g coordinates, becomes

$$2(f+g)V_{fg} + V_f + V_g = 0 \tag{3.47}$$

which is the well known Euler-Poisson-Darboux equation. There exists a large class of solutions of this equation [2]. Szekeres has obtained the solution

$$V = -2k_1 \tanh^{-1}\left(\frac{\frac{1}{2} - f}{\frac{1}{2} + g}\right)^{1/2} - 2k_2 \tanh^{-1}\left(\frac{\frac{1}{2} - g}{\frac{1}{2} + f}\right)^{1/2} , \qquad (3.48)$$

which contains two arbitrary constants k_1 and k_2. With this expression for V, the remaining equations (3.44)-(3.46) may be integrated to give

$$\begin{aligned} M = & -\log(cf_u g_v) - \frac{(k_1{}^2 + k_2{}^2 + 2k_1k_2 - 1)}{2}\log(f+g) + \frac{k_1{}^2}{2}\log(\frac{1}{2} - f) \\ & + \frac{k_2{}^2}{2}\log(\frac{1}{2} + f) + \frac{k_1{}^2}{2}\log(\frac{1}{2} + g) + \frac{k_2{}^2}{2}\log(\frac{1}{2} - g) \\ & + 2k_1k_2 \log(\sqrt{\frac{1}{2} - f}\sqrt{\frac{1}{2} - g}\sqrt{\frac{1}{2} + f}\sqrt{\frac{1}{2} + g}) \end{aligned} \qquad (3.49)$$

where c is constant. This expression contains the necessary multiples of $\log(\frac{1}{2} - f)$ and $\log(\frac{1}{2} - g)$ that are required to cancel the effects of the unbounded term $\log f_u g_v$ on the boundary. Szekeres has given definitions for f and g as

$$f = \tfrac{1}{2} - (c_1 u)^{n_1} , \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} . \qquad (3.50)$$

Then the terms $\frac{k_1{}^2}{2}\log(\frac{1}{2} - f)$ and $\frac{k_2{}^2}{2}\log(\frac{1}{2} - g)$ in (3.48) exactly cancel the term $\log(cf_u g_v)$ when $c = (c_1n_1c_2n_2)^{-1}$. In this case, M becomes continuous and is given by

$$\begin{aligned} M = & -\frac{(k_1{}^2 + k_2{}^2 + 2k_1k_2 - 1)}{2}\log(f+g) + \frac{k_2{}^2}{2}\log(\frac{1}{2} + f) \\ & + \frac{k_1{}^2}{2}\log(\frac{1}{2} + g) + 2k_1k_2 \log(\sqrt{\tfrac{1}{2} - f}\sqrt{\tfrac{1}{2} - g}\sqrt{\tfrac{1}{2} + f}\sqrt{\tfrac{1}{2} + g}) \end{aligned} \qquad (3.51)$$

where $k_1^2 = 2(1-1/n_1)$, $k_2^2 = 2(1-1/n_2)$, $n_i \geq 2$ $(i=1,2)$. Therefore, the metric is at least piecewise C^2 and the metric functions satisfy the required Lichnerowicz junction conditions. The above solution includes the Khan and Penrose solution for colliding impulsive waves when $n_1 = n_2 = 2$, $k_1 = k_2 = 1$.

Having obtained an exact solution in region IV, the question is to find the initial conditions which give rise to it. We obtain the corresponding solutions in region II simply by replacing g by 1/2. Then the equation (3.42) gives

$$U = -\log(f + \tfrac{1}{2}). \tag{3.52}$$

Using this equality, it can be seen that the solution in region II must have the line element

$$ds^2 = 2e^{-M} du dv + (f + \tfrac{1}{2})(e^V dx^2 + e^{-V} dy^2), \tag{3.53}$$

where, retaining the coordinate freedom in u. Then we can rewrite the exact solutions for region II as

$$e^V = \left(\frac{1 - \sqrt{\frac{1}{2} - f}}{1 + \sqrt{\frac{1}{2} - f}} \right)^{k_1}, \tag{3.54}$$

$$e^{-M} = -\frac{f'(\frac{1}{2} + f)^{(k_1^2 - 1)/2}}{c_1 n_1 (\frac{1}{2} - f)^{k_1^2/2}}. \tag{3.55}$$

In order to interpret this solution as a plane wave, we have to transform the metric (3.53) to the Brinkmann form (2.38) by doing the transformations:

$$\tilde{x} = (\tfrac{1}{2} + f)^{1/2} e^{V/2} x = e^{-(U-V)/2} x,$$

$$\tilde{y} = (\tfrac{1}{2} + f)^{1/2} e^{-V/2} y = e^{-(U+V)/2} y,$$

$$r = v + \tfrac{1}{4}(U_u - V_u) e^{M} \tilde{x}^2 + \tfrac{1}{4}(U_u + V_u) e^{M} \tilde{y}^2$$

$$= v + \tfrac{1}{4}(U_u - V_u) e^{M-(U-V)} x^2 + \tfrac{1}{4}(U_u + V_u) e^{M-(U+V)} y^2,$$

$$\tilde{u} = -\int \frac{(\tfrac{1}{2} + f)^{(k_1{}^2 - 1)/2}}{c_1 n_1 (\tfrac{1}{2} - f)^{k_1{}^2/2}} \frac{df}{du} du = \int e^{-M} du. \tag{3.56}$$

where the profile function is found to be

$$H(u, Y, Z) = \tfrac{1}{2} e^{2M} (U_u V_u - V_{uu} - V_u M_u) . \tag{3.57}$$

We need to calculate the profile function for given values of n_1 and k_1. Using (3.52), (3.54) and (3.55), we can simply find that

$$\text{for } n_1 = 2 \quad : \quad H(\tilde{u}) = c_1 \delta(\tilde{u}),$$

$$\text{for } n_1 > 2 \quad : \quad H(\tilde{u}) = c_1{}^2 n_1{}^2 \frac{k_1}{4} (1 - \frac{2}{n_1}) \frac{(\tfrac{1}{2} - f)^{1/2 - 2/n_1}}{(\tfrac{1}{2} + f)^{3 - 2/n_1}} \theta(\tilde{u}) . \tag{3.58}$$

Thus it can be clearly seen that approaching gravitational wave in region II is an impulsive plane wave if $n_1 = 2$ and a step plane wave if $n_1 > 2$.

We need to consider the nature of the space-time singularity as in the Bell-Szekeres solution. Using the non-zero components of the Riemann tensor (3.3) for the Szekeres metric, we can calculate the quadratic Riemann invariant (R_4) as

$$
\begin{aligned}
R_4 &= 4R^{uxux}R_{uxux} + 4R^{uyuy}R_{uyuy} + 4R^{vxvx}R_{vxvx} \\
&\quad + 4R^{vyvy}R_{vyvy} + 4R^{uvuv}R_{uvuv} + 8R^{uxvx}R_{uxvx} + 8R^{uyvy}R_{uyvy} \\
&= e^{2M}(4(U_u)^2(U_v)^2 + 4U_{uu}U_{vv} + 4(V_{uv})^2 \\
&\quad + 2((U_u)^2 - U_{uu})(V_v)^2 + 2((U_v)^2 - U_{vv})(V_u)^2 + (M_{uv})^2 \\
&\quad - 6(U_v)^2U_{uu} - 6(U_u)^2U_{vv} + 4M_vU_{uu}U_u + 4M_uU_{vv}U_v \\
&\quad + 8M_uM_v(U_uU_v - V_uV_v) - 2M_uU_u(U_v^2 + V_v^2) - 2M_uU_u(U_v^2 + V_v^2))
\end{aligned}
\tag{3.59}
$$

Here we consider the most singular term, then we find

$$
R_4 \cong f_u^2 g_v^2 (f + g)^{-\alpha-4} \ , \tag{3.60}
$$

where $\alpha = (k_1{}^2 + k_2{}^2 + 2k_1k_2 - 1)$. It can be said that there is a curvature singularity in region IV on the surface on which $f + g = 0$.

CHAPTER 4

COLLIDING PLANE GRAVITATIONAL WAVES IN DILATON GRAVITY

In this chapter, we discuss the collision of two plane gravitational waves in dilaton gravity.

Einstein-Maxwell-dilaton gravity is derivable from a variational principle with the action

$$S = \int d^4x\sqrt{-g}\left(R - (\nabla\psi)^2 - \frac{1}{4}e^{a\psi}F^2 \right), \tag{4.1}$$

where a is the dilaton coupling constant and $\psi = \psi(u,v)$ is the dilaton field. Dilaton fields coupled to Einstein-Maxwell fields appear as a result of a dimensional reduction of the Kaluza-Klein Lagrangian [30]. The field equations are

$$R_{bc} = \partial_b\psi\partial_c\psi + \frac{1}{2}e^{a\psi}\left[F_b^d F_{cd} - \frac{1}{4}F^2 g_{bc} \right], \tag{4.2}$$

$$\nabla_b(\sqrt{-g}e^{a\psi}F^{bc}) = 0, \tag{4.3}$$

$$\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}g^{bc}\partial_c\psi) = \frac{a\sqrt{-g}}{8}e^{a\psi}F^2. \tag{4.4}$$

The metric for the action (4.1) is given by

$$ds^2 = 2e^{-M}dudv + e^{-U}(e^V dx^2 + e^{-V}dy^2)\,. \tag{4.5}$$

The gauge potential has a single non-zero component $A_b = (0,0,A,0)$ where A_b is a function of u and v. Then the field equations turn out to be

$$-2M_u U_u - 2U_{uu} + U_u^2 + V_u^2 + 2\psi_u^2 + e^{U+V+a\psi} A_u^2 = 0\,, \tag{4.6}$$

$$-2M_v U_v - 2U_{vv} + U_v^2 + V_v^2 + 2\psi_v^2 + e^{U+V+a\psi} A_v^2 = 0\,, \tag{4.7}$$

$$-2M_{uv} - 2U_{uv} + U_u U_v + V_u V_v + 2\psi_u \psi_v = 0, \tag{4.8}$$

$$U_{uv} - U_u U_v = 0, \tag{4.9}$$

$$2V_{uv} - U_u V_v - U_v V_u - e^{U+V+a\psi} A_u A_v = 0. \tag{4.10}$$

The equations of motion for dilaton and 1-form potentials are

$$2A_{uv} + (V_u + a\psi_u) A_v + (V_v + a\psi_v) A_u = 0, \tag{4.11}$$

$$2\psi_{uv} - U_u \psi_v - U_v \psi_u - \frac{a}{2} e^{U+V+a\psi} A_u A_v = 0 \tag{4.12}$$

From (4.9) we can be integrated to give

$$U = -\log(f(u) + g(v))\,, \tag{4.13}$$

as in the previous chapters. Here $f(u)$ and $g(v)$ are arbitrary functions and satisfy $f(0) + g(0) = 1$. We set, without loss of generality, $f(0) = g(0) = 1/2$. It is useful to change the variables (u, v) to (f, g) and V, ψ to E, X defined as follows:

$$E = -V - a\psi, \qquad\qquad X = \psi - \tfrac{1}{2} aV\,. \tag{4.14}$$

With this choice in term of f and g, the equations (4.10), (4.11), and (4.12) take the form

$$(f + g) X_{fg} + \tfrac{1}{2}(X_f + X_g) = 0, \tag{4.15}$$

$$2A_{fg} - E_f A_g - E_g A_f = 0, \tag{4.16}$$

$$(f+g)E_{fg} + \tfrac{1}{2}(E_f + E_g) = -\tfrac{\alpha}{2} e^{-E} A_f A_g, \tag{4.17}$$

where

$$\alpha = 1 + \tfrac{1}{2} a^2. \tag{4.18}$$

The equations (4.6) and (4.7) can be integrated to give M in terms of (f, g). They can be written as

$$S_f + \tfrac{1}{2} e^{-E} A_f^2 + \tfrac{1}{2\alpha}(f+g)E_f^2 + \tfrac{1}{\alpha}(f+g)X_f^2 = 0, \tag{4.19}$$

$$S_g + \tfrac{1}{2} e^{-E} A_g^2 + \tfrac{1}{2\alpha}(f+g)E_g^2 + \tfrac{1}{\alpha}(f+g)X_g^2 = 0, \tag{4.20}$$

where

$$S = M - \tfrac{1}{2}\log(f+g) + \log(f_u g_v). \tag{4.21}$$

Now we consider the X-equation (4.15). It can be solved by the Khan-Penrose-Szekeres method:

$$X = \frac{a}{2}\log\frac{w-p}{w+p} + \frac{a}{2}\log\frac{r-q}{r+q}, \tag{4.22}$$

where w, p, r and q are given in (3.27).

We make an ansatz for E and A in this form [21]:

$$E = \log\frac{rw+pq}{rw-pq}, \qquad A = \gamma(pw - rq), \tag{4.23}$$

which solves (4.16) automatically and from (4.17) we find that

$$\gamma^2 = \frac{4}{\alpha}. \tag{4.24}$$

After integrating (4.19) and (4.20) with X given by (4.22) to get

$$S = b_1 \log(1-2f)(1+2g) + b_2 \log(1+2f)(1-2g) + (b_3 - 1/2)\log(f+g) + \\ + \frac{a^2}{2\alpha}\log\left(\frac{1}{2} + 2fg + 2pqrw\right), \tag{4.25}$$

where

$$b_1 = b_2 = \frac{a^2+2}{4\alpha}, \qquad b_3 = \frac{\alpha - 1 + 2a^2}{2}, \tag{4.26}$$

and using (4.21) we find that

$$e^{-M} = f_u g_v [(1-2f)(1+2g)]^{-b_1} [(1+2f)(1-2g)]^{-b_2} (f+g)^{-b_3} \times \\ \times \left[\frac{1}{2} + 2fg + 2pqrw\right]^{-\frac{2a^2}{4\alpha}}. \tag{4.27}$$

Here, e^{-M} is not continuous across the boundaries because of the terms $(1-2f)^{-b_1}$ and $(1-2g)^{-b_2}$. To make it so we assume that the functions f and g take the forms (3.48)

$$f = \tfrac{1}{2} - (c_1 u)^{n_1}, \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2}.$$

Then the terms $(1-2f)^{-b_1}$ and $(1-2g)^{-b_2}$ in (4.28) cancel the term $f_u g_v$ when $n_1 = n_2 = 2$. So, e^{-M} becomes continuous across the boundaries and can be written as

$$e^{-M} = [(1+2g)(1+2f)]^{-b_1} (f+g)^{-b_3} \times \left[\frac{1}{2} + 2fg + 2pqrw\right]^{-\frac{2a^2}{4\alpha}} . \qquad (4.28)$$

In this case, M is given by

$$M = -b_1 \log[(1+2g)(1+2f)] - b_3 \log(f+g) - \frac{2a^2}{4\alpha} \log\left[\frac{1}{2} + 2fg + 2pqrw\right]. \qquad (4.29)$$

Therefore, the metric is piecewise C^1 and metric functions satisfy the required O'Brien-Synge junction conditions.

The other components of the metric are given by

$$e^{-(U+V)} = (f+g)\left(\frac{rw+pq}{rw-pq}\right)^{\frac{1}{\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\frac{a}{2}} \left(\frac{r-q}{r+q}\right)^{\frac{a}{2}}\right]^{\frac{a}{\alpha}}, \qquad (4.30)$$

$$e^{-U+V} = (f+g)\left(\frac{rw+pq}{rw-pq}\right)^{-\frac{1}{\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\frac{a}{2}} \left(\frac{r-q}{r+q}\right)^{\frac{a}{2}}\right]^{-\frac{a}{\alpha}}, \qquad (4.31)$$

and the dilaton field is given by

$$e^{\psi} = \left(\frac{rw+pq}{rw-pq}\right)^{\frac{a}{2\alpha}} \left[\left(\frac{w-p}{w+p}\right)^{\frac{a}{2}} \left(\frac{r-q}{r+q}\right)^{\frac{a}{2}}\right]^{\frac{1}{\alpha}} . \qquad (4.32)$$

The above (4.22), (4.23)-(4.29) solve the equations of motion for the region IV. When $a=0$ goes to zero, the above solution reduces to the well-known Bell-Szekeres solution.

We need to calculate the quadratic Riemann invariant R_4 by using (4.13), (4.14), (4.22), (4.23), (4.29), and the Riemann tensors (3.3) and the Ricci tensors (3.4) as

$$\begin{aligned}
R_4 &= 4R^{uxux}R_{uxux} + 4R^{uyuy}R_{uyuy} + 4R^{vxvx}R_{vxvx} \\
&\quad + 4R^{vyvy}R_{vyvy} + 4R^{uvuv}R_{uvuv} + 8R^{uxvx}R_{uxvx} + 8R^{uyvy}R_{uyvy} \\
&= e^{2M}\,(4(U_u)^2(U_v)^2 + 4U_{uu}U_{vv} + 4(V_{uv})^2 \\
&\quad + 2((U_u)^2 - U_{uu})(V_v)^2 + 2((U_v)^2 - U_{vv})(V_u)^2 + (M_{uv})^2 \\
&\quad - 6(U_v)^2U_{uu} - 6(U_u)^2U_{vv} + 4M_vU_{uu}U_u + 4M_uU_{vv}U_v \\
&\quad + 8M_uM_v(U_uU_v - V_uV_v) - 2M_uU_u(U_v^2 + V_v^2) - 2M_uU_u(U_v^2 + V_v^2))
\end{aligned} \tag{4.33}$$

Here we consider the most singular term, then we find (4.33) as

$$R_4 \cong b_3\, f_u^2 g_v^2 (f+g)^{2b_3-4}\,. \tag{4.34}$$

It can be said that there is a curvature singularity in region IV on the surface on which $f+g=0$.

CHAPTER 5

CONCLUSION

In this thesis, we studied the collision of pure electromagnetic plane waves with collinear polarization in Einstein-Maxwell theory which is known as the Bell-Szekeres solution. It has been found that, in the Bell-Szekeres solution, the collision of two step electromagnetic plane waves always generates impulsive gravitational waves along the null boundaries. Then, the Szekeres solution has been studied which describes the collision of two gravitational plane waves. We have found that there is a curvature singularity in the interaction region on the surface on which $f + g = 0$. We have given a solution for the collision of two plane gravitational waves in dilaton gravity. We have seen that the solution reduces to the well-known Bell-Szekeres solution when dilaton coupling constant becomes zero and there is a curvature singularity in the interaction region on the surface which $f + g = 0$.

REFERENCES

[1] R. Penrose, Rev. Mod. Phys., **37**, 215 (1965).

[2] J. B. Griffiths, *Colliding Plane Waves in General Relativity* (Clarendon Press, Oxford, 1991).

[3] P. Szekeres, Nature, **228**, 1183 (1970).

[4] K. Khan and R. Penrose, Nature, **229**, 185 (1971).

[5] Y. Nutku and M. Halil, Pyhs. Rev. Lett., **39**, 1379 (1977).

[6] P. Bell and P. Szekeres, Gen. Relativ. Gravit. **5**, 275 (1974).

[7] M. Halil, J. Math. Phys., **20**, 120 (1979).

[8] M. Gürses and M. Halilsoy, Lett. Nuovo Cimento, **34**, 588 (1982).

[9] J. B. Griffiths, Phys. Lett. A, **54**, 269 (1975).

[10] S. Chandrasekhar and B. C. Xanthopoulos, Proc. Roy. Doc. A, **398**, 223 (1985).

[11] Yu. G. Sbytov, Zh. Eksp. Teor. Fiz., **71**, 2001 (1976).

[12] F. J. Tipler, Phys. Rev. D, **22**, 2929 (1980).

[13] W. B. Bonnor and P. A. Wickers, Gen. Rel. Grav., **13**, 29 (1981).

[14] R. Güven, Phys. Lett. B, **191**, 275 (1987).

[15] M. Gürses and E. Sermutlu, Phys. Rev. D, **52**, 809 (1995).

[16] M. Halil and I. Sakalli, Class. Quantum Grav., **20**, 1417 (2003).

[17] M. Gürses and A. Karasu, Class. Quantum Grav., **18**, 509 (2001).

[18] M. Gürses, E. O. Kahya and A. Karasu, Phys. Rev. D, **66**, 024029 (2002).

[19] M. Gürses, Y. İpekoğlu, A. Karasu and Ç. Şentürk, Phys. Rev. D, **68**, 084007 (2003).

[20] A. Das, J. Maharana and A. Melikyan, Phys. Lett. B **518**, 306-314 (2001).

[21] B. Chen, Chong-Sun Chu, Ko Furuta, Feng-Li Lin, JHEP **0402**, 020 (2004).

[22] B. Chen, JHEP **0412**, 016 (2004).

[23] Ç. Şentürk, *Higher Dimensional Bell-Szekeres Metric,* Ms. Thesis (Middle East Technical University, 2003).

[24] Ö. Gürtuğ, *New Extensions of the Colliding Wave Solutions in General Relativity,* Ph.D. Thesis (Eastern Mediterranean University, 1997).

[25] R. D'Inverno, *Introducing Einstein's Relativity* (Clarendon Press, Oxford, 1995).

[26] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of General Relativity* (Princeton, New York, 1972).

[27] R. A. Matzner and F. J. Tipler, Phys. Rev. D, **29**, 1575 (1984).

[28] C. J. S. Clarke and S. A. Hayward, Class. Quantum Grav., **16**, 615 (1989).

[29] T. M. Helliwell and D. A. Konkowski, Class. Quantum Grav., **16**, 2709 (1999).

[30] M. J. Duff, *Kaluza Klein Theory in Perspective*, arxiv: hep-th/9410046.