HILBERT FUNCTIONS OF GORENSTEIN MONOMIAL CURVES

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ABSTRACT

HILBERT FUNCTIONS OF GORENSTEIN MONOMIAL CURVES

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The aim of this thesis is to study the Hilbert function of a one-dimensional Gorenstein local ring of embedding dimension four in the case of monomial curves. We show that the Hilbert function is non-decreasing for some families of Gorenstein monomial curves in affine 4-space. In order to prove this result, under some arithmetic assumptions on generators of the defining ideal, we determine the minimal generators of their tangent cones by using the standard basis and check the Cohen-Macaulayness of them. Later, we determine the behavior of the Hilbert function of these curves, and we extend these families to higher dimensions by using a method developed by Morales. In this way, we obtain large families of local rings with non-decreasing Hilbert function.

Keywords: Monomial curves, Standard basis, Hilbert function, Gorenstein rings

ÖZ

GORENSTEIN TEKTERİMLİ EĞRİLERİNİN HILBERT FONKSİYONLARI

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Bu tezde amacımız, tekterimli eğriler durumu için 4. gömme boyutunda bir boyutlu yerel Gorenstein halkalarının Hilbert fonksiyonunu çalışmaktır. 4 boyutlu afin uzayda, Gorenstein tekterimli eğrilerinin bazı ailelerinin Hilbert fonksiyonunun azalmayan olduğunu gösterdik. Bu sonucu ispatlamak için, tanımlayan idealin üreteçleri üzerine bazı aritmetik kabuller altında standart baz kullanmak suretiyle bunların teğet konilerinin minimal üreteçlerini belirleyerek bu konilerin Cohen-Macaulay olup olmadığını test ettik. Ayrıca, bu eğrilerin Hilbert fonksiyonlarının davranışını belirledik ve bu eğri ailelerini Morales'in geliştirdiği metot ile daha yüksek boyutlara genişlettik. Bu şekilde, Hilbert fonksiyonu azalmayan olan, geniş yerel halka aileleri elde ettik.

Anahtar Kelimeler: Tekterimli eğriler, Standart bazları, Hilbert fonksiyonu, Gorenstein halkaları

To my love, Ersen

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CHAPTER 1

INTRODUCTION

The behavior of the Hilbert function $H_R(n)$ of a local ring (R, m) has remained as an important problem to be determined for a long time. The importance of this problem comes from the fact that the Hilbert function $H_R(n)$ of a local ring (R, m) is a good measure of the singularity of (R, m) at m [29]. Nevertheless, relatively little is known about the behavior of the Hilbert function $H_R(n)$ of a local ring (R, m). Since the Hilbert function $H_R(n)$ of a local ring (R, m) is the Hilbert function of the associated graded ring,

$$\mathrm{gr}R = \mathrm{gr}_m R = R/m \oplus m/m^2 \oplus m^2/m^3 \oplus \cdots$$

we need to find $\mathrm{gr}R$, in order to determine the Hilbert function of R. The toughness to get $\mathrm{gr}R$ for a given local ring (R, m) makes the direct computation of $H_R(n)$ difficult. If $\mathrm{gr}R$ is Cohen-Macaulay, the computation of the Hilbert function of a local ring can be reduced to a computation of the Hilbert function of an Artin local ring, which has a Hilbert function with finite number of nonzero values. Besides, Cohen-Macaulayness of the associated graded ring guarantees that the Hilbert function is non-decreasing [39]. In this case we need to check the Cohen-Macaulayness, which assures this reduction.

In spite of the fact that the Hilbert function of a graded k-algebra is well-understood, when it is Cohen-Macaulay, very little is known in the local case even if this has a close relation with the well developed theory of singularities. This stems from the fact that the associated graded ring of a local Cohen-Macaulay ring can be very bad [41, p44].

The main related open problem was stated by Sally as the following conjecture [35]:

> *Sally's conjecture.* If R is a one-dimensional Cohen-Macaulay local ring with small enough embedding dimension, then $H_R(n)$ is non-decreasing.

This conjecture is well known for embedding dimension less than or equal to 2. Matlis proved the Sally's conjecture for the embedding dimension two [28]. Moreover, the non-decreasing property of the Hilbert function of a local ring for the embedding dimension 3 case was also proved by Juan Elias [15]. In the literature, there are several examples showing that Cohen-Macaulayness of a local ring does not necessarily assure the non-decreasing behavior of its Hilbert function. The first examples of local rings with decreasing Hilbert function were given by Herzog-Waldi [23] and Eakin-Sathaye [13]. Besides, Orecchia showed that for all $b \geq 5$ there exists a reduced one-dimensional local ring of embedding dimension b with decreasing Hilbert function [31]. Thus, the Cohen-Macaulayness of a local ring does not guarantee that its Hilbert function is non-decreasing. Moreover, nothing is known even for Gorenstein local rings, which are special Cohen-Macaulay rings, so Sally's conjecture can be restated for Gorenstein rings: If R is a one-dimensional Gorenstein local ring with small enough embedding dimension, then $H_R(n)$ is non-decreasing.

The aim of this thesis is to study the Hilbert function of a one-dimensional Gorenstein local ring of embedding dimension four in the case of monomial curves.

A monomial affine curve C in the affine space $\mathbb{A}_k^d$ over a field k is given parametrically by $x_i = t^{n_i}$, that is , we have

$$C = \{(t^{n_1}, \ldots, t^{n_d}) \in \mathbb{A}_k^d \mid t \in k\}$$

where $n_1, n_2, \ldots, n_d$ are positive integers with $\gcd(n_1, n_2, \ldots, n_d) = 1$. The additive semigroup , which is denoted by $< n_1, n_2, \ldots, n_d >$, generated minimally by $n_1, n_2, \ldots, n_d$ and is defined as $< n_1, n_2, \ldots, n_d >= \{n \mid n = \sum_{i=1}^{d} a_i n_i,\ a_i$'s are non-negative integers$\}$. There is a connection between a monomial curve and the additive semigroup depending on the semigroup ring $k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]]$. From this standpoint, monomial curves can be seen as a common ground where geometric, algebraic and arithmetical techniques apply.

In order to proceed, it is possible to consider either the associated graded ring of $R = k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]]$ which is denoted by $\mathrm{gr}R = \mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]])$, or the ring $k[x_1, x_2, \ldots, x_d]/I(C)_*$. We prefer the latter, namely, the ring to study the problem, since we have tools to find the generators of $I(C)_*$. This ring is called coordinate ring of the tangent cone of the monomial curve C. We also have methods to check the Cohen-Macaulayness of the associated graded ring by using standard basis theory, which provides crucial information about the Hilbert function.

Knowing the defining ideal explicitly by the work of Bresinsky [8], we investigate and determine the behavior of the Hilbert function of Gorenstein monomial curves in the case of embedding dimension four. Based on the standard basis theory, we find the minimal generators of the tangent cone of a monomial curve C in affine space $\mathbb{A}^4$ under some arithmetic conditions on the generators of defining ideal. By using the Cohen-Macaulayness of the tangent cone, we determine the Hilbert function of the associated graded ring of these curves.

In chapter 2, we introduce basic concepts required to study our problem by giving the definitions of the Hilbert function of a local ring and of the Gorenstein ring, and we give a survey of the literature. Then, we describe the relation between Hilbert function and Gorenstein rings and also Cohen-Macaulay rings, respectively.

Chapter 3 is devoted to the monomial curves and Cohen-Macaulayness of the tangent cone. We give the definitions of monomial curve and symmetric semigroup, and mention some very important results. After defining the tangent cone, we establish the criteria for determining the Cohen-Macaulayness of the tangent cone of a monomial curve.

In chapter 4, we first present the results about the generators of the defining ideals of Gorenstein monomial curves in affine $\mathbb{A}^4$ space. We find the minimal generators of the tangent cone of Gorenstein monomial curves C under some arithmetic assumptions on their defining ideals. Showing the Cohen-Macaulayness of the tangent cones of these families gives us the opportunity to obtain families of Gorenstein local rings with non-decreasing Hilbert function. We also consider families of Gorenstein monomial curves, whose associated graded rings are not Cohen-Macaulay. By extending the Gorenstein monomial curves to higher dimensions, we obtain large families of local rings with non-decreasing Hilbert function.

CHAPTER 2

COHEN-MACAULAY RINGS AND HILBERT FUNCTION OF A LOCAL COHEN-MACAULAY RING

In this chapter, we introduce the basic definitions of Cohen-Macaulay and Gorenstein rings. After giving the main related facts about the Hilbert functions of a graded ring, we also give some results about the Hilbert functions of Cohen-Macaulay and Gorenstein rings respectively. Then we define the Hilbert function of a local ring and give a literature summary of the main problem: Hilbert function of a Cohen-Macaulay local ring.

2.1 Cohen-Macaulay and Gorenstein Rings

First, we need some definitions to define a Cohen-Macaulay ring.

Definition 2.1.1. Let R be a ring. A regular sequence on R(or an R-sequence) is a set $a_1, a_2, \ldots, a_n$ of elements or R with the following properties:

(i) $R \neq (a_1, a_2, \ldots, a_n)R$,

(ii) The jth element a_j is not a zero divisor on the ring $R/(a_1, a_2, \ldots, a_{j-1})R$ for $j = 1, 2, \ldots, n$, where for $j = 1$, we set $(a_1, a_2, \ldots, a_n)$ to be the zero ideal.

The lengths of all the maximal R-sequences(where R is Noetherian) in an ideal are the same. This gives us the opportunity to define the depth of an ideal of a Noetherian ring.

Definition 2.1.2. Let R be a Noetherian ring. The depth of an ideal I is the length of any maximal R-sequences in I.

The definition of height of any ideal is given as follows:

Definition 2.1.3. Let R be a commutative ring, and P be a prime ideal. The height of P is the supremum of the lengths l of strictly descending chains

$$P = P_0 \supset P_1 \supset \ldots \supset P_\ell$$

of prime ideals. The height of any ideal I is the infimum of the heights of the prime ideals containing I.

Now, we can define the Cohen-Macaulay ring.

Definition 2.1.4. A Noetherian ring R is Cohen-Macaulay, if depth(I)=height(I) for each ideal I of R.

We can now give the definition of Gorenstein rings, which are special Cohen-Macaulay rings. Gorenstein rings constitute an important class of local rings. These are distinguished from the others by their ubiquity and various characterizations of rings which belong to this class. From the point of such properties, they have been considered as the nicest rings.

Bass made invaluable contributions to algebra and geometry such that he is the first to characterize Gorenstein rings in terms of the type [4]. Hence, he is given respect for developing the theory of these rings [5].

In [16] Foxby gives a remarkable characterization of Gorenstein rings by proving the conjecture of Vasconcelos :

Suppose (R, m, k) is a Noetherian local ring of dimension d containing a field; then R is a Gorenstein ring if $\mu_d(m, R) = 1$.

where $\mu_d(m, R)$ is d^{th} Bass number of R with respect to m.

Geometrically, Gorenstein rings are common and significant as the title of Bass' foundational paper, [14, p526].

The characterization of the class of Gorenstein rings usually involves homological algebra. The fundamental definition states that a Noetherian local ring R is a Gorenstein ring if it is of finite injective dimension [10, p94]. On the other hand, we prefer another approach that follows an elementary way to the most important facts of the theory by avoiding the use of structure theorems of injective modules and duality theorems. In order to define Gorenstein rings based upon our approach, we need to give some required definitions.

Definition 2.1.5. [3, p35] Let (R, m) be a local ring of dimension $d = \dim R$. Any d-element set of generators of an m-primary ideal is called a system of parameters (or a set of parameters) of the local ring (R, m).

Definition 2.1.6. [3, p137] An irreducible ideal is a proper ideal which can not be expressed as an intersection of two ideals properly containing it.

Then the definition of Gorenstein rings reads,

Definition 2.1.7. A local ring (R, m) is Gorenstein if and only if every system of parameters of the ring R generates an irreducible ideal.

Alternatively, one can define Gorenstein rings to be the rings of type equal to 1. This is the main homological characterization. It can be shown that this is equivalent to the one in the Definition 2.1.7. In order to make the equivalence clear we need to give the definition of the type.

Definition 2.1.8. If (R, m) is a local Cohen-Macaulay ring of dimension d, then the number

$$\dim_{R/m} \operatorname{Ext}_R^d(Rm, R)$$

is called as the type $r(R)$ of the ring R.

Now the following theorem shows the equivalence.

Theorem 2.1.9. *[3, p140] The following properties are equivalent for a local ring (R, m):*

(i) The ring R is Cohen-Macaulay and there exists a system of parameters of R generating an irreducible ideal, i.e. R is a Cohen-Macaulay ring of type 1,

(ii) Every system of parameters of the ring R generates an irreducible ideal.

Proof. See [3, p140]. □

We will be interested in monomial curves, therefore we will consider the ring $R = [[t^{n_1}, \dots, t^{n_d}]]$ of dimension one. As a result, R is Gorenstein, if every principal ideal (r) generated by an element $r \in R$ with $\sqrt{(r)} = (t^{n_1}, \dots, t^{n_d})$ is irreducible.

Remark 2.1.10. Knowing that Gorenstein rings are Cohen-Macaulay rings, one way to check if a ring is Gorenstein is to first check its Cohen-Macaulayness. If it is a Cohen-Macaulay ring, the procedure follows as to find a system of parameters, to kill it and then to compute the Socle. In order to get the type, it is required to determine how many linearly independent elements are in the Socle.

Example 2.1.11. Let $A = k[X,Y]/(X^2, XY, Y^2)$, which has dimension zero, and hence A is a Cohen-Macaulay ring. Now, $\mathrm{Soc}(A) = (0 : m)_A = \{z \in A \mid mz = (0)\} = m$. Since there are two linearly independent elements are in the socle, type of A is greater than 1. Therefore, A can not be Gorenstein.

2.2 Hilbert Function of a Graded Algebra

Before we define the Hilbert function of a local ring, we need to define the Hilbert function of a graded k-algebra.

Definition 2.2.1. The Hilbert function $H_R(n)$ of a graded k-algebra R is defined as

$$H_R(n) = \dim_k R_n$$

where R_n is the homogeneous piece consisting of elements of degree n.

The assertion of Hilbert is that it is possible to extract all the information contained in infinitely many values of the function $H(n)$ using a set of finite number of its values. The following theorem shows Hilbert's approach :

Theorem 2.2.2. *(Hilbert) If R is a finitely generated algebra over $k[x_1, \ldots, x_r]$, then $H_R(n)$ agrees, for large S, with a polynomial of degree $\leq r - 1$*

Definition 2.2.3. The polynomial $P_R(n)$ satisfying $P_R(n) = H_R(n)$ for sufficiently large n is the Hilbert polynomial of R.

There is more to say about the Hilbert polynomial.

Theorem 2.2.4. *[42, p342] Let the graded ring R have dimension d*

(i)

$$H_R(t) = \frac{h_R(t)}{(1-t)^d} \ ,$$

where $h_R(t)$ is a polynomial

(ii) the Hilbert polynomial $P_R(n)$ of R is of degree $d-1$ with leading coefficient $h_R(1)/(d-1)!$.

Proof. See [42, p342] □

Definition 2.2.5. The multiplicity of a graded ring R is defined to be $h_R(1)$ and it is denoted by $e(R)$. The polynomial $h_R(t)$ is called the h-polynomial of R.

Definition 2.2.6. [10, p169] Let R be a positively graded k-algebra where k is a field. Then the degree of the Hilbert function of R is denoted by $a(R)$ and called the a-invariant of R.

Before giving the first criteria for checking the Cohen-Macaulayness of a graded ring, we need the following definition:

Definition 2.2.7. Let A be a graded ring of dimension d. A system of parameters for A is a set of homogeneous elements $a_1, \ldots, a_d \in A$ such that $\dim\left(A/(a_1, \ldots, a_d)\right)$ is zero.

Proposition 2.2.8. [42, p56] Suppose that $a_1, \ldots, a_d$ is a homogeneous system of parameters for a graded ring A. Then A is a Cohen-Macaulay if and only if $a_1, \ldots, a_d$ is a regular sequence. Moreover, if $a_1, \ldots, a_d$ are of degree 1, and if $H_A(t) = (h_0 + h_1 + \ldots + h_r t^r)/(1-t)^d$, then the polynomial $h_0 + h_1 + \ldots + h_r t^r$ is the Hilbert series of the Artin ring $A/(a_1, \ldots, a_d)$. In particular, $h_i \geq 0$.

The following proposition is another useful test to check the Cohen-Macaulayness of a graded ring which is in the form $k[x_1, \ldots, x_n]/I$, where I is a homogeneous ideal.

Proposition 2.2.9. [6, p117] Let $A = k[x_1, \ldots, x_n]/I$, where I is a homogeneous ideal, and let $\dim A = d$. Then, the ring A is Cohen-Macaulay if and only if $e(A) = \dim_k(A/(a_1, \ldots, a_d))$, for some (and hence all) system of parameters $a_1, \ldots, a_d$ of degree 1.

It would not be surprising that additional properties of R will put further constraints on the Hilbert function. Here, it is suitable to draw attention to a discussion on this feature for Gorenstein properties.

Theorem 2.2.10. *(Stanley's Gorenstein Criterion) [10, p170] Let k be a field, R is a d-dimensional C-M positively graded k-algebra. Suppose that R has the Hilbert series*

$$H_R(t) = \frac{\sum_{i=0}^{s} h_i t^i}{\prod_{j=1}^{d}(1 - t^{a_j})}.$$

(i) If R is Gorenstein, then $H_R(t) = (-1)^d t^{a(R)} H_R(t^{-1})$.

(ii) Suppose R is a domain, and $H_R(t) = (-1)^d t^q H_R(t^{-1})$ for some integer q. Then R is Gorenstein.

Proof. See [10, p170]. □

Remark 2.2.11. Assume that the positively graded k-algebra R is Gorenstein, and

$$H_R(t) = \frac{h_R(t)}{\prod_{i=1}^{d}(1 - t^{a_i})}$$

where $a_1, \ldots, a_d \in \mathbb{Z}^+$. Then, the functional equation in Theorem 2.2.10(i) for $H_R(t)$ is equivalent to the equation $h_R(t) = t^{\deg h_R} h_R(t^{-1})$, that is, to the symmetry of the polynomial $h_R(t)$.

2.3 Hilbert Function of a Local Ring

The Hilbert function of a d-dimensional local ring (R, m) proves to be important since it is a good measure of the singularity at (R, m). Sally summarizes the idea behind this fact by saying that the Hilbert function measures the degree to which R deviates from a regular local ring, or equivalently measures the degree to which gr$R$ deviates from a polynomial ring over R/m [36].

Definition 2.3.1. The Hilbert function of (R, m) is the function $H(n)$ which is the dimension over R/m of the n^{th} component of the associated graded ring

$$\mathrm{gr}_m R = R/m \oplus m/m^2 \oplus m^2/m^3 \oplus \cdots$$

Thus, $H_R(n) = \dim_{R/m} m^n/m^{n+1}$ for all $n \in \mathbb{Z}_{\geq 0}$. Hilbert series of R is defined as

$$H_R(t) = \sum_{n \in \mathbb{Z}_{\geq 0}} H_R(n) t^n$$

Here, $\dim_{R/m}$ represents the ordinary vector space dimension over the field R/m. This graded algebra $\mathrm{gr}_m R$ corresponds to a relevant geometric structure. If R is the localization at the origin of the coordinate ring of an affine variety passing through 0, then the associated graded ring $\mathrm{gr}_m R$ turns out to be the coordinate ring of the tangent cone of this variety at the origin, which is the cone containing all lines that are the limiting positions of secant lines to the variety in 0. Consequently, they both have the same Hilbert function, as well as the Hilbert series.

Remark 2.3.2. It is very difficult to get the associated graded ring, gr$R$, for a given local ring since gr$m = m/m^2 \oplus m^2/m^3 \oplus \ldots$ may involve zero divisors. Now, in this case, all the R/m-vector space homomorphisms of m^n/m^{n+1} to m^{n+1}/m^{n+2} given by multiplication by elements of m have nontrivial kernels for some n. As will be seen below, the Hilbert function of a one-dimensional local domain even may satisfy $H(2) < H(1)$.

2.3.1 Literature

In general, very little is known about the Hilbert function of a Cohen-Macaulay local ring, since the associated graded ring of a local Cohen-Macaulay ring can be very bad.

The first obvious question was to understand, whether the Hilbert function of a local Cohen-Macaulay ring was non-decreasing, and this was stated by Sally as the following conjecture [35]:

> *Sally's conjecture.* If R is a one-dimensional Cohen-Macaulay local ring with small enough embedding dimension, then $H_R(n)$ is non-decreasing.

Matlis proved the Sally's conjecture for the embedding dimension two [28]. Elias proved Sally's conjecture for the embedding dimension three in the equicharacteristic case [15].

The very first examples of rings with decreasing Hilbert function were given by Herzog-Waldi [23] and Eakin-Sathaye [13] in the literature. In the first example, $R = k[t^{30}, t^{35}, t^{42}, t^{47}, t^{148}, t^{153}, t^{157}, t^{169}, t^{181}, t^{193}]$ is the ring of regular functions of a monomial curve with embedding dimension 10 and the corresponding Hilbert function is $H_R = \{1, 10, 9, 16, 25, \ldots\}$. The second one in [13] is the ring of regular functions $R = k[t^{15}, t^{21}, t^{23}, t^{47}, t^{48}, t^{49}, t^{50}, t^{52}, t^{54}, t^{55}, t^{56}, t^{58}]$ of a monomial curve of embedding dimension 12 with decreasing Hilbert function $H_R = \{1, 12, 11, 13, 15, \ldots\}$.

Moreover, these two rings are one-dimensional Cohen-Macaulay local rings, whose associated graded rings are not Cohen-Macaulay. In [31], Orecchia showed that for all $b \geq 5$ there exists a reduced one-dimensional local ring of embedding dimension b with decreasing Hilbert function. Later, ordinary singularities with decreasing Hilbert function and embedding dimension at least 7 were constructed by Roberts [32]. Gupta and Roberts gave examples of one-dimensional local rings with decreasing Hilbert functions with embedding dimension $b \geq 4$ in [20].

These examples show us that the Cohen-Macaulayness of a local ring does not necessarily assure that its Hilbert function is non-decreasing. Therefore Sally's conjecture can be restated as, if R is a one-dimensional Gorenstein local ring with small enough embedding dimension, then $H_R(n)$ is non-decreasing.

2.3.2 Hilbert Functions and Cohen-Macaulayness

It is important to discover which local rings (R, m) have Cohen-Macaulay associated graded rings, since Cohen-Macaulayness reduce the computation of the Hilbert function of (R, m) to the computation of the Hilbert function of an Artin local ring, the latter function having only a finite number of nonzero values. Sometimes the key to whether $\mathrm{gr}(R)$ is Cohen-Macaulay lies in the embedding dimension, which is $H(1)$, of the local ring. One may find several articles in the literature about this approach [36]. Besides, knowing which local rings have Cohen-Macaulay associated graded rings with respect to the maximal ideal enables us to make a rough classification of singularities such as Gorenstein singularities, normal singularities, etc. Also, the varieties all of whose local rings are Cohen-Macaulay manifest some special properties [27].

Suppose that R is a graded algebra of dimension d. (If $R_0 = k$ is a field and so that R is a k-algebra, then one can say that R is a graded algebra). R is said to be Cohen-Macaulay if and only if R has a homogeneous R-sequence of length d [39, p66]. If depth(R)≥ 1, we have $H(0) \leq H(1) \leq H(2) \leq \ldots$. This is clearly seen, because multiplication by a homogeneous non-zero divisor of degree one is a monomorphism from R_n into R_{n+1}.

Remark 2.3.3. If gr$m = m/m^2 \oplus m^2/m^3 \oplus \ldots$ includes a zero divisor, then it, naturally, contains a homogeneous nonzero divisor $\bar{x} \in m^t/m^{t+1}$ for some $t \geq 1$. Also, multiplication by $\bar{x}$ is a one-to-one vector space homomorphism of m^n/m^{n+1} to m^{n+t}/m^{n+t+1} for all $n \geq 0$. Thus, if x is any lifting of $\bar{x}$ to R, then $\mathrm{gr}_m(R)/(\bar{x}) \cong \mathrm{gr}_m(R/(x))$ where $\dim(R/(x)) = \dim R - 1$. If $\mathrm{gr}_m(R)$ is Cohen-Macaulay and $\dim R$ =d, then gr$m$ contains a regular sequence $\bar{x}_1, \ldots, \bar{x}_d$ of length d. By the argument above, if $x_1, \ldots, x_d$ are liftings of $\bar{x}_1, \ldots, \bar{x}_d$, then $\mathrm{gr}_m(R)/(\bar{x}_1, \ldots, \bar{x}_d) \cong \mathrm{gr}_m(R/(x_1, \ldots, x_d))$. From Proposition 2.2.8, $H_R(t) = H_{R/(x_1,\ldots,x_d)}(t)/(1-t)^d$ where $H_R(t)$ is the Hilbert series of the ring R and $H_{R/(x_1,\ldots,x_d)}(t)$ is the Hilbert series of the Artin local ring $R/(x_1, \ldots, x_d)$. This relation manifests how the Cohen-Macaulayness of the associated graded ring of a local ring with respect to the maximal ideal reduces the computation of the Hilbert function of a local ring to a computation of the Hilbert function of an Artin local ring.

CHAPTER 3

MONOMIAL CURVES IN AFFINE SPACE

Our main geometric objects of interest in this thesis are the monomial curves. These form an important class of curves in the sense that they interconnect geometry, algebra and combinatorics. This is a direct consequence of the relationship between the monomial curves and semigroups generated by integers. The close relation between numerical semigroups and monomial curves let us use the algebraic geometry terminology in studying numerical semigroups.

In this chapter we, firstly, present some basic facts about monomial curves and symmetric semigroups. Then, we discuss the Cohen-Macaulayness of the tangent cone of a monomial curve.

3.1 Defining Equations of Monomial Curves

A monomial affine curve C in the affine space $\mathbb{A}_k^d$ over a field k is given parametrically by $x_i = t^{n_i}$, i.e., we have

$$\Gamma = \{(t^{n_1}, \ldots, t^{n_d}) \in \mathbb{A}_k^d \mid t \in k\}$$

Let $R = k[x_1, \ldots, x_n]$ and $k[t]$ be two polynomial rings over k, graded by $\deg(x_i) = n_i$ for all i and $\deg(t) = 1$, respectively. Let Φ be the graded homomorphism of k-algebras

$$\Phi : R \to k[t], x_i \to t^{n_i}.$$

Now, the image of Φ, denoted by $k[S]$ or $k[\Gamma]$, is the semigroup ring. The additive semigroup S generated minimally by $n_1, n_2, \ldots, n_d$ is represented by $< n_1, n_2, \ldots, n_d >$ and is defined as

$$< n_1, n_2, \ldots, n_d >= \Big\{ n \mid n = \sum_{i=1}^{d} a_i n_i,\ a_i{'}\text{s are non} - \text{negative integers} \Big\}. \qquad (3.1)$$

$I(C)$ is the defining ideal of C consisting of

$$\{ f(x_1, \ldots, x_d) \mid f(x_1, \ldots, x_d) \in k[x_1, \ldots, x_d],\quad f(t^{n_1}, \ldots, t^{n_d}) = 0,$$

$$t \text{ transcendental over } k \}\,.$$

Φ gives an isomorphism for $1 \leq i \leq d$

$$k[x_1, \ldots, x_n]/I(C) \cong k[t^{n_1}, \ldots, t^{n_d}].$$

As a result, this isomorphism exhibits the relationship between the monomial curve and the semigroup leading to isomorphism of local rings,

$$(k[x_1, \ldots, x_n]/I(C))_{(x_1,\ldots,x_d)} \cong k[t^{n_1}, \ldots, t^{n_d}]_{(t^{n_1},\ldots,t^{n_d})}$$

which, subsequently, yields that the completions of the local rings give

$$(k[[x_1, \ldots, x_n]]/I(C)) \cong k[[t^{n_1}, \ldots, t^{n_d}]].$$

3.2 Symmetric Semigroups

It is considered to be a classical problem to study the subsemigroups of $\mathbb{N}$. This is equivalent to investigate the sets of natural solutions of linear equations with coefficients in $\mathbb{N}$. The study of the subsemigroups of $\mathbb{N}$ has also gained motivation by its interaction with algebraic geometry and commutative algebra after the works of [21, 26] and others. The observation is that the properties of the semigroup S

imply properties on the semigroup ring $R = k[S]$. For instance, Kunz showed that R is Gorenstein if and only if S is symmetric [26].

A numerical semigroup S, as defined in equation (3.1), is said to be symmetric if and only if the number of gaps is equal to the number of nongaps. A gap is given by

$$n \notin < n_1, n_2, \ldots, n_d >, \ \ 0 < n \leq c\,,$$

and a nongap is,

$$n \in < n_1, n_2, \ldots, n_d >, \ \ 0 \leq n < c$$

where the number c is the greatest integer not in S and is called the Frobenius number of S. The problem of Frobenius number refers to determination of the integer c. The problem of computing and estimating the Frobenius number has been examined by several authors in terms of the generators for different classes of semigroups [18] and the references therein. The formula for the Frobenius number $c = n_1n_2 - n_1 - n_2$ first proved in [40], if S is generated by two elements, $S =< n_1, n_2 >$.

For the case that S is generated by three elements, an algorithm is proposed by Johnson [25] which incorporates subsequent improvements in [11], [37], and [34]. Fröberg derived an easier expression for c for three-generated semigroups $< n_1, n_2, n_3 >$ and for four-generated $< n_1, n_2, n_3, n_4 >$ symmetric semigroups. These are the following theorems :

Theorem 3.2.1. *[17] Let $S =< n_1, n_2, n_3 >$ be non-symmetric. Then the Frobenius number of S is $c = max\{n_2\alpha_{12} + n_3\alpha_3 - (n_1+n_2+n_3), n_2\alpha_2 + n_3\alpha_{13} - (n_1+n_2+n_3)\} = max\{n_2\alpha_{12} + n_3(\alpha_{13} + \alpha_{23}) - (n_1 + n_2 + n_3), n_2(\alpha_{12} + \alpha_{32}) + n_3\alpha_{13} - (n_1 + n_2 + n_3)\}$.*

Theorem 3.2.2. *[17] Let $S =< n_1, n_2, n_3, n_4 >$ be symmetric but $k[S]$ not a complete intersection. Then the Frobenius number of S is $c = n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_{14} - (n_1 +$*

$n_2 + n_3 + n_4) = n_2(\alpha_{32} + \alpha_{42}) + n_3(\alpha_{13} + \alpha_{43}) + n_4\alpha_{14} - (n_1 + n_2 + n_3 + n_4)$.

Based on the findings of Herzog and Kunz [22], Stanley realized that the following theorem of Kunz on numerical semigroup rings can easily be derived from the Stanley's Gorenstein criterion 2.2.10.

Theorem 3.2.3. *[10, p171] Let S be a numerical semigroup with Frobenius number c. The following conditions are equivalent*

(a) $k[S]$ is Gorenstein;

(b) The semigroup S is symmetric, i.e., for all i with $O \leq i \leq c-1$ one has $i \in S$ if and only if $c - i - 1 \notin S$.

Proof. Write $R = k[S]$. Then

$$H_R(t) = \sum_{j \in S} t^j = 1/(1-t) - \sum_{i \in \mathbb{N} \setminus S} t^i \,,$$

and subsequently,

$$-H_R(t^{-1}) = t/(1-t) + \sum_{i \in \mathbb{N} \setminus S} t^{-i} \,.$$

Suppose $H_R(t) = -t^r H_R(t^{-1})$ which necessitates $r = c - 1$, and

$$1/(1-t) - \sum_{i \in \mathbb{N} \setminus S} t^i = t^c/(1-t) + \sum_{i \in \mathbb{N} \setminus S} t^{c-1-i} \,.$$

Hence $H_R(t) = -t^{c-1} H_R(t^{-1})$ if and only if S is symmetric, and the assertion follows from Theorem 2.2.10. □

This result gives us the opportunity that if we have a monomial curve corresponding to a symmetric semigroup, we can call it a Gorenstein monomial curve.

The results show that there is a connection between the symmetric semigroups and the number of generators of the defining ideals of corresponding monomial curves in affine 3- and 4-spaces. For a monomial curve C with embedding dimension 3, Herzog shows that the defining ideal $I(C)$ has 2 generators if and only if the semigroup $< n_1, n_2, n_3 >$ is symmetric [21]. This shows that every Gorenstein monomial curve with embedding dimension three is a complete intersection. In the same paper, he also show that if $S =< n_1, n_2, n_3 >$ and not symmetric, then

$$k[S] \simeq k[x_1, x_2, x_3]/(x_1^{\alpha_1} - x_2^{\alpha_{12}} x_3^{\alpha_{13}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_3^{\alpha_{23}}, x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}})$$

and $\alpha_{ij} < \alpha_j$ for all i, j. For the embedding dimensions greater than or equal to 4, every Gorenstein monomial curve is not a complete intersection, which was shown in [21] and [44]. Bresinsky not only proved that $I(C)$ is generated by 3 or 5 elements for a monomial curve C with symmetric semigroups of integers generated by four elements, but he also gave an explicit description of the generators of $I(C)$ [8]. We use Bresinsky's results extensively in this thesis. Herzog and Bresinsky's results show that in embedding dimensions 3 and 4, symmetry implies the existence of a finite upper bound on the number of generators of the ideals of Gorenstein monomial curves. This is not true for general monomial curves in embedding dimensions greater than or equal to 4, as it was shown by Bresinsky [7]. In other words, in embedding dimension 4, there are monomial curves with arbitrary large number of generators. It is still an open question, whether symmetry always implies the existence of a finite upper bound on the number of generators of the ideals of Gorenstein monomial curves. Bresinsky has some contribution for embedding dimension 5 case, but even this case has not been completed yet [9].

3.3 Tangent Cone of a Monomial Curve

The tangent cone of a variety at a point, which refers to the approximation of the variety at this point, comes as a very important and useful geometric object. It is possible to get local information, particularly, when the point is a singular one. Naturally, a monomial curve C has a singular point at the origin if $n_i > 1$ for all $1 \leq i \leq d$. In this case, the tangent cone of a monomial curve at the origin is important to understand more about the monomial curves.

Definition 3.3.1. Let I_* be the ideal which is generated by the polynomials f_*, where the homogeneous summand of f of least degree, for $f \in I$. The geometric tangent cone $C_p(V)$ at p is $V(I_*)$, and the tangent cone is the pair $(V(I_*), k[x_1, \ldots, x_d]/I_*)$.

One can study the tangent cone of the variety V at the origin with a different approach by using the associated graded ring of the coordinate ring $k[x_1, \ldots, x_d]/I(V)$ of a variety V with respect to the maximal ideal m. We can investigate the tangent cone of a monomial curve C at the origin by studying either the associated graded ring of $k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]]$ with respect to the maximal ideal $m = (t^{n_1}, t^{n_2}, \ldots, t^{n_d})$ which is denoted by $\mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]])$, or the ring $k[x_1, x_2, \ldots, x_d]/I(C)_*$. For the details, see [1].

3.4 Cohen-Macaulayness of Tangent Cone

As we have mentioned before, it is an important problem to determine if the tangent cone of the associated graded ring of a local ring is Cohen-Macaulay, (CM), because this guarantees the non-decreasing property of the Hilbert function.

Since our main geometric objects are monomial curves, we will pay attention to the Cohen-Macaulayness of the tangent cone of a monomial curve. In this section, we will discuss some results about checking the Cohen-Macaulayness of the tangent cone of a monomial curve C by considering the ideal $I(C)_*$. For details, see [1].

Lemma 3.4.1. *[19, Theorem7]* $\mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]])$ *is Cohen-Macaulay if and only if* t^{n_1} *is not a zero divisor in* $\mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]])$.

Then the useful result follows:

Theorem 3.4.2. *[2, Theorem 2.1] Let C be a monomial curve. Let $g_1, \ldots, g_s$ be a minimal Gröbner basis for $I(C)_*$ with respect to a reverse lexicographical order that makes x_1 the lowest variable. Then* $\mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]])$ *is Cohen-Macaulay if and only if $x_1 \nmid in(g_i)$ for $1 \leqslant i \leqslant s$, where $in(g_i)$ is the leading term of g_i.*

CHAPTER 4

HILBERT FUNCTIONS OF GORENSTEIN MONOMIAL CURVES

Our main aim is to study the Hilbert function of a one-dimensional Gorenstein local ring of embedding dimension four which corresponds to a monomial curve. We have mentioned that, it is a longstanding problem to determine the behavior of the Hilbert function of a local ring, and we have restated the Sally's famous conjecture as:

> *Sally's conjecture.* If R is a one-dimensional Gorenstein local ring with small enough embedding dimension, then $H_R(n)$ is non-decreasing.

This conjecture is open even for monomial curves in embedding dimension 4. In order to study this problem in monomial curve case, we consider monomial curves for which their semigroups are symmetric. We recall that a numerical semigroup $< n_1, n_2, \ldots, n_d >= \{n \mid n = \sum_{i=1}^{d} a_i n_i,\ a_i\text{'s are non-negative integers}\}$ is symmetric if and only if the number of gaps is equal to the number of nongaps ($n \notin < n_1, n_2, \ldots, n_d >, 0 < n \leq c$ is called a gap, and $n \in < n_1, n_2, \ldots, n_d >, 0 \leq n < c$ is called a nongap). Kunz [26] gives an algebraic characterization of symmetric semigroups by showing that $< n_1, n_2, \ldots, n_d >$ is symmetric if and only if $k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]]$ is Gorenstein. Bresinsky shows that for a monomial curve C with symmetric semigroups of integers generated by four elements, $I(C)$ is generated by 3 (complete intersection case) or 5 elements [8]. He also gives description of the defining ideal and arithmetic conditions for the generators of $I(C)$.

Since the tangent cone may not be Cohen-Macaulay even in the complete intersection case, almost nothing is known about the Hilbert function. For example, the monomial curve having parametrization $x_1 = t^{12}, x_2 = t^{14}, x_3 = t^{21}, x_4 = t^{30}$ is a complete intersection, but the associated graded ring of $k[[t^{12}, t^{14}, t^{21}, t^{30}]]$ is not Cohen-Macaulay, so that we cannot immediately conclude that its Hilbert function is non-decreasing even in the complete intersection case.

Hence, our main aim is to investigate and try to determine the behavior of the Hilbert function of Gorenstein monomial curves in the case of embedding dimension four. Based on the standard basis theory, we find the minimal generators of the tangent cone of a monomial curve C with defining ideal as described in [8] under some arithmetic assumptions. By using the Cohen-Macaulayness of these tangent cones, we determine that the Hilbert function of these curves are non-decreasing under these assumptions.

We can summarize the notation as the following: A monomial affine curve C has parametrization

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ \ldots,\ x_d = t^{n_d}$$

where $n_1, n_2, \ldots, n_d \in \mathbb{Z}^+$ with $\gcd(n_1, n_2, \ldots, n_d) = 1$ and $n_1, n_2, \ldots, n_d$ generate minimally the semigroup $< n_1, n_2, \ldots, n_d >$. $I(C)$ is the defining ideal of C consisting of

$$\{f(x_1, \ldots, x_d) \mid f(x_1, \ldots, x_d) \in k[x_1, \ldots, x_d],\ \ f(t^{n_1}, \ldots, t^{n_d}) = 0,$$
$$t \text{ transcendental over } k\},$$

and $I(C)_*$ is the ideal generated by the polynomials f_* for f in $I(C)$, where f_* is the homogeneous summand of f of least degree, and $\mu(I(C)_*)$ is the minimal number of generators of ideal $I(C)_*$ which is also called the tangent cone of the monomial curve C. In order to study the tangent cone of a monomial curve C at the origin, it is possible to consider either the associated graded ring of $k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]]$ with respect to the maximal ideal $m = (t^{n_1}, t^{n_2}, \ldots, t^{n_d})$ which is denoted by $\mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]])$, or the ring $k[x_1, x_2, \ldots, x_d]/I(C)_*$.

4.1 Generators of Tangent Cone

In this section, we find the minimal generators of the tangent cone of monomial curve C having the defining ideal as in Theorem 3 in [8] under some arithmetic assumptions on the generators. First, we recall the Bresinsky's theorem which gives the explicit description of a Gorenstein monomial curve [8].

Theorem 4.1.1. (Bresinsky) [8, Theorem 3] $S =< n_1, n_2, n_3, n_4 >$ *symmetric and* $I(C)$ *is not necessarily generated by 3 elements if and only if* $I(C) = (f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}})$ *where the polynomials* $f_i, 1 \leq i \leq 4$, *are unique up to isomorphism and* $0 < \alpha_{ij} < \alpha_j$.

Remark 4.1.2. Here, $\alpha_i n_i \in < n_1, \ldots, \hat{n}_i, \ldots, n_4 >$ such that α_i, $1 \leq i \leq 4$, are minimal.

Remark 4.1.3. Since the polynomials f_i are unique up to isomorphism by Lemma 4 in [8], there are six isomorphic possible permutations which can be considered within three cases :

Case 1: $f_1 = (1, (3, 4))$. Then

either $f_2 = (2, (1, 4)), f_3 = (3, (1, 2)), f_4 = (4, (2, 3))$

or $f_2 = (2, (1, 3)), f_3 = (3, (2, 4)), f_4 = (4, (1, 2))$

Case 2: $f_1 = (1, (2, 3))$. Then

either $f_2 = (2, (3, 4)), f_3 = (3, (1, 4)), f_4 = (4, (1, 2))$

or $f_2 = (2, (1, 4)), f_3 = (3, (2, 4)), f_4 = (4, (1, 3))$

Case 3: $f_1 = (1, (2, 4))$. Then

either $f_2 = (2, (1, 3)), f_3 = (3, (1, 4)), f_4 = (4, (2, 3))$

or $f_2 = (2, (3, 4)), f_3 = (3, (1, 2)), f_4 = (4, (1, 3))$

In these cases the generators $f_i = (i, (j, k))$ are denoted symbolically by the notation $x_i^{\alpha_i} - x_j^{\alpha_{ij}} x_k^{\alpha_{ik}}$.

Now we want to show the relation between the defining ideal of Gorenstein monomial curves generated by 5 elements and their tangent cone. In order to do this, we construct examples by using the main characterization of [26]. First, we obtain monomial curves whose semigroups are symmetric. Later, we compute their defining ideals by using *Macaulay2*, then we find their tangent cones and determine their Hilbert functions.

The examples presented in tables 4.1 and 4.2 and tested by the *Macaulay2* code implied that the generators of tangent cone can be obtained provided that some restrictions are imposed on generators of the defining ideal. Generally speaking, one can compute a set of generators of $I(C)_*$ using the algorithm which is known as the "tangent cone algorithm" (see [12, p.467]) by merely knowing the description of the ideal $I(C)$. This algorithm utilizes the Gröbner basis which confines the

Table 4.1: Some examples...

[Case 1(a)] $f_1 = (1,(3,4))$, $f_2 = (2,(1,4))$, $f_3 = (3,(1,2))$, $f_4 = (4,(2,3))$			
C	(11, 16, 18, 26)	(16, 17, 22, 26)	
$I(C)$	$(x^4 - zw, y^3 - x^2w, z^3 - x^2y^2, w^2 - yz^2, x^2z^2 - y^2w)$	$(x^3 - zw, y^4 - xw^2, z^3 - x^2y^2, w^3 - y^2z^2, xz^2 - y^2w)$	
$I(C)_*$	$(w^2, zw, z^3, y^2w, y^3 - x^2w)$	$(w^3, zw, z^3, y^2w^2, y^4z - x^4w, y^6 - x^5z, xw^2, xz^2 - y^2w)$	
$H(n)$	$1, 4, 8, 10, 11, \dots$	$1, 4, 9, 12, 14, 16, \dots$	
[Case 1(b)] $f_1 = (1,(3,4))$, $f_2 = (2,(1,3))$, $f_3 = (3,(2,4))$, $f_4 = (4,(1,2))$			
C	(11, 14, 15, 18)	(7, 8, 17, 18)	(9, 10, 11, 23)
$I(C)$	$(x^3 - zw, y^4 - xz^3, z^4 - y^3w, w^2 - x^2y, yz - xw)$	$(x^5 - zw, y^3 - xz, z^2 - y^2w, w^2 - x^4y, yz - xw)$	$(x^5 - z^2w, y^2 - xz, z^3 - yw, w^2 - x^4y, yz^2 - xw)$
$I(C)_*$	$(w^2, zw, z^4 - y^3w, yz - xw, y^4 - xz^3)$	$(w^2, zw, z^2, yz - xw, y^3w, y^7, xz)$	$(w^2, z^2w, z^5, yw, yz^4, y^2 - xz, xw)$
$H(n)$	$1, 4, 7, 10, 11, \dots$	$1, 4, 5, 6, 6, 6, 7, \dots$	$1, 4, 6, 7, 9, \dots$
[Case 2(a)] $f_1 = (1,(2,3))$, $f_2 = (2,(3,4))$, $f_3 = (3,(1,4))$, $f_4 = (4,(1,2))$			
C	(14, 17, 22, 24)		
$I(C)$	$(x^4 - y^2z, y^4 - z^2w, z^3 - x^3w, w^2 - xy^2, xz^2 - y^2w)$		
$I(C)_*$	$(w^2, z^2w, z^3, y^2z, y^4w, y^6 - x^4zw, xz^2 - y^2w)$		
$H(n)$	$1, 4, 9, 12, 13, 14, \dots$		
[Case 2(b)] $f_1 = (1,(2,3))$, $f_2 = (2,(1,4))$, $f_3 = (3,(2,4))$, $f_4 = (4,(1,3))$			
C	(5, 12, 13, 14)	(10, 11, 19, 24)	
$I(C)$	$(x^5 - yz, y^2 - x^2w, z^2 - yw, w^2 - x^3z, x^3y - zw)$	$(x^3 - yz, y^4 - x^2w, z^3 - y^3w, w^2 - xz^2, xy^3 - zw)$	
$I(C)_*$	$(w^2, zw, z^2 - yw, yz, y^2)$	$(w^2, zw, z^3, yz, y^5w, y^9 - x^8z, x^2w, x^3z^2 - y^4w)$	
$H(n)$	$1, 4, 5, \dots$	$1, 4, 7, 8, 9, 9, 9, 9, 10, \dots$	

Table 4.2: Some examples continued...

[Case 3(a)] $f_1 = (1, (2, 4)),\ f_2 = (2, (1, 3)),\ f_3 = (3, (1, 4)),\ f_4 = (4, (2, 3))$		
C	(7, 8, 9, 13)	(13, 14, 17, 38)
$I(C)$	$(x^3 - yw, y^2 - xz, z^3 - x^2w,$ $w^2 - yz^2, x^2y - zw)$	$(x^4 - yw, y^4 - x^3z, z^3 - xw,$ $w^2 - y^3z^2, xy^3 - zw)$
$I(C)_*$	$(w^2, zw, z^3 - x^2w, yw, y^2 - xz)$	$(w^2, zw, z^4, yw, yz^3, y^4 - x^3z, xw)$
$H(n)$	$1, 4, 6, 7, \ldots$	$1, 4, 6, 10, 12, 13, \ldots$
[Case 3(b)] $f_1 = (1, (2, 4)),\ f_2 = (2, (3, 4)),\ f_3 = (3, (1, 2)),\ f_4 = (4, (1, 3))$		
C	(9, 11, 14, 16)	(8, 9, 22, 23)
$I(C)$	$(x^3 - yw, y^4 - z^2w, z^3 - xy^3,$ $w^2 - x^2z, yz - xw)$	$(x^4 - yw, y^5 - zw, z^2 - xy^4,$ $w^2 - x^3z, yz - xw)$
$I(C)_*$	$(w^2, z^2w, z^3, yw, yz - xw, y^5 - x^3z^2)$	$(w^2, zw, z^2, yw, yz - xw, y^8, x^4z)$
$H(n)$	$1, 4, 7, 8, 9, \ldots$	$1, 4, 5, 6, 7, 7, 7, 8, \ldots$

computation only to global orderings, and this method was used by many authors, see [1]. However, in our case, this method is not efficient and one must go to local orderings to avoid imposing relatively larger number of restrictions and, therefore, to achieve a better generalization. Hence, we use standard basis theory and for local orderings, we use Mora's tangent cone algorithm based on standard basis.

The standard basis of an ideal or a module is just a special set of generators, which makes the computation of various invariants of the corresponding ideal or module possible via only its leading monomials. A Gröbner basis will be referred as the standard basis for a global ordering. We use Buchberger‘s algorithm for the computation of Gröbner bases. The algorithm for computing the standard basis is the same for any monomial ordering. Yet, only the normal form algorithm differs for well-orderings, called as the global orderings and for non-global orderings, called as the local, respectively mixed, orderings. The reader should see the reference [24] for the details of local orderings, normal form (NF), standard basis and Mora's tangent cone algorithm.

We give only the definition of the negative degree reverse lexicographical ordering among the other local orderings.

Definition 4.1.4. [24, p.14] (negative degree reverse lexicographical ordering)

$$\begin{aligned} x^\alpha >_{ds} x^\beta :\Longleftrightarrow \deg x^\alpha < \deg x^\beta, \text{ where } \deg x^\alpha = \alpha_1 + \ldots + \alpha_n, \\ \text{or } (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \\ \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i). \end{aligned}$$

Proposition 4.1.5. Let C be a Gorenstein monomial curve with defining ideal $I(C) = (f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}})$ as in Theorem 4.1.1. Then, under the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, the defining ideal $I(C)_*$ of the tangent cone is generated by a set S_* consisting of the least homogeneous summands of f_i's in $I(C)$ for $1 \leq i \leq 5$,

$$S_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 < \alpha_{21} + \alpha_{24}$ and

$$S_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 = \alpha_{21} + \alpha_{24}$.

Remark 4.1.6. A set of restrictions appear naturally from the structure of the symmetric group $S =< n_1, n_2, n_3, n_4 >$. Then, we have the following extra conditions

$\alpha_1 > \alpha_{13} + \alpha_{14}$ and $\alpha_4 < \alpha_{42} + \alpha_{43}$, since $n_1 < n_2 < n_3 < n_4$. There is another restriction $\alpha_3 < \alpha_{31} + \alpha_{32}$ which appears as a consequence of the inequality $n_3\alpha_3 = n_1\alpha_{31} + n_2\alpha_{32} < n_3\alpha_{31} + n_3\alpha_{32}$.

Remark 4.1.7. The restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$ implies another condition on the generator f_5 so that $\alpha_{43} + \alpha_{21} > \alpha_{32} + \alpha_{14}$.

In order to prove this proposition, we need the following remark and lemmas. First, we give some definitions.

Definition 4.1.8. [24, p.46] (normal form) Let $\mathcal{G}$ denote the set of all finite lists $G \subset R$, where the ring $R = k[x_1, \ldots, x_n]_>$ is the localization of $k[x_1, \ldots, x_n]$ with respect to a monomial ordering $>$. A map

$$NF : R \times \mathcal{G} \to R, (f, G) \mapsto NF(f \mid G),$$

is called a normal form on R if, for all $G \in \mathcal{G}$,

(0) $NF(0 \mid G) = 0$,

and, for all $f \in R$ and $G \in \mathcal{G}$,

(1) $NF(0 \mid G) \neq 0 \Rightarrow LM(NF(f \mid G)) \notin L(G)$, where $LM(NF(f \mid G))$ and $L(G)$ are the leading monomial of normal form of f with respect to G and the leading ideal of G, respectively.

(2) If $G = \{g_1, \ldots, g_s\}$, then $r := f - NF(f \mid G)$ has a standard representation with respect to G, that is, either $r = 0$, or

$$r = \textstyle\sum_{i=1}^{s} a_i g_i, a_i \in R,$$

satisfying $LM(f) \geq LM(a_i g_i)$ for all i such that $a_i g_i \neq 0$.

Definition 4.1.9. [24, p.120] (ecart of a polynomial f) For a monomial $x^\alpha e_i \in K[x]^r$ set

$$\deg x^\alpha e_i := \deg x^\alpha = \alpha_1 + ... + \alpha_n$$

For $f \in K[x]^r \backslash \{0\}$, let $\deg f$ be the maximal degree of all monomials occurring in f. We define the *ecart* of f as

$$\operatorname{ecart}(f) := \deg f - \deg LM(f).$$

Remark 4.1.10. *The Product Criterion* [24, p.63]: Let $f, g \in K[x_1, \ldots, x_n]$ be polynomials such that $\operatorname{lcm}(LM(f), LM(g)) = LM(f).LM(g)$, then

$$NF(\operatorname{spoly}(f, g) \mid \{f, g\}) = 0.$$

Lemma 4.1.11. *The set $S = \{f_1, f_2, f_3, f_4, f_5\}$ where the polynomials f_i's are as in Proposition 4.1.5. $I(C)$ is a standard basis with respect to the negative degree reverse lexicographical ordering $>_{ds}$, under the condition $\alpha_2 \leq \alpha_{21} + \alpha_{24}$*

Proof. We apply standard basis algorithm (Mora's tangent cone algorithm) to the set $\{f_1, f_2, f_3, f_4, f_5\}$. We begin with f_1 and f_2. Since

$NF(\operatorname{spoly}(f_1, f_2) \,|\, \{f_1, f_2\}) = 0$ by remark 4.1.10, then

$NF(\operatorname{spoly}(f_1, f_2) \,|\, \{f_1, f_2, f_3, f_4, f_5\}) = 0$ and the algorithm terminates. Next, we choose f_1 and f_3. $\operatorname{spoly}(f_1, f_3) = x_1^{\alpha_1} x_3^{\alpha_{43}} - x_1^{\alpha_{31}} x_2^{\alpha_{32}} x_4^{\alpha_{14}}$ and since

$\alpha_1 + \alpha_{43} > \alpha_{31} + \alpha_{32} + \alpha_{14}$, $LM\{h = \operatorname{spoly}(f_1, f_3) = x_1^{\alpha_{31}} x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$. Then,

$T_h = \{f_5 = x_3^{\alpha_{43}} x_1^{\alpha 21} - x_2^{\alpha_{32}} x_4^{\alpha_{14}} \,|\, LM(f_5) \mid LM(h)\} \neq \varnothing$.

ecart(f_5)=$\alpha_{43} + \alpha_{21} - (\alpha_{32} + \alpha_{14})$

ecart(h)=$\alpha_1 + \alpha_{43} - (\alpha_{31} + \alpha_{32} + \alpha_{14})$

Since ecart(f_5)=ecart(h) no h will be added to the finite list. spoly$(f_5, h) = 0$ causes the iterative steps to stop at this stage of the algorithm. In the same manner, the algorithm is applied to all the 2-tuples in the list.

$\square$

Lemma 4.1.12. *[24, p.296] Let $I \subset \langle x \rangle \subset K[x]$, $x = (x_1, \ldots, x_r)$, be an ideal, and let $\{f_1, \ldots, f_s\}$ be a standard basis of I with respect to a local degree ordering $>$. Then $I_* = \langle f_{1*}, \ldots, f_{s*} \rangle$, where I_* is the defining ideal of the tangent cone f_{i*} is the homogeneous summand of least degree of f_i.*

Proposition 4.1.13. [24, p.296] Let $I \subset \langle x \rangle \subset K[x]$, $x = (x_1, \ldots, x_r)$, be an ideal, let $A := K[x]_{\langle x \rangle}/I$, and let m be the maximal ideal of A. Then

$$\mathrm{gr}_m(A) \cong K[x]/I_*.$$

We can now prove the Proposition 4.1.5.

Proof of Proposition 4.1.5. By the Lemma 4.1.11, $S = \{f_1, f_2, f_3, f_4, f_5\}$, where $f_i's$ are as in theorem 4.1.1, is a standard basis of $I(C)$ with respect to a local degree ordering $>_{ds}$. Then,

$$I(C)_* = \langle f_{1*}, \ldots, f_{5*} \rangle = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 < \alpha_{21} + \alpha_{24}$ and

$$I(C)_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 = \alpha_{21} + \alpha_{24}$ by the Lemma 4.1.12. Finally, from the Proposition 4.1.13 and

$$\mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]]) \cong k[x_1, x_2, \ldots, x_d]/I(C)_*,$$

$I(C)_*$ is generated by

$$S_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 < \alpha_{21} + \alpha_{24}$ and

$$S_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 = \alpha_{21} + \alpha_{24}$.

□

Remark 4.1.14. The same method can be applied to Case 1(b) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{23}$ and $\alpha_3 \leq \alpha_{32} + \alpha_{34}$, Case 2(b) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$, $\alpha_3 \leq \alpha_{32} + \alpha_{34}$ and Case 3(a) with the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{23}$, $\alpha_3 \leq \alpha_{31} + \alpha_{34}$. Thus, in all these cases, $I(C)_*$ is generated by a set of five elements, which are the homogeneous summands of least degree of the five generators of $I(C)$.

Now, we can determine the behavior of the Hilbert function of the tangent cone of C, with its defining ideal $I(C)$ as in the Theorem 4.1.1 with the restriction of Proposition 4.1.5.

Theorem 4.1.15. *The Gorenstein monomial curve C having parametrization*

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ x_3 = t^{n_3}\ x_4 = t^{n_4}$$

and with the defining ideal $I(C)$ as in Theorem 4.1.1 under the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$ has Cohen-Macaulay tangent cone at the origin. Therefore, the Hilbert function of the corresponding Gorenstein local ring is non-decreasing.

Proof. We can apply the Theorem 3.4.2 to the generators of the tangent cone which are given by the set

$$S_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 < \alpha_{21} + \alpha_{24}$ and

$$S_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}\}$$

for $\alpha_2 = \alpha_{21} + \alpha_{24}$. Both of these sets are Gröbner basis with respect to the reverse lexicographic order with $x_1 > x_4 > x_2 > x_3$. Since x_1 does not divide the leading monomial of any element in S_*, the ring $k[x_1, x_2, x_3, x_4]/I(C)_*$ is Cohen-Macaulay from Theorem 3.4.2. Since $R = \text{gr}_m(k[[t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}]]) \cong k[x_1, x_2, x_3, x_4]/I(C)_*$ is Cohen-Macaulay, multiplication by a homogeneous non-zero divisor of degree one is a monomorphism from R_n into R_{n+1}. This means $H(0) \leq H(1) \leq H(2) \leq \dots$. $\square$

Remark 4.1.16. This result can be proved similarly when the defining ideal $I(C)$ is as in other isomorphic possible cases.

As a result, we can now give the general theorem:

Theorem 4.1.17. *Let C be a Gorenstein monomial curve having parametrization*

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ x_3 = t^{n_3}\ x_4 = t^{n_4}$$

with defining ideal $I(C) = (f_1 = x_1^{\alpha_1} - m_1, f_2 = x_2^{\alpha_2} - m_2, f_3 = x_3^{\alpha_3} - m_3, f_4 = x_4^{\alpha_4} - m_4, f_5)$, where m_1, m_2, m_3 and m_4 are monomials, and $n_1 < n_2 < n_3 < n_4$. If $\alpha_2 \leq$ total degree of m_2 and $\alpha_3 \leq$ total degree of m_3, then the Hilbert function of the local Gorenstein ring $k[[t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}]]$ is non-decreasing.

4.2 Gorenstein Monomial Curves whose Tangent Cones are non-CM

In this section, we try to understand what we can say about the Cohen-Macaulayness of the tangent cone if we do not have the restriction on the generators of the defining ideal $I(C)$ as given in Theorem 4.1.1. Hence, we only consider the remaining case $\alpha_2 > \alpha_{21} + \alpha_{24}$ and we obtain the following Lemma:

Lemma 4.2.1. *Let C be a Gorenstein monomial curve with defining ideal* $I(C) = (f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}})$ *as in Theorem 4.1.1. Then, under the restriction $\alpha_2 > \alpha_{21} + \alpha_{24}$, its tangent cone at the origin is not Cohen-Macaulay.*

Proof. In this case, since $\alpha_2 > \alpha_{21} + \alpha_{24}$ the generators of $I(C)_*$ must either contain the homogeneous summand of least degree of $f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}$, which is $x_1^{\alpha_{21}} x_4^{\alpha_{24}}$ or this term must be a zero element in the ring $k[x_1, x_2, x_3, x_4]/I(C)_*$. But this is only possible, if either there exists a generator $x_1^a x_4^b$ of $I(C)_*$ satisfying $x_1^a x_4^b \mid x_1^{\alpha_{21}} x_4^{\alpha_{24}}$ or a generator x_4^c of $I(C)_*$ satisfying $x_4^c \,|\, x_4^{\alpha_{24}}$. The second case is not possible, because $x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}$ is the binomial in $I(C)$ with α_4, the smallest possible value, and $\alpha_{24} < \alpha_4$. Thus, x_1 is a zero divisor of the ring $k[x_1, x_2, x_3, x_4]/I(C)_*$ and this shows that $R = \mathrm{gr}_m(k[[t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}]]) \cong k[x_1, x_2, x_3, x_4]/I(C)_*$ is not Cohen-Macaulay □

Remark 4.2.2. Same observations show us that we can obtain same results for the other three permutations:

$f_1 = (1, (3, 4)), f_2 = (2, (1, 3)), f_3 = (3, (2, 4)), f_4 = (4, (1, 2)),$

$f_1 = (1, (2, 3)), f_2 = (2, (1, 4)), f_3 = (3, (2, 4)), f_4 = (4, (1, 3)),$

$f_1 = (1, (2, 4)), f_2 = (2, (1, 3)), f_3 = (3, (1, 4)), f_4 = (4, (2, 3)).$

Remark 4.2.3. In other two cases

$f_1 = (1, (2, 3)), f_2 = (2, (3, 4)), f_3 = (3, (1, 4)), f_4 = (4, (1, 2))$ and

$f_1 = (1, (2, 4)), f_2 = (2, (3, 4)), f_3 = (3, (1, 2)), f_4 = (4, (1, 3)),$

we produced several examples of Gorenstein monomial curves. While some of their tangent cones are CM, the others come out to be not CM.

4.3 A Conjecture for Higher Dimensions

In this section, we consider the same problem first in embedding dimension 5 and then in all higher dimensions. In order to do this, we construct examples using the results of [9] and [33]. We obtain monomial curves in 5-space whose semigroups are symmetric, compute generators of their defining ideals and tangent cone by using *Singular* and determine the Hilbert function.

The examples 4.3 which are tested by *Singular* [38] show that our problem can be extended to embedding dimension 5 within the same approach. We remind that these examples are selected such that we impose restrictions on α_2, α_3 and α_4 which are the exponents of the generators starting with x_2, x_3 and x_4, respectively. For the generators $x_2^{\alpha_2} - m_2$, $x_3^{\alpha_3} - m_3$ and $x_4^{\alpha_4} - m_4$, we have the restrictions $\alpha_2 \leq$ total degree of m_2, $\alpha_3 \leq$ total degree of m_3 and $\alpha_4 \leq$ total degree of m_4, and these conditions can be considered as the generalization of the conditions in Theorem 4.1.17. If we consider these examples, it is interesting to observe that the tangent cones are Cohen-Macaulay. Thus, these examples suggest that this can be stated as a new conjecture even though we could not prove it since the explicit description of the defining ideals in higher dimensions are unknown.

Table 4.3: Examples in 5-space.

C	(6, 7, 8, 9, 10)	(10, 11, 13, 14, 17)
$I(C)$	$(x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_1x_5,$ $x_2x_4 - x_1x_5, x_3x_4 - x_2x_5, x_4^2 - x_3x_5,$ $x_3x_5 - x_1^3, x_4x_5 - x_1^2x_2, x_5^2 - x_1^2x_3)$	$(x_2x_3 - x_1x_4, x_3x_4 - x_1x_5, x_4^2 - x_2x_5$ $x_3x_5 - x_1^3, x_4x_5 - x_1^2x_2, x_5^2 - x_1^2x_4$ $x_2^3 - x_1^2x_3, x_1x_3^2 - x_2^2x_4, x_3^3 - x_2^2x_5)$
$I(C)_*$	$(x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_1x_5,$ $x_2x_4 - x_1x_5, x_3x_4 - x_2x_5, x_4^2 - x_3x_5,$ $x_3x_5, x_4x_5, x_5^2)$	$(x_2x_3 - x_1x_4, x_3x_4 - x_1x_5,$ $x_4^2 - x_2x_5, x_3x_5, x_4x_5, x_5^2,$ $x_2^3 - x_1^2x_3, x_1x_3^2 - x_2^2x_4, x_3^3 - x_2^2x_5)$
$H(n)$	$1, 5, 6, \ldots$	$1, 5, 9, 10, \ldots$
C	(10, 11, 12, 13, 14)	(18, 19, 20, 21, 22)
$I(C)$	$(x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4,$ $x_2x_4 - x_1x_5, x_3x_4 - x_2x_5, x_4^2 - x_3x_5,$ $x_3x_5^2 - x_1^4, x_4x_5^2 - x_1^3x_2, x_5^3 - x_1^3x_3$	$(x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4$ $x_2x_4 - x_1x_5, x_3x_4 - x_2x_5, x_4^2 - x_3x_5$ $x_3x_5^4 - x_1^6, x_4x_5^4 - x_1^5x_2, x_5^5 - x_1^5x_3)$
$I(C)_*$	$(x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4,$ $x_2x_4 - x_1x_5, x_3x_4 - x_2x_5, x_4^2 - x_3x_5,$ $x_3x_5^2, x_4x_5^2, x_5^3$	$(x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4$ $x_2x_4 - x_1x_5, x_3x_4 - x_2x_5, x_4^2 - x_3x_5$ $x_3x_5^4, x_4x_5^4, x_5^5)$
$H(n)$	$1, 5, 9, 10, \ldots$	$1, 5, 9, 13, 17, 18, \ldots$
C	(19, 23, 29, 31, 37)	(19, 27, 28, 31, 32)
$I(C)$	$(x_3x_4 - x_2x_5, x_2^3 - x_1^2x_4,$ $x_2^2x_3 - x_1^2x_5, x_1x_3^2 - x_2^2x_4,$ $x_2x_3^2 - x_1x_4^2, x_3^3 - x_1x_4x_5,$ $x_2x_4^2 - x_1x_3x_5, x_4^3 - x_1x_5^2,$ $x_3^2x_5 - x_1^5, x_4^2x_5 - x_1^4x_2,$ $x_3x_5^2 - x_1^3x_2^2, x_4x_5^2 - x_1^4x_3,$ $x_5^3 - x_1^3x_2x_4)$	$(x_3x_4 - x_2x_5, x_2^3 - x_1x_4^2,$ $x_2^2x_3 - x_1x_4x_5, x_2x_3^2 - x_1x_5^2,$ $x_3^3 - x_1^3x_2, x_2^2x_4 - x_1^3x_3,$ $x_2x_4^2 - x_1^3x_5, x_4^3 - x_1^2x_2x_3,$ $x_3^2x_5 - x_1^3x_4, x_4^2x_5 - x_1^2x_3^2,$ $x_3x_5^2 - x_1^2x_2^2, x_4x_5^2 - x_1^5)$
$I(C)_*$	$(x_3x_4 - x_2x_5, x_2^3 - x_1^2x_4,$ $x_2^2x_3 - x_1^2x_5, x_1x_3^2 - x_2^2x_4,$ $x_2x_3^2 - x_1x_4^2, x_3^3 - x_1x_4x_5,$ $x_2x_4^2 - x_1x_3x_5, x_4^3 - x_1x_5^2,$ $x_3^2x_5, x_4^2x_5, x_3x_5^2, x_4x_5^2, x_5^3)$	$(x_3x_4 - x_2x_5, x_2^3 - x_1x_4^2,$ $x_2^2x_3 - x_1x_4x_5, x_2x_3^2 - x_1x_5^2,$ $x_3^3, x_2^2x_4, x_2x_4^2, x_4^3, x_3^2x_5,$ $x_4^2x_5, x_3x_5^2, x_4x_5^2, x_5^3)$
$H(n)$	$1, 5, 14, 18, 19, \ldots$	$1, 5, 14, 18, 19, \ldots$

Conjecture. Let C be a Gorenstein monomial curve having parametrization

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ \ldots\ x_\ell = t^{n_\ell},\ \ \ell \geq 5$$

with defining ideal $I(C)$ containing the generators

$$f_2 = x_2^{\alpha_2} - m_2, \ldots, f_{\ell-1} = x_{\ell-1}^{\alpha_{\ell-1}} - m_{\ell-1}$$

where α_i's, $1 \leq i \leq \ell$ are minimal, $m_2, \ldots, m_{\ell-1}$ are monomials and $n_1 < n_2 < \ldots < n_\ell$. If $\alpha_2 \leq$ total degree of m_2, $\ldots$,$\alpha_{\ell-1} \leq$ total degree of $m_{\ell-1}$, then the Hilbert function of the local Gorenstein ring $k[[t^{n_1}, t^{n_2}, \ldots, t^{n_\ell}]]$ is non-decreasing.

4.4 Generalization to Higher Dimensions

In this section, we connect the ideas in [30] and [8] to obtain families of monomial curves with Cohen-Macaulay associated graded rings in higher dimensions.

Theorem 4.4.1. *Let C be a Gorenstein monomial curve having parametrization*

$$x_1 = t^{n_1},\ x_2 = t^{n_2},\ x_3 = t^{n_3}\ x_4 = t^{n_4}$$

and with the defining ideal $I(C)$ as in Theorem 4.1.1 under the restriction $\alpha_2 \leq \alpha_{21} + \alpha_{24}$. Let C' be a monomial curve in 5-space having parametrization,

$$x_1 = t^{pn_1},\ x_2 = t^{pn_2},\ x_3 = t^{pn_3}\ x_4 = t^{pn_4}\ \ x_5 = t^{n_5}$$

where $n_5 = a_1n_1 + a_2n_2 + a_3n_3 + a_4n_4 \neq 0$ is any element of the numerical semigroup generated by n_1, n_2, n_3, n_4 and p is a prime satisfying $\gcd(n_5, p) = 1$ and $p \leq a_1 + a_2 + a_3 + a_4$. Then the Hilbert function of the local ring $k[[t^{pn_1}, t^{pn_2}, t^{pn_3}, t^{pn_4}, t^{n_5}]]$ is non-decreasing.

To prove this theorem, we need the following proposition.

Proposition 4.4.2. $I(C')$ is generated by the set $S = \{f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}, f_6 = x_5^p - x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}\}$.

Remark 4.4.3. Note that we start with the Gorenstein monomial curves in affine 4-space but we give up the Goreinsteinness property of the new curves in dimension 5.

To prove the above proposition, we need the following lemma:

Lemma 4.4.4. *[30, Lemma 3.2] Let C be a curve having parametrization*

$$x_1 = \varphi_1(t), ..., x_{\ell-1} = \varphi_{\ell-1}(t), x_\ell = t^a.$$

Let β be a positive integer such that $\gcd(a, \beta) = 1$, and let $\tilde{C}$ be a curve having parametrization

$$x_1 = \varphi_1(t^\beta), ..., x_{\ell-1} = \varphi_{\ell-1}(t^\beta), x_\ell = t^a.$$

For any $f(x_1, ..., x_\ell) \in k[x_1, ..., x_\ell]$, we denote by $\tilde{f}$ the element $f(x_1, ..., x_{\ell-1}, x_\ell^\beta)$. Then if $f_1, ..., f_s$ is a set of generators for $I(C)$, then $\tilde{f}_1, ..., \tilde{f}_s$ is a set of generators for $I(\tilde{C})$.

Proof of Proposition 4.4.2. The proof is a direct consequence of the above lemma with the generators of $I(C)$ as in Theorem 4.1.1.

□

We can now prove the Theorem 4.4.1.

Proof of Theorem 4.4.1. The standard basis of the set S in Proposition 4.4.2 with respect to the negative degree reverse lexicographical ordering with $x_5 < x_1 < x_2 < x_3 < x_4$ is itself. Thus, the tangent cone is generated by the set S_* consisting of just the homogeneous summands of least degree of the elements in S. If we compute the Gröbner basis with respect to the reverse lexicographic order with $x_1 > x_5 > x_4 > x_2 > x_3$, and take the leading monomials of the elements in the Gröbner basis with respect to this order, we obtain the set

$$S'_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}, x_5^{p}\}.$$

for $\alpha_2 < \alpha_{21} + \alpha_{24}$ and

$$S'_* = \{x_3^{\alpha_{13}} x_4^{\alpha_{14}}, x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, x_3^{\alpha_3}, x_4^{\alpha_4}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}, x_5^{p}\}$$

for $\alpha_2 = \alpha_{21} + \alpha_{24}$. Since x_1 does not divide any element in S'_*, the coordinate ring of tangent cone of the monomial curve C' , $k[x_1, x_2, x_3, x_4, x_5]/I(C')_*$, is Cohen-Macaulay from Theorem 3.4.2. Since $R = \mathrm{gr}_m(k[[t^{pn_1}, t^{pn_2}, t^{pn_3}, t^{pn_4}, t^{n_5}]]) \cong k[x_1, x_2, x_3, x_4, x_5]/I(C')_*$ is Cohen-Macaulay, the Hilbert function of the local ring $k[[t^{pn_1}, t^{pn_2}, t^{pn_3}, t^{pn_4}, t^{n_5}]]$ is non-decreasing.

□

The same method can be used in the same manner to obtain monomial curves in higher dimensions.

CHAPTER 5

CONCLUSION

In this thesis, we studied the Sally's famous conjecture for Gorenstein monomial curves and showed that this conjecture holds for such curves in embedding dimension four. We used the ring $k[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]/I(C)_*$, since we tried to utilize standard basis theory to find the generators of $I(C)_*$. This computational approach allowed us to have a greater look over the problem. The results drawn in this thesis would not have been obtained by solely considering the semigroup ring. Because it does not give the generators of $I(C)_*$. On the other hand, our approach determines the generators of tangent cone which enables us to investigate its Cohen-Macaulayness. Then, we have the ability to study the behavior of the Hilbert function of the corresponding ring.

We also determined families of Gorenstein monomial curves, whose tangent cone is not Cohen-Macaulay, but all the examples of this type of monomial curves showed that their associated graded ring has still non-decreasing Hilbert function.

Thus, we solved part of the Sally's restated conjecture for the Gorenstein monomial curves in embedding dimension 4 case, and for the remaining part we found many examples to support the conjecture.

We realized that the procedure which we used in embedding dimension four can be applied to higher dimensions. We obtained examples and saw that our claim is still true for some of these examples which obey certain restrictions.

Starting with Gorenstein monomial curves in embedding dimension four we achieved to extend the same problem to fifth and higher dimensions by giving up being Gorenstein, and in this manner we obtained large families of local rings with non-decreasing Hilbert function.

Finally, we can say the computations done in this thesis show that computational methods can be applied to geometric problems to obtain a broader point of view.

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