EXACT SOLUTIONS OF THE SUPERSYMMETRIC QUANTUM MECHANICS

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ABSTRACT

EXACT SOLUTIONS OF THE SUPERSYMMETRIC QUANTUM MECHANICS

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The supersymmetric solutions of PT-/non-PT symmetric and Hermitian/non-Hermitian forms of quantum systems are obtained by solving the Schrödinger equation with the deformed Morse, Hulthén, Pöschl-Teller, Hyperbolic Kratzer-like, Screened Coulomb, and Exponential-Cosine Screened Coulomb (ECSC) potentials. The Hamiltonian hierarchy method is used to get the real energy eigenvalues and corresponding wave functions.

Keywords: Supersymmetric Quantum Mechanics, Exactly Solvable Potentials, PT-Symmetric Quantum Mechanics, Hierarchy of Hamiltonian.

ÖZ

SÜPERSİMETRİK KUANTUM MEKANIĞIN TAM ÇÖZÜMLERİ

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Hermitik veya hermitik olmayan ve PT simetrik veya PT simetrik olmayan kuantum sistemlerinin süpersimetrik çözümleri, değiştirilmiş Morse, Hulthén, Pöschl-Teller, Hiperbolik Kratzer-like, Screened Coulomb ve Exponential-Cosine Screened Coulomb (ECSC) potansiyelleri için Schrödinger denkleminin çözülmesiyle elde edildi.

Anahtar Kelimeler: Süpersimetrik Kuantum Mekaniği, Tam Çözülebilir Potansiyeller, Parite-Zaman Simetrik Kuantum Mekaniği, Kademeli Hamiltonyen.

...MY FAMILY

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CHAPTER 1

INTRODUCTION

PT-symmetric quantum systems have acquired much interest in recent years [1]. About twelve years ago, Bessis suggested that the eigenvalue spectrum of complex-valued Hamiltonians is real and positive. Bender and Boettcher claimed that this result is due to the property of PT-symmetry where P and T are the parity and time reversal operators respectively. This is neither a necessary nor a sufficient condition for a Hamiltonian to have real spectrum. In particular, the spectrum of the Hamiltonian is real if PT-symmetry is not spontaneously broken which means exactness. Thus, the property of exactness guarantees the real eigenvalues. Recently, Mostafazadeh introduced another concept for a class of PT-invariant Hamiltonians called $\eta - pseudo - Hermiticity$ [2]. In fact, Hamiltonians of this type satisfy the transformation $\eta\ \hat{H}\ \eta^{-1} = \hat{H}^{\dagger}$ [3]. Moreover, completeness and orthonormality conditions for the eigenstates of such potentials are proposed [4]. Various techniques have been applied in the analysis of PT-invariant potentials such as variational methods [5], numerical approaches [6], Fourier analysis [7], semi-classical estimates [8], quantum field theory [9] and

group theoretical approaches with the Lie algebra [10, 11, 12, 13]. It is pointed out in the Lie algebraic applications that a generalization of the symmetry concept is encountered in the supersymmetric quantum mechanics (SUSYQM), namely, PT-supersymmetric quantum mechanics (PT-SUSYQM) [14]. Various numbers of PT-symmetric examples can be found using the SUSYQM techniques[15-20]. Furthermore, one can get more examples of the PT-symmetric and non-PT-symmetric non-Hermitian potential cases such as oscillator type potentials [21], flat, step and double square-well like potentials within the framework of SUSYQM [22, 23], exponential type potentials [24], quasi/conditionally exactly solvable ones [25], complex Hénon-Heiles potentials [26], periodic isospectral potentials [27] and some others[28, 29].

The aim of the present work is to calculate the real and complex-valued energy eigenvalues and corresponding eigenfunctions of some certain potentials of physical interest through the Hamiltonian hierarchy method [30] within the framework of the PT-SUSYQM. The solution method is also known as the factorization method introduced by Schrödinger [31] and later developed by Infeld and Hull [32]. This method is a useful technique to obtain the equivalent energy spectra for different potentials in non-relativistic quantum mechanics [33, 34, 35].

The thesis is organized as follows: In chapter 2, we give a brief review of the SUSYQM. Chapter 3 includes the method of the Hamiltonian hierarchy based

on the SUSYQM. In chapter 4, we present our calculations for the supersymmetric solutions of non-deformed and deformed, PT-/non-PT symmetric and Hermitian/non-Hermitian forms of some well-known potentials by using this method. In chapter 5, the conclusion and discussion of the results are presented.

CHAPTER 2

The SUSYQM

In the past decades, the ideas of supersymmetry have been profitably applied to many non-relativistic quantum mechanical problems. In particular, there is now a much deeper understanding of why certain potentials are analytically solvable and an array of powerful new approximation methods for handling potentials which are not exactly solvable. In this chapter, we review the theoretical formulation of the supersymmetric quantum mechanics (SUSYQM).

2.1 Hamiltonian Formulation of the SUSYQM

We will calculate the possible forms of the energy eigenvalues and eigenfunctions with two partner Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ respectively. Let us assume that one dimensional Schrödinger equation for the ground state wave function $\Psi_0(x)$ is

$$\hat{H}_1\Psi_0(x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)\right]\Psi_0(x) = 0, \tag{2.1}$$

where we choose the ground state energy $E_0^{(1)}$ of $\hat{H}_1$ to be zero. Solving the Eq. (2.1) for $V_1(x)$, we have

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\Psi_0''(x)}{\Psi_0(x)}. \tag{2.2}$$

Thus, we can globally reconstruct of $V_1(x)$ from the ground state wave function that has no nodes. Using a Hermitian positive semi-definite operator in which $\hat{\Omega}^\dagger$ is the Hermitian adjoint of the operator $\hat{\Omega}$, one can factorize the first partner Hamiltonian as

$$\hat{H}_1(x) = \hat{\Omega}^\dagger\ \hat{\Omega}, \tag{2.3}$$

where

$$\hat{\Omega}^\dagger = W(x) - \frac{i}{\sqrt{2m}}\hat{p}, \quad \hat{\Omega} = W(x) + \frac{i}{\sqrt{2m}}\hat{p}, \tag{2.4}$$

with $\hat{p} = -i\hbar\frac{d}{dx}$. In the Eq. (2.4), $W(x)$ is known as the Witten superpotential. Therefore, the first partner Hamiltonian in terms of this superpotential leads to the well-known nonlinear Riccati equation:

$$V_1(x) - W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) = 0. \tag{2.5}$$

Here, $W'(x)$ is the first order derivative of the superpotential $W(x)$. The Witten superpotential is given through the ground state wave function as

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\left(\frac{\Psi_0'}{\Psi_0}\right) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ln[\Psi_0(x)], \tag{2.6}$$

where

$$\Psi_0(x) = N\exp\left[-\frac{\sqrt{2m}}{\hbar}\int^x W(x')dx'\right]. \tag{2.7}$$

As a second step, one can construct the other partner Hamiltonian $\hat{H}_2$ for $V_2(x)$,

$$\hat{H}_2 = \hat{\Omega}\hat{\Omega}^\dagger = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x)\right], \tag{2.8}$$

and this yields

$$V_2(x) - W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) = 0. \tag{2.9}$$

The potentials $V_1(x)$ and $V_2(x)$ which are connected by $W(x)$ are known as supersymmetric partner potentials. We also note that the quadratic terms $W^2(x)$ in the Eqs. (2.5) and (2.9) are the average of the partner potentials $V_1(x)$ and $V_2(x)$.

Now, let us assume that $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ denote the eigenfunctions of the bosonic Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ with the energy eigenvalues $E_n^{(1)}$ and $E_n^{(2)}$ respectively. In the wave function Ψ_n, the integer n represents the number of nodes and it takes the value from 0 to ∞. Here, we will prove that the partner potentials V_1 and V_2 have identical bound-state spectra except for the ground state energy $E_0^{(1)} = 0$. In SUSYQM approach, we consider a pair of Schrödinger equations for $n > 0$ as,

$$\begin{aligned}
\hat{H}_1\Psi_n^{(1)} &= \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)\right]\Psi_n^{(1)} \\
&= \hat{\Omega}^\dagger\hat{\Omega}\Psi_n^{(1)} \\
&= E_0^{(1)}\Psi_n^{(1)}.
\end{aligned} \tag{2.10}$$

Similarly, one can introduce the other supersymmetric Hamiltonian by iterating

process leading to

$$\begin{aligned}\hat{H}_2\Psi_n^{(2)} &= \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+V_2(x)\right]\Psi_n^{(2)} \\ &= \hat{\Omega}\hat{\Omega}^\dagger\Psi_n^{(2)} \\ &= E_0^{(2)}\Psi_n^{(2)}, \end{aligned} \tag{2.11}$$

Especially, assuming that $\Psi_n^{(1)}$ is the eigenfunction of $\hat{H}_1$ with the eigenvalue $E_n^{(1)}$, then $\hat{\Omega}\Psi_n^{(1)}$ is an eigenfunction of $\hat{H}_R$ the same eigenvalue. Therefore, the Eqs. (2.15) and (2.16) yield to

$$\begin{aligned}\hat{H}_2\left[\hat{\Omega}\Psi_n^{(1)}\right] &= \hat{\Omega}\hat{\Omega}^\dagger(\hat{\Omega}\Psi_n^{(1)}) \\ &= \hat{\Omega}(\hat{H}_1\Psi_n^{(1)}) \\ &= E_n^{(1)}(\hat{\Omega}\Psi_n^{(1)}). \end{aligned} \tag{2.12}$$

Similarly, if $\Psi_n^{(2)}$ is an eigenfunction of $\hat{H}_{(2)}$ with the eigenvalue $E_n^{(2)}$, then $\hat{\Omega}^\dagger\Psi_n^{(2)}$ is also an eigenfunction of $\hat{H}_{(1)}$ with the same eigenvalue:

$$\begin{aligned}\hat{H}_1\left[\hat{\Omega}^\dagger\Psi_n^{(2)}\right] &= \hat{\Omega}^\dagger\hat{\Omega}(\hat{\Omega}^\dagger\Psi_n^{(2)}) \\ &= \hat{\Omega}^\dagger(\hat{H}_2\Psi_n^{(2)}) \\ &= E_n^{(2)}(\hat{\Omega}^\dagger\Psi_n^{(2)}). \end{aligned} \tag{2.13}$$

Their normalized eigenfunctions are related by the following transformations:

$$\Psi_n^{(2)} = \frac{\hat{\Omega}\Psi_{n+1}^{(1)}(x)}{\sqrt{E_n^{(2)}}}, \tag{2.14}$$

and

$$\Psi_{n+1}^{(1)} = \frac{\hat{\Omega}^{\dagger}\Psi_{n}^{(2)}(x)}{\sqrt{E_{n+1}^{(1)}}}. \tag{2.15}$$

The normalized eigenfunctions of $\hat{H}_1$ and $\hat{H}_2$ are also connected by the relation as

$$\Psi_{n}^{(2)} = \frac{\hat{\Omega}\Psi_{n+1}^{(1)}(x)}{\sqrt{E_{n+1}^{(2)} - E_{0}^{(2)}}}. \tag{2.16}$$

For unbroken SUSY case in which $E_0^{(1)} = 0$ and $\hat{\Omega}\Psi_0^{(1)} = 0$, the eigenstate of $\hat{H}_2$ corresponding to $\Psi_0^{(1)}$ is not available and the relation

$$E_{n+1}^{(1)} = E_{n}^{(2)}, \tag{2.17}$$

holds for $n = 0, 1, 2 \ldots$. The last successive equations (2.19)-(2.21) imply that if the wave function $\Psi_{n+1}^{(1)}$ of $\hat{H}_1$ is normalizable, then $\Psi_n^{(2)}$ of $\hat{H}_2$ is also normalizable. Furthermore, when the operators $\hat{\Omega}$ and $\hat{\Omega}^{\dagger}$ are operated on the eigenfunctions $\hat{H}_1$ and $\hat{H}_2$ respectively, they convert the eigenfunctions of $\hat{H}_2$ to that of $\hat{H}_1$ or vice versa with the same energy spectra.

The underlying reason for the degeneracy of the spectra of $\hat{H}_1$ and $\hat{H}_2$ can be understood most easily from the properties of the SUSY algebra. That is, we can consider a matrix SUSY Hamiltonian of the form,

$$H = \begin{bmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{bmatrix} \tag{2.18}$$

which contains both $\hat{H}_1$ and $\hat{H}_2$. This matrix Hamiltonian is part of a closed algebra which contains both bosonic and fermionic operators with commutation

and anti-commutation relations. We consider the operators,

$$Q = \begin{bmatrix} 0 & 0 \\ \hat{\Omega} & 0 \end{bmatrix} \tag{2.19}$$

$$Q^{\dagger} = \begin{bmatrix} 0 & \hat{\Omega}^{\dagger} \\ 0 & 0 \end{bmatrix}, \tag{2.20}$$

in conjunction with H. The following commutation and anti-commutation relations then describe the closed superalgebra as,

$$[H, Q] = \left[H, Q^{\dagger}\right] = 0, \qquad \left\{Q, Q^{\dagger}\right\} = H, \qquad \{Q, Q\} = \left\{Q^{\dagger}, Q^{\dagger}\right\} = 0 \tag{2.21}$$

The fact that the supercharges Q and $Q^{\dagger}$ commute with H is responsible for the degeneracy. The operators Q and $Q^{\dagger}$ can be interpreted as operators which change bosonic degrees of freedom into fermionic ones and vice versa. This will be elaborated further below using the example of the SUSY harmonic oscillator. Let us look at a well-known potential; namely, the infinite square well and determine its SUSY partner potential. Consider a particle of mass m in an infinite square well potential of width L as,

$$\begin{aligned} V(x) &= 0 \quad 0 \leq x \leq L \\ &= \infty \quad otherwise \end{aligned} \tag{2.22}$$

The ground state wave function is known to be,

$$\Psi_0^{(1)} = (2/L)^{1/2} \sin(\pi x/L), \qquad 0 \leq x \prec L \tag{2.23}$$

and the ground state energy is $E_0 = \frac{\hbar^2\pi^2}{2mL^2}$.

Subtracting off the ground state energy so that we can factorize the Hamiltonian as $H_1 = H - E_0$ we get the energy eigenvalues as,

$$E_n^{(1)} = \frac{n(n+2)}{2mL^2}\hbar^2\pi^2, \tag{2.24}$$

and the eigenfunctions are,

$$\Psi_n^{(1)} = (2/L)^{1/2} \sin\frac{(n+1)\pi x}{L}, \qquad 0 \leq x \leq L \tag{2.25}$$

The superpotential for this problem can be obtained through the Eq.(2.6) as,

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\pi}{L}\cot(\pi x/L) \tag{2.26}$$

and hence the supersymmetric partner potential V_2 is,

$$V_2(x) = \frac{\hbar^2\pi^2}{2mL^2}\left[2\cos ec^2(\pi x/L) - 1\right] \tag{2.27}$$

The wave function for H_2 are obtained by applying the operator A to the wave functions of H_1. In particular we find that,

$$\Psi_0^{(2)} \propto \sin^2(\pi x/L), \qquad \Psi_1^{(2)} \propto \sin(\pi x/L)\sin(2\pi x/L). \tag{2.28}$$

Thus, using SUSY it is shown that two rather different potentials corresponding to H_1 and H_2 have exactly the same spectra for the fact that H_2 has one fewer bound state.

2.2 SUSYQM and Reflection and Transmission Coefficients

SUSYQM also allows one to relate the reflection and transmission coefficients in situations where the two partner potentials have continuum spectra. In order for scattering to take place in both of the partner potentials, it is necessary that the potentials $V_{1,2}$ are finite as $x \to -\infty$ or as $x \to +\infty$ or both. Let us define:

$$W(x \to \pm\infty) = W_{\pm}. \tag{2.29}$$

Then,

$$V_{1,2} \to W_{\pm}^2 \; x \to \pm\infty \tag{2.30}$$

Let us consider an incident plane wave e^{ikx} of energy E coming from the direction $x \to -\infty$. As a result of scattering from the potentials $V_{1,2}(x)$ one would obtain transmitted waves $T_{1,2}(k)e^{ik'x}$ and reflected waves $R_{1,2}\,(k)\,e^{-ikx}$. Thus we have,

$$\Psi^{(1,2)}(k, x \to -\infty) \to e^{ikx} + R_{1,2}e^{-ikx}, \qquad \Psi^{(1,2)}\left(k', x \to +\infty\right) \to T_{1,2}e^{ik'x} \tag{2.31}$$

SUSY connects continuum wave functions of H_1 and H_2 having the same energy analogously to what happens in the discrete spectrum. Thus we have the relationships:

$$e^{ikx} + R_1 e^{-ikx} = N\left[(-ik + W_-)\,e^{ikx} + (ik + W_-)\,e^{-ikx}R_2\right], \tag{2.32}$$

and,

$$T_1 e^{ik\acute{}x} = N\left[\left(-ik\acute{} + W_+\right) e^{ik\acute{}x} T_2\right], \tag{2.33}$$

where N is an overall normalization constant. On equating terms with the same exponent and eliminating N, we find,

$$R_1(k) = \left(\frac{W_- + ik}{W_- - ik}\right) R_2(k), \qquad T_1(k) = \left(\frac{W_+ - ik\acute{}}{W_- - ik}\right) T_2(k), \tag{2.34}$$

where k and $k\acute{}$ are given by,

$$k = \left(E - W_-^2\right)^{1/2}, \qquad k\acute{} = \left(E - W_+^2\right)^{1/2}. \tag{2.35}$$

A few remarks are now in order at this stage:

(1) Clearly $|R_1|^2 = |R_2|^2$ and $|T_1|^2 = |T_2|^2$, that is the partner potentials have identical reflection and transmission probabilities.

(2) $R_1(T_1)$ and $R_2(T_2)$ have the same poles in the complex plane except that $R_1(1)$ has an extra pole at $k = -iW_-$. This pole is on the positive imaginary axis only if $W_- < 0$ in which case it corresponds to a zero energy bound state.

(3) In the special case that $W_+ = W_-$, we have that $T_1(k) = T_2(k)$.

(4) When $W_- = 0$ then $R_1(k) = -R_2(k)$.

It is clear from these remarks that if one of the partner potentials is a constant potential (i.e. a free particle), then the order partner will be of necessity reflectionless. In this way, we can understand the reflectionless potential of the form $V(x) = A\,\mathrm{sec}\,h^2(\alpha x)$ which play a critical role in understanding the solutions of

the hierarchy. Let us consider the superpotential,

$$W(x) = A \tanh \alpha x. \tag{2.36}$$

The two partner potentials are,

$$V_1 = A^2 - A\left(A + \alpha\frac{\hbar}{\sqrt{2m}}\right) \sec h^2 \alpha x, \tag{2.37}$$

and,

$$V_2 = A^2 - A\left(A - \alpha\frac{\hbar}{\sqrt{2m}}\right) \sec h^2 \alpha x, \tag{2.38}$$

It can be seen that for $A = \alpha\hbar/\sqrt{2m}$, $V_2(x)$ corresponds to a constant potential so that the corresponding V_1 is a reflectionless potential. It is worth noting that V_1 is $\hbar$-dependent.

2.3 **Broken SUSYQM**

For SUSY to be a good symmetry, the operators Q and $Q^\dagger$ must annihilate the vacuum. Thus, the ground state energy of the super-Hamiltonian must be zero since,

$$H = \{Q^\dagger, Q\}. \tag{2.39}$$

Witten [38] proposed an index to determine whether SUSY is broken in supersymmetric field theories. The index is defined by,

$$\triangle = Tr(-1)^F, \tag{2.40}$$

where the trace is over all the bound states and continuum states of the super-Hamiltonian. The fermion number $n_F \equiv F$ is defined by $\frac{1}{2}[1-\sigma_3]$ and we can represent $(-1)^F$ by the matrix σ_3. If we write the eigenstates of H as the vector:

$$\psi_n(x) = \begin{bmatrix} \psi_n^{(+)}(x) \\ \\ \psi_n^{-}(x) \end{bmatrix}. \tag{2.41}$$

Then, the $\pm$ corresponds to the eigenvalues of $(-1)^F$ being ± 1 . For our convention the eigenvalue $+1$ corresponds to H_1 and the bosonic sector and the eigenvalue -1 corresponds to H_2 and the fermionic sector. Since the bound states of H_1 and H_2 are paired, except for the case of unbroken SUSY where there is an extra state in the bosonic sector with $E = 0$, we expect for the quantum mechanic situation that $\triangle = 0$ for broken SUSY and $\triangle = 1$ for unbroken SUSY. Witten gives arguments that in general the index measures $N_+(E=0) - N_-(E=0)$ which is the difference $\triangle$ between the number of Bose states and Fermi states of zero energy. The Witten index needs to be regulated to be well defined so that one considers instead,

$$\triangle(\beta) = Tr(-1)^F e^{-\beta H}, \tag{2.42}$$

which for SUSY quantum mechanics becomes,

$$\triangle(\beta) = Tr[e^{-\beta H_1} - e^{-\beta H_2}]. \tag{2.43}$$

It is quite hard to determine if SUSY is broken non-perturbatively, thus SUSYQM became a testing ground for finding different methods to understand non-perturbative

SUSY breaking. In the case of quantum mechanics, the breakdown of SUSY is related to the question of whether there is a normalizable wave function solution to the equation $Q|0\rangle = 0|0\rangle$ which implies,

$$\Psi_0(x) = Ne^{-\int W(x)dx}. \tag{2.44}$$

If this candidate ground state wave function does not fall off fast enough at $\pm\infty$ then Q does not annihilate the vacuum and SUSY is spontaneously broken. Let us, using a trivial calculation, show that for two simple polynomial potentials the Witten index does indeed provide the correct answer to the question of SUSY breaking. Let us consider,

$$\Delta(\beta) = Tr\sigma_3 \int \left[\frac{dpdx}{2\pi}\right] e^{-\beta\left[p^2/2+W^2/2-\sigma_3 W^{'}(x)/2\right]}. \tag{2.45}$$

Expanding the term proportional to σ_3 in the exponent and taking the trace, we obtain,

$$\Delta(\beta) = \int \left[\frac{dpdx}{\pi}\right] e^{-\beta\left[p^2/2+W^2/2\right]} \sinh(\beta W^{'}(x)/2). \tag{2.46}$$

We are interested in the regulated index as β tends to 0, so that practically we need to evaluate,

$$\Delta(\beta) = \int \left[\frac{dpdx}{2\pi}\right] e^{-\beta\left[p^2/2+W^2/2\right]} (\beta W^{'}(x)/2). \tag{2.47}$$

If we directly evaluate this integral for any potential of the form $W(x) = gx^{2n+1}$, which leads to a normalizable ground state wave function, then are the integrals are gamma functions and we explicitly obtain $\triangle = 1$. If instead $W(x) = gx^{2n}$ so that the candidate ground state wave function is not normalizable, then the integrand becomes an odd function of x and therefore vanishes. Thus, it can be seen from these simple cases that the Witten index immediately coincides with the direct method available in the quantum mechanic case.

2.4 SUSYQM and PT-Symmetry

In quantum mechanics PT-symmetry requires the invariance of a potential under the simultaneous action of the spatial, P, and time reflection, T, operations. For one-dimensional potentials of non-relativistic quantum mechanics this invariance requires $V^*(-x) = V(x)$, therefore the real and imaginary components of a PT-invariant potential has to be an even and odd function of x, respectively. These potentials represent a rather peculiar class among complex potentials, since it was found that their discrete energy eigenvalues are real [39]. This is different from complex potentials appearing, e.g., in nuclear physics imitating the absorption of particles in a nuclear reaction, since the discrete energy eigenvalues were found to be complex in that case.

It soon turned out that PT is neither a necessary nor a sufficient condition for

having real energy eigenvalues in a complex potential. A typical feature appearing in most PT-symmetric potentials was the pairwise merging of real energy eigenvalues and their re-emergence as complex conjugate pairs as some potential parameter was tuned. At the same time the corresponding solutions ceased to be eigenfunctions of the PT operator, so this phenomenon was interpreted as the spontaneous breakdown of PT-symmetry [39]. Besides the real energy eigenvalues, there were further signs indicating that PT-symmetric potentials share some features with Hermitian problems. For example, with the modification of the inner product the orthogonality of the energy eigenstates could be restored, and also a modified continuity equation could be derived. However, the pseudo-norm defined this way had an indefinite sign, questioning the probabilistic interpretation of the wave functions. These unusual results have finally been interpreted in terms of pseudo-Hermitian Hamiltonians and anti-linear operators [40], as PT-symmetry represents a special case of these.

The first example of PT-symmetric potentials have been derived in numerical studies, but soon the PT-symmetric versions of solvable potentials have also been constructed. In a systematic study, the PT-symmetric versions of solvable potentials have been constructed and conditions have been formulated for having real [41] or complex [42] energy eigenvalues in their spectra, i.e. for having unbroken or spontaneously broken PT symmetry.

SUSYQM has been combined with PT-symmetry in various other ways. Fundamental mathematical aspects have been discussed in [40,43]; and the reality of the spectrum of PT-symmetric SUSY partners of real potentials has been pointed out in [44].

Here, our aim is to analyze how the complex structure of the $W(x)$ superpotential influences the PT-symmetry of the SUSY partner potentials $V_1(x)$ and $V_2(x)$. For this, we separate the superpotential into real and imaginary components and then split both of them into even and odd functions of x as:

$$W(x) = W_R(x) + iW_I(x) = W_{Re}(x) + W_{Ro}(x) + iW_{Ie}(x) + iW_{Io}(x). \qquad (2.48)$$

Separating the $V_1(x)$ potential in a similar fashion, one gets,

$$V_1(x) = V_{1Re}(x) + V_{1Ro}(x) + iV_{1Ie}(x) + iV_{1Io}(x). \qquad (2.49)$$

with,

$$\begin{aligned}
V_{1Re}(x) &= W_{Re}^2(x) + W_{Ro}^2(x) - W_{Ie}^2(x) - W_{Io}^2(x) - W'_{Ro}(x) + Re(\varepsilon), \\
V_{1Ro}(x) &= 2W_{Re}(x)W_{Ro}(x) - 2W_{Ie}(x)W_{Io}(x) - W'_{Re}(x), \\
V_{1Ie}(x) &= 2W_{Re}(x)W_{Ie}(x) + 2W_{Ro}(x)W_{Io}(x) - W'_{Io}(x) + Im(\varepsilon), \\
V_{1Io}(x) &= 2W_{Re}(x)W_{Io}(x) + 2W_{Ro}(x)W_{Ie}(x) - W'_{Ie}(x). \qquad (2.50)
\end{aligned}$$

The PT-symmetry requirement $V^*(-x) = V(x)$ means for $V_1(x)$ that $Re[V_1(x)]$ has to be an even function of x, while $Im[V_1(x)]$ has to be odd, and this leads to

two coupled first-order differential equations:

$$\begin{aligned} W'_{Re}(x) - 2W_{Ro}(x)W_{Re}(x) + 2W_{Ie}(x)W_{Io}(x) &= 0, \\ W'_{Io}(x) - 2W_{Re}(x)W_{Ie}(x) - 2W_{Ro}(x)W_{Io}(x) &= Im(\varepsilon). \end{aligned} \tag{2.51}$$

This can be considered as a system of inhomogeneous (or nonhomogeneous) linear first-order differential equations for the functions $W_{Re}(x)$ and $W_{Io}(x)$, where the inhomogeneity is represented by a constant in only one of the equations; furthermore, the coefficients appearing in the two equations are expressed in terms of the same two functions, $W_{Ro}(x)$ and $W_{Ie}(x)$. Before discussing the general problem of solving (2.51), we first derive formulae analogous to (2.50) for the SUSY partner $V_2(x)$:

$$\begin{aligned} V_{2Re}(x) &= W^2_{Re}(x) + W^2_{Ro}(x) - W^2_{Ie}(x) - W^2_{Io}(x) + W'_{Ro}(x) + Re(\varepsilon), \\ V_{2Ro}(x) &= 4W_{Re}(x)W_{Ro}(x) - 4W_{Ie}(x)W_{Io}(x), \\ V_{2Ie}(x) &= 4W_{Re}(x)W_{Ie}(x) + 4W_{Ro}(x)W_{Io}(x) + 2Im(\varepsilon), \\ V_{2Io}(x) &= 2W_{Re}(x)W_{Io}(x) + 2W_{Ro}(x)W_{Ie}(x) + W'_{Ie}(x). \end{aligned} \tag{2.52}$$

Here, we eliminated the derivatives from $V_{2Ro}(x)$ and $V_{2Ie}(x)$ using (2.51). Let us now discuss the solution of (2.51) for increasingly complex situations and its implication on the PT symmetry of $V_2(x)$.

(a) $W_{Re}(x) = W_{Io}(x) = 0$. This case represents a trivial solution. It necessarily leads to $Im(\varepsilon) = 0$, reducing (2.51) to a homogeneous system of differential

equations. It is straightforward to prove that in this case $V_2(x)$ is also PT-symmetric.

(b) $W_{Io}(x) = 0$, and $W_{Re}(x) \neq 0$. In this case the integration of (2.51) is a straightforward task, leading to,

$$W_{Re}(x) = C \exp\left(2\int_0^x W_{Ro}(x')dx'\right),$$

$$W_{Ie}(x) = -\frac{Im(\varepsilon)}{2} W_{Re}^{-1}(x), \tag{2.53}$$

where we have already taken the parity requirement for $W_{Re}(x)$ into account through the boundary conditions. It can be proved that in this case the imaginary condition of $V_2(x)$ is odd; however, its real component does not have definite parity, therefore the SUSY partner is not PT-symmetric.

(c) $W_{Re}(x) = 0, W_{Io}(x) \neq 0$. Here, the solution of (2.51) becomes,

$$W_{Io}(x) = Im(\varepsilon) \exp\left(2\int_0^x W_{Ro}(x')dx'\right) \int_0^x \left[\exp\left(-2\int_0^{x'} W_{Ro}(x'')dx''\right)\right] dx', \tag{2.54}$$

and,

$$W_{Ie}(x) = 0, \tag{2.55}$$

where we have again considered appropriate boundary conditions rendering $W_{Io}(x)$ to an odd function of x. In this case, $V_1(x)$ is an even real function (a special case of PT-symmetry), while its SUSY partner,$V_2(x)$ is an even imaginary function, so it can not be PT-symmetric.

(d) *The general case.* The general solution of (2.51) can be given in the following form [45],

$$W_{Re}(x) = \exp\left(2\int_0^x W_{Ro}(x')dx'\right)\cos\left(-2\int_0^x W_{Ie}(x')dx'\right)$$

$$+ \; Im(\varepsilon)\int_0^x \exp\left(2\int_s^x W_{Ro}(x')dx'\right)\sin\left(-2\int_s^x W_{Ie}(x')dx'\right)ds. \quad (2.56)$$

$$W_{Io}(x) = -\exp\left(2\int_0^x W_{Ro}(x')dx'\right)\sin\left(-2\int_0^x W_{Ie}(x')dx'\right)$$

$$+ \; Im(\varepsilon)\int_0^x \exp\left(2\int_s^x W_{Ro}(x')dx'\right)\cos\left(-2\int_s^x W_{Ie}(x')dx'\right)ds. \quad (2.57)$$

Now, $W_{Re}(x)$ is indeed an even function of x, while $W_{Io}(x)$ is odd. The SUSY partner $V_2(x)$ does not have PT-symmetry in this case in general.

Taking a real factorization energy ε which turns (2.51) into a homogeneous system reduces the complexity of the solution in all faces. In case (d) it also appears in $W_{Re}(x)$, and when it is zero, $W_{Re}(x)$ and $W_{Io}(x)$ become proportional to each other:

$$W_{Re}(x) = t(x)W_{Io}(x) \quad (2.58)$$

where,

$$t(x) = cot\left(2\int^x W_{Ie}(x')dx'\right). \quad (2.59)$$

This $t(x)$ function can also be obtained in a different way, when one derives a particular solution of the homogeneous version of (2.51) by substituting (2.58) into it.

For the most well-known PT-symmetric potentials, unbroken PT-symmetry occurs for the trivial $W_{Re}(x) = W_{Io}(x) = 0$ solutions of (2.51), which reduce to a homogeneous system of differential equations. The formulae presented above allow more general construction of $W(x)$ superpotentials leading to a PT-symmetric $V_1(x)$ potential, and a generally non-PT-symmetric SUSY partner.

The general construction of solving equations (2.51) works for analytically and numerically solvable potentials too. So, it is possible to generate potentials which are PT-symmetric by construction by selecting some $W_{Ro}(x)$ and $W_{Ie}(x)$ functions and deriving from them the full superpotential. It would then be an interesting task to investigate situations in which the PT-symmetry of the $V_1(x)$ potential is intact or is spontaneously broken, and also to study how this affects the PT-symmetry of the SUSY partners. This would be an interesting task even if the solutions cannot be determined in an explicit form (apart from the ground state of $V_1(x)$, which is determined by the $W(x)$ superpotential).

2.5 SUSYQM, Shape Invariance and Solvable Potentials

In quantum mechanics, the one dimensional harmonic oscillator problem can be elegantly solved using the raising and lowering operator method. Using the ideas of SUSYQM and an integrability condition called the shape invariance condition as,

$$\Psi_n^{(2)} = [E_n^{(1)}]^{-1/2} \hat{\Omega} \Psi_n^{(1)}, \tag{2.60}$$

we now show that the operator method for the harmonic oscillator can be generalized to the whole class of shape invariant potentials (SIP) which include all the popular, analytically solvable potentials. For such potentials, the generalized operator method quickly yields all the bound state energy eigenvalues, eigenfunctions as well as the scattering matrix. It turns out that this approach is essentially equivalent to Schrödinger's method of factorization, although the language of SUSY is more appealing.

Let us explain precisely what one means by shape invariance. If the pair of SUSY partner potentials $V_1(x)$ and $V_2(x)$ are similar in shape and differ only in the parameters that appear in them, then they are said to be shape invariant. More precisely, if the partner potentials $V_1(x; a_1)$ and $V_2(x; a_1)$ satisfy the condition,

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1), \tag{2.61}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$ and the remainder $R(a_1)$ is independent of x, then $V_1(x; a_1)$ and $V_2(x; a_1)$ are said to be shape invariant. The shape invariance condition(2.68) is an integrability condition. Using this condition and the hierarchy of Hamiltonians, one can easily obtain the energy eigenvalues and eigenfunctions of any SIP when SUSY is unbroken.

One constructs a hierarchy of Hamiltonians as,

$$H_n = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x; a_n) + \sum_{s=2}^{n} R(a_s), \tag{2.62}$$

where $a_s = f^s(a_1)$, i.e., the function f applied s times. In view of Eqs. (2.68) and (2.69), we have

$$H_{n+1} = -\frac{1}{2}\frac{d^2}{dx^2} + V_1(x; a_{n+1}) + \sum_{s=2}^{n+1} R(a_s) \tag{2.63}$$

$$= -\frac{1}{2}\frac{d^2}{dx^2} + V_2(x; a_n) + \sum_{s=2}^{n} R(a_s). \tag{2.64}$$

Comparing (2.69), (2.70), and (2.71), we immediately note that H_n and H_{n+1} are SUSY partner Hamiltonians with identical energy spectra except for the ground state level,

$$E_n^{(0)} = \sum_{s=2}^{n} R(a_s) \tag{2.65}$$

of H_n, which follows from Eq.(2.69) and the normalization that for any $V_1(x; a)$, $E_1^{(0)}$. Thus, Eq.(2.17) get translate simply, letting $n \to n + 1$, to

$$E_1^n = E_2^{n-1} = ... = E_{n+1}^{(0)} = \sum_{s=2}^{n+1} R(a_s), \qquad n = 1, 2, ... \tag{2.66}$$

and for the wave functions,

$$\Psi_1^{(n)} \propto \hat{\Omega}_1(x; a_1)\hat{\Omega}_2(x; a_2)...\hat{\Omega}_n(x; a_n)\Psi_{n+1}^{(0)}(x; a_{n+1}). \tag{2.67}$$

Equations (2.73) and (2.74) successively express the SUSY algebraic generalization, for various shape-invariant potentials of physical interest of the method of constructing energy eigenfunctions $\Psi_{osc}^{(n)}$ for the usual oscillator problem. When $a_1 = a_2 = ... = a_n = a_{n+1}$, we obtain,

$$\Psi_{osc}^{(n)} \propto (\hat{a})^n \Psi_1^{(0)}, \qquad \hat{\Omega}_n = \hat{a}, \tag{2.68}$$

and

$$\Psi^0_{osc} = \Psi^{(0)}_{n+1} = \Psi^{(0)}_1 \propto \exp(-\frac{\omega x^2}{2}), \tag{2.69}$$

where ω is the angular frequency.

2.6 Shape Invariance in More Than One Step

The list of solvable potentials can be expanded by extending the shape invariance idea to the more general concept of shape invariance in two and even multi-steps. In this way, it is possible to go much beyond the factorization method and obtain a huge class of new solvable potentials.

Consider the unbroken SUSY case of two superpotentials as $W(x; a_1)$ and $\tilde{W}(x; a_1)$ such that $V_2(x; a_1)$ and $V_1(x; a_1)$ are same up to an additive constant, i.e.

$$V_2(x; a_1) = \tilde{V}_1(x; a_1) + R(a_1) \tag{2.70}$$

or equivalently,

$$W^2(x; a_1) + W'(x; a_1) = \tilde{W}^2(x; a_1) - \tilde{W}'(x; a_1) + R(a_1). \tag{2.71}$$

Shape invariance in two steps means that

$$\tilde{V}_2(x; a_1) = V_1(x; a_2) + \tilde{R}(a_1), \tag{2.72}$$

that is,

$$\tilde{W}^2(x; a_1) + \tilde{W}'(x; a_1) = W^2(x; a_2) - W'(x; a_2) + \tilde{R}(a_1). \tag{2.73}$$

We now show that when this condition holds, the energy eigenvalues and eigenfunctions of the potential $V_1(x;a_1)$ can be obtained algebraically. First of all, let us notice that unbroken SUSY implies zero energy ground states for the potentials $V_1(x;a_1)$ and $\tilde{V}_1(x;a_1)$:

$$E_0^{(1)}(a_1) = 0, \qquad \tilde{E}_0^{(1)}(a_1) = 0. \tag{2.74}$$

The degeneracy of the energy levels for the SUSY partner potentials yields,

$$E_n^{(2)}(a_1) = E_{n+1}^{(1)}(a_1), \qquad \tilde{E}_n^{(2)}(a_1) = \tilde{E}_{n+1}^{(1)}(a_1). \tag{2.75}$$

From Eq.(2.77) it follows that,

$$E_n^{(2)}(a_1) = \tilde{E}_n^{(1)}(a_1) + R(a_1), \tag{2.76}$$

so that for $n = 0$, these two equations yield,

$$E_1^{(1)}(a_1) = R(a_1). \tag{2.77}$$

Also, the shape invariance condition (2.79) yields,

$$\tilde{E}_n^{(2)}(a_1) = E_n^{(1)}(a_2) + \tilde{R}(a_1), \tag{2.78}$$

From the above equations, one can then show that,

$$E_{n+1}^{(1)}(a_1) = E_{n-1}^{(1)}(a_2) + R(a_1) + \tilde{R}(a_1). \tag{2.79}$$

On solving these equations recursively, we obtain $(n = 0, 1, 2, ...)$,

$$E_{2n}^{(1)} = \sum_{k=1}^{n} \left[R(a_k) + \tilde{R}(a_k) \right], \tag{2.80}$$

$$E_{2n+1}^{(1)} = \sum_{k=1}^{n} \left[R(a_k) + \tilde{R}(a_k) \right] + R(a_{n+1}). \tag{2.81}$$

We now show that the bound state wavefunctions $\Psi_n^{(1)}(x; a_1)$ can also be easily obtained in terms of the ground state wave functions $\Psi_0^{(1)}(x; a_1)$ and $\tilde{\Psi}_0^{(1)}(x; a_1)$ which in turn are known in terms of the superpotentials W and $\tilde{W}$. In particular from Eq.(2.77), it follows that,

$$\Psi_{n+1}^{(1)}(x; a_1) \propto \hat{\Omega}^{\dagger}(x; a_1)\Psi_n^{(2)}(x; a_1) \propto \hat{\Omega}^{\dagger}(x; a_1)\tilde{\Psi}_n^{(1)}(x; a_1) \tag{2.82}$$

while from Eq.(2.79), we have,

$$\tilde{\Psi}_{n+1}^{(1)}(x; a_1) \propto \tilde{\hat{\Omega}}^{\dagger}(x; a_1)\tilde{\Psi}_n^{(2)}(x; a_1) \propto \tilde{\hat{\Omega}}^{\dagger}(x; a_1)\Psi_n^{(1)}(x; a_2) \tag{2.83}$$

Hence on combining the two equations, we have the identity,

$$\Psi_{n+2}^{(1)}(x; a_1) \propto \hat{\Omega}^{\dagger}(x; a_1)\tilde{\hat{\Omega}}^{\dagger}(x; a_1)\Psi_n^{(1)}(x; a_2). \tag{2.84}$$

Recursive application of the above identity yields,

$$\Psi_{2n}^{(1)}(x; a_1) \propto \left[\hat{\Omega}^{\dagger}(x; a_1)\tilde{\hat{\Omega}}^{\dagger}(x; a_1) \right] ... \left[\hat{\Omega}^{\dagger}(x; a_n)\tilde{\hat{\Omega}}^{\dagger}(x; a_n) \right] \Psi_0^{(1)}(x; a_{n+1}), \tag{2.85}$$

$$\Psi^{(1)}_{2n+1}(x;a_1) \propto \left[\hat{\Omega}^{\dagger}(x;a_1)\tilde{\hat{\Omega}}^{\dagger}(x;a_1)\right] ... \left[\hat{\Omega}^{\dagger}(x;a_n)\tilde{\hat{\Omega}}^{\dagger}(x;a_n)\right] \hat{\Omega}^{\dagger}(x;a_{n+1})\tilde{\Psi}^{(1)}_0(x;a_{n+1}), \tag{2.86}$$

where we have used the fact that,

$$\Psi^{(1)}_1(x;a_1) \propto \hat{\Omega}^{\dagger}(x;a_1)\tilde{\Psi}^{(1)}_0(x;a_1). \tag{2.87}$$

It is clear that this procedure can be easily generalized and one can consider multi-step shape invariant potentials and therefore obtain the eigenfunctions algebraically.

CHAPTER 3

THE HAMILTONIAN HIERARCHY METHOD

The central idea of the factorization method for ordinary differential equations, introduced by Schrödinger [1] and Infeld and Hull [2] to solve problems in quantum mechanics is the recognition that once the ground state energy and wave function of a one-dimensional potential problem are known, then the potential is determined as well as the factors of the Hamiltonian. Recently, Gozzi, Reuter, and Thacker [3] proposed a simple technique for solving such one-dimensional potential problems, based on utilizing the hierarchy of Hamiltonians that are related by supersymmetry (SUSY).

The radial Schrödinger equation for some specific potentials can be solved analytically only for the states with zero angular momentum [4, 5]. However, in supersymmetric quantum mechanics one can deal with the hierarchy problem by using effective potentials for non-zero angular momentum states in order to solve the Schrödinger equation analytically. Hamiltonian hierarchy method suggests a hierarchy problem in the frame of the SUSYQM in which the adjacent members are the supersymmetric partners that share the same eigenvalue spectrum except

for the missing ground state. We apply this method to solve the Schrödinger equation with PT-symmetric and non-PT symmetric non-Hermition forms for some exactly solvable potentials [52]. Here, we briefly review the formalism for describing the hierarchy of Hamiltonians.

According to the supersymmetric algebra, Hamiltonian can be written as [6] ,

$$H_{\pm} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{\pm}(x), \tag{3.1}$$

where, the supersymmetric partner potentials $V_{\pm}(x)$ in terms of the superpotential $W\left(x\right)$ are given by,

$$V_{\pm}(x) = W^2 \pm \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}. \tag{3.2}$$

The superpotential is defined as,

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\left[\frac{d}{dx}\ln\Psi_0^{(0)}(x)\right], \tag{3.3}$$

where, $\Psi_0^{(0)}(x)$ denotes the ground state wave function that satisfies the relation,

$$\Psi_0^{(0)}(x) = N\exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(x^{'})dx^{'}\right). \tag{3.4}$$

The Hamiltonian $H_{\pm}$ can be written in terms of the bosonic operaters A^{-} and A^{+} as,

$$H_{\pm} = A^{\mp}A^{\pm}, \tag{3.5}$$

where,

$$A^{\pm} = \pm \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \tag{3.6}$$

The Hamiltonians $H_{\pm}$ are said to be "supersymmetric" partners of each other due to the fact that the energy eigenvalues of H_{+} and H_{-} are identical except for the ground state. In the case of unbroken supersymmetry, the ground state energy of the Hamiltonian H_{-} is zero ($E_0^{(0)} = 0$) [6] . The Hamiltonian of the Schrödinger equation can always be factorized in the form of the equation (3.5) as,

$$H_1(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) = (A_1^{+} A_1^{-}) + E_1^{(0)}, \tag{3.7}$$

such that,

$$H_1 \Psi_{(1)}^n(x) = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) \right] \Psi_{(1)}^n(x) = E_{(1)}^n \Psi_{(1)}^n(x), \tag{3.8}$$

where, $V_1(x)$ is chosen such that the ground state $\Psi_{(1)}^0(x)$ has energy equal to zero. The Riccati equation for the the superpotential $W_1(x)$ can be obtained via the Eq. (3.7) as,

$$W_1^2 - \frac{dW_1}{dx} = \frac{2m}{\hbar^2} \left[V_1(x) - E_1^{(0)} \right], \tag{3.9}$$

and as a result,

$$V_1(x) = \frac{\hbar^2}{2m} \left[W_1^2 - \frac{dW_1}{dx} \right]. \tag{3.10}$$

As long as Eq. (3.9) has a solution; namely W_1, the one-dimentional Schrödinger equation can be made supersymmetric by the construction given in Eqs. (3.5) and (3.6). The challenge then in using the SUSYQM techniques is not in the mechanics the construction of just any SUSY Hamiltonian, but in finding a suitable superpotential (or $V_1(x)$) to construct a SUSY Hamiltonian which will be relevant to the problem at hand. It is a common practice to choose or pose as an ansatz, W_1, to solve a physical problem [7, 8].

The SUSY partner of H_1; namely H_2 is then given by,

$$H_2(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x) = (A_2^- A_2^+) + E_2^{(0)}. \tag{3.11}$$

Therefore, the Riccati equation will be,

$$W_2^2 + \frac{dW_2}{dx} = \frac{2m}{\hbar^2}\left[V_2(x) - E_2^{(0)}\right], \tag{3.12}$$

and consequently,

$$V_2(x) = \frac{\hbar^2}{2m}\left[W_1^2 + \frac{dW_1}{dx}\right]. \tag{3.13}$$

Note that H_2 is altogether a new Hamiltonian. One can repeat the procedure in constructing H_2 from $H_1$to construct an H_3 from H_2 such that,

$$H_2 = A_2^+ A_2^-, \tag{3.14}$$

with

$$A_2^+ = \frac{\hbar}{\sqrt{2m}}\left[\frac{d}{dx} + W_2(x)\right], \tag{3.15}$$

$$A_2^- = -\frac{\hbar}{\sqrt{2m}}\left[\frac{d}{dx} + W_2(x)\right], \tag{3.16}$$

and with a new Riccati equation as,

$$V_2(x) = \frac{\hbar^2}{2m}\left[W_2^2 - \frac{dW_2}{dx}\right]. \tag{3.17}$$

W_2 in the Eq.(3.17) is then solved to construct $A_2^{\pm}$. H_3 can then be constructed as,

$$H_3(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_3(x) = (A_2^- A_2^+) + E_3^{(0)}. \tag{3.18}$$

We can evidently construct a "hierarchy" of SUSY-partner Hamiltonians as H_1, H_2, $H_3, ..., H_n$ starting from H_1.

In conclusion, the Riccati equation and Hamiltonian by iteration will be,

$$W_n^2 \pm \frac{dW_n}{dx} = \frac{2m}{\hbar^2}\left[V_n(x) - E_n^{(0)}\right] = \left(A_n^{\pm} A_n^{\mp}\right) + E_n^{(0)} \tag{3.19}$$

and

$$H_n(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_n(x) = (A_n^- A_n^+) + E_n^{(0)}, \qquad n = 1, 2, 3, ..., \tag{3.20}$$

where,

$$A_n^{\pm} = \pm\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \frac{d}{dx}\left[\ln \Psi_n^{(0)}(x)\right]. \tag{3.21}$$

Because of the SUSY unbroken case, the partner Hamiltonians satisfy the following expressions [6] ,

$$E_{n+1}^{(0)} = E_n^{(1)}, \qquad with \qquad E_0^{(0)} = 0, \qquad n = 0, 1, 2, 3, ... \tag{3.22}$$

and also the wave function with the same eigenvalue can be written as [6] ,

$$\Psi_n^{(1)} = \frac{A^- \Psi_{n+1}^{(0)}}{\sqrt{E_n^{(0)}}}. \tag{3.23}$$

with

$$\Psi_{n+1}^{(0)} = \frac{A^+ \Psi_n^{(1)}}{\sqrt{E_n^{(0)}}}. \tag{3.24}$$

Herein lies a very important consequence of SUSYQM. The energy eigenfunctions of the hierarchy of Hamiltonians are related by the SUSY operators $A^{\pm}$. Thus, if one knows the eigenvalues and eigenfunctions of a particular H_n, then one can get the eigenvalues and eigenfunctions of its SUSY partner.

The interesting question is how accurate is such a scheme? All the Hamiltonians of the variational hierarchy are not the real Hamiltonians of the hierarchy but approximate ones related to the variational approximation of the ground state wave function of the original problem. Thus, we do not know that whether we are accurately calculating the energy eigenvalues of the original Hamiltonian or not. We only know that we are accurately determining the energies of the approximate Hamiltonians of the hierarchy. Thus, for this scheme to work, one

needs an extremely accurate method of finding the ground state wave function and energy of H_1. Also, in this scheme the excited wave functions of H_1 are obtained by repeated derivatives of the ground state wave functions of the approximate Hamiltonians. Thus, we expect a loss in accuracy as we go to the excited wave functions.

CHAPTER 4

CALCULATIONS

In this chapter, in the first case, the exact supersymmetric solutions of the Schrödinger equation with the deformed Morse, Hulthén, Pöschl-Teller and Kratzer-like potentials are obtained. The Hamiltonian hierarchy method is used to get the real energy eigenvalues and corresponding eigenfunctions. Numerical results are computed for the different values of the deforming parameter q. In the second case, the supersymmetric solutions of PT-symmetric and Hermitian/non-Hermitian forms of quantum systems are obtained by solving the Schrödinger equation with the deformed Morse and Pöschl-Teller potentials. In the third case, the eigenvalues and eigenfunctions of the screened and exponential-cosine screened Coulomb potentials are obtained via the Hamiltonian hierarchy method.

4.1 Some q-deformed Potentials

4.1.1 The Morse Potential

The Morse potential is a convenient model for a diatomic molecule such as HCl. It is used for developing a firm understanding about the relationship of

each parameter to molecular properties such as bond length and dissociation energy. For example, in the harmonic oscillator as a representation of molecular vibration, one notices that a diatomic molecule which was actually bound using a harmonic potential would never dissociate. The Morse potential realistically leads to dissociation, making it more useful than the harmonic potential. Consequently, the Morse potential is a model just as the harmonic oscillator is a model for a chemical bond.

This potential, because of its exponential component follows the shape of experimentally determined potential functions over a wider range of internuclear distances than does the harmonic oscillator [46]. The Morse potential also accounts for the fact that the energy levels get closer together as the potential energy increases for a quantum mechanical oscillator. Because of its exponential component, the Morse potential as a model has a range of applicability for real systems. In the case of diatomic molecules, the Morse potential starts to fail when the bond length is stretched beyond 0.5 to 1.0 Angstrom. For iodine, this corresponds to about a 20% increase in bond length. An excellent figure comparing the experimentally determined potential and Morse potential for iodine can be found in the paper by Verma (1960) [47].

In conclusion, the Morse potential is the simplest representative of the potential between two nuclei in which dissociation is possible. This potential is much used in spectroscopic applications as it is possible to solve the Scrödinger equation for

this system [48]. It is also used in several branches of chemical physics including collision theory, theory of intermolecular energy transfer, theory of photodissociation, etc. [49-50].

For the diatomic system, the three dimensional deformed Morse oscillation can be written as,

$$V_M = D\ \left(e^{-2a(r-r_e)} - 2qe^{-a(r-r_e)}\right), \tag{4.1}$$

where D is the dissociation energy, r_e is the equilibrium internuclear distance, a is the range parameter and q is the deformation factor. The first step to be taken is to look for an effective potential similar to the original deformed Morse potential. Inspired by the SUSYQM and [1], we propose an ansatz for the superpotential as,

$$W_{(l+1)}(r) = -\frac{\sqrt{2mD}}{a\hbar}\ e^{-a(r-r_e)} + (\frac{q\sqrt{2mD}}{a\hbar} - \frac{2l+1}{2}). \tag{4.2}$$

This equation can be rewritten as,

$$W_{(l+1)}(r) = -\lambda e^{-a(r-r_e)} + (\lambda q - \frac{2l+1}{2}), \tag{4.3}$$

where $\lambda^2 = \frac{2mD}{a^2\hbar^2}$ and $(2l+1)$ denotes the partner number with $l = 0, 1, 2, ...$, and the parameter m is the reduced mass of the molecule. This superpotential choice leads to the $(l+1)$th member of the Hamiltonian hierarchy through,

$$V_{(l+1)}(r) - E^{n=0}_{(l+1)} = W^2_{(l+1)}(r) - \frac{1}{a}\frac{d}{dr}W_{(l+1)}(r), \tag{4.4}$$

which yields,

$$V_{(l+1)}(r) = \lambda^2(e^{-2(r-r_e)} - 2qe^{-a(r-r_e)}) + 2l\lambda e^{-a(r-r_e)}. \tag{4.5}$$

Considering the shape invariance requirement, an expression for the bound-state energies of the above potential is obtained as,

$$E^n_{(l+1)} = -(\lambda q - \frac{2l+n+1}{2})^2, \qquad n = 0, 1, 2, \ldots, \tag{4.6}$$

The corresponding eigenfunction for the lowest state is related to the superpotential W as,

$$\Psi^{n=0}_{(l+1)}(r) = N \exp\left(-\int_0^r W_{(l+1)}(\acute{r})d\acute{r}\right). \tag{4.7}$$

Therefore, the corresponding ground-state for this case will be,

$$\Psi(r) = Nexp\left[-\frac{\lambda}{a}e^{-a(r-r_e)} - \left(\lambda q - \frac{(2l+n+1)}{2}\right)r\right]. \tag{4.8}$$

It should be noted that for $l = 0$ and $q = 1$ the potential obtained in Eq. (6) reduces to the original deformed Morse potential.

4.1.2 **The Hulthén Potential**

The Hulthén potential is a useful interaction model used in different areas of physics, including nuclear [37] and atomic physics [38]. It yields to the analytic solution for the s waves [39, 40] . At small values of the radial coordinate r, the Hulthén potential behaves like a Coulomb potential $V_c(r)$,whereas for large values of r it decreases exponentially, so that its capacity for bound state is smaller than $V_c(r)$. The Coulomb problem is analytically solvable for all energies and all

angular momenta. In contrast to this, the Hulthén potential can be solved for s waves only.

Because of this similarity and points of contrast between Coulomb and Hulthén potentials, it may be of considerate interest to generate the supersymmetric partners of the latter and study their eigensolutions, which have important applications in quantum scattering theory [51].

The deformed Hulthén Potential is given by,

$$V_H = -Ze^2 \frac{\delta e^{-\delta r}}{1 - qe^{-\delta r}}. \tag{4.9}$$

The superpotential similar to the deformed Hulthén potential can be suggested as,

$$W_{(l+1)}(r) = -\frac{\hbar}{\sqrt{2m}} \frac{(l+1)\,\delta e^{-\delta r}}{1 - qe^{-\delta r}} + \sqrt{\frac{m}{2}} \frac{e^2}{\hbar} \left[\frac{1}{(l+1)} - \frac{(l+1)\,\beta}{2} \right], \tag{4.10}$$

where $(l+1)$ denotes the partner number with $l = 0, 1, 2, \ldots$ and $\beta = \frac{\hbar^2 \delta}{me^2}$ which is a dimensionless quantity. According to the Riccati equation, the $(l+1)$th member of the Hamiltonian hierarchy for this superpotential leads to,

$$V_{(l+1)}(r) = \frac{\hbar^2}{2m} \frac{\delta^2 e^{-2\delta r}}{\left(1 - qe^{-\delta r}\right)^2} l\,(l+1) - E^0_{(l+1)} - e^2 \frac{\delta e^{-\delta r}}{1 - qe^{-\delta r}} \left[1 - l\,(l+1)\,\frac{\beta}{2} \right]. \tag{4.11}$$

Considering the shape invariance requirement, the bound-state energies of the above potential can be obtained as,

$$E^n_{(l+1)} = -\frac{q^2 me^4}{2\hbar^2} \left[\frac{1}{(n+l+1)} - \frac{(n+l+1)}{2} \beta \right]^2, \qquad n = 0, 1, 2, \ldots \tag{4.12}$$

and as a result, the corresponding ground-state eigenfunction becomes,

$$\Psi_{(l+1)}^{n=0} = N(1-qe^{-\delta r})^{l+1} \exp\left\{-\frac{me^2}{\hbar^2}\left[\frac{1}{(l+1)} - \frac{(l+1)}{2}\beta\right] r\right\}. \tag{4.13}$$

For $l = 0$ and $q = 1$ the potential in Eq.(12) reduces to the usual Hulthén potential. It has an interesting property such that when the angular momentum is zero it is not shape invariant [1] .

4.1.3 **The Pöschl-Teller Potential**

The Pöschl-Teller potential is one of the exactly solvable one-dimensional quantum-mechanical potentials. It is extensively applied in the investigation of many physical systems in condensed matter, nuclear physics, quantum optics, and solid-state physics. The Pöschl-Teller potential has always been explored for its short-range properties. Its generalized coherent states[41], nonlinear properties[42], and supersymmetric extension[43] have also been extensively studied. It is known that the local behavior of most solvable potentials reduces either to the harmonic oscillator or to the Pöschl-Teller potential [44] .

The deformed Pöschl-Teller potential is given by,

$$V_q(r) = -4V_0 \frac{e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}. \tag{4.14}$$

The superpotential similar to $V_q(r)$ can be proposed as,

$$W_{(l+1)}(r) = -\frac{\sqrt{mV_0}}{\hbar}e^{-\alpha r} + \left[\frac{\sqrt{mV_0}}{\hbar q} - \frac{\alpha}{2}(2l+1)\right] \tag{4.15}$$

where $l = 0, 1, 2, ...$ Following the same procedure, the $(l+1)$th member of the Hamiltonian hierarchy becomes,

$$V_{(l+1)}(r) = \lambda^2(e^{-2\alpha r} - 2e^{-\alpha r}) + 2l\alpha\lambda e^{-\alpha r}. \tag{4.16}$$

Therefore, the bound-state energy will be,

$$E^n_{(l+1)} = -\frac{\alpha^2}{4}\left[-(2l+n+1) + \sqrt{\frac{4mV_0}{\alpha^2 q\hbar}}\right]^2, \qquad n = 0, 1, 2, \ldots \tag{4.17}$$

The corresponding ground-state eigenfunction becomes,

$$\Psi^{n=0}_{(l+1)}(r) = N\exp\left\{-\sqrt{\frac{mV_0}{\alpha^2\hbar^2}}e^{-\alpha r} - \left[\sqrt{\frac{mV_0}{q^2\hbar^2}} - \frac{\alpha}{2}(2l+1)\right]r\right\}. \tag{4.18}$$

It is worth considering once more that for $l = 0$ and $q = 1$, the Eq.(4.17) reduces to the bound-state energies for the usual Pöschl-Teller potential.

4.1.4 **The Hyperbolic Kratzer-like Potential**

The Hyperbolic Kratzer-like potential has proved its great importance in many branches of physics such as laser theory as well as zero-dimensional field theory[45] .

This potential has received much attention. It provides a basis for expanding more realistic cases and also for its solutions to test analytical methods. The hyperbolic Kratzer-like potential also gives good results for diatomic molecules [46]. The deformed hyperbolic Kratzer-like potential can be given as,

$$V_q(r) = -V_1 q \frac{e^{-2ar}}{1-qe^{-2ar}} + V_2 q \frac{e^{-2ar}}{(1-qe^{-2ar})^2}. \tag{4.19}$$

The superpotential similar to this potential can be proposed as,

$$W_{(l+1)}(r) = -\frac{\hbar}{\sqrt{2m}} \frac{(l+1)\, e^{-2ar}}{1-qe^{-2ar}} (1+\frac{1}{1-qe^{-2ar}}) + \sqrt{\frac{m}{2}} \frac{q}{\hbar} \left[\frac{1}{l+1} - \frac{(l+1)}{2}\beta\right] (V_1+V_2)\,. \tag{4.20}$$

Therefore, the $(l+1)$th member of the deformed Kratzer-like potential yields,

$$\begin{aligned} V_{(l+1)}(r) &= \frac{\hbar^2}{2m} \frac{(l+1)^2 e^{-4ar}}{(1-qe^{-ar})^2} \left[1 + \frac{1}{1-qe^{-2ar}}\right]^2 - q \frac{(l+1)e^{-2ar}}{(1-qe^{-2ar})} \left[1 + \frac{1}{1-qe^{-2ar}}\right] \\ &\quad \times \left[\frac{1}{l+1} - \frac{(l+1)}{2}\beta\right] - \frac{a\hbar^2}{m} \frac{(l+1)e^{-2ar}}{(1-qe^{-2ar})^2} \left[1 + \frac{q^2 e^{-3ar} - 1}{(1-qe^{-2ar})^2}\right]. \end{aligned} \tag{4.21}$$

Thus, the bound-state energies of this potential will be,

$$E^n_{(l+1)} = -\frac{mq^2}{2\hbar^2} \left[\frac{1}{(n+l+1)} - \frac{(n+l+1)}{2}\beta\right]^2 (V_1+V_2)^2, \tag{4.22}$$

and the corresponding ground-state eigenfunctions will become,

$$\Psi_{(l+1)}^{n=0} = N\ exp\left\{-\frac{\hbar(l+1)}{\sqrt{2m}}\frac{e^{2ar}}{2aq^2}\left[(1-qe^{-2ar})+\frac{1}{1-qe^{-2ar}}\right]\right.$$

$$\left. + \sqrt{\frac{m}{2}}\frac{q}{\hbar}\left[\frac{1}{l+1}-\frac{(l+1)}{2}\beta\right]\cdot(V_1+V_2)r\right\}. \tag{4.23}$$

For q=1 and l=0 the result will yield for the usual non-deformed Kratzer-like potential energy.

4.2 PT-/Non-PT and Non-Hermitian Potentials

4.2.1 Generalized Morse Potential

The general Morse potential is given by,

$$V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x}. \tag{4.24}$$

To apply the Hamiltonian hierarchy method, the Eq. (3) can be rewritten as,

$$V(x) = V(e^{-2\alpha x} - qe^{-\alpha x}), \tag{4.25}$$

where, $V_1 = V$, and $\frac{V_2}{V_1} = q$. Inspired by the SUSYQM, we may propose an ansatz for the superpotential as,

$$W_{(l+1)}(x) = -\frac{\sqrt{2mV}}{a\hbar}e^{-\alpha x} + (\frac{q\sqrt{2mV}}{a\hbar} - \frac{2l+1}{2}), \tag{4.26}$$

or simply,

$$W_{(l+1)}(x) = -\lambda e^{-\alpha x} + (\lambda q - \frac{2l+1}{2}), \tag{4.27}$$

where, $\lambda^2 = \frac{2mV}{a^2\hbar^2}$, and $(2l+1)$ denotes the partner number with $l = 0, 1, 2, ...$, and the parameter m is the reduced mass of a diatomic molecule. The superpotential chosen in Eq.(4.28) leads to the $(l+1)th$ member of the Hamiltonian hierarchy through the Riccati equation as,

$$V_{(l+1)}(x) - E^0_{(l+1)} = W^2_{(l+1)}(x) - \frac{1}{\alpha}\frac{dW_{(l+1)}(x)}{dx}, \tag{4.28}$$

which yields,

$$V_{(l+1)}(x) = \lambda^2(e^{-2\alpha x} - qe^{-\alpha x}) + 2l\lambda e^{-\alpha x}. \tag{4.29}$$

Now, considering the shape invariance requirement, the energy eigenvalues become,

$$E^0_{(l+1)} = -(\lambda q - \frac{2l+1}{2})^2, \tag{4.30}$$

and for any n-th state,

$$E^n_{(l+1)} = -(\lambda q - \frac{2l+n+1}{2})^2, \qquad n = 0, 1, 2, \ldots \tag{4.31}$$

The corresponding eigenfunctions for the lowest state is obtained through the Eq.(4.7), and therefore, the corresponding ground-state will be,

$$\Psi_{(l+1)}(x) = N \exp\left[-\frac{\lambda}{\alpha}e^{-\alpha x} - (\lambda q - \frac{2l+1}{2})x\right], \tag{4.32}$$

where, N is the normalization constant.

4.2.2 Non-PT Symmetric and Non-Hermitian Morse Case

Defining the potential parameters as $V_1 = (A+iB)^2, \quad V_2 = (2C+1)(A+iB)$, and $\alpha = 1$, one gets,

$$V(x) = (A+iB)^2 e^{-2x} - (2C+1)(A+iB) e^{-x}, \tag{4.33}$$

where, A, B and C are arbitrary real parameters, and $i = \sqrt{-1}$. Taking the parameters as $A+iB = i\omega$, $(A+iB)^2 = -\omega^2$, $2C+1 = K$, we have,

$$V(x) = -\frac{\omega^2}{K}\left[Ke^{-2x} - \frac{K^2}{i\omega}e^{-x}\right]. \tag{4.34}$$

To get the final compact form, we define $\frac{\omega^2}{K} = G$, and $\frac{K^2}{\omega} = t$, and $GK = D$, and also $\frac{t}{K} = P$. As a result, the Eq.(4.35) becomes,

$$V(x) = -D\left[e^{-2x} + iPe^{-x}\right]. \tag{4.35}$$

Proposing an ansatz for the superpotential as,

$$W_{(l+1)}(x) = -i\ \frac{\sqrt{2mD}}{a\hbar}e^{-x} + \left(\frac{\sqrt{2mD}}{a\hbar} - \frac{2l+1}{2}\right), \tag{4.36}$$

and introducing $\lambda^2 = \frac{2mD}{a^2\hbar^2}$, we will have,

$$W_{(l+1)}(x) = -i\ \lambda e^{-x} + (\lambda - \frac{2l+1}{2}). \tag{4.37}$$

Consequently, according to Hamiltonian hierarchy method, we get,

$$V_{(l+1)}(x) = -\lambda^2(e^{-2x} + 2i\ e^{-x}) + 2il\lambda e^{-x}, \tag{4.38}$$

and by substituting it in the Riccati equation, the corresponding eigenvalues and eigenfunctions will be,

$$E^{n}_{(l+1)} = -(\lambda - \frac{n+2l+1}{2})^2, \qquad n = 0, 1, 2, \ldots, \tag{4.39}$$

$$\Psi^{n=0}_{(l+1)}(x) = N \exp\left[-i\lambda e^{-x} - (\lambda - \frac{2l+1}{2})x\right], \tag{4.40}$$

where, N is a normalization constant.

4.2.3 The First Type of PT-Symmetric and Non-Hermitian Morse Case

The general Morse potential has the form,

$$V(x) = (A+iB)^2 e^{-2ix} - (2C+1)(A+iB) e^{-ix}, \tag{4.41}$$

where, $V_1 = (A+iB)^2$, $V_2 = (2C+1)(A+iB)$, and $\alpha = 1$. Following the same procedure, by proposing an ansatz for the superpotential as,

$$W_{(l+1)}(x) = -\lambda e^{-ix} + (\lambda - \frac{2l+1}{2}), \tag{4.42}$$

the Hamiltonian hierarchy method yields,

$$V_{(l+1)}(x) = \lambda^2(e^{-2ix} - e^{-ix}) + 2l\lambda e^{-ix}. \tag{4.43}$$

Consequently, this form of potential will yield the same eigenvalues as in the Eq.(4.40).

4.2.4 The Second Type of PT-Symmetric and Non-Hermitian Morse Case

This type of Morse case is given as,

$$V(x) = V_1 e^{-2i\alpha x} - V_2 e^{-i\alpha x}. \tag{4.44}$$

When $\alpha = i\alpha$, and V_1 and V_2 are real. If we take the parameters as $V_1 = -\omega^2$, and $V_2 = D$ for $V_1 \Longrightarrow 0$, we get no real spectra for this kind of PT-symmetric Morse potentials. The superpotential which can be proposed for this potential is,

$$W_{(l+1)}(x) = -e^{-i\alpha x} + (2l + 1 + \frac{D}{2\omega}). \tag{4.45}$$

By applying the Hamiltonian hierarchy method, we get,

$$V_{(l+1)}(x) = e^{-2i\alpha x} - 2\left[(2l+1) + \frac{D}{2\omega} + \frac{i\alpha}{2}\right] e^{-i\alpha x}, \tag{4.46}$$

and the corresponding eigenvalues are,

$$E^0_{(l+1)} = -(2l + 1 + \frac{D}{2\omega})^2. \tag{4.47}$$

Also, for any n-th state,

$$E^n_{(l+1)} = -(2l + n + 1 + \frac{D}{2\omega})^2. \tag{4.48}$$

4.2.5 The Pöschl-Teller Potential

The Pöschl-Teller potential is given as,

$$V(x) = -4V_0 \frac{e^{-2\alpha x}}{\left(1 + qe^{-2\alpha x}\right)^2}. \tag{4.49}$$

In the framework of the SUSYQM, the superpotential similar to this potential can be proposed as,

$$W_{(l+1)}(x) = -\frac{\sqrt{mV_0}}{\hbar} e^{-\alpha x} + \left[\frac{\sqrt{mV_0}}{\hbar q} - \frac{\alpha}{2}(2l+1)\right] \tag{4.50}$$

where $l = 0, 1, 2, ...$ Following the same procedure, the $(l+1)$th member of the Hamiltonian hierarchy becomes,

$$V_{(l+1)}(x) = \lambda^2(e^{-2\alpha x} - 2e^{-\alpha x}) + 2l\alpha\lambda e^{-\alpha x}. \tag{4.51}$$

Therefore, the bound-state energy will be,

$$E^n_{(l+1)} = -\frac{\alpha^2}{4}\left[-(2l+n+1) + \sqrt{\frac{4mV_0}{\alpha^2 q\hbar}}\right]^2, \qquad n = 0, 1, 2, \ldots \tag{4.52}$$

The corresponding ground-state eigenfunction becomes,

$$\Psi^{n=0}_{(l+1)}(x) = N \exp\left\{-\sqrt{\frac{mV_0}{\alpha^2\hbar^2}} e^{-\alpha x} - \left[\sqrt{\frac{mV_0}{q^2\hbar^2}} - \frac{\alpha}{2}(2l+1)\right] x\right\}. \tag{4.53}$$

where, N is a normalization constant.

4.2.6 Non-PT Symmetric and Non-Hermitian Pöschl-Teller Case

This time, V_0 and q are complex parameters as $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$, but α is an arbitrary real parameter. Although the potential is complex and the corresponding Hamiltonian is non-Hermitian and also non-PT symmetric, there may be real spectra if and only if $V_{0I}\ q_R = V_{0R}\ q_I$. When both parameters V_0, and q are taken pure imaginary, the potential turns out to be,

$$V(x) = -4V_0 \frac{2qe^{-4\alpha x} + i(1 - q^2 e^{-4\alpha x})}{(1 + q^2 e^{-4\alpha x})^2}. \tag{4.54}$$

Here, V_0 and q have been used instead of V_{0I} and q_I. To obtain the energy eigenvalues via the Hamiltonian hierarchy method, the proposed superpotential can be,

$$W_{(l+1)}(x) = -\frac{\sqrt{mV_0}}{\hbar} e^{-2\alpha x} + \left[\frac{\sqrt{mV_0}}{\hbar q} - \frac{\alpha}{2}(2l+1) \right] \tag{4.55}$$

and therefore, by substituting this equation into the Riccati equation, we get the same energy eigenvalues as in Eq.(4.52).

4.2.7 PT-Symmetric and Non-Hermitian Pöschl-Teller Case

We choose the parameters V_0 and q as arbitrary real, and $\alpha \Longrightarrow i\alpha$. As a result, the potential becomes,

$$V(x) = -4V_0 \frac{2qe^{-4i\alpha x} + i(1 - q^2 e^{-4i\alpha x})}{(1 + q^2 e^{-4i\alpha x})^2}. \tag{4.56}$$

Here, we propose the superpotential similar to this potential as,

$$W_{(l+1)}(x) = -\frac{\sqrt{mV_0}}{\hbar}e^{-2i\alpha x} + \left[\frac{\sqrt{mV_0}}{\hbar q} - \frac{\alpha}{2}(2l+1)\right] \tag{4.57}$$

and hence, by applying the Hamiltonian hierarchy method the corresponding eigenvalues can be obtained as in the Eq.(4.52).

4.2.8 **The Screened Coulomb Potential**

The screened Coulomb potential is given by,

$$V_{SC}(r) = -\frac{a}{r}\left[1 + (1+br)\, e^{-2br}\right], \tag{4.58}$$

where b is the screened length. In order to determine an effective potential similar to this potential, inspired by the SUSYQM the following ansatz to the superpotential is suggested,

$$W(r) = -(l+1)\, ab\frac{e^{-br}}{1-e^{-br}} + \frac{1}{l+1} - \frac{b}{2}, \tag{4.59}$$

where $(2l+1)$ denotes the partner number with $l = 0, 1, 2, ...$ This superpotential choice leads to the $(l+1)\, th$ member of the Hamiltonian hierarchy through the Riccati equation as,

$$V_{(l+1)}(r) = \frac{ab^2e^{-br}}{\left(1-e^{-br}\right)^2}\left[ae^{-br}(l+1) - 1\right](l+1) - \frac{abe^{-br}}{1-e^{-br}}(2-b), \tag{4.60}$$

Considering the shape invariance requirement, the energy eigenvalues for any n-th

state become,

$$E^{n}_{(l+1)} = \frac{1}{(n+l+1)^2} - \frac{b}{n+l+1} + \frac{b^2}{4}, \quad n = 0, 1, 2, ... \tag{4.61}$$

The corresponding eigenfunction for the lowest state is obtained through the Eq.(4.7) as,

$$\Psi^{n=0}_{(l+1)} = N \exp\left[\frac{r}{l+1} - \frac{br}{2} + a\,(l+1)\ln\left(1-e^{-br}\right)\right], \tag{4.62}$$

where, N is a normalization constant.

4.2.9 Non-PT Symmetric and Non-Hermitian Screened Coulomb Case

The non-PT and non-Hermitian Coulomb potential can be defined as,

$$V_{SC}(r) = -\frac{a}{r}\left[1 + (1+ibr)\,e^{-2br}\right], \tag{4.63}$$

where $i = \sqrt{-1}$. An ansatz for the superpotential can be proposed as,

$$W(r) = -\,(l+1)\,ab\frac{ie^{-br}}{1-e^{-br}} + \frac{1}{l+1} - \frac{b}{2}. \tag{4.64}$$

As a result, following the Hamiltonian Hierarchy method we can get,

$$V_{(l+1)}(r) = ab^2\frac{ie^{-br}}{\left(1-e^{-br}\right)^2}\left[ae^{-br}\,(l+1) - 1\right](l+1) - ab\frac{ie^{-br}}{1-e^{-br}}\,(2-b)\,, \tag{4.65}$$

Consequently, this form of potential will yield the same eigenvalues as in the Eq.(4.64), and the corresponding eigenfunctions will be,

$$\Psi^{n=0}_{(l+1)} = N \exp\left[\frac{r}{l+1} - \frac{br}{2} + ia\,(l+1)\ln\left(1-e^{-br}\right)\right], \tag{4.66}$$

where, N is a normalization constant.

4.2.10 PT-Symmetric and Non-Hermitian Screened Coulomb Case

The PT symmetric and non-Hermitian screened Coulomb potential can be introduced as,

$$V_{SC}(r) = -\frac{a}{r}\left[1 + (1+br)\, e^{-2ibr}\right], \tag{4.67}$$

This time, an ansatz for the superpotential is proposed as,

$$W(r) = -\,(l+1)\, ab\frac{e^{-ibr}}{1-e^{-ibr}} + \frac{1}{l+1} - \frac{b}{2}. \tag{4.68}$$

and by substituting it in the Ricatti equation we get

$$V_{(l+1)}(r) = ab^2\frac{e^{-ibr}}{(1-e^{-ibr})^2}\left[ae^{-ibr}\,(l+1) - 1\right](l+1) - ab\frac{e^{-ibr}}{1-e^{-ibr}}\,(2-b)\,, \tag{4.69}$$

The eigenvalues for this type of potential will be the same eigenvaules as in Eq. (4.64), and the corresponding eigenfunctions as,

$$\Psi^{n=0}_{(l+1)} = N\exp\left[\frac{r}{l+1} - \frac{br}{2} + ia\,(l+1)\ln\left(1-e^{-ibr}\right)\right], \tag{4.70}$$

4.2.11 The Exponential-Cosine Screened Coulomb Potential (ECSC)

The Exponential-cosine screened coulomb potential (ECSC) is presented as,

$$V(r) = -\frac{q}{r}e^{-\lambda r}\cos(\mu r). \tag{4.71}$$

Substituting $\cos(\mu r) = \frac{e^{i\mu r}+e^{-i\mu r}}{2}$ in the above potential, we get,

$$V(r) = -q\frac{e^{-\lambda r}}{r}(\frac{e^{i\mu r}+e^{-i\mu r}}{2}). \tag{4.72}$$

or,

$$V(r) = -\frac{q}{2}\left[\frac{e^{(i\mu-\lambda)r} + e^{-(i\mu+\lambda)r}}{r}\right]. \tag{4.73}$$

In order to avoid the cumbersome calculations and for simplicity let us take $q = 2$, therefore,

$$V(r) = -\frac{e^{-(\lambda-i\mu)r}}{r} - \frac{e^{-(\lambda+i\mu)r}}{r}. \tag{4.74}$$

This potential can be considered as two separate parts as,

$$V_1(r) = -\frac{e^{-(\lambda-i\mu)r}}{r}, \tag{4.75}$$

and,

$$V_2(r) = -\frac{e^{-(\lambda+i\mu)r}}{r}. \tag{4.76}$$

Let us define $\lambda - i\mu = \alpha$ and $\lambda + i\mu = \beta$. Thus, the superpotential which can be proposed as an ansatz for the $V_1(r)$ potential is,

$$W_{1(l+1)}(r) = -(l+1)\frac{\alpha e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{1}{l+1} - \frac{\alpha}{2}. \tag{4.77}$$

According to the Hamiltonian hierarchy method, the corresponding eigenfunction for this superpotential will be,

$$\Psi_{01}(r) = \left(1 - e^{-\alpha r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\alpha}{2})r}. \tag{4.78}$$

Assuming that the radial trial wave function is given by (4.79), replacing α by the variational parameter μ_1, i.e.,

$$\Psi_{\mu_1}(r) = \left(1 - e^{-\mu_1 r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\mu_1}{2})r}. \tag{4.79}$$

The variational energy is given by,

$$E_{\mu_1} = \frac{\int_0^\infty \Psi_{\mu_1}(r) \left[-\frac{1}{2}\frac{d^2}{dr^2} - \frac{e^{-\alpha r}}{r} + \frac{l(l+1)}{2r^2}\right] \Psi_{\mu_1}(r) dr}{\int_0^\infty \Psi_{\mu_1}(r)^2 dr}, \tag{4.80}$$

The superpotential which can be proposed as an ansatz for the $V_2(r)$ potential is,

$$W_{2(l+1)}(r) = -(l+1)\frac{\beta e^{-\beta r}}{1-e^{-\beta r}} + \frac{1}{l+1} - \frac{\beta}{2}, \tag{4.81}$$

and the corresponding eigenfunction for this superpotential is,

$$\Psi_{02}(r) = \left(1 - e^{-\beta r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\beta}{2})r}. \tag{4.82}$$

Again, assuming that the radial trial wave function is given by (4.83), replacing β by the variational parameter μ_2, i.e.,

$$\Psi_{\mu_2}(r) = \left(1 - e^{-\mu_2 r}\right)^{l+1} e^{-(\frac{1}{l+1} - \frac{\mu_2}{2})r}. \tag{4.83}$$

The variational energy is given by,

$$E_{\mu_2} = \frac{\int_0^\infty \Psi_{\mu_2}(r) \left[-\frac{1}{2}\frac{d^2}{dr^2} - \frac{e^{-\beta r}}{r} + \frac{l(l+1)}{2r^2}\right] \Psi_{\mu_2}(r) dr}{\int_0^\infty \Psi_{\mu_2}(r)^2 dr}, \tag{4.84}$$

Thus, by minimizing the energies E_{μ_1} and E_{μ_2} with respect to the variational parameter μ_1 and μ_2, one obtains the best estimate for the energy of the exponential-screened Coulomb potential.

As the exponential-cosine screened Coulomb potential is not exactly solvable, the superpotentials given by Eqs.(4.78) and (4.82) do not satisfy the Riccati equation, but they do satisfy for effective potentials instead, V_{1eff} and V_{2eff} as,

$$V_{1eff}(r) = \frac{\bar{W}_1^2 - \bar{W}\prime_1}{2} + E(\bar{\mu_1}). \tag{4.85}$$

and

$$V_{2eff}(r) = \frac{\bar{W}_2^2 - \bar{W}\prime_2}{2} + E(\bar{\mu_2}), \tag{4.86}$$

where $\bar{W}_1 = W_1(\alpha = \bar{\mu_1})$ and $\bar{W}_2 = W_2(\beta = \bar{\mu_2})$. $\bar{\mu_1}$ and $\bar{\mu_2}$ are the parameters that minimize the energy expectation values (4.81) and (4.85). They are given by,

$$V_{1eff}(r) = -\frac{\alpha e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{l(l+1)}{2}\frac{\alpha^2 e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} + \frac{1}{2}(\frac{1}{l+1} - \frac{\alpha}{2})^2 + E(\alpha), \tag{4.87}$$

and

$$V_{2eff}(r) = -\frac{\beta e^{-\beta r}}{1 - e^{-\beta r}} + \frac{l(l+1)}{2}\frac{\beta^2 e^{-2\beta r}}{(1 - e^{-\beta r})^2} + \frac{1}{2}(\frac{1}{l+1} - \frac{\beta}{2})^2 + E(\beta), \tag{4.88}$$

By substituting the values of α and β in (4.88) and (4.89), one can obtain the bound state energies of the exponential-cosine screened Coulomb potential as,

$$E = -\frac{1}{2}\left[\frac{1}{(l+1)^2} + \frac{\lambda^2 - \mu^2}{4} - \frac{\lambda}{l+1}\right]. \tag{4.89}$$

It is interesting to notice that for $\mu = 0$, the exponential-cosine screened Coulomb potential reduces to the general screened Coulomb potential and that Eq.(4.90) reduces to,

$$E = -\frac{1}{2}\left[\frac{1}{(l+1)} - \frac{\lambda}{2}\right]^2. \tag{4.90}$$

which are the bound state energies for the general screened Coulomb potential.

4.2.12 Non-PT Symmetric and non-Hermitian Exponential-Cosine Screened Coulomb Case

The non-PT and non-Hermitian cosine screened Coulomb potential can be defined as,

$$V(r) = \frac{iq}{r} e^{-\lambda r} \cos(\mu r). \tag{4.91}$$

or simply,

$$V(r) = \frac{iq}{2}\left(\frac{e^{-\alpha r}}{r} + \frac{e^{-\beta r}}{r}\right). \tag{4.92}$$

In this case the proposed superpotentials can be,

$$W_{1(l+1)}(r) = -(l+1)\frac{i\alpha e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{1}{l+1} - \frac{\alpha}{2}. \tag{4.93}$$

and

$$W_{2(l+1)}(r) = -(l+1)\frac{i\beta e^{-\beta r}}{1-e^{-\beta r}} + \frac{1}{l+1} - \frac{\beta}{2}, \tag{4.94}$$

Though the superpotentials are complex, following the same method will yield the same energy eigenvalues as in (4.90).

4.2.13 *PT* Symmetric and non-Hermitian Exponential-Cosine Screened Coulomb Case

The PT symmetric and non-Hermitian cosine screened Coulomb potential can be introduced as,

$$V(r) = -\frac{q}{r}e^{-i\lambda r}\cos(\mu r). \tag{4.95}$$

or,

$$V(r) = -\frac{q}{2}\left[\frac{e^{-i(\lambda-\mu)r} + e^{-i(\lambda+\mu)r}}{r}\right]. \tag{4.96}$$

Taking, $(\lambda - \mu) = -\alpha_0$ and $\lambda + \mu = \beta_0$, we will have,

$$V(r) = -\frac{q}{2}(\frac{e^{-i\alpha_0 r}}{r} + \frac{e^{-i\beta_0 r}}{r}), \tag{4.97}$$

and as a result the superpotentials can be proposed as,

$$W_{1(l+1)}(r) = -(l+1)\frac{\alpha_0 e^{-i\alpha_0 r}}{1-e^{-i\alpha_0 r}} + \frac{1}{l+1} - \frac{\alpha_0}{2}. \tag{4.98}$$

and

$$W_{2(l+1)}(r) = -(l+1)\frac{\beta_0 e^{-i\beta_0 r}}{1-e^{-i\beta_0 r}} + \frac{1}{l+1} - \frac{\beta_0}{2}, \tag{4.99}$$

In conclusion, by applying the method the same energy eigenvalues will be obtained as in (4.90).

CHAPTER 5

CONCLUSIONS AND REMARKS

We have applied the Hamiltonian hierarchy method to solve the Schrödinger equation for the deformed Morse, Hulthén, Pöschl-Teller, Hyperbolic Kratzer-like, screened Coulomb and exponential-screened Coulomb potentials in the framework of the SUSYQM. Consequently, the energy eigenvalues and corresponding wave functions of these potentials were calculated in details. It was pointed out that for the case $l = 0$, the partner potentials obtained for the deformed forms of these potentials are identical to their usual non-deformed forms. Moreover, it was identified that for the case $q = 1$, the bound-state energies and corresponding ground-state eigenfunctions for the deformed forms of these potentials reduce to the bound-state energies and the corresponding ground-state eigenfunctions for the non-deformed forms of these potentials. Consequently, we can remark that the results presented in this work suggest that the Hamiltonian hierarchy method within the framework of the SUSYQM provides an appropriate approach to get acceptable good results for both non-deformed and deformed systems.

We have also applied the PT-symmetric formulation to solve the Schrödinger

equation with more general Morse and Pöschl-Teller potentials. The Hamiltonian hierarchy method within the framework of the SUSYQM is used. We have obtained the energy eigenvalues and the corresponding eigenfunctions and have considered many different forms of the complex forms of these potentials. As the energy spectrum of the PT-invariant complex-valued non-Hermitian potentials may be real or complex depending on the parameters, we have clarified that there are some restrictions on the potential parameters for the bound states in PT-symmetric, or more generally, in non-Hermitian quantum mechanics. It is also indicated that the interesting features of the quantum expectation theory for the PT-violating potentials may be affected by changing them from complex to real systems. Furthermore, it is pointed out that the superpotentials, their superpartners and the corresponding ground state eigenfunctions satisfy the PT-symmetry condition. As a final conclusion, we can add that our exact results may increase the number of applications in the study of different quantum systems.

Table 1: Energy eigenvalues ($in\ eV$) of the Morse potential for H_2, HCl, CO and LiH molecules. Comparisons are made with the results from references.

	State	Our Work	Variational *Ref.[35]*	Shifted 1/N exp. *Ref.[56]*
$D = 47446\ cm^{-1}$,	$a = 1.9426\ \mathring{A}^{-1}$,	$r_e = 0.741\ \mathring{A}$,	$m = 0.50391\ amu$,	$\lambda = 2.6156$
	l=0	-4.4758	-4.4758	-4.4749
H_2	l=5	-4.2563	-4.2563	-4.2589
	l=10	-3.7187	-3.7187	-3.7247
$D = 32755\ cm^{-1}$,	$a = 1.8677\ \mathring{A}^{-1}$,	$r_e = 1.2746\ \mathring{A}$,	$m = 0.9801045\ amu$,	$\lambda = 2.60618$
	l=0	-4.4360	-4.4360	–4.4352
HCl	l=5	-4.3971	-4.3971	-4.3967
	l=10	-4.2940	-4.2940	-4.2939
$D = 90540\ cm^{-1}$,	$a = 2.2994\ \mathring{A}^{-1}$,	$r_e = 1.1283\ \mathring{A}$,	$m = 6.8606719\ amu$,	$\lambda = 3.8306$
	l=0	-11.0928	-11.0928	-11.0910
CO	l=5	-11.085	-11.0850	-11.840
	l=10	-11.0660	-11.0660	-11.0650
$D = 20287\ cm^{-1}$,	$a = 1.1280\ \mathring{A}^{-1}$,	$r_e = 1.5956\ \mathring{A}$,	$m = 0.8801221\ amu$,	$\lambda = 2.058$
	l=0	-2.4291	-2.4291	-2.4278
LiH	l=5	-2.4014	-2.4014	-2.4000
	l=10	-2.3287	-2.3261	-2.3261

Table 2: Energy eigenvalues of the Hulthén potential as a function of screening parameter for various states in atomic units. Comparisons are made with the results from references.

State	δ	**Our Work**	**Variational** *Ref.[57]*	**Numerical** *Ref.[58]*
2p	0.025	-0.1127605	-0.1127600	-0.1127605
	0.050	-0.1010425	-0.1010420	-0.1010425
3p	0.025	-0.0437068	-0.0436010	-0.0430690
	0.050	-0.0331632	-0.0327480	-0.0331645
3d	0.025	-0.0436030		
	0.050	-0.0327532		
4p	0.025	-0.0199480		
	0.050	-0.0110430		
4d	0.025	-0.0198460		
	0.050	-0.0106609		
4f	0.025	-0.0196911		
	0.050	-0.0100618		

Table 3: Energy eigenvalues of the Pöschl–Teller potential as a function of screening parameter α for some different states in atomic units. Comparisons are made with the results from references.

State	α	**Our Work**	**Numerical** *Ref.[59]*	**Apprpox. Method** *Ref.[60]*
	7	-7.692	-7.702	-7.718
	9	-17.879	-17.999	-17.701
1p	11	-31.568	-31.686	-31.669
	15	-71.593	-71.680	-71.684
	7	-3.555	-3.672	-4.002
	9	-12.102	-12.195	-11.802
1d	11	-23.438	-23.586	-23.716
	15	-59.352	-59.557	-59.635
	7	-0.538	-0.546	—
	9	-7.111	-7.124	-7.256
1f	11	-16.217	-16.238	-16.860
	15	-48.101	-48.155	-48.509
	7	-0.068	—	—
	9	-2.862	-2.907	—
1g	11	-9.217	-9.698	-11.560
	15	-37.335	-37.499	-38.522

Table 4 : Energy eigenvalues as a function of the screening parameter b for $1s$, $2p$, $3d$ and $4f$ states in Rydberg units of energy.

State 1s				
Screening	**SUSYQM**	**SUSYQM** [**53**]	**NR – QM** [**36**]	**Exact** [**54**]
b	*Our Work*	*Variational*	*Variational*	*Numerical*
0.001	-0.499020	-0.499012	-0.499000	—
0.002	-0.498020	-0.498012	-0.498000	-0.498000
0.005	-0.495040	-0.495040	-0.495020	-0.495000
0.010	-0.490090	-0.490090	-0.490070	-0.490100
0.020	-0.480310	-0.480310	-0.480300	-0.480300
0.025	-0.475480	-0.475480	-0.475460	-0.475500
0.030	-0.470680	-0.470690	-0.460660	—
0.040	-0.461190	-0.461190	-0.461170	—
0.050	-0.451800	-0.451800	-0.451820	-0.451800
0.060	-0.442590	-0.442590	-0.442600	—
0.070	-0.433510	-0.433510	-0.433520	—
0.080	-0.424560	-0.424560	-0.424570	—
0.090	-0.415740	-0.415740	-0.415750	—

(continued)

State 1s				
Screening	**SUSYQM**	**SUSYQM** [53]	**NR – QM** [36]	**Exact** [54]
b	*Our Work*	*Variational*	*Variational*	*Numerical*
0.100	-0.407050	-0.407050	-0.470600	-0.407100
0.200	-0.326810	-0.326810	-0.326810	-0.326800
0.250	-0.290920	-0.290920	-0.290920	-0.290900
0.300	-0.257640	-0.257640	-0.257630	—
0.400	-0.198420	-0.198420	-0.198360	—
0.500	-0.148060	-0.148060	-0.148080	-0.148100
0.600	-0.106077	-0.106077	-0.106080	—
0.700	-0.071750	-0.071750	-0.071740	—
0.800	-0.044590	-0.044590	-0.044590	—
0.900	-0.024200	-0.024200	-0.024180	—
1.000	-0.010260	-0.010260	-0.010160	-0.010290
1.050	-0.005680	-0.005680	-0.005440	—

(continued)

	State 2p		
	SUSYQM (Our Work)	**Variational[53]**	**Numerical[54]**
0.001	-0.12475	-0.24800	-0.05539
0.005	-0.123753	—	—
0.010	-0.122512	-0.2305	-0.2305
0.020	-0.12005	-0.2119	-0.2119
0.025	-0.118828	-0.2030	-0.2030
0.050	-0.112812	-0.1615	-0.03374
0.100	-0.10125	-0.09289	-0.09307

	State 3d		
	SUSYQM (Our Work)	**Variational[53]**	**Numerical[54]**
0.001	-0.05539	-0.10910	-0.10910
0.005	-0.05473	—	—
0.010	-0.05390	-0.09212	-0.091212
0.020	-0.05228	-0.07503	-0.07503
0.025	-0.051468	-0.06714	-0.06715
0.050	-0.47535	-0.03374	-0.03383
0.100	-0.40139	—	—

(continued)

	State 4f		
	SUSYQM (Our Work)	**Variational[53]**	**Numerical[54]**
0.001	-0.03112	-0.06051	-0.06052
0.005	-0.03062	-0.52930	-0.05294
0.010	-0.03001	-0.04419	-0.04420
0.020	-0.0289	-0.02897	-0.02898
0.025	-0.0282	—	—
0.050	-0.02531	—	—
0.100	-0.020	—	—

Table 5: Energy eigenvalues as a function of the screening parameter λ for $\lambda = \mu$, for $1s$, $2p$, $3d$ and $4f$ states in Rydberg units of energy.

State 1s				
Screening	**SUSYQM**	**Hypervirial**	**NR – QM**	**Exact**
λ	*Our Work*	*Solution [55]*	*Variational [36]*	*Numerical [54]*
0.020	-0.480310	-0.480310	-0.480300	-0.480300
0.050	-0.451800	-0.451800	-0.451820	-0.451800
0.080	-0.424560	-0.424560	-0.424570	—
0.100	-0.407050	-0.407050	-0.470600	-0.407100
State 2p				
Screening	**SUSYQM**	**Hypervirial**	**NR – QM**	**Exact**
λ	*Our Work*	*Solution [55]*	*Variational [36]*	*Numerical [54]*
0.020	-0.120000	-0.105890	-0.211900	-0.211900
0.050	-0.112500	-0.080400	-0.161500	—
0.080	-0.100500	-0.046000	—	—
0.100	-0.100000	-0.008000	-0.092890	-0.093070

(continued)

State 3d				
Screening	**SUSYQM**	**Hypervirial**	**NR – QM**	**Exact**
λ	*Our Work*	*Solution [55]*	*Variational [36]*	*Numerical [54]*
0.020	-0.052220	-0.037500	-0.075030	-0.075030
0.050	-0.047220	-0.017340	-0.033740	-0.033830
0.080	-0.042220	-0.008000	—	—
0.100	-0.038889	—	—	—

State 4f				
Screening	**SUSYQM**	**Hypervirial**	**NR – QM**	**Exact**
λ	*Our Work*	*Solution [55]*	*Variational [36]*	*Numerical [54]*
0.020	-0.028750	-0.014700	-0.028970	-0.089800
0.050	-0.025000	-0.003200	—	—
0.080	-0.021250	-0.175000	—	—
0.100	-0.018700	—	—	—

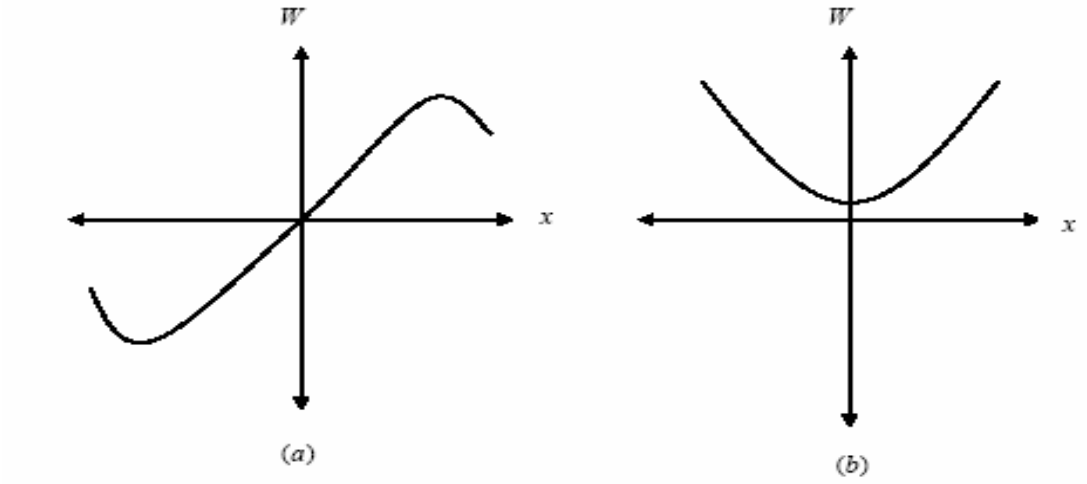

Figure 1. (a) shows an example of the superpotential W(x) for which the ground state energy is zero, $E_0 = 0$ (exact SUSY), (b) corresponds to $E_0 > 0$ (unbroken SUSY case).

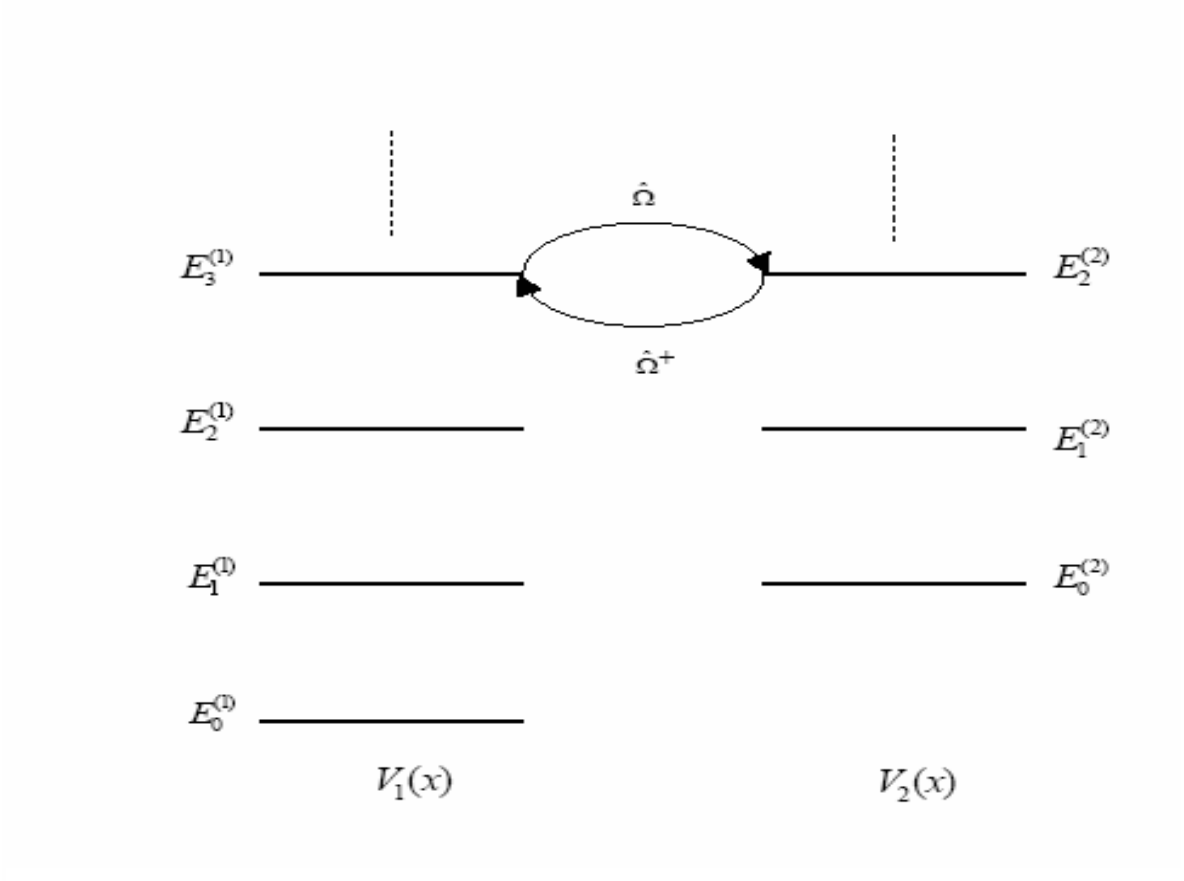

Figure 2. The energy levels of two supersymmetric partner potential V_1 and V_2. The figure corresponds to unbroken SUSY

REFERENCES

[1] C. M. Bender, S. Boettcher, Phys. Rev. Lett. **80**, 5243(1998),
C. M. Bender, J. Math. Phys. **40**, 2201(1999),
C. M. Bender, D. C. Brody, H. F. Jones, Phys. Rev. Lett. **89**, 270401(2002),
C. M. Bender, P. N. Meisinger, Q. Wang, J. Phys. A**36**,6791(2003)
C. M. Bender, D. C. Brody, H. F. Jones, Am. J. Phys. **71**, 1095(2003),
C. M. Bender, P. N. Meisinger, Q. Wang, J. Phys A**36**, 1029(2003),
C. M. Bender, M. V. Berry, A. Mandilara, J. Phys. A**35**,L467(2002)

[2] A. Mostafazadeh, J. Math. Phys. **43**, 205(2002), ibid **43**, 2814(2002) , ibid **43**, 3944(2002), J. Phys. A**36**, 7081(2003)

[3] Z. Ahmed, Phys. Lett. A**290**, 19(2001), ibid **310**, 139(2003), J. Phys. A**36**, 10325(2003)

[4] S. Weigert, Phys. Rev. A**68**, 062111(2003)

[5] C. M. Bender, F. Cooper, P. N. Meisinger, M. V. Savage, Phys. Lett. A**259**, 224(1999)

[6] C. M. Bender, G. V. Dunne, P. N. Meisinger, Phys. Lett. A**252**, 272(1999)

[7] V. Buslaev, V. Grecchi, J. Phys. A**36**, 5541(1993)

[8] E. Delabaere, F. Pham, Phys. Lett. A**250**, 25(1998); ibid, **250**, 29(1998)

[9] C. M. Bender, K. A. Milton, V. M. Savage, Phys. Rev. D**62**, 085001(2000),
C. M. Bender, S. Boettcher, H. F. Jones, P. N. Meisinger, J. Math. Phys. **42**, 1960(2001),
C. M. Bender, S. Boettcher, H. F. Jones, P. N. Meisinger, M. Şimşek, Phys. Lett. A**291**, 197(2001),
C. M. Bender, Czech. J. Phys. **54**, 13(2004)

[10] B. Bagchi, C. Quesne, Phys. Lett. A**273**, 285(2000) 285

[11] B. Bagchi, C. Quesne, Phys. Lett. A**300**, 18(2002)

[12] G. Levai, F. Cannata, A. Ventura, J.Phys. A**34**, 839 (2001)

[13] G. Levai, F. Cannata, A. Ventura, J. Phys. A**35**, 5041 (2002)

[14] F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. **251**, 267(1995), hep-th/0209062

[15] F. Cannata, G. Junker, J. Trost, Phys. Lett. A**246** (1998) 219

[16] B. Bagchi, S. Mallik, C. Quesne, Mod. Phys. Lett. A**17**, 1651(2002)

[17] B. Bagchi, R. Roychoudhury, J. Phys. A**33** , L1(2000)

[18] A. A. Adrianov, et al., Int. J. Mod. Phys. A**14**, 2675(1999)

[19] S. Vincenzo, V. Alonso, Phys. Lett. A**298** , 98(2002)

[20] J. S. Petrovic, V. Milanovic, Z. Ikonić, Phys. Lett. A**300**, 595(2002)

[21] F. M. Fernandez, R. Guardiola, J. Ros, M. Znojil, J. Phys. A**31**, 10105(1998),
G. A. Merzinescu, J. Phys. A**33**, 4911(2000),
O. Mustafa, M. Znojil, J. Phys. A**35**, 8929(2002),
M. Znojil, F. Gemperle, O. Mustafa, J. Phys. A**35**, 5781(2002),
M. Znojil, J. Phys. A**35**, 2341(2002),
M. Znojil, Phys. Lett. A**271**, 327(2000)

[22] V. Milanovic, Z. Ikonic, Phys. Lett. A**293**, 29 (2002)

[23] C. S. Jia, X. L. Zeng, L. T. Sun, Phys. Lett. A**294**, 185 (2002),
M. Znojil, J. Phys. A**36**, 7639(2003),
M. Znojil, J. Phys. A**36**, 7825(2003),
M. Znojil, G. Levai, Mod. Phys. Lett. A**16**, 2273(2001),
M. Znojil, Phys. Lett. A**285**, 7(2001), hep-th/0404213

[24] M. Znojil, Phys. Lett. A**264** (1999) 108,
Ö. Yeşiltaş, M. Şimşek, R. Sever, C. Tezcan, Phys. Scripta **T67**, 472(2003),
D. T. Barclay, R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta, C. S. Jia, Y. Sun, Y. Li, Phys. Lett. A**305**, 231(2002),
G. Levai, M. Znojil, J. Phys. A**35**, 8793(2002)

[25] C. M. Bender, S. Boettcher, J. Phys. A**31**, L273(1998),
C. M. Bender, S. Boettcher, H. F. Jones, V. M. Savage, J. Phys. A**32**, 6771(1999),
M. Znojil, J. Phys. A**33**, 4203(2000),
A. Sinha, G. Levai, P. Roy, Phys. Lett. A**322**, 78(2004)

[26] C. M. Bender, G. V. Dunne, P. N. Meisinger, M. Şimşek, Phys. Lett. A**281**, 311(2001)

[27] A. Khare, U. Sukhatme, Phys. Lett. A**324**, 406(2004)

[28] G. Lévai, M. Znojil, J. Phys. A **33**, 7165(2000)

[29] Parthasarathi, D. Parashar, R. S. Kaushal, J. Phys. A**37**, 781(2004)

[30] B. Gönül, O. Özer, Y. Cançelik, M. Koçak, Phys. Lett. A**275**, 238(2000)

[31] E. Schrödinger, Proc. R. Irish Acad. A**46**, 9(1940); ibid, **46**, 183(1940); ibid, **47**, 53(1941)

[32] L. Infeld, T. E. Hull, Mod. Phys. **23**, 21(1951)

[33] M. Ioffe, A. I. Neelov, J. Phys. A**36**, 2193(2003)

[34] V. M. Tkachuk, T. V. Fityo, J. Phys. A**34**, 8673(2001)

[35] E. D. Filho, R. M. Ricotta, Phys. Lett. A**269**, 269(2000)

[36] C. S. Lam, Y. P. Varshni, Phys. Rev. A **4** (1971) 1875

[37] S. Flügge, Practical Quantum Mechanics, Springer, Berlin (1974)

[38] E.Witten, Nucl.Phys. B **202**, 253(1982).

[39] Bender C. M. and Boettcher S. 1998, Phys. Rev. Lett. **80** 5243.

[40] Mostafazadeh A 2002 *J. Math. Phys.* **43** (205, 2814, 3944)

[41] Levai G. and Znojil M. 2000 *J.Phys. A: Math. Gen.***33** 7165.

[42] Levai G. and Znojil M. 2001 *Mod. Phys. Lett.* A **30** 1973.

[43] Dorey P., Dunning C. and Tateo R., 2001 *J. Phys. A: Math. Gen.* **34** L 391.

[44] Andrianov A. A, Ioffe M. V, Cannata F. and Dedonder J. P, 1999 *Int.J. Mod. Phys.* A **14** 2675.

[45] J. Kurzweil, 1986, *Ordinary differential equations; Studies in Applied Mathematics*, Vol. **13** (Amsterdam: Elsevier).

[46] Theresa Julia Zielinski,*MorsePotential.mcd*, Modified: July 5, 1998, p 1-6.

[47] Verma, R. D. Ultraviolet Spectrum of the Iodine Molecule. J. Chem. Phys. 1960, 32, pp 738-749.

[48] Atkins, P. W. Physical Chemistry, 5th ed.; W. H. Freeman: New York, 1994; p 572.

[49] I. Cooper, Chem. Phys. 112, 67 (1987).

[50] J.N. Huffaker and P.H. Dwivedi, J. Chem. Phys. **68**, 1303 (1978).

[51] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[52] G. Faridfathi, R. Sever, Metin Aktaş, arXiv: quant-ph/0407016

[53] E. D. Filho, R. M. Ricotta, arXiv: hep-th/0002015.

[54] R. L. Greene, C. Aldrich, Phys. Rev. A **14** (1976) 2363.

[55] R. Sever, C. Tezcan, Tr. J. of Physics **17** (1993) 459-464

[56] M. Bag, M. M. Panja, R. Dutt, Y. P. Varshni, Phys. Rev. A **46**, 6059 (1992).

[57] E. Drigo Filho, R. Maria Ricotta, Mod. Phys. Lett. A **10**, 1613 (1995), arXiv: hep-th/9507132.

[58] Y. P. Varshni, Phys. Rev. A **41**, 4682 (1990).

[59] V. C. Aguilera-Navarro, E. Ley-Koo, and S. Mateos-Cortes, *Rev. Mex. Fisica* **43**, 413 (1998).

[60] F. Z. Ighezou, and R. J. Lombard, Annals of Physics **278**, 265-279 (1999).

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