AUTOMORPHISMS OF COMPLEXES OF CURVES ON ODD GENUS
NONORIENTABLE SURFACES

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ABSTRACT

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Let N be a connected nonorientable surface of genus g with n punctures. Suppose that g is odd and $g + n \geqslant 6$. We prove that the automorphism group of the complex of curves of N is isomorphic to the mapping class group $\mathcal{M}_N$ of N.

Keywords: Mapping class group, complex of curves, nonorientable surface.

ÖZ

TEK SAYIDA DELİKLİ YÖNLENDİRİLEMEYEN YÜZEYLER ÜZERİNDE EĞRİLERİN KOMPLEKSLERİNİN OTOMORFİZMİ

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N g delikli n işaretli noktası olan bağlantılı yönlendirilemeyen yüzey olsun. N nin eğrilerinin kompleksinin otomorfizm grubunun N nin $\mathcal{M}_N$ gönderim sınıfları grubuna izomorfik olduğunu kanıtlıyoruz.

Anahtar Kelimeler: Yüzeylerin gönderim sınıfları grubu, eğrilerin kompleksi, yönlendirilemeyen yüzey.

To my mother and the memory of my father

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

1.1 Introduction and statement of results

Let N be a connected nonorientable surface of genus g with n punctures and let $\mathcal{M}_N$ denote the mapping class group of N, the group of isotopy classes of all diffeomorphisms $N \to N$. The complex of curves $C(N)$ on N is defined to be the abstract simplicial complex whose vertices are the isotopy classes of nontrivial simple closed curves. A set of vertices $\{v_0, v_1, \ldots, v_q\}$ forms a $q - simplex$ if and only if $v_0, v_1, \ldots, v_q$ have pairwise disjoint representatives.

Each diffeomorphism $N \to N$ acts on the set of nontrivial simple closed curves preserving the disjointness of simple closed curves. It follows that the mapping class group of N acts on $C(N)$ as simplicial automorphisms. In other words, there is a natural group homomorphism $\mathcal{M}_N \to \operatorname{Aut} C(N)$. The purpose of this thesis is to prove the following theorem.

Theorem 1.1. *Let N be a connected nonorientable surface of genus g with n punctures. Suppose that g is odd and $g + n \geqslant 6$. Then the natural map $\mathcal{M}_N \to \operatorname{Aut} C(N)$ is an isomorphism.*

The complex of curves on an orientable surface S was introduced by Harvey [6]. It was shown by Ivanov [13] and Korkmaz [15] that all automorphisms of the complex of curves on S are induced by diffeomorphisms of the surface S,

with a few exception. Another proof of this result was also obtained by Luo [17]. As a consequence of this result, it was proved that any isomorphism between two subgroups of finite index in the mapping class group of S is given by the conjugation with a mapping class (cf. [13], [15]). Ivanov also gave another proof of the fact that the isometries of the Teichmüller space are induced by diffeomorphisms of S.

Schaller [20] showed that the extended mapping class group of a hyperbolic surface is isomorphic to the automorphism group of the graph. The set of vertices of the graph is the set of nonseparating simple closed geodesic and the edges consisting of pairs of those nonseparating curves satisfying the property that the two curves in each pair intersect exactly once. Margalit [18] proved that the automorphism group of the pants complex is isomorphic to the extended mapping class group. Irmak [8, 9, 10] defined a superinjective simplicial map and showed that a superinjective simplicial map of the complex of curves and the complex of nonseparating curves is induced by a homeomorphism of an orientable surface. Irmak and Korkmaz [11] proved that the automorphism group of the Hatcher-Thurston complex is isomorphic to the extended mapping class group modulo its center. Brendle and Margalit [3] showed that any injection of a finite index subgroup of $\mathcal{K}$, generated by the Dehn twists about separating curves, into the Torelli group $\mathcal{I}$ of a closed orientable surface is induced by a homeomorphism, conforming a conjecture of Farb that $Comm(\mathcal{K}) \cong \mathrm{Aut}(\mathcal{K}) \cong \mathcal{M}_S$. Behrstock and Margalit [1] proved that for a torus with at least 3 punctures or a surface of genus 2 with at most 1 puncture, every injection of a finite index subgroup of the extended mapping class group into the extended mapping class group is the restriction of an inner automorphism.

All the above results are about orientable surfaces. For nonorientable surfaces

we prove that the automorphism group of the complex of curves of a nonorientable surface of odd genus is isomorphic to its mapping class group. First, we show that the natural group homomorphism is injective. Second, we prove that the natural group homomorphism is surjective for the punctured projective plane using the results and the ideas of Korkmaz's analogous work on the punctured sphere [15] and a result by Scharlemann [21]. For higher genus, we use induction and some ideas contained in Irmak's analogous work [9].

CHAPTER 2

PRELIMINARIES AND NOTATIONS

2.1 Introduction

Let N be a connected nonorientable surface of genus g with n marked points. We call these marked points punctures. Recall that the genus of a nonorientable surface is the maximum number of projective planes in a connected sum decomposition.

2.1.1 Circles and arcs

If a is a circle on N, by which we mean a simple closed curve, then according to whether a regular neighborhood of a is an annulus or a Mobius strip, we call a two-sided or one-sided simple closed curve, respectively.

We say that a circle is nontrivial if it bounds neither a disc nor annulus together with a boundary component, nor a disc with one puncture, nor a Mobius band on N.

If a is a circle, then we denote by N_a the surface obtained by cutting N along a. A circle a is called nonseparating if N_a is connected and separating otherwise. If a is separating, then N_a has two connected components. If a is separating and if one of the components of N_a is a disc with k punctures, then we say that a is a k-separating circle.

We denote circles by the lowercase letters a, b, c and their isotopy classes by α, β, γ. An embedded arc connecting a puncture to itself or two different punctures will be denoted by a', b', c' and their isotopy classes by α', β', γ'.

Let α be the isotopy class of a circle a. We say that α is nonseparating (respectively separating) if a is nonseparating (respectively separating). Similarly, we say that α is one-sided, two-sided or k-separating vertex if a is one-sided, two-sided or k-separating circle, respectively.

The geometric intersection number $i(\alpha, \beta)$ of two isotopy classes α and β is defined to be the infimum of the cardinality of $a \cap b$ with $a \in \alpha$, $b \in \beta$. The geometric intersection numbers $i(\alpha, \beta')$ and $i(\alpha', \beta')$ are defined similarly.

The following lemma is proved in [5].

Lemma 2.1. *Let S be a sphere with 3 punctures. Then*

   (i) *up to isotopy, there exists a unique nontrivial embedded arc joining a puncture P to itself, or P to another puncture Q,*

  (ii) *all circles on S are trivial.*

2.1.2   The complex of curves

An abstract simplicial complex is defined as follows (cf. [19]): Let V be a nonempty set. An abstract simplicial complex K with vertices V is a collection of nonempty finite subsets of V, called simplices, such that if $v \in V$, then $\{v\} \in K$, and if $\sigma \in K$ and $\sigma' \subset \sigma$ is a nonempty subset of V, then $\sigma' \in K$. The dimension $\dim \sigma$ of a simplex σ is $\operatorname{card} \sigma - 1$, where $\operatorname{card} \sigma$ is the cardinality of σ. A simplex σ is called a q-simplex if $\dim \sigma = q$. The supremum of the dimension of the simplices of K is called the dimension of K.

A subcomplex L of an absract simplicial complex K is called a full subcomplex if whenever a set of vertices of L is a simplex in K, it is also a simplex in L.

The complex of curves $C(S)$ on an orientable surface S is the abstract simplicial complex whose vertices are the isotopy classes of nontrivial simple closed curves. Similarly, the complex of curves $C(N)$ on a nonorientable surface N is the abstract simplicial complex whose vertices are the isotopy classes of nontrivial simple closed curves. In this complex of curves, we take one-sided vertices as well as two-sided vertices. Clearly, the complex of curves of a surface of genus g with n punctures and with b boundary components, and that of a surface of genus g with $n + b$ punctures are isomorphic. Therefore, sometimes we regard boundary components and the punctures the same.

Two distinct vertices $\alpha, \beta \in C(N)$ are joined by an edge if and only if their geometric intersection number is zero. More generally, a set of vertices $\{v_0, v_1, \ldots, v_q\}$ forms a q-simplex if and only if $i(v_j, v_k) = 0$ for all $0 \leqslant j, k \leqslant q$.

Dimension

Clearly, the dimension of $C(N)$ is $n - 2$ if N is a projective plane with n punctures. If S is a sphere with n punctures, then the dimension of $C(S)$ is $n - 4$. For higher genus, if N is a connected nonorientable surface of genus $g \geqslant 2$ with n punctures such that the Euler characteristic of N is negative and if $g = 2r + 1$, then the dimension of $C(N)$ is $4r + n - 2$ and if $g = 2r + 2$, then the dimension of $C(N)$ is $4r + n$ (see Section 2.1.5). If S is a connected orientable surface of genus g with n punctures such that $2g + n \geqslant 4$, then the dimension of $C(S)$ is $3g + n - 4$.

Links and dual links

Let α be a vertex in the complex of curves. We define the link $L(\alpha)$ of α to be the full subcomplex of the complex of curves whose vertices are those of the complex of curves which are joined to α by an edge in the complex of curves. The dual link $L^d(\alpha)$ of α is the graph whose vertices are those of $L(\alpha)$ such that two vertices of $L^d(\alpha)$ are joined by edge if and only if they are not joined by an edge in the complex of curves (or in $L(\alpha)$).

Pentagons

A pentagon is an ordered five-tuple $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, defined up to cyclic permutations and inversion, of vertices of the complex of curves such that $i(\gamma_i, \gamma_{i+1}) = 0$ for $i = 1, 2, \ldots, 5$ and $i(\gamma_i, \gamma_j) \neq 0$ otherwise.

2.1.3   Curve complexes of low dimensions

Obviously, if S is a sphere with $\leqslant 3$, then there are no nontrivial circles on S. Therefore, $C(S)$ is empty. If S is a sphere with four punctures, then $C(S)$ is infinite discrete. If N is a projective plane or a projective plane with one puncture then $C(N)$ consists of a unique vertex. If N is a projective plane with two punctures, then $C(N)$ is finite (cf. [21]). It consists of two vertices, the isotopy classes of the circles c_1 and c_2 of the Figure 2.1.

2.1.4   The arc complex $B(N)$

We now define another abstract simplicial complex $B(N)$ as follows. The vertices of $B(N)$ are the isotopy classes of nontrivial embedded arcs on N connecting

Figure 2.1: The isotopy classes of circles on twice-punctured $\mathbb{R}P^2$

two punctures. A set of vertices of $B(N)$ forms a simplex if and only if these vertices have pairwise disjoint representatives.

Arcs and 2-separating circles

If a is 2-separating circle on N, there exists up to isotopy a unique nontrivial embedded arc a' on the twice-punctured disc component of N_a joining two punctures by Lemma 1.1 in [15] and in [5]. On the other hand, an arc a' connecting two different punctures of N determines uniquely a 2-separating circle up to isotopy, that is, the boundary of a regular neighborhood of the arc. So, we have a one-to-one correspondence between the set of 2-separating isotopy classes and the set of isotopy classes of embedded arcs connecting two different punctures.

Example

If N is a projective plane with one puncture, then $B(N)$ consists of a unique vertex. Scharlemann studied the arc complex of a twice-punctured projective plane. He showed that the vertices of the arc complex consist of the isotopy classes of arcs shown in the Figure 2.2.
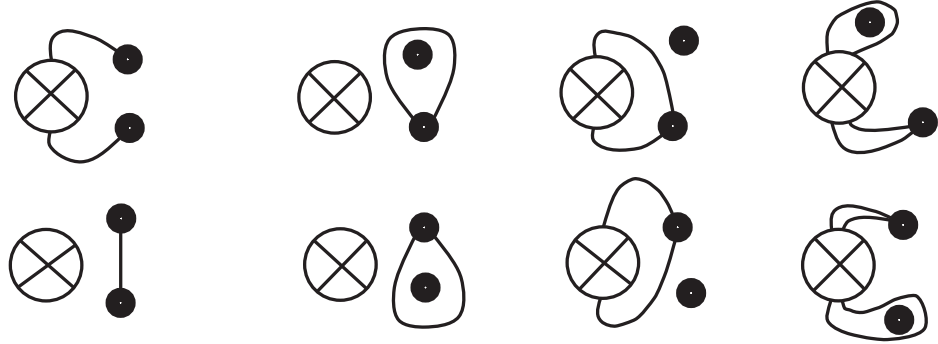
Figure 2.2: The isotopy classes of arcs on twice-punctured $\mathbb{R}P^2$

Simple pairs and chains

If a and b are two 2-separating circles, and α, β are their isotopy classes, such that the corresponding arcs a' and b' can be chosen disjoint with exactly one common endpoint P, then we say that a and b constitute a simple pair of circles and denote it by $\langle a; b \rangle$ (see Figure 2.3(a)). Similarly, we say that $\langle a'; b' \rangle$ is a simple pair of arcs. We also say $\langle \alpha; \beta \rangle$ and $\langle \alpha'; \beta' \rangle$ simple pairs.
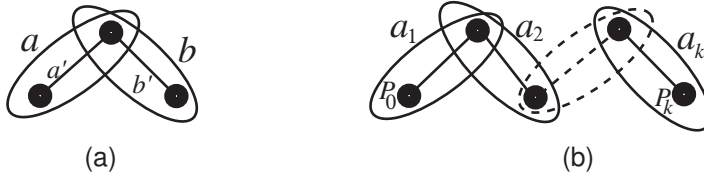
Figure 2.3: A simple pair and a chain

Let $a'_1, a'_2, \ldots, a'_k$ be embedded pairwise disjoint arcs, P_i and P_{i+1} the endpoints of a'_i, with $P_i \neq P_j$ for $i \neq j$, $0 \leqslant i, j \leqslant k$. Therefore, $\langle a'_i; a'_{i+1} \rangle$ is a simple pair of arcs for each $1 \leqslant i \leqslant k - 1$. Let $a_1, a_2, \ldots, a_k$ be the corresponding circles. We say that $\langle a'_1; a'_2; \ldots; a'_k \rangle$ is a chain of arcs (see Figure 2.3(b)). Similarly,

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$\langle a_1; a_2; \ldots; a_k \rangle$ is a chain of circles.

2.1.5 Maximal simplices in the curve complex

We recall that the maximum number of disjoint pairwise nonisotopic nontrivial circles on a connected orientable surface S of genus r with n boundary components is $3r - 3 + n$, whenever the Euler characteristic $\chi(S)$ of S is negative.

Lemma 2.2. *Let S be a connected orientable surface of genus g with n punctures. Suppose that $2g + n \geqslant 4$. Then all maximal simplices in $C(S)$ have the same dimension $3g + n - 4$.*

Lemma 2.3. *Let N be a real projective plane with $n \geqslant 2$ punctures. All maximal simplices in $C(N)$ have the same dimension $n - 2$.*

Proof. Let $n = 2$. The complex $C(N)$ consists of only two vertices, hence all simplices are of dimension 0.

Let $n \geqslant 3$. We consider a maximal simplex σ of dimension ℓ. Hence, σ contains $\ell + 1$ elements only one of which is a one-sided vertex. By cutting N along this one-sided simple closed curve, we get sphere with $n + 1$ punctures. By Lemma 2.2, all maximal simplices in the complex of curves on the sphere with $n + 1$ punctures have the same dimension $n - 3$. It follows that $\ell = n - 2$. $\quad\square$

Proposition 2.1. *Let N be a connected nonorientable surface of genus $g \geqslant 2$ with n punctures such that the Euler characteristic of N is negative. Let $a_r = 3r + n - 2$ and $b_r = 4r + n - 2$ if $g = 2r + 1$, and $a_r = 3r + n - 3$ and $b_r = 4r + n - 4$ if $g = 2r$. Then there is a maximal simplex of dimension ℓ in $C(N)$ if and only if $a_r \leqslant \ell \leqslant b_r$.*

Proof. For each integer ℓ satisfying $a_r \leqslant \ell \leqslant b_r$, the maximal simplicies are shown in Figure 2.4 and Figure 2.5 for closed nonorientable surface of odd genus. One can draw similar figures for nonorientable surface of odd genus with punctures and nonorientable surface of even genus. Moreover, Figure 2.6 helps to see the maximal simplex of dimension ℓ between a_r and b_r.
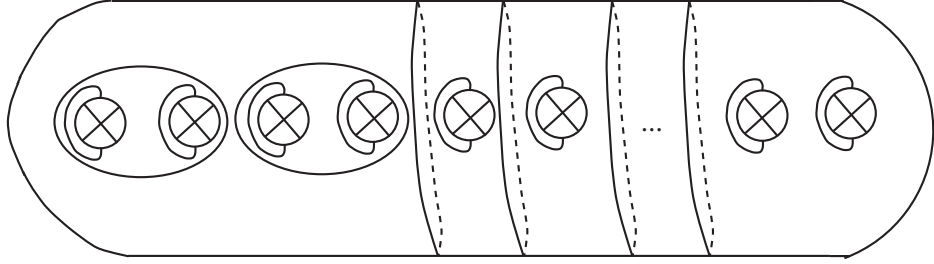
Figure 2.4: maximum number of disjoint pairwise nonisotopic nontrivial circles
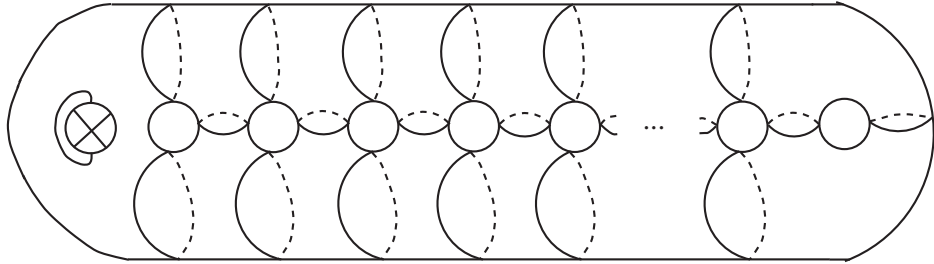
Figure 2.5: maximum number of disjoint pairwise nonisotopic nontrivial circles

We now prove the converse. Let us consider a maximal simplex σ of dimension ℓ. Hence, σ contains $\ell + 1$ elements. Choose pairwise disjoint simple closed curves representing elements of σ, and let N_σ denote the surface obtained by cutting N along these simple closed curves.

Suppose that the number of one-sided simple closed curves is m, so that we have $\ell + 1 - m$ two-sided elements in σ. The surface N_σ is a disjoint union of k

Figure 2.6:

pair of pants for some positive integer k. By the Euler characteristic argument, it can be seen that $k = g + n - 2$. The number of boundary components and punctures on N_σ is $3k$. By counting the the contribution of one-sided curves and two-sided curves to the boundary of N_σ, one can easily see that

$$3k = n + m + 2(\ell + 1 - m). \tag{2.1}$$

Suppose first that $g = 2r + 1$. In this case $1 \leq m \leq 2r + 1$. From the equality (2.1), it is easy to see that m must be odd and $\ell = 3r + n - 2 + \frac{m-1}{2}$.

Suppose now that $g = 2r$. In this case $1 \leq m \leq 2r$. From the equality (2.1), it is easy to see that m must be even and $\ell = 3r + n - 4 + \frac{m}{2}$.

The proposition follows from these. $\qquad\square$

2.1.6 Centralizer of certain subgroups

Let $\mathcal{T}'$ be the subgroup of mapping class group of N such that $\mathcal{T}'$ is generated by the Dehn twist of two-sided nonseperating circles as below shown in Figure 2.7.

Proposition 2.2. *Let N be a connected nonorientable surface of odd genus g and $g + n \geqslant 5$. Suppose that C is a collection of two-sided nonseparating circles in Figure 2.7 such that $\mathcal{T}'$ is generated by the Dehn twist t_{c_i} along the circles c_i of C. Then the centralizer $\mathcal{C}_{\mathcal{M}_N}(\mathcal{T}')$ is trivial.*
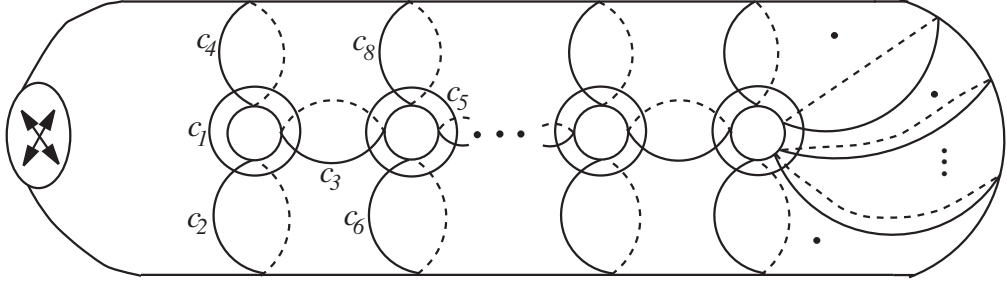
Figure 2.7:

Proof. Let $[f] \in \mathcal{C}_{\mathcal{M}_N}(\mathcal{T}')$. Unoriented circles will be denoted by $\bar{c}$. Then, $f(\bar{c}_i) = \bar{c}_i$ for $c_i \in C$. Let a be a one-sided circle such that N_a is an orientable surface. We observe that $f(\bar{a}) = \bar{a}$. Because there is one isotopy class of one-sided circle which does not intersect circles c_i in C. Therefore, by cutting N along a, we get diffeomorphism $f_| : N_a \to N_a$. Moreover, $f_|$ is orientation preserving. To see this, assume that c_j is dual to c_i, we know that $f_|$ preserves the orientation of a regular neighborhood of c_i if and only if it preserves the orientation of c_j. Recall that tubular neighborhood of $c_i \cup c_j$ is a torus with one boundary component. Since the product of the orientations of c_i and c_j gives the orientation of the torus, $f_|$ preserves the orientation of tubular neighborhood. Using this argument for these circles in Figure 2.7, we deduce that $f_|$ preserves the orientation of the surface N_a. Since $\mathcal{C}_{\mathcal{M}_{N_a}}(\mathcal{P}M_{N_a}) = \{1\}$ in [14], $f_|$ is isotopic to identity. Since $f(a) = a$, we obtain that f is isotopic to identity on N. Hence, $\mathcal{C}_{\mathcal{M}_N}(\mathcal{T}') = \{1\}$. $\square$

CHAPTER 3

INJECTIVITY OF $\mathcal{M}_N \to \mathrm{Aut}\, C(N)$

We first show that the natural map $\mathcal{M}_N \to \mathrm{Aut}\, C(N)$ is injective.

Theorem 3.1. *Let N be a connected nonorientable surface of genus g with n punctures. Suppose that $g + n \geqslant 6$. Then the natural map $\mathcal{M}_N \to \mathrm{Aut}\, C(N)$ is injective.*

Proof. Suppose that $[f] \in \mathcal{M}_N$ acts trivially on $C(N)$. Then $f : N \to N$, $f_* : C(N) \to C(N)$, $f_* = id$; i.e., $f_*(\nu) = \nu$ for all $\nu \in C(N)$. First, suppose that $g = 2r + 1$ is odd. Let a be a one-sided circle such that N_a is orientable surface of genus r with $n + 1$ boundary components. We denote by α the isotopy class of a. Then $f_*(\alpha) = \alpha$. This implies $f(a)$ is isotopic to a. Then there exists a diffeomorphism $g \simeq id$, $g(f(a)) = a$. Let $h = g \circ f$. We observe that $h(a) = a$ and $h_* = g_* \circ f_* = id \circ id = id$. Therefore, we have a diffeomorphism $h_| : N_a \longrightarrow N_a$ such that $(h_|)_* = id$. Using Ivanov's Theorem in [13] (for $r = 0$ and 1, using Theorem 1 in [15]), we get $h_| \simeq id$. Since $h(a) = a$, we see that $h \simeq id$ on N. In other words, it descends to a diffeomorphism of N. So, we have $g \circ f \simeq id$. Since $g \simeq id$, we get $f \simeq id$. Hence, the natural map $\mathcal{M}_N \to \mathrm{Aut}\, C(N)$ is injective.

Now, suppose that $g = 2r + 2$. Let b be a nonseparating two-sided circle such that N_b is orientable surface of genus r with $n + 2$ boundary components. Let β be the isotopy class of b. Then $f_*(\beta) = \beta$. This implies $f(b)$ is isotopic to b. Then there exists a diffeomorphism $g \simeq id$, $g(f(b)) = b$. Let $h = g \circ f$.

We see that $h(b) = b$ and $h_* = g_* \circ f_* = id \circ id = id$. Therefore, we have a diffeomorphism $h_| : N_b \longrightarrow N_b$ such that $(h_|)_* = id$. Using Ivanov's Theorem in [13] (for $r = 0$ and 1, using Theorem 1 in [15]), we get $h_| \simeq id$. Since $h(b) = b$, $h \simeq id$ on N. In other words, it descends to a diffeomorphism of N. Therefore, we obtain that $g \circ f \simeq id$. Since $g \simeq id$, we get $f \simeq id$. Hence, the natural map $\mathcal{M}_N \to \operatorname{Aut} C(N)$ is injective. $\qquad\square$

CHAPTER 4

SURJECTIVITY OF $\mathcal{M}_N \to \operatorname{AUT} C(N)$

4.1 Punctured $\mathbb{R}P^2$

Throughout this section unless otherwise stated, N will denote a real projective plane with $n \geqslant 5$ punctures. We need at least 5 punctures for the proof of the Lemma 4.2.

In this section, we prove that the natural homomorphism $\mathcal{M}_N \to \operatorname{Aut} C(N)$ is surjective. Hence it will be an isomorphism. For this, we first prove that automorphisms of $C(N)$ preserve the topological type of the vertices of $C(N)$ and that certain pairs of vertices of $C(N)$ can be realized in the complex of curves. We conclude that automorphisms of $C(N)$ preserve these pairs of vertices. Next, we show that every automorphism of $C(N)$ induces an automorphism of the complex $B(N)$ in a natural way. The automorphisms of $B(N)$ are determined by their action on a maximal simplex. Then, we use the relation between maximal simplices of $B(N)$ and isotopy classes of ideal triangulations of N and in conclusion, we show that an automorphism of $B(N)$ induced by some automorphism of $C(N)$ agrees with a mapping class.

We remind that if N is a projective plane with $n \geqslant 2$ punctures, then up to diffeomorphism there is only one nonseparating one-sided circle and also there is no nonseparating two-sided circle. The other circles are k-separating for some k.

Lemma 4.1. *Let $n \geqslant 2$ and $k \geqslant 4$. If N is a projective plane with n punctures and S is a sphere with k punctures, then $C(N)$ and $C(S)$ are not isomorphic.*

Proof. The complexes $C(N)$ and $C(S)$ have dimensions $n - 2$ and $k - 4$, respectively. If $k \neq n + 2$, since these complexes of curves have different dimensions, $C(N)$ and $C(S)$ are not isomorphic. If $k = n + 2$, we proceed as follows.

Let $n = 2$. Then $C(N)$ is finite (see [21]), however, $C(S)$ is infinite discrete since S is a sphere with 4 punctures. Therefore, they are not isomorphic.

Now, assume that $n \geqslant 3$ and $C(N)$ and $C(S)$ are not isomorphic when N has $n - 1$ punctures. We need to show that these complexes are not isomorphic if N has n punctures. Assume that there is an isomorphism $\varphi : C(N) \to C(S)$. Note that for a vertex γ of $C(N)$, the dual link $L^d(\gamma)$ of γ is connected if and only if γ is either one-sided or 2-separating. For a vertex δ of $C(S)$, the dual link $L^d(\delta)$ of δ is connected if and only if δ is 2-separating. From this, it follows that the image of the union of the set of one-sided vertices and the set of 2-separating vertices of $C(N)$ is precisely the set of 2-separating vertices of $C(S)$. Let γ be a 2-separating vertex of $C(N)$. Then φ takes γ to a 2-separating vertex δ of $C(S)$ and induces an isomorphism $L(\gamma) \to L(\delta)$.

Clearly, $L(\gamma)$ is isomorphic to the complex of curves of a real projective plane with $n - 1$ punctures and $L(\delta)$ is isomorphic to the complex of curves of a sphere with $n + 1$ punctures. By assumption, these complexes are not isomorphic. We get a contradiction. Hence, $C(N)$ and $C(S)$ are not isomorphic. $\square$

Theorem 4.1. *The group $\operatorname{Aut} C(N)$ preserves the topological type of the vertices of $C(N)$.*

Proof. Let φ be an automorphism of $C(N)$. Note that for a vertex γ of $C(N)$, the dual link $L^d(\gamma)$ of γ is connected if and only if γ is either one-sided vertex or 2-separating. Therefore, φ cannot take a one-sided vertex or a 2-separating to a k-separating vertex with $k > 2$.

Assume that $\varphi(\alpha) = \beta$ is a 2-separating vertex for some one-sided vertex α. Let $a \in \alpha$ be a circle. Then N_a is disc with n punctures. Let $b \in \beta$. Then N_b is homeomorphic to $\mathbb{R}P^2$ with $n - 2$ punctures and with one boundary component. Clearly, $L(\alpha)$ is isomorphic to $C(N_a)$. Similarly, $L(\beta)$ is isomorphic to $C(N_b)$. Since the complexes $C(N_a)$ and $C(N_b)$ are not isomorphic by Lemma 4.1, β cannot be 2-separating. It also follows from this that φ maps a 2-separating vertex to a 2-separating vertex.

Let γ be a k-separating vertex for some $3 \leqslant k \leqslant n - 1$. Then $\varphi(\gamma) = \delta$ is an l-separating vertex for some $3 \leqslant l \leqslant n - 1$. We must show that $k = l$. Let $c \in \gamma$ and $d \in \delta$ be circles. N_c and N_d have two connected components. Let $N_c = N_0 \sqcup N_1$ and $N_d = N_0' \sqcup N_1'$ such that N_i and N_i' have genera i. The dual link $L^d(\gamma)$ has exactly two connected components. Let us denote these components by $L_0^d(\gamma)$ and $L_1^d(\gamma)$; $L^d(\gamma) = L_0^d(\gamma) \sqcup L_1^d(\gamma)$. We name $L_i(\gamma)$ so that the vertices of $L_i^d(\gamma)$ are the isotopy classes of circles on N_i. Let $L_i(\gamma)$ be the full subcomplex of $C(N)$ with vertices $L_i^d(\gamma)$. It follows that $(L_i(\gamma))^d = L_i^d(\gamma)$. Then $L_i(\gamma)$ is isomorphic to $C(N_i)$. Clearly, the dimension of $C(N_0)$ is $k - 3$ since N_0 is a sphere with punctures and the dimension of $C(N_1)$ is $n - k - 2$. Similarly, we define $L_i^d(\delta)$ and $L_i(\delta)$ so that $L_i(\delta)$ is isomorphic to $C(N_i')$. If $\varphi(\gamma) = \delta$, then φ restricts to an isomorphism $L(\gamma) \to L(\delta)$, which induces an isomorphism $L^d(\gamma) \to L^d(\delta)$. Then either $\varphi(L_0^d(\gamma)) = L_0^d(\delta)$ or $\varphi(L_0^d(\gamma)) = L_1^d(\delta)$. Hence, $\varphi(L_0(\gamma)) = L_0(\delta)$ or $\varphi(L_0(\gamma)) = L_1(\delta)$. However, $\varphi(L_0(\gamma)) = L_1(\delta)$ is not possible by Lemma 4.1. Therefore, $\varphi(L_0(\gamma)) = L_0(\delta)$. It follows that their

dimensions are equal: $k - 3 = l - 3$. Hence, $k = l$.

The proof of the theorem is now complete. $\qquad\square$

Theorem 4.2 (Korkmaz). *Let α and β be two 2-separating vertices of $C(S)$. Then $\langle \alpha; \beta \rangle$ is a simple pair if and only if there exist vertices $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{n-2}$ of $C(S)$ satisfying the following conditions.*

(i) $(\gamma_1, \gamma_2, \alpha, \gamma_3, \beta)$ *is a pentagon in $C(S)$,*

(ii) γ_1 *and γ_{n-2} are 2-separating, γ_2 is 3-separating, and γ_k and γ_{n-k} are k-separating for $3 \leqslant k \leqslant n/2$,*

(iii) $\{\alpha, \gamma_3\} \cup \sigma$, $\{\alpha, \gamma_2\} \cup \sigma$, $\{\beta, \gamma_3\} \cup \sigma$ *and $\{\gamma_1, \gamma_2\} \cup \sigma$ are codimension-zero simplices, where $\sigma = \{\gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$.*

Theorem 4.3. *Let α and β be two 2-separating vertices of $C(N)$. Then $\langle \alpha; \beta \rangle$ is a simple pair if and only if there exist vertices $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{n-1}, \delta$ of $C(N)$ satisfying the following conditions.*

(i) $(\gamma_1, \gamma_2, \alpha, \gamma_3, \beta)$ *is a pentagon in $C(N)$,*

(ii) γ_1 *is 2-separating, γ_2 is 3-separating, and γ_k is k-separating for $3 \leqslant k \leqslant n - 1$, δ is one-sided,*

(iii) $\{\alpha, \gamma_3\} \cup \sigma \cup \{\delta\}$, $\{\alpha, \gamma_2\} \cup \sigma \cup \{\delta\}$, $\{\beta, \gamma_3\} \cup \sigma \cup \{\delta\}$ *and $\{\gamma_1, \gamma_2\} \cup \sigma \cup \{\delta\}$ are codimension-zero simplices, where $\sigma = \{\gamma_4, \gamma_5, \ldots, \gamma_{n-1}\}$.*

Proof. Suppose that $\langle \alpha; \beta \rangle$ is a simple pair. Let $a \in \alpha$ and $b \in \beta$ so that $\langle a; b \rangle$ is a simple pair. It is clear that any two simple pairs of circles are topologically equivalent, that is; if $\langle c; d \rangle$ is any other simple pair, then there exists a diffeomorphism $F : N \to N$ such that $\langle F(c); F(d) \rangle = \langle a; b \rangle$. So, we can assume that a and

b are the circles illustrated in Figure 4.1. In the figure, we think of the sphere as the one point compactification of the plane and the cross inside the circle means that we delete open disc and identify the antipodal boundary points so that we get a real projective plane with punctures. The isotopy classes γ_i of the circles c_i and the isotopy class δ of the one-sided circle d satisfy (i)-(iii).
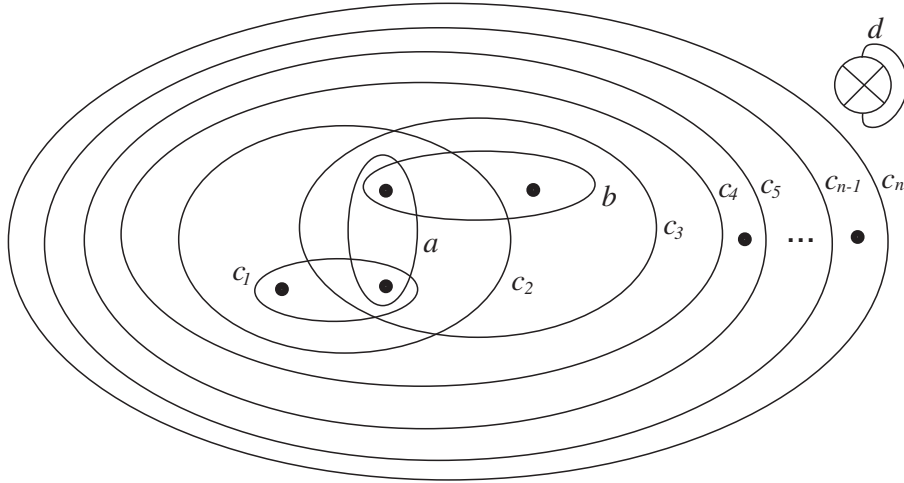
Figure 4.1:

Now, we prove the converse. Assume that conditions (i)-(iii) above hold. Let $d \in \delta$ be a one-sided circle. Deleting $\{\delta\}$ from conditions (ii) and (iii), we have codimension-one simplices $\{\alpha, \gamma_3\} \cup \sigma$, $\{\alpha, \gamma_2\} \cup \sigma$, $\{\beta, \gamma_3\} \cup \sigma$ and $\{\gamma_1, \gamma_2\} \cup \sigma$. However, these simplicies are codimension-zero simplices in the complex $C(N_d)$. By Theorem 4.2, we see that $\langle \alpha; \beta \rangle$ is a simple pair on the sphere and N_d is a sphere with $n + 1 \geqslant 6$ punctures. Let us say that a puncture is inside a if it is one of the two punctures on the disc bounded by a. Similarly for b. There are three possibilities the boundary component d' (we see it as a puncture from point of view of the curve complex) resulting from cutting along d, as illustrated in

Figure 4.2. In the figure, this boundary component d' is drawn as an oval. The case are: (1) it may be outside of both a and b, (2) it may be inside, say, a and outside of b, or (3) it may be the unique puncture inside both a and b.
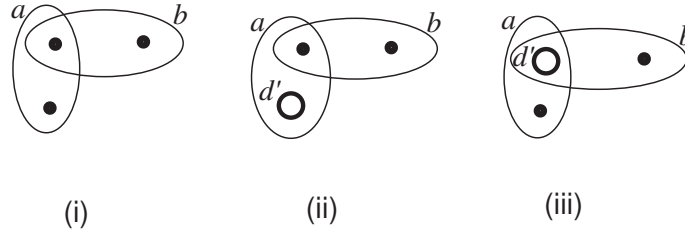
Figure 4.2:

After the identifying the antipodal boundary points of the oval, in order to get both a and b as 2-separating circles, we must have the case in Figure 4.2(i). Hence, we see that $\langle \alpha; \beta \rangle$ is a simple pair on N as well. □

Corollary 4.1. *Let φ be an automorphism of $C(N)$. If $\langle \alpha; \beta \rangle$ is a simple pair, so is $\langle \varphi(\alpha); \varphi(\beta) \rangle$. Similarly, the image of a chain in $C(N)$ under φ is also a chain.*

Proof. The conditions (i) and (iii) of Theorem 4.3 are invariant under the automorphisms of $C(N)$. By Theorem 4.1, the condition (ii) is also invariant under the automorphisms of $C(N)$. Corollary follows from these. □

Action of $\operatorname{Aut} C(N)$ on punctures

We can define an action of the group $\operatorname{Aut} C(N)$ on punctures of N as follows. For $\varphi \in \operatorname{Aut} C(N)$ and for a puncture P of N, choose two isotopy classes α', β' of embedded arcs such that $\langle \alpha'; \beta' \rangle$ is a simple pair with the common endpoint P.

Define $\varphi(P)$ to be the common endpoint of the simple pair $\langle \varphi(\alpha'); \varphi(\beta') \rangle$. We note that by the one-to-one correspondence between the set of 2-separating vertices of $C(N)$ and the set of those vertices of $B(N)$ which join different punctures, $\operatorname{Aut} C(N)$ has a well-defined action on the latter set.

Next two lemmas below can be proved in the same way as Lemma 3.5 and Lemma 3.6 in [15].

Lemma 4.2. *The definition of the action of* $\operatorname{Aut} C(N)$ *on the punctures of N is independent of the choice of the simple pair.*

Lemma 4.3. *Let $\varphi \in \operatorname{Aut} C(N)$, α be a k-separating vertex of $C(N)$ and $a \in \alpha$. If N'_a denotes the k-punctured disc component of N_a and N''_a denotes $(n-k)$-punctured $\mathbb{R}P^2$ with one boundary component, then $\varphi(\mathcal{P}(N'_a)) = \mathcal{P}(N'_{\varphi(a)})$ and $\varphi(\mathcal{P}(N''_a)) = \mathcal{P}(N''_{\varphi(a)})$.*

Action of $\operatorname{Aut} C(N)$ **on arcs**

We define an action of $\operatorname{Aut} C(N)$ on the vertices of $B(N)$ as follows. Let $\varphi \in \operatorname{Aut} C(N)$, α' a vertex of $B(N)$ and let $a' \in \alpha'$. If a' is joining two different punctures, then $\varphi(\alpha')$ is already defined by the correspondence between the 2-separating vertices of $C(N)$ and the action of $\operatorname{Aut} C(N)$ on $C(N)$. The other words, $\varphi(\alpha')$ is the isotopy class of the arc, which is unique up to isotopy, joining two punctures on the twice-punctured disc component of $N_{\varphi(a)}$ for $\varphi(a) \in \varphi(\alpha)$.

Suppose now that the arc a' joins a puncture P to itself and it is a two-sided loop. Then the action is same as in [15]. To be more precise, let a_1 and a_2 be the boundary components of a regular neighborhood of $a' \cup \{P\}$ and α_1 and α_2 be their isotopy classes, respectively. Since a' cannot be deformed to P, at most one of a_1 and a_2 is trivial.

If a_1 is trivial, then a_1 bounds either a disc with one puncture or a Möbius band. If it bounds a disc with a puncture Q, then a_2 bounds a disc with two punctures P and Q. By Theorem 4.1, $\varphi(\alpha_2)$ is 2-separating. So for a representative $\varphi(a_2)$ of $\varphi(\alpha_2)$, one of the components, say $N'_{\varphi(a_2)}$ of $N_{\varphi(a_2)}$ is a twice-punctured disc, with punctures $\varphi(P)$ and $\varphi(Q)$ by Lemma 4.3. Define $\varphi(\alpha')$ to be the isotopy class of a nontrivial simple arc on $N'_{\varphi(a_2)}$ joining $\varphi(P)$ to itself. Such an arc is unique up to isotopy by Lemma 2.1.

If a_1 bounds a Möbius band, then a_2 bounds a projective plane with one puncture P. By Theorem 4.1, $\varphi(\alpha_2)$ bounds a projective plane with one puncture $\varphi(P)$. Therefore, for a representative $\varphi(a_2)$ of $\varphi(\alpha_2)$, one of components, say $N'_{\varphi(a_2)}$, of $N_{\varphi(a_2)}$ is a projective plane with one puncture and one boundary component. The puncture on $N'_{\varphi(a_2)}$ is $\varphi(P)$ by Lemma 4.3. There is only one nontrivial two-sided loop (arc) joining a puncture $\varphi(P)$ to itself by Example 2.1.4. We define $\varphi(\alpha')$ to be the isotopy class of this two-sided loop (arc) joining $\varphi(P)$ to itself.

If neither of a_1 and a_2 is trivial, then a_1 and a_2 bound an annulus with a puncture P. We claim that $\varphi(a_1)$ and $\varphi(a_2)$ also bound a once-punctured annulus with only one puncture $\varphi(P)$. Here, $\varphi(a_i)$ is a representative of $\varphi(\alpha_i)$ for $i = 1, 2$. To see this, let N'_{a_i} be the subsurface of N bounded by a_i not containing the puncture P. Similarly we define $N'_{\varphi(a_i)}$ to be the component of $N_{\varphi(a_i)}$ not containing $\varphi(P)$. Now, assume that the set of punctures on N'_{a_1} and N'_{a_2} are $\mathcal{P}(N'_{a_1}) = \{P_1, \ldots, P_k\}$ and $\mathcal{P}(N'_{a_2}) = \{P_{k+1}, \ldots, P_{n-1}\}$, respectively. Then $P_i \neq P_j$ for all i, j. By Lemma 4.3, $\mathcal{P}(N'_{\varphi(a_1)}) = \{\varphi(P_1), \ldots, \varphi(P_k)\}$ and $\mathcal{P}(N'_{\varphi(a_2)}) = \{\varphi(Q_1), \ldots, \varphi(Q_{n-k})\}$. We deduce that since $\varphi(a_1)$ and $\varphi(a_2)$ are disjoint and nonisotopic, they must bound an annulus with only one puncture $\varphi(P)$. The class $\varphi(\alpha')$ is defined to be the isotopy class of the unique arc up to

isotopy on this annulus joining $\varphi(P)$ to itself.

Suppose finally that a' is a one-sided loop (arc) joining a puncture P to itself. Let a be the boundary component of a regular neighborhood of $a' \cup \{P\}$ and let α be the isotopy class of a. The circle a bounds a Möbius band with a puncture P. By Theorem 4.1, $\varphi(a) \in \varphi(\alpha)$ bounds a Möbius band M with a puncture, say Q. By Lemma 4.3, $Q = \varphi(P)$. By Example 2.1.4, there is up to isotopy a unique one-sided loop b' on M joining $\varphi(P)$ to itself. We define $\varphi(\alpha')$ to be the isotopy class of b'.

Lemma 4.4. *Let φ be an automorphism of $C(N)$ and α' and β' be two distinct vertices of $B(N)$ such that $i(\alpha', \beta') = 0$. Then $i(\varphi(\alpha'), \varphi(\beta')) = 0$. Therefore, every automorphism of $C(N)$ yields an automorphism of $B(N)$.*

Proof. Let a' and b' be two disjoint representatives of α' and β', respectively. There are thirteen possible cases as illustrated in Figure 4.3. In each figure, we assume that the arc on the left is a' and the other is b'.

If a' (respectively b') is joining two different punctures, we denote by α (respectively β) the 2-separating vertex of $C(N)$ corresponding to α' (respectively β'), and by a (respectively b) a representative of α(respectively β).

If a' (respectively b') is a two-sided arc joining a puncture P to itself, we denote by a_1 and a_2 (respectively b_1 and b_2) the boundary components of a regular neighborhood of $a' \cup \{P\}$ (respectively $b' \cup \{P\}$). If a regular neighborhood of $a' \cup \{P\}$ (respectively $b' \cup \{P\}$) is a once-punctured Möbius band M, we denote the boundary component of M by a_3 (respectively b_3). We also denote representatives of $\varphi(\alpha)$, $\varphi(\alpha')$, $\varphi(\alpha_1)$, $\varphi(\alpha_2)$, $\varphi(\alpha_3)$ by $\varphi(a)$, $\varphi(a')$, $\varphi(a_1)$, $\varphi(a_2)$, $\varphi(a_3)$, respectively. If, say, a_1 is trivial, then a_1 bounds either a disc with a puncture or a Möbius band. In the first case, we think of a_1 as the puncture its

bounds. In the second case, we think of a_1 as the core of the Möbius band it bounds. Aut $C(N)$ has a well defined action on the isotopy classes of these trivial simple closed curves.

We now examine each of the thirteen cases.
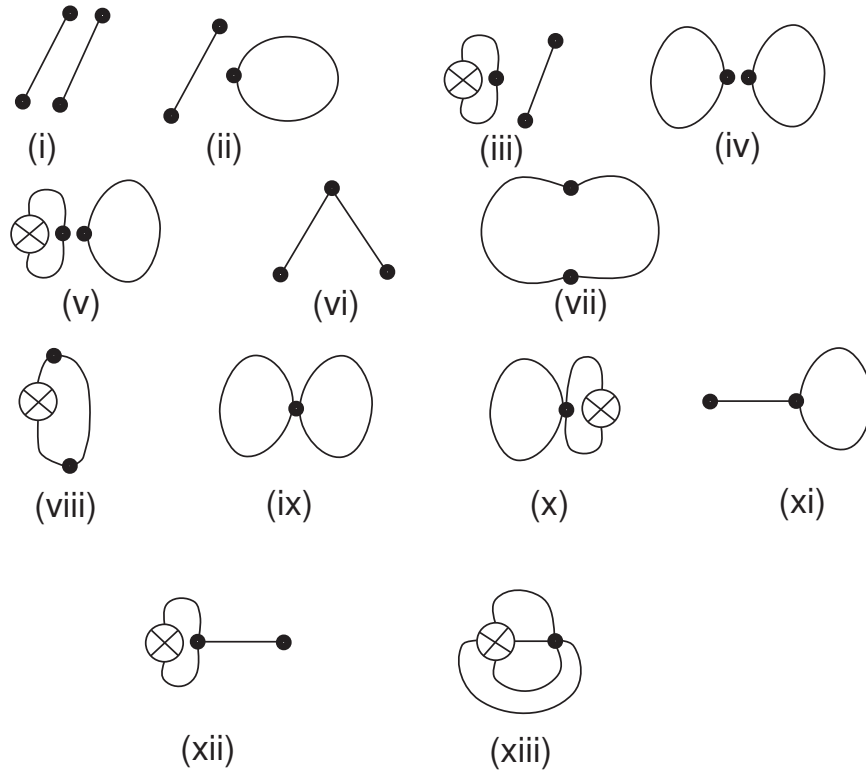
Figure 4.3:

The proof is similar to that of Lemma 3.7 in [15] for (i), (ii), (iv), (vii), (ix), (xi) and we will not repeat them here.

(iii) In this case, $\varphi(a_3)$ bounds a Möbius band M with a puncture and $\varphi(b)$ bounds a disc D with two punctures. Since φ is an automorphism, and since a_3 and b are disjoint and nonisotopic, $\varphi(a_3)$ and $\varphi(b)$ are disjoint and nonisotopic. Then M does not intersect D. Since $\varphi(a')$ is on M and $\varphi(b')$ is on D, it follows

that $\varphi(a')$ is disjoint from $\varphi(b')$.

(v) The once-punctured annulus bounded by b_1 and b_2 and the once-punctured Möbius band bounded by a_3 are disjoint. Since b_1, b_2 and a_3 are pairwise disjoint, so are $\varphi(b_1)$, $\varphi(b_2)$ and $\varphi(a_3)$. So the annulus A bounded by $\varphi(b_1)$, $\varphi(b_2)$, and the once-punctured Möbius band M bounded by $\varphi(a_3)$ are disjoint. Since $\varphi(a')$ is on M and $\varphi(b')$ is on A, they are disjoint.

(vi) This case follows from Corollary 4.1.

(viii) Suppose that a' and b' are joining the punctures P and Q. Let R be any other puncture and let c' be an arc from P to R disjoint from $a' \cup b'$. Let D denote a regular neighborhood of $b' \cup c' \cup \{P, Q, R\}$, so that D is a disc with three punctures. Let d' denote the unique arc on D joining Q to R such that d' does not intersect $a' \cup b' \cup c'$. Let e be the boundary of D. So any two arcs in the set $\{b', c', d'\}$ is a simple pair. Thus any two arcs in the set $\{\varphi(b'), \varphi(c'), \varphi(d')\}$ is a simple pair, and $\varphi(b')$, $\varphi(c')$ and $\varphi(d')$ are contained in the three punctured disk component of $\varphi(e)$. It follows that any arc disjoint from $\varphi(c')$ and $\varphi(d')$ is also disjoint from $\varphi(b')$. Since $\varphi(a')$ is disjoint from $\varphi(c')$ and $\varphi(d')$ by (vi), it is also disjoint from $\varphi(b')$.

(x) Let P be the common endpoint of two-sided loop a' and one-sided loop b', so that both arcs connect P to itself. By Theorem 4.1 and Lemma 4.3, $\varphi(a_1)$ and $\varphi(a_2)$ are boundaries of an annulus with one puncture $\varphi(P)$. Since $\varphi(b')$ is disjoint from $\varphi(a_2)$, $\varphi(b')$ is also disjoint from $\varphi(a')$.

(xii) Assume that a' is joining P to itself such that a' is a one-sided loop, and b' is connecting P to Q. Let $P_1, ..., P_{n-2}$ be the punctures other than P and Q. Choose a chain $\langle c'_1, ..., c'_{n-2} \rangle$ such that c'_i joins P_{i-1} to P_i for $1 \leqslant i \leqslant n-2$, where $P_0 = Q$. We consider a two-sided loop d' joining P to P disjoint from a', b' and $\langle c'_1, ..., c'_{n-2} \rangle$ such that one of the components of the complement of d' is a

Möbius band and the other is a disc with punctures $Q, P_1, P_2, ..., P_{n-2}$. By (vi), $\langle \varphi(c_1'), ..., \varphi(c_{n-2}') \rangle$ is also chain and disjoint from $\varphi(b')$. By (xi), $\varphi(d')$ is disjoint from $\varphi(b')$ and the chain $\langle \varphi(c_1'), ..., \varphi(c_{n-2}') \rangle$. Note that one of the components of the complement of $\varphi(d')$ contains $\varphi(b')$ and the chain $\langle \varphi(c_1'), ..., \varphi(c_{n-2}') \rangle$. Since $\varphi(a')$ is disjoint from $\varphi(d')$ by (x), we obtain that $\varphi(a')$ is also disjoint from $\varphi(b')$.

(xiii) Let P be the common endpoint one-sided loops a' and b'. The complement of a regular neighborhood of $a' \cup b'$ is the union of two discs D_1 and D_2 with $n - k - 1$ and k punctures for some k with $1 \leqslant k \leqslant n - 2$. Let $b_1', ..., b_{n-k-2}'$ be a chain on D_1 so that each b_i' is disjoint from $a' \cup b'$. Let $P_1, ..., P_k$ be the punctures on D_2. We can choose pairwise disjoint arcs c_i' connecting P_{i-1} and P_i such that each c_i' is also disjoint from $a' \cup b' \cup \partial D_1$, where $P_0 = P_{k+1} = P$ and $1 \leqslant i \leqslant k+1$.

It follows that a regular neighborhood of $c_1' \cup \cdots \cup c_{k+1}'$ is a Möbius band with $k + 1$ punctures. Then a regular neighborhood of $\varphi(c_1') \cup \cdots \cup \varphi(c_{k+1}')$ is also a Möbius band with $k + 1$ punctures.

Now the surface obtained from N by cutting along $\varphi(c_1') \cup \cdots \cup \varphi(c_{k+1}')$ and $\varphi(\partial D_1)$ is an annulus A. The puncture $\varphi(P)$ gives rise to two punctures R_1 and R_2 on the same component of ∂A. The arcs $\varphi(a')$ and $\varphi(b')$ live on A. In order to get a one-sided arc, each must connect R_1 to R_2. Up to isotopy there are two arcs from R_1 and R_2 which are disjoint. These two arcs must be $\varphi(a')$ and $\varphi(b')$. $\square$

Lemma 4.5. *The natural map* $\operatorname{Aut} C(N) \to \operatorname{Aut} B(N)$ *is injective.*

Proof. From the lemma above, every element of $\operatorname{Aut} C(N)$ yields an element of $\operatorname{Aut} B(N)$. This gives a homomorphism. Now, we need to show that the kernel of this homomorphism is trivial. In other words, if an automorphism of $C(N)$ induces the identity automorphism of $B(N)$, then this automorphism must be

the identity.

Let φ be an automorphism of $C(N)$. Suppose that φ induces the identity automorphism of $B(N)$. We recall that there is a one-to-one correspondence between 2-separating vertices of $C(N)$ and the vertices of $B(N)$ joining different punctures. It follows that φ is the identity on 2-separating vertices of $C(N)$.

Let α be a one-sided vertex of $C(N)$ and let $a \in \alpha$. Let us denote by $P_1, \ldots, P_n$ the punctures on the connected component of N_a. Let us take any chain $\langle c'_1, \ldots, c'_{n-1} \rangle$ disjoint from a such that c'_i connects P_i to P_{i+1}. Let γ'_i be the isotopy class of c'_i. Since $i(\gamma_i, \alpha) = 0$, we have $i(\gamma_i, \varphi(\alpha)) = 0$ and hence $i(\gamma'_i, \varphi(\alpha)) = 0$. Let $C' = c'_1 \cup \cdots \cup c'_{n-1}$. The surface $N_{C'}$ obtained from N by cutting along C' is a projective plane with one boundary component. Up to isotopy, there is only one one-sided simple closed curve on $N_{C'}$. Both α and $\varphi(\alpha)$ are on $N_{C'}$. So, we must have $\varphi(\alpha) = \alpha$.

Let α be a k-separating vertex of $C(N)$ with $3 \leqslant k \leqslant n-1$ and let $a \in \alpha$. Let us denote by $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_{n-k}$ the punctures on the two connected components of N_a. Let us take any two chains $\langle b'_1, \ldots, b'_{k-1} \rangle$ and $\langle c'_1, \ldots, c'_{n-k-1} \rangle$ disjoint from a such that b'_i connects P_i to P_{i+1} and c'_j connects Q_j to Q_{j+1}. Let β'_i and γ'_j be the isotopy classes of b'_i and c'_j, respectively. Let d' be a one-sided loop joining to Q_1 to itself disjoint from the other arcs. Let δ' be the isotopy class of d'. Because $i(\beta_i, \alpha) = 0$, $i(\gamma_j, \alpha) = 0$ and $i(\delta, \alpha) = 0$, we have $i(\beta_i, \varphi(\alpha)) = i(\gamma_j, \varphi(\alpha)) = 0$ and $i(\delta, \varphi(\alpha)) = 0$. Then $i(\beta'_i, \varphi(\alpha)) = i(\gamma'_j, \varphi(\alpha)) = 0$ and $i(\delta', \varphi(\alpha)) = 0$. Let $A' = b'_1 \cup \cdots \cup b'_{k-1} \cup \delta' \cup c'_1 \cup \cdots \cup c'_{n-k-1}$. The surface $N_{A'}$ obtained from N by cutting along A' is an annulus. Since, up to isotopy, there is only one two-sided simple closed curve on $N_{A'}$, we must have $\varphi(\alpha) = \alpha$.

$\square$

Ideal triangulations of N and maximal simplices of $B(N)$

All maximal simplices in $B(N)$ have the same dimension, and there is a well-defined action of the group $\operatorname{Aut} B(N)$ on maximal simplices. Any realization of a maximal simplex is an ideal triangulation of N. An ideal triangulation is a triangulation of N whose vertex set is the set of punctures on N in the sense that vertices of a triangle can coincide as can a pair of edges. Note that isotopy class of any ideal triangulation forms a maximal simplex in $B(N)$. The converse of that is also true. So, $\operatorname{Aut} B(N)$ acts on the isotopy classes of ideal triangulations.

We quote the following definition from [15].

Definition 4.1. A good ideal triangle is a set $\{a', b', c'\}$ of nontrivial embedded disjoint arcs such that a', b' and c' connect P_1 to P_2, P_2 to P_3 and P_3 to P_1, respectively, for three different punctures P_1, P_2 and P_3, and such that $a' \cup b' \cup c'$ bounds a disc in N.

The following lemma can be proven similar to the proof Corollary in [7].

Lemma 4.6. *Let N be a projective plane with at least five punctures. Then given any two maximal simplices σ and σ' of $B(N)$, there exists a sequence of maximal simplices $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k = \sigma'$ such that $\sigma_i \cap \sigma_{i+1}$ is a codimension-one simplex for each i.*

The lemmas below are analogous to Lemma 3.9 and Lemma 3.10 in [15].

Lemma 4.7. *Let $\tilde{\varphi} \in \operatorname{Aut} B(N)$, $\triangle = \{a', b', c'\}$ be a good ideal triangle and let α', β', γ' be the isotopy classes of a', b', c', respectively. Then $\{\alpha', \beta', \gamma'\}$ and, hence $\{\tilde{\varphi}(\alpha'), \tilde{\varphi}(\beta'), \tilde{\varphi}(\gamma')\}$ is a 2-simplex in $B(N)$. If $\tilde{\varphi}(\triangle) = \{\tilde{\varphi}(a'), \tilde{\varphi}(b'), \tilde{\varphi}(c')\}$ is a realization of the latter simplex, then it is a good ideal triangle on N.*

Lemma 4.8. *Let $\tilde{\varphi}$ and $\tilde{\psi}$ be two automorphisms of $B(N)$. If they agree on a codimension-zero simplex, then they agree on all of $B(N)$.*

Proof of Theorem 1.1 for punctured $\mathbb{R}P^2$

In Section 3, we showed that the natural map $\mathcal{M}_N \to \operatorname{Aut} C(N)$ is injective. We show that this natural homomorphism is onto. Let $\varphi \in \operatorname{Aut} C(N)$ and let σ be the isotopy class of a good ideal triangulation of N. So, σ is a maximal simplex of $B(N)$. By Lemma 4.7, $\tilde{\varphi} \in \operatorname{Aut} B(N)$, the automorphism induced by φ, takes a good ideal triangle to a good ideal triangle and $\tilde{\varphi}$ can be realized by a homeomorphism. Also, because each edge of a good ideal triangulation is an edge of two good ideal triangles, the homeomorphism of these triangles gives a homeomorphism Φ of N. By replacing Φ by a diffeomorphism isotopic to Φ if necessary, we may assume that $\Phi : N \to N$ is a diffeomorphism. If $[\Phi]$ is the isotopy class of Φ, then $\tilde{\varphi}$ agrees with $\tilde{\Phi}_*$ the automorphism induced by Φ, on the maximal simplex σ of $B(N)$. From Lemma 4.8, they agree on all of $B(N)$. Thus, $\tilde{\varphi} = \tilde{\Phi}_*$. Since the map $\operatorname{Aut} C(N) \to \operatorname{Aut} B(N)$ is injective, we get $\varphi = \Phi_*$.

The proof of the theorem for punctured $\mathbb{R}P^2$ is now complete.

4.2   Surfaces of Higher Genus

Throughout this section unless otherwise stated, N will denote a connected nonorientable surface of genus g with n punctures where g is odd and $g + n \geqslant 6$.

In this section, we first show that automorphisms of $C(N)$ preserve the topological type of the vertices of $C(N)$. We then prove that every automorphism of

$C(N)$ is induced by a diffeomorphism of N. We prove this by induction on r, where $g = 2r + 1$.

Lemma 4.9. *Let φ be an automorphism of $C(N)$. Let α and β be nonseparating two-sided vertices. If $i(\alpha, \beta) = 1$, then $i(\varphi(\alpha), \varphi(\beta)) = 1$.*

Proof. Let γ_1, γ_2,..., γ_{2r-1} be pairwise disjoint one-sided vertices such that each γ_i is disjoint from α and β. Consider the link $L(\gamma_1, ..., \gamma_{2r-1})$ of these vertices. Then φ restricts to an automorphism $\varphi_| : L(\gamma_1, ..., \gamma_{2r-1}) \rightarrow L(\gamma_1, ..., \gamma_{2r-1})$. $L(\gamma_1, ..., \gamma_{2r-1})$ is isomorphic to the complex of curves of a torus with $n + 2r - 1$ punctures. Observe that α and β vertices are contained in $L(\gamma_1, ..., \gamma_{2r-1})$ and by assumption $i(\alpha, \beta) = 1$. Since $n + 2r - 1 \geqslant 3$, by Theorem 1 in [15]. $\varphi_|$ is induced by a diffeomorphism. In particular, $i(\varphi(\alpha), \varphi(\beta)) = 1$. $\qquad \square$

The following lemma can be proven using similar techniques as in Lemma 3.8. in [9] or Lemma 3 in [11].

Lemma 4.10. *Let a be a two-sided nonseparating circle. If c and d are two two-sided nonseparating circles, both intersecting a transversely once, then there is a sequence $c = c_0, c_1, ..., c_n = d$ of two-sided nonseparating circles such that each c_i intersects the circle a transversely once and c_i is disjoint from c_{i+1} for each $i = 0, 1, ..., n - 1$.*

Lemma 4.11. *Let $n \geqslant 2$ and $k \geqslant 1$. If N is a projective plane with n punctures and T is a torus with k punctures, then $C(N)$ and $C(T)$ are not isomorphic.*

Proof. The complexes $C(N)$ and $C(T)$ have dimensions $n - 2$ and $k - 1$, respectively. If $k \neq n - 1$, since these complexes of curves have different dimensions, $C(N)$ and $C(T)$ are not isomorphic. If $k = n - 1$, we proceed as follows.

Let $n = 2$. Thus N is a projective plane with two punctures and T is a torus with one puncture. In this case, $C(N)$ is finite ([21]), however, $C(T)$ is infinite discrete since T is a torus with one puncture. Therefore, they are not isomorphic.

Now, assume that $C(N)$ and $C(T)$ are not isomorphic when N has $n - 1$ punctures. Since there are $n - 1$ punctures on N, there are $n - 2$ punctures on T. We need to show that the complexes $C(N)$ and $C(T)$ are not isomorphic if N has n punctures. Assume the contrary that there is an isomorphism $\varphi : C(N) \to C(T)$. For any vertex γ in $C(N)$, φ induces an isomorphisms $L(\gamma) \to L(\varphi(\gamma))$ and $L^d(\gamma) \to L^d(\varphi(\gamma))$.

Note that for a vertex γ of $C(N)$, the dual link $L^d(\gamma)$ of γ is connected if and only if γ is either one-sided or 2-separating. For a vertex δ of $C(T)$, the dual link $L^d(\delta)$ of δ is connected if and only if δ is either nonseparating or 2-separating vertex. From this, it follows that the image of the union of the set of one-sided vertices and the set of 2-separating vertices of $C(N)$ is precisely the union of the set of nonseparating vertices and the set of 2-separating vertices of $C(T)$. Let γ be a 2-separating vertex of $C(N)$. Then $\varphi(\gamma)$ is either a nonseparating vertex or a 2-separating vertex of $C(T)$. First, we assume that $\varphi(\gamma)$ is a nonseparating vertex δ of $C(T)$. Clearly, $L(\gamma)$ is isomorphic to the complex of curves of a projective plane with $n - 1$ punctures and $L(\delta)$ is isomorphic to the complex of curves of a sphere with $n + 1$ punctures. By Lemma 4.1, these complexes are not isomorphic. Therefore, $\varphi(\gamma)$ cannot be a nonseparating vertex δ of $C(T)$. Now, we assume that $\varphi(\gamma)$ is a 2-separating vertex μ of $C(T)$. Then $L(\mu)$ is isomorphic to the complex of curves of a torus with $n - 2$ punctures. By assumption, these complexes are not isomorphic. Therefore, we get a contradiction. Hence, $C(N)$ and $C(T)$ are not isomorphic.

$\square$

Lemma 4.12. *Let N be a connected nonorientable surface of genus $g = 2r + 1$, $r \geqslant 1$ with $n \geqslant 0$ boundary components. The group $\operatorname{Aut} C(N)$ preserves the topological type of the vertices of $C(N)$.*

Proof. Let φ be an automorphism of $C(N)$. Note that for a vertex γ of $C(N)$, the dual link $L^d(\gamma)$ of γ is connected if and only if γ is either one-sided vertex or nonseparating two-sided vertex or 2-separating. Therefore, φ cannot take a one-sided vertex or a nonseparating two-sided vertex or a 2-separating to a k-separating vertex with $k > 2$ or separating vertex.

Let α be a one-sided vertex of $C(N)$ and let $a \in \alpha$ such that N_a is an orientable surface. Clearly, $L(\alpha)$ is isomorphic to the complex of curves of an orientable surface of genus r with $n + 1$ boundary components $S_{r,n+1}$. Let β be a one-sided vertex and let $b \in \beta$ such that N_b is a nonorientable surface of genus $2r$ with $n+1$ boundary components. Obviously, $L(\beta)$ is isomorphic to the complex of curves of a nonorientable surface of genus $2r$ with $n + 1$ boundary components $N_{2r,n+1}$. Let γ be a nonseparating two-sided vertex of $C(N)$ and $L(\gamma)$ is isomorphic to the complex of curves of a nonorientable surface of genus $2r - 1$ with $n + 2$ boundary components $N_{2r-1,n+2}$. Let δ be a 2-separating vertex of $C(N)$ and $L(\delta)$ is isomorphic to the complex of curves of a nonorientable surface of genus $2r + 1$ with $n - 1$ boundary components $N_{2r+1,n-1}$. Since there are maximal simplices of different dimensions in the complexes of curves $C(N_{2r,n+1})$, $C(N_{2r-1,n+2})$ and $C(N_{2r+1,n-1})$, these complexes are not isomorphic to $C(S_{r,n+1})$. Moreover, the dimensions of $C(N_{2r,n+1})$, $C(N_{2r-1,n+2})$ and $C(N_{2r+1,n-1})$ are $4r+n-3$, $4r+n-4$ and $4r + n - 3$, respectively. Therefore, the complex of curves $C(N_{2r-1,n+2})$ is not isomorphic to $C(N_{2r,n+1})$ and $C(N_{2r+1,n-1})$. Furthermore, although these complexes of curves $C(N_{2r,n+1})$ and $C(N_{2r+1,n-1})$ have the same dimensions, these complexes are not isomorphic. Because there is a maximal simplex of dimension

$3r+n-3$ in the complex of curves $C(N_{2r+1,n-1})$, whereas, there is no any maximal simplex of dimension $3r + n - 3$ in the complex of curves $C(N_{2r,n+1})$. Hence, the complex of curves $C(N_{2r+1,n-1})$ is not isomorphic to $C(N_{2r,n+1})$.

Let λ be a separating vertex and let $e \in \lambda$ be circle. N_e has two connected components. Let $N_e = N_0 \sqcup N_1$ such that N_0 and N_1 are nonorientable surfaces. More precisely, N_0 is a nonorientable surface of genus l for some $1 \leqslant l \leqslant 2r$ with $k+1$ punctures for some $0 \leqslant k \leqslant n$ denoted by $N_{l,k+1}$, then N_1 is a nonorientable surface of genus $2r + 1 - l$ with $n - k + 1$ punctures denoted by $N_{2r+1-l,n-k+1}$. The dual link $L^d(\lambda)$ has exactly two connected components. Let us denote these components by $L_0^d(\lambda)$ and $L_1^d(\lambda)$; $L^d(\lambda) = L_0^d(\lambda) \sqcup L_1^d(\lambda)$. We name $L_i(\lambda)$ so that the vertices of $L_i^d(\lambda)$ are the isotopy classes of circles on N_i. Let $L_i(\lambda)$ be the full subcomplex of $C(N)$ with vertices $L_i^d(\lambda)$. It follows that $(L_i(\lambda))^d = L_i^d(\lambda)$. Then $L_i(\lambda)$ is isomorphic to $C(N_i)$. Let μ be a separating vertex and $w \in \mu$ be circle. Similarly, we define $L_i^d(\mu)$ and $L_i(\mu)$ so that $L_i(\mu)$ is isomorphic to $C(N_i')$. In other words, $L_0(\mu)$ is isomorphic to the complex of curves of a orientable surface of genus l for some $0 \leqslant l \leqslant r$ with $k + 1$ punctures for some $0 \leqslant k \leqslant n$ denoted by $S_{l,k+1}$, then $L_1(\mu)$ is isomorphic to the complex of curves of a nonorientable surface of genus $2(r - l) + 1$ with $n - k + 1$ denoted by $N_{2(r-l)+1,n-k+1}$. Since $L_0(\mu)$ is isomorphic to $C(S_{l,k+1})$ and all maximal simplices in $C(S_{l,k+1})$ have the same dimension, $L^d(\mu)$ is not isomorphic to $L^d(\lambda)$. Hence, λ cannot be mapped μ under φ.

The proof of the lemma is complete. $\qquad\square$

Remark 4.1. *In the above proof, in case of $r = 1$ and $n \geqslant 0$, one can see that the complex of curves of a torus with $n + 1$ boundary components and the complex of curves of a projective plane with $n + 2$ boundary components are not isomorphic by Lemma 4.11.*

Theorem 4.4. *Let N be a connected nonorientable surface of odd genus $g = 2r + 1$ with n punctures. Suppose that $g + n \geqslant 6$. Then φ agrees with a map $h_* : C(N) \to C(N)$ which is induced by a diffeomorphism $h : N \to N$.*

Proof. Let $r = 0$. Then N is a projective plane with $n \geqslant 5$ punctures. In Section 4, we showed that φ is induced by a diffeomorphism of N.

We assume that every automorphism $C(N) \to C(N)$ is induced by a diffeomorphism of N if N is of odd genus $g \leqslant 2r - 1$. Now, we show that φ is induced by a diffeomorphism of N for genus $g = 2r + 1$.

Let c be any two-sided nonseparating circle and γ denote its isotopy class. By Lemma 4.12, φ takes γ to a two-sided nonseparating vertex, say γ', and let $c' \in \gamma'$. There is a diffeomorphism f such that $f(c) = c'$. Then $f_*^{-1}\varphi(\gamma) = \gamma$ where $f_* : C(N) \to C(N)$ the automorphism induced by f. By replacing φ by $f_*^{-1}\varphi$, we can assume that $\varphi(\gamma) = \gamma$. φ restricts to an automorphism $\varphi_c : L(\gamma) \to L(\gamma)$. Since $L(\gamma)$ is isomorphic to the complex of curves $C(N_c)$ of a nonorientable surface N_c of genus $2r - 1$ with $n + 2$ boundary components, we get an automorphism φ_c of the complex of curves of a nonorientable surface of genus $2r - 1$ with $n + 2$ boundary components. By induction, we can assume that φ_c is equal to a map $(\overline{g}_c)_* : C(N_c) \to C(N_c)$ which is induced by a diffeomorphism $\overline{g}_c : N_c \to N_c$. By gluing two boundary components of N_c obtained c in a convenient way g_c induces a diffeomorphism $g_c : N \to N$. It follows that $g_c(c) = c$. Therefore, φ agrees with $(g_c)_*$ on every element of $L(\gamma)$. The composition $(g_c^{-1})_*$ with φ, $(g_c^{-1})_* \circ \varphi$ fixes γ and every element of $L(\gamma)$. We may replace $(g_c^{-1})_* \circ \varphi$ by φ. Now, φ is an automorphism of $C(N)$ such that it is identity on $\gamma \cup L(\gamma)$.

Let d be a two-sided nonseparating circle dual to c. In other words, d intersects c transversely once and there is no other intersection. Let T be a regular neighborhood of $c \cup d$. The surface T is a torus with one boundary component.

Let e be the boundary component of T. Let δ and ϵ denote the isotopy class of d and e, respectively. Obviously, since $\epsilon \in L(\gamma)$, $\varphi(\epsilon) = \epsilon$. Also, $\varphi(\gamma) = \gamma$. Since $i(\delta, \epsilon) = 0$, $i(\varphi(\delta), \varphi(\epsilon)) = i(\varphi(\delta), \epsilon) = 0$. Since $i(\gamma, \delta) = 1$ by Lemma 4.9, there exists an integer n such that $\varphi(\delta) = (t_c^n)_*(\delta)$.

Let $D(\gamma)$ be the set of isotopy classes of two-sided nonseparating circles which are dual to c on N. Let d_1 be a two-sided nonseparating circle which is disjoint from d and dual to c. Similarly, there exists an integer n_1 such that $\varphi(\delta_1) = (t_c^{n_1})_*(\delta_1)$, where δ_1 is the isotopy class of d_1. Since $i(\delta, \delta_1) = 0$, $i(\varphi(\delta), \varphi(\delta_1)) = 0$. If $n \neq n_1$, then $i((t_c^n)_*(\delta), (t_c^{n_1})_*(\delta_1)) \neq 0$ since both d and d_1 are dual to c. Thus, we have $i(\varphi(\delta), \varphi(\delta_1)) \neq 0$. This is a contradiction. Therefore, we must have $n = n_1$. Now for any two-sided nonseparating circle s which is dual to c, by Lemma 4.10 we can find a sequence of two-sided nonseparating circles dual to c, connecting d to s such that each consecutive pair is disjoint. It follows that φ agrees with $(t_c^n)_*$ on every element of $D(\gamma)$. Therefore, φ agrees with $(t_c^n)_*$ on γ and on $L(\gamma) \cup D(\gamma)$. Let $h_c = t_c^n$. In the following, we will denote $(h_c)_*$ simply by h_*.

If u and v are any other two-sided nonseparating circles dual to each other, then there exists a diffeomorphism $N \rightarrow N$ mapping (u, v) to (c_1, c_2), where c_1 and c_2 are the circles in Figure 4.4. The set $\{c_1, c_2\}$ can be completed to a set C of two-sided circles except a as shown in Figure 4.4. Let γ_i be the isotopy class of c_i.

We orient tubular neighborhoods of elements of C in such way that these orientation agree with an orientation of a regular neighborhood M of $\cup c_i$. We note that M is an orientable surface.

The isotopy classes of Dehn twists about elements of C generate a subgroup $\mathcal{T}'$. As shown above, the restriction of φ to $L(\gamma_i) \cup D(\gamma_i)$ agrees with the induced

map $(h_i)_*$ of a diffeomorphism $h_i : N \to N$.

Any circle in C is either disjoint from c_1 or is dual to c_1, $\varphi(\gamma_i) = (h_1)_*(\gamma_i)$ for all i. Similarly, $\varphi(\gamma_i) = (h_2)_*(\gamma_i)$ for all i. Hence, $(h_1^{-1} \circ h_2)_*(\gamma_i) = \gamma_i$. In particular, $(h_1^{-1} \circ h_2)_*(c_i)$ is isotopic to c_i and

$$(h_1^{-1} \circ h_2)t_{c_i}(h_1^{-1} \circ h_2)^{-1} = t_{c_i}^{\epsilon_i},$$

where $\epsilon_i = \pm 1$. Let us denote $h_1^{-1} \circ h_2$ by h. For a circle $c_i \in C$ dual to c_1, we have the braid relation

$$t_{c_1}t_{c_i}t_{c_1} = t_{c_i}t_{c_1}t_{c_i}.$$

By conjugating with h, we get

$$t_{c_1}^{\epsilon_1}t_{c_i}^{\epsilon_i}t_{c_1}^{\epsilon_1} = t_{c_i}^{\epsilon_i}t_{c_1}^{\epsilon_1}t_{c_i}^{\epsilon_i}.$$

But this relation holds if and only if ϵ_1 and ϵ_i has the same sign. Now if c_i is dual to c_1 and if c_j is dual to c_i, by similar reasoning we get ϵ_j and ϵ_1 have the same sign. It follows that all ϵ_i have the same sign, as we can pass from c_1 to any circle in C through dual circles.
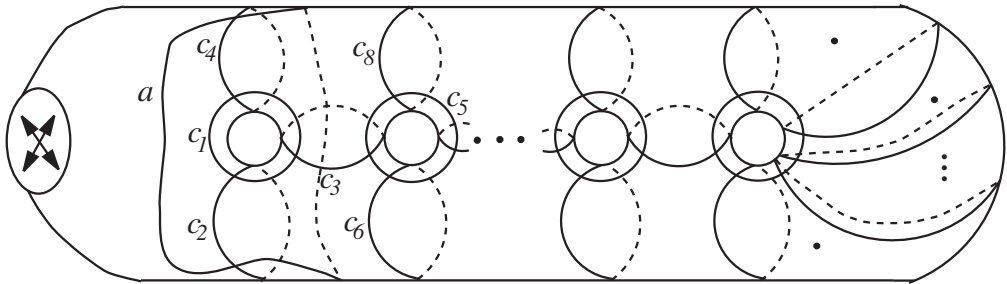

Figure 4.4: Identify antipodal points on the boundary component on the left hand side

Suppose that $\epsilon_i = -1$, so that $ht_{c_i}h^{-1} = t_{c_i}^{-1}$ for all $c_i \in C$. Thus h reverses the orientation of tubular neighborhoods of c_i. Let ρ be the reflection in $yz-$plane as shown in Figure 4.4. The reflection ρ leaves each $c_i \in C$ invariant, but reverses the orientations of tubular neighborhoods of c_i. Here $\rho t_{c_i} \rho^{-1} = t_{c_i}^{-1}$ for all i. Thus

$$(\rho \circ h)t_{c_i}(\rho \circ h)^{-1} = t_{c_i}.$$

In other words, $\rho \circ h \in \mathcal{C}_{\mathcal{M}_N}(\mathcal{T}')$. Since $\mathcal{C}_{\mathcal{M}_N}(\mathcal{T}')$ is trivial by Proposition 2.2, we get that

$$h = \rho^{-1} = \rho.$$

On the other hand, there exists a two-sided nonseparating circle a as shown in Figure 4.4 such that the class α of a is contained in $L(\gamma_1) \cap D(\gamma_2)$ and $h(\alpha) = \rho(\alpha) \neq \alpha$. However, this is a contradiction since $h(\alpha) = \varphi(\alpha) = \alpha$.

Therefore, we must have $\epsilon_i = 1$ for all i. Thus, $ht_{c_i}h = t_{c_i}$ for all i. In particular $h = h_1^{-1} \circ h_2 \in \mathcal{C}_{\mathcal{M}_N}(\mathcal{T}') = 1$. Consequently, $h_1 = h_2$.

Let d be any two-sided nonseparating circle and δ denote the isotopy class of d on N. Since N is a nonorientable surface of genus $g \geqslant 3$, c_1 and d are dually equivalent by Theorem 3.1 of [16]. In other words, there exists a sequence of two-sided nonseparating circles $a_1, \ldots, a_k$ on N such that $a_1 = c_1$, $a_k = d$ and the circles a_i and a_{i+1} are dual. Using this sequence, we obtain that $(h_1)_* = (h')_* = \varphi$. Indeed, for any two-sided nonseparating circle d, $(h_1)_*$ agrees with φ. Furthermore, since the isotopy classes of every separating circle and of any one-sided circle are in $L(\zeta)$ where ζ is the isotopy class of some two-sided nonseparating circle s, we obtain that $(h_1)_*$ agrees with φ on $C(N)$. $\qquad \square$

Proof of Theorem 1.1 for nonorientable surface of odd genus

It is shown that $\mathcal{M}_N \to \operatorname{Aut} C(N)$ is injective in Section 3. Theorem 4.4 implies that automorphisms of $C(N)$ are induced by diffeomorphisms of N. Hence, this completes the proof Theorem 1.1.

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