GÖDEL SPACETIME

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MEHMET KAVUK

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Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Sinan Bilikmen
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Özgür Sarıoğlu
Supervisor

Examining Committee Members

Prof. Dr. Ayşe Karasu   (METU, PHYS)

Assoc. Prof. Dr. Özgür Sarıoğlu (METU, PHYS)

Assoc. Prof. Dr. Yusuf İpekoğlu (METU, PHYS)

Assist. Prof. Dr. Hakan Öktem   (METU, IAM)

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Mehmet Kavuk

ABSTRACT

GÖDEL SPACETIME

Kavuk, Mehmet

MS., Department of Physics

Supervisor: Özgür Sarıoğlu

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In this thesis properties of the Gödel spacetime are analyzed and it is explicitly shown that there exist closed timelike curves in this spacetime. Geodesic motions for massive particles and light rays are investigated. One observes the focusing effect as a result of the solution of the geodesic equations. The time it takes for a free particle released from a point to come back to its starting point is calculated. A geometrical interpretation of the Gödel spacetime is given and the question of what the Gödel spacetime looks like is answered.

Keywords: Gödel spacetime, Null geodesics, Timelike geodesics, Time travel, Rotation of spacetime.

ÖZ

GÖDEL UZAY-ZAMANI

Kavuk, Mehmet

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Bu tezde Gödel uzay-zamanının özellikleri çözümlendi ve kapalı zamansı eğrilerin varlığı gösterildi. Jeodezik denklemi çözülerek kütleli parçacıkların ve ışık demetlerinin jeodezik hareketleri incelendi ve odaklanma etkisi görüldü. Bir noktadan yollanan bir ışık ışınının ve serbest bir parçacığın ne kadar sürede aynı yere döneceği hesaplandı. Gödel evreninin geometrik bir yorumu verildi ve bu evrenin neye benzediği sorusu cevaplandı.

Anahtar kelimeler: Gödel uzay-zamanı, Null jeodezikler, Zamansı jeodezikler, Zamanda yolculuk, Uzay zamanın dönmesi.

To my family and to my girlfriend Görkem Öcalan…

ACKNOWLEDGMENTS

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

General Relativity theory helps us to determine possible models for the large scale structure of spacetime. Each of the models is a pair $(M, g_{\mu\nu})$. Here M is a four dimensional manifold which represents the totality of all point-event locations; and $g_{\mu\nu}$ is a geometric object which represents the metric structure of spacetime. We can think of it, simply, as a function which assigns lengths to vectors at points of M, but which assigns negative and zero lengths to vectors as well as positive ones. It thus partitions the vectors at any point into three classes and determines a light cone structure. A vector is said to be *timelike, null,* or *spacelike* according to whether its length is negative, zero or positive. Timelike vectors fall inside the cone whereas null ones fall on the boundary. Of course, these null cones have an immediate physical significance. It is absolutely fundamental to General Relativity theory that there is an upper bound to the speeds with which particles can travel (as measured by any observer). If we think of vectors at a point as velocity vectors, then the light cone can be interpreted as a marking of that upper bound [1]. Massive particles must have timelike velocity vectors; those with zero mass must have null ones. The most important geometrical notion that turns out to be the tool to travel back in time is *closed timelike curves*. They are simply closed curves on the spacetime manifold whose tangent vectors at every point are timelike and point to the future. These future directed timelike curves, of course, have various magnitudes due to the spacetime metric, in particular, length and acceleration. The length is usually called *elapsed proper time*. Timelike curves with no accerelation are called *timelike geodesics* [1]. In General Relativity, Gödel spacetime is important because it is the first example of a spacetime that is a solution of Einstein's field equations and that has closed timelike curves, which are classically forbidden. In this thesis, we will review the Gödel spacetime.

In the late 1940s, Kurt Gödel took an interest in Einstein's theory of General Relativity. He was looking for an answer to the question of what the nature of time

is. Eventually he succeeded in finding a new solution to the field equations of Einstein's theory of General Relativity in 1949 [2] which influenced all the notions about time known thus far. This new solution suggests a spacetime with strange properties. First of all, the spacetime is filled with incoherent pressureless fluid matter in a state of uniform, rigid rotation. Secondly, the particles with no force on them follow a path similar to the paths that boomerangs follow. This means that the particles released from a point of the spacetime recede from their starting point until they reach a critical distance and then move back to their starting point at a later time. Finally, and the most important of all, there is a possibility of traveling back in time in the Gödel spacetime. As a matter of fact, all of us are travelling in time during our daily life on earth, but here by "travelling" we mean that a particle starting its motion from a spacetime point is able to come back to its starting point again. So there is no such definite notion as past, future and present in the Gödel spacetime.

In fact, Gödel's work did not attract much attention in the beginning. In 1956, Kundt [3] in Hamburg, Germany, calculated the geodesic world lines in Gödel's spacetime. He used Killing vectors to find the first integrals of the geodesic equations as suggested by Felix Pirani. Then in 1961, S. Chandrasekhar and J.P. Wright [4] calculated the timelike geodesics of the Gödel spacetime and concluded that there wasn't any closed timelike geodesic, so that Gödel's remark on this issue was incorrect. However, Gödel had never claimed that there were closed timelike curves which were geodesics. Nobody noticed this misunderstanding until 1970, when Howard Stein [5] showed that Chandrasekhar and Wright had been mistaken.

In fact, all the calculations done to find the geodesic world line of the Gödel spacetime began with either the metric given in (2.26), whose solution of the geodesic equations has a complicated form and is not very enlightening to understand the geometrical picture of the spacetime, or the metric conformal to the Gödel one. Actually Gödel himself applied a transformation to his metric (2.26) to write it in cylindrical coordinates so as to show the existence of closed timelike curves easily. In this thesis we calculate the timelike and null geodesics of the Gödel spacetime by using the Gödel metric written in cylindrical coordinates.

There was no geometrical figure showing the behavior of the geodesics of the Gödel spacetime until 1973 when Hawking and Ellis pictured Gödel spacetime in

their monograph entitled *The Large Scale Structure of Spacetime* [6]. This was the first time that geometrical insight was employed in understanding the properties of the Gödel spacetime. In 2003, I. Ozsvath and E. Schucking [7] published a paper where they asked the question as to how Hawking and Ellis drew this picture, *i.e.* what are the equations of the null geodesics that lead to this picture? They believed that some "intrigue" had been made while drawing this picture. By using a metric conformal to the Gödel metric, they calculated the null geodesics and reconstructed the picture of Hawking and Ellis. Even though they stated a problem related with the matter world line which is drawn as spacelike in the picture of Hawking and Ellis (see Figure 4.1), they did not give the equations of the timelike geodesics showing that they really must be inside the critical radius. In this thesis the problems of the picture in [6] are stated again when necessary and the ways to correct it are shown. In the second chapter, we begin with an ansatz and find the conditions under which this ansatz satisfies the field equations of cosmological Einstein's theory of General Relativity coupled with a pressureless perfect fluid. We first get the most general form of the Gödel metric with three integration constants. However A. K. Raychaudhuri and S. N. Thakurta [8] showed in their paper that all the solutions obtained for different values of the integration constants can be transformable to the Gödel metric. Hence we decide on how to choose the integration constants so as to get the original metric that Gödel used in his calculations. In the third chapter, we investigate some physical properties of the Gödel spacetime and choose a coordinate system to transform the metric into cylindrical coordinates which is useful in demonstrating the existence of closed timelike curves. We also find the Killing vectors of the Gödel spacetime. In the fourth chapter, null and timelike geodesics are calculated and it is shown that these equations are the ones from which Figure 4.1 can be obtained.

CHAPTER 2

DERIVATION OF THE GÖDEL SPACETIME

In this chapter we begin with an ansatz and a specific energy-momentum tensor, namely that of a pressureless perfect fluid source, for the spacetime and find under what conditions this ansatz satisfies the field equations of cosmological Einstein's theory of General Relativity. We end up with the general solution of the field equations with three integration constants. As it has been stated before, in [8] it is shown that all the solutions obtained for different values of the integration constants can be transformable to the Gödel metric. By appropriate choice of constants we get the metric that Gödel found in his paper.

2.1 The Metric

In 1949, Gödel found a new cosmological solution of Einstein's field equations of the form [2]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dz^2 + u(x)dy^2 + 2v(x)dt\,dy \,. \qquad (2.1)$$

In this model, spacetime is full of a pressureless perfect fluid which has an energy-momentum tensor of the form $T_{\mu\nu} = \rho V_\mu V_\nu$. Here $\rho > 0$ is the energy density of the fluid and V^μ is the fluid four-velocity. To find under what conditions (2.1) is a solution to the field equations of cosmological Einstein's theory of General Relativity with the corresponding energy-momentum tensor stated, *i.e.* to find the unknown functions $u(x)$ and $v(x)$, we have to start by calculating the components of the Christoffel symbol and the Ricci tensor corresponding to this metric. The metric can be written in matrix form as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & v & 0 \\ 0 & 1 & 0 & 0 \\ v & 0 & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ with an inverse } g^{\mu\nu} = \begin{pmatrix} \dfrac{-u}{v^2+u} & 0 & \dfrac{v}{v^2+u} & 0 \\ 0 & 1 & 0 & 0 \\ \dfrac{v}{v^2+u} & 0 & \dfrac{1}{v^2+u} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By using the definition of the Christoffel symbol

$$\Gamma^{\alpha}_{\ \mu\nu} \equiv \frac{1}{2}g^{\alpha\gamma}\left(g_{\gamma\nu,\mu} + g_{\mu\gamma,\nu} - g_{\mu\nu,\gamma}\right), \tag{2.2}$$

we can calculate the nontrivial components of it easily as

$$\Gamma^{0}_{10} = \frac{1}{2}\frac{v}{(v^2+u)}\dot{v}, \quad \Gamma^{0}_{12} = \frac{1}{2}\frac{v\dot{u}-u\dot{v}}{(v^2+u)}, \quad \Gamma^{1}_{20} = -\frac{1}{2}\dot{v},$$

$$\Gamma^{2}_{10} = \frac{1}{2}\frac{1}{(v^2+u)}\dot{v}, \quad \Gamma^{2}_{12} = \frac{1}{2}\frac{v\dot{v}+\dot{u}}{(v^2+u)}, \quad \Gamma^{1}_{22} = -\frac{1}{2}\dot{u}.$$

Here "dot" denotes derivative with respect to x. We will use the explicit form of it (d/dx) when necessary. In the same manner, by using the definition of the Ricci tensor [9]

$$R_{\mu\rho} \equiv \frac{\partial \Gamma^{\nu}_{\ \mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\nu}_{\ \nu\rho}}{\partial x^{\mu}} + \Gamma^{\alpha}_{\ \mu\rho}\Gamma^{\nu}_{\ \alpha\nu} - \Gamma^{\alpha}_{\ \nu\rho}\Gamma^{\nu}_{\ \alpha\mu}, \tag{2.3}$$

we can easily obtain the nontrivial components of the Ricci tensor as

$$R_{00} = \frac{1}{2}\frac{\dot{v}^2}{v^2+u}, \tag{2.4}$$

$$R_{11} = \frac{2(v^2 - u)\dot{v}^2 - 4(v^2 + u)v\ddot{v} + 4v\dot{v}\dot{u} - 2(v^2 + u)\ddot{u} + \dot{u}^2}{4(v^2 + u)^2} \,, \qquad (2.5)$$

$$R_{20} = R_{02} = -\frac{1}{2}\dot{v}^2 + \frac{1}{4}\frac{\dot{v}\dot{u}}{v^2 + u} \,, $$

$$R_{22} = -\frac{1}{2}\ddot{u} + \frac{1}{4}\frac{\dot{u}^2 - 2u\dot{v}^2 + 2v\dot{v}\dot{u}}{v^2 + u} \,. \qquad (2.6)$$

These components of the Christoffel symbol and of the Ricci tensor are the only ones that do not vanish. The components of the Einstein tensor are given in Appendix A. The Ricci scalar is obtained easily

$$R = R_{\mu\nu} g^{\mu\nu} = \frac{(v^2 - 3u)2\dot{v}^2 - 4(v^2 + u)\ddot{u} + 2\dot{u}^2 + 8v\dot{v}\dot{u} - 8v(v^2 + u)\ddot{v}}{4(v^2 + u)^2} \,. \qquad (2.7)$$

If we put all of these things together in the field equations of Einstein's General Relativity with a cosmological constant[1]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2}\rho V_\mu V_\nu \qquad (2.8)$$

and use a co-moving coordinate system for the energy-momentum tensor, *i.e.* take

$$V^\mu = (1, 0, 0, 0) = \delta^\mu{}_0 \,, \qquad (2.9)$$

so that

_____

[1] Here G is Newton's gravitational constant, c is the speed of light and Λ is the cosmological constant. For simplicity, we use units such that $G = c = 1$.

$$V_\mu = g_{\mu\nu}V^\nu = (-1, 0, v(x), 0) = g_{0\mu} , \qquad (2.10)$$

we get, first of all, that

$$G_{33} = R_{33} - \frac{1}{2}Rg_{33} = 8\pi\rho V_3 V_3 - \Lambda g_{33} \qquad \Rightarrow \qquad R = 2\Lambda , \qquad (2.11)$$

where we use the fact that R_{33} and V_3 are zero. If we use (2.11) and put it into (2.8), we obtain

$$R_{\mu\nu} = 8\pi\rho V_\mu V_\nu . \qquad (2.12)$$

Then we can easily obtain the rest of the equations

$$R_{00} = 8\pi\rho V_0 V_0 \qquad \Rightarrow \qquad R_{00} = 8\pi\rho , \qquad (2.13)$$

$$R_{11} = 8\pi\rho V_1 V_1 \qquad \Rightarrow \qquad R_{11} = 0 , \qquad (2.14)$$

$$R_{22} = 8\pi\rho V_2 V_2 \qquad \Rightarrow \qquad R_{22} = 8\pi\rho[v(x)]^2 , \qquad (2.15)$$

$$R_{02} = 8\pi\rho V_0 V_2 \qquad \Rightarrow \qquad R_{02} = R_{20} = -8\pi\rho v(x) . \qquad (2.17)$$

From (2.5) and (2.14), we obtain

$$\ddot{k} - \frac{\dot{k}^2}{2k} = \dot{v}^2 , \qquad (2.17)$$

where $k = v^2 + u$.

We also find from (2.4) and (2.13) that

$$\frac{\dot{v}^2}{2k} = 8\pi\rho .$$

(2.18)

Combining (2.17) together with (2.18), we get

$$\ddot{k} - \frac{\dot{k}^2}{2k} = 16\pi\rho k .$$

(2.19)

Now let us solve (2.19). First note that

$$\frac{d}{dx}(k^{-1/2}) = -\frac{1}{2}k^{-3/2}\dot{k} \qquad \Rightarrow \qquad \dot{k}k^{1/2}\frac{d}{dx}(k^{-1/2}) = -\frac{\dot{k}^2}{2k} .$$

Then (2.19) becomes

$$\ddot{k} + \dot{k}k^{1/2}\frac{d}{dx}(k^{-1/2}) = 16\pi\rho k ,$$

and multiplying this by $k^{-1/2}$, one finds

$$k^{-1/2}\ddot{k} + \dot{k}\frac{d}{dx}(k^{-1/2}) = 16\pi\rho k^{1/2} ,$$

which can be written as

$$\frac{d}{dx}(k^{-1/2}\dot{k}) = c_0 k^{1/2} ,$$

(2.20)

where $c_0 = 16\pi\rho > 0$. Let

$$k^{1/2} = p ,$$

(2.21)

then (2.20) becomes

$$2\ddot{p} = c_0 p ,$$

which is easy to handle since it is a second order homogeneous equation with constant coefficients. The most general solution is given by

$$p(x) = c_1 e^{\sqrt{c_0/2}x} + c_2 e^{-\sqrt{c_0/2}x} ,$$

where c_1 and c_2 are integration constants. So by using (2.21), we get

$$k(x) = (c_1 e^{\sqrt{c_0/2}x} + c_2 e^{-\sqrt{c_0/2}x})^2 . \qquad (2.22)$$

From (2.18), we have

$$\dot{v}(x) = \pm\sqrt{c_0}(c_1 e^{\sqrt{c_0/2}x} + c_2 e^{-\sqrt{c_0/2}x}) ,$$

whose solution is simply

$$v(x) = \pm\sqrt{2}(c_1 e^{\sqrt{c_0/2}x} - c_2 e^{-\sqrt{c_0/2}x}) + c_3 , \qquad (2.23)$$

where c_3 is another integration constant. Let us define the parameter

$$\Omega^2 = c_0/4 = 4\pi\rho > 0 ,$$

then (2.23) becomes

$$v(x) = \pm\sqrt{2}(c_1 e^{\sqrt{2}\Omega x} - c_2 e^{-\sqrt{2}\Omega x}) + c_3 . \qquad (2.24)$$

The meaning of Ω will become clear in the next section. In fact, with (2.22) and (2.24), we now have all we need since $u(x)$ can easily be found by using $k = v^2 + u$. The metric functions $u(x)$ and $v(x)$ satisfy the field equations with no further constraints on the integration constants.

2.2 The Choice of the Integration Constants

Now we have a solution with three integration constants. We now limit our considerations to the specific choice of integration constants to get the metric that Gödel presented in his paper. In fact, Raychaudhuri and Thakurta [8] showed that all the solutions obtained for different values of c_1, c_2 and c_3 can be transformable to the Gödel metric. Since we are after the Gödel spacetime, we will choose appropriate constants so as to get the Gödel metric. If we take the negative value of $v(x)$ and choose integration constants as

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = c_3 = 0,$$

then (2.22) and (2.24) become

$$k(x) = \frac{1}{2} e^{2\sqrt{2}\Omega x}, \qquad v(x) = -e^{\sqrt{2}\Omega x}.$$

Remember that $k = v^2 + u$, so $u(x)$ is easily obtained as

$$u(x) = -\frac{1}{2} e^{2\sqrt{2}\Omega x}.$$

Now we have the metric that Gödel used and all the relevant quantities corresponding to this metric, *i.e.*

$$ds^2 = -dt^2 + dx^2 + dz^2 - \frac{1}{2}e^{2\sqrt{2}\Omega x}dy^2 - 2e^{\sqrt{2}\Omega x}dtdy \, , \qquad (2.25)$$

$$\Gamma^0_{10} = \sqrt{2}\Omega \, , \qquad \Gamma^0_{12} = \frac{\sqrt{2}\Omega}{2}e^{\sqrt{2}\Omega x} \, , \qquad \Gamma^1_{20} = \frac{\sqrt{2}\Omega}{2}e^{\sqrt{2}\Omega x} \, , \qquad \Gamma^1_{22} = \frac{\sqrt{2}\Omega}{2}e^{2\sqrt{2}\Omega x}$$

$$, \qquad \Gamma^2_{10} = -\sqrt{2}\Omega e^{-\sqrt{2}\Omega x} \, , \qquad \Gamma^2_{12} = 0 \, ,$$

$$\Omega^2 = 4\pi\rho > 0 \, , \quad \frac{R}{2} = \Lambda = -4\pi\rho < 0 \, ,$$

where the second equation in the last line follows immediately from (2.11) and (2.12) *i.e.*

$$2\Lambda = R = g^{\mu\nu}R_{\mu\nu} = 8\pi\rho \; g^{\mu\nu}V_\mu V_\nu = 8\pi\rho V^\nu V_\nu = -8\pi\rho \, .$$

This four dimensional spacetime can be viewed as the sum of a three dimensional spacetime M_1 whose metric g_1 is given by

$$ds_1{}^2 = -dt^2 + dx^2 - \frac{1}{2}e^{2\sqrt{2}\Omega x}dy^2 - 2e^{\sqrt{2}\Omega x}dtdy \qquad (2.26)$$

on the geometry $M_1 = R^3$ defined by the coordinates (t,x,y), and a one dimensional spacelike line with a metric g_2 given by

$$ds_2{}^2 = dz^2$$

on the manifold $M_2 = R$ defined by the coordinate z. The metric on M_1 does not depend on z and the $z = $ constant lines are all orthogonal to M_1. Since closed

timelike lines exist for constant z in the M_1, we do not need to consider the full four dimensional spacetime , instead we can restrict ourselves to just M_1 [6,7]

CHAPTER 3

PROPERTIES OF THE GÖDEL SPACETIME

In this chapter, we investigate some physical properties of the Gödel spacetime and choose a coordinate system to transform the metric into cylindrical coordinates which is useful in demonstrating the existence of closed timelike curves.

3.1 Transformation to Cylindrical Coordinates

In his paper Gödel made the transformation to his metric to demonstrate the cylindrical symmetry of the spacetime more explicitly. The existence of closed timelike curves can easily be shown for the metric written in cylindrical coordinates.

With the transformation

$$e^{\sqrt{2}\Omega x} = Cosh2r + Cos\phi \; Sinh2r \, ,$$

$$\Omega y e^{\sqrt{2}\Omega x} = Sin\phi \; Sinh2r \, ,$$

$$Tan\frac{1}{2}(\phi + \Omega t - \sqrt{2}t') = e^{-2r}Tan\frac{1}{2}\phi \, , \qquad \frac{\left|\Omega t - \sqrt{2}t'\right|}{2} < \frac{\pi}{2} \, ,$$

the metric (2.26) can be transformed to the form (see Appendix B)

$$ds^2 = 2\Omega^{-2}(-dt'^2 + dr^2 - (Sinh^4r - Sinh^2r)d\phi^2 - 2\sqrt{2}Sinh^2r \, d\phi \, dt') \, , \qquad (3.1)$$

where $-\infty < t' < \infty$, $0 \le r < \infty$ and $0 \le \phi \le 2\pi$, and $\phi = 0$ is identified with $\phi = 2\pi$. We in fact have written the metric in a form which is explicitly cylindrically symmetric, since it does not depend on ϕ. Let us use the advantage of this symmetry condition and take a circle $C \equiv \{(r,\phi) \; r = \text{const.}, \; 0 \le \phi \le 2\pi\}$ in the $t' = 0$ plane to investigate the characteristic of such a circle; *i.e.* let us examine under what circumstances it is timelike, spacelike or null.
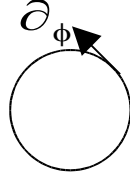
Figure 3.1: The circle C on the $t' = 0$ plane which has a tangent vector ∂_ϕ.

We see that the tangent vector $\gamma^\mu = (\partial_\phi)^\mu$ to the circle C has the length squared

$$\gamma \cdot \gamma = (\partial_\phi)^2 = g_{\mu\nu} \gamma^\mu \gamma^\nu = g_{\phi\phi} = -Sinh^2 r(Sinh^2 r - 1). \qquad (3.2)$$

Then, if

$$
\begin{array}{ll}
> 1, & \text{then } C \text{ is timelike} \\
Sinh^2 r = 1, & \text{then } C \text{ is null} \\
< 1, & \text{then } C \text{ is spacelike}
\end{array}
$$

(see Figure 3.2). The acceleration vector ζ^μ is given by

$$\zeta^\mu = \gamma^\mu{}_{;\nu} \gamma^\nu, \qquad\qquad \gamma^\nu \gamma_\nu = -1,$$

for the timelike unit vector [7]

$$\gamma^\nu = \delta^\nu{}_\phi |g_{\phi\phi}|^{-1/2}.$$

An explicit calculation yields

$$\zeta^\mu = \delta^\nu{}_\phi |g_{\phi\phi}|^{-1/2} \nabla_\nu (\gamma^\mu) = |g_{\phi\phi}|^{-1/2} \nabla_\phi (\gamma^\mu) = |g_{\phi\phi}|^{-1/2} \Gamma^\mu{}_{\phi\alpha} \gamma^\alpha = |g_{\phi\phi}|^{-1} \Gamma^\mu{}_{\phi\phi}$$

14

$$\zeta_\mu = \left| g_{\phi\phi} \right|^{-1} g_{\mu r} \Gamma^r{}_{\phi\phi} .$$

We then get the magnitude of the acceleration vector defined as

$$(\zeta)^2 = \zeta^\nu \zeta_\nu$$

to be

$$\zeta = \frac{1}{2} \frac{Coshr}{Sinhr} \left(\frac{2Sinh^2 r - 1}{Sinh^2 r - 1} \right) . \tag{3.3}$$

It is easily seen that ζ does not vanish for $Sinh^2 r > 1$. Thus, with the help of (3.2), we see that we have closed timelike curves (but not geodesics since its acceleration is not zero) for $r > \ln(1 + \sqrt{2})$.

3.2 Killing vectors

It is easily seen from the metrics (2.25) and (3.1) that the former admits three translations along the y , t and z axes, while the latter admits rotation around the ϕ axis. As a matter of fact, by using the Killing equation in Cartesian coordinates

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 , \tag{3.4}$$

we can easily obtain four of the Killing vectors as

$$A = \frac{\partial}{\partial t} , \qquad\qquad B = \frac{\partial}{\partial z} , \qquad\qquad C = \frac{\partial}{\partial y} ,$$

$$D = \frac{\partial}{\partial x} - \sqrt{2}\Omega y \frac{\partial}{\partial y} .$$

15

We also have a Killing vector $\dfrac{\partial}{\partial \phi}$ in cylindrical coordinates. Transforming back to the Cartesian coordinates;

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} \, , \qquad \partial_{\phi} = \frac{\partial t}{\partial \phi} \partial_t + \frac{\partial x}{\partial \phi} \partial_x + \frac{\partial y}{\partial \phi} \partial_y \, ,$$

we get the fifth Killing vector as

$$E = \partial_{\phi} = \frac{1}{\Omega} e^{-\sqrt{2}\Omega x} \frac{\partial}{\partial t} - \frac{y}{\sqrt{2}} \frac{\partial}{\partial x} + \frac{1}{2} \left(\Omega y^2 - \frac{1}{\Omega} e^{-2\sqrt{2}\Omega x} \right) \frac{\partial}{\partial y}$$

in Cartesian coordinates.

Now we have five independent Killing vectors with the following Lie algebra.

$$[A,B]=0 \, , \qquad [A,C]=0 \, , \qquad [A,D]=0 \, , \qquad [A,E]=0 \, ,$$

$$[B,C]=0 \, , \qquad [B,D]=0 \, , \qquad [B,E]=0 \, , \qquad [C,D]=-\sqrt{2}\Omega C \, ,$$

$$[C,E]=-\frac{1}{\sqrt{2}} D \, , \qquad\qquad [D,E]=-\sqrt{2}\Omega E \, .$$

A, B and *C* are the trivial generators corresponding to the translation invariance of the metric (2.25) in the *t, z* and *y* directions, respectively.

We see that *A* and *B* commute with every Killing vector. Another feature is that the commutator of *D* with the Killing vectors C and D is proportional to the Killing vector itself. Since we have five Killing vectors, there is a five dimensional group of isometries in this spacetime which is also transitive. (Transitive means that no point remains unchanged by a finite transformation.) Hence Gödel spacetime is homogeneous and admits the following four parameter group of transformations:

$$\text{(i)} \quad t' = t + \alpha \ ,$$

$$\text{(ii)} \quad y' = ye^{-\sqrt{2}\Omega\delta} + \beta \ ,$$

$$\text{(iii)} \quad z' = z + \varepsilon \ ,$$

$$\text{(iv)} \quad x' = x + \delta \ ,$$

where α, β, δ and ε are the parameters of the group.

3.3 Rotation, Shear and Expansion

If V is the four-velocity of a fluid, then ∇V can be decomposed as

$$V_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha V_\beta \ , \tag{3.5}$$

where a is the "four-acceleration" of the fluid

$$a_\alpha \equiv V_{\alpha;\beta}V^\beta \ , \tag{3.6}$$

θ is the "expansion" of the fluid world lines

$$\theta \equiv \nabla \cdot V = V^\alpha{}_{;\alpha} \ , \tag{3.7}$$

$\omega_{\alpha\beta}$ is the "rotation 2-form" of the fluid and $\sigma_{\alpha\beta}$ is the "shear tensor"

$$\omega_{\alpha\beta} \equiv \frac{1}{2}(V_{\alpha;\mu}P^\mu{}_\beta - V_{\beta;\mu}P^\mu{}_\alpha) \ , \tag{3.8}$$

$$\sigma_{\alpha\beta} \equiv \frac{1}{2}(V_{\alpha;\mu}P^\mu{}_\beta + V_{\beta;\mu}P^\mu{}_\alpha) - \frac{1}{3}\theta P_{\alpha\beta} \ . \tag{3.9}$$

Here P is the projection tensor

$$P_{\alpha\beta} \equiv g_{\alpha\beta} + V_\alpha V_\beta \qquad \Rightarrow \qquad P^\alpha{}_\beta \equiv \delta^\alpha{}_\beta + V^\alpha V_\beta , \qquad (3.10)$$

which projects a vector onto the 3-surface perpendicular to V [10]. Rotation 2-form and the shear tensor are antisymmetric and symmetric, respectively. Now since we are co-moving observers sitting on the flow of the fluid, our four-velocity tangent to the world lines of the fluid has the simplest form given in (2.9) and (2.10):

$$V^\mu = \delta^\mu{}_0 , \qquad V_\mu = g_{0\mu} .$$

First, let us examine the acceleration by using (3.6). Straightforward calculations show that it is zero:

$$a_\alpha = V^\beta \nabla_\beta V_\alpha = \delta^\beta{}_0 \nabla_\beta g_{\alpha 0} = \nabla_0 g_{\alpha 0} = 0 .$$

This means that the flow of the fluid moves on the geodesics. The expansion θ is also easy to calculate with the help of (3.7)

$$\theta = \nabla_\alpha V^\alpha = \nabla_\alpha \delta^\alpha{}_0 = 0 .$$

So Gödel spacetime is not expanding which is different than the observations of our spacetime. Our spacetime is expanding as it is obvious from the red-shifted light rays coming towards us from the distant galaxies.

Now a little bit of work is required to find the rotation 2-form of the fluid:
By using the definition of the covariant derivative

$$V_{\alpha;\mu} \equiv \partial_\mu V_\alpha - \Gamma^\nu{}_{\alpha\mu} V_\nu \qquad (3.11)$$

and (2.10), we get

$$V_{\alpha;\mu} = \partial_\mu g_{0\alpha} - \Gamma^\nu{}_{\alpha\mu} g_{0\nu} \,.$$

From the definition of the Christoffel symbol (2.2), we find

$$\Gamma_{\beta\mu\nu} \equiv g_{\alpha\beta} \Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}(g_{\beta\nu,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}) \,. \tag{3.12}$$

Multiplication of (3.10) and (3.11) gives

$$V_{\alpha;\mu} P^\mu{}_\beta = \partial_\beta g_{0\alpha} + g_{0\beta} \partial_0 g_{0\alpha} - \Gamma_{0\alpha\beta} - g_{0\beta} \Gamma_{0\alpha 0} \,,$$

where the second term in the right hand side is zero because there is no time dependency in the metric. Then we have

$$V_{\alpha;\mu} P^\mu{}_\beta = \partial_\beta g_{0\alpha} - \Gamma_{0\alpha\beta} - g_{0\beta} \Gamma_{0\alpha 0} \,, \tag{3.13}$$

$$V_{\beta;\mu} P^\mu{}_\alpha = \partial_\alpha g_{0\beta} - \Gamma_{0\alpha\beta} - g_{0\alpha} \Gamma_{0\beta 0} \,, \tag{3.14}$$

where in the last line we have interchanged the indices α and β. Now subtract these two equations and put them into (3.9) to get

$$\omega_{\alpha\beta} = \frac{1}{2}(\partial_\beta g_{0\alpha} - \partial_\alpha g_{0\beta} - g_{0\beta} \Gamma_{0\alpha 0} + g_{0\alpha} \Gamma_{0\beta 0}) \,. \tag{3.15}$$

From (3.12) we find

$$\Gamma_{0\beta 0} = \frac{1}{2} \partial_\beta g_{00} = 0 \,, \tag{3.17}$$

19

where we use the fact that $g_{00} = \text{constant}$. Then (3.15) becomes

$$\omega_{\alpha\beta} = \frac{1}{2}(\partial_\beta g_{0\alpha} - \partial_\alpha g_{0\beta}).$$

Now we see that the only component which survives is ω_{12}

$$\omega_{12} = \frac{\sqrt{2}}{2}\Omega e^{\sqrt{2}\Omega x} \qquad \Rightarrow \qquad \omega^{12} = g^{11}g^{22}\omega_{12} = \sqrt{2}\Omega e^{-\sqrt{2}\Omega x},$$

and the magnitude of the rotation 2-form is

$$\omega^2 = \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} = \omega_{12}\omega^{12} = \Omega^2.$$
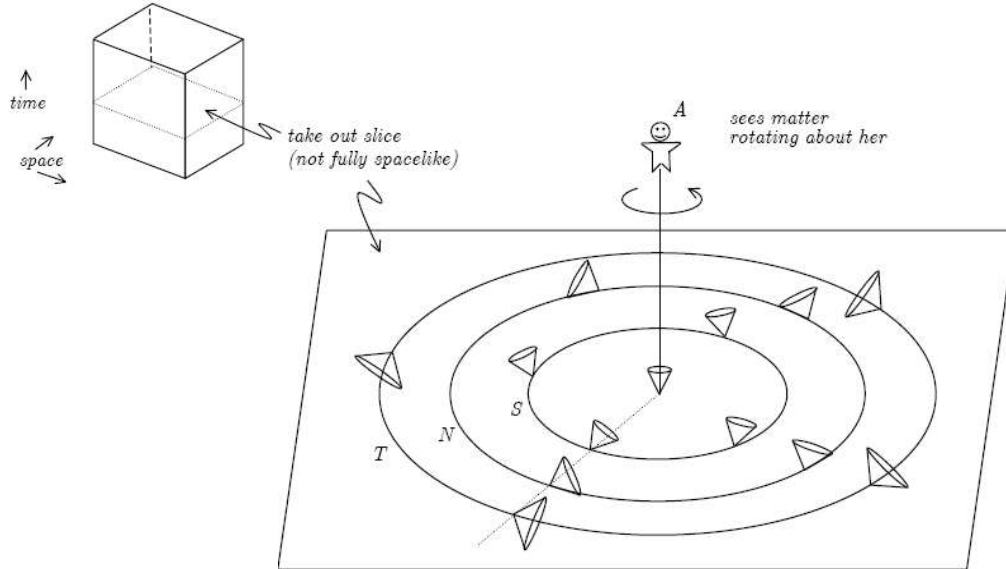
Figure 3.2: $t' = 0$ plane. Light cones tip over and open out as we proceed away from *A* along concentric circles. Initially, such circles are closed spacelike curves like *S*. Eventually light cones tip to create closed null curves like *N*, and then tip further to create closed timelike curves like *T*. Since spacetime is stationary and rotationally symmetric about each matter world line, all observers see what *A* sees; all see matter spinning about them.

Now Ω can be interpreted as the rotation of the pressureless matter fluid in the Gödel spacetime. In section 4.2 we will show that the central matter world line is a timelike geodesic, so if we move freely on this world line, we see that the matter fluid around us is rotating with the angular velocity Ω. In fact, all the fluid matter content is rotating with respect to every point in space, because of the homogeneity of the Gödel spacetime.

Now let us digress here a little bit and think about the meaning of having non-vanishing rotation. Actually this spacetime is stationary, since it admits a timelike Killing vector field, but it is not static because in order for this spacetime to be static, in addition to being stationary, there should exist a hypersurface which is

orthogonal to the congruence of timelike Killing vector fields. However a straightforward calculation shows that

$$\omega_{\alpha\beta} = 0$$

if and only if (see Appendix C for the proof)

$$V_{[\alpha}V_{\nu;\mu]} = 0, \qquad (3.18)$$

which is the condition for hypersurface orthogonality.

Hence we can conclude that our world lines are not hypersurface orthogonal, since we have non-vanishing rotation. Let us now show that the condition (3.18) is necessary and sufficient for a vector to be orthogonal to a one-parameter hypersurface. Suppose that there is a one-parameter family f of hypersurfaces given by $f = $ constant. A normal vector ξ to a hypersurface can be written as follows;

$$\xi \propto \nabla f \qquad \Rightarrow \qquad \xi_\mu = hf_{,\mu}, \qquad (3.19)$$

where h is an arbitrary scalar function. Now let us construct from an arbitrary vector the completely antisymmetric tensor

$$a_{\mu\nu\gamma} \equiv \xi_{[\mu}\xi_{\nu;\gamma]}$$

$$\equiv \frac{1}{3!}\left\{\xi_\mu(\xi_{\nu;\gamma} - \xi_{\gamma;\nu}) + \xi_\nu(\xi_{\gamma;\mu} - \xi_{\mu;\gamma}) + \xi_\gamma(\xi_{\mu;\nu} - \xi_{\nu;\mu})\right\}.$$

This tensor is actually zero for the specific vector given by (3.19), as can be shown as follows:

$$\xi_{\mu;\nu} = h_{,\nu}f_{,\mu} + hf_{,\mu;\nu} \quad \Rightarrow \quad \xi_\gamma\xi_{\mu;\nu} = hf_{,\gamma}h_{,\nu}f_{,\mu} + h^2 f_{,\gamma}f_{,\mu;\nu},$$

$$a_{\mu\nu\gamma} \equiv \frac{1}{3!} \{hf_{,\mu} h_{,\gamma} f_{,\nu} - hf_{,\nu} h_{,\gamma} f_{,\mu} - hf_{,\mu} h_{,\nu} f_{,\gamma} + hf_{,\gamma} h_{,\nu} f_{,\mu} + hf_{,\nu} h_{,\mu} f_{,\gamma} - hf_{,\gamma} h_{,\mu} f_{,\nu}$$

$$+ h^2 f_{,\mu} f_{,\nu;\gamma} - h^2 f_{,\mu} f_{,\gamma;\nu} + h^2 f_{,\nu} f_{,\gamma;\mu} - h^2 f_{,\nu} f_{,\mu;\gamma} + h^2 f_{,\gamma} f_{,\mu;\nu} - h^2 f_{,\gamma} f_{,\nu;\mu} \} = 0,$$

where the first term in the parentheses cancels out with the second, the third term with the fourth, *etc...* Thus a necessary condition for a vector field ξ_μ to be everywhere orthogonal to a one-parameter family f of three dimensional hypersurfaces is

$$a_{\mu\nu\gamma} \equiv \xi_{[\mu} \xi_{\nu;\gamma]} = 0.$$

Hence our world lines are not hypersurface orthogonal, because (3.18) does not hold for our situation. Therefore we have a stationary but non-static spacetime.

To find the shear tensor, we just add (3.13) and (3.14) up and find

$$\sigma_{\alpha\beta} = \frac{1}{2} \left(\partial_\beta g_{0\alpha} + \partial_\alpha g_{0\beta} - 2\Gamma_{0\beta\alpha} - g_{0\beta} \Gamma_{0\alpha 0} - g_{0\alpha} \Gamma_{0\beta 0} \right). \tag{3.20}$$

The last two terms are obviously zero due to (3.17). By using (3.16), we get

$$\Gamma_{0\beta\alpha} = \frac{1}{2} (\partial_\beta g_{0\alpha} + \partial_\alpha g_{\beta 0} - \partial_0 g_{\beta\alpha}).$$

If we put this equation into (3.20), we find

$$\sigma_{\alpha\beta} = 0. \tag{3.21}$$

None of the components of the shear tensor survives, so its magnitude is zero. (3.21) represents the fact that, there is no distortion in the shape of our collection of test particles. That is, suppose some portion of our test particles occupy a volume in

space in the shape of a sphere, then, as time goes on, the sphere will remain as a sphere as opposed to the case where there is a non-vanishing shear. In the case of a non-vanishing shear, the sphere would have become an ellipsoid as time passed by. Hence, there is only a rotation in Gödel spacetime which actually prevents us from choosing a *cosmic time* coordinate to agree on events that are simultaneous.

CHAPTER 4

GEODESICS

In this chapter geodesic motion in Gödel spacetime is analyzed and the geometrical picture of the spacetime is given.

4.1 Null Geodesics

To study the null geodesics, we use the variational method for the metric given in (3.1) and recapitulate the arguments that lead to Figure 4.1 below. (This figure is reproduced from [6].) In their paper [7] Ozsvath and Schucking drew attention to the fact that the equations used in drawing Figure 4.1 are not given and stated their belief that an "intrigue" has been made during the drawing of it. So they calculated the null geodesics of the Gödel spacetime by using the metric conformal to the Gödel metric to reconstruct it again. Since null geodesics are not affected by such a conformal transformation and since the timelike or null nature of a vector is left invariant under such a transformation, there is no loss of generality in doing so. Here we find the parametric equations of the null geodesics by using the original Gödel metric and try to show that they are the ones used in drawing Figure 4.1.
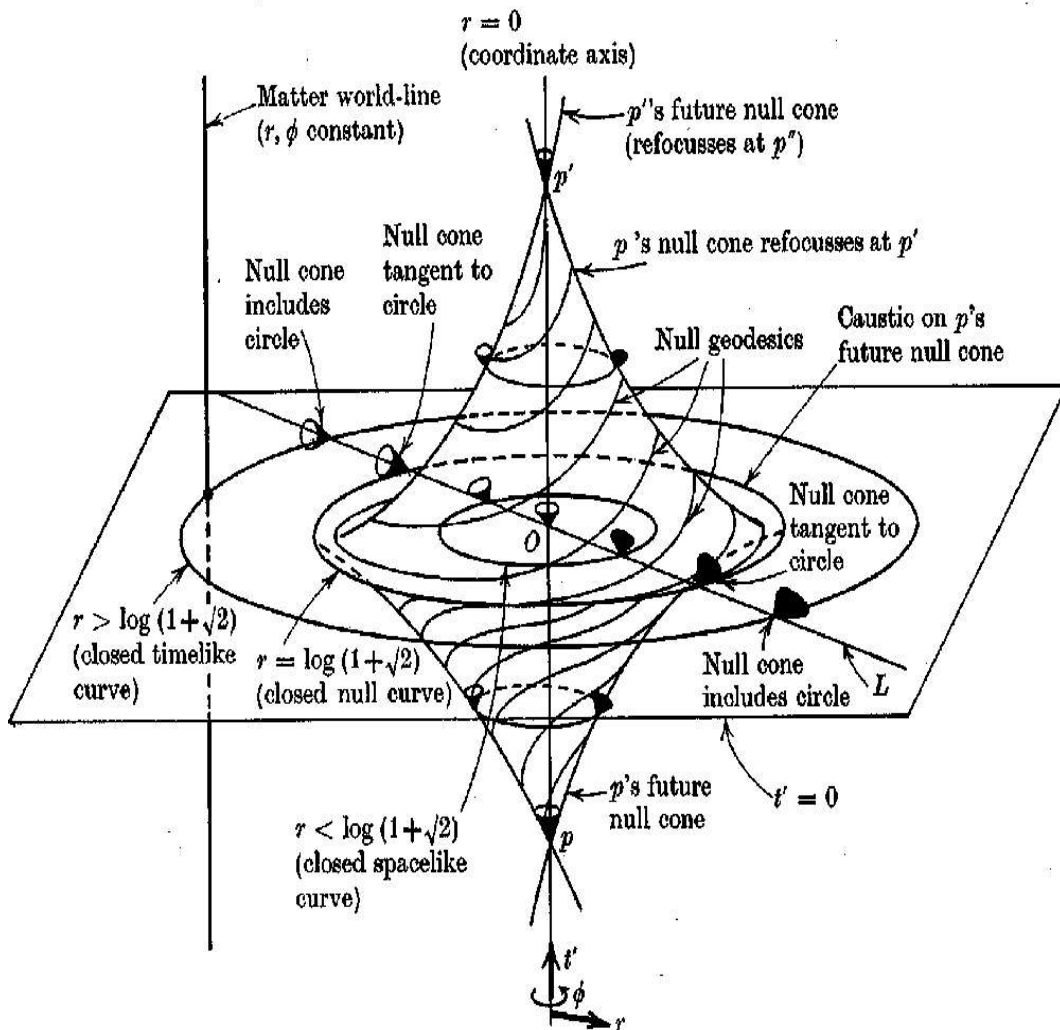
Figure 4.1: Gödel's spacetime with the irrelevant coordinate *z* suppressed. The space is rotationally symmetric about any point; the diagram represents correctly the rotational symmetry about the *r*=0 axis, and the time invariance. The light cone opens out and tips over as *r* increases (see line *L*) resulting in closed timelike curves. The diagram does not correctly represent the fact that *all* points are in fact equivalent.

Consider the following extremization related to the three dimensional metric (3.1) which yields the geodesics:

$$\delta \int L d\lambda = 0 \,,$$

$$L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = -\dot{t}^2 + \dot{r}^2 - (Sinh^4 r - Sinh^2 r)\dot{\phi}^2 - 2\sqrt{2}(Sinh^2 r)\dot{\phi}\,\dot{t}\,,$$

where

$$\dot{t} = \frac{dt}{d\lambda}\,, \qquad \dot{r} = \frac{dr}{d\lambda}\,, \qquad \dot{\phi} = \frac{d\phi}{d\lambda}\,,$$

with the affine parameter λ that can be used for null geodesics.

Since the Lagrangian does not depend on t and ϕ, we have from the Euler-Lagrange equations the two integrals

$$\frac{\partial L}{\partial \dot{t}} = \dot{t} + \sqrt{2}(Sinh^2 r)\dot{\phi} = a = \text{constant}\,, \tag{4.1}$$

$$\frac{\partial L}{\partial \dot{\phi}} = (Sinh^4 r - Sinh^2 r)\dot{\phi} + \sqrt{2}(Sinh^2 r)\dot{t} = b = \text{constant}\,. \tag{4.2}$$

Here a can be interpreted as the total energy and b is interpreted as the total angular momentum of the photons.

Since we are trying to find the null geodesics, we also have

$$L = -\dot{t}^2 + \dot{r}^2 - (Sinh^4 r - Sinh^2 r)\dot{\phi}^2 - 2\sqrt{2}(Sinh^2 r)\dot{\phi}\,\dot{t} = 0\,. \tag{4.3}$$

If we substitute (4.1) into (4.3), we find

$$\dot{r}^2 + Cosh^2 r\,Sinh^2 r\,\dot{\phi}^2 - a^2 = 0\,. \tag{4.4}$$

Moreover from (4.1) and (4.2), we also get

$$\dot{t} = \frac{a + \sqrt{2}b - a\,Sinh^2 r}{Cosh^2 r}\,, \qquad (4.5)$$

$$\dot{\phi} = \frac{-b + a\sqrt{2}\,Sinh^2 r}{Sinh^2 r\,Cosh^2 r}\,. \qquad (4.6)$$

If we put (4.6) into (4.4), we find the integral

$$\lambda - \lambda_0 = \int \frac{(Cosh r)(Sinh r)dr}{\left\{a^2 Cosh^2 r\,Sinh^2 r - (-b + a\sqrt{2}Sinh^2 r)^2\right\}^{1/2}}\,. \qquad (4.7)$$

(4.7) gives (see Appendix D)

$$Sinh^2 r = \sqrt{\Delta}Sin^2(a(\lambda - \lambda_0)) + u_1\,, \qquad \text{where}$$

$$\Delta = 4\frac{b^2}{a^2} + 4\sqrt{2}\frac{b}{a} + 1 \quad,\quad u_1 = \frac{1}{2} + \sqrt{2}\frac{b}{a} - \frac{\sqrt{\Delta}}{2} \qquad \text{and } \lambda_0 \text{ is an integration constant.}$$

If we put this solution into (4.5) and (4.6), we get the integrals

$$\int dt = \int \frac{\{-a\sqrt{\Delta}Sin^2(a(\lambda - \lambda_0)) + a(1 - u_1) + \sqrt{2}b\}d\lambda}{\sqrt{\Delta}Sin^2(a(\lambda - \lambda_0)) + u_1 + 1}\,, \qquad (4.8)$$

$$\int d\phi = \int \frac{a\sqrt{2}d\lambda}{\sqrt{\Delta}Sin^2(a(\lambda - \lambda_0)) + u_1 + 1}$$

$$-\int \frac{bd\lambda}{\{\sqrt{\Delta}Sin^2(a(\lambda - \lambda_0)) + u_1 + 1\}\{\sqrt{\Delta}Sin^2(a(\lambda - \lambda_0)) + u_1\}}\,.$$

The first one of these two integrals is easier to handle compared to the second one, so first we find the solution of the first integral and then deal with the second one later. After some calculations, we get the solution of the first integral as follows (see Appendix D):

$$t - t_0 = -a\lambda + \frac{2a + \sqrt{2}b}{a\sqrt{u_1 + 1}\sqrt{\sqrt{\Delta} + u_1 + 1}} ArcTan\left[\frac{\sqrt{\sqrt{\Delta} + u_1 + 1}}{\sqrt{u_1 + 1}} Tan(a(\lambda - \lambda_0))\right]. \quad (4.9)$$

Now we can investigate the geodesic motion for three distinct cases: $b > 0$, $b = 0$ and $b < 0$. However, since we are after Figure 4.1 and since all the null geodesics drawn in it come from $r = 0$, we must choose $b = 0$. Now it is much simpler to integrate the second equation, because after inserting $b = 0$, the remaining integration becomes trivial. Notice that if we set $b = 0$, then $\Delta = 1$, $u_1 = 0$ and (4.12) becomes

$$t - t_0 = -a\lambda + \sqrt{2} ArcTan(\sqrt{2} Tan(a(\lambda - \lambda_0))),$$

whereas the second integral becomes

$$\int d\phi = \int \frac{a\sqrt{2}d\lambda}{Sin^2(a(\lambda - \lambda_0)) + 1}. \quad (4.10)$$

(4.10) gives

$$\phi - \phi_0 = ArcTan(\sqrt{2} Tan(a(\lambda - \lambda_0))).$$

If we write them together, we now have the parametric representation of the null geodesics parameterized by λ :

$$Sinhr = \pm Sin(a(\lambda - \lambda_0)), \qquad\qquad (4.11)$$

$$t - t_0 = -a\lambda + \sqrt{2}ArcTan(\sqrt{2}Tan(a(\lambda - \lambda_0))), \qquad\qquad (4.12)$$

$$\phi - \phi_0 = ArcTan(\sqrt{2}Tan(a(\lambda - \lambda_0))), \qquad\qquad (4.13)$$

where t_0 and ϕ_0 are integration constants. These equations give the correct representation of the null geodesics drawn in Figure 4.1. For every ϕ_0 value, we have a different null geodesic. Suppose that one sends out a light signal from point p at some time of t_p in the direction ϕ equals zero, and let the constants $\phi_0 = 0$ and $t_0 = 0$ as well. Then as the angle increases, the distance between the light signal and the world line $r = 0$ increases until the light signal reaches its maximum distance $r = \ln(1 + \sqrt{2})$ from the world line $r = 0$, in which case the angle is equal to $\pi/2$. One can take the time the signal reaches the point $r = \ln(1 + \sqrt{2})$ as $t = 0$. Continuing on its path, the light signal returns to its starting point at the time $t_{p'}$. At point p, $r = 0$ and (4.11) gives $\lambda = \lambda_0$, whereas (4.12) gives $t_p = -a\lambda_0$. When the light signal reaches the point $r = \ln(1 + \sqrt{2})$ at time $t = 0$, (4.11) gives $a(\lambda - \lambda_0) = \pi/2$. Then from (4.12), we find

$$0 = -a\lambda_0 - \frac{\pi}{2} + \sqrt{2}\frac{\pi}{2}.$$

The time t_p becomes

$$t_p = -\frac{\pi}{2}(\sqrt{2} - 1).$$

In fact, this is the time for half of the trip. When the light signal returns to its starting point p' at a later time $t_{p'}$, (4.11) gives $a(\lambda - \lambda_0) = \pi$ and (4.12) yields

$$t_{p'} = -a\lambda_0 - \pi + \sqrt{2}\pi$$

i.e.

$$t_{p'} = \frac{\pi}{2}(\sqrt{2} - 1).$$

All the rays sent out simultaneously in different directions of the plane return later at the same moment. At the critical distance $r = \ln(1 + \sqrt{2})$ from the world line $r = 0$, we have circles for $t = \text{constant}$, because every null geodesic touches the $t = \text{constant}$ planes in one point. In their papers, Ozsvath and Schucking [7] and Malament [1] state that the diagram of the Gödel model in [6] is not correct about the matter lines, since they are drawn as spacelike. However they do not give the timelike geodesics equations which show that all the timelike geodesics must be in the circle with radius $r = \ln(1 + \sqrt{2})$. In the next section we show this property of the timelike geodesics. Moreover, the structure of the light cones, *i.e.* what happens to the light cones as we move away from the world line $r = 0$, is given in section 4.3.

4.2 Timelike Geodesics

Timelike geodesics can be calculated with the same method used in calculating the null geodesics but this time our Lagrangian must be equal to -1. Obviously, (4.1) and (4.2) hold also for this situation. However this time the affine parameter λ can be chosen as the proper time τ measured by a co-moving observer. Again we set the value of b to zero because of the same reason used in the previous section. Hence we have

31

$$\dot{\phi} = \frac{a\sqrt{2}}{Cosh^2 r} , \qquad (4.14)$$

$$\dot{t} = \frac{a(1 - Sinh^2 r)}{Cosh^2 r} , \qquad (4.15)$$

$$L = -\dot{t}^2 + \dot{r}^2 - (Sinh^4 r - Sinh^2 r)\dot{\phi}^2 - 2\sqrt{2}(Sinh^2 r)\dot{\phi}\,\dot{t} = -1 . \qquad (4.16)$$

If we put (4.14) and (4.15) into (4.16), we obtain

$$d\tau = \frac{Cosh r \; dr}{\left(a^2 - a^2 Sinh^2 r - Cosh^2 r\right)^{1/2}} , \qquad (4.17)$$

whose solution is (see Appendix D)

$$Sinh r = \Psi Sin[\chi(\tau - \tau_0)], \qquad (4.18)$$

where $\Psi = \dfrac{\eta}{\chi}$, $\eta = \sqrt{a^2 - 1}$ and $\chi = \sqrt{a^2 + 1}$ with the condition $a \geq 1$.

If we substitute (4.18) into (4.14) and (4.15), we get the integrals

$$\int dt = \int \frac{\{a - a\Psi^2 Sin^2[\chi(\tau - \tau_0)]\}d\tau}{\Psi^2 Sin^2[\chi(\tau - \tau_0)] + 1} , \qquad (4.19)$$

$$\int d\phi = \int \frac{a\sqrt{2}d\tau}{\Psi^2 Sin^2[\chi(\tau - \tau_0)] + 1} . \qquad (4.20)$$

These are easily evaluated as

$$\phi - \phi_0 = ArcTan\left[\frac{\sqrt{2}a}{\sqrt{a^2+1}}Tan\left[\sqrt{a^2+1}(\tau - \tau_0)\right]\right], \qquad (4.21)$$

$$t - t_0 = -a\tau + \sqrt{2}ArcTan\left[\frac{\sqrt{2}a}{\sqrt{a^2+1}}Tan\left[\sqrt{a^2+1}(\tau - \tau_0)\right]\right]. \qquad (4.22)$$

For every value of a we have a different timelike geodesic.

When $a = 1$, (4.18), (4.21) and (4.22) become

$$Sinh\, r = 0 \qquad \Rightarrow \qquad r = 0 ,$$

$$\phi - \phi_0 = \sqrt{2}(\tau - \tau_0),$$

$$t - t_0 = \tau - 2\tau_0,$$

respectively, which corresponds to the central matter world line. Suppose that at time $t = t_p$, a particle is released from point p in Figure 4.1 in the direction $\phi = \phi_0$. Then this particle will move on the central matter world line (notice that this particle moves only in time) and will reach the point O at time $t = 0$ when $\phi - \phi_0 = \frac{\pi}{2}$. At the end the particle reaches the point p' at time $t = t_{p'}$ when $\phi - \phi_0 = \pi$. Then one can easily obtain

$$t_p = -\frac{\sqrt{2}}{4}\pi , \qquad\qquad t_{p'} = \frac{\sqrt{2}}{4}\pi .$$

For all values of a, r is always smaller than $\ln(1 + \sqrt{2})$. Since a is interpreted as the energy per unit mass, then no massive particle $-$ no matter what its energy is $-$ can reach this critical radius. All the particles released from the point p with different energies will reach some maximum value of r smaller than $\ln(1 + \sqrt{2})$, and

then reconverge to p' which shows that all the matter lines must be inside the cylinder with radius $\ln(1+\sqrt{2})$.

4.3 The Structure of the Light Cones

To study the structure of the light cones, we need to find the closed-form solution of the null geodesics equations. After setting $b=0$, we now come back to (4.4), (4.5) and (4.6). Take (4.6)

$$\dot{\phi} = \frac{\sqrt{2}a}{Cosh^2 r}$$

and substitute this expression into (4.4) to obtain

$$\frac{d\phi}{dr} = \frac{\sqrt{2}}{Coshr\sqrt{1-Sinh^2 r}} . \qquad (4.23)$$

(4.23) gives

$$Sinhr = \frac{Sin(\phi+\phi_0)}{\sqrt{1+Cos^2(\phi+\phi_0)}} \qquad (4.24)$$

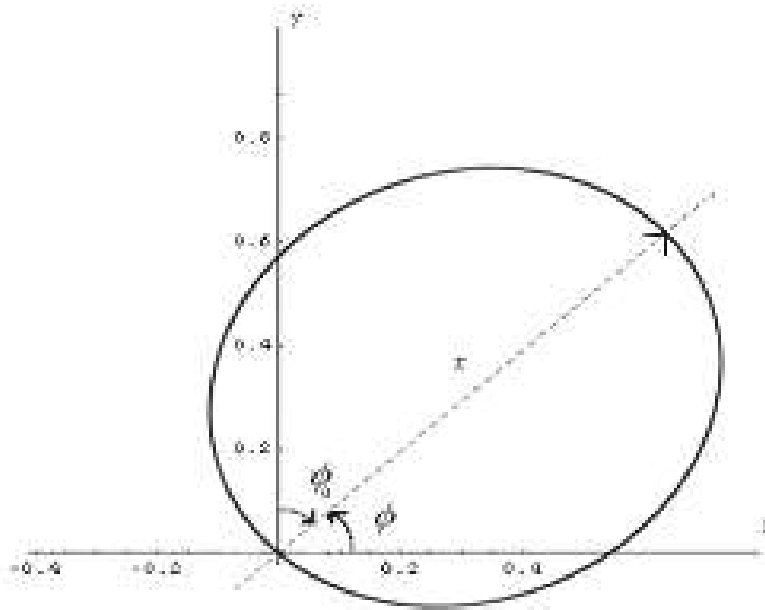with the integration constant ϕ_0 .

Figure 4.2: Path of the photon in the (r, ϕ) plane. For every value of ϕ_0, there is a different ellipse passing through the origin. These ellipses form an envelope, which is a circle with radius $r = \ln(1 + \sqrt{2})$ around the origin.

We have from (4.2) that

$$\frac{dt}{d\phi} = \frac{1 - Sinh^2 r}{\sqrt{2}}, \qquad (4.25)$$

or using (4.24)

$$\frac{dt}{d\phi} = \frac{\sqrt{2}}{2 + Tan^2(\phi + \phi_0)}. \qquad (4.26)$$

If we integrate (4.26), we get

$$t - t_0 = \sqrt{2}\,(\phi + \phi_0) - ArcTan(\frac{1}{\sqrt{2}}\,Tan(\phi + \phi_0))\ . \qquad\qquad (4.27)$$

We have two equations to think about, which are (4.24) and (4.25). From these two equations, we can of course obtain Figure 4.1 again. Although all of the comments made in section 4.1 hold also for these two equations, since there is no distinction between them, it will be useful to consider the situation again. For every ϕ_0 value, we have a different null geodesic. Let us see what happens to the light rays originating from point p in Figure 4.1. Let us think of the case where ϕ_0 is zero. We see that the left hand side of (4.24) is maximum when $\phi = \pi/2$, the null geodesic reaches its largest distance $r = \ln(1 + \sqrt{2})$; and its distance is zero when $\phi = 0$ and $\phi = \pi$. The light ray starts to move from point p when $\phi = 0$ and $t = t_0$, and reaches to its largest distance from the world line $r = 0$ and then goes to the point p'. The time taken by the light ray during the first half of its trip is

$$t(\pi/2) - t(0) = \frac{\pi}{2}\,(\sqrt{2} - 1)\ .$$

All the light rays originated from point p in different directions return later at the same moment to the point p'. Of course, the time interval between the point p and p' is

$$t(\pi) - t(0) = \pi\,(\sqrt{2} - 1)\ .$$

At the distance $r = \ln(1 + \sqrt{2})$ from the world line $r = 0$, we have circles for $t = \mathrm{constant}$. These circles are null curves as we see from (3.2). However they are not geodesics. They form a cylinder with an axis at $r = 0$. What happens to the light cones when we move away from the axis $r = 0$? To answer this question, we need dr/dt and it is sufficient to consider one radial direction (see line L in Figure 4.1). From (4.23) and (4.25), we obtain

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{Coshr(1-Sinh^2r)^{1/2}}{\sqrt{2}}\frac{\sqrt{2}}{(1-Sinh^2r)} = \frac{Coshr}{(1-Sinh^2r)^{1/2}}.$$

At the origin $r=0$, the slope of $r(t)$ is equal to one. This means that we have a light cone with an angle $\pi/4$. That is, on the central matter line the cones are upright and oriented at $\pi/4$ to the indicated horizontal plane. The last expression becomes infinite for $r = \ln(1+\sqrt{2})$. It means, as one moves outward, that the cones open up and tilt over in a counter-clockwise direction. At a critical radius $r = \ln(1+\sqrt{2})$, they become tangent to the plane.

4.4 Some Remarks About the Gödel Spacetime

For a given point P in the Gödel spacetime, there is a cylinder associated with P which separates the spacetime into two regions in a given coordinate system in which P rests on the $r=0$ line. All of the points, lying inside this cylinder, have radial distances from P which is less than $\ln(1+\sqrt{2})$. In fact this cylinder is constructed by the circles with radius $r = \ln(1+\sqrt{2})$ which are also null curves. Circles with radius $r > \ln(1+\sqrt{2})$ are closed timelike curves. For any value of energy a, the photons originating from P are confined inside this cylinder. This result holds also for massive particles. However massive particles are never able to reach the cylindrical surface whereas the photons are. Since there are no closed timelike curves within this cylinder, time t always increases along every future directed timelike curve. Beyond $r = \ln(1+\sqrt{2})$ the time t runs backward which can be seen from (4.15)

$$\dot{t} = \frac{a(1-Sinh^2r)}{Cosh^2r}. \tag{4.15}$$

This expression is zero for $r = \ln(1 + \sqrt{2})$ for which the time t changes its sign. For $r > \ln(1 + \sqrt{2})$, (4.15) becomes negative which means t decreases. This shows that there can not be a cosmic time coordinate t in this spacetime which increases along every future directed timelike or null curve. Moreover, our observer sitting on the $r = 0$ line will observe only some part of the spacetime since all of the geodesics are confined inside the cylinder. That is, you can not send a light signal beyond the radius $r = \ln(1 + \sqrt{2})$. Since there is no way to go beyond that cylinder, then there is no permission for the light rays, coming from distant galaxies, to enter inside the cylinder either.

CHAPTER 5

CONCLUSION

We have shown that by using the original Gödel metric in calculating the null geodesics of Gödel spacetime, Figure 4.1 can be obtained very easily. This answers the question that Ozsvath and Schucking have posed in their paper [7]. Moreover, the problem of Figure 4.1, that is pointed out by Ozsvath and Schucking [7] and Malament [1], is corrected with the help of timelike geodesics equations. By investigating the structure of light cones, we saw how the closed timelike curves are created in Gödel spacetime. Gödel spacetime is interesting in two features: Firstly, all the light rays and the free particles follow the trajectories which are closed in space (but not in spacetime, of course). This is called the *boomerang effect*. As it is easily seen from Figure 4.2 (for the case where $\phi_0 = 0$), if we stand at the origin and hold a flashlight pointing to the east, then all the light rays follow a closed path and come back to the origin from the west. Everything would be quite messy if we were to live in a Gödel spacetime. Secondly, the existence of closed timelike curves was first shown in Gödel spacetime and such curves allow one to travel back in time.

So far we have not referred to the notion of "*time travel*" even though the existence of closed timelike curves has been explicitly shown in Gödel spacetime. Since these closed timelike curves are not geodesics (the acceleration is not zero), one has to do an accelerated motion to follow these curves. The energy required for a time traveler to move on a closed timelike curve was calculated by Malament [11]. Since this energy issue is beyond the scope of this thesis, we have not studied numerical results of his calculations. It is just enough to say that a huge amount of energy, that is impossible to produce with our present technology, is required for such an accelerated motion to travel back in time. However the consequences of time travel would be important. There is no satisfactory answer to the question of what happens if one can travel backwards in time and meet his/her younger self. The issue remains obscure even though there are lots of scenarios to eliminate the problem that such a meeting can cause. There would certainly be a paradox if the person killed

his/her younger self. Some people tend to leave the Gödel solution aside due to such a possibility. We can not exclude the Gödel spacetime immediately just because it allows for time travel and the fact that the results of time travel are paradoxical to our minds. The second reason for the tendency of leaving Gödel spacetime aside is that it is not expanding while our universe is. This result alone makes it possible to rule out the Gödel spacetime as a possible model for describing our universe. However the issues that arise by the existence of closed timelike curves is still standing.

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APPENDIX A

THE COMPONENTS OF THE EINSTEIN TENSOR

For the sake of completeness, here we list the nontrivial components of the Einstein tensor of the Gödel spacetime corresponding to the metric (2.1).

$$G_{00} = \frac{(3v^2 - u)\dot{v}^2 - 2\ddot{u}(v^2 + u) + \dot{u}^2 + 4v\dot{v}\dot{u} - 4(v^3 + uv)\ddot{v}}{4(v^2 + u)^2} ,$$

$$G_{02} = \frac{2(v^4 - u^2)\ddot{v} + (-3v^2 + u)\dot{u}\dot{v} + (3uv - v^3)\dot{v}^2 + 2(v^3 + uv)\ddot{u} - v\dot{u}^2}{4(v^2 + u)^2} ,$$

$$G_{11} = \frac{\dot{v}^2}{4(v^2 + u)^2} ,$$

$$G_{22} = \frac{2(-v^4 - uv^2)\ddot{u} + v^2\dot{u}^2 + 2(v^3 - uv)\dot{u}\dot{v} + (-3uv^2 + u^2)\dot{v}^2 + 4(uv^3 + u^2v)\ddot{v}}{4(v^2 + u)^2} ,$$

$$G_{33} = \frac{3u\dot{v}^2 + 2\ddot{u}(v^2 + u) - \dot{u}^2 - 4v\dot{v}\dot{u} + 4\ddot{v}(v^3 + uv) - v^2\dot{v}^2}{4(v^2 + u)^2} .$$

APPENDIX B

TRANSFORMATION TO CYLINDRICAL COORDINATES

In this appendix, we show the details of the calculation that transforms the three dimensional part of the Gödel metric (2.26) to its form (3.1) in cylindrical coordinates. For this purpose let us start with (2.26)

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2} e^{2\sqrt{2}\Omega x} dy^2 - 2e^{\sqrt{2}\Omega x} dt dy .$$ (2.26)

Let us write (2.26) as

$$ds^2 = -(dt + e^{\sqrt{2}\Omega x} dy)^2 + dx^2 + \frac{1}{2} e^{2\sqrt{2}\Omega x} dy^2 ,$$ (B.1)

and then apply the following transformations to (B.1)

$$e^{\sqrt{2}\Omega x} = Cosh 2r + Cos\phi\ Sinh 2r ,$$ (B.2)

$$\Omega y e^{\sqrt{2}\Omega x} = Sin\phi\ Sinh 2r ,$$ (B.3)

$$Tan\frac{1}{2}(\phi + \Omega t - \sqrt{2}t') = e^{-2r} Tan\frac{1}{2}\phi , \qquad \frac{\left|\Omega t - \sqrt{2}t'\right|}{2} < \frac{\pi}{2} .$$ (B.4)

Let us first define new coordinates such that

$$2r = \rho , \qquad \phi = 2\delta .$$

Then (B.2), (B.3) and (B.4) can simply be written as

$$e^{\sqrt{2}\Omega x} = e^\rho Cos^2\delta + e^{-\rho} Sin^2\delta ,$$ (B.5)

$$\Omega y e^{\sqrt{2}\Omega x} = Sin\delta\ Cos\delta\ (e^\rho - e^{-\rho}) ,$$ (B.6)

$$Tan(\delta + \frac{\Omega}{2}t - \frac{\sqrt{2}}{2}t') = e^{-\rho}\,Tan\delta \ . \tag{B.7}$$

Now let us differentiate (B.5):

$$\sqrt{2}\Omega e^{\sqrt{2}\Omega x}dx = (e^{\rho}Cos^2\delta - e^{-\rho}Sin^2\delta)d\rho - 2Sin\delta\ Cos\delta\,(e^{\rho} - e^{-\rho})d\delta \ ,$$

i.e.

$$dx = (\frac{e^{\rho}Cos^2\delta - e^{-\rho}Sin^2\delta}{\sqrt{2}\Omega e^{\sqrt{2}\Omega x}})d\rho - 2Sin\delta\ Cos\delta\,(\frac{e^{\rho} - e^{-\rho}}{\sqrt{2}\Omega e^{\sqrt{2}\Omega x}})d\delta \ . \tag{B.8}$$

Meanwhile the differentiation of (B.6) gives:

$$\Omega e^{\sqrt{2}\Omega x}dy + \sqrt{2}\Omega^2 y e^{\sqrt{2}\Omega x}dx = (Cos^2\delta - Sin^2\delta)(e^{\rho} - e^{-\rho})d\delta$$
$$+ Sin\delta\ Cos\delta\,(e^{\rho} + e^{-\rho})d\rho. \tag{B.9}$$

If we use (B.8) in (B.6), we get

$$\sqrt{2}\Omega^2 y e^{\sqrt{2}\Omega x}dx = \frac{1}{e^{\sqrt{2}\Omega x}}\Big((e^{\rho}Cos^2\delta - e^{-\rho}Sin^2\delta)Sin\delta\ Cos\delta\,(e^{\rho} - e^{-\rho})d\rho\Big)$$
$$- \frac{2}{e^{\sqrt{2}\Omega x}}\Big(Sin^2\delta\ Cos^2\delta\,(e^{\rho} - e^{-\rho})^2\,d\delta\Big) \tag{B.10}$$

Now subtract (B.10) from (B.9) to obtain

$$\Omega e^{\sqrt{2}\Omega x}dy = \frac{2Cos\delta\ Sin\delta}{e^{\sqrt{2}\Omega x}}d\rho + \frac{(e^{\rho}Cos^2\delta - e^{-\rho}Sin^2\delta)(e^{\rho} - e^{-\rho})}{e^{\sqrt{2}\Omega x}}d\delta \ . \tag{B.11}$$

Taking the square of (B.11) gives

45

$$e^{2\sqrt{2}\Omega x}dy^2 = \frac{4Cos^2\delta\ Sin^2\delta}{\Omega^2 e^{2\sqrt{2}\Omega x}}d\rho^2 + \frac{(e^\rho Cos^2\delta - e^{-\rho}Sin^2\delta)^2(e^\rho - e^{-\rho})^2}{2\Omega^2 e^{2\sqrt{2}\Omega x}}d\delta^2$$

$$+ \frac{4Cos\delta\ Sin\delta}{e^{\sqrt{2}\Omega x}}\frac{(e^\rho Cos^2\delta - e^{-\rho}Sin^2\delta)(e^\rho - e^{-\rho})}{\Omega^2 e^{\sqrt{2}\Omega x}}d\delta\ d\rho.$$

(B.12)

In the same manner, taking the square of (B.8) gives

$$dx^2 = (\frac{e^\rho Cos^2\delta - e^{-\rho}Sin^2\delta}{\sqrt{2}\Omega e^{\sqrt{2}\Omega x}})^2 d\rho^2 + 4Sin^2\delta\ Cos^2\delta(\frac{e^\rho - e^{-\rho}}{\sqrt{2}\Omega e^{\sqrt{2}\Omega x}})^2 d\delta^2$$

$$- 2(\frac{e^\rho Cos^2\delta - e^{-\rho}Sin^2\delta}{\Omega e^{\sqrt{2}\Omega x}})Sin\delta\ Cos\delta(\frac{e^{2r} - e^{-2r}}{\Omega e^{\sqrt{2}\Omega x}})d\delta\ d\rho$$

(B.13)

The combination $dx^2 + \frac{1}{2}e^{2\sqrt{2}\Omega x}dy^2$ in the metric (B.1) can easily be calculated as (notice that the $d\delta\ d\rho$ terms in (B.12) and (B.13) are equal and opposite)

$$dx^2 + \frac{1}{2}e^{2\sqrt{2}\Omega x}dy^2 = \frac{\Omega^{-2}}{2}\left[d\rho^2 + (e^\rho - e^{-\rho})^2 d\delta^2\right].$$

(B.14)

Now the differentiation of (B.7) gives:

$$\left(1 + Tan^2(\delta + \frac{\Omega}{2}t - \frac{\sqrt{2}}{2}t')\right)(d\delta + \frac{\Omega}{2}dt - \frac{\sqrt{2}}{2}dt') = -e^{-\rho}Tan\delta\ dr + e^{-\rho}Sec^2\delta\ d\delta$$ (B.15)

and by using (B.8), (B.16) can be written as

$$d\delta + \frac{\Omega}{2} dt - \frac{\sqrt{2}}{2} dt' = \frac{-e^{-\rho} Tan\delta}{1 + e^{-2\rho} Tan^2\delta} d\rho + \frac{e^{-\rho} Sec^2\delta}{1 + e^{-2\rho} Tan^2\delta} d\delta \ .$$

Next, let us multiply and divide the right hand side of the above equation with $e^{\rho} Cos^2\delta$:

$$d\delta + \frac{\Omega}{2} dt - \frac{\sqrt{2}}{2} dt' = \frac{-Sin\delta \ Cos\delta}{e^{\rho} Cos^2\delta + e^{-\rho} Sin^2\delta} d\rho + \frac{1}{e^{\rho} Cos^2\delta + e^{-\rho} Sin^2\delta} d\delta \ ,$$

i.e.

$$dt = \Omega^{-1}\left(\frac{-2Sin\delta \ Cos\delta}{e^{\sqrt{2}\Omega x}} d\rho + 2(\frac{1}{e^{\sqrt{2}\Omega x}} - 1)d\delta + \sqrt{2}dt' \right). \tag{B.16}$$

Now add (B.11) and (B.16) up

$$dt + e^{\sqrt{2}\Omega x} dy = \Omega^{-1}\left(2(\frac{1}{e^{\sqrt{2}\Omega x}} - 1)d\phi + \sqrt{2}dt' + \frac{(e^{\rho} Cos^2\delta - e^{-\rho} Sin^2\delta)(e^{\rho} - e^{-\rho})}{e^{\sqrt{2}\Omega x}} d\delta \right),$$

i.e.

$$dt + e^{\sqrt{2}\Omega x} dy = \Omega^{-1}\left[\sqrt{2}dt' + \left(\frac{(e^{\rho} Cos^2\delta - e^{-\rho} Sin^2\delta)(e^{\rho} - e^{-\rho}) + 2}{e^{\sqrt{2}\Omega x}} - 2 \right)d\delta \right]$$

and we have,

$$dt + e^{\sqrt{2}\Omega x} dy = \Omega^{-1}\left[\sqrt{2}dt' + (e^{\rho/2} - e^{-\rho/2})^2 d\delta \right]. \tag{B.17}$$

Taking the square of (B.17) gives

$$(dt + e^{\sqrt{2}\Omega x} dy)^2 = \Omega^{-2} \left[2dt'^2 + (e^{\rho/2} - e^{-\rho/2})^4 d\delta^2 + 2\sqrt{2}(e^{\rho/2} - e^{-\rho/2})^2 dt'd\delta \right]. \quad \text{(B.18)}$$

Now let us put (B.14) and (B.18) into (B.1)

$$ds^2 = \Omega^{-2} \left[\frac{d\rho^2}{2} + \frac{(e^{\rho} - e^{-\rho})^2 - 2(e^{\rho/2} - e^{-\rho/2})^4}{2} d\delta^2 - 2\sqrt{2}(e^{\rho/2} - e^{-\rho/2})^2 dt'd\delta - 2dt'^2 \right].$$

If we write the above equation in terms of r, ϕ and t' again, we find

$$ds^2 = \Omega^{-2} \left[2dr^2 + \frac{(e^{2r} - e^{-2r})^2 - 2(e^{r} - e^{-r})^4}{8} d\phi^2 - \sqrt{2}(e^{r} - e^{-r})^2 dt'd\phi - 2dt'^2 \right].$$

It is now straightforward to obtain (3.1)

$$ds^2 = 2\Omega^{-2}(-dt'^2 + dr^2 - (Sinh^4 r - Sinh^2 r)d\phi^2 - 2\sqrt{2}Sinh^2 r \, d\phi \, dt'). \quad \text{(3.1)}$$

APPENDIX C

PROOF OF (3.18)

In this appendix, we show the proof of the statement that ($\omega_{\alpha\beta} = 0$ if and only if $V_{[\alpha}$ $V_{\nu;\mu]} = 0$). For this purpose, let us first use (3.11) and then (3.10) to obtain

$$V_{\alpha;\mu} P^{\mu}{}_{\beta} = V_{\alpha;\mu} (\delta^{\mu}{}_{\beta} + V^{\mu} V_{\beta}),$$

i.e.

$$V_{\alpha;\mu} P^{\mu}{}_{\beta} = V_{\alpha;\beta} + V^{\mu} V_{\beta} V_{\alpha;\mu}. \tag{C.1}$$

Now substituting (C.1) into (3.8)

$$\omega_{\alpha\beta} \equiv \frac{1}{2} (V_{\alpha;\beta} + V^{\mu} V_{\beta} V_{\alpha;\mu} - V_{\beta;\alpha} - V^{\mu} V_{\alpha} V_{\beta;\mu}),$$

we can obtain

$$\omega_{\alpha\beta} \equiv \frac{1}{2} V^{\mu} (V_{\beta} V_{\alpha;\mu} - V_{\beta} V_{\mu;\alpha} + V_{\beta} V_{\mu;\alpha} - V_{\alpha} V_{\beta;\mu} + V_{\alpha} V_{\mu;\beta} - V_{\alpha} V_{\mu;\beta}) + V_{[\alpha;\beta]} = 0, \tag{C.2}$$

where we have added and subtracted some terms. (C.2) can be written as

$$\omega_{\alpha\beta} = V_{[\alpha;\beta]} + \frac{1}{2} V^{\mu} (2 V_{\beta} V_{[\alpha;\mu]} + 2 V_{\alpha} V_{[\mu;\beta]} + V_{\beta} V_{\mu;\alpha} - V_{\alpha} V_{\mu;\beta}),$$

i.e.

$$\omega_{\alpha\beta} = V_{[\alpha;\beta]} + V^{\mu}(V_{\beta} \, V_{[\alpha;\mu]} + V_{\alpha} \, V_{[\mu;\beta]}) + \frac{1}{2}V^{\mu} \, V_{\beta} \, V_{\mu;\alpha} - \frac{1}{2}V^{\mu}V_{\alpha} \, V_{\mu;\beta}.$$

Notice that the terms in parentheses can be written as follows:

$$\omega_{\alpha\beta} = V_{[\alpha;\beta]} + V^{\mu}(-V_{\mu} \, V_{[\beta;\alpha]} + 3 \, V_{[\alpha}V_{\mu;\beta]}) + \frac{1}{2}V^{\mu} \, V_{\beta} \, V_{\mu;\alpha} - \frac{1}{2}V^{\mu}V_{\alpha} \, V_{\mu;\beta},$$

or

$$\omega_{\alpha\beta} = 3V^{\mu} \, V_{[\alpha}V_{\mu;\beta]} + \frac{1}{2}V^{\mu} \, V_{\beta} \, V_{\mu;\alpha} - \frac{1}{2}V^{\mu}V_{\alpha} \, V_{\mu;\beta}. \tag{C.3}$$

Recall that V^{μ} is timelike, then $V^{\mu}V_{\mu} = -1$. If we take the covariant derivative of $V^{\mu}V_{\mu}$, we get

$$\nabla_{\alpha}(V^{\mu}V_{\mu}) = 0,$$

i.e.

$$V_{\mu}\nabla_{\alpha}V^{\mu} + V^{\mu}\nabla_{\alpha}V_{\mu} = 2V_{\mu}\nabla_{\alpha}V^{\mu} = 0,$$

then (C.3) becomes

$$\omega_{\alpha\beta} = 3V^{\mu} \, V_{[\alpha}V_{\mu;\beta]}.$$

Thus $\omega_{\alpha\beta} = 0$ if and only if $V_{[\alpha}V_{\nu;\mu]} = 0$.

APPENDIX D

CALCULATIONS OF THE INTEGRALS

D.1 How to Integrate (4.7)

Recall

$$\int d\lambda = \int \frac{(Coshr)(Sinhr)dr}{\left\{a^2 Cosh^2 r\ Sinh^2 r -(-b+a\sqrt{2}Sinh^2 r)^2\right\}^{1/2}}\ .$$

(4.7)

If we make the substitution

$$u = Sinh^2 r\ ,$$

(D.1)

we get

$$\int d\lambda = \frac{1}{2a}\int \frac{du}{\left\{-u^2 +(1+2\sqrt{2}\frac{b}{a})u -\frac{b^2}{a^2}\right\}^{1/2}}\ .$$

Now let us find the discriminant of the quadratic equation in the denominator

$$\Delta = (1+2\sqrt{2}\frac{b}{a})^2 -4(-1)(-\frac{b^2}{a^2}) = 4\frac{b^2}{a^2} +4\sqrt{2}\frac{b}{a} +1\ .$$

Hence the roots of the denominator are

$$u_1 = \frac{1}{2} +\sqrt{2}\frac{b}{a} -\frac{\sqrt{\Delta}}{2}\ ,\qquad\qquad u_2 = \frac{1}{2} +\sqrt{2}\frac{b}{a} +\frac{\sqrt{\Delta}}{2}\ .$$

We now have

$$\int d\lambda = \frac{1}{2a} \int \frac{du}{(u-u_1)^{1/2}(-u+u_2)^{1/2}} \, . \tag{D.2}$$

Now we need another substitution

$$u - u_1 = (u_2 - u_1) Sin^2\theta \, . \tag{D.3}$$

Differentiation of (D.3) gives

$$\begin{aligned} du &= (u_2 - u_1) \, Sin\, 2\theta \ d\theta \\ &= 2(u_2 - u)^{1/2}(u-u_1)^{1/2} \, d\theta \, . \end{aligned} \tag{D.4}$$

Then by using (D.3) and (D.4), (D.2) can be written as

$$\int d\lambda = \frac{1}{a}\int d\theta \qquad \Rightarrow \qquad \lambda - \lambda_0 = \frac{1}{a}\theta \, \, .$$

After using the inverses of the substitutions (D.3) and (D.1), respectively, we get the desired result

$$Sinh^2 r = \sqrt{\Delta} Sin^2 (a(\lambda - \lambda_0)) + u_1 \, .$$

D.2 How to Integrate (4.8)

Recall

$$\int dt = \int \frac{\{-a\sqrt{\Delta} Sin^2 (a(\lambda - \lambda_0)) + a(1-u_1) + \sqrt{2}b\}d\lambda}{\sqrt{\Delta} Sin^2 (a(\lambda - \lambda_0)) + u_1 + 1} \, . \tag{4.8}$$

(4.8) can be written as

$$\int dt = -a\lambda + \int \frac{(2a+\sqrt{2}b)d\lambda}{\sqrt{\Delta}Sin^2(a(\lambda-\lambda_0))+u_1+1} .$$ (D.5)

After the substitution

$$q = Tan(a(\lambda-\lambda_0)),$$ (D.6)

(D.4) becomes

$$\int dt = -a\lambda + (2a+\sqrt{2}b)\int \frac{dq}{a(1+q^2)\left[\sqrt{\Delta}\dfrac{q^2}{1+q^2}+u_1+1\right]},$$

$$\int dt = -a\lambda + \frac{(2a+\sqrt{2}b)}{a}\int \frac{dq}{(\sqrt{\Delta}+u_1+1)q^2+(u_1+1)} .$$ (D.7)

Now we need another substitution

$$q = \sqrt{\frac{u_1+1}{\sqrt{\Delta}+u_1+1}}\ Tan\Phi ,$$ (D.8)

$$dq = \sqrt{\frac{u_1+1}{\sqrt{\Delta}+u_1+1}}(1+Tan^2\Phi)d\Phi ,$$

which takes (D.7) to

$$\int dt = -a\lambda + \frac{1}{a}\frac{(2a + \sqrt{2}b)}{\sqrt{\sqrt{\Delta} + u_1 + 1}\sqrt{u_1 + 1}}\int d\Phi,$$

$$t - t_0 = -a\lambda + \frac{1}{a}\frac{(2a + \sqrt{2}b)}{\sqrt{u_1 + 1}\sqrt{\sqrt{\Delta} + u_1 + 1}}\Phi.$$

After using the inverses of the substitutions (D.8) and (D.6) respectively, we get

$$t - t_0 = -a\lambda + \frac{1}{a}\frac{(2a + \sqrt{2}b)}{\sqrt{u_1 + 1}\sqrt{\sqrt{\Delta} + u_1 + 1}}ArcTan\left[\frac{\sqrt{\sqrt{\Delta} + u_1 + 1}}{\sqrt{u_1 + 1}}Tan(a(\lambda - \lambda_0))\right].$$

D.3 How to Integrate (4.17)

(4.17) yields

$$\int d\tau = \int \frac{Coshr\ dr}{\left(a^2 - a^2 Sinh^2r - Cosh^2r\right)^{1/2}}.$$

The substitution

$$n = Sinhr \qquad\qquad (D.9)$$

yields

$$\int d\tau = \int \frac{dn}{\left(-(a^2+1)n^2 + a^2 - 1\right)^{1/2}} \; ,$$

which can be written as

$$\int d\tau = \int \frac{dn}{(a^2-1)^{1/2}\left(-\dfrac{(a^2+1)}{(a^2-1)}n^2 + 1\right)^{1/2}} \; .$$

Then, after the second substitution

$$n = \sqrt{\frac{a^2-1}{a^2+1}} \, Sin\alpha \; , \tag{D.10}$$

we have

$$\int d\tau = \int \frac{d\alpha}{(a^2+1)^{1/2}} \; ,$$

$$\tau - \tau_0 = \frac{1}{\sqrt{a^2+1}}\alpha \; .$$

Now using the inverses of the substitutions (D.10) and (D.9), respectively, we get the desired result

$$Sinhr = \frac{\sqrt{a^2-1}}{\sqrt{a^2+1}} Sin\left[\sqrt{a^2+1}(\tau - \tau_0)\right].$$