GÖDEL’S METRIC AND ITS GENERALIZATION

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Prof. Dr. Canan Özgen
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Sinan Bilikmen
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Atalay Karasu
Supervisor

Examining Committee Members

Assoc. Prof. Dr. Bayram Tekin (METU, PHYS) ____________________

Prof. Dr. Atalay Karasu (METU, PHYS) ____________________

Prof. Dr. Ayşe Karasu (METU, PHYS) ____________________

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ABSTRACT

GÖDEL’S METRIC AND ITS GENERALIZATION

Özgören, Kıvanç
M. S. Department of Physics
Supervisor: Prof. Dr. Atalay Karasu

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In this thesis, firstly the original Gödel’s metric is examined in detail. Then a more general class of Gödel-type metrics is introduced. It is shown that they are the solutions of Einstein field equations with a physically acceptable matter distribution provided that some conditions are satisfied. Lastly, some examples of the Gödel-type metrics are given.

Keywords: Gödel-type metrics, Einstein-Maxwell field equations

ÖZ

GÖDEL METRİĞİ VE GENELLEMESİ

Özgören, Kıvanç
Yüksek Lisans Fizik Bölümü
Tez Yöneticisi: Prof. Dr. Atalay Karasu

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Bu tezde, öncelikle orijinal Gödel metriği ayrıntılı şekilde incelenmektedir. Ardından genel bir sınıf olarak Gödel-tipi metrikler tanıtılmaktadır. Belirli koşullar sağlandığında, bunların, Einstein alan denklemlerinin fiziksel olarak kabul edilebilir bir madde dağılımı için olan çözümleri olduğu gösterilmektedir. Son olarak Gödel-tipi metrikler hakkında bazı örnekler verilmektedir.

Anahtar kelimeler: Gödel-tipi metrikler, Einstein-Maxwell alan denklemleri

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CHAPTER 1

INTRODUCTION

The Gödel's metric is first introduced by Kurt Gödel in 1949 [1]. It has an importance because it is one of the solutions of the Einstein field equations with a homogeneous matter distribution. However, it is not an isotropic solution. For a homogeneous and isotropic matter distribution, General Relativity (without a cosmological constant) gives cosmological models such that the universe will either expand forever or collapse onto itself depending on its density. To avoid this, Einstein put a cosmological constant to the equations and obtained a static universe which is called as Einstein's static universe. The Gödel's metric is another stationary solution of the Einstein field equations with the same stress-energy tensor. However, the Gödel universe has some interesting properties such as it contains closed timelike or null curves (but not geodesics [3]).

Following the Gödel's paper, lots of papers were published on this subject. Two of the most importants are the followings: In 1980, Raychaudhuri and Thakura [6] investigated the homogeneity conditions of a class of cylindrically symmetric metrics to which the Gödel's metric belongs. In 1983, Rebouças and Tiomno [7] made a definition for the Gödel-type metrics in four dimensions and examined their homogeneity conditions. In addition to these, in 2003, Ozsvath and Schucking [9] investigated the light cone structure of the Gödel universe.

In this thesis, firstly, the original paper [1] of Gödel will be examined in detail in chapter 2. Some calculations which are not shown there will be given. Moreover, the computer plots of some simple geodesics of the Gödel universe will be presented in appendix B.

In chapter 3, a general class of Gödel-type metrics will be introduced. In literature, some metrics that show some of the characteristics of Gödel's metric are already called as Gödel-type metrics. So a general definition of Gödel-type metrics will be done. The key ingredient of this definition will be a ($D-1$)-dimensional metric which acts as a background to the Gödel-type metrics [2].

In chapter 4, the metrics with flat backgrounds will be examined. It will be shown that they are the solutions of the Einstein-Maxwell field equations for a charged dust distribution provided that a simple equation is satisfied which is the source-free Euclidean Maxwell's equation in $D-1$ dimensions. Similarly, the geodesic equation will turn out to be the Lorentz force equation again in $D-1$ dimensions.

In chapter 5, the metrics with non-flat backgrounds will be examined. Again it will be shown that the Einstein tensor may correspond to a physically acceptable matter distribution if the ($D-1$)-dimensional source-free Maxwell's equation is satisfied.

Lastly, some examples of the Gödel-type metrics will be given in chapter 6.

CHAPTER 2

GÖDEL'S METRIC

The new solution [1] introduced by Kurt Gödel to the Einstein field equations is for an incoherent (i.e. homogeneous) matter field at rest in a four-dimensional manifold M such as Einstein's static universe. It has some interesting properties and philosophical meaning which will be stated later. But firstly, it is better to describe how it satisfies the Einstein field equations.

2.1 The Original Solution

In accordance with the sign convention used in this thesis, the Gödel's metric defined in [1] can be written as

$$ds^2 = a^2\left[-dx_0^2 + dx_1^2 - \frac{e^{2x_1}}{2}dx_2^2 + dx_3^2 - 2e^{x_1}dx_0 dx_2\right] \tag{2.1}$$

in a four-dimensional manifold. Here x_0, x_1, x_2 and x_3 are local coordinates and a is a real constant. This can also be written in the following form:

$$ds^2 = a^2\left[-(dx_0 + e^{x_1}dx_2)^2 + dx_1^2 + \frac{e^{2x_1}}{2}dx_2^2 + dx_3^2\right]. \tag{2.2}$$

So the metric is:

$$g_{\mu\nu} = a^2\begin{bmatrix} -1 & 0 & -e^{x_1} & 0 \\ 0 & 1 & 0 & 0 \\ -e^{x_1} & 0 & -e^{2x_1}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mu,\nu = 0,1,2,3\,. \tag{2.3}$$

As a note, here and upto the end of this chapter, the Greek indices run from 0 to 3.

The determinant of the metric can be found as $g = -a^8 e^{2x_1}/2$ and the inverse of the metric can be obtained from $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$ (where δ^μ_ν is the Kronecker delta) as

$$g^{\mu\nu} = \frac{1}{a^2}\begin{bmatrix} 1 & 0 & -2e^{-x_1} & 0 \\ 0 & 1 & 0 & 0 \\ -2e^{-x_1} & 0 & 2e^{-2x_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.4}$$

Now the Christoffel symbols can be calculated from the following relation:

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right]. \tag{2.5}$$

Note that only $\partial_1 g_{02} = -a^2 e^{x_1}$ and $\partial_1 g_{22} = -a^2 e^{2x_1}$ are nonzero. So it is quite easy to find and compute the nonzero Christoffel symbols:

$$\begin{aligned}
\Gamma^0{}_{01} &= \frac{1}{2} g^{02}(\partial_1 g_{02}) = 1, \\
\Gamma^0{}_{12} &= \frac{1}{2}\left[g^{00}(\partial_1 g_{20}) + g^{02}(\partial_1 g_{22})\right] = \frac{e^{x_1}}{2}, \\
\Gamma^1{}_{02} &= \frac{1}{2} g^{11}(-\partial_1 g_{02}) = \frac{e^{x_1}}{2}, \\
\Gamma^1{}_{22} &= \frac{1}{2} g^{11}(-\partial_1 g_{22}) = \frac{e^{2x_1}}{2}, \\
\Gamma^2{}_{01} &= \frac{1}{2} g^{22}(\partial_1 g_{02}) = -e^{-x_1}.
\end{aligned} \tag{2.6}$$

The Ricci tensor can be obtained directly from:

$$R_{\mu\rho} = \partial_\nu \Gamma^\nu{}_{\mu\rho} - \partial_\mu \Gamma^\nu{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\nu{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\nu{}_{\alpha\mu}. \tag{2.7}$$

This equation can be simplified by using the fact that $\Gamma^\nu{}_{\nu\rho} = \delta^0_\rho$. Furthermore, only ∂_1 produces nonzero results. Then the equation reduces to:

$$R_{\mu\rho} = \partial_1 \Gamma^1{}_{\mu\rho} + \Gamma^1{}_{\mu\rho} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\nu{}_{\alpha\mu}. \tag{2.8}$$

Now it is easy to compute the nonzero components of the Ricci tensor:

$$\begin{aligned}
R_{00} &= -\Gamma^2{}_{10}\Gamma^1{}_{20} - \Gamma^1{}_{20}\Gamma^2{}_{10} = 1, \\
R_{02} &= \partial_1 \Gamma^1{}_{02} + \Gamma^1{}_{02} - \Gamma^1{}_{22}\Gamma^2{}_{10} - \Gamma^1{}_{02}\Gamma^0{}_{10} = e^{x_1}, \\
R_{22} &= \partial_1 \Gamma^1{}_{22} + \Gamma^1{}_{22} - \Gamma^1{}_{02}\Gamma^0{}_{12} - \Gamma^0{}_{12}\Gamma^1{}_{02} = e^{2x_1}.
\end{aligned} \tag{2.9}$$

So the Ricci tensor in the matrix form is

$$R_{\mu\nu} = \begin{bmatrix} 1 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{x_1} & 0 & e^{2x_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (2.10)$$

The Ricci scalar can be calculated from here which turns out to be a constant:

$$R = R_{\mu}{}^{\mu} = R_{\mu\nu} g^{\mu\nu} = -\frac{1}{a^2}. \qquad (2.11)$$

For an incoherent matter field at rest, the stress energy tensor is given as [5]

$$T_{\mu\nu} = \rho u_\mu u_\nu , \qquad (2.12)$$

where ρ is the density of the matter field and $u^\mu = (1/a, 0,0,0)$ is the unit vector in the direction of x_0 lines. So

$$u_\mu = u^\nu g_{\mu\nu} = (-a, 0, -ae^{x_1}, 0) \qquad (2.13)$$

and

$$T_{\mu\nu} = \rho a^2 \begin{bmatrix} 1 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{x_1} & 0 & e^{2x_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \rho a^2 R_{\mu\nu}. \qquad (2.14)$$

The Einstein field equation (with a cosmological term Λ) is:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (2.15)$$

So, it can easily be seen that the above equation is satisfied if $a^2 = 1/8\pi\rho$ and $\Lambda = -1/2a^2 = -4\pi\rho$.

2.2 Properties of the Gödel Universe

The manifold M that is defined by the Gödel's metric has the following properties:

First of all, M is homogeneous (i.e. all points of M are equivalent to each other) since it admits the following transformations seperately:

$$
\begin{aligned}
&\text{i) } x_0' = x_0 + b_0 ; \\
&\text{ii) } x_1' = x_1 + b_1 \text{ , } x_2' = x_2 e^{-b_1} ; \qquad (2.16)\\
&\text{iii) } x_2' = x_2 + b_2 ; \\
&\text{iv) } x_3' = x_3 + b_3 ,
\end{aligned}
$$

where b_0, b_1, b_2 and b_3 are some constants.

Furthermore, *M* is rotationally symmetric. If proper coordinates are used (which are given in appendix A), the metric can be converted to the following form:

$$ds^2 = 4a^2\left[-dt^2 + dr^2 + dz^2 - (\sinh^4 r - \sinh^2 r)d\varphi^2 - 2\sqrt{2}\sinh^2 r d\varphi dt\right]. \quad (2.17)$$

Here *r*, φ and *t* are cylindrical coordinates in subspaces $z = const$. So the rotational symmetry can be seen easily since the metric $g_{\mu\nu}$ does not depend on the angular coordinate φ.

In *M*, there is not any absolute time coordinate. In other words, the worldlines are not everywhere orthogonal to a one-parameter family of three-dimensional hypersurfaces (because otherwise a co-moving Gaussian coordinate system can be constructed in which an absolute time coordinate can be defined [5]). To prove this statement, suppose the contrary: Suppose that there exists such a family defined as

$$F(x^\mu) - \lambda = 0 , \quad (2.18)$$

where F is a fixed function and λ is the parameter. If a vector dx^μ is entirely in this surface, then $dF = \partial_\mu F dx^\mu = 0$. This means $\partial_\mu F$ is normal to the surface. So any vector field v_μ that is orthogonal to these family of surfaces can be written in terms of $\partial_\mu F$ as

$$v_\mu = l\partial_\mu F , \quad (2.19)$$

where *l* is an arbitrary scalar function. If a completely antisymmetric tensor

$$a_{\mu\nu\gamma} = v_{[\mu}\nabla_\gamma v_{\nu]} = \frac{1}{3!}\left[v_\mu(\nabla_\gamma v_\nu - \nabla_\nu v_\gamma) + v_\nu(\nabla_\mu v_\gamma - \nabla_\gamma v_\mu) + v_\gamma(\nabla_\nu v_\mu - \nabla_\mu v_\nu)\right]$$

(2.20)

is introduced, it can be calculated that $a_{\mu\nu\gamma} = 0$ for $v_\mu = l\partial_\mu F$. However, for the case of Gödel's solution, $u_\mu = (-a,0,-ae^{x_1},0)$ and $a_{\mu\nu\gamma}$ is not identically zero:

$$a_{\mu\nu\gamma} = -\frac{1}{6}a^2 e^{x_1}\varepsilon_{\mu\nu\gamma}, \tag{2.21}$$

which completes the proof.

Another property of M is that it contains closed timelike circles. So it is possible to travel in the Gödel universe and arrive to the starting point of the voyage. It is also possible to travel into the past. Remember that the Gödel's metric in cylindrical coordinates is

$$ds^2 = 4a^2\left[-dt^2 + dr^2 + dz^2 - (\sinh^4 r - \sinh^2 r)d\varphi^2 - 2\sqrt{2}\sinh^2 r d\varphi dt\right]. \tag{2.22}$$

Now, as an example of a closed timelike curve, take the one defined by $r = R$, $t = z = 0$. Then (2.22) becomes

$$ds^2 = -4a^2(\sinh^4 r - \sinh^2 r)d\varphi^2. \tag{2.23}$$

It can be seen from here that if $\sinh^4 R - \sinh^2 R > 0$ or $R > \ln(1+\sqrt{2})$, the curve is timelike. However, keep in mind that this curve is not a geodesic. Actually there is not any closed timelike geodesics in the Gödel universe [3]. So some acceleration is needed to follow the curve defined above.

Lastly, in the Gödel universe, the matter rotates with an angular velocity of $2\sqrt{\pi\rho}$. To prove this, lets intoduce the following vector which is defined in terms of $a_{\mu\nu\gamma}$:

$$\Omega^\beta = \frac{\varepsilon^{\beta\mu\nu\gamma}}{\sqrt{|\mathrm{g}|}}a_{\mu\nu\gamma}. \tag{2.24}$$

In a flat space with the usual coordinates, it can be seen that Ω^β is twice the angular velocity (see [5] for more details). Calculating Ω^β for the Gödel's metric gives $(0,0,0,\sqrt{2}/a^2)$. So the angular velocity is

$$a\frac{1}{2}\frac{\sqrt{2}}{a^2} = 2\sqrt{\pi\rho}. \tag{2.25}$$

It can be asked how the entire universe can rotate and with respect to what it rotates. Suppose a test particle is thrown in the x_1 direction. If there is no external force on it, it must follow a straight line. However, in the Gödel universe, this is not the case; it follows a circular path. The tangent to this path can be called as “the compass of inertia”. So the bulk matter in the Gödel universe rotates with respect to it.

2.3 Philosophical Considerations

First of all, Gödel’s solution permits travelling into the past. Although such a voyage would be extremely long, this breaks the causal structure of the spacetime and brings some paradoxes such as the one that a person can go back and kill himself.

Secondly, it was longly belived that an inertial frame is determined by the distant stars (in other words, the bulk matter of the universe). Some philosophers like Mach went one step forward and argued that the inertial forces in an accelerated frame may arise from the relative accelerated motions of bulk matter in the universe with respect to that frame. Although Einstein found this idea useful, after the discovery of General Relativity, it turned out that this idea is not correct. Nevertheless, it is expected that the compass of inertia is determined by the bulk matter of the universe and they should not rotate relative to each other. However, this is not the case in the Gödel’s solution.

Lastly, it is expected that the matter distribution in the universe should determine its structure uniquely. However, for the same stress-energy tensor, there are two solutions namely Gödel’s solution and Einstein’s static universe. So it can be said that General Relativity does not fit to this expectation unless the cosmological constant is not used or some boundary conditions are imposed.

CHAPTER 3

GÖDEL-TYPE METRICS

In this chapter, the Gödel-type metrics will be defined as a general class in a *D*-dimensional manifold. But to be able to generalize the Gödel's metric, the first thing to do is to investigate some of its mathematical characteristics.

First of all, it can be seen from (2.3) that the metric $g_{\mu\nu}$ (which was defined in a $D=4$ dimensional manifold) can be written as $g_{\mu\nu}=h_{\mu\nu}-u_{\mu}u_{\nu}$, where $h_{\mu\nu}$ is a degenerate $D\times D$ matrix with rank equal to $D-1$ such that $h_{0\mu}=0$. (Here and at the rest of this thesis, the Greek indices run from 0 to $D-1$ and the Latin indices run from 1 to $D-1$.) Actually, any metric can be written in this form if u^{μ} is a unit vector such that $u_{\mu}u^{\mu}=-1$ and taken as $u^{\mu}=-\delta_{0}^{\mu}/u_{0}$. Secondly, the Ricci tensor was obtained as $R_{\mu\nu}=u_{\mu}u_{\nu}/a^{2}$ where a is a real constant. This leads the Ricci scalar to be a constant and the Einstein tensor to correspond to a physically acceptable source. Also, it can be seen that $\partial_{0}g_{\mu\nu}=0$ which is another important property. Now, by considering these facts, lets try to define the Gödel-type metrics.

3.1 Definition of the Gödel-Type Metrics

Let *M* be a *D*-dimensional manifold with a metric of the form

$$g_{\mu\nu}=h_{\mu\nu}-u_{\mu}u_{\nu}. \tag{3.1}$$

In this thesis, the metrics of this form will be called as Gödel-type metrics if the following conditions are satisfied:

i) $h_{\mu\nu}$ is a degenerate $D \times D$ matrix with rank equal to $D-1$ such that $h_{k\mu}=0$. Here x^k is a fixed coordinate and *k* can be chosen as $0 \le k \le D-1$.

ii) $h_{\mu\nu}$ is a metric of a ($D-1$)-dimensional Riemannian manifold which can be thought as a "background" from which the Gödel-type metrics arise.

iii) $h_{\mu\nu}$ does not depend on the fixed coordinate ($\partial_k h_{\mu\nu}=0$).

iv) u^μ is a timelike unit vector such that $u_\mu u^\mu = -1$.

v) u^μ is chosen as $u^\mu = -\delta_k^\mu / u_k$.

vi) u_μ does not depend on the fixed coordinate ($\partial_k u_\mu = 0$).

Also, some other conditions on $h_{\mu\nu}$ and u_μ will be clarified in the next chapters when the Einstein tensor is forced to correspond to a physically acceptable source. Moreover, throughout this thesis, the fixed coordinate will be taken as x^0. Then $u_k = u_0$ and u_0 will be taken as $u_0 = 1$.

If the literature is investigated, it can be found that there are some classifications of the metrics similar to the Gödel-type metrics defined above. For example, Geroch [10] took $u^\mu = \xi^\mu / \sqrt{|\xi^\alpha \xi_\alpha|}$ (where ξ^μ is a Killing vector field to start with) and reduced the vacuum Einstein field equations to a scalar, complex, Ernst-type non-linear differential equation and developed a technique for generating new solutions of vacuum Einstein field equations from vacuum spacetimes. Also, (3.1) looks like the Kerr-Schild metrics ($g_{\mu\nu} = \eta_{\mu\nu} - \ell_\mu \ell_\nu$ where ℓ^μ is a null vector) and the metrics used in Kaluza-Klein reductions in string theories [11]. However, there are some major differences between these metrics and the Gödel-type metrics [2].

CHAPTER 4

GÖDEL-TYPE METRICS WITH FLAT BACKGROUNDS

In this chapter, the simplest choice for $h_{\mu\nu}$ will be examined that is

$$h_{ij} = \bar{\delta}_{ij}, \tag{4.1}$$

where $\bar{\delta}_{ij}$ is the $(D-1)$-dimensional Kronecker delta symbol. (Note that it can be written as $\bar{\delta}_{\mu\nu} = \delta_{\mu\nu} - \delta_{\mu 0}\delta_{\nu 0}$ in D dimensions.) Then it is easy to see that

$$\partial_\alpha h_{\mu\nu} = 0. \tag{4.2}$$

The inverse of the metric can be found as

$$g^{\mu\nu} = \bar{h}^{\mu\nu} + (-1 + \bar{h}^{\alpha\beta} u_\alpha u_\beta) u^\mu u^\nu + u^\mu (\bar{h}^{\nu\alpha} u_\alpha) + u^\nu (\bar{h}^{\mu\alpha} u_\alpha), \tag{4.3}$$

where $\bar{h}^{\mu\nu}$ is the $(D-1)$-dimensional inverse of $h_{\mu\nu}$ (i.e. $\bar{h}^{\mu\nu} h_{\nu\alpha} = \bar{\delta}^\mu_\alpha$). Now it is possible to calculate the Christoffel symbols:

$$\begin{aligned}
\Gamma^\mu{}_{\alpha\beta} &= \frac{1}{2} g^{\mu\sigma} (\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}) \\
&= \frac{1}{2} g^{\mu\sigma} (-\partial_\alpha u_\sigma u_\beta - \partial_\beta u_\sigma u_\alpha + \partial_\sigma u_\alpha u_\beta) \\
&= \frac{1}{2} g^{\mu\sigma} (-u_\sigma \partial_\alpha u_\beta - u_\beta \partial_\alpha u_\sigma - u_\sigma \partial_\beta u_\alpha - u_\alpha \partial_\beta u_\sigma + u_\alpha \partial_\sigma u_\beta + u_\beta \partial_\sigma u_\alpha) \\
&= \frac{1}{2} g^{\mu\sigma} \left[u_\alpha (\partial_\sigma u_\beta - \partial_\beta u_\sigma) + u_\beta (\partial_\sigma u_\alpha - \partial_\alpha u_\sigma) - u_\sigma (\partial_\alpha u_\beta + \partial_\beta u_\alpha) \right].
\end{aligned} \tag{4.4}$$

At this point, lets introduce $f_{\mu\nu} = \partial_\mu u_\nu - \partial_\nu u_\mu$ which will be very useful in the remaining calculations. Then (4.4) can be written as

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} (u_\alpha f^\mu{}_\beta + u_\beta f^\mu{}_\alpha) - \frac{1}{2} u^\mu (\partial_\alpha u_\beta + \partial_\beta u_\alpha), \tag{4.5}$$

where $f^\mu{}_\beta = g^{\mu\alpha} f_{\alpha\beta}$.

Before continuing further, lets give some useful identities which can be derived by using the newly introduced tensor $f_{\mu\nu}$. Firstly, $f_{\mu\nu}$ is an antisymmetric tensor so

$$f_{\mu\nu} = -f_{\nu\mu}\,. \tag{4.6}$$

Secondly,

$$f_{\mu 0} = \partial_\mu u_0 - \partial_0 u_\mu = 0\,, \tag{4.7}$$

which leads

$$u^\mu f_{\mu\nu} = u_\mu f^\mu{}_\nu = 0\,. \tag{4.8}$$

Now lets look the covariant derivative of u_μ:

$$\nabla_\alpha u_\beta = \partial_\alpha u_\beta - \Gamma^\mu{}_{\alpha\beta} u_\mu = \partial_\alpha u_\beta - \frac{1}{2} u_\mu (u_\alpha f^\mu{}_\beta + u_\beta f^\mu{}_\alpha) + \frac{1}{2} u_\mu u^\mu (\partial_\alpha u_\beta + \partial_\beta u_\alpha)$$

$$= \partial_\alpha u_\beta - \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) = \frac{1}{2} f_{\alpha\beta}\,, \tag{4.9}$$

which means the vector field u^μ satisfies the Killing vector equation,

$$\nabla_\alpha u_\beta + \nabla_\beta u_\alpha = 0\,, \tag{4.10}$$

and is a Killing vector. Furthermore, (4.9) leads

$$u^\beta \nabla_\alpha u_\beta = \frac{1}{2} u^\beta f_{\alpha\beta} = 0\,, \tag{4.11}$$

and

$$u^\alpha \nabla_\alpha u_\beta = \frac{1}{2} u^\alpha f_{\alpha\beta} = 0\,. \tag{4.12}$$

So the vector field u^μ is tangent to a geodesic curve.

By using these, the Ricci tensor can be obtained from:

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma{}_{\mu\nu} - \partial_\mu \Gamma^\sigma{}_{\sigma\nu} + \Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\rho\sigma} - \Gamma^\rho{}_{\sigma\mu}\Gamma^\sigma{}_{\rho\nu}\,. \tag{4.13}$$

This can be simplified since

$$\Gamma^\alpha{}_{\alpha\beta} = \frac{1}{2}(u_\alpha f^\alpha{}_\beta + u_\beta f^\alpha{}_\alpha) - \frac{1}{2} u^\alpha (\partial_\alpha u_\beta + \partial_\beta u_\alpha) = 0\,. \tag{4.14}$$

Then (4.13) becomes

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma{}_{\mu\nu} - \Gamma^\rho{}_{\sigma\mu}\Gamma^\sigma{}_{\rho\nu}\,. \tag{4.15}$$

The first term can be calculated as

$$\partial_\sigma \Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}\partial_\sigma (u_\mu f^\sigma{}_\nu + u_\nu f^\sigma{}_\mu) - \frac{1}{2}\partial_\sigma \left[u^\sigma (\partial_\mu u_\nu + \partial_\nu u_\mu)\right]$$

$$= \frac{1}{2}\left[\partial_\sigma (u_\mu f^\sigma{}_\nu) + \partial_\sigma (u_\nu f^\sigma{}_\mu) - u^\sigma \partial_\sigma (\partial_\mu u_\nu + \partial_\nu u_\mu)\right]$$

$$= \frac{1}{2}(f^\sigma{}_\nu \partial_\sigma u_\mu - u_\mu j_\nu + f^\sigma{}_\mu \partial_\sigma u_\nu - u_\nu j_\mu), \qquad (4.16)$$

where $j_\mu = \partial_\alpha f_\mu{}^\alpha$ and the second term of (4.15) is

$$\Gamma^\rho{}_{\sigma\mu}\Gamma^\sigma{}_{\rho\nu} = \left[\frac{1}{2}(u_\sigma f^\rho{}_\mu + u_\mu f^\rho{}_\sigma) - \frac{1}{2}u^\rho (\partial_\sigma u_\mu + \partial_\mu u_\sigma)\right]$$

$$\left[\frac{1}{2}(u_\rho f^\sigma{}_\nu + u_\nu f^\sigma{}_\rho) - \frac{1}{2}u^\sigma (\partial_\rho u_\nu + \partial_\nu u_\rho)\right]$$

$$= \frac{1}{4}\left[(u_\sigma f^\rho{}_\mu + u_\mu f^\rho{}_\sigma)(u_\rho f^\sigma{}_\nu + u_\nu f^\sigma{}_\rho)\right.$$

$$+ u^\rho u^\sigma (\partial_\sigma u_\mu + \partial_\mu u_\sigma)(\partial_\nu u_\rho + \partial_\rho u_\nu)$$

$$- u^\rho (\partial_\sigma u_\mu + \partial_\mu u_\sigma)(u_\rho f^\sigma{}_\nu + u_\nu f^\sigma{}_\rho)$$

$$\left. - u^\sigma (\partial_\nu u_\rho + \partial_\rho u_\nu)(u_\sigma f^\rho{}_\mu + u_\mu f^\rho{}_\sigma)\right]$$

$$= \frac{1}{4}\left[u_\sigma u_\rho f^\rho{}_\mu f^\sigma{}_\nu + u_\sigma u_\nu f^\rho{}_\mu f^\sigma{}_\rho + u_\mu u_\rho f^\rho{}_\sigma f^\sigma{}_\nu + u_\mu u_\nu f^\rho{}_\sigma f^\sigma{}_\rho\right.$$

$$\left. + f^\sigma{}_\nu (\partial_\mu u_\sigma + \partial_\sigma u_\mu) + f^\rho{}_\mu (\partial_\nu u_\rho + \partial_\rho u_\nu)\right]$$

$$= \frac{1}{4}\left[-u_\mu u_\nu f^2 + f^\sigma{}_\nu (\partial_\mu u_\sigma + \partial_\sigma u_\mu) + f^\rho{}_\mu (\partial_\nu u_\rho + \partial_\rho u_\nu)\right], \qquad (4.17)$$

where $f^2 = f^{\alpha\beta} f_{\alpha\beta}$. By combining these two quantities, the Ricci tensor can be obtained:

$$R_{\mu\nu} = \frac{1}{2}(f^\sigma{}_\nu \partial_\sigma u_\mu - u_\mu j_\nu + f^\sigma{}_\mu \partial_\sigma u_\nu - u_\nu j_\mu)$$

$$- \frac{1}{4}\left[-u_\mu u_\nu f^2 + f^\sigma{}_\nu (\partial_\mu u_\sigma + \partial_\sigma u_\mu) + f^\rho{}_\mu (\partial_\nu u_\rho + \partial_\rho u_\nu)\right]$$

$$= -\frac{1}{2}(u_\mu j_\nu + u_\nu j_\mu) + \frac{1}{2}f^\sigma{}_\nu \partial_\sigma u_\mu + \frac{1}{2}f^\sigma{}_\mu \partial_\sigma u_\nu + \frac{1}{4}u_\mu u_\nu f^2$$

$$-\frac{1}{4}f^{\sigma}{}_{\nu}(\partial_{\mu}u_{\sigma}+\partial_{\sigma}u_{\mu})-\frac{1}{4}f^{\rho}{}_{\mu}(\partial_{\nu}u_{\rho}+\partial_{\rho}u_{\nu})$$

$$=-\frac{1}{2}(u_{\mu}j_{\nu}+u_{\nu}j_{\mu})+\frac{1}{4}f^{2}u_{\mu}u_{\nu}-\frac{1}{4}f^{\sigma}{}_{\nu}f_{\mu\sigma}-\frac{1}{4}f^{\sigma}{}_{\mu}f_{\nu\sigma}$$

$$=\frac{1}{2}f_{\mu}{}^{\sigma}f_{\nu\sigma}-\frac{1}{2}(u_{\mu}j_{\nu}+u_{\nu}j_{\mu})+\frac{1}{4}f^{2}u_{\mu}u_{\nu}. \tag{4.18}$$

Then the Ricci scalar can be found easily:

$$R=R_{\mu\nu}g^{\mu\nu}=\frac{1}{4}f^{2}-u^{\mu}j_{\mu}. \tag{4.19}$$

The Einstein tensor is

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$$

$$=\frac{1}{2}f_{\mu}{}^{\sigma}f_{\nu\sigma}-\frac{1}{2}(u_{\mu}j_{\nu}+u_{\nu}j_{\mu})+\frac{1}{4}f^{2}u_{\mu}u_{\nu}-\frac{1}{2}g_{\mu\nu}\left(\frac{1}{4}f^{2}-u^{\sigma}j_{\sigma}\right). \tag{4.20}$$

This tensor should be equal to the stress-energy tensor of a physically acceptable source. To be so, j_{μ} should be something like $j_{\mu}=ku_{\mu}$ where k is a constant. But it can be seen that $j_{0}=0$ while $u_{0}\neq 0$. So $k=0$ which means $j_{\mu}=0$. Then (4.20) becomes

$$G_{\mu\nu}=\frac{1}{2}f_{\mu}{}^{\sigma}f_{\nu\sigma}+\frac{1}{4}f^{2}u_{\mu}u_{\nu}-\frac{1}{8}g_{\mu\nu}f^{2}. \tag{4.21}$$

The Maxwell energy-momentum tensor for $f_{\mu\nu}$ is given as

$$T^{f}_{\mu\nu}=f_{\mu}{}^{\sigma}f_{\nu\sigma}-\frac{1}{4}g_{\mu\nu}f^{2}. \tag{4.22}$$

Hence,

$$G_{\mu\nu}=\frac{1}{2}T^{f}_{\mu\nu}+\frac{1}{4}f^{2}u_{\mu}u_{\nu}, \tag{4.23}$$

which implies that (3.1) is the solution of a charged dust field with density $\rho=f^{2}/4$ provided that $j_{\mu}=0$. Since $j_{0}=0$ already and j_{μ} does not depend on x_{0}, this means

$$j_{i}=\partial_{j}f_{i}^{\ j}=0, \tag{4.24}$$

which is the flat ($D-1$)-dimensional Euclidean source-free Maxwell's equation. Note that $f_i^{\ j}$ can be written as

$$f_i^{\ j} = f_{ik} g^{kj}, \tag{4.25}$$

where $g^{kj} = \bar{h}^{kj} = \bar{\delta}^{kj}$ in this case. Hence

$$f_i^{\ j} = f_{ij}, \tag{4.26}$$

and (4.24) becomes

$$\partial_j f_{ij} = 0. \tag{4.27}$$

This means that the Gödel-type metrics with the conditions given at the beginning of this chapter are the solutions of Einstein field equations with a matter distribution of a charged dust with density $\rho = f^2/4$ provided that the above equation is satisfied.

4.1 Geodesics

Now lets investigate the geodesics of this case. The geodesic equation is given as

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0, \tag{4.28}$$

where a dot represents the derivative with respect to an affine parameter τ. Substituting (4.5) to this equation gives

$$\ddot{x}^\mu + \frac{1}{2}(u_\alpha f^\mu{}_\beta \dot{x}^\alpha \dot{x}^\beta + u_\beta f^\mu{}_\alpha \dot{x}^\alpha \dot{x}^\beta) - \frac{1}{2}u^\mu(\partial_\alpha u_\beta \dot{x}^\alpha \dot{x}^\beta + \partial_\beta u_\alpha \dot{x}^\alpha \dot{x}^\beta) = 0. \tag{4.29}$$

Here α and β are dummy indices so this can be written as

$$\ddot{x}^\mu + u_\alpha f^\mu{}_\beta \dot{x}^\alpha \dot{x}^\beta - u^\mu \dot{x}^\alpha(\partial_\beta u_\alpha \dot{x}^\beta) = 0. \tag{4.30}$$

Using the fact that

$$\partial_\beta u_\alpha \dot{x}^\beta = \frac{\partial u_\alpha}{\partial x^\beta}\dot{x}^\beta = \dot{u}_\alpha, \tag{4.31}$$

the geodesic equation becomes

$$\ddot{x}^\mu + u_\alpha f^\mu{}_\beta \dot{x}^\alpha \dot{x}^\beta - u^\mu \dot{x}^\alpha \dot{u}_\alpha = 0. \tag{4.32}$$

Contracting this with u_μ gives

$$u_\mu \ddot{x}^\mu + u_\mu u_\alpha f^\mu{}_\beta \dot{x}^\alpha \dot{x}^\beta - u_\mu u^\mu \dot{x}^\alpha \dot{u}_\alpha = 0, \tag{4.33}$$

and since the second term vanishes,

$$u_\mu \ddot{x}^\mu + \dot{x}^\alpha \dot{u}_\alpha = 0\,, \tag{4.34}$$

which implies

$$u_\mu \dot{x}^\mu = const. = -e\,. \tag{4.35}$$

Remembering that $u_0 = 1$, this can be written as

$$\dot{x}^0 + u_i \dot{x}^i = -e\,. \tag{4.36}$$

Furthermore, if (4.35) is substituted to (4.32):

$$\ddot{x}^\mu - ef^\mu{}_\beta \dot{x}^\beta - u^\mu \dot{x}^\alpha \dot{u}_\alpha = 0\,. \tag{4.37}$$

Since $u^\mu = -\delta_0^\mu$, this can be written by replacing μ with *i* as

$$\ddot{x}^i - ef^i{}_\beta \dot{x}^\beta = 0\,. \tag{4.38}$$

Lastly by using $f^i{}_0 = 0$, the following equation is obtained:

$$\ddot{x}^i - ef^i{}_j \dot{x}^j = 0\,. \tag{4.39}$$

This is the ($D-1$)-dimensional Lorentz force equation for a charged particle with the charge/mass ratio *e*. Contracting this by $\dot{x}^i$ and using (4.26) gives

$$\dot{x}^i \ddot{x}^i - ef^i{}_j \dot{x}^i \dot{x}^j = \dot{x}^i \ddot{x}^i - ef_{ij} \dot{x}^i \dot{x}^j = \dot{x}^i \ddot{x}^i = 0\,. \tag{4.40}$$

So another constant of motion is found as

$$\dot{x}^i \dot{x}^i = const. = l^2\,. \tag{4.41}$$

Since

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - (u_\mu \dot{x}^\mu)^2 = l^2 - e^2\,, \tag{4.42}$$

the nature of the geodesics necessarily depends on the sign of $l^2 - e^2$.

CHAPTER 5

GÖDEL-TYPE METRICS WITH NON-FLAT BACKGROUNDS

In chapter 4, the simplest choice for h_{ij} ($h_{ij} = \bar{\delta}_{ij}$) was examined. Now lets see what will happen if h_{ij} is not determined at the beginning. Again the inverse of the metric is given by (4.3) but the new Christoffel symbols are:

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma}(\partial_{\alpha} g_{\sigma\beta} + \partial_{\beta} g_{\sigma\alpha} - \partial_{\sigma} g_{\alpha\beta})$$

$$= \frac{1}{2} g^{\mu\sigma}(\partial_{\alpha} h_{\sigma\beta} + \partial_{\beta} h_{\sigma\alpha} - \partial_{\sigma} h_{\alpha\beta}) + \frac{1}{2} g^{\mu\sigma}(-\partial_{\alpha} u_{\sigma} u_{\beta} - \partial_{\beta} u_{\sigma} u_{\alpha} + \partial_{\sigma} u_{\alpha} u_{\beta}) .$$

(5.1)

Note that the second term is equal to the $\Gamma^{\mu}{}_{\alpha\beta}$ given by (4.5). If $g^{\mu\sigma}$ is substituted from (4.3),

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2}\left[\bar{h}^{\mu\sigma} + (-1 + \bar{h}^{\rho\gamma} u_{\rho} u_{\gamma}) u^{\mu} u^{\sigma} + u^{\mu}(\bar{h}^{\sigma\rho} u_{\rho}) + u^{\sigma}(\bar{h}^{\mu\rho} u_{\rho})\right]$$

$$(\partial_{\alpha} h_{\sigma\beta} + \partial_{\beta} h_{\sigma\alpha} - \partial_{\sigma} h_{\alpha\beta}) + \Gamma^{\mu}{}_{\alpha\beta} . \tag{5.2}$$

The terms of $g^{\mu\sigma}$ containing u^{σ} vanish. Then

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2}(\bar{h}^{\mu\sigma} + u^{\mu}(\bar{h}^{\sigma\rho} u_{\rho}))(\partial_{\alpha} h_{\sigma\beta} + \partial_{\beta} h_{\sigma\alpha} - \partial_{\sigma} h_{\alpha\beta}) + \Gamma^{\mu}{}_{\alpha\beta}$$

$$= \hat{\Gamma}^{\mu}{}_{\alpha\beta} + u^{\mu} u_{\rho} \hat{\Gamma}^{\rho}{}_{\alpha\beta} + \Gamma^{\mu}{}_{\alpha\beta} , \tag{5.3}$$

where $\hat{\Gamma}^{\mu}{}_{\alpha\beta}$ are the Christoffel symbols of $h_{\mu\nu}$ given by

$$\hat{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{1}{2} \bar{h}^{\mu\sigma}(\partial_{\alpha} h_{\sigma\beta} + \partial_{\beta} h_{\sigma\alpha} - \partial_{\sigma} h_{\alpha\beta}) . \tag{5.4}$$

The covariant derivative associated with $\hat{\Gamma}^{\mu}{}_{\alpha\beta}$ can be defined as

$$\hat{\nabla}_{\beta} u_{\alpha} = \partial_{\beta} u_{\alpha} - \hat{\Gamma}^{\mu}{}_{\alpha\beta} u_{\mu} , \tag{5.5}$$

so (5.3) can also be written as

$$\tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \hat{\Gamma}^{\mu}{}_{\alpha\beta} + \frac{1}{2}(u_{\alpha} f^{\mu}{}_{\beta} + u_{\beta} f^{\mu}{}_{\alpha}) - \frac{1}{2} u^{\mu} (\hat{\nabla}_{\alpha} u_{\beta} + \hat{\nabla}_{\beta} u_{\alpha}) . \tag{5.6}$$

Lastly, the covariant derivative associated with $\tilde{\Gamma}^{\mu}{}_{\alpha\beta}$ can be defined as

$$\tilde{\nabla}_{\alpha} u_{\beta} = \partial_{\alpha} u_{\beta} - \tilde{\Gamma}^{\mu}{}_{\alpha\beta} u_{\mu} , \tag{5.7}$$

and can be calculated by using (5.3) as

$$\tilde{\nabla}_{\alpha} u_{\beta} = \partial_{\alpha} u_{\beta} - (u_{\mu} \hat{\Gamma}^{\mu}{}_{\alpha\beta} + u_{\mu} u^{\mu} u_{\rho} \hat{\Gamma}^{\rho}{}_{\alpha\beta} + u_{\mu} \Gamma^{\mu}{}_{\alpha\beta}) = \frac{1}{2} f_{\alpha\beta} , \tag{5.8}$$

which means u^{μ} is still a Killing vector. Also it is still tangent to a geodesic curve since

$$u^{\alpha} \tilde{\nabla}_{\alpha} u_{\beta} = 0 . \tag{5.9}$$

Using these results, the Ricci tensor can be calculated from:

$$\tilde{R}_{\mu\nu} = \partial_{\sigma} \tilde{\Gamma}^{\sigma}{}_{\mu\nu} - \partial_{\mu} \tilde{\Gamma}^{\sigma}{}_{\sigma\nu} + \tilde{\Gamma}^{\rho}{}_{\mu\nu} \tilde{\Gamma}^{\sigma}{}_{\rho\sigma} - \tilde{\Gamma}^{\rho}{}_{\sigma\mu} \tilde{\Gamma}^{\sigma}{}_{\rho\nu} . \tag{5.10}$$

However, now $\tilde{\Gamma}^{\alpha}{}_{\alpha\beta} \neq 0$. By using (5.3),

$$\tilde{\Gamma}^{\alpha}{}_{\alpha\beta} = \hat{\Gamma}^{\alpha}{}_{\alpha\beta} + u^{\alpha} u_{\rho} \hat{\Gamma}^{\rho}{}_{\alpha\beta} + \Gamma^{\alpha}{}_{\alpha\beta} . \tag{5.11}$$

Since $\Gamma^{\alpha}{}_{\alpha\beta} = 0$ and $u^{\alpha} \hat{\Gamma}^{\rho}{}_{\alpha\beta} = 0$,

$$\tilde{\Gamma}^{\alpha}{}_{\alpha\beta} = \hat{\Gamma}^{\alpha}{}_{\alpha\beta} . \tag{5.12}$$

Substituting (5.12) and then (5.3) to (5.10) gives

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= \partial_{\sigma} \tilde{\Gamma}^{\sigma}{}_{\mu\nu} - \partial_{\mu} \hat{\Gamma}^{\sigma}{}_{\sigma\nu} + \tilde{\Gamma}^{\rho}{}_{\mu\nu} \hat{\Gamma}^{\sigma}{}_{\rho\sigma} - \tilde{\Gamma}^{\rho}{}_{\sigma\mu} \tilde{\Gamma}^{\sigma}{}_{\rho\nu} \\
&= \partial_{\sigma} \hat{\Gamma}^{\sigma}{}_{\mu\nu} + \partial_{\sigma} \left(u^{\sigma} u_{\rho} \hat{\Gamma}^{\rho}{}_{\mu\nu} \right) + \partial_{\sigma} \Gamma^{\sigma}{}_{\mu\nu} - \partial_{\mu} \hat{\Gamma}^{\sigma}{}_{\sigma\nu} \\
&\quad + \hat{\Gamma}^{\beta}{}_{\mu\nu} \hat{\Gamma}^{\sigma}{}_{\sigma\beta} + u^{\beta} u_{\rho} \hat{\Gamma}^{\rho}{}_{\mu\nu} \hat{\Gamma}^{\sigma}{}_{\sigma\beta} + \Gamma^{\beta}{}_{\mu\nu} \hat{\Gamma}^{\sigma}{}_{\sigma\beta} \\
&\quad - \hat{\Gamma}^{\beta}{}_{\sigma\mu} \hat{\Gamma}^{\sigma}{}_{\beta\nu} - u^{\sigma} u_{\gamma} \hat{\Gamma}^{\gamma}{}_{\beta\nu} \hat{\Gamma}^{\beta}{}_{\sigma\mu} - \Gamma^{\sigma}{}_{\beta\nu} \hat{\Gamma}^{\beta}{}_{\sigma\mu} \\
&\quad - u^{\beta} u_{\rho} \hat{\Gamma}^{\rho}{}_{\sigma\mu} \hat{\Gamma}^{\sigma}{}_{\beta\nu} - u^{\beta} u^{\sigma} u_{\rho} u_{\gamma} \hat{\Gamma}^{\gamma}{}_{\beta\nu} \hat{\Gamma}^{\rho}{}_{\sigma\mu} - u^{\beta} u_{\rho} \Gamma^{\sigma}{}_{\beta\nu} \hat{\Gamma}^{\rho}{}_{\sigma\mu} \\
&\quad - \Gamma^{\beta}{}_{\sigma\mu} \hat{\Gamma}^{\sigma}{}_{\beta\nu} - u^{\sigma} u_{\gamma} \hat{\Gamma}^{\gamma}{}_{\beta\nu} \Gamma^{\beta}{}_{\sigma\mu} - \Gamma^{\sigma}{}_{\beta\nu} \Gamma^{\beta}{}_{\sigma\mu} .
\end{aligned} \tag{5.13}$$

After some cancelations:

$$\widetilde{R}_{\mu\nu} = \hat{R}_{\mu\nu} + \left[\partial_\sigma (u^\sigma u_\rho \hat{\Gamma}^\rho{}_{\mu\nu}) + \partial_\sigma \Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\beta\nu}\Gamma^\beta{}_{\sigma\mu}\right] + \Gamma^\beta{}_{\mu\nu}\hat{\Gamma}^\sigma{}_{\sigma\beta}$$
$$- \Gamma^\sigma{}_{\beta\nu}\hat{\Gamma}^\beta{}_{\sigma\mu} - \Gamma^\beta{}_{\sigma\mu}\hat{\Gamma}^\sigma{}_{\beta\nu} - u^\beta u_\rho \hat{\Gamma}^\rho{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - u^\sigma u_\gamma \hat{\Gamma}^\gamma{}_{\beta\nu}\Gamma^\beta{}_{\sigma\mu}\,, \qquad (5.14)$$

where $\hat{R}_{\mu\nu}$ is the Ricci tensor of $h_{\mu\nu}$ and the terms in the bracket give $R_{\mu\nu}$ which is given in (4.18). By rearranging the dummy indices and using $u^\beta \Gamma^\mu{}_{\alpha\beta} = -f^\mu{}_\alpha/2$, the above equation simplifies to

$$\widetilde{R}_{\mu\nu} = \hat{R}_{\mu\nu} + \left[\frac{1}{2} f_\mu{}^\sigma f_{\nu\sigma} - \frac{1}{2}(u_\mu j_\nu + u_\nu j_\mu) + \frac{1}{4} f^2 u_\mu u_\nu\right]$$
$$+ \frac{1}{2}(u_\mu f^\beta{}_\nu + u_\nu f^\beta{}_\mu)\hat{\Gamma}^\sigma{}_{\sigma\beta} - \frac{1}{2} u_\nu f^\sigma{}_\beta \hat{\Gamma}^\beta{}_{\sigma\mu} - \frac{1}{2} u_\mu f^\sigma{}_\beta \hat{\Gamma}^\beta{}_{\sigma\nu}$$
$$= \hat{R}_{\mu\nu} + \frac{1}{2} f_\mu{}^\sigma f_{\nu\sigma} + \frac{1}{4} f^2 u_\mu u_\nu + \frac{1}{2} u_\mu (-j_\nu - f^\sigma{}_\beta \hat{\Gamma}^\beta{}_{\sigma\nu} + f^\beta{}_\nu \hat{\Gamma}^\sigma{}_{\sigma\beta})$$
$$+ \frac{1}{2} u_\nu (-j_\mu - f^\sigma{}_\beta \hat{\Gamma}^\beta{}_{\sigma\mu} + f^\beta{}_\mu \hat{\Gamma}^\sigma{}_{\sigma\beta})\,. \qquad (5.15)$$

Now lets define $\widetilde{j}_\mu$ as

$$\widetilde{j}_\mu = \hat{\nabla}_\sigma f^\sigma{}_\mu = \partial_\sigma f^\sigma{}_\mu - f^\sigma{}_\beta \hat{\Gamma}^\beta{}_{\sigma\mu} + f^\beta{}_\mu \hat{\Gamma}^\sigma{}_{\sigma\beta}\,. \qquad (5.16)$$

Since $j_\mu = \partial_\sigma f_\mu{}^\sigma = -\partial_\sigma f^\sigma{}_\mu$, the Ricci tensor becomes

$$\widetilde{R}_{\mu\nu} = \hat{R}_{\mu\nu} + \frac{1}{2} f_\mu{}^\sigma f_{\nu\sigma} + \frac{1}{2}(u_\mu \widetilde{j}_\nu + u_\nu \widetilde{j}_\mu) + \frac{1}{4} f^2 u_\mu u_\nu\,, \qquad (5.17)$$

and the Ricci scalar is

$$\widetilde{R} = \hat{R} + \frac{1}{4} f^2 + u^\mu \widetilde{j}_\mu\,. \qquad (5.18)$$

By setting $\widetilde{j}_\mu = 0$ again, the Einstein tensor can be found as

$$\widetilde{G}_{\mu\nu} = \hat{R}_{\mu\nu} + \frac{1}{2} f_\mu{}^\sigma f_{\nu\sigma} + \frac{1}{4} f^2 u_\mu u_\nu - \frac{1}{2} g_{\mu\nu}\hat{R} - \frac{1}{8} g_{\mu\nu} f^2\,, \qquad (5.19)$$

and in terms of $T^f_{\mu\nu}$ given by (4.22), it can be written as

$$\widetilde{G}_{\mu\nu} = \hat{R}_{\mu\nu} + \frac{1}{2} T^f_{\mu\nu} + \frac{1}{4} f^2 u_\mu u_\nu - \frac{1}{2} g_{\mu\nu}\hat{R}\,, \qquad (5.20)$$

or

$$\widetilde{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} h_{\mu\nu}\hat{R} + \frac{1}{2} T^f_{\mu\nu} + \left(\frac{1}{4} f^2 + \frac{1}{2}\hat{R}\right) u_\mu u_\nu\,. \qquad (5.21)$$

Now lets turn back to $\tilde{j}_\mu = 0$. This equation means

$$\bar{h}^{\mu\nu}\tilde{j}_\nu = \bar{h}^{\mu\nu}\hat{\nabla}_\alpha(\bar{h}^{\alpha\beta} f_{\beta\nu}) = \hat{\nabla}_\alpha(\bar{h}^{\mu\nu}\bar{h}^{\alpha\beta} f_{\beta\nu})$$

$$= \partial_\alpha(\bar{h}^{\mu\nu}\bar{h}^{\alpha\beta} f_{\beta\nu}) + \hat{\Gamma}^\alpha{}_{\alpha\rho}(\bar{h}^{\mu\nu}\bar{h}^{\rho\beta} f_{\beta\nu}) + \hat{\Gamma}^\mu{}_{\alpha\rho}(\bar{h}^{\rho\nu}\bar{h}^{\alpha\beta} f_{\beta\nu}) = 0 \,. \quad (5.22)$$

Here, the last term vanishes and using $\hat{\Gamma}^\alpha{}_{\alpha\rho} = \partial_\rho h / 2h$ gives

$$\sqrt{h}\partial_\alpha(\bar{h}^{\mu\nu}\bar{h}^{\alpha\beta} f_{\beta\nu}) + (\bar{h}^{\mu\nu}\bar{h}^{\rho\beta} f_{\beta\nu})\frac{1}{2\sqrt{h}}\partial_\rho h = 0 \,, \quad (5.23)$$

or

$$\partial_\alpha(\bar{h}^{\mu\nu}\bar{h}^{\alpha\beta}\sqrt{h} f_{\beta\nu}) = 0 \,. \quad (5.24)$$

Using $\bar{h}^{\mu 0} = 0$ and $f_{\mu 0} = 0$, this can be written as

$$\partial_i(\bar{h}^{ik}\bar{h}^{jl}\sqrt{h} f_{kl}) = 0 \,. \quad (5.25)$$

Note that, this is the source-free Maxwell equation in $D-1$ dimensions. If this equation is satisfied, the Einstein tensor may correspond to a physically acceptable stress-energy tensor depending on the suitable choice of $h_{\mu\nu}$.

CHAPTER 6

SOME EXAMPLES OF THE GÖDEL-TYPE METRICS

In this chapter some examples of the Gödel-type metrics will be presented.

6.1 The Original Gödel's Metric as a Gödel-type Metric

First of all, lets represent the original Gödel's metric as a Gödel-type metric and see which conditions does it satisfy. Remember that the Gödel's metric was:

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1 & 0 & -e^{x_1} & 0 \\ 0 & 1 & 0 & 0 \\ -e^{x_1} & 0 & -e^{2x_1}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.1}$$

Furthermore, lets take $a = -1$ for simplicity. Since $u^{\mu} = -\delta_0^{\mu}$, it can be found that $u_{\mu} = g_{\mu\nu} u^{\nu} = (1, 0, e^{x_1}, 0)$ and

$$u_{\mu} u_{\nu} = \begin{bmatrix} 1 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & 0 \\ e^{x_1} & 0 & e^{2x_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.2}$$

If the metric is written in the form of $g_{\mu\nu} = h_{\mu\nu} - u_{\mu} u_{\nu}$, it can be seen that

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2x_1}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.3}$$

Its determinant is $h = e^{2x_1}/2$ and inverse is

$$\bar{h}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2e^{-2x_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.4}$$

Note that these satisfies (5.25):

$$\partial_i(\bar{h}^{ik}\bar{h}^{jl}\sqrt{h}f_{kl}) = \partial_1(\bar{h}^{11}\bar{h}^{22}\sqrt{h}f_{12}) = \partial_1(2e^{-2x_1}\sqrt{e^{2x_1}/2}(-e^{x_1})) = 0. \tag{6.5}$$

So it can be concluded that (5.25) is one of the conditions for (3.1) to be a Gödel-type metric.

6.2 An Example with a Flat Background

In chapter 4, $h_{\mu\nu}$ was taken as $h_{ij} = \bar{\delta}_{ij}$ and a condition on u_μ was searched for such that the resultant Einstein tensor corresponds to a physically acceptable matter distribution. Then it was found that

$$\partial_i f_{ij} = 0. \tag{6.6}$$

Now lets take u_i in the form of

$$u_i = Q_{ij}x^j, \tag{6.7}$$

and check if it satisfies the above equation. Here, Q_{ij} is an antisymmetric tensor with constant components. Substituting this to (6.6) gives

$$\partial_i\left[\partial_i(Q_{jk}x^k) - \partial_j(Q_{ik}x^k)\right] = \partial_i(Q_{ji} - Q_{ij}) = \partial_i(-2Q_{ij}) = 0, \tag{6.8}$$

which means u_i satisfies (6.6). For the remaining part, lets take $D = 4$ and let the only nonzero component of Q_{ij} be Q_{12}. Then

$$u_\mu dx^\mu = u_0 dx^0 + u_i dx^i = dx^0 + Q_{12}(x^2dx^1 - x^1dx^2), \tag{6.9}$$

or in cylindrical coordinates,

$$u_\mu dx^\mu = dt - Q_{12}\rho^2 d\phi. \tag{6.10}$$

Substituting this to (3.1):

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 - (dt - Q_{12}\rho^2 d\phi)^2. \tag{6.11}$$

Now consider the curve defined as

$$C = \{(t, \rho, \phi, z) \mid t = t_0, \rho = \rho_0, z = z_0\}. \tag{6.12}$$

Then (6.11) becomes

$$ds^2 = \rho_0^2 (1 - (Q_{12})^2 \rho_0^2) d\phi^2. \tag{6.13}$$

From here, it can be seen that this spacetime contains closed timelike and null curves for $\rho_0 \geq 1/|Q_{12}|$.

6.3 An Example with a Non-flat Background

In chapter 5, h_{ij} was not choosen at the beginning and naturally it was found that the Einstein tensor of $g_{\mu\nu}$ explicitly depends on the Ricci tensor and scalar of h_{ij} in the following way:

$$\tilde{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \hat{R} + \frac{1}{2} T^f_{\mu\nu} + \left(\frac{1}{4} f^2 + \frac{1}{2} \hat{R} \right) u_\mu u_\nu. \tag{6.14}$$

Depending on h_{ij}, $\hat{R}_{ij}$ may be very complex and may not allow the Einstein tensor to correspond any physically acceptable matter distribution. However, if h_{ij} is choosen as a metric of a ($D-1$)-dimensional Einstein space (i.e. $\hat{R}_{ij} = kh_{ij}$ where k is a constant), then

$$\hat{R} = kh_{ij}\bar{h}^{ij} = (D-1)k, \tag{6.15}$$

and the Einstein tensor becomes

$$\tilde{G}_{\mu\nu} = \left(\frac{3-D}{2} \right) k g_{\mu\nu} + \frac{1}{2} T^f_{\mu\nu} + \left(\frac{1}{4} f^2 + k \right) u_\mu u_\nu, \tag{6.16}$$

which describes a charged perfect fluid with pressure

$$p = \frac{1}{2}(3-D)k, \tag{6.17}$$

and density

$$\rho = \frac{1}{4} f^2 + \frac{1}{2}(D-1)k. \tag{6.18}$$

Note that $k<0$ (for $D\geq 4$) in order to have a positive pressure. Now lets choose $D=4$ as before. According to [4] (see appendix D of it), if h_{ij} is choosen as a conformally flat metric (i.e. $h_{ij}=e^{2\psi}\bar{\delta}_{ij}$ where ψ is a smooth function), then

$$\hat{R}_{ij}=-\partial_i\partial_j\psi-\bar{\delta}_{ij}\bar{\delta}^{kl}\partial_k\partial_l\psi+(\partial_i\psi)(\partial_j\psi)-\bar{\delta}_{ij}\bar{\delta}^{kl}(\partial_k\psi)(\partial_l\psi)\,. \qquad (6.19)$$

At this point, to simplify the calculations, lets assume that ψ is a function of only one of the coordinates; $\psi=\psi(x^3)=\psi(z)$. In addition to the equation above, the Einstein space condition implies

$$\hat{R}_{ij}=ke^{2\psi}\bar{\delta}_{ij}\,. \qquad (6.20)$$

Hence, the following equations can be obtained from the above ones:

$$-2\psi''=ke^{2\psi}\,, \qquad \text{(for } i=j=3\text{)} \qquad (6.21)$$

$$-\psi''-(\psi')^2=ke^{2\psi}\,, \qquad \text{(for } i=j\neq 3\text{)} \qquad (6.22)$$

where a prime denotes the derivative with respect to *z*. Combining these two equations,

$$\psi'=e^{\psi}\sqrt{\frac{-k}{2}}\,, \qquad (6.23)$$

and ψ can be obtained here as

$$\psi=\ln\sqrt{\frac{-2}{k(z+a)^2}}\,, \qquad (6.24)$$

where *a* is an integration constant. So this means

$$h_{ij}=\frac{-2}{k(z+a)^2}\bar{\delta}_{ij}\,. \qquad (6.25)$$

Remember that all of these calculations are meaningful if (5.25) is satisfied which can be simplified to the following form:

$$\partial_i(\bar{h}^{ik}\bar{h}^{jl}\sqrt{h}f_{kl})=\partial_i\left(\frac{k(z+a)^2}{-2}\bar{\delta}^{ik}\,\frac{k(z+a)^2}{-2}\bar{\delta}^{jl}\sqrt{\frac{-8}{k^3(z+a)^6}}f_{kl}\right)$$

$$=\partial_i((z+a)f_{ij})=0\,. \qquad (6.26)$$

To solve this equation, lets assume $u_i=s(x,y,z)\bar{\delta}_i^3$. Then

$$f_{ij}=\bar{\delta}_j^3\partial_i s-\bar{\delta}_i^3\partial_j s\,. \qquad (6.27)$$

When $j = 3$, (6.26) gives

$$\partial_\ell((z+a)\partial_\ell s) = 0, \qquad \ell = 1,2. \tag{6.28}$$

When $j \neq 3$, it gives

$$\partial_3((z+a)\partial_\ell s) = 0. \tag{6.29}$$

So it can be seen that s can be choosen as

$$s(x,y,z) = s(z) = \frac{b}{z+a}, \tag{6.30}$$

where b is another constant. Substituting these into (3.1), the line element in cylindrical coordinates can be found as

$$ds^2 = \frac{-2}{k(z+a)^2}(d\rho^2 + \rho^2 d\phi^2 + dz^2) - \left(dt + \frac{b}{z+a}dz\right)^2. \tag{6.31}$$

Again considering the curve C defined by (6.12), (6.31) becomes

$$ds^2 = \frac{-2\rho_0^2 d\phi^2}{k(z_0+a)^2}. \tag{6.32}$$

So this spacetime contains no closed timelike curves since $k < 0$.

CHAPTER 7

CONCLUSION

As a summary, at the beginning, the original Gödel metric was introduced, and at the rest of the thesis, it was tried to be generalized. Starting with a general metric form (3.1), the conditions needed to call this metric as a Gödel-type metric were investigated. It turned out that if (5.25) is satisfied and the background geometry is an Einstein space or at least if its Ricci scalar is constant, the metric becomes a solution to the Einstein field equations with a physical matter distribution.

The Gödel-type metrics introduced in this thesis can also be used in obtaining exact solutions to various supergravity theories and some examples can be found in [2]. Remember that u_k was chosen as a constant in chapter 3 and for this case, the Gödel-type metrics were found to be the solutions of the Einstein-Maxwell field equations. Then, for the case of non-constant u_k, it is expected that they are the solutions of the Einstein-Maxwell dilaton 3-form field equations. In fact, it is shown in [12] that the conformally transformed Gödel-type metrics can be used in solving a rather general class of Einstein-Maxwell-dilaton-3-form field theories in $D \geq 6$ dimensions.

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APPENDIX A

GÖDEL'S METRIC IN CYLINDRICAL COORDINATES

According to (2.2) Gödel's metric is given as

$$ds^2 = a^2\left[-(dx_0 + e^{x_1}dx_2)^2 + dx_1^2 + \frac{e^{2x_1}}{2}dx_2^2 + dx_3^2\right]. \qquad \text{(a1.1)}$$

It can be converted to

$$ds^2 = 4a^2\left[-dt^2 + dr^2 + dz^2 - (\sinh^4 r - \sinh^2 r)d\varphi^2 - 2\sqrt{2}\sinh^2 r d\varphi dt\right] \quad \text{(a1.2)}$$

if the following transformations are made:

$$e^{x_1} = \cosh 2r + \cos\varphi \sinh 2r\,, \qquad \text{(a1.3)}$$

$$x_2 e^{x_1} = \sqrt{2}\sin\varphi \sinh 2r\,, \qquad \text{(a1.4)}$$

$$\tan\left(\frac{\varphi}{2} + \frac{x_0 - 2t}{2\sqrt{2}}\right) = e^{-2r}\tan\frac{\varphi}{2}, \quad \text{where} \quad \left|\frac{x_0 - 2t}{2\sqrt{2}}\right| < \frac{\pi}{2}, \qquad \text{(a1.5)}$$

$$x_3 = 2z\,. \qquad \text{(a1.6)}$$

If these equations are differentiated:

$$e^{x_1}dx_1 = 2\sinh 2r dr - \sin\varphi \sinh 2r d\varphi + 2\cos\varphi \cosh 2r dr\,, \qquad \text{(a1.7)}$$

$$e^{x_1}dx_2 + x_2 e^{x_1}dx_1 = \sqrt{2}\cos\varphi \sinh 2r d\varphi + 2\sqrt{2}\sin\varphi \cosh 2r dr\,, \qquad \text{(a1.8)}$$

$$\left[1 + \tan^2\left(\frac{\varphi}{2} + \frac{x_0 - 2t}{2\sqrt{2}}\right)\right]\left(\frac{d\varphi}{2} + \frac{dx_0 - 2dt}{2\sqrt{2}}\right) = -2e^{-2r}\tan\frac{\varphi}{2}dr + \frac{e^{-2r}}{2}\left(1 + \tan^2\frac{\varphi}{2}\right)d\varphi, \qquad \text{(a1.9)}$$

$$dx_3 = 2dz\,. \qquad \text{(a1.10)}$$

Substituting (a1.4) and (a1.7) to (a1.8) gives

$$\mathrm{e}^{x_1}dx_2 = \sqrt{2}\cos\varphi \sinh 2r d\varphi + 2\sqrt{2}\sin\varphi \cosh 2r dr - \mathrm{e}^{-x_1}\sqrt{2}\sin\varphi \sinh 2r$$
$$(2\sinh 2r dr - \sin\varphi \sinh 2r d\varphi + 2\cos\varphi \cosh 2r dr)\,. \qquad \text{(a1.11)}$$

After some cancelations:

$$e^{2x_1}dx_2 = \sqrt{2}\cos\varphi\sinh 2r\cosh 2rd\varphi + 2\sqrt{2}\sin\varphi dr + \sqrt{2}\sinh^2 2rd\varphi . \quad (a1.12)$$

And its square is:

$$\begin{aligned} e^{4x_1}dx_2^2 &= 2\cos^2\varphi\sinh^2 2r\cosh^2 2rd\varphi^2 + 8\sin^2\varphi dr^2 + 2\sinh^4 2rd\varphi^2 \\ &+ 8\sin\varphi\cos\varphi\sinh 2r\cosh 2rd\varphi dr + 4\cos\varphi\sinh^3 2r\cosh 2rd\varphi^2 \\ &+ 8\sin\varphi\sinh^2 2rdrd\varphi . \end{aligned} \quad (a1.13)$$

From (a1.7):

$$\begin{aligned} e^{2x_1}dx_1^2 &= 4\sinh^2 2rdr^2 + \sin^2\varphi\sinh^2 2rd\varphi^2 + 4\cos^2\cosh 2rdr^2 \\ &- 4\sin\varphi\sinh^2 2rdrd\varphi + 8\cos\varphi\sinh 2r\cosh 2rdr^2 \\ &- 4\sin\varphi\cos\varphi\sinh 2r\cosh 2rdrd\varphi . \end{aligned} \quad (a1.14)$$

So

$$\begin{aligned} e^{2x_1}dx_1^2 + \frac{1}{2}e^{4x_1}dx_2^2 &= 4\sinh^2 2rdr^2 + \sin^2\varphi\sinh^2 2rd\varphi^2 + 4\cos^2\varphi\cosh^2 2rdr^2 \\ &+ 8\cos\varphi\sinh 2r\cosh 2rdr^2 + \cos^2\varphi\sinh^2 2r\cosh^2 2rd\varphi^2 \\ &+ 4\sin^2\varphi dr^2 + \sinh^4 2rd\varphi^2 + 2\cos\varphi\sinh^3 2r\cosh 2rd\varphi^2 . \end{aligned}$$

(a1.15)

Note that

$$\begin{aligned} e^{2x_1} &= (\cosh 2r + \cos\varphi\sinh 2r)^2 \\ &= \cosh^2 2r + 2\cos\varphi\sinh 2r\cosh 2r + \cos^2\varphi\sinh^2 2r \\ &= (1+\sinh^2 2r) + 2\cos\varphi\sinh 2r\cosh 2r + (\cos^2\varphi\cosh^2 2r - \cos^2\varphi) \\ &= \sinh^2 2r + 2\cos\varphi\sinh 2r\cosh 2r + \cos^2\varphi\cosh^2 2r + \sin^2\varphi . \end{aligned} \quad (a1.16)$$

Combining these two results:

$$dx_1^2 + \frac{e^{2x_1}}{2}dx_2^2 = 4dr^2 + \sinh^2 2rd\varphi^2 = 4dr^2 + 4\sinh^2 r\cosh^2 rd\varphi^2 . \quad (a1.17)$$

From (a1.9):

$$dx_0 = \frac{-4\sqrt{2}\tan\frac{\varphi}{2}dr + \sqrt{2}\left(1+\tan^2\frac{\varphi}{2}\right)d\varphi}{e^{2r} + e^{-2r}\tan^2\frac{\varphi}{2}} + (-\sqrt{2}d\varphi + 2dt) \quad (a1.18)$$

Note that

$$e^{2r} + e^{-2r}\tan^2\frac{\varphi}{2} = \cosh 2r + \sinh 2r + (\cosh 2r - \sinh 2r)\sin^2\frac{\varphi}{2}\cos^{-2}\frac{\varphi}{2}$$

$$= \cos^{-2}\frac{\varphi}{2}\left[\cosh 2r + \left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)\sinh 2r\right]$$

$$= \cos^{-2}\frac{\varphi}{2}e^{x_1} \qquad \text{(a1.19)}$$

Then

$$e^{x_1}dx_0 = -4\sqrt{2}\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}dr + \sqrt{2}\left(\cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}\right)d\varphi + e^{x_1}(-\sqrt{2}d\varphi + 2dt)$$

$$= -2\sqrt{2}\sin\varphi dr + \sqrt{2}d\varphi + (\cosh 2r + \cos\varphi\sinh 2r)(-\sqrt{2}d\varphi + 2dt)\,. \qquad \text{(a1.20)}$$

Using this and (a1.12),

$$e^{x_1}dx_0 + e^{2x_1}dx_2 = \sqrt{2}d\varphi + (\cosh 2r + \cos\varphi\sinh 2r)(-\sqrt{2}d\varphi + 2dt)$$

$$+ \sqrt{2}\cos\varphi\sinh 2r\cosh 2rd\varphi + \sqrt{2}(\cosh^2 2r - 1)d\varphi$$

$$= -\sqrt{2}e^{x_1}d\varphi + \sqrt{2}e^{x_1}\cosh 2rd\varphi + 2e^{x_1}dt\,. \qquad \text{(a1.21)}$$

Dividing this by e^{x_1} gives

$$dx_0 + e^{x_1}dx_2 = \sqrt{2}(\cosh 2r - 1)d\varphi + 2dt = 2\sqrt{2}\sinh^2 rd\varphi + 2dt\,. \qquad \text{(a1.22)}$$

And its square is:

$$(dx_0 + e^{x_1}dx_2)^2 = 8\sinh^4 rd\varphi^2 + 4dt^2 + 8\sqrt{2}\sinh^2 rd\varphi dt\,. \qquad \text{(a1.23)}$$

Substituting this result and (a1.17) to (a1.1):

$$ds^2 = a^2\left[-8\sinh^4 rd\varphi^2 - 4dt^2 - 8\sqrt{2}\sinh^2 rd\varphi dt\right.$$

$$\left. + 4dr^2 + 4\sinh^2 r\cosh^2 rd\varphi^2 + 4dz^2\right] \qquad \text{(a1.24)}$$

and using $\cosh^2 r = \sinh^2 r + 1$, the desired result is obtained:

$$ds^2 = 4a^2\left[-dt^2 + dr^2 + dz^2 - (\sinh^4 r - \sinh^2 r)d\varphi^2 - 2\sqrt{2}\sinh^2 rd\varphi dt\right]. \qquad \text{(a1.25)}$$

APPENDIX B

GEODESICS OF THE GÖDEL UNIVERSE

In this appendix, the simplest geodesics of the Gödel universe will be given which are the null geodesics passing through the origin. According to appendix A, Gödel's metric in cylindrical coordinates is given as

$$ds^2 = 4a^2\left[-dt^2 + dr^2 + dz^2 - (\sinh^4 r - \sinh^2 r)d\varphi^2 - 2\sqrt{2}\sinh^2 r d\varphi dt\right]. \quad \text{(a2.1)}$$

Simply the Lagrangian can be taken as

$$L = -\dot{t}^2 + \dot{r}^2 + \dot{z}^2 - (\sinh^4 r - \sinh^2 r)\dot{\varphi}^2 - 2\sqrt{2}(\sinh^2 r)\dot{\varphi}\dot{t}, \quad \text{(a2.2)}$$

where a dot denotes the derivative with respect to an affine parameter λ. Here the z coordinate can be omitted because it has nothing to do with the following calculations. The Euler-Lagrange equations are

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} - \frac{\partial L}{\partial t} = 0, \quad \text{(a2.3)}$$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0. \quad \text{(a2.4)}$$

The Euler-Lagrange equation for r is too complicated and will not be used so it is not given above. The above ones give the following equations:

$$2\dot{t} + 2\sqrt{2}(\sinh^2 r)\dot{\varphi} = 2a, \quad \text{(a2.5)}$$

$$2\dot{\varphi}(\sinh^4 r - \sinh^2 r) + 2\sqrt{2}(\sinh^2 r)\dot{t} = 2b \quad \text{(a2.6)}$$

where a and b are some constants. For the geodesics that passes through the origin (i.e. $r = 0$), it can be seen that b must be equal to zero. Furthermore, for the null geodesics, the following equation holds:

$$-\dot{t}^2 + \dot{r}^2 - (\sinh^4 r - \sinh^2 r)\dot{\varphi}^2 - 2\sqrt{2}(\sinh^2 r)\dot{\varphi}\dot{t} = 0. \quad \text{(a2.7)}$$

Combining this with (a2.6) while taking $b = 0$:

$$-\dot{t}^2 + \dot{r}^2 - \sqrt{2}(\sinh^2 r)\dot{\varphi}\dot{t} = 0 . \qquad \text{(a2.8)}$$

As a summary, the following equations are in hand:

$$\dot{t} + \sqrt{2}\sinh^2 r\dot{\varphi} = a , \qquad \text{(a2.9)}$$

$$\left(\sinh^4 r - \sinh^2 r\right)\dot{\varphi} + \sqrt{2}\sinh^2 r\dot{t} = 0 . \qquad \text{(a2.10)}$$

$$\dot{t}^2 - \dot{r}^2 + \sqrt{2}\sinh^2 r\dot{\varphi}\dot{t} = 0 , \qquad \text{(a2.11)}$$

From (a2.9) and (a2.10)

$$a = \frac{\cosh^2 r}{\sqrt{2}}\dot{\varphi} . \qquad \text{(a2.12)}$$

From (a2.9) and (a2.11)

$$\dot{r}^2 - a\dot{t} = 0 . \qquad \text{(a2.13)}$$

Combining (a2.9), (a2.12) and (a2.13)

$$\frac{dr}{d\varphi} = \frac{\cosh r}{\sqrt{2}}\sqrt{1 - \sinh^2 r} . \qquad \text{(a2.14)}$$

From (a2.10)

$$\frac{dt}{d\varphi} = \frac{1 - \sinh^2 r}{\sqrt{2}} . \qquad \text{(a2.15)}$$

Using these two results

$$\frac{dr}{dt} = \frac{\cosh r}{\sqrt{1 - \sinh^2 r}} . \qquad \text{(a2.16)}$$

If these three equations are plotted by a computer (by defining $x = r\cos\varphi$ and $y = r\sin\varphi$), the following figures can be obtained (the units are the geometrized units in which $G = c = 1$ as usual):

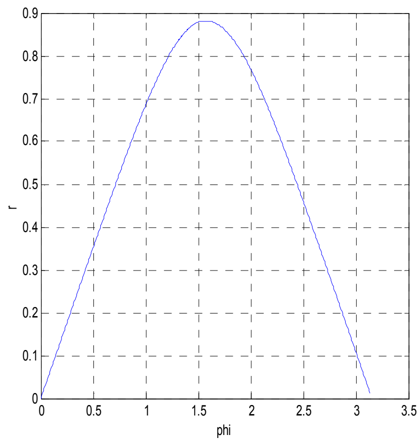

Fig 1: r vs φ

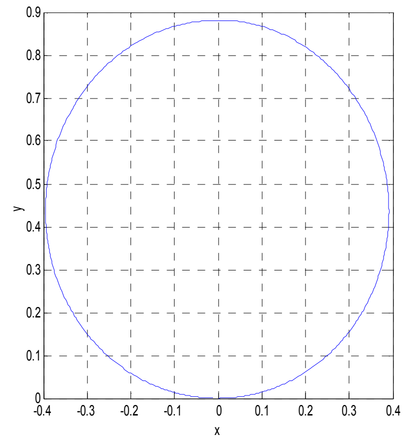

Fig 2: y vs x

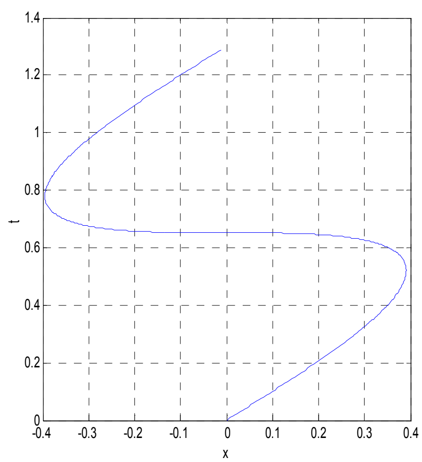

Fig 3: t vs x

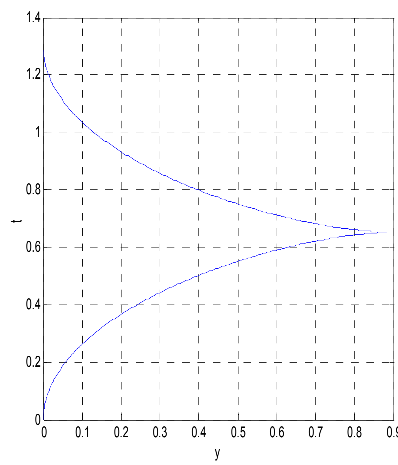

Fig 4: t vs y

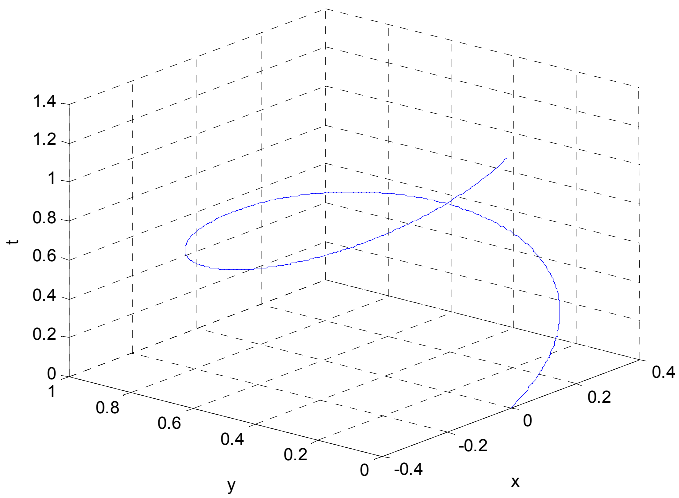

Fig 5: 3-dimensional view

It can be seen that, when a light signal is sent in the +x direction, it follows nearly a circular path and comes back to its source after a certain time from the –x direction.