MIXED –MODE FRACTURE ANALYSIS
OF
ORTHOTROPIC FUNCTIONALLY GRADED MATERIALS

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

MIXED –MODE FRACTURE ANALYSIS
OF
ORTHOTROPIC FUNCTIONALLY GRADED MATERIALS

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Functionally graded materials processed by the thermal spray techniques such as electron beam physical vapor deposition and plasma spray forming are known to have an orthotropic structure with reduced mechanical properties. Debonding related failures in these types of material systems occur due to embedded cracks that are perpendicular to the direction of the material property gradation. These cracks are inherently under mixed-mode loading and fracture analysis requires the extraction of the modes I and II stress intensity factors. The present study aims at developing semi-analytical techniques to study embedded crack problems in graded orthotropic media under various boundary conditions. The cracks are assumed to be aligned parallel to one of the principal axes of orthotropy. The problems are formulated using the averaged constants of plane orthotropic elasticity and reduced to two coupled integral equations with Cauchy type dominant singularities. The equations are solved numerically by adopting an expansion - collocation technique. The main results of the analyses are the mixed mode stress intensity factors and the energy release rate as functions of the material nonhomogeneity and orthotropy parameters. The effects of the boundary conditions on the mentioned fracture parameters are also duly discussed.

Keywords : Orthotropic Functionally Graded Materials, Embedded Crack, Singular Integral Equations, Stress Intensity Factor, Energy Release Rate.

ÖZ

ORTOTROPİK FONKSİYONEL DERECELENDİRİLMİŞ MALZEMELERİN KARIŞIK MOD KIRILMA ANALİZİ

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Elektron ışınlamayla fiziksel buhar biriktirme ve plazma spreyiyle oluşturma gibi ısıl sprey teknikleriyle üretilen Fonksiyonel Derecelendirilmiş Malzemelerin, indirgenmiş mekanik özellikli bir ortotropik yapıya sahip oldukları bilinmektedir. Bu malzeme sistemlerinde bağ kopmasına bağlı kırılmalar, malzeme özellik derecelenmesine dik yöndeki gömülü çatlaklardan ötürü meydana gelir. Bu çatlaklar doğal olarak karışık mod yükleme altındadır ve kırılma analizi, mod 1 ve mod 2 gerilme şiddeti çarpanlarının hesaplanmasını gerektirir. Bu çalışma, çeşitli sınır şartlarında derecelendirilmiş ortotropik ortamdaki gömülü çatlak problemlerini çözebilmek için yarı-analitik teknikler geliştirmeyi hedeflemektedir. Çatlakların, ortotropi ana eksenlerinden birisine paralel olarak hizalandıkları varsayılmıştır. Problemler, düzlem ortotropik elastisite ortalama sabitleri kullanılarak formüle edilmiş ve Cauchy tip etkin tekillik içeren iki adet integral denklemine indirgenmiştir. Denklemler, açılım-düzenleme tekniği kullanılarak nümerik olarak çözülmüştür. Analizlerin asıl sonuçları, karışık mod gerilme şiddeti çarpanları ve malzeme homojen olmama ve ortotropi parametrelerinin fonksiyonları olarak enerji bırakma miktarıdır. Sınır şartlarının bahsedilen kırılma parametrelerine olan etkileri de beklendiği gibi tartışılmıştır.

Anahtar Kelimeler : Ortotropik Fonksiyonel Olarak Derecelenmiş Malzemeler, Gömülü Çatlak, Tekil İntegral Denklemleri, Gerilme Şiddeti Çarpanı, Enerji Bırakma Miktarı .

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$a\sqrt{\delta_0}$	: Half crack length in physical domain.
c	: Half crack length in physical domain.
a	: Half crack length in the transformed domain.
h_1	: Thickness of the medium 1.
h_2	: Thickness of the medium 2.
h_1/c	: Normalized thickness of the medium 1.
h_2/c	: Normalized thickness of the medium 2.
E	: Effective stiffness.
κ	: Shear parameter.
δ	: Stiffness ratio.
ν	: Effective Poisson's ratio.
σ_{ij}	: Stress components in physical domain $(i, j = 1,2)$.
σ_{ij}	: Stress components in transformed domain $(i, j = x, y)$.
ε_{ij}	: Strain components in physical domain $(i, j = 1,2)$.
ε_{ij}	: Strain components in transformed domain $(i, j = x, y)$.
u_1, u_2	: Displacement components in x_1- and x_2- directions.
u, v	: Displacement components in x- and y- directions.
β	: Nonhomogeneity constant in physical domain.
γ	: Nonehomogeneity constant in transformed domain.

βc	: Normalized nonhomogeneity constant.
E_0	: Effective stiffness at $x_2 = 0$.
ω	: Fourier Transform variable.
n_j, s_j, r_j	: Roots of the characteristic equations $(j = 1, 2, 3, 4)$.
$M_j(\omega)$, $G_j(\omega)$	: Unknown functions $(j = 1, 2, 3, 4)$.
$p(x_1)$	: Arbitrary normal traction applied to the crack surfaces in physical domain.
$q(x_1)$	: Arbitrary shear traction applied to the crack surfaces in physical domain.
$\overline{p}(x)$	: Arbitrary normal traction applied to the crack surfaces in transformed domain.
$\overline{q}(x)$	: Arbitrary shear traction applied to the crack surfaces in transformed domain.
$\phi_1(t)$, $\phi_2(t)$	: Unknown functions.
a_j, b_j, c_j,....,h_j	: Coefficients of the asymptotic expansions $(j = 0, 1, 2, 3, 4)$.
A_{ij}	: Integration cut-off points $(i, j = 1, 2)$.
γ_0	: Euler number.
T_n	: Chebyshev polynomial of the first kind of order n.
$U_n(s)$	: Chebyshev polynomial of the second kind.
A_n, B_n	: Unknown constants.
$k_1(a\sqrt{\delta_0})$, $k_1(-a\sqrt{\delta_0})$	: Mode I stress intensity factors at the crack tips $x_1 = \pm a\sqrt{\delta_0}$
$k_2(a\sqrt{\delta_0})$, $k_2(-a\sqrt{\delta_0})$	: Mode II stress intensity factors at the crack tips $x_1 = \pm a\sqrt{\delta_0}$
$G\left(a\sqrt{\delta_0}\right)$, $G\left(-a\sqrt{\delta_0}\right)$	: Energy release rates at the crack tips $x_1 = \pm a\sqrt{\delta_0}$
$\dfrac{k_1(c)}{k_0}$	: Normalized mode I stress intensity factor

$\dfrac{k_2(c)}{k_0}$: Normalized mode II stress intensity factor.

$\dfrac{G(c)}{G_0}$: Normalized energy release rate.

CHAPTER 1

INTRODUCTION

1.1 Functionally Graded Materials

Developing applications in aerospace, power generation, microelectronics and bioengineering demand properties that are unobtainable in any single material. These properties should include both resistance against thermal and mechanical stresses. Structural ceramic materials are especially used in high temperature environments, due to their high refractoriness, wear resistance and corrosion resistance where metallic or organic materials can not survive. However, ceramics cannot withstand the mechanical stresses which the metals easily overcome. On the other hand, properties offered by metals are fracture strength, fracture toughness, wear resistance and corrosion resistance, but they should be shielded from excessive heat under operating conditions. Correspondingly, composite or layered materials are developed to bring the desirable characteristics together of each phases in order to meet such requirements. However, the internal stresses caused by the elastic and thermal differences at the interface between two different materials can prevent the successful application of such composites. Also the strength of bonding between the two layers is generally very poor. A material designed with a soft transition from the metallic core to the ceramic surface would avoid these thermal and thermomechanical stresses.

The Functionally Gradient Materials (FGM) overcomes the problems arising in the composite or bulk layered materials due to their sharp-grading structures, by bringing the desirable properties of metals and ceramics more effectively. It has been shown that graded material properties reduce the magnitude of residual stresses, significantly increase the bonding strength and increase the fracture toughness in thickness direction.

In practice, the nature of processing techniques of some FGMs may lead to loss of isotropy. For example, graded materials processed by the *electron beam physical vapor deposition* technique have a columnar structure [1] (see Fig.1.1(b)), which leads to a higher stiffness in the thickness direction and weak fracture planes perpendicular to the boundary. Furthermore, graded materials processed by *a plasma spray technique* generally have a lamellar structure [1] (see Fig.1.1(a)), where flattened splats and relatively weak splat boundaries create an oriented material with higher stiffness and weak cleavage planes parallel to the boundary. Embedded cracks that can initiate at the weak cleavage planes are inherently under mixed-mode mechanical or thermal loading. One of the approaches to examine fracture mechanics problems in this type of structures is to model the functionally graded medium as orthotropic with principal directions of orthotropy parallel and perpendicular to the free surface.

In this study, mixed–mode crack problems of orthotropic functionally graded materials are examined using an analytical method. The behavior of the embedded cracks is investigated under various boundary conditions.

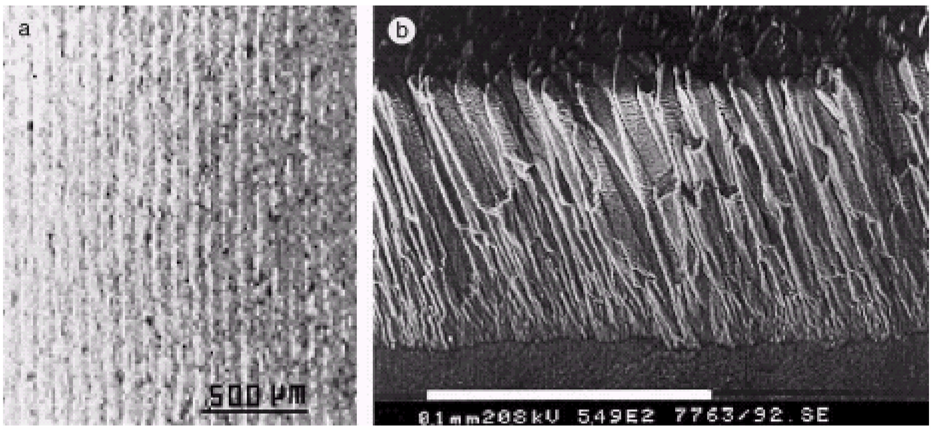

Figure 1.1 Cross-section microscopy of FGMs: **(a)** lamellar NiCrAlY–PSZ FGM processed by plasma spray technique [1]; **(b)** columnar ZrO2–Y2O3 thermal barrier coating with graded porosity processed by electron beam physical vapor deposition [1].

1.2 Literature Survey

Up to now various material models have been used to solve crack problems in FGMs and nonhomogeneous materials. Kim and Paulino [1] consider the interaction integral for fracture analysis of orthotropic FGMs. In this study stress intensity factors for mode I and mixed-mode two-dimensional problems are evaluated by means of the interaction integral and the finite element method. Extensive computational experiments have been performed to validate the proposed formulation. The accuracy of numerical results is discussed by comparisons to available analytical, semi-analytical, or numerical solutions.

Interface crack problems in graded orthotropic media are considered by Dağ et al. [2]. In this study the authors examine the problem using analytical and computational techniques. In the analytical formulation an interface crack between a graded orthotropic coating and a homogeneous orthotropic substrate is considered. The principal axes of orthotropy are assumed to be parallel and perpendicular to the crack plane. Mechanical properties of the medium are assumed to be continuous with discontinuous derivatives at the interface. The problem is formulated in terms of the averaged constants of plane orthotropic elasticity and reduced to a pair of singular integral equations which are solved numerically to compute the mixed mode stress intensity factors and the energy release rate. In the second part of the study, enriched finite elements are formulated and implemented for graded orthotropic materials. Comparisons of the finite element and analytical results show that enriched finite element technique is capable of producing highly accurate results for crack problems in graded orthotropic media. Finally, periodic interface cracking and the four point bending test for graded orthotropic solids are modeled using enriched finite elements.

A surface crack in a semi-infinite elastic graded medium under general loading is studied by Dağ and Erdoğan [3]. In this study it is assumed that first by solving the problem in the absence of a crack it is reduced to a local perturbation problem with arbitrary self-equilibrating crack surface tractions. The local problem is then solved by approximating the normal and shear tractions on the crack surfaces by polynomials and the normalized modes I and II stress intensity factors are given. As an example the results for a graded half-plane loaded by a sliding rigid circular stamp are presented.

Kadıoğlu et al. [4] consider internal and edge crack problems for an FGM layer attached to an elastic foundation. This model can be used to simulate circumferential crack problem for a thin walled cylinder. It is assumed that the Young' s modulus of the layer varies in thickness direction exponentially. Because of its insignificant effect on stress intensity factors, Poisson's ratio is assumed to be constant. Crack is assumed to be perpendicular to the surfaces. Mode I stress intensity factors are calculated for various values of the nonhomogeneity parameter.

Circumferential crack problem for an FGM cylinder under thermal stresses is studied by Dağ et al. [5]. In this study the stress intensity factors associated with a circumferential crack in a thin-walled cylinder subjected to quasi-static thermal loading are determined. In order to make the problem analytically tractable, the thin-walled cylinder is modeled as a layer on an elastic foundation whose thermal and mechanical properties are exponential functions of the thickness coordinate. Hence a plane strain crack problem is obtained. First temperature and thermal stress distributions for a crack-free layer are determined. Then using these solutions, the crack problem is reduced to a local perturbation problem where the only nonzero loads are the crack surface tractions. Both internal and edge cracks are considered. Stress intensity factors are computed as functions of crack geometry, material properties, and time.

In the paper by Yıldırım et al. [6], three dimensional surface crack problems in functionally graded coatings subjected to mode I mechanical or transient thermal loading are examined. The surface cracks are assumed to have a semi-elliptical crack front profile of arbitrary aspect ratio. The cracks are embedded in the functionally graded material (FGM) coating which is perfectly bonded to a homogeneous substrate. A three dimensional finite element method is used to solve the thermal and structural problems. The stress intensity factors are computed by using the displacement correlation technique. Four different coating types are considered in the analyses which have homogeneous, ceramic-rich (CR), metal-rich (MR) and linear variation (LN) material composition profiles. The stress intensity factors calculated for FGM plates are in good agreement with the previously published results on three dimensional surface cracks. The new results provided show that maximum stress intensity factors computed during transient thermal loading period for the FGM coatings are lower than those of the homogeneous ceramic ones.

Dağ and Erdoğan [7] consider the coupled problem of crack / contact mechanics in a nonhomogeneous medium and investigate the behavior of a surface crack in a functionally graded medium loaded by a sliding rigid stamp in the presence of friction. In this study, the dimensions of the graded medium are assumed to be very large in comparison with the local length parameters of the crack/contact region. Thus in formulating the problem the graded medium is assumed to be semi-infinite. Contact stresses, the in-plane component of the surface stress and stress intensity factors at the crack tip are determined. The results are presented for various combinations of friction coefficient, material nonhomogeneity constant and crack/contact length parameters.

Mode I crack problem in an inhomogeneous orthotropic medium is considered by Öztürk and Erdoğan [8]. In this study the symmetric crack problem is considered and the material is both oriented and graded. The mode I crack problem for the inhomogeneous orthotropic plane is formulated and the solution is obtained for various loading conditions and material parameters.

The mixed mode crack problem in plane elasticity for a graded and oriented material is considered by Öztürk and Erdoğan [9]. It is assumed that the crack is located in a plane perpendicular to the direction of property grading and the principal axes of orthotropy are parallel and perpendicular to the crack plane. The problem is formulated in terms of the averaged constants of plane orthotropic elasticity and reduced to a system of singular integral equations which is solved for various loading conditions and material parameters. The results presented consist of the strain energy release rate, the stress intensity factors and the crack opening displacements. It is found that generally the stress intensity factors increase with increasing material inhomogeneity parameter, shear parameter and decreasing stiffness ratio.

In the study by Kim and Paulino [10], a finite element methodology is developed for fracture analysis of orthotropic functionally graded materials (FGMs) where cracks are arbitrarily oriented with respect to the principal axes of material orthotropy. The graded and orthotropic material properties are smooth functions of spatial coordinates, which are integrated into the element stiffness matrix using the isoparametric concept and special graded finite elements. Stress intensity factors (SIFs) for mode I and mixed-mode two-dimensional problems are evaluated and compared by means of the modified crack closure (MCC) and the displacement correlation technique (DCT) especially tailored for orthotropic FGMs. An accurate technique to evaluate SIFs by means of the MCC is presented using a simple two-step (predictor–corrector) process in which the SIFs are first predicted (e.g. by the DCT) and then corrected by Newton iterations. The effects of boundary conditions, crack tip mesh discretization and material properties on fracture behavior are investigated in detail. Many numerical examples are given to validate the proposed methodology. The accuracy of results is discussed by comparison to available (semi-) analytical or numerical solutions.

Mode I crack problem for a functionally graded orthotropic strip is considered by Guo et al. [11]. In this study, internal and edge cracks perpendicular to the boundaries are investigated. The elastic property of the material is assumed to vary continuously along the thickness direction. The principal directions of orthotropy are parallel and perpendicular to the boundaries of the strip. The singular integral equation for solving the problem and the corresponding asymptotic expression of the singular kernel are obtained. Three different loading conditions, namely crack surface pressure, fixed-grip loading and bending, are considered during the analysis. The influences of parameters such as the material constants and the geometry parameters on the stress intensity factors (SIFs) are studied.

A functionally graded material strip containing an embedded or a surface crack perpendicular to its boundaries is considered by Wang et al. [12]. In this study the graded medium is divided into a large number of layers in the thickness direction, with each layer being a homogeneous material. Surface crack in the functionally graded material is considered for arbitrarily distributed material properties in the thickness direction. In the numerical examples, the graded medium is subjected to two different loading types, a uniform mechanical pressure on the crack surfaces and a non-uniform thermal stress distribution. Using these loads, the mode I stress intensity factors are computed for different crack lengths and property distributions.

Guo et al. consider the static crack problem of a functionally graded coating-substrate structure with a crack perpendicular to the interface in [13] and the transient problem of the same case in [14].

1.3 Scope of the Study

The aim of this study is to examine the effects of, material nonhomogeneity, orthotropy and boundary conditions on fracture mechanics parameters for a crack in an orthotropic functionally graded layer under various boundary conditions using a semi-analytical technique. The geometry of the crack problem is shown in Figure 1.2 The problem having the same geometry and properties is examined for three different boundary conditions (free – free, fixed – fixed and fixed – free) to obtain the effect of boundary conditions on fracture mechanics parameters. The problem is formulated using the four averaged constants of plane orthotropic elasticity which are the effective stiffness, shear parameter, stiffness ratio and effective Poisson's ratio. The governing partial differential equations are obtained in terms of the displacement components and then they are reduced to a pair of singular integral equations using Fourier transforms. The integral equations are solved numerically using an expansion-collocation technique to compute the modes I and II stress intensity factors and the energy release rate at the crack tips.

This thesis contains four chapters. Introduction, literature survey and the scope of the study are given in the present chapter. Problem definition and formulation are given in Chapter 2. The computed results are presented in Chapter 3. Finally, discussion of the results and conclusions are given in Chapter 4.

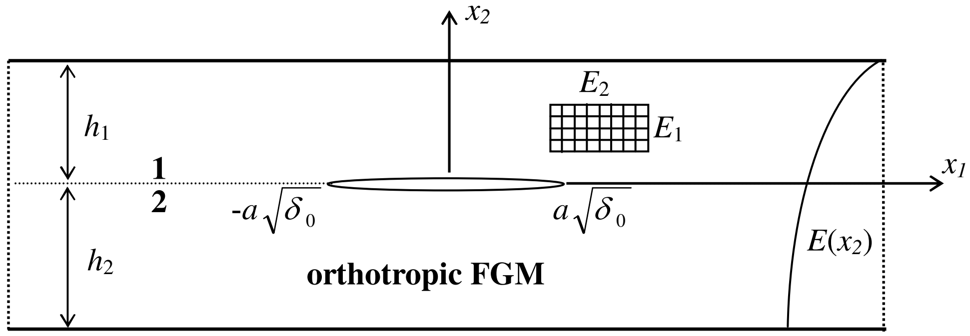

Figure 1.2 An embedded crack in an orthotropic functionally graded layer

CHAPTER 2

PROBLEM DEFINITION and FORMULATION

2.1 Problem Definition

In this study, embedded crack problems in orthotropic FGMs under mixed mode loading are considered. Three different boundary conditions are studied which are free – free, fixed – fixed and fixed – free as shown as Figures 2.1-2.3.

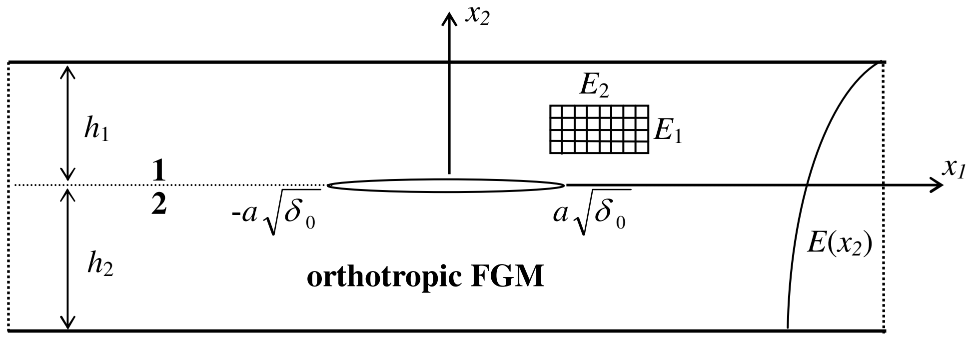

Figure 2.1 An embedded crack in an orthotropic functionally graded layer under free – free boundary condition (Case I)

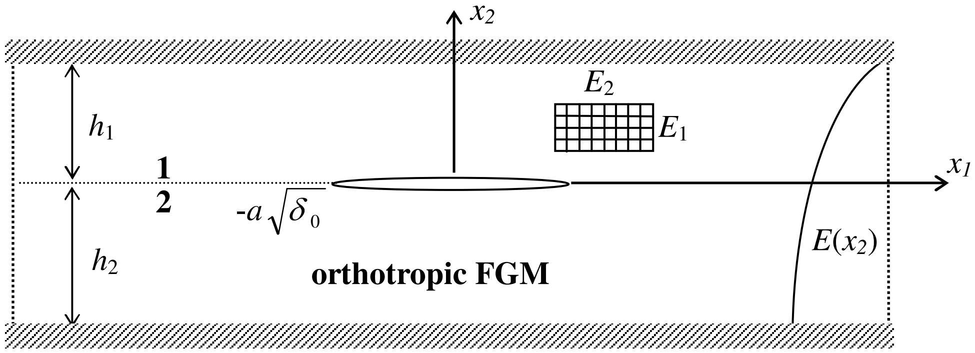

Figure 2.2 An embedded crack in an orthotropic functionally graded layer under fixed – fixed boundary condition (Case II)

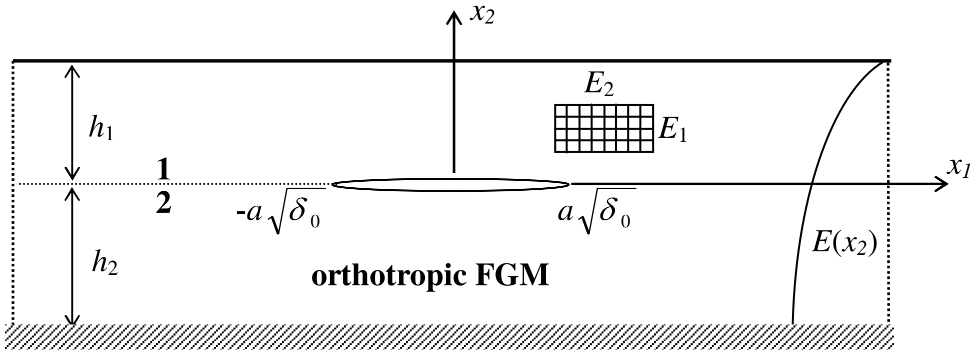

Figure 2.3 An embedded crack in an orthotropic functionally graded layer under fixed – free boundary condition (Case III)

In these figures x_1 and x_2 are the principal directions of orthotropy. The layer is graded in x_2 direction and contains an embedded crack of length $2a\sqrt{\delta_0}$ at $x_2 = 0$. δ_0 is the stiffness ratio as defined in section 2.2. Crack length is taken as $2a\sqrt{\delta_0}$ to simplify the formulation of the problem. The thicknesses of medium 1 and medium 2 are given as h_1 and h_2, respectively. The crack is loaded by arbitrary normal and/or shear tractions which are applied at $x_2 = 0^+$ and $x_2 = 0^-$ for $|x_1| < a\sqrt{\delta_0}$.

2.2 Formulation

The crack problem is formulated using the averaged constants of plane orthotropic elasticity which are first introduced by Krenk [15]. Using these averaged constants, relationship between the strain and the stress components can be expressed as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E(x_1, x_2)} \begin{bmatrix} \delta^{-2} & -\nu & 0 \\ -\nu & \delta^{-2} & 0 \\ 0 & 0 & 2(\kappa+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{2.1}$$

Where E is the effective stiffness, κ is the shear parameter, δ is the stiffness ratio and ν is the effective Poisson's ratio. These parameters are given in terms of the engineering constants for both generalized plane stress and plane strain cases as follows

$$E = \begin{cases} \sqrt{E_{11}\ E_{22}} & , \textit{generalized plane stress} \\ \sqrt{\dfrac{E_{11}\ E_{22}}{(1-\nu_{13}\ \nu_{31})\,(1-\nu_{23}\ \nu_{32})}} & , \textit{plane strain} \end{cases} \tag{2.2a}$$

$$\nu = \begin{cases} \sqrt{\nu_{12}\ \nu_{21}} & , \textit{generalized plane stress} \\ \sqrt{\dfrac{(\nu_{12}+\nu_{13}\nu_{32})\,(\nu_{21}+\nu_{23}\nu_{31})}{(1-\nu_{13}\nu_{31})\,(1-\nu_{23}\nu_{32})}} & , \textit{plane strain} \end{cases} \tag{2.2b}$$

$$\delta^{4} = \begin{cases} (E_{11}/E_{22}) = (\nu_{12}/\nu_{21}) & , \textit{generalized plane stress} \\ \dfrac{E_{11}}{E_{22}}\,\dfrac{1-\nu_{23}\nu_{32}}{1-\nu_{13}\nu_{31}} & , \textit{plane strain} \end{cases} \tag{2.2c}$$

$$\kappa = \left\{ E\,/\,(2\,G_{12}) - \nu, \right. \quad \textit{for both of generalized plane stress and plane strain} \tag{2.2d}$$

By using transformations for the coordinates, displacements and stresses it is possible to reduce the strain – stress relation given by (2.1) to a form that does not contain δ. These transformation are given by

$$x = x_1 / \sqrt{\delta} \quad , \qquad y = x_2 \sqrt{\delta} \tag{2.3a}$$

$$u(x,y) = u_1(x_1, x_2)\sqrt{\delta} \quad , \quad v(x,y) = v_2(x_1, x_2)/\sqrt{\delta} \tag{2.3b}$$

$$\sigma_{xx}(x,y) = \sigma_{11}(x_1,x_2)/\delta \,,\ \sigma_{yy}(x,y) = \sigma_{22}(x_1,x_2)\,\delta \,,\ \sigma_{xy}(x,y) = \sigma_{12}(x_1,x_2) \tag{2.3c}$$

where (x, y), (u, v) and $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ are the coordinates, displacements and stress components in the transformed domain, respectively. Using (2.3), constitutive and kinematic relations can be expressed as

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix} = \frac{1}{E^*(x,y)} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(\kappa+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \tag{2.4a}$$

$$\varepsilon_{xx}(x,y) = \frac{1}{2}\left(\frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial x} u(x,y) \right) = \frac{\partial}{\partial x} u(x,y) \tag{2.4b}$$

$$\varepsilon_{yy}(x,y) = \frac{1}{2}\left(\frac{\partial}{\partial y} v(x,y) + \frac{\partial}{\partial y} v(x,y) \right) = \frac{\partial}{\partial y} v(x,y) \tag{2.4c}$$

$$\varepsilon_{xy}(x,y) = \frac{1}{2}\left(\frac{\partial}{\partial y} u(x,y) + \frac{\partial}{\partial x} v(x,y) \right) \tag{2.4d}$$

the strain – stress relations can be inverted to obtain the stress displacement relations as follows

$$\sigma_{xx}(x,y)=\frac{E^*(x,y)}{1-\nu^2}\left\{\frac{\partial}{\partial x}u(x,y)+\nu\frac{\partial}{\partial y}v(x,y)\right\} \tag{2.5a}$$

$$\sigma_{yy}(x,y)=\frac{E^*(x,y)}{1-\nu^2}\left\{\nu\frac{\partial}{\partial x}u(x,y)+\frac{\partial}{\partial y}v(x,y)\right\} \tag{2.5b}$$

$$\sigma_{xx}(x,y)=\frac{E^*(x,y)}{2(\kappa+\nu)}\left\{\frac{\partial}{\partial y}u(x,y)+\frac{\partial}{\partial x}v(x,y)\right\} \tag{2.5c}$$

In the absence of body forces equations of equilibrium in the transformed coordinate system are expressed as,

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0 \tag{2.6a}$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0 \tag{2.6b}$$

Substituting equations (2.5) into equations (2.6), it then follows that

$$\frac{\partial^2 u}{\partial y^2}+\beta_1\frac{\partial^2 u}{\partial x^2}+\beta_2\frac{\partial^2 v}{\partial x\partial y}+\frac{\beta_1}{E^*}\frac{\partial E^*}{\partial x}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)+\frac{1}{E^*}\frac{\partial E^*}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0 \tag{2.7a}$$

$$\frac{\partial^2 v}{\partial x^2}+\beta_1\frac{\partial^2 v}{\partial y^2}+\beta_2\frac{\partial^2 u}{\partial x\partial y}+\frac{\beta_1}{E^*}\frac{\partial E^*}{\partial y}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)+\frac{1}{E^*}\frac{\partial E^*}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0 \tag{2.7b}$$

where

$$\beta_1 = \frac{2(\kappa + \nu)}{1 - \nu^2}, \qquad \beta_2 = 1 + \nu\,\beta_1 \tag{2.8}$$

Note that in general the averaged properties E, κ, δ and ν can be functions of x and y. In this study, in order to make the problem analytically tractable we make some simplifying assumptions regarding material property distribution in the graded orthotropic medium. First, we assume that the stiffnesses E_{11}, E_{22} and G_{12} vary in x_2-direction. The variations in the stiffness coefficients are assumed to be proportional. It was previously shown that the effect of the gradation in the Poisson's ratio is rather insignificant in a crack problem [2]. Therefore, Poisson's ratios are assumed to be constant. These assumptions imply that κ, δ and ν are constants. Furthermore, it is assumed that variation in the effective stiffness in the graded orthotropic medium can be fitted to an exponential function. Under these assumptions, material coefficients are given in the following form

$$E(x_1, x_2) = E(x_2) = E_0 \exp(\beta x_2), \quad -h_2 < x_2 < h_1, \quad -\infty < x_1 < \infty \tag{2.9}$$

$$\kappa(x_1, x_2) = \kappa_0, \;\; \delta(x_1, x_2) = \delta_0, \;\; \nu(x_1, x_2) = \nu_0, \;\; -h_2 < x_2 < h_1, \;\; -\infty < x_1 < \infty \tag{2.10}$$

where E_0 is the effective stiffness at $x_2 = 0$ and β is a nonhomogeneity constant. Using (2.3), variation of the effective stiffness in the transformed coordinate system can be expressed as

$$E^*(x, y) = E_0 \exp(\gamma y), \quad -\sqrt{\delta_0}\, h_2 < y < \sqrt{\delta_0}\, h_1, \quad -\infty < x < \infty \tag{2.11}$$

where $\gamma = \beta / \sqrt{\delta_0}$

From equation (2.7a), (2.7b) and (2.11) it may then be seen that

$$\frac{\partial^2 u}{\partial y^2} + \beta_1 \frac{\partial^2 u}{\partial x^2} + \beta_2 \frac{\partial^2 v}{\partial x \partial y} + \gamma \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0 \qquad (2.12a)$$

$$\frac{\partial^2 v}{\partial x^2} + \beta_1 \frac{\partial^2 v}{\partial y^2} + \beta_2 \frac{\partial^2 u}{\partial x \partial y} + \beta_1 \gamma \left(\frac{\partial v}{\partial y} + \nu_0 \frac{\partial u}{\partial x} \right) = 0 \qquad (2.12b)$$

$$\beta_1 = \frac{2(\kappa_0 + \nu_0)}{1 - \nu_0^2} \quad , \quad \beta_2 = 1 + \nu_0 \beta_1 \qquad (2.13)$$

Note that for $\gamma = 0$, equations (2.12a) and (2.12b) would reduce to the differential equations for a homogeneous medium. The general solution of (2.12) can be obtained by employing Fourier transformations in x-direction. The general solutions for the displacements in the graded orthotropic medium can be expressed as

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(\omega, y) \exp(i \omega x) \, d\omega \qquad (2.14a)$$

$$v(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V(\omega, y) \exp(i \omega x) \, d\omega \qquad (2.14b)$$

Substituting (2.14) in (2.12) following differential equations are obtained:

$$\frac{d^2}{d y^2} U(\omega, y) - \beta_1 \omega^2 U(\omega, y) + i \omega \beta_2 \frac{d}{d y} V(\omega, y)$$
$$+ \gamma \left(\frac{d}{dy} U(\omega, y) + i \omega V(\omega, y) \right) = 0 \qquad (2.15a)$$

$$-\omega^2 V(\omega, y) + \beta_1 \frac{d^2}{dy^2} V(\omega, y) + i\omega\beta_2 \frac{d}{dy} U(\omega, y)$$
$$+ \beta_1 \gamma \left(\frac{d}{dy} V(\omega, y) + \nu_0 \, i\omega U(\omega, y) \right) = 0 \qquad (2.15b)$$

First the system for $U(\omega, y)$ and $V(\omega, y)$ i.e., equations (2.15a) and (2.15b) will be considered. By defining, the differential operators $D = \frac{d}{dy}$ and $D^2 = \frac{d^2}{dy^2}$ equations (2.15a) and (2.15b) can be written in the following matrix form,

$$\begin{bmatrix} \left(D^2 + \gamma D - \beta_1 \omega^2\right) & \left(i\omega\beta_2 D + \gamma i\omega\right) \\ \left(i\omega\beta_2 D + \beta_1 \gamma \nu_0 i\omega\right) & \left(\beta_1 D^2 + \beta_1 \gamma D - \omega^2\right) \end{bmatrix} \begin{bmatrix} U(\omega, y) \\ V(\omega, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2.16)$$

Defining the determinant of the coefficient matrix by Δ equation (2.16) can be uncoupled as follows

$$\Delta \cdot U(\omega, y) = 0 \qquad (2.17a)$$

$$\Delta \cdot V(\omega, y) = 0 \qquad (2.17b)$$

At this point one can assume a solution of the form $\exp(n\,y)$. If one substitutes the solution into equations (2.17) following characteristic equation will be obtained,

$$n^4 + 2\gamma n^3 + (-2\omega^2 \kappa_0 + \gamma^2) n^2 - 2\gamma \omega^2 \kappa_0 n + (\omega^2 + \gamma^2 \nu_0)\omega^2 = 0 \qquad (2.18)$$

Roots of the characteristic equation are given by

$$n_1 = -(\gamma/2) - \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 + \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(n_1) < 0 \qquad (2.19a)$$

$$n_2 = -(\gamma/2) - \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 - \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(n_2) < 0 \qquad (2.19b)$$

$$n_3 = -(\gamma/2) + \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 + \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(n_3) > 0 \qquad (2.19c)$$

$$n_4 = -(\gamma/2) + \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 - \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(n_4) > 0 \qquad (2.19d)$$

The general solution can be expressed as

$$U(\omega, y) = A_j(\omega) \exp(n_j\, y) \qquad (2.20a)$$

$$V(\omega, y) = A_j(\omega)\, B_j(\omega)\, \exp(n_j\, y) \qquad (2.20b)$$

where $A_j(\omega)$, (j = 1,..., 4), are unknown functions. n_j are the characteristic roots , $B_j(\omega)$ can be determined by substituting equations (2.20) into equation (2.15a) and they are found as,

$$B_j = \frac{-i\left(s_j^2\left(\nu_0^2 - 1\right) + 2\omega^2\left(\kappa_0 + \nu_0\right) - \gamma\, s_j\left(1 - \nu_0^2\right)\right)}{\omega\left(s_j\left(1 + \nu_0^2 + 2\kappa_0\, \nu_0\right) + \gamma\left(1 - \nu_0^2\right)\right)} \quad , \quad (j = 1, \ldots\ldots, 4) \qquad (2.21)$$

The layers 1 and 2 shown in Figures 2.1 – 2.3 are both graded orthotropic, so the solution for media 1 and 2 are the same but they are represented by different letters for simplicity and they can be expressed as follows

$$u^{(1)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{4}M_j(\omega)\exp(s_j\,y+i\,\omega\,x)\,d\,\omega \tag{2.22a}$$

$$v^{(1)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{4}M_j(\omega)\,N_j(\omega)\exp(s_j\,y+i\,\omega\,x)\,d\,\omega \tag{2.22b}$$

where superscript (1) refers to medium 1 and $M_j(\omega)$, (j = 1,..., 4), are unknown functions. The characteristic roots s_j and the functions N_j are given as

$$s_1=-(\gamma/2)-\sqrt{(\gamma/2)^2+\omega^2\kappa_0+\omega\sqrt{\omega^2\kappa_0^2-\omega^2-\gamma^2\nu_0}}\ ,\quad \Re(s_1)<0 \tag{2.23a}$$

$$s_2=-(\gamma/2)-\sqrt{(\gamma/2)^2+\omega^2\kappa_0-\omega\sqrt{\omega^2\kappa_0^2-\omega^2-\gamma^2\nu_0}}\ ,\quad \Re(s_2)<0 \tag{2.23b}$$

$$s_3=-(\gamma/2)+\sqrt{(\gamma/2)^2+\omega^2\kappa_0+\omega\sqrt{\omega^2\kappa_0^2-\omega^2-\gamma^2\nu_0}}\ ,\quad \Re(s_3)>0 \tag{2.23c}$$

$$s_4=-(\gamma/2)+\sqrt{(\gamma/2)^2+\omega^2\kappa_0-\omega\sqrt{\omega^2\kappa_0^2-\omega^2-\gamma^2\nu_0}}\ ,\quad \Re(s_4)>0 \tag{2.23d}$$

$$N_j=\frac{-i\left(s_j^2\left(\nu_0^2-1\right)+2\omega^2\left(\kappa_0+\nu_0\right)-\gamma\,s_j\left(1-\nu_0^2\right)\right)}{\omega\left(s_j\left(1+\nu_0^2+2\kappa_0\,\nu_0\right)+\gamma\left(1-\nu_0^2\right)\right)}\ ,\qquad (j=1,.......,4) \tag{2.24}$$

Substituting equations (2.22) into equations (2.5) the general solutions for the stresses in medium 1 can also be determined as follows,

$$\sigma_{xx}^{(1)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{1-\nu_0^2} \times$$

$$\times \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{4} M_j \left(i\omega + \nu_0 N_j s_j \right) \exp(s_j\, y) \right) \exp(i\omega x)\, d\omega \right\} \tag{2.25a}$$

$$\sigma_{yy}^{(1)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{1-\nu_0^2}$$

$$\times \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{4} M_j \left(\nu_0 i\omega + N_j s_j \right) \exp(s_j\, y) \right) \exp(i\omega x)\, d\omega \right\} \tag{2.25b}$$

$$\sigma_{xy}^{(1)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{2\,(\kappa_0 + \nu_0)} \times$$

$$\times \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{4} M_j \left(N_j i\omega + s_j \right) \exp(s_j\, y) \right) \exp(i\omega x)\, d\omega \right\} \tag{2.25c}$$

The solutions for medium 2 can be expressed as

$$u^{(2)}(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{4} G_j(\omega) \exp(r_j\, y + i\omega x)\, d\omega \tag{2.26a}$$

$$v^{(2)}(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{4} G_j(\omega) H_j(\omega) \exp(r_j\, y + i\omega x)\, d\omega \tag{2.26b}$$

where superscript (2) refers to medium 2 and $G_j(\omega)$, (j = 1,..., 4), are unknown functions. The characteristic roots r_j and the functions H_j are given as

$$r_1 = -(\gamma/2) + \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 + \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(r_1) > 0 \tag{2.27a}$$

$$r_2 = -(\gamma/2) + \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 - \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(r_2) > 0 \tag{2.27b}$$

$$r_3 = -(\gamma/2) - \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 + \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(r_3) < 0 \tag{2.27c}$$

$$r_4 = -(\gamma/2) - \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 - \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}} \quad , \quad \Re(r_4) < 0 \tag{2.27d}$$

$$H_j = \frac{-i\left(r_j^2(\nu_0^2 - 1) + 2\omega^2(\kappa_0 + \nu_0) - \gamma\, r_j (1 - \nu_0^2)\right)}{\omega\left(r_j(1 + \nu_0^2 + 2\kappa_0 \nu_0) + \gamma(1 - \nu_0^2)\right)} \quad , \quad (j = 1, \dots\dots, 4) \tag{2.28}$$

Substituting equations (2.26) into equations (2.5) the general solutions for the stresses in medium 2 can also be determined as follows,

$$\sigma_{xx}^{(2)}(x,y) = \frac{E_0 \exp(\gamma y)}{1 - \nu_0^2} \times$$
$$\times \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{4} G_j \left(i\omega + \nu_0 H_j r_j \right) \exp(r_j y) \right) \exp(i\omega x)\, d\omega \right\} \tag{2.29a}$$

$$\sigma_{yy}^{(2)}(x,y) = \frac{E_0 \exp(\gamma y)}{1 - \nu_0^2} \times$$
$$\times \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{j=1}^{4} G_j \left(\nu_0 i\omega + H_j r_j \right) \exp(r_j y) \right) \exp(i\omega x)\, d\omega \right\} \tag{2.29b}$$

$$\sigma_{xy}^{(2)}(x,y)=\frac{E_0\exp(\gamma y)}{2(\kappa_0+\nu_0)}\times$$
$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{4}G_j\left(H_j\, i\omega+r_j\right)\exp(r_j\,y)\right)\exp(i\omega x)\,d\omega\right\} \qquad (2.29c)$$

In the solution of the crack problem, the derivatives of the relative displacements of the crack surfaces are used as the primary unknown functions. These unknown functions are given as

$$\frac{\partial}{\partial x_1}\left(u_2^{(1)}(x_1,0)-u_2^{(2)}(x_1,0)\right)=\begin{cases}f_1(x_1) & , \ |x_1|<a\sqrt{\delta_0}\\ 0 & , \ |x_1|>a\sqrt{\delta_0}\end{cases} \qquad (2.30a)$$

$$\frac{\partial}{\partial x_1}\left(u_1^{(1)}(x_1,0)-u_1^{(2)}(x_1,0)\right)=\begin{cases}f_2(x_1) & , \ |x_1|<a\sqrt{\delta_0}\\ 0 & , \ |x_1|>a\sqrt{\delta_0}\end{cases} \qquad (2.30b)$$

where $a\sqrt{\delta_0}$ is the half crack length. The general solutions for the media 1 and 2 have to satisfy boundary and continuity conditions. The boundary conditions for Case I (free – free) are given in the following form

$$\sigma_{22}^{(1)}(x_1,h_1)=0 \quad , \quad -\infty<x_1<\infty \qquad (2.31a)$$

$$\sigma_{12}^{(1)}(x_1,h_1)=0 \quad , \quad -\infty<x_1<\infty \qquad (2.31b)$$

$$\sigma_{22}^{(1)}(x_1,0)=\sigma_{22}^{(2)}(x_1,0) \quad , \quad -\infty<x_1<\infty \qquad (2.31c)$$

$$\sigma_{12}^{(1)}(x_1,0)=\sigma_{12}^{(2)}(x_1,0) \quad , \quad -\infty<x_1<\infty \qquad (2.31d)$$

$$\sigma_{22}^{(2)}(x_1,-h_2)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31e}$$

$$\sigma_{12}^{(2)}(x_1,-h_2)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31f}$$

The boundary conditions for Case II (fixed – fixed) are given as

$$u_1(x_1,h_1)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31g}$$

$$u_2(x_1,h_1)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31h}$$

$$\sigma_{22}^{(1)}(x_1,0)=\sigma_{22}^{(2)}(x_1,0) \quad , \quad -\infty< x_1<\infty \tag{2.31i}$$

$$\sigma_{12}^{(1)}(x_1,0)=\sigma_{12}^{(2)}(x_1,0) \quad , \quad -\infty< x_1<\infty \tag{2.31j}$$

$$u_1(x_1,-h_2)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31k}$$

$$u_2(x_1,-h_2)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31l}$$

The boundary conditions for Case III (fixed – free) are also given as

$$\sigma_{22}^{(1)}(x_1,h_1)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31m}$$

$$\sigma_{12}^{(1)}(x_1,h_1)=0 \quad , \quad -\infty< x_1<\infty \tag{2.31n}$$

$$\sigma_{22}^{(1)}(x_1,0)=\sigma_{22}^{(2)}(x_1,0) \quad , \quad -\infty< x_1<\infty \tag{2.31o}$$

$$\sigma_{12}^{(1)}(x_1,0)=\sigma_{12}^{(2)}(x_1,0) \quad , \quad -\infty< x_1<\infty \tag{2.31p}$$

$$u_1(x_1, -h_2) = 0 \qquad , \qquad -\infty < x_1 < \infty \tag{2.31q}$$

$$u_2(x_1, -h_2) = 0 \qquad , \qquad -\infty < x_1 < \infty \tag{2.31r}$$

The final two boundary conditions valid for all cases represent the arbitrary normal and shear tractions applied to the crack surfaces :

$$\begin{aligned} \sigma_{22}^{(1)}(x_1, 0) &= -p(x_1) \qquad , \qquad |x_1| < a\sqrt{\delta_0} \\ \sigma_{12}^{(1)}(x_1, 0) &= -q(x_1) \qquad , \qquad |x_1| < a\sqrt{\delta_0} \end{aligned} \tag{2.32}$$

After expressing equations (2.31) and (2.30) in the transformed coordinate system and taking Fourier transforms of both sides of these equations, the unknown functions of the general solutions (2.22) and (2.26), i.e., $M_j(\omega)$ and $G_j(\omega)$, (j = 1,..., 4) can be solved by using the following 8×8 complex system of linear equations,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} & a_{78} \\ 0 & 0 & 0 & 0 & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix} \begin{bmatrix} M_1(\omega) \\ M_2(\omega) \\ M_3(\omega) \\ M_4(\omega) \\ G_1(\omega) \\ G_2(\omega) \\ G_3(\omega) \\ G_4(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F_1(\omega) \\ F_2(\omega) \\ 0 \\ 0 \end{bmatrix} \tag{2.33}$$

The entries of the coefficient matrix are given in Appendix A for each case. The solution can now be expressed in the following form.

$$M_j(\omega) = V_j(\omega)\ F_1(\omega) + W_j(\omega)\ F_2(\omega) \qquad (j=1,...,4) \tag{2.34a}$$

$$G_j(\omega) = Y_j(\omega)\ F_1(\omega) + Z_j(\omega)\ F_2(\omega) \qquad (j=1,...,4) \tag{2.34b}$$

where

$$F_1(\omega) = \int_{-a}^{a} \phi_1(t)\ \exp(-i\,\omega\, t)\ dt \tag{2.35a}$$

$$F_2(\omega) = \int_{-a}^{a} \phi_2(t)\ \exp(-i\,\omega\, t)\ dt \tag{2.35b}$$

and $\phi_1(t) = f_1\left(\sqrt{\delta_0}\, t\right)$, $\phi_2(t) = \delta_0\, f_2\left(\sqrt{\delta_0}\, t\right)$. Equations (2.32) can be used to derive the singular integral equations and to solve for the unknown functions $\phi_1(t)$ and $\phi_2(t)$. Substituting $\sigma_{yy}(x, y)$ and $\sigma_{xy}(x, y)$ from equations (2.29b) and (2.29c) into equation (2.32) and after some standard manipulations, the problem can be reduced to two integral equations which can be written as

$$\sigma_{yy}^{(2)}(x, y) = \frac{E_0 \exp(\gamma\, y)}{1 - \nu_0^2} \left\{ \int_{-a}^{a} \sum_{j=1}^{2} k_{ij}(x, y, t)\ \phi_j(t)\, dt \right\} = -\delta_0\ p\left(x\sqrt{\delta_0}\right), \ (i = 1) \tag{2.36a}$$

$$\sigma_{xy}^{(2)}(x, y) = \frac{E_0 \exp(\gamma\, y)}{2(\kappa_0 + \nu_0)} \left\{ \int_{-a}^{a} \sum_{j=1}^{2} k_{ij}(x, y, t)\ \phi_j(t)\, dt \right\} = -q\left(x\sqrt{\delta_0}\right), \ (i=2) \tag{2.36b}$$

where

$$k_{ij}(x,y,t)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{\infty} K_{ij}(\omega,y)\,\sin(\omega\,(x-t))\,d\omega\,, & (i=j)\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_0^{\infty} K_{ij}(\omega,y)\,\cos(\omega\,(x-t))\,d\omega\,, & (i\neq j)\end{cases}\,,\quad (i,j=1,2) \tag{2.37}$$

$$K_{11}(\omega,y) = 2i\left\{\sum_{j=1}^{4}\left(i\omega\nu_0 + r_j H_j\right)Y_j(\omega)\exp(r_j\,y)\right\} \tag{2.38a}$$

$$K_{12}(\omega,y) = 2\left\{\sum_{j=1}^{4}\left(i\omega\nu_0 + r_j H_j\right)Z_j(\omega)\exp(r_j\,y)\right\} \tag{2.38b}$$

$$K_{21}(\omega,y) = 2\left\{\sum_{j=1}^{4}\left(i\omega H_j + r_j\right)Y_j(\omega)\exp(r_j\,y)\right\} \tag{2.38c}$$

$$K_{22}(\omega,y) = 2i\left\{\sum_{j=1}^{4}\left(i\omega H_j + r_j\right)Z_j(\omega)\exp(r_j\,y)\right\} \tag{2.38d}$$

Note that equations given above are valid for all cases. The integrands in (2.37) are bounded and continuous for $\omega < \infty$ and integrable for $\omega = 0$. The singular nature of the kernels k_{ij} can, therefore, be determined by examining the asymptotic behavior of K_{ij} as ω approaches infinity. Details of the asymptotic analysis of the terms K_{ij} are described in Appendix B. The asymptotic forms are obtained as follows

$$K_{11}^{\infty}(\omega, y) = K_{111}^{\infty}(\omega)\exp(r_1\, y) + K_{112}^{\infty}(\omega)\exp(r_2\, y) \tag{2.39a}$$

$$K_{12}^{\infty}(\omega, y) = K_{121}^{\infty}(\omega)\exp(r_1\, y) + K_{122}^{\infty}(\omega)\exp(r_2\, y) \tag{2.39b}$$

$$K_{21}^{\infty}(\omega, y) = K_{211}^{\infty}(\omega)\exp(r_1\, y) + K_{212}^{\infty}(\omega)\exp(r_2\, y) \tag{2.39c}$$

$$K_{22}^{\infty}(\omega, y) = K_{221}^{\infty}(\omega)\exp(r_1\, y) + K_{222}^{\infty}(\omega)\exp(r_2\, y) \tag{2.39d}$$

where

$$K_{111}^{\infty}(\omega) = a_0 + \frac{a_1}{\omega} + \frac{a_2}{\omega^2} + \frac{a_3}{\omega^3} + \frac{a_4}{\omega^4} \tag{2.40a}$$

$$K_{112}^{\infty}(\omega) = b_0 + \frac{b_1}{\omega} + \frac{b_2}{\omega^2} + \frac{b_3}{\omega^3} + \frac{b_4}{\omega^4} \tag{2.40b}$$

$$K_{121}^{\infty}(\omega) = c_0 + \frac{c_1}{\omega} + \frac{c_2}{\omega^2} + \frac{c_3}{\omega^3} + \frac{c_4}{\omega^4} \tag{2.40c}$$

$$K_{122}^{\infty}(\omega) = d_0 + \frac{d_1}{\omega} + \frac{d_2}{\omega^2} + \frac{d_3}{\omega^3} + \frac{d_4}{\omega^4} \tag{2.40d}$$

$$K_{211}^{\infty}(\omega) = e_0 + \frac{e_1}{\omega} + \frac{e_2}{\omega^2} + \frac{e_3}{\omega^3} + \frac{e_4}{\omega^4} \tag{2.40e}$$

$$K_{212}^{\infty}(\omega) = f_0 + \frac{f_1}{\omega} + \frac{f_2}{\omega^2} + \frac{f_3}{\omega^3} + \frac{f_4}{\omega^4} \tag{2.40f}$$

$$K_{221}^{\infty}(\omega) = g_0 + \frac{g_1}{\omega} + \frac{g_2}{\omega^2} + \frac{g_3}{\omega^3} + \frac{g_4}{\omega^4} \tag{2.40g}$$

$$K_{222}^{\infty}(\omega) = h_0 + \frac{h_1}{\omega} + \frac{h_2}{\omega^2} + \frac{h_3}{\omega^3} + \frac{h_4}{\omega^4} \tag{2.40h}$$

where superscript (∞) stands for the asymptotic expansion as $\omega \to \infty$. The coefficients of the expansions a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j $(j=0, 1,...,4)$ are lengthy functions of the constants $\gamma, \kappa_0, \nu_0, h_1, h_2$ and they are not reproduced here. It is noted that $c_0 + d_0 = e_0 + f_0 = 0$, i.e., the leading coefficients of the coupling terms are equal to zero. One can now write left hand side of the equations (2.36a) and (2.36b) in the following form

$$\sigma_{yy}^{(2)}(x,y) = \frac{E_0 \exp(\gamma y)}{1-\nu_0^2} \left\{ \int_{-a}^{a} \phi_1(t)\, dt\, \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega(x-t))\, d\omega \right.$$
$$\left. + \int_{-a}^{a} \phi_2(t)\, dt\, \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y) \cos(\omega(x-t))\, d\omega \right\} \tag{2.41a}$$

$$\sigma_{xy}^{(2)}(x,y) = \frac{E_0 \exp(\gamma y)}{2(\kappa_0 + \nu_0)} \left\{ \int_{-a}^{a} \phi_1(t)\, dt\, \frac{1}{2\pi} \int_0^{\infty} K_{21}(\omega, y) \cos(\omega(x-t))\, d\omega \right.$$
$$\left. + \int_{-a}^{a} \phi_2(t)\, dt\, \frac{1}{2\pi} \int_0^{\infty} K_{22}(\omega, y) \sin(\omega(x-t))\, d\omega \right\} \tag{2.41b}$$

At this point, four integral appearing in equations (2.41a) and (2.41b) will be considered. First we consider the first integral in equation (2.41a) which is given as

$$\frac{1}{2\pi}\int_0^\infty K_{11}(\omega, y)\sin(\omega(x-t))\,d\omega \tag{2.42}$$

the leading terms of equation (2.39a) can be subtracted from the integrands in equation (2.42) , then equation (2.42) is expressed as,

$$\frac{1}{2\pi}\int_0^\infty K_{11}(\omega, y)\sin(\omega(x-t))\,d\omega=$$
$$=\frac{1}{2\pi}\int_0^\infty \left[K_{11}(\omega, y)-\left(a_0 e^{r_1 y}+b_0 e^{r_2 y}\right)\right]\sin(\omega(x-t))\,d\omega +$$
$$+\frac{1}{2\pi}\int_0^\infty \left[a_0 e^{r_1 y}+b_0 e^{r_2 y}\right]\sin(\omega(x-t))\,d\omega \tag{2.43}$$

By using symbolic manipulator MAPLE r_1 and r_2 can be expanded asymptotically as follows

$$r_1 = \alpha_1 \omega \tag{2.44a}$$

$$r_2 = \alpha_2 \omega \tag{2.44b}$$

where

$$\alpha_1 = \sqrt{\kappa_0 + \sqrt{\kappa_0^2 - 1}} \qquad \Re(\alpha_1)>0 \tag{2.45a}$$

$$\alpha_2 = \sqrt{\kappa_0 - \sqrt{\kappa_0^2 - 1}} \qquad \Re(\alpha_2)>0 \tag{2.45b}$$

then equation (2.43) can be expressed in the following form

$$\frac{1}{2\pi}\int_0^\infty K_{11}(\omega, y)\sin(\omega\,(x-t))\,d\omega =$$

$$= \frac{1}{2\pi}\int_0^\infty \left[K_{11}(\omega, y) - \left(a_0\, e^{r_1 y} + b_0\, e^{r_2 y}\right)\right]\sin(\omega\,(x-t))\,d\omega +$$

$$+ \frac{1}{2\pi}\int_0^\infty \left[a_0\, e^{\alpha_1 y\omega} + b_0\, e^{\alpha_2 y\omega}\right]\sin(\omega\,(x-t))\,d\omega \qquad (2.46)$$

It is assumed that $y < 0$ and $\Re(\alpha_j) > 0$. Under these assumptions following equation is obtained

$$\frac{1}{2\pi}\int_0^\infty K_{11}(\omega, y)\sin(\omega\,(x-t))\,d\omega =$$

$$= \frac{1}{2\pi}\int_0^\infty \left[K_{11}(\omega, y) - \left(a_0\, e^{r_1 y} + b_0\, e^{r_2 y}\right)\right]\sin(\omega\,(x-t))\,d\omega +$$

$$+ \frac{1}{2\pi}\left\{\frac{a_0\,(x-t)}{\alpha_1^2 y^2 + (x-t)^2} + \frac{b_0\,(x-t)}{\alpha_2^2 y^2 + (x-t)^2}\right\} \qquad (2.47)$$

Now, the limit is taken as $y \to 0$ and the following expression is obtained,

$$\lim_{y\to 0}\frac{1}{2\pi}\int_0^\infty K_{11}(\omega, y)\sin(\omega\,(x-t))\,d\omega =$$

$$= \frac{1}{2\pi}\int_0^\infty \left[K_{11}(\omega, 0) - \left(a_0 + b_0\right)\right]\sin(\omega\,(x-t))\,d\omega + \frac{1}{2\pi}\left\{\frac{a_0 + b_0}{(x-t)}\right\} \qquad (2.48)$$

From equations (2.39a), (2.40a) and (2.40b) $K_{11}^{\infty}(\omega, 0)$ is obtained as

$$K_{11}^{\infty}(\omega, 0) = (a_0 + b_0) + \frac{(a_1 + b_1)}{\omega} + \frac{(a_2 + b_2)}{\omega^2} + \frac{(a_3 + b_3)}{\omega^3} + \frac{(a_4 + b_4)}{\omega^4} \tag{2.49}$$

The second term of equation (2.49) can be subtracted from the integrands in equation (2.48)

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega (x - t))\, d\omega =$$
$$= \frac{1}{2\pi} \int_0^{\infty} \left[K_{11}(\omega, 0) - (a_0 + b_0) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega (x - t))\, d\omega +$$
$$+ \frac{1}{2\pi} \int_0^{\infty} \frac{(a_1 + b_1)}{\omega} \sin(\omega (x - t))\, d\omega + \frac{1}{2\pi} \left\{ \frac{a_0 + b_0}{(x - t)} \right\} \tag{2.50}$$

Then

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega (x - t))\, d\omega =$$
$$= \frac{1}{2\pi} \int_0^{\infty} \left[K_{11}(\omega, 0) - (a_0 + b_0) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega (x - t))\, d\omega +$$
$$+ \frac{1}{2\pi} \left\{ \frac{a_0 + b_0}{(x - t)} \right\} + \frac{1}{2\pi} \left\{ (a_1 + b_1) \frac{\pi}{2} \operatorname{sign}(x - t) \right\} \tag{2.51}$$

The expressions used in the evaluation of integrals are given in Appendix C. Using integration cut-off points; equation (2.51) is expressed as

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega (x-t))\, d\omega =$$

$$= \frac{1}{2\pi} \int_0^{A_{11}} \left[K_{11}(\omega,0) - \left(a_0 + b_0\right) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega (x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \int_{A_{11}}^{\infty} \left[K_{11}(\omega,0) - \left(a_0 + b_0\right) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega (x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \left\{ \frac{a_0 + b_0}{(x-t)} + (a_1 + b_1) \frac{\pi}{2} \operatorname{sign}(x-t) \right\} \tag{2.52}$$

where A_{11} is the integration cut-off point. Following equation is obtained by subtracting the remaining terms of equation (2.49) from the integrands in equation (2.52)

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega (x-t))\, d\omega =$$

$$= \frac{1}{2\pi} \int_0^{A_{11}} \left[K_{11}(\omega,0) - \left(a_0 + b_0\right) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega (x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \int_{A_{11}}^{\infty} \left[K_{11}(\omega,0) - \left(a_0 + b_0\right) - \frac{(a_1 + b_1)}{\omega} - \cdots - \frac{(a_4 + b_4)}{\omega^4} \right] \sin(\omega (x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \int_{A_{11}}^{\infty} \left[\frac{(a_2 + b_2)}{\omega^2} + \frac{(a_3 + b_3)}{\omega^3} + \frac{(a_4 + b_4)}{\omega^4} \right] \sin(\omega (x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \left\{ \frac{a_0 + b_0}{(x-t)} + (a_1 + b_1) \frac{\pi}{2} \operatorname{sign}(x-t) \right\} \tag{2.53}$$

In summary,

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega\ (x-t))\ d\omega =$$

$$= \frac{1}{2\pi} \left\{ \frac{a_0 + b_0}{(x-t)} + (a_1 + b_1) \frac{\pi}{2} \operatorname{sign}\ (x-t) + \right.$$

$$+ \int_0^{A_{11}} \left[K_{11}(\omega,0) - \left(a_0 + b_0 \right) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega\ (x-t))\ d\omega\ +$$

$$+ \int_{A_{11}}^{\infty} \left[K_{11}(\omega,0) - K_{11}^{\infty}(\omega,0) \right] \sin(\omega\ (x-t))\ d\omega +$$

$$\left. + \int_{A_{11}}^{\infty} K_{11}^{s}(\omega) \sin(\omega\ (x-t))\ d\omega \right\} \tag{2.54}$$

where

$$K_{11}^{s}(\omega) = \frac{(a_2 + b_2)}{\omega^2} + \frac{(a_3 + b_3)}{\omega^3} + \frac{(a_4 + b_4)}{\omega^4} \tag{2.55}$$

Secondly, consider the following term in equation (2.41a)

$$\frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y) \cos(\omega\ (x-t))\ d\omega \tag{2.56}$$

First, subtract the leading terms of equation (2.39b) from the integrands in equation (2.56), and then equation (2.56) is reduced to,

$$\frac{1}{2\pi}\int_0^\infty K_{12}(\omega,y)\cos(\omega\,(x-t))\,d\omega =$$

$$= \frac{1}{2\pi}\int_0^\infty \left[K_{12}(\omega,y) - \left(c_0\,e^{r\,1\,y} + d_0\,e^{r\,2\,y}\right)\right]\cos(\omega\,(x-t))\,d\omega +$$

$$+ \frac{1}{2\pi}\int_0^\infty \left[c_0\,e^{r\,1\,y} + d_0\,e^{r\,2\,y}\right]\cos(\omega\,(x-t))\,d\omega \qquad (2.57)$$

Substituting equations (2.44) into equation (2.57), equation (2.57) can be expressed as follows,

$$\frac{1}{2\pi}\int_0^\infty K_{12}(\omega,y)\cos(\omega\,(x-t))\,d\omega =$$

$$= \frac{1}{2\pi}\int_0^\infty \left[K_{12}(\omega,y) - \left(c_0\,e^{r\,1\,y} + d_0\,e^{r\,2\,y}\right)\right]\cos(\omega\,(x-t))\,d\omega +$$

$$+ \frac{1}{2\pi}\int_0^\infty \left[c_0\,e^{\alpha\,1\,y\omega} + d_0\,e^{\alpha 2\,y\omega}\right]\cos(\omega\,(x-t))\,d\omega \qquad (2.58)$$

Under these assumptions that $y<0$ and $\Re\left(\alpha_{\;j}\right)>0$ following result is obtained

$$\frac{1}{2\pi}\int_0^\infty K_{12}(\omega,y)\cos(\omega\,(x-t))\,d\omega =$$

$$= \frac{1}{2\pi}\int_0^\infty \left[K_{12}(\omega,y) - \left(c_0\,e^{r\,1\,y} + d_0\,e^{r\,2\,y}\right)\right]\cos(\omega\,(x-t))\,d\omega -$$

$$- \frac{1}{2\pi}\left\{\frac{c_0\,\alpha_1\,y}{\alpha_1^2\,y^2 + (x-t)^2} + \frac{b_0\,\alpha_2\,y}{\alpha_2^2\,y^2 + (x-t)^2}\right\} \qquad (2.59)$$

Now, the limit is taken as $y \to 0$ and the following is obtained

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y) \cos(\omega (x-t)) \, d\omega =$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left[K_{12}(\omega, 0) - \left(c_0 + d_0 \right) \right] \cos(\omega (x-t)) \, d\omega - \frac{1}{2\pi} \left\{ \left(c_0 + d_0 \right) \pi \, \delta(x-t) \right\} \quad (2.60)$$

Using MAPLE, it can be shown that $c_0 + d_0 = 0$. Hence

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y) \cos(\omega (x-t)) \, d\omega = \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, 0) \cos(\omega (x-t)) \, d\omega \quad (2.61)$$

Using integration cut-off points, equation (2.61) is then expressed as

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y) \cos(\omega (x-t)) \, d\omega =$$

$$= \frac{1}{2\pi} \int_0^{A_{12}} K_{12}(\omega, 0) \cos(\omega (x-t)) \, d\omega + \frac{1}{2\pi} \int_{A_{12}}^{\infty} K_{12}(\omega, 0) \cos(\omega (x-t)) \, d\omega \quad (2.62)$$

From equations (2.39b), (2.40c) and (2.40d) $K_{12}^{\infty}(\omega, 0)$ is obtained as

$$K_{12}^{\infty}(\omega, 0) = \frac{(c_1 + d_1)}{\omega} + \frac{(c_2 + d_2)}{\omega^2} + \frac{(c_3 + d_3)}{\omega^3} + \frac{(c_4 + d_4)}{\omega^4} \quad (2.63)$$

The first terms of equation (2.63) can be subtracted from the integrands in equation (2.62)

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^\infty K_{12}(\omega, y) \cos(\omega(x-t))\, d\omega =$$

$$= \frac{1}{2\pi} \int_0^{A_{12}} K_{12}(\omega, 0) \cos(\omega(x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \int_{A_{12}}^\infty \left[K_{12}(\omega, 0) - \frac{(c_1 + d_1)}{\omega} \right] \cos(\omega(x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \int_{A_{12}}^\infty \frac{(c_1 + d_1)}{\omega} \cos(\omega(x-t))\, d\omega \tag{2.64}$$

Then,

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^\infty K_{12}(\omega, y) \cos(\omega(x-t))\, d\omega =$$

$$= \frac{1}{2\pi} \int_0^{A_{12}} K_{12}(\omega, 0) \cos(\omega(x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} \int_{A_{12}}^\infty \left[K_{12}(\omega, 0) - \frac{(c_1 + d_1)}{\omega} \right] \cos(\omega(x-t))\, d\omega +$$

$$+ \frac{1}{2\pi} (c_1 + d_1) \left\{ -\gamma_0 - \ln\left(A_{12} \left| x-t \right| \right) - \int_0^{A_{12}|x-t|} \frac{\cos\alpha - 1}{\alpha} d\alpha \right\} \tag{2.65}$$

where A_{12} is the integration cut-off point and γ_0 is the Euler number. The expressions used in the evaluation of integrals are given in Appendix C. Following equation is obtained by subtracting the remaining terms of equation (2.63) from the integrands in equation (2.65)

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y) \cos(\omega (x-t))\, d\omega =$$

$$= \frac{1}{2\pi} \left\{ -(c_1+d_1) \ln\left(A_{12}\,|x-t|\right) + \frac{1}{2\pi} \int_0^{A_{12}} K_{12}(\omega,0) \cos(\omega (x-t))\, d\omega \; + \right.$$

$$+ \int_{A_{12}}^{\infty} \left[K_{12}(\omega,0) - K_{12}^{\infty}(\omega,0)\right] \cos(\omega (x-t))\, d\omega \; +$$

$$\left. -(c_1+d_1)\left(\gamma_0 + \int_0^{A_{12}|x-t|} \frac{\cos\alpha - 1}{\alpha} d\alpha \right) + \int_{A_{12}}^{\infty} K_{12}^{s}(\omega) \cos(\omega (x-t))\, d\omega \right\}$$

(2.66)

where

$$K_{12}^{s}(\omega) = \frac{(c_2+d_2)}{\omega^2} + \frac{(c_3+d_3)}{\omega^3} + \frac{(c_4+d_4)}{\omega^4} \qquad (2.67)$$

Now consider the first integral in equation (2.41b)

$$\frac{1}{2\pi} \int_0^{\infty} K_{21}(\omega, y) \cos(\omega (x-t))\, d\omega \qquad (2.68)$$

This term is similar to the second term in equation (2.41a), so the same procedure can be followed. By using MAPLE it can be shown that $e_0 + f_0 = 0$ and one can write

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{21}(\omega, y) \cos(\omega\,(x-t))\, d\omega$$

$$= \frac{1}{2\pi} \left\{ -(e_1+f_1) \ln\left(A_{21}\,|x-t|\right) + \frac{1}{2\pi} \int_0^{A21} K_{21}(\omega,0) \cos(\omega\,(x-t))\, d\omega + \right.$$

$$+ \int_{A21}^{\infty} \left[K_{21}(\omega,0) - K_{21}^{\infty}(\omega,0)\right] \cos\,(\omega\,(x-t))\, d\omega +$$

$$\left. -(e_1+f_1)\left(\gamma_0 + \int_0^{A21\,|x-t|} \frac{\cos\alpha - 1}{\alpha} d\alpha\right) + \int_{A21}^{\infty} K_{21}^{s}(\omega) \cos\,(\omega\,(x-t))\, d\omega \right\}$$

(2.69)

where

$$K_{21}^{\infty}(\omega,\, 0) = \frac{(e_1+f_1)}{\omega} + \frac{(e_2+f_2)}{\omega^2} + \frac{(e_3+f_3)}{\omega^3} + \frac{(e_4+f_4)}{\omega^4} \qquad (2.70)$$

$$K_{12}^{s}(\omega) = \frac{(e_2+f_2)}{\omega^2} + \frac{(e_3+f_3)}{\omega^3} + \frac{(e_4+f_4)}{\omega^4} \qquad (2.71)$$

Finally the second integral in equation (2.41b) is considered

$$\frac{1}{2\pi} \int_0^{\infty} K_{22}(\omega, y) \sin\,(\omega\,(x-t))\, d\omega \qquad (2.72)$$

This term is similar to the first term in equation (2.41a), so the same procedure can be followed and one can write

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{22}(\omega, y) \sin(\omega (x-t))\, d\omega =$$

$$= \frac{1}{2\pi} \left\{ \frac{g_0 + h_0}{(x-t)} + (g_1 + h_1) \frac{\pi}{2} \operatorname{sign}(x-t) + \right.$$

$$+ \int_0^{A_{22}} \left[K_{22}(\omega,0) - (g_0 + h_0) - \frac{(g_1 + h_1)}{\omega} \right] \sin(\omega (x-t))\, d\omega +$$

$$+ \int_{A_{22}}^{\infty} \left[K_{22}(\omega,0) - K_{22}^{\infty}(\omega,0) \right] \sin(\omega (x-t))\, d\omega +$$

$$\left. + \int_{A_{22}}^{\infty} K_{22}^{s}(\omega) \sin(\omega (x-t))\, d\omega \right\} \tag{2.73}$$

where

$$K_{22}^{\infty}(\omega, 0) = (g_0 + h_0) + \frac{(g_1 + h_1)}{\omega} + \frac{(g_2 + h_2)}{\omega^2} + \frac{(g_3 + h_3)}{\omega^3} + \frac{(g_4 + h_4)}{\omega^4} \tag{2.74}$$

$$K_{22}^{s}(\omega) = \frac{(g_2 + h_2)}{\omega^2} + \frac{(g_3 + h_3)}{\omega^3} + \frac{(g_4 + h_4)}{\omega^4} \tag{2.75}$$

The equations for the integrals can be rearranged as

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{11}(\omega, y) \sin(\omega (x-t))\, d\omega =$$

$$= \frac{1}{\pi} \frac{(a_0 + b_0)/2}{(x-t)} + \frac{(a_1 + b_1)}{4} \operatorname{sign}(x-t) + H_{11}(x,t) \tag{2.76}$$

where

$$H_{11}(x,t) = \frac{1}{2\pi}\left\{ \int_0^{A_{11}} \left[K_{11}(\omega,0) - \left(a_0 + b_0\right) - \frac{(a_1 + b_1)}{\omega} \right] \sin(\omega\,(x-t))\, d\omega \, + \right.$$

$$+ \int_{A_{11}}^{\infty} \left[K_{11}(\omega,0) - K_{11}^{\infty}(\omega,0)\right] \sin(\omega\,(x-t))\, d\omega \, +$$

$$\left. + \int_{A_{11}}^{\infty} K_{11}^{s}(\omega)\, \sin(\omega\,(x-t))\, d\omega \right\} \tag{2.77}$$

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{12}(\omega, y)\, \cos(\omega\,(x-t))\, d\omega =$$

$$= -\frac{1}{\pi} \left(\frac{c_1 + d_1}{2} \right) \ln\left(A_{12}\, |x-t| \right) + H_{12}(x,t) \tag{2.78}$$

where

$$H_{12}(x,t) = \frac{1}{2\pi}\left\{ \int_0^{A_{12}} K_{12}(\omega,0)\, \cos(\omega\,(x-t))\, d\omega \, + \right.$$

$$+ \int_{A_{12}}^{\infty} \left[K_{12}(\omega,0) - K_{12}^{\infty}(\omega,0)\right] \cos(\omega\,(x-t))\, d\omega \, +$$

$$- (c_1 + d_1) \left(\gamma_0 + \int_0^{A_{12}|x-t|} \frac{\cos\alpha - 1}{\alpha}\, d\alpha \right) +$$

$$\left. + \int_{A_{12}}^{\infty} K_{12}^{s}(\omega) \cos(\omega\,(x-t))\, d\omega \right\} \tag{2.79}$$

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{21}(\omega, y) \cos(\omega (x-t))\, d\omega$$

$$= -\frac{1}{\pi} \left(\frac{e_1 + f_1}{2} \right) \ln \left(A_{21} \, | x - t | \right) + H_{21}(x,t) \qquad (2.80)$$

where

$$H_{21}(x,t) = \frac{1}{2\pi} \left\{ \int_0^{A_{21}} K_{21}(\omega,0) \cos(\omega (x-t))\, d\omega \right.$$

$$+ \int_{A_{21}}^{\infty} \left[K_{21}(\omega,0) - K_{21}^{\infty}(\omega,0) \right] \cos(\omega (x-t))\, d\omega$$

$$- (e_1 + f_1) \left(\gamma_0 + \int_0^{A_{21} | x-t |} \frac{\cos\alpha - 1}{\alpha} d\alpha \right)$$

$$\left. + \int_{A_{21}}^{\infty} K_{21}^{s}(\omega) \cos(\omega (x-t))\, d\omega \right\} \qquad (2.81)$$

$$\lim_{y \to 0} \frac{1}{2\pi} \int_0^{\infty} K_{22}(\omega, y) \sin(\omega (x-t))\, d\omega$$

$$= \frac{1}{\pi} \frac{(g_0 + h_0)/2}{(x-t)} + \frac{(g_1 + h_1)}{4} \operatorname{sign}(x-t) + H_{22}(x,t) \qquad (2.82)$$

where

$$H_{22}(x,t) = \frac{1}{2\pi}\left\{ \int_{0}^{A_{22}} \left[K_{22}(\omega,0) - \left(g_0 + h_0\right) - \frac{(g_1 + h_1)}{\omega} \right] \sin(\omega(x-t))\, d\omega \; + \right.$$

$$+ \int_{A_{22}}^{\infty} \left[K_{22}(\omega,0) - K_{22}^{\infty}(\omega,0) \right] \sin(\omega(x-t))\, d\omega +$$

$$\left. + \int_{A_{22}}^{\infty} K_{22}^{s}(\omega) \sin(\omega(x-t))\, d\omega \right\} \tag{2.83}$$

Now the integral equations can be written as

$$\sigma_{yy}(x,0) = \frac{E_0}{1-\nu_0^2}\left\{ \int_{-a}^{a} \left[\frac{1}{\pi}\frac{(a_0+b_0)/2}{(x-t)} + \frac{(a_1+b_1)}{4}\,\text{sign}\,(x-t) + H_{11}(x,t) \right] \phi_1(t)\, dt + \right.$$

$$\left. + \int_{-a}^{a} \left[-\frac{1}{\pi}\left(\frac{c_1+d_1}{2}\right) \ln\left(A_{12}\,|x-t|\right) + H_{12}(x,t) \right] \phi_2(t)\, dt \right\} = \overline{p}(x)\ ,$$

$$-a < x < a$$

(2.84a)

$$\sigma_{xy}(x,0) = \frac{E_0}{2(\kappa_0 + \nu_0)}\left\{ \int_{-a}^{a} \left[-\frac{1}{\pi}\left(\frac{e_1+f_1}{2}\right) \ln\left(A_{21}\,|x-t|\right) + H_{21}(x,t) \right] \phi_1(t)\, dt \; + \right.$$

$$\left. + \int_{-a}^{a} \left[\frac{1}{\pi}\frac{(g_0+h_0)/2}{(x-t)} + \frac{(g_1+h_1)}{4}\,\text{sign}\,(x-t) + H_{22}(x,t) \right] \phi_2(t)\, dt \right\} = \overline{q}(x)\ ,$$

$$-a < x < a$$

(2.84b)

where $\overline{p}(x)$ and $\overline{q}(x)$ expressed in the transformed coordinate system are arbitrary normal and shear tractions applied to the crack surface.

$$\overline{p}(x) = -\delta_0\, p\left(x\sqrt{\delta_0}\right) , \qquad -a < x < a , \qquad (2.85a)$$

$$\overline{q}(x) = -q\left(x\sqrt{\delta_0}\right) , \qquad -a < x < a , \qquad (2.85b)$$

and a is the half crack length in the transformed coordinate system. Equations (2.84) can also be written as

$$\int_{-a}^{a}\left[\frac{1}{\pi}\frac{(a_0+b_0)/2}{(x-t)} + \frac{(a_1+b_1)}{4}\,\text{sign}\,(x-t) + H_{11}(x,t)\right]\phi_1(t)\,dt$$
$$+\int_{-a}^{a}\left[-\frac{1}{\pi}\left(\frac{c_1+d_1}{2}\right)\ln\left(A_{12}\,|x-t|\right) + H_{12}(x,t)\right]\phi_2(t)\,dt = p^*(x) \qquad , \quad -a < x < a$$

(2.86a)

$$\int_{-a}^{a}\left[-\frac{1}{\pi}\left(\frac{e_1+f_1}{2}\right)\ln\left(A_{21}\,|x-t|\right) + H_{21}(x,t)\right]\phi_1(t)\,dt$$
$$+\int_{-a}^{a}\left[\frac{1}{\pi}\frac{(g_0+h_0)/2}{(x-t)} + \frac{(g_1+h_1)}{4}\,\text{sign}\,(x-t) + H_{22}(x,t)\right]\phi_2(t)\,dt = q^*(x),$$

$$-a < x < a$$

(2.86b)

where

$$p^*(x) = \frac{1-\nu_0^2}{E_0}\,\overline{p}(x) \tag{2.87a}$$

$$q^*(x) = \frac{2(\kappa_0 + \nu_0)}{E_0}\,\overline{q}(x) \tag{2.87b}$$

At this point, the integrals in equations (2.86) and the intervals $|x| < a$, are normalized by using the following transformations

$$x = a\,s\,, \qquad -a < x < a\,, \qquad -1 < s < 1 \tag{2.88a}$$

$$t = a\,r\,, \qquad -a < x < a\,, \qquad -1 < t < 1 \tag{2.88b}$$

Using these transformations, equations (2.86) take the following form

$$\int_{-1}^{1}\left[\frac{1}{\pi}\frac{(a_0 + b_0)/2}{(s-r)} + \frac{(\hat{a}_1 + \hat{b}_1)}{4}\,\text{sign}\,(s-r) + a\,H_{11}(a\,s, a\,r)\right]\phi_1(a\,r)\,dr$$
$$+\int_{-1}^{1}\left[-\frac{1}{\pi}\left(\frac{\hat{c}_1 + \hat{d}_1}{2}\right)\ln\left(A_{12}\,a|s-r|\right) + a\,H_{12}(a\,s, a\,r)\right]\phi_2(a\,r)\,dr = p^*(a\,s),$$
$$-1 < s < 1$$
$$\tag{2.89a}$$

$$\int_{-1}^{1}\left[-\frac{1}{\pi}\left(\frac{\hat{e}_1 + \hat{f}_1}{2}\right)\ln\left(A_{21}\,a|s-r|\right) + a\,H_{21}(a\,s, a\,r)\right]\phi_1(a\,r)\,dr$$
$$+\int_{-1}^{1}\left[\frac{1}{\pi}\frac{(g_0 + h_0)/2}{(s-r)} + \frac{(\hat{g}_1 + \hat{h}_1)}{4}\,\text{sign}\,(s-r) + a\,H_{22}(a\,s, a\,r)\right]\phi_2(a\,r)\,dr = q^*(a\,s),$$
$$-1 < s < 1$$
$$\tag{2.89b}$$

where in the terms with the sign ^ , nonhomogeneity constant γ is replaced by $\gamma\, a$.

Now, examine following terms in equations (2.89)

- $a H_{11}(a s, a r)$

$$a H_{11}(a s, a r) = \frac{1}{2\pi}\left\{ \int_0^{A_{11}} \left[K_{11}(\omega,0) - \left(a_0 + b_0\right) - \frac{(a_1 + b_1)}{\omega} \right] \sin(a\omega\,(s-r))\,(a\,d\omega) + \int_{A_{11}}^{\infty} \left[K_{11}(\omega,0) - K_{11}^{\infty}(\omega,0)\right] \sin(a\omega\,(s-r))\,(a\,d\omega) + \int_{A_{11}}^{\infty} K_{11}^{s}(\omega)\,\sin(a\omega\,(s-r))\,(a\,d\omega) \right\} \tag{2.90}$$

Using the transformation $a\omega = \alpha$, one may obtain

$$a H_{11}(a s, a r) = \frac{1}{2\pi}\left\{ \int_0^{A_{11}a} \left[K_{11}(\alpha/a,0) - \left(a_0 + b_0\right) - \frac{(\hat{a}_1 + \hat{b}_1)}{\alpha} \right] \sin(\alpha\,(s-r))\,d\alpha + \int_{A_{11}a}^{\infty} \left[K_{11}(\alpha/a,0) - K_{11}^{\infty}(\alpha/a,0)\right] \sin(\alpha\,(s-r))\,d\alpha + \int_{A_{11}a}^{\infty} K_{11}^{s}(\alpha/a)\,\sin(\alpha\,(s-r))\,d\alpha \right\} \tag{2.91}$$

Normalizing the integrands $a H_{11}(a s, a r)$ and $K_{11}(\omega,0)$ can be written as follows

$$aH_{11}(as,ar) = \frac{1}{2\pi}\left\{\int_0^{A_{11}^*}\left[\hat{K}_{11}(\alpha,0)-\left(a_0+b_0\right)-\frac{(\hat{a}_1+\hat{b}_1)}{\alpha}\right]\sin(\alpha\,(s-r))\;d\alpha\;+\right.$$

$$+\int_{A_{11}^*}^{\infty}\left[\hat{K}_{11}(\alpha,0)-\hat{K}_{11}^{\infty}(\alpha,0)\right]\sin(\alpha\;(s-r))\;d\alpha+$$

$$\left.+\int_{A_{11}^*}^{\infty}\hat{K}_{11}^{s}(\alpha)\sin(\alpha\;(s-r))\;d\alpha\right\} = \hat{H}_{11}(s,r) \tag{2.92}$$

where

$$\hat{K}_{11}(\alpha,0) = 2i\left\{\sum_{j=1}^{4}\left(i\alpha\nu_0+\hat{r}_j\,\hat{H}_j\right)\hat{Y}_j(\alpha)\right\} \tag{2.93}$$

$$A_{11}^{*} = A_{11}\,a \tag{2.94}$$

After this case one can follow the same procedure which is applied to find $aH_{11}(as,ar)$ to obtain $aH_{12}(as,ar)$, $aH_{21}(as,ar)$ and $aH_{22}(as,ar)$. These terms are determined as follows

$$a H_{12}(as, ar) = \frac{1}{2\pi}\left\{ \int_0^{A_{12}^*} \hat{K}_{12}(\alpha,0)\cos(\alpha(s-r))\,d\alpha + \right.$$

$$+ \int_{A_{12}^*}^{\infty} \left[\hat{K}_{12}(\alpha,0) - \hat{K}_{12}^{\infty}(\alpha,0)\right]\cos(\alpha(s-r))\,d\alpha +$$

$$-(\hat{c}_1 + \hat{d}_1)\left(\gamma_0 + \int_0^{A_{12}^*|s-r|} \frac{\cos\alpha - 1}{\alpha}d\alpha\right) +$$

$$\left. + \int_{A_{12}^*}^{\infty} \hat{K}_{12}^s(\alpha)\cos(\alpha(s-r))\,d\alpha \right\} = \hat{H}_{12}(s,r) \qquad (2.95)$$

where

$$\hat{K}_{12}(\alpha,0) = 2\left\{ \sum_{j=1}^{4}\left(i\alpha\nu_0 + \hat{r}_j\,\hat{H}_j\right)\hat{Z}_j(\alpha) \right\} \qquad (2.96)$$

$$A_{12}^* = A_{12}\,a \qquad (2.97)$$

$$a H_{21}(as, ar) = \frac{1}{2\pi}\left\{ \int_0^{A_{21}^*} \hat{K}_{21}(\alpha,0)\cos(\alpha(s-r))\,d\alpha + \right.$$

$$+ \int_{A_{21}^*}^{\infty} \left[\hat{K}_{21}(\alpha,0) - \hat{K}_{21}^{\infty}(\alpha,0)\right]\cos(\alpha(s-r))\,d\alpha +$$

$$-(\hat{e}_1 + \hat{f}_1)\left(\gamma_0 + \int_0^{A_{21}^*|s-r|} \frac{\cos\alpha - 1}{\alpha}d\alpha\right) +$$

$$\left. + \int_{A_{21}^*}^{\infty} \hat{K}_{21}^s(\alpha)\cos(\alpha(s-r))\,d\alpha \right\} = \hat{H}_{21}(s,r) \qquad (2.98)$$

where

$$\hat{K}_{21}(\alpha,0) = 2\left\{\sum_{j=1}^{4}\left(i\alpha\,\hat{H}_j + \hat{r}_j\right)\hat{Y}_j(\alpha)\right\} \tag{2.99}$$

$$A^*_{21} = A_{21}\,a \tag{2.100}$$

$$\begin{aligned} a\,H_{22}(as,ar) = \frac{1}{2\pi}\Bigg\{ & \int_0^{A^*_{22}}\left[\hat{K}_{22}(\alpha,0) - \left(g_0 + h_0\right) - \frac{(\hat{g}_1 + \hat{h}_1)}{\alpha}\right]\sin(\alpha\,(s-r))\,d\alpha + \\ & + \int_{A^*_{22}}^{\infty}\left[\hat{K}_{22}(\alpha,0) - \hat{K}^{\infty}_{22}(\alpha,0)\right]\sin(\alpha\,(s-r))\,d\alpha + \\ & + \int_{A^*_{22}}^{\infty}\hat{K}^{s}_{22}(\alpha)\,\sin(\alpha\,(s-r))\,d\alpha\Bigg\} = \hat{H}_{22}(s,r) \end{aligned} \tag{2.101}$$

where

$$\hat{K}_{22}(\alpha,0) = 2i\left\{\sum_{j=1}^{4}\left(i\alpha\,\hat{H}_j + \hat{r}_j\right)\hat{Z}_j(\alpha)\right\} \tag{2.102}$$

$$A^*_{22} = A_{22}\,a \tag{2.103}$$

Then integral equations can be written as

$$\int_{-1}^{1}\left[\frac{1}{\pi}\,\frac{(a_0+b_0)/2}{(s-r)}+\frac{(\hat{a}_1+\hat{b}_1)}{4}\,\text{sign}\,(s-r)+\hat{H}_{11}(s,r)\right]\hat{\phi}(r)\,dr$$

$$+\int_{-1}^{1}\left[-\frac{1}{\pi}\left(\frac{\hat{c}_1+\hat{d}_1}{2}\right)\ln\left(A^*_{12}\,|s-r|\right)+\hat{H}_{12}(s,r)\right]\hat{\varphi}(r)\,dr=p^*(a\,s),$$

$$-1< s <1$$

(2.104a)

$$\int_{-1}^{1}\left[-\frac{1}{\pi}\left(\frac{\hat{e}_1+\hat{f}_1}{2}\right)\ln\left(A^*_{21}\,|s-r|\right)+\hat{H}_{21}(s,r)\right]\hat{\phi}(r)\,dr$$

$$+\int_{-1}^{1}\left[\frac{1}{\pi}\,\frac{(g_0+h_0)/2}{(s-r)}+\frac{(\hat{g}_1+\hat{h}_1)}{4}\,\text{sign}\,(s-r)+\hat{H}_{22}(s,r)\right]\hat{\varphi}(r)\,dr=q^*(a\,s),$$

$$-1< s <1$$

(2.104b)

where

$$\hat{\phi}(r)=\phi_1(a\,r) \qquad (2.105a)$$

$$\hat{\varphi}(r)=\phi_2(a\,r) \qquad (2.105b)$$

It is observed that the dominant singularity in the integral equations is the Cauchy type singularity. The singular integral equations given by equations (2.104) are solved using an expansion–collocation technique. In the analysis of the problem Chebyshev polynomials will be used. The solution can be expressed in the following form.

$$\hat{\phi}(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n=0}^{\infty} A_n T_n(r) \tag{2.106a}$$

$$\hat{\varphi}(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n=0}^{\infty} B_n T_n(r) \tag{2.106b}$$

T_n, in this equation, is the Chebyshev polynomial of the first kind of order n and A_n, B_n are unknown constants. Another condition that has to be satisfied in the problem is the single valuedness condition. First, the equations for the crack opening displacements are considered. Using the equation (2.30) crack opening displacement at any point x_1 $\left(|x_1| < a\sqrt{\delta_0}\right)$ is expressed as follows

$$\int_{-a\sqrt{\delta_0}}^{x_1} \left[\frac{\partial}{\partial x_1} \left(u_2^{(1)}(x_1,0) - u_2^{(2)}(x_1,0) \right) \right] dx_1 = \int_{-a\sqrt{\delta_0}}^{x_1} f_1(x_1)\, dx_1$$
$$= \left[u_2^{(1)}(x_1,0) - u_2^{(2)}(x_1,0) \right] - \left[u_2^{(1)}\left(-a\sqrt{\delta_0},0\right) - u_2^{(2)}\left(-a\sqrt{\delta_0},0\right) \right] \tag{2.107}$$

Hence, crack opening displacements are expressed as follows

$$\int_{-a\sqrt{\delta_0}}^{x_1} f_1(x_1)\, dx_1 = \left[u_2^{(1)}(x_1,0) - u_2^{(2)}(x_1,0) \right] \tag{2.108a}$$

$$\int_{-a\sqrt{\delta_0}}^{x_1} f_2(x_1)\, dx_1 = \left[u_1^{(1)}(x_1,0) - u_1^{(2)}(x_1,0) \right] \tag{2.108b}$$

If $x_1 = a\sqrt{\delta_0}$ equations (2.108) can be written as

$$\int_{-a\sqrt{\delta_0}}^{a\sqrt{\delta_0}} f_1(x_1)\, dx_1 = 0 \qquad (2.109a)$$

$$\int_{-a\sqrt{\delta_0}}^{a\sqrt{\delta_0}} f_2(x_1)\, dx_1 = 0 \qquad (2.109b)$$

which are the so called single valuedness conditions. One can write these equations in transformed coordinate system as follows.

$$\int_{-a}^{a} \phi_1(x)\, dx = 0 \qquad (2.110a)$$

$$\int_{-a}^{a} \phi_2(x)\, dx = 0 \qquad (2.110b)$$

Substituting equation (2.88a) into equations (2.110), one can obtain

$$\int_{-1}^{1} \phi_1(as)\, ds = 0 \qquad (2.111a)$$

$$\int_{-1}^{1} \phi_2(as)\, ds = 0 \qquad (2.111b)$$

Substituting equations (2.105) into equations (2.111) one may obtain

$$\int_{-1}^{1} \hat{\phi}\,(s)\, d\,s \;=\; 0 \tag{2.112a}$$

$$\int_{-1}^{1} \hat{\varphi}\;(s)\, d\,s \;=\; 0 \tag{2.112b}$$

and substituting equations (2.106) into equations (2.112), one can obtain

$$\sum_{n=0}^{\infty} A_n \int_{-1}^{1} \frac{T_n(s)}{\sqrt{1-s^2}}\, d\,s = 0 \tag{2.113a}$$

$$\sum_{n=0}^{\infty} B_n \int_{-1}^{1} \frac{T_n(s)}{\sqrt{1-s^2}}\, d\,s = 0 \tag{2.113b}$$

Using equations (2.113) and orthogonality property of Chebyshev polynomials given in Appendix D, equation (D3), it can easily be shown that

$$A_0 = B_0 = 0 \tag{2.114}$$

Hence, using single – valuedness conditions, equations (2.106) can be expressed as follows

$$\hat{\phi}\,(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n=1}^{\infty} A_n\, T_n\,(r) \tag{2.115a}$$

$$\hat{\varphi}\,(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n=1}^{\infty} B_n T_n(r) \tag{2.115b}$$

Substituting equations (2.115) into equations (2.104), integral equations are now determined as

$$\sum_{n=1}^{\infty} A_n \left\{ \int_{-1}^{1} \frac{1}{\pi} \frac{(a_0 + b_0)/2}{(s-r)} \frac{T_n(r)}{\sqrt{1-r^2}} dr + \right.$$
$$\left. + \int_{-1}^{1} \frac{(\hat{a}_1 + \hat{b}_1)}{4} \text{ sign } (s-r) \frac{T_n(r)}{\sqrt{1-r^2}} dr + \int_{-1}^{1} \hat{H}_{11}(s,r) \frac{T_n(r)}{\sqrt{1-r^2}} dr \right\} +$$
$$+ \sum_{n=1}^{\infty} B_n \left\{ \int_{-1}^{1} -\frac{1}{\pi} \left(\frac{\hat{c}_1 + \hat{d}_1}{2} \right) \ln \left(A_{12}^{*} |s-r| \right) \frac{T_n(r)}{\sqrt{1-r^2}} dr + \right.$$
$$\left. + \int_{-1}^{1} \hat{H}_{12}(s,r) \frac{T_n(r)}{\sqrt{1-r^2}} dr \right\} = p^{*}(as), \qquad -1 < s < 1$$

(2.116a)

$$\sum_{n=1}^{\infty} A_n \left\{ \int_{-1}^{1} -\frac{1}{\pi} \left(\frac{\hat{e}_1 + \hat{f}_1}{2} \right) \ln \left(A_{21}^{*} \; |s-r| \right) \frac{T_n(r)}{\sqrt{1-r^2}} dr + \int_{-1}^{1} \hat{H}_{21}(s,r) \frac{T_n(r)}{\sqrt{1-r^2}} dr \right\} +$$
$$+ \sum_{n=1}^{\infty} B_n \left\{ \int_{-1}^{1} \frac{1}{\pi} \frac{(g_0 + h_0)/2}{(s-r)} \frac{T_n(r)}{\sqrt{1-r^2}} dr \right.$$
$$\left. + \int_{-1}^{1} \frac{(\hat{g}_1 + \hat{h}_1)}{4} \text{ sign } (s-r) \frac{T_n(r)}{\sqrt{1-r^2}} dr + \int_{-1}^{1} \hat{H}_{22}(s,r) \frac{T_n(r)}{\sqrt{1-r^2}} dr \right\} = q^{*}(as),$$
$$-1 < s < 1$$

(2.116b)

Singular terms in equations (2.116) can be evaluated using the following properties

$$\int_{-1}^{1} \frac{1}{\pi} \frac{(a_0 + b_0)/2}{(s-r)} \frac{T_n(r)}{\sqrt{1-r^2}} dr = -\left(\frac{a_0 + b_0}{2}\right) U_{n-1}(s) \ , \quad |s| < 1, \quad n \geq 1 \qquad (2.117a)$$

$$\int_{-1}^{1} \frac{(\hat{a}_1 + \hat{b}_1)}{4} \text{ sign } (s-r) \frac{T_n(r)}{\sqrt{1-r^2}} dr = -\left(\frac{\hat{a}_1 + \hat{b}_1}{2}\right) \frac{1}{n} \sqrt{1-s^2}\, U_{n-1}(s) ,$$

$$|s| < 1, \quad n \geq 1 \qquad (2.117b)$$

$$\int_{-1}^{1} -\frac{1}{\pi} \left(\frac{\hat{c}_1 + \hat{d}_1}{2}\right) \ln\left(A_{12}^{*} |s-r|\right) \frac{T_n(r)}{\sqrt{1-r^2}} dr = \left(\frac{\hat{c}_1 + \hat{d}_1}{2}\right) \frac{T_n(s)}{n} ,$$

$$|s| < 1, \quad n \geq 1$$

$$(2.117c)$$

$$\int_{-1}^{1} -\frac{1}{\pi} \left(\frac{\hat{e}_1 + \hat{f}_1}{2}\right) \ln\left(A_{21}^{*} \ |s-r|\right) \frac{T_n(r)}{\sqrt{1-r^2}} dr = \left(\frac{\hat{e}_1 + \hat{f}_1}{2}\right) \frac{T_n(s)}{n} ,$$

$$|s| < 1, \quad n \geq 1$$

$$(2.117d)$$

$$\int_{-1}^{1} \frac{1}{\pi} \frac{(g_0 + h_0)/2}{(s-r)} \frac{T_n(r)}{\sqrt{1-r^2}} dr = -\left(\frac{g_0 + h_0}{2}\right) U_{n-1}(s) , \quad |s| < 1, \quad n \geq 1 \qquad (2.117e)$$

$$\int_{-1}^{1} \frac{(\hat{g}_1 + \hat{h}_1)}{4} \text{ sign } (s-r) \frac{T_n(r)}{\sqrt{1-r^2}} dr = -\left(\frac{\hat{g}_1 + \hat{h}_1}{2}\right) \frac{1}{n} \sqrt{1-s^2}\, U_{n-1}(s) ,$$

$$|s| < 1, \quad n \geq 1 \qquad (2.117f)$$

where $U_{n-1}(s)$ is the Chebyshev polynomial of the second kind. The identities used in the evaluation of integrals are also given in Appendix D. The integral equations can now be rewritten in the following form.

$$\sum_{n=1}^{\infty} A_n \left\{ -\left(\frac{a_0 + b_0}{2}\right) U_{n-1}(s) - \left(\frac{\hat{a}_1 + \hat{b}_1}{2}\right) \frac{1}{n} \sqrt{1-s^2}\, U_{n-1}(s) + k_{11n}(s) \right\}$$

$$+ \sum_{n=1}^{\infty} B_n \left\{ \left(\frac{\hat{c}_1 + \hat{d}_1}{2}\right) \frac{T_n(s)}{n} + k_{12n}(s) \right\} = p^*(a\,s), \qquad -1 < s < 1$$

(2.118a)

$$\sum_{n=1}^{\infty} A_n \left\{ \left(\frac{\hat{e}_1 + \hat{f}_1}{2}\right) \frac{T_n(s)}{n} + k_{21n}(s) \right\}$$

$$+ \sum_{n=1}^{\infty} B_n \left\{ -\left(\frac{g_0 + h_0}{2}\right) U_{n-1}(s) - \left(\frac{\hat{g}_1 + \hat{h}_1}{2}\right) \frac{1}{n} \sqrt{1-s^2}\, U_{n-1}(s) + k_{22n}(s) \right\} = q^*(a\,s),$$

$$-1 < s < 1$$

(2.118b)

where

$$k_{11n}(s) = \int_{-1}^{1} \hat{H}_{11}(s, r) \frac{T_n(r)}{\sqrt{1-r^2}}\, dr \qquad (2.119a)$$

$$k_{12n}(s) = \int_{-1}^{1} \hat{H}_{12}(s, r) \frac{T_n(r)}{\sqrt{1-r^2}}\, dr \qquad (2.119b)$$

$$k_{21n}(s) = \int_{-1}^{1} \hat{H}_{21}(s, r) \frac{T_n(r)}{\sqrt{1-r^2}}\, dr \qquad (2.119c)$$

$$k_{22n}(s) = \int_{-1}^{1} \hat{H}_{22}(s,r) \frac{T_n(r)}{\sqrt{1-r^2}} dr \tag{2.119d}$$

The roots of the Chebyshev polynomials given in Appendix D in equation (D8) are used to locate the collocation points. By using these collocation points integral equations are converted to a system of linear algebraic equations of size $2N \times 2N$ and are solved for A_n and B_n, $(n=1,...,N)$. After solving the system of equations, one can calculate the fracture mechanics variables. The fracture mechanics variables that are going to be calculated for this problem are

1. Stress Intensity Factors $\left(k_1\left(a\sqrt{\delta_0}\right), k_2\left(a\sqrt{\delta_0}\right), k_1\left(-a\sqrt{\delta_0}\right), k_2\left(-a\sqrt{\delta_0}\right)\right)$
2. Energy Release Rates $\left(G\left(a\sqrt{\delta_0}\right), G\left(-a\sqrt{\delta_0}\right)\right)$

The modes I and II stress intensity factors at the crack tips $x_1 = \pm a\sqrt{\delta_0}$ are defined by

$$k_1(a\sqrt{\delta_0}) = \lim_{x_1 \to \left(a\sqrt{\delta_0}\right)^+} \sqrt{2(x_1 - a\sqrt{\delta_0})}\, \sigma_{22}(x_1,0) \tag{2.120a}$$

$$k_2(a\sqrt{\delta_0}) = \lim_{x_1 \to \left(a\sqrt{\delta_0}\right)^+} \sqrt{2(x_1 - a\sqrt{\delta_0})}\, \sigma_{12}(x_1,0) \tag{2.120b}$$

$$k_1(-a\sqrt{\delta_0}) = \lim_{x_1 \to \left(-a\sqrt{\delta_0}\right)^-} \sqrt{2(-x_1 - a\sqrt{\delta_0})}\, \sigma_{22}(x_1,0) \tag{2.120c}$$

$$k_2(-a\sqrt{\delta_0}) = \lim_{x_1 \to \left(-a\sqrt{\delta_0}\right)^-} \sqrt{2(-x_1 - a\sqrt{\delta_0})}\, \sigma_{12}(x_1,0) \tag{2.120d}$$

Dominant parts of the stress components σ_{yy} and σ_{xy} can be written as

$$\sigma_{yy}(x,0) \cong \frac{E_0}{1-\nu_0^2}\left(\frac{a_0+b_0}{2}\right)\frac{1}{\pi}\int_{-a}^{a}\frac{\phi_1(t)}{x-t}\,dt \qquad (2.121a)$$

$$\sigma_{xy}(x,0) \cong \frac{E_0}{2(\kappa_0+\nu_0)}\left(\frac{g_0+h_0}{2}\right)\frac{1}{\pi}\int_{-a}^{a}\frac{\phi_2(t)}{x-t}\,dt \qquad (2.121b)$$

$$x = a\,s,\quad t = a\,r,\qquad -\infty < s < \infty \qquad \Rightarrow$$

$$\sigma_{yy}(a\,s,0) \cong \frac{E_0}{1-\nu_0^2}\left(\frac{a_0+b_0}{2}\right)\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_1(a\,r)}{s-r}\,dr\ ,\qquad -\infty < s < \infty \qquad (2.122a)$$

$$\sigma_{xy}(a\,s,0) \cong \frac{E_0}{2(\kappa_0+\nu_0)}\left(\frac{g_0+h_0}{2}\right)\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_2(a\,r)}{s-r}\,dr\,,\qquad -\infty < s < \infty \qquad (2.122b)$$

Substituting equations (2.115) into equations (2.122) and then using the equation (D7) given in Appendix D, the dominant parts of the stress components can be written as follows

$$\sigma_{yy}(a\,s,0) \cong \frac{E_0}{1-\nu_0^2}\left(\frac{a_0+b_0}{2}\right)\frac{|s|}{s\sqrt{s^2-1}}\sum_{n=1}^{\infty}A_n\left(s-(|s|/s)\sqrt{s^2-1}\right)^n\ ,$$

$$|s| > 1$$

(2.123a)

$$\sigma_{xy}(as,0) \cong \frac{E_0}{2(\kappa_0 + \nu_0)} \left(\frac{g_0 + h_0}{2}\right) \frac{|s|}{s\sqrt{s^2-1}} \sum_{n=1}^{\infty} B_n \left(s - (|s|/s)\sqrt{s^2-1}\right)^n ,$$

$$|s| > 1$$

(2.123b)

For $s > 1$ and $s < -1$ one can write

- $s > 1$

$$\sigma_{yy}(as,0) \cong \frac{E_0}{1-\nu_0^2} \left(\frac{a_0 + b_0}{2}\right) \frac{1}{\sqrt{s^2-1}} \sum_{n=1}^{\infty} A_n \left(s - \sqrt{s^2-1}\right)^n \quad (2.124a)$$

$$\sigma_{xy}(as,0) \cong \frac{E_0}{2(\kappa_0 + \nu_0)} \left(\frac{g_0 + h_0}{2}\right) \frac{1}{\sqrt{s^2-1}} \sum_{n=1}^{\infty} B_n \left(s - \sqrt{s^2-1}\right)^n \quad (2.124b)$$

- $s < -1$

$$\sigma_{yy}(as,0) \cong -\frac{E_0}{1-\nu_0^2} \left(\frac{a_0 + b_0}{2}\right) \frac{1}{\sqrt{s^2-1}} \sum_{n=1}^{\infty} A_n \left(s + \sqrt{s^2-1}\right)^n \quad (2.125a)$$

$$\sigma_{xy}(as,0) \cong -\frac{E_0}{2(\kappa_0 + \nu_0)} \left(\frac{g_0 + h_0}{2}\right) \frac{1}{\sqrt{s^2-1}} \sum_{n=1}^{\infty} B_n \left(s + \sqrt{s^2-1}\right)^n \quad (2.125b)$$

Equations (2.124) and (2.125) can be written in physical coordinate system as follows

- $x_1 > a\sqrt{\delta_0}$

$$\sigma_{22}(x_1,0) = \frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0+b_0}{2}\right)\frac{1}{\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2-1}}\times$$

$$\times\sum_{n=1}^{\infty}A_n\left(\left(\frac{x_1}{a\sqrt{\delta_0}}\right)-\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2-1}\right)^n \qquad (2.126a)$$

$$\sigma_{12}(x_1,0) = \frac{E_0}{2(\kappa_0+\nu_0)}\left(\frac{g_0+h_0}{2}\right)\frac{1}{\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2-1}}\times$$

$$\times\sum_{n=1}^{\infty}B_n\left(\left(\frac{x_1}{a\sqrt{\delta_0}}\right)-\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2-1}\right)^n \qquad (2.126b)$$

- $x_1 < -a\sqrt{\delta_0}$

$$\sigma_{22}(x_1,0) = -\frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0+b_0}{2}\right)\frac{1}{\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2-1}}\times$$

$$\times\sum_{n=1}^{\infty}A_n\left(\left(\frac{x_1}{a\sqrt{\delta_0}}\right)+\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2-1}\right)^n \qquad (2.127a)$$

$$\sigma_{12}(x_1,0) = -\frac{E_0}{2(\kappa_0 + \nu_0)}\left(\frac{g_0 + h_0}{2}\right)\frac{1}{\sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2 - 1}}\times$$

$$\times\sum_{n=1}^{\infty} B_n\left(\left(\frac{x_1}{a\sqrt{\delta_0}}\right) + \sqrt{\left(\frac{x_1}{a\sqrt{\delta_0}}\right)^2 - 1}\right)^n \qquad (2.127b)$$

Substituting equations (2.126a) ,(2.126b), (2.127a) and (2.127b) into equations (2.120a) ,(2.120b), (2.120c) and (2.120d), respectively the stress intensity factors can be determined as

$$k_1(a\sqrt{\delta_0}) = \frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0 + b_0}{2}\right)\left(a\sqrt{\delta_0}\right)^{1/2}\sum_{n=1}^{\infty} A_n \qquad (2.128a)$$

$$k_2(a\sqrt{\delta_0}) = \frac{E_0}{2(\kappa_0 + \nu_0)}\left(\frac{g_0 + h_0}{2}\right)\left(a\sqrt{\delta_0}\right)^{1/2}\sum_{n=1}^{\infty} B_n \qquad (2.128b)$$

$$k_1(-a\sqrt{\delta_0}) = -\frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0 + b_0}{2}\right)\left(a\sqrt{\delta_0}\right)^{1/2}\sum_{n=1}^{\infty} A_n(-1)^n \qquad (2.128c)$$

$$k_2(-a\sqrt{\delta_0}) = -\frac{E_0}{2(\kappa_0 + \nu_0)}\left(\frac{g_0 + h_0}{2}\right)\left(a\sqrt{\delta_0}\right)^{1/2}\sum_{n=1}^{\infty} B_n(-1)^n \qquad (2.128d)$$

The crack closure energy method can be used to derive the expressions for the energy release rates at the crack tips. In graded orthotropic solids, principal axes of material orthotropy are preferred cleavage planes [9]. Hence, generally the growth path of an embedded crack will be confined to the direction along the x_1 axis.

Consequently, in applying the crack closure energy method, crack extension can be assumed to be along the x_1 axis and the work done by the normal and shear stresses can be summed to calculate the total closure work. Under fixed-grip conditions, the expressions for the energy release rates at the crack tips $x_1 = \pm a\sqrt{\delta_0}$ are defined by

$$G = \lim_{\Delta a \to 0} \frac{\Delta W_{closure}}{B \Delta a} \tag{2.129}$$

where B is the thickness.

At the tip $x_1 = a\sqrt{\delta_0}$, one can write

$$\Delta W_{closure} = \int_{a\sqrt{\delta_0}}^{a\sqrt{\delta_0}+\Delta a} \frac{1}{2}\sigma_{22}(x_1,0) B\left(u_2^{(1)}(x_1,0) - u_2^{(2)}(x_1,0) \right) dx_1 + \int_{a\sqrt{\delta_0}}^{a\sqrt{\delta_0}+\Delta a} \frac{1}{2}\sigma_{12}(x_1,0) B\left(u_1^{(1)}(x_1,0) - u_1^{(2)}(x_1,0) \right) dx_1 \tag{2.130a}$$

and at $x_1 = -a\sqrt{\delta_0}$,

$$\Delta W_{closure} = \int_{-a\sqrt{\delta_0}-\Delta a}^{-a\sqrt{\delta_0}} \frac{1}{2}\sigma_{22}(x_1,0) B\left(u_2^{(1)}(x_1,0) - u_2^{(2)}(x_1,0) \right) dx_1 + \int_{-a\sqrt{\delta_0}-\Delta a}^{-a\sqrt{\delta_0}} \frac{1}{2}\sigma_{12}(x_1,0) B\left(u_1^{(1)}(x_1,0) - u_1^{(2)}(x_1,0) \right) dx_1 \tag{2.130b}$$

First integral in equations (2.130a) and (2.130b) is the closure work due to mode I loading. Second is the closure work due to mode II loading. Expressing equations (2.121) in the physical coordinate system and then substituting $t = s \big/ \sqrt{\delta_0}$ in these equations the dominant parts of the expressions for the stresses can be written as follows

$$\sigma_{22}(x_1,0) = -\frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0+b_0}{2}\right)\frac{1}{\pi}\int_{-a\sqrt{\delta_0}}^{a\sqrt{\delta_0}}\frac{f_1(s)}{s-x_1}\,ds \qquad (2.131a)$$

$$\sigma_{12}(x_1,0) = -\frac{E_0\,\delta_0}{2(\kappa_0+\nu_0)}\left(\frac{g_0+h_0}{2}\right)\frac{1}{\pi}\int_{-a\sqrt{\delta_0}}^{a\sqrt{\delta_0}}\frac{f_2(t)}{s-x_1}\,ds \qquad (2.131b)$$

Using asymptotic behavior of the Cauchy integral defined in Appendix D, f_1 and f_2 can be written as

$$f_1(x_1) = G_1(x_1)\,(x_1+a\sqrt{\delta_0})^{-1/2}\,(a\sqrt{\delta_0}-x_1)^{-1/2} \qquad (2.132a)$$

$$f_2(x_1) = G_2(x_1)\,(x_1+a\sqrt{\delta_0})^{-1/2}\,(a\sqrt{\delta_0}-x_1)^{-1/2} \qquad (2.132b)$$

Substituting these equations into equations (2.131) one may obtain

$$\sigma_{22}(x_1,0) = -\frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0+b_0}{2}\right)\Big\{G_1(-a\sqrt{\delta_0})\,(-a\sqrt{\delta_0}-x_1)^{-1/2}\,(2a\sqrt{\delta_0})^{-1/2}$$
$$-G_1(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(x_1-a\sqrt{\delta_0})^{-1/2}\Big\}$$

(2.133a)

$$\sigma_{12}(x_1,0) = -\frac{E_0\,\delta_0}{2(\kappa_0+\nu_0)}\left(\frac{g_0+h_0}{2}\right)\Big\{G_2(-a\sqrt{\delta_0})\,(-a\sqrt{\delta_0}-x_1)^{-1/2}\,(2a\sqrt{\delta_0})^{-1/2}$$
$$-G_2(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(x_1-a\sqrt{\delta_0})^{-1/2}\Big\}$$

(2.133b)

Asymptotic expression of the stresses near $x_1 = a\sqrt{\delta_0}$ can now be expressed as follows

$$\sigma_{22}(x_1,0) = \frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0+b_0}{2}\right)G_1(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(x_1-a\sqrt{\delta_0})^{-1/2},$$

$$x_1 > a\sqrt{\delta_0}$$

(2.134a)

$$\sigma_{12}(x_1,0) = \frac{E_0\,\delta_0}{2(\kappa_0+\nu_0)}\left(\frac{g_0+h_0}{2}\right)G_2(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(x_1-a\sqrt{\delta_0})^{-1/2}$$

$$x_1 > a\sqrt{\delta_0}$$

(2.134b)

Near $x_1 = a\sqrt{\delta_0}$ f_1 and f_2 are reduced to

$$f_1(x_1) = G_1(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(a\sqrt{\delta_0}-x_1)^{-1/2}, \qquad x_1 < a\sqrt{\delta_0} \qquad (2.135a)$$

$$f_2(x_1) = G_2(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(a\sqrt{\delta_0}-x_1)^{-1/2}, \qquad x_1 < a\sqrt{\delta_0} \qquad (2.135b)$$

the equations for the crack opening displacements are given as

$$u_2^{(1)}(x_1,0)-u_2^{(2)}(x_1,0)=\int_{-a\sqrt{\delta_0}}^{x_1} f_1(x_1)\,dx_1=$$

$$=\int_{-a\sqrt{\delta_0}}^{a\sqrt{\delta_0}} f_1(x_1)\,dx_1-\int_{x_1}^{a\sqrt{\delta_0}} f_1(x_1)\,dx_1=-\int_{x_1}^{a\sqrt{\delta_0}} f_1(x_1)\,dx_1 \quad (2.136a)$$

$$u_1^{(1)}(x_1,0)-u_1^{(2)}(x_1,0)=\int_{-a\sqrt{\delta_0}}^{x_1} f_2(x_1)\,dx_1=$$

$$=\int_{-a\sqrt{\delta_0}}^{a\sqrt{\delta_0}} f_2(x_1)\,dx_1-\int_{x_1}^{a\sqrt{\delta_0}} f_2(x_1)\,dx_1=-\int_{x_1}^{a\sqrt{\delta_0}} f_2(x_1)\,dx_1 \quad (2.136b)$$

Substituting equations (2.135a) and (2.135b) into equations (2.136a) and (2.136b), respectively and then taking the integrals, following equations are obtained.

$$u_2^{(1)}(x_1,0)-u_2^{(2)}(x_1,0)=-2G_1(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(a\sqrt{\delta_0}-x_1)^{1/2},\quad x_1<a\sqrt{\delta_0} \quad (2.137a)$$

$$u_1^{(1)}(x_1,0)-u_1^{(2)}(x_1,0)=-2G_2(a\sqrt{\delta_0})\,(2a\sqrt{\delta_0})^{-1/2}\,(a\sqrt{\delta_0}-x_1)^{1/2},\quad x_1<a\sqrt{\delta_0} \quad (2.137b)$$

Substituting equations (2.134a) and (2.134b) into equations (2.120a) and (2.120b) respectively, the stress intensity factors at the crack tips $x_1=a\sqrt{\delta_0}$ can be expressed in the following form

$$k_1(a\sqrt{\delta_0})=\frac{E_0}{\delta_0\left(1-\nu_0^2\right)}\left(\frac{a_0+b_0}{2}\right)(a\sqrt{\delta_0})^{-1/2}\,G_1(a\sqrt{\delta_0}) \quad (2.138a)$$

$$k_2(a\sqrt{\delta_0}) = \frac{E_0\,\delta_0}{2(\kappa_0 + \nu_0)}\left(\frac{g_0 + h_0}{2}\right)(a\sqrt{\delta_0})^{-1/2}\,G_2(a\sqrt{\delta_0}) \qquad (2.138b)$$

From equations (2.138) following equations can be obtained easily

$$G_1(a\sqrt{\delta_0}) = \frac{\delta_0\left(1-\nu_0^2\right)}{E_0}\left(\frac{2}{a_0 + b_0}\right)(a\sqrt{\delta_0})^{1/2}\,k_1(a\sqrt{\delta_0}) \qquad (2.139a)$$

$$G_2(a\sqrt{\delta_0}) = \frac{2(\kappa_0 + \nu_0)}{E_0\,\delta_0}\left(\frac{2}{g_0 + h_0}\right)(a\sqrt{\delta_0})^{1/2}\,k_2(a\sqrt{\delta_0}) \qquad (2.139b)$$

Substituting equations (2.139a) and (2.139b) into equations (2.134a) and (2.134b) respectively, following equations can be obtained

$$\sigma_{22}(x_1,0) = k_1(a\sqrt{\delta_0})\left(2(x_1 - a\sqrt{\delta_0})\right)^{-1/2}, \qquad x_1 > a\sqrt{\delta_0} \qquad (2.140a)$$

$$\sigma_{12}(x_1,0) = k_2(a\sqrt{\delta_0})\left(2(x_1 - a\sqrt{\delta_0})\right)^{-1/2}, \qquad x_1 > a\sqrt{\delta_0} \qquad (2.140b)$$

Now substituting equations (2.139a) and (2.139b) into equations (2.137a) and (2.137b), respectively, following equations are obtained

$$u_2^{(1)}(x_1,0) - u_2^{(2)}(x_1,0) = -\frac{\delta_0\left(1-\nu_0^2\right)}{E_0}\left(\frac{2}{a_0 + b_0}\right)k_1(a\sqrt{\delta_0})\left(2(a\sqrt{\delta_0} - x_1)\right)^{1/2},$$

$$x_1 < a\sqrt{\delta_0}$$

(2.141a)

$$u_1^{(1)}(x_1,0)-u_1^{(2)}(x_1,0)=-\frac{2(\kappa_0+\nu_0)}{E_0\,\delta_0}\left(\frac{2}{g_0+h_0}\right)k_2(a\sqrt{\delta_0})\left(2(a\sqrt{\delta_0}-x_1)\right)^{1/2},$$

$$x_1 < a\sqrt{\delta_0}$$

(2.141b)

Using equations (2.140) and (2.141) in equations (2.130a), $\Delta W_{closure}$ is expressed as

$$\begin{aligned}\Delta W_{closure} = &-\frac{B\,\Delta a\,\pi}{4}\,k_1(a\sqrt{\delta_0})\,k_1(a\sqrt{\delta_0}+\Delta a)\frac{\delta_0\left(1-\nu_0^2\right)}{E_0}\left(\frac{2}{a_0+b_0}\right)\\ &-\frac{B\,\Delta a\,\pi}{4}\,k_2(a\sqrt{\delta_0})\,k_2(a\sqrt{\delta_0}+\Delta a)\frac{2(\kappa_0+\nu_0)}{E_0\,\delta_0}\left(\frac{2}{g_0+h_0}\right)\end{aligned}$$

(2.142)

Substituting equations (2.142) into equations (2.129), the energy release rate at the crack tips $x_1 = a\sqrt{\delta_0}$ can be obtained in the following form

$$G\left(a\sqrt{\delta_0}\right)=-\frac{\pi\,\delta_0\left(1-\nu_0^2\right)}{2\,E_0\,(a_0+b_0)}\,k_1^2(a\sqrt{\delta_0})\;-\;\frac{\pi\,(\kappa_0+\nu_0)}{E_0\,\delta_0\,(g_0+h_0)}\,k_2^2(a\sqrt{\delta_0}) \quad (2.143)$$

The same procedure can be applied to get the energy release rate at the crack tips $x_1 = -\,a\sqrt{\delta_0}$:

$$G\left(-\,a\sqrt{\delta_0}\right)=-\frac{\pi\,\delta_0\left(1-\nu_0^2\right)}{2\,E_0\,(a_0+b_0)}\,k_1^2(a\sqrt{\delta_0})\;-\;\frac{\pi\,(\kappa_0+\nu_0)}{E_0\,\delta_0\,(g_0+h_0)}\,k_2^2(a\sqrt{\delta_0}) \quad (2.144)$$

CHAPTER 3

NUMERICAL RESULTS

In order to implement the numerical methods, a computer program is developed using Visual Fortran language. In this program after obtaining the coefficients of the asymptotic expansions integrals are computed and then by using linear equation solver unknown functions are obtained. Finally the energy release rate and stress intensity factors are calculated. To obtain these twenty-three collocation points are used. The numerical results obtained are presented in this chapter. The effects of the orthotropy parameters κ_0, δ_0^4, ν_0, nonhomogeneity parameter $\beta a\sqrt{\delta_0}$ and boundary conditions on the energy release rate, mode I and mode II stress intensity factors are examined in Figures 3.1 – 3.36 for a crack subjected to uniform normal stress. In all the results presented crack length in the physical coordinated is denoted by c, i.e, $c = a\sqrt{\delta_0}$. In these figures βc is the normalized nonhomogeneity parameter, $k_0 = \sigma_0 c^{1/2}$ and $G_0 = \pi k_0^2 / E_0$ which are the mode I stress intensity factor and the energy release rate for a crack embedded in a homogeneous isotropic medium, respectively are the normalization constants.

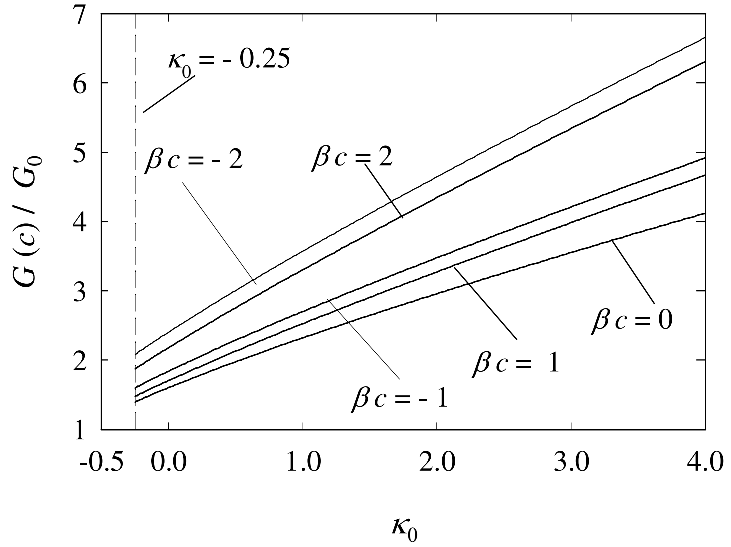

Figure 3.1 Normalized energy release rate versus the shear parameter κ_0 and the nonhomogeneity parameter $\beta\ c$ for an embedded crack (see Figure 2.1,i.e.,Case I) subjected to uniform tension, $\delta_0^4=2$, $\nu_0=0.3$, $h_1/c=1$, $h_2/c=5$, $p(x_1)=\sigma_0$, $q(x_1)=0$.

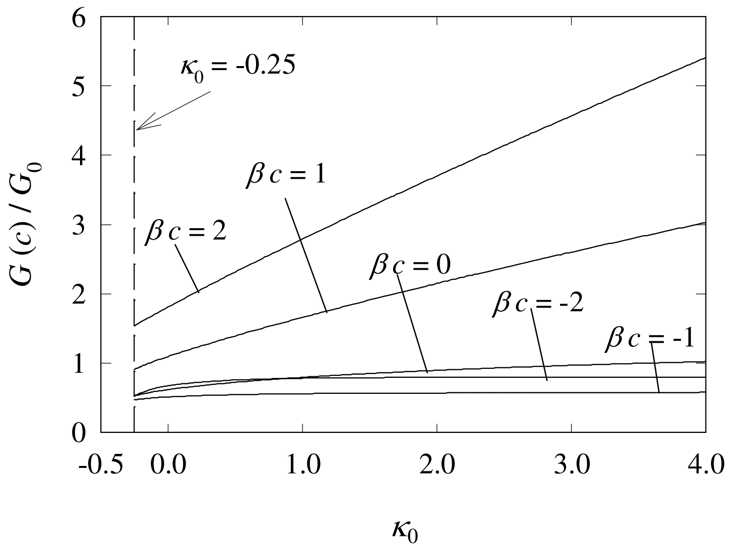

Figure 3.2 Normalized energy release rate versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2,i.e.,Case II) subjected to uniform tension, $\delta_0^4=2$, $\nu_0=0.3$, $h_1/c=1$, $h_2/c=5$, $p(x_1)=\sigma_0$, $q(x_1)=0$.

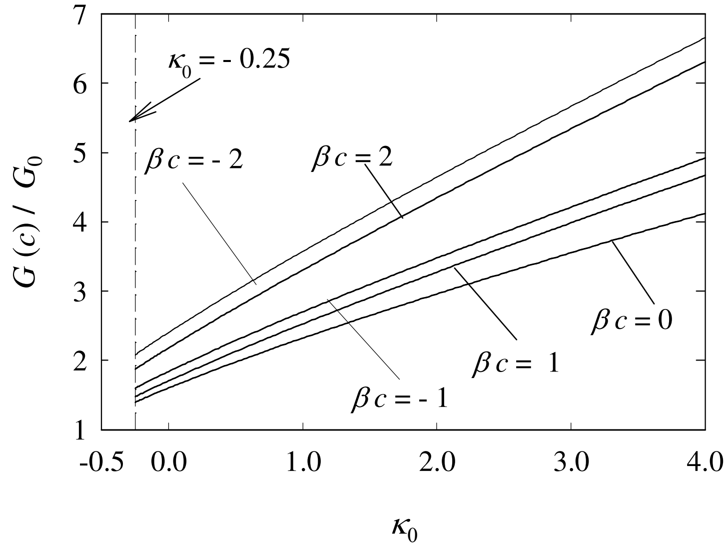

Figure 3.3 Normalized energy release rate versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3,i.e.,Case III) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

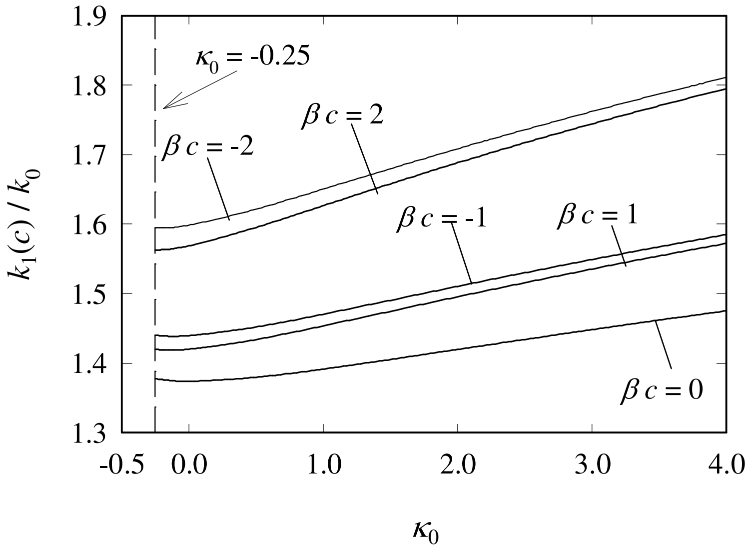

Figure 3.4 Normalized mode I SIF versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

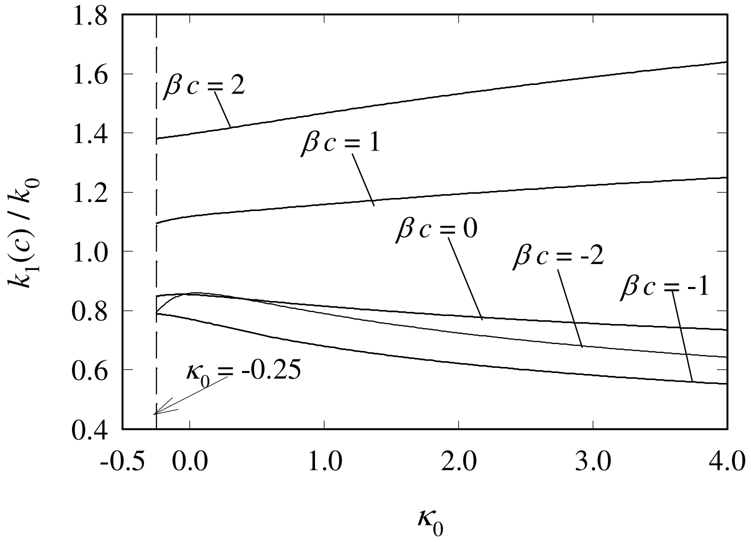

Figure 3.5 Normalized mode I SIF versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

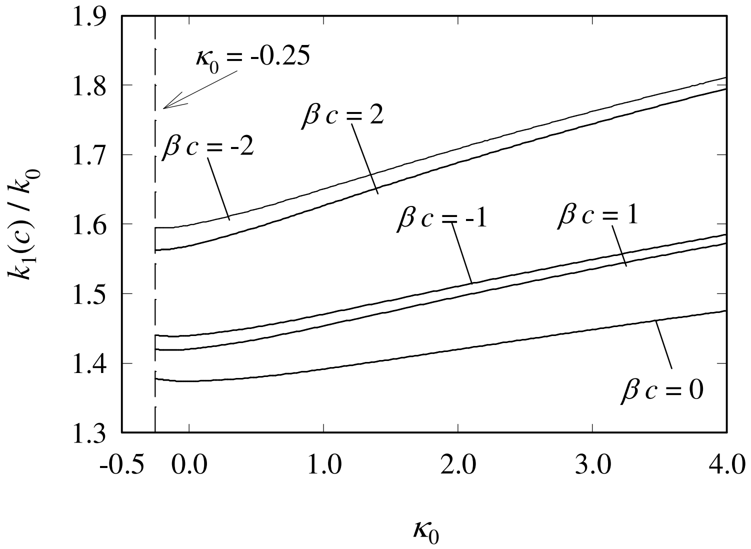

Figure 3.6 Normalized mode I SIF versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

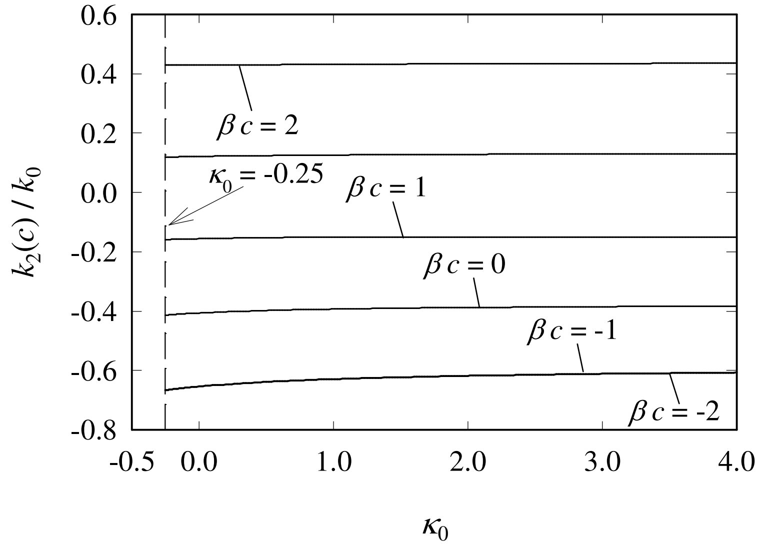

Figure 3.7 Normalized mode II SIF versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

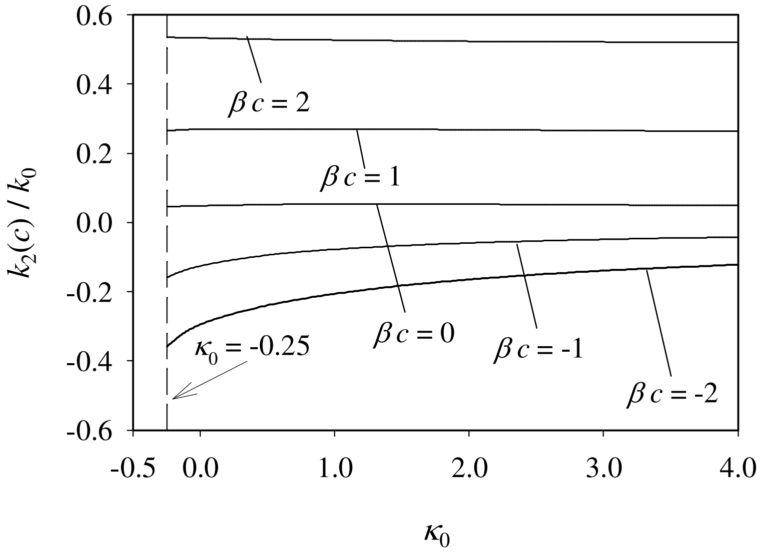

Figure 3.8 Normalized mode II SIF versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

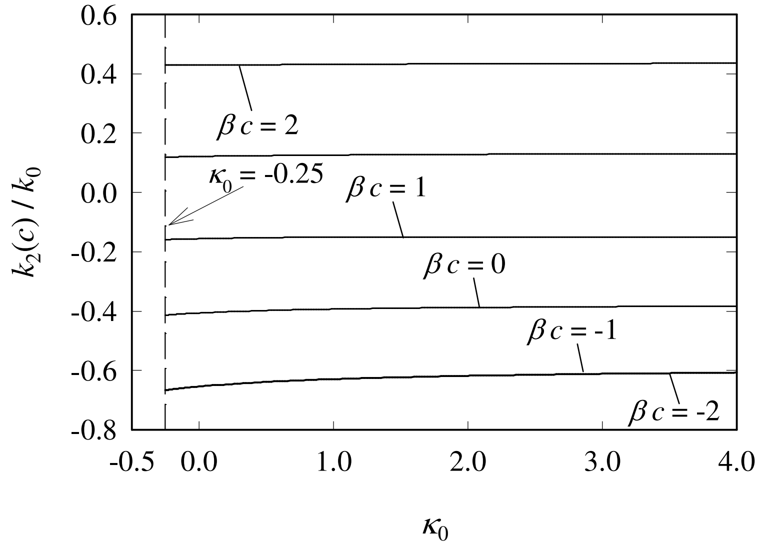

Figure 3.9 Normalized mode II SIF versus the shear parameter κ_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

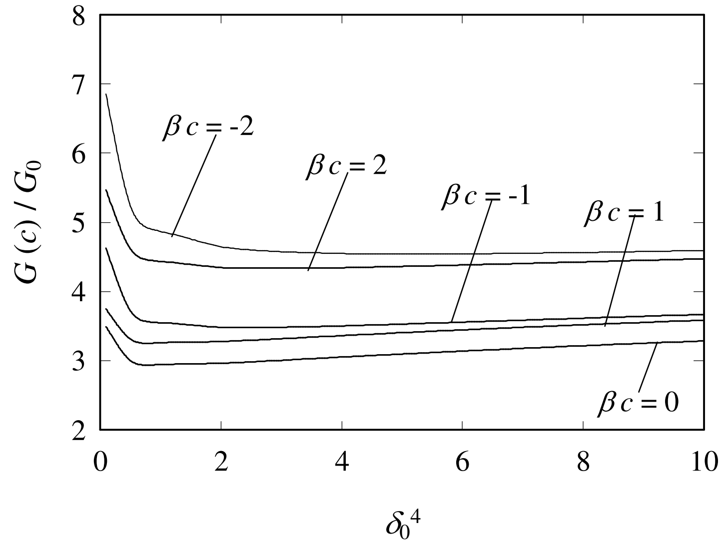

Figure 3.10 Normalized energy release rate versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

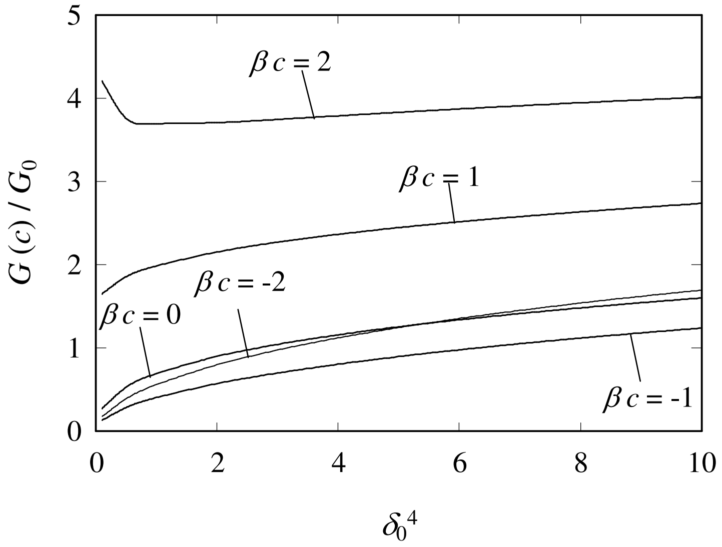

Figure 3.11 Normalized energy release rate versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

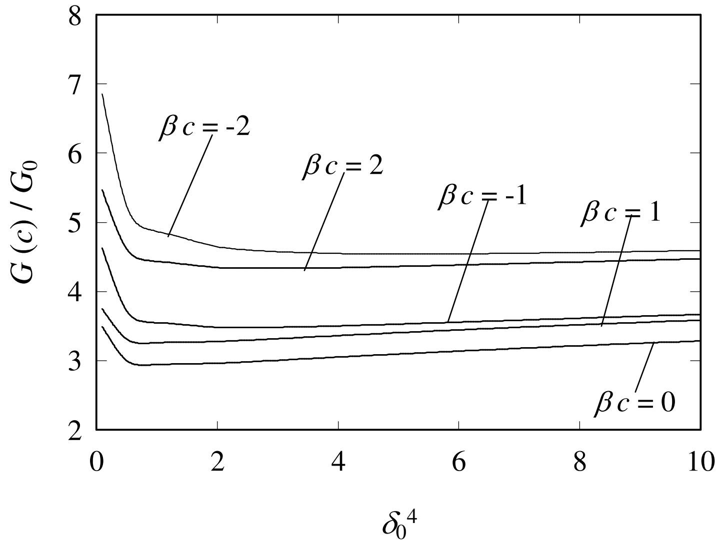

Figure 3.12 Normalized energy release rate versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

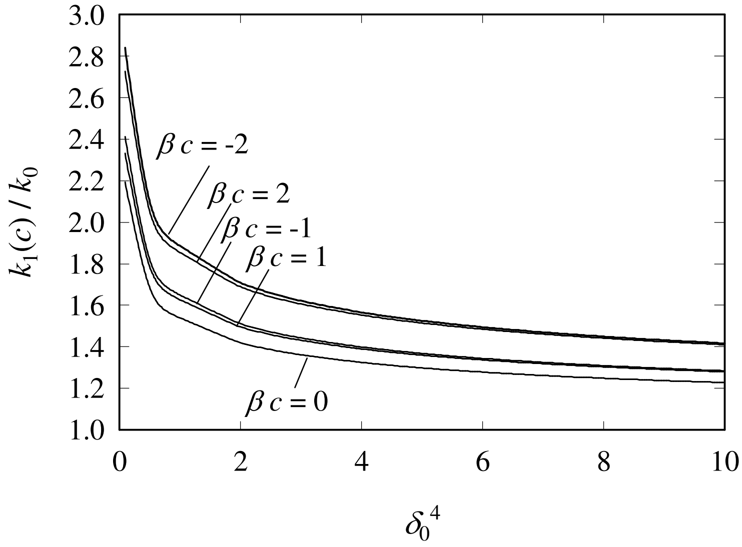

Figure 3.13 Normalized mode I SIF versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

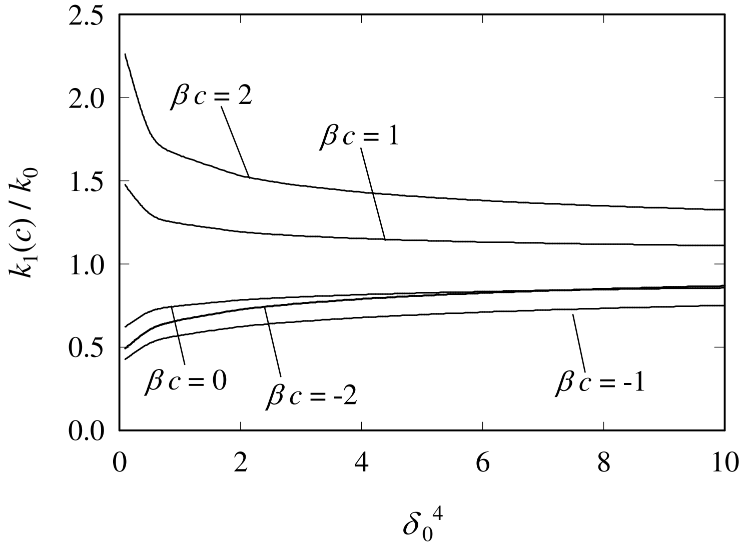

Figure 3.14 Normalized mode I SIF versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

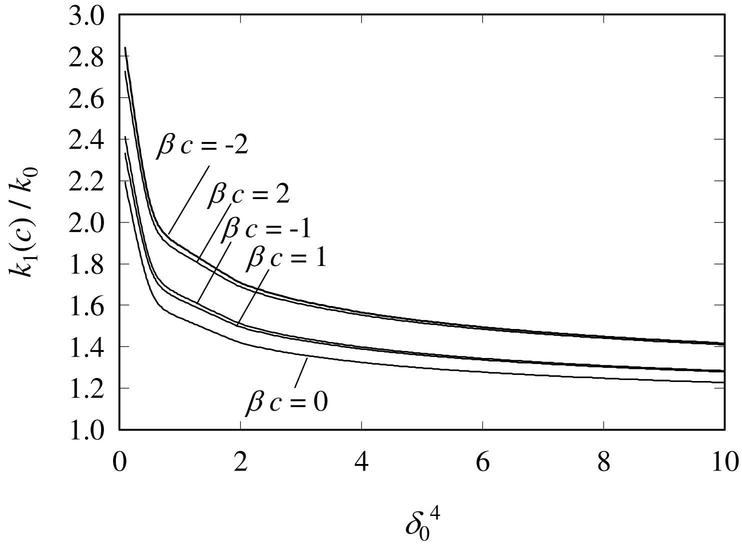

Figure 3.15 Normalized mode I SIF versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

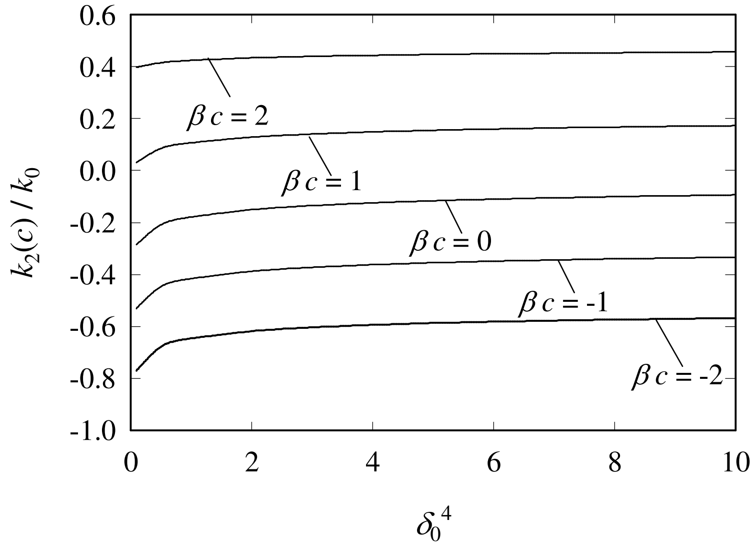

Figure 3.16 Normalized mode II SIF versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

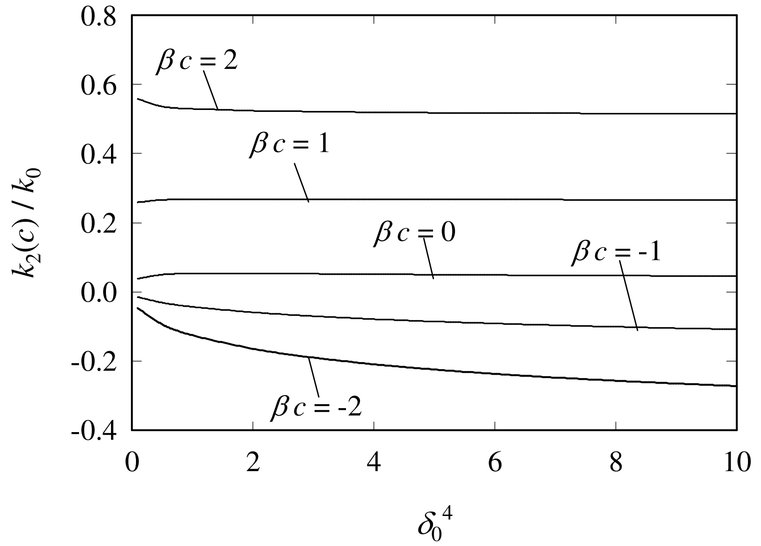

Figure 3.17 Normalized mode II SIF versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

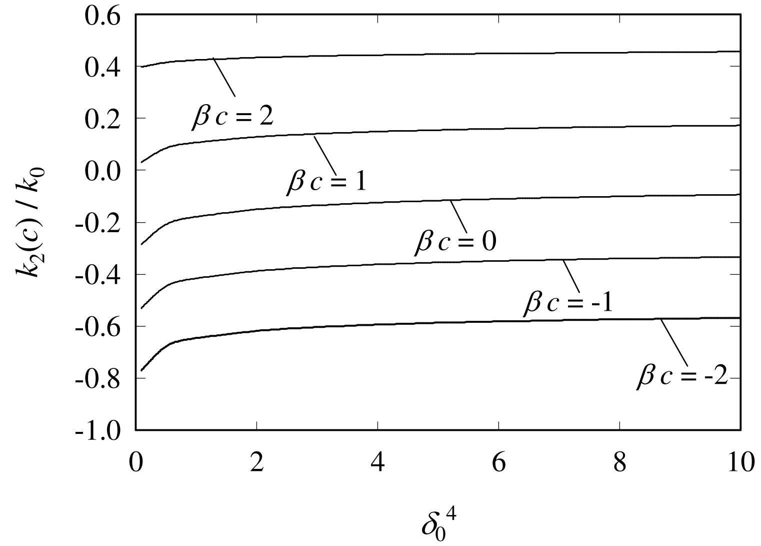

Figure 3.18 Normalized mode II SIF versus the stiffness ratio δ_0^4 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

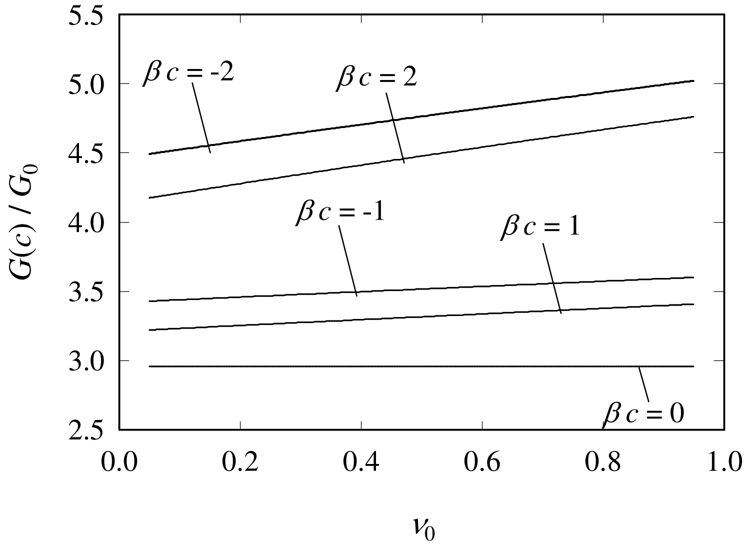

Figure 3.19 Normalized energy release rate versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

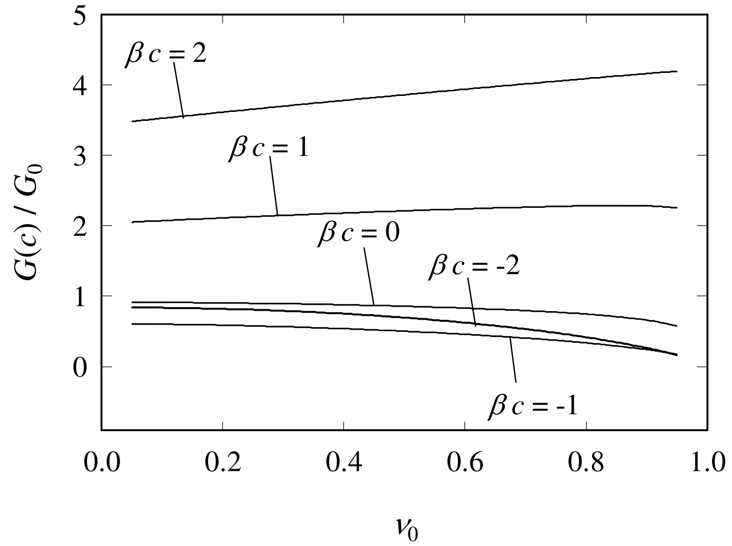

Figure 3.20 Normalized energy release rate versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2,i.e.,Case II) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

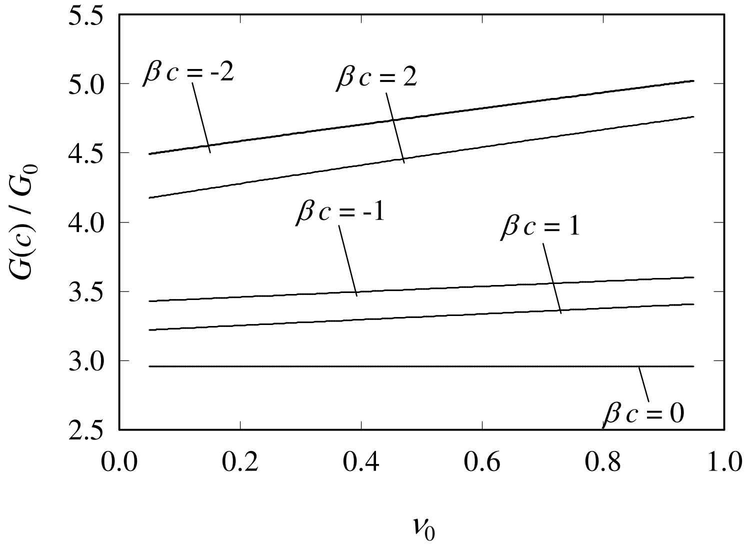

Figure 3.21 Normalized energy release rate versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

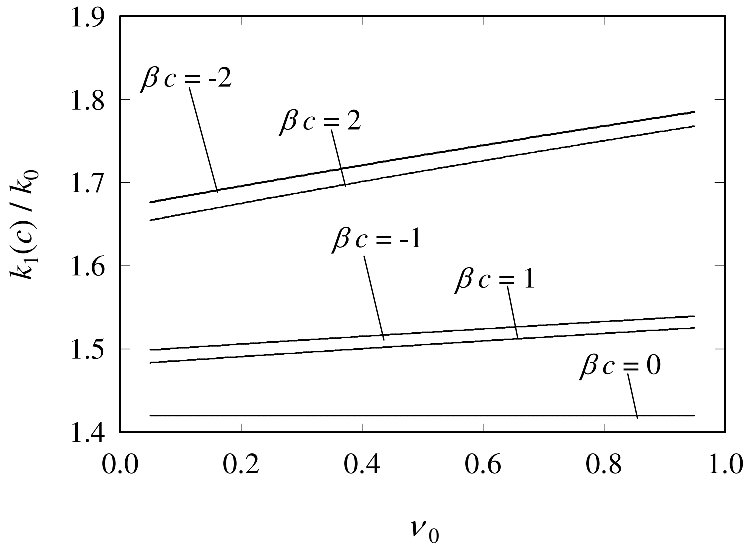

Figure 3.22 Normalized mode I SIF versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

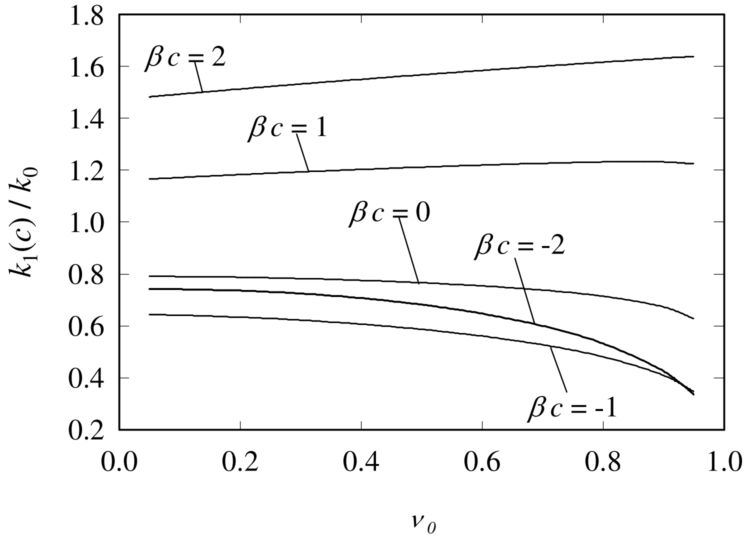

Figure 3.23 Normalized mode I SIF versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

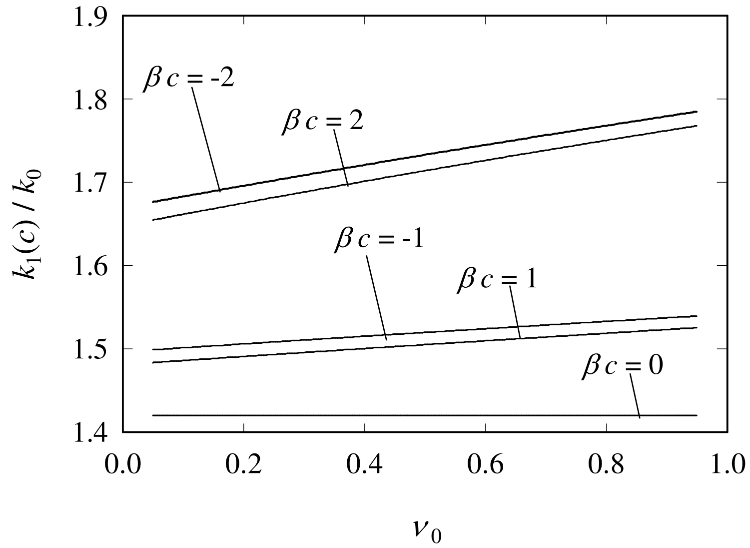

Figure 3.24 Normalized mode I SIF versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

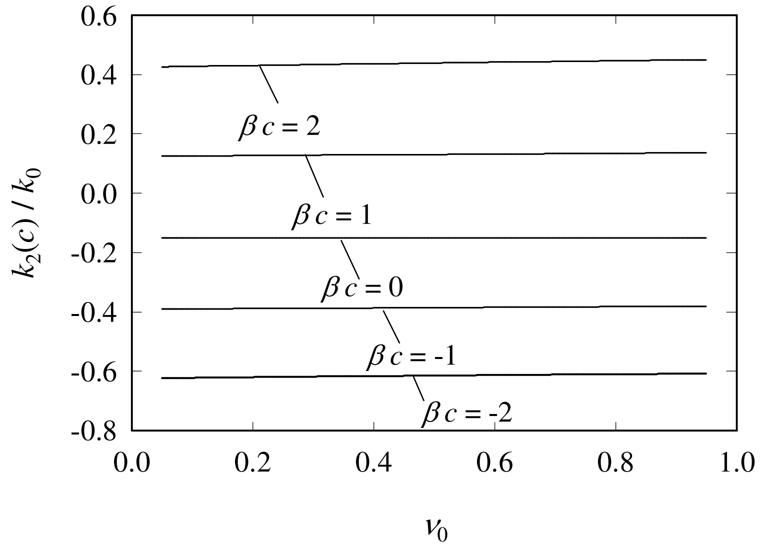

Figure 3.25 Normalized mode II SIF versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.1, i.e., Case I) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

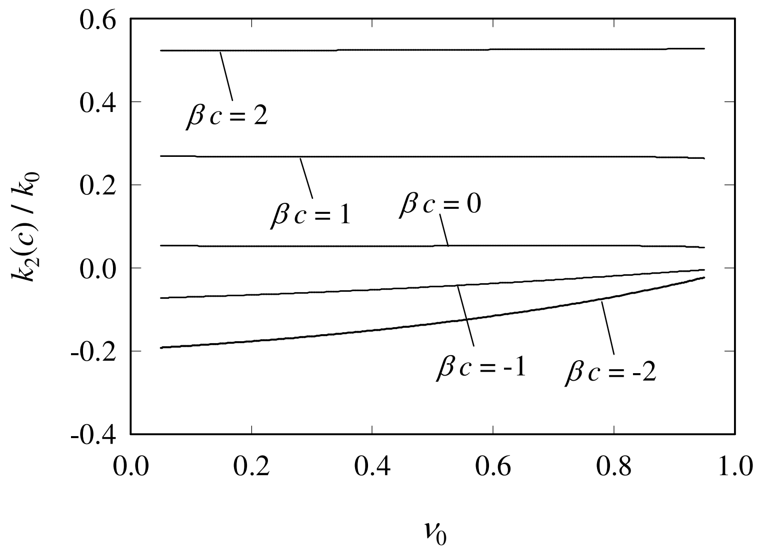

Figure 3.26 Normalized mode II SIF versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.2, i.e., Case II) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

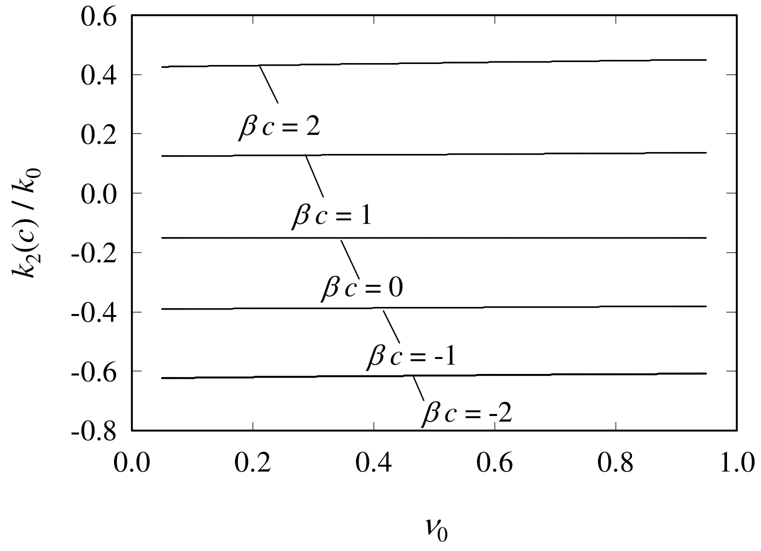

Figure 3.27 Normalized mode II SIF versus the effective Poisson's ratio ν_0 and the nonhomogeneity parameter βc for an embedded crack (see Figure 2.3, i.e., Case III) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $h_1/c = 1$, $h_2/c = 5$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

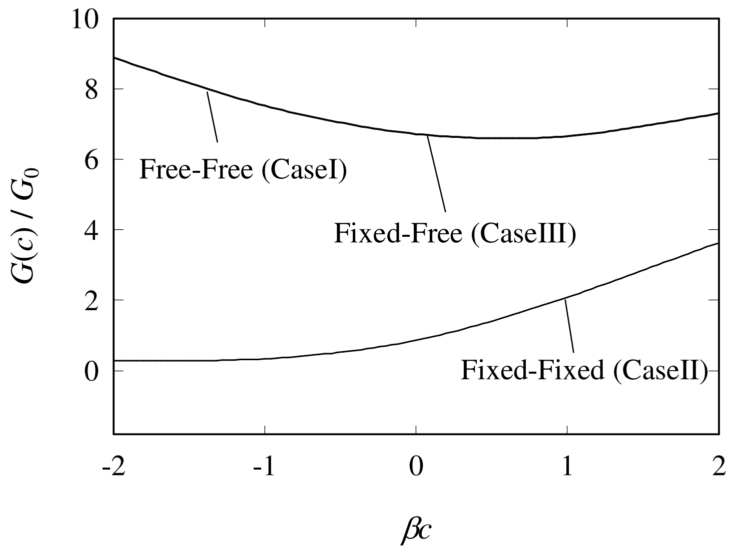

Figure 3.28 Normalized energy release rate versus the nonhomogeneity parameter βc and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

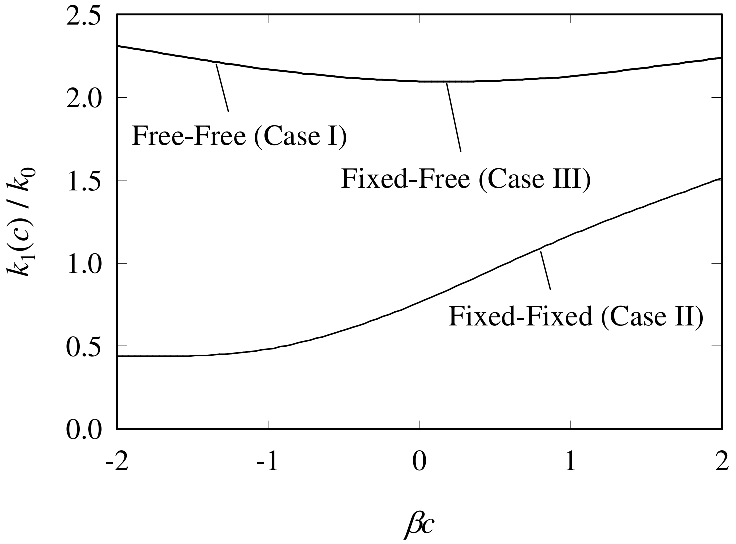

Figure 3.29 Normalized mode I SIF versus the nonhomogeneity parameter βc and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

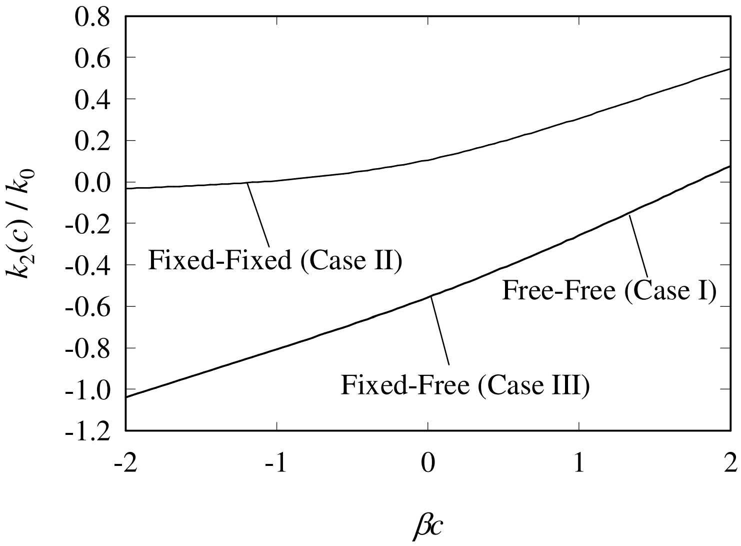

Figure 3.30 Normalized mode II SIF versus the nonhomogeneity parameter βc and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\delta_0^4 = 2$, $\kappa_0 = 2$, $\nu_0 = 0.3$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

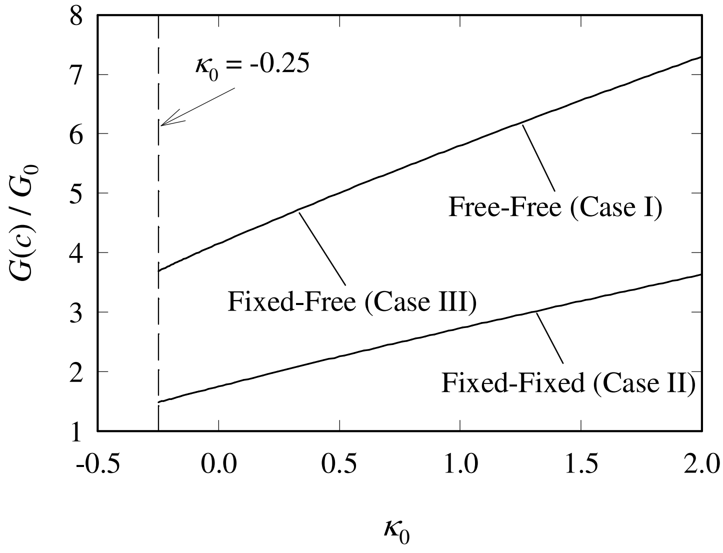

Figure 3.31 Normalized energy release rate versus the shear parameter κ_0 and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $\beta c = 2$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

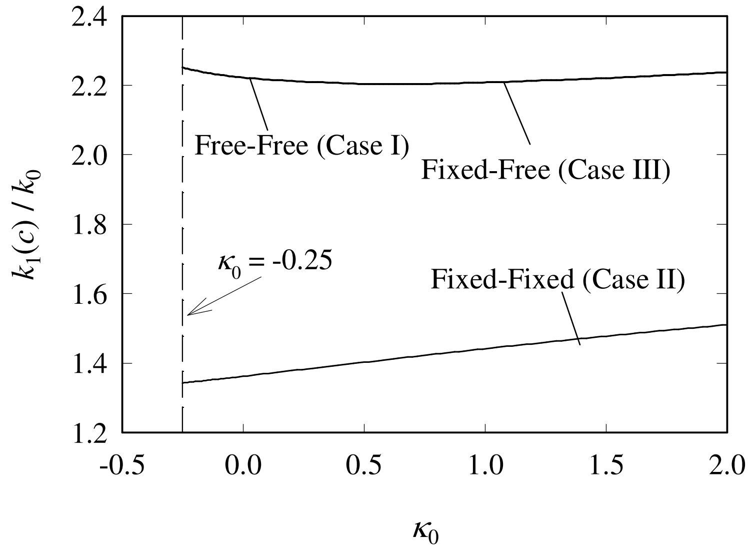

Figure 3.32 Normalized mode I SIF versus the shear parameter κ_0 and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $\beta c = 2$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

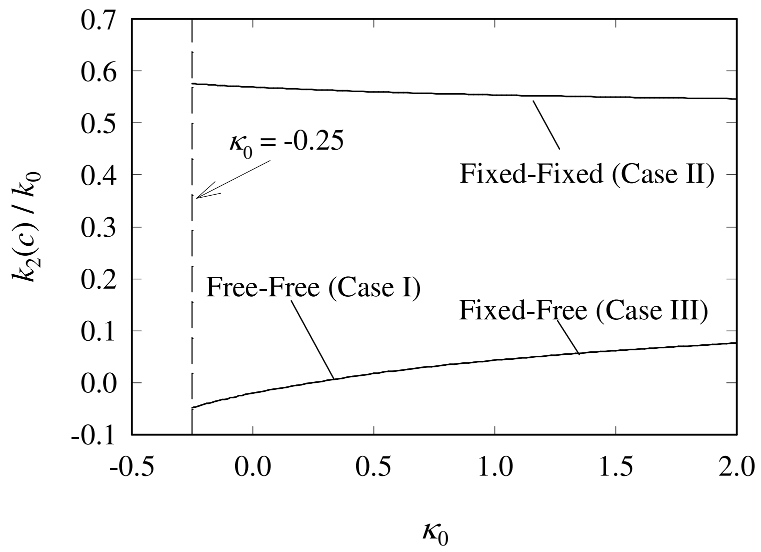

Figure 3.33 Normalized mode II SIF versus the shear parameter κ_0 and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\delta_0^4 = 2$, $\nu_0 = 0.3$, $\beta c = 2$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

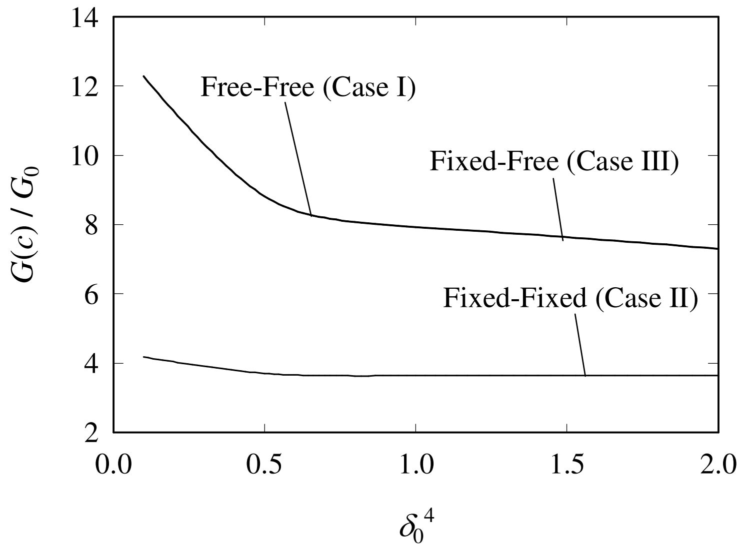

Figure 3.34 Normalized energy release rate versus the stiffness ratio δ_0^4 and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $\beta c = 2$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

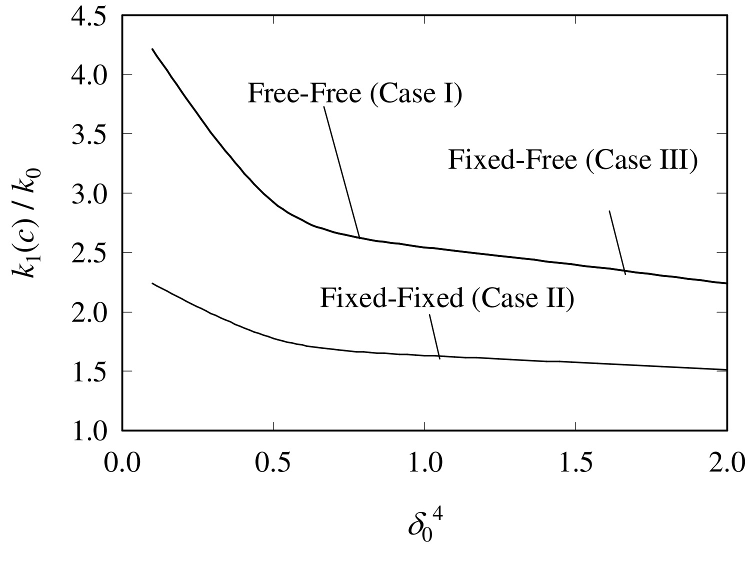

Figure 3.35 Normalized mode I SIF versus the stiffness ratio δ_0^4 and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $\beta c = 2$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

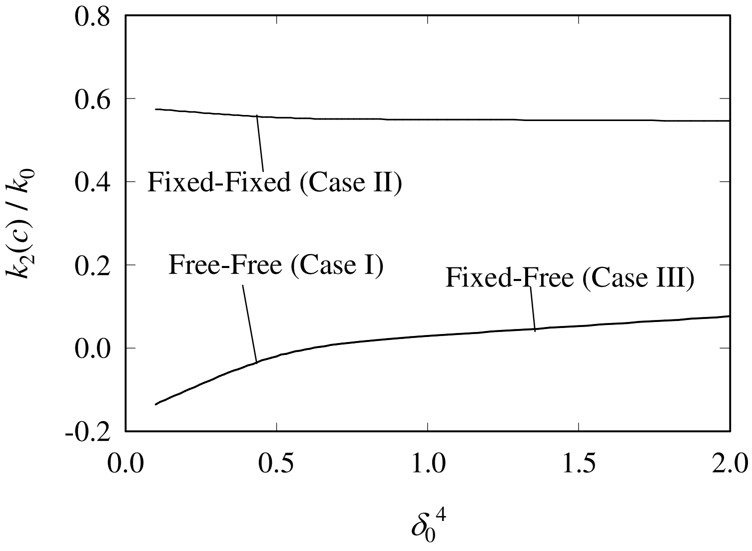

Figure 3.36 Normalized mode II SIF versus the stiffness ratio δ_0^4 and the boundary conditions for an embedded crack (see Figures 2.1, 2.2, 2.3) subjected to uniform tension, $\kappa_0 = 2$, $\nu_0 = 0.3$, $\beta c = 2$, $h_1/c = 0.5$, $h_2/c = 10$, $p(x_1) = \sigma_0$, $q(x_1) = 0$.

CHAPTER 4

DISCUSSION AND CONCLUSIONS

4.1 Discussion of The Results

The coupled effect of the shear parameter and the nonhomogeneity constant on the energy release rate is depicted in Figures 3.1-3.3. As can be seen in all three figures, the energy release rate is an increasing function of κ_0. In Figures 3.1, 3.3 it is seen that if $\beta c = 0$ energy release rate reaches the minimum value for any κ_0. But in Figure 3.2, as the nonhomogenity constant is increased from 0 to 2, the energy release rate increases. Figures 3.4-3.6 depict the effects of shear parameter and the nonhomogeneity constant on the mode I stress intensity factor. In Figures 3.4, 3.6, the mode I stress intensity factor is an increasing function of κ_0 for all values of βc, but in Figure 3.5 it is an increasing function of κ_0 only when βc is equal to 1 and 2. Due to the nonhomogeneity of the medium, the stress intensity factors exhibit mixed mode condition even though the loading is uniform tension, so in Figures 3.7-3.9 the influence of shear parameter on the mode II stress intensity factor is presented. From the Figures 3.4-3.9 it can be concluded that the mode II stress intensity factor is of secondary importance as compared to mode I under uniform normal stress. This is an expected result. It is also noted that in an orthotropic medium the sum $\kappa_0 + \nu_0$ should be a positive number in order to have a positive value for the ratio E_0/G_{120} (see for example equation (2.2d)).

In Figures 3.10-3.12 the energy release rate vs. the stiffness ratio are ploted at different values of nonhomogeneity parameter. In Figures 3.10, 3.12 energy release rate increases significantly as δ_0^4 approaches zero and for $\beta c = 0$ it is minimum.

The influence of the stiffness ratio on the mode I stress intensity factor is shown in Figures 3.13-3.15. In Figures 3.13, 3.15 it is seen that also as δ_0^4 approaches zero mode I stress intensity factor increases and for equal absolute values of nonhomogenity parameter the magnitude of mode I stress intensity factors are very close to each other. In Figure 3.14 for $\beta c = 1$ and 2, as δ_0^4 approaches zero mode I stress intensity factor again increases but for the other values of βc it decreases. Figures 3.16-3.18 give the effect of the stiffness ratio on the mode II stress intensity factors. In these figures as the nonhomogenity constant is increased the mode II stress intensity factor increases. In Figures 3.16, 3.18 as δ_0^4 approaches zero mode II stress intensity factor decreases.

The influence of the effective Poisson's ratio on the energy release rate and on the mode I stress intensity factor are given in Figures 3.19-3.21 and in Figures 3.22-3.24, respectively. In Figures 3.19, 3.21 and Figures 3.22, 3.24 as the absolute value of nonhomogeneity constant is increased the energy release rate and mode I stress intensity factor increase. It is also noted that for free boundaries energy release rate and mode I stress intensity factor are independent of the value of ν_0 if the crack is in a homogeneous orthotropic medium ($\beta c = 0$). Figures 3.25-3.27 show the effect of the effective Poisson's ratio on the mode II stress intensity factor. In these figures it is seen that as the value of nonhomogenity constant is increased the mode II stress intensity factor increases. In Figures 3.25, 3.27 for all values of βc and in Figure 3.26 except the negative value of βc, mode II stress intensity factor is almost constant while Poisson's ratio is changing. It can be concluded that the influence of the effective Poisson's ratio is not as much as the effects of the shear parameter and the stiffness ratio.

Finally the effect of boundary conditions on fracture mechanics parameters are examined. This has not been considered in the literature. In the results given in Figures 3.28–3.36 the crack is located closer to the top surface so the effect of bottom condition is negligible. Therefore the graphs of case I and case III coincide in all Figures 3.28-3.36. In these figures it can be seen that the case of free top surface has greater value of energy release rate and mode I stress intensity factor but less value of mode II stress intensity factor than the fixed one. The effect of the nonhomogenity parameter on energy release rate is shown in Figure 3.28. In Figure 3.29 the effect of the nonhomogenity parameter on mode I stress intensity factor is given. In Figures 3.28, 3.29, for case II energy release rate and mode I stress intensity factor are increasing functions of βc. In Figure 3.30 the influence of the nonhomogenity parameter on mode II stress intensity factor is given. As can be seen in all three cases, the mode II stress intensity factor is an increasing function of βc. The effect of the shear parameter on energy release rate, mode I stress intensity factor and mode II stress intensity factor are given in Figure 3.31, Figure 3.32 and Figure 3.33, respectively. In Figure 3.31 the energy release rate is an increasing function of κ_0 for all cases. In Figure 3.33 for case I and case III the mode II stress intensity factor is an increasing function of κ_0, for case II it is a decreasing function of κ_0. Figures 3.34-3.36 give the effect of stiffness ratio on energy release rate, mode I stress intensity factor and mode II stress intensity factor, respectively. In Figures 3.34, 3.35 it is seen that energy release rate and mode I stress intensity factor increases as δ_0^4 approach zero especially for case I and case III. In Figure 3.36 mode II stress intensity factor decreases as δ_0^4 approach zero for case I and case III. For case II mode II stress intensity factor is an decreasing function of δ_0^4.

4.2 Concluding Remarks

In this study, semi-analytical techniques are developed to examine embedded crack problems in orthotropic functionally graded material under mixed–mode loading. The problem is formulated using the averaged constants of plane orthotropic elasticity. Using transform techniques, the crack problem is reduced to a couple of singular integral equations which are solved numerically. Calculated results consists of modes I and II stress intensity factors and energy release rate at the crack tips. In this study same problem is examined for three different boundary conditions. The effects of the boundary conditions, material nonhomogenity and orthotropy on fracture mechanics parameters are determined.

The results provided in this study show that the energy release rate is an increasing function of the shear parameter for all cases and the mode I stress intensity factor is also an increasing function of the shear parameter for case I and case III. The mode II stress intensity factor is of secondary importance as compared to mode I stress intensity factor under uniform normal stress. Furthermore, the influence of the effective Poisson's ratio is not as pronounced as the effects of shear parameter and stiffness ratio. The energy release rate and mode I stress intensity factor for the free boundaries are calculated to be higher than those calculated for the fixed-fixed boundary condition. Hence, the probability for the crack to grow is higher when the upper surface is free.

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APPENDIX A

The Entries Of The Coefficient Matrix

The entries of the coefficient matrix given in (2.33) are given as follows

Case I (Free – Free)

$$a_{1j} = \left(\nu_0\, i\omega + N_j\, s_j\right)\exp(s_j \sqrt{\delta_0}\, h_1) \qquad (j=1,...,4) \qquad \text{(A1a)}$$

$$a_{2j} = \left(N_j\, i\omega + s_j\right)\exp(s_j \sqrt{\delta_0}\, h_1) \qquad (j=1,...,4) \qquad \text{(A1b)}$$

$$a_{3j} = \nu_0\, i\omega + N_j\, s_j \qquad (j=1,...,4) \qquad \text{(A1c)}$$

$$a_{3j} = -\nu_0\, i\omega - H_{(j-4)}\, r_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A1d)}$$

$$a_{4j} = N_j\, i\omega + s_j \qquad (j=1,...,4) \qquad \text{(A1e)}$$

$$a_{4j} = -H_{(j-4)}\, i\omega - r_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A1f)}$$

$$a_{5j} = i\omega N_j \qquad (j=1,...,4) \qquad \text{(A1g)}$$

$$a_{5j} = -i\omega H_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A1h)}$$

$$a_{6j} = i\omega \qquad (j=1,...,4) \qquad \text{(A1i)}$$

$$a_{6j} = -i\omega \qquad (j=5,...,8) \qquad \text{(A1j)}$$

$$a_{7j} = \left(\nu_0\, i\omega + H_{(j-4)}\, r_{(j-4)}\right)\exp(-r_{(j-4)}\sqrt{\delta_0}\, h_2) \qquad (j=5,...,8) \qquad \text{(A1k)}$$

$$a_{8j} = \left(H_{(j-4)}\, i\omega + r_{(j-4)}\right)\exp(-r_{(j-4)}\sqrt{\delta_0}\, h_2) \qquad (j=5,...,8) \qquad \text{(A1l)}$$

Case II (Fixed – Fixed)

$$a_{1j} = \exp(s_j\sqrt{\delta_0}\, h_1) \qquad (j=1,...,4) \qquad \text{(A2a)}$$

$$a_{2j} = N_j \exp(s_j\sqrt{\delta_0}\, h_1) \qquad (j=1,...,4) \qquad \text{(A2b)}$$

$$a_{3j} = \nu_0\, i\omega + N_j\, s_j \qquad (j=1,...,4) \qquad \text{(A2c)}$$

$$a_{3j} = -\nu_0\, i\omega - H_{(j-4)}\, r_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A2d)}$$

$$a_{4j} = N_j\, i\omega + s_j \qquad (j=1,...,4) \qquad \text{(A2e)}$$

$$a_{4j} = -H_{(j-4)}\, i\omega - r_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A2f)}$$

$$a_{5j} = i\omega N_j \qquad (j=1,...,4) \qquad \text{(A2g)}$$

$$a_{5j} = -i\omega H_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A2h)}$$

$$a_{6j} = i\omega \qquad (j=1,...,4) \qquad \text{(A2i)}$$

$$a_{6j} = -i\omega \qquad (j=5,...,8) \qquad \text{(A2j)}$$

$$a_{7j} = \exp(-r_{(j-4)}\sqrt{\delta_0}\,h_2) \qquad (j=5,...,8) \qquad \text{(A2k)}$$

$$a_{8j} = H_{(j-4)}\exp(-r_{(j-4)}\sqrt{\delta_0}\,h_2) \qquad (j=5,...,8) \qquad \text{(A2l)}$$

Case III (Fixed – Free)

$$a_{1j} = \left(\nu_0\, i\omega + N_j\, s_j\right)\exp(s_j\sqrt{\delta_0}\,h_1) \qquad (j=1,...,4) \qquad \text{(A3a)}$$

$$a_{2j} = \left(N_j\, i\omega + s_j\right)\exp(s_j\sqrt{\delta_0}\,h_1) \qquad (j=1,...,4) \qquad \text{(A3b)}$$

$$a_{3j} = \nu_0\, i\omega + N_j\, s_j \qquad (j=1,...,4) \qquad \text{(A3c)}$$

$$a_{3j} = -\nu_0\, i\omega - H_{(j-4)}\, r_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A3d)}$$

$$a_{4j} = N_j\, i\omega + s_j \qquad (j=1,...,4) \qquad \text{(A3e)}$$

$$a_{4j} = -H_{(j-4)}\, i\omega - r_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A3f)}$$

$$a_{5j} = i\omega N_j \qquad (j=1,...,4) \qquad \text{(A3g)}$$

$$a_{5j} = -i\omega H_{(j-4)} \qquad (j=5,...,8) \qquad \text{(A3h)}$$

$$a_{6j} = i\omega \qquad (j=1,...,4) \qquad \text{(A3i)}$$

$$a_{6j} = -i\omega \qquad (j=5,...,8) \qquad \text{(A3j)}$$

$$a_{7j} = \exp(-r_{(j-4)}\sqrt{\delta_0}\,h_2) \qquad (j=5,...,8) \qquad \text{(A3k)}$$

$$a_{8j} = H_{(j-4)}\exp(-r_{(j-4)}\sqrt{\delta_0}\,h_2) \qquad (j=5,...,8) \qquad \text{(A3l)}$$

APPENDIX B

Asymptotic Expansions

In order to determine the asymptotic terms it is sufficient to consider two half-planes as shown in Figure B.1.

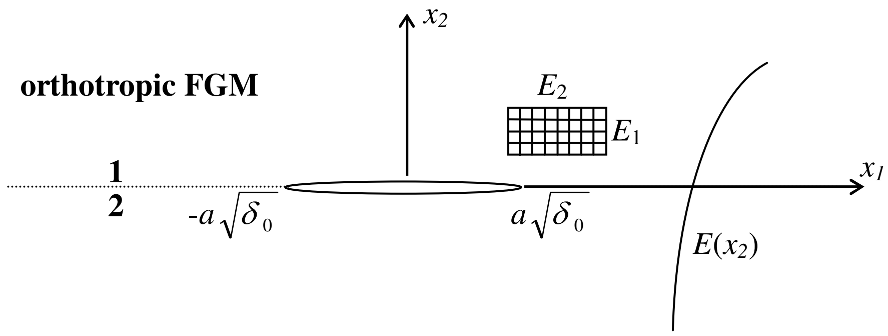

Figure B.1 A crack in a graded orthotropic infinite medium

The displacements for $y>0$ can be expressed as

$$u^{(1)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{2}M_j(\omega)\exp(s_j\,y+i\,\omega\,x)\,d\,\omega \qquad \text{(B1a)}$$

$$v^{(1)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{2}M_j(\omega)\,N_j(\omega)\exp(s_j\,y+i\,\omega\,x)\,d\,\omega \qquad \text{(B1b)}$$

where

$$s_1 = -(\gamma/2) - \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 + \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}}\ , \quad \Re(s_1) < 0 \tag{B2a}$$

$$s_2 = -(\gamma/2) - \sqrt{(\gamma/2)^2 + \omega^2 \kappa_0 - \omega\sqrt{\omega^2 \kappa_0^2 - \omega^2 - \gamma^2 \nu_0}}\ , \quad \Re(s_2) < 0 \tag{B2b}$$

$$N_j = \frac{-i\left(s_j^2\left(\nu_0^2 - 1\right) + 2\omega^2(\kappa_0 + \nu_0) - \gamma\, s_j\left(1 - \nu_0^2\right)\right)}{\omega\left(s_j\left(1 + \nu_0^2 + 2\kappa_0\, \nu_0\right) + \gamma\left(1 - \nu_0^2\right)\right)}\ , \quad (j = 1,\ 2) \tag{B3}$$

Substituting equations (B1) into equations (2.5) the general solutions for the stresses in medium 1 can also be determined as follows,

$$\sigma_{xx}^{(1)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{1 - \nu_0^2} \times$$

$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{2} M_j\left(i\omega + \nu_0\, N_j\, s_j\right)\exp(s_j\, y)\right)\exp(i\omega x)\, d\omega\right\} \tag{B4a}$$

$$\sigma_{yy}^{(1)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{1 - \nu_0^2} \times$$

$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{2} M_j\left(\nu_0\, i\omega + N_j\, s_j\right)\exp(s_j\, y)\right)\exp(i\omega x)\, d\omega\right\} \tag{B4b}$$

$$\sigma_{xy}^{(1)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{2\,(\kappa_0 + \nu_0)} \times$$

$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{2} M_j\left(N_j\, i\omega + s_j\right)\exp(s_j\, y)\right)\exp(i\omega x)\, d\omega\right\} \tag{B4c}$$

Similarly, the solution for medium 2, is written as

$$u^{(2)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{2}G_j(\omega)\,\exp(r_j\,y+i\,\omega\,x)\,d\,\omega \tag{B5a}$$

$$v^{(2)}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=1}^{2}G_j(\omega)\,H_j(\omega)\exp(r_j\,y+i\,\omega\,x)\,d\,\omega \tag{B5b}$$

where

$$r_1=-(\gamma/2)+\sqrt{(\gamma/2)^2+\omega^2\,\kappa_0+\omega\sqrt{\omega^2\,\kappa_0^2-\omega^2-\gamma^2\,\nu_0}}\,,\quad \Re(r_1)>0 \tag{B6a}$$

$$r_2=-(\gamma/2)+\sqrt{(\gamma/2)^2+\omega^2\,\kappa_0-\omega\sqrt{\omega^2\,\kappa_0^2-\omega^2-\gamma^2\,\nu_0}}\,,\quad \Re(r_2)>0 \tag{B6b}$$

$$H_j=\frac{-i\left(r_j^2\left(\nu_0^2-1\right)+2\,\omega^2\left(\kappa_0+\nu_0\right)-\gamma\,r_j\left(1-\nu_0^2\right)\right)}{\omega\left(r_j\left(1+\nu_0^2+2\kappa_0\,\nu_0\right)+\gamma\left(1-\nu_0^2\right)\right)}\,,\qquad (j=1,2) \tag{B7}$$

Substituting equations (B5) into equations (2.5) the general solutions for the stresses in medium 2 can also be determined as follows,

$$\sigma_{xx}^{(2)}(x,y)=\frac{E_0\exp(\gamma\,y)}{1-\nu_0^2}\times$$

$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{2}G_j\left(i\,\omega+\nu_0\,H_j\,r_j\right)\exp(r_j\,y)\right)\exp(i\,\omega\,x)\,d\,\omega\right\} \tag{B8a}$$

$$\sigma_{yy}^{(2)}(x,y)=\frac{E_0\exp(\gamma y)}{1-\nu_0^2}\times$$

$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{2}G_j\left(\nu_0\,i\omega+H_j\,r_j\right)\exp(r_j\,y)\right)\exp(i\omega x)\,d\omega\right\} \qquad \text{(B8b)}$$

$$\sigma_{xy}^{(2)}(x,y)=\frac{E_0\exp(\gamma y)}{2(\kappa_0+\nu_0)}\times$$

$$\times\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{2}G_j\left(H_j\,i\omega+r_j\right)\exp(r_j\,y)\right)\exp(i\omega x)\,d\omega\right\} \qquad \text{(B8c)}$$

In order to determine the unknown functions M_1, M_2, G_1, G_2 the boundary conditions given bellow must be used.

$$\sigma_{22}^{(1)}(x_1,0)=\sigma_{22}^{(2)}(x_1,0) \quad , \quad -\infty<x_1<\infty \qquad \text{(B9a)}$$

$$\sigma_{12}^{(1)}(x_1,0)=\sigma_{12}^{(2)}(x_1,0) \quad , \quad -\infty<x_1<\infty \qquad \text{(B9b)}$$

$$\frac{\partial}{\partial x_1}\left(u_2^{(1)}(x_1,0)-u_2^{(2)}(x_1,0)\right)=\begin{cases} f_1(x_1) & , \ |x_1|<a\sqrt{\delta_0} \\ 0 & , \ |x_1|>a\sqrt{\delta_0}\end{cases} \qquad \text{(B9c)}$$

$$\frac{\partial}{\partial x_1}\left(u_1^{(1)}(x_1,0)-u_1^{(2)}(x_1,0)\right)=\begin{cases} f_2(x_1) & , \ |x_1|<a\sqrt{\delta_0} \\ 0 & , \ |x_1|>a\sqrt{\delta_0}\end{cases} \qquad \text{(B9d)}$$

After expressing equations (B9) in the transformed coordinate system and taking Fourier transform of both sides of these equations, the unknown functions $M_j(\omega)$

and $G_j(\omega)$, (j = 1,..., 2) can be solved for using the following 4×4 complex system of linear equations,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} M_1(\omega) \\ M_2(\omega) \\ G_1(\omega) \\ G_2(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_1(\omega) \\ F_2(\omega) \end{bmatrix} \qquad \text{(B10)}$$

The entries of the coefficient matrix are given as

$$a_{1j} = v_0\, i\omega + N_j\, s_j \qquad (j=1,2) \qquad \text{(B11a)}$$

$$a_{1j} = -v_0\, i\omega - H_{(j-2)}\, r_{(j-2)} \qquad (j=3,4) \qquad \text{(B11b)}$$

$$a_{2j} = N_j\, i\omega + s_j \qquad (j=1,2) \qquad \text{(B11c)}$$

$$a_{2j} = -H_{(j-2)}\, i\omega - r_{(j-2)} \qquad (j=3,4) \qquad \text{(B11d)}$$

$$a_{3j} = i\omega N_j \qquad (j=1,2) \qquad \text{(B11e)}$$

$$a_{3j} = -i\omega H_{(j-2)} \qquad (j=3,4) \qquad \text{(B11f)}$$

$$a_{4j} = i\omega \qquad (j=1,2) \qquad \text{(B11g)}$$

$$a_{4j} = -i\omega \qquad (j=3,4) \qquad \text{(B11h)}$$

The solution can now be expressed in the following form.

$$M_j(\omega)=V_j(\omega)\ F_1(\omega) + W_j(\omega)\ F_2(\omega) \qquad (j=1,2) \qquad \text{(B12a)}$$

$$G_j(\omega)=Y_j(\omega)\ F_1(\omega) + Z_j(\omega)\ F_2(\omega) \qquad (j=1,2) \qquad \text{(B12b)}$$

where

$$F_1(\omega) = \int_{-a}^{a} \phi_1(t)\ \exp(-i\,\omega\, t)\ dt \qquad \text{(B13a)}$$

$$F_2(\omega) = \int_{-a}^{a} \phi_2(t)\ \exp(-i\,\omega\, t)\ dt \qquad \text{(B13b)}$$

and $\phi_1(t) = f_1\left(\sqrt{\delta_0}\, t\right)$ and $\phi_2(t) = \delta_0\, f_2\left(\sqrt{\delta_0}\, t\right)$. $V_j(\omega)$, $Y_j(\omega)$, $W_j(\omega)$ and $Z_j(\omega)$, $(j=1,2)$ are related the unknown functions and they can be obtained from

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} V_1(\omega) \\ V_2(\omega) \\ Y_1(\omega) \\ Y_2(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{(B14a)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} W_1(\omega) \\ W_2(\omega) \\ Z_1(\omega) \\ Z_2(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(B14b)}$$

where $a_{ij}\ (i=1,...,4),(j=1,...,4)$ are given in the equations (B11). Stresses are obtained in the following form

$$\sigma_{yy}^{(2)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{1-\nu_0^2} \left\{ \int_{-a}^{a} \sum_{j=1}^{2} k_{ij}(x,y,t)\, \phi_j(t)\, dt \right\} \quad , \quad (i=1) \tag{B15a}$$

$$\sigma_{xy}^{(2)}(x,y) = \frac{E_0 \exp(\gamma\, y)}{2(\kappa_0 + \nu_0)} \left\{ \int_{-a}^{a} \sum_{j=1}^{2} k_{ij}(x,y,t)\, \phi_j(t)\, dt \right\} , \quad (i=2) \tag{B15b}$$

where

$$k_{ij}(x,y,t) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{\infty} K_{ij}(\omega,y) \sin(\omega\,(x-t))\, d\omega \quad , \quad (i=j) \\ \\ \dfrac{1}{2\pi} \displaystyle\int_0^{\infty} K_{ij}(\omega,y) \cos(\omega\,(x-t))\, d\omega \quad , \quad (i \neq j) \end{cases} , \; (i,j=1,2) \tag{B16}$$

$$K_{11}(\omega,y) = 2i \left\{ \sum_{j=1}^{2} \left(i\,\omega\,\nu_0 + r_j H_j \right) Y_j(\omega) \exp(r_j\, y) \right\} \tag{B17a}$$

$$K_{12}(\omega,y) = 2 \left\{ \sum_{j=1}^{2} \left(i\,\omega\,\nu_0 + r_j H_j \right) Z_j(\omega) \exp(r_j\, y) \right\} \tag{B17b}$$

$$K_{21}(\omega,y) = 2 \left\{ \sum_{j=1}^{2} \left(i\,\omega\, H_j + r_j \right) Y_j(\omega) \exp(r_j\, y) \right\} \tag{B17c}$$

$$K_{22}(\omega,y) = 2i \left\{ \sum_{j=1}^{2} \left(i\,\omega\, H_j + r_j \right) Z_j(\omega) \exp(r_j\, y) \right\} \tag{B17d}$$

The integrands in (B16) are bounded and continuous for $\omega < \infty$ and integrable $\omega = 0$. The singular nature of the kernels k_{ij} can, therefore, be determined by examining the asymptotic behavior of K_{ij} as ω approaches to infinity. Asymptotic expansions can be expressed in the following form

$$K_{11}^{\infty}(\omega, y) = K_{111}^{\infty}(\omega)\exp(r_1\, y) + K_{112}^{\infty}(\omega)\exp(r_2\, y) \tag{B18a}$$

$$K_{12}^{\infty}(\omega, y) = K_{121}^{\infty}(\omega)\exp(r_1\, y) + K_{122}^{\infty}(\omega)\exp(r_2\, y) \tag{B18b}$$

$$K_{21}^{\infty}(\omega, y) = K_{211}^{\infty}(\omega)\exp(r_1\, y) + K_{212}^{\infty}(\omega)\exp(r_2\, y) \tag{B18c}$$

$$K_{22}^{\infty}(\omega, y) = K_{221}^{\infty}(\omega)\exp(r_1\, y) + K_{222}^{\infty}(\omega)\exp(r_2\, y) \tag{B18d}$$

Using the symbolic manipulator MAPLE, five - term asymptotic expansions of the integrands are obtained in the following form

$$K_{111}^{\infty}(\omega) = a_0 + \frac{a_1}{\omega} + \frac{a_2}{\omega^2} + \frac{a_3}{\omega^3} + \frac{a_4}{\omega^4} \tag{B19a}$$

$$K_{112}^{\infty}(\omega) = b_0 + \frac{b_1}{\omega} + \frac{b_2}{\omega^2} + \frac{b_3}{\omega^3} + \frac{b_4}{\omega^4} \tag{B19b}$$

$$K_{121}^{\infty}(\omega) = c_0 + \frac{c_1}{\omega} + \frac{c_2}{\omega^2} + \frac{c_3}{\omega^3} + \frac{c_4}{\omega^4} \tag{B19c}$$

$$K_{122}^{\infty}(\omega) = d_0 + \frac{d_1}{\omega} + \frac{d_2}{\omega^2} + \frac{d_3}{\omega^3} + \frac{d_4}{\omega^4} \tag{B19d}$$

$$K_{211}^{\infty}(\omega) = e_0 + \frac{e_1}{\omega} + \frac{e_2}{\omega^2} + \frac{e_3}{\omega^3} + \frac{e_4}{\omega^4} \tag{B19e}$$

$$K_{212}^{\infty}(\omega) = f_0 + \frac{f_1}{\omega} + \frac{f_2}{\omega^2} + \frac{f_3}{\omega^3} + \frac{f_4}{\omega^4} \quad \text{(B19f)}$$

$$K_{221}^{\infty}(\omega) = g_0 + \frac{g_1}{\omega} + \frac{g_2}{\omega^2} + \frac{g_3}{\omega^3} + \frac{g_4}{\omega^4} \quad \text{(B19g)}$$

$$K_{222}^{\infty}(\omega) = h_0 + \frac{h_1}{\omega} + \frac{h_2}{\omega^2} + \frac{h_3}{\omega^3} + \frac{h_4}{\omega^4} \quad \text{(B19h)}$$

where superscript (∞) stands for the asymptotic expansion as $\omega \to \infty$. The coefficients of the expansions a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j, $(j=0, 1,...,4)$ are lengthy functions of the constants $\gamma, \kappa_0, \nu_0, h_1, h_2$ and they are not reproduced here. Note that $c_0 + d_0 = e_0 + f_0 = 0$, i.e., the leading coefficients of the coupling terms are equal to zero.

APPENDIX C

Integral Formulae

Here, the expressions that are used to evaluate the integrals involving the asymptotic expansions of the integrands of the kernels are given. The integrals to be evaluated are in the following form

$$C_n = \int_A^{\infty} \frac{1}{\rho^n} \cos(\rho\, u)\, d\rho \qquad n = 1, 2, 3, \ldots, N. \qquad \text{(C1a)}$$

$$S_n = \int_A^{\infty} \frac{1}{\rho^n} \sin(\rho\, u)\, d\rho \qquad n = 1, 2, 3, \ldots, N. \qquad \text{(C1b)}$$

For $n = 1$, following results are obtained using maple

$$C_1 = -\mathrm{Ci}\left(A|u|\right), \qquad \text{(C2a)}$$

$$S_1 = \mathrm{sign}(u)\left(\frac{\pi}{2} - \mathrm{Si}\left(A|u|\right)\right) \qquad \text{(C2b)}$$

where Ci and Si are cosine and sine integrals, respectively, and they are defined by

$$\mathrm{Ci}(x) = \gamma_0 + \ln(x) + \int_0^{x} \frac{\cos(\alpha) - 1}{\alpha}\, d\alpha \qquad \text{(C3a)}$$

$$\text{Si}(x) = \int_0^x \frac{\sin(\alpha)}{\alpha}\, d\alpha \tag{C3b}$$

and the Euler constant is $\gamma_0 = 0.5772156649$. For $n>1$, Integrating (C1a) and (C1b) by parts the following general recursive relations can be obtained [16],

$$C_n = -\frac{1}{1-n}\frac{\cos(uA)}{A^{n-1}} + \frac{u}{1-n} S_{n-1} \ , \qquad n > 1 \tag{C4a}$$

$$S_n = -\frac{1}{1-n}\frac{\sin(uA)}{A^{n-1}} - \frac{u}{1-n} C_{n-1} \ , \qquad n > 1 \tag{C4b}$$

Following result is also used in the integration of asymptotic expressions,

$$\int_0^\infty \frac{\sin(\rho\, u)}{\rho}\, d\rho = \frac{\pi}{2}\,\text{sign}(u) \tag{C5}$$

APPENDIX D

Chebyshev Polynomials

Chebyshev polynomials are defined as

$$T_n(x) = \cos(n\cos^{-1} x) \quad , \qquad n = 0, 1, 2, \ldots \tag{D1}$$

Explicitly these polynomials are defined as follows

$$\begin{aligned} &T_0(r) = 1 \\ &T_1(r) = r \\ &T_2(r) = 2r^2 - 1 \\ &T_4(r) = 4r^3 - 3r \\ &. \\ &. \\ &. \end{aligned} \tag{D2}$$

Orthogonality property of Chebyshev polynomials are given as follows [17]

$$\frac{1}{\pi}\int_{-1}^{1} T_i(t)\, T_k(t)\, \frac{dt}{\sqrt{1-t^2}} = \begin{cases} 0 & , \quad i \neq k \\ 1 & , \quad i = k = 0 \\ 1/2 & , \quad i = k > 0 \end{cases} \tag{D3}$$

The singular integral equations are regularized by using the following properties given in [17].

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(t)}{\sqrt{1-t^2}}\,\frac{dt}{t-x}=U_{n-1}(x) \qquad |x|<1 \tag{D4}$$

$$\int_{-1}^{x}\frac{T_n(t)}{\sqrt{1-t^2}}\,dt=-\frac{1}{n}\sqrt{1-x^2}\;U_{n-1}(x) \qquad |x|<1 \tag{D5}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(t)\,\ln|t-x|}{\sqrt{1-t^2}}\,dt=-\frac{T_n(x)}{n} \qquad n\geq 1 \tag{D6}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(r)}{\sqrt{1-r^2}}\,\frac{dr}{s-r}=\frac{|s|}{s}\,\frac{\left(s-\left(|s|/s\right)\sqrt{s^2-1}\right)^n}{\sqrt{s^2-1}}\;, \qquad |s|>1 \tag{D7}$$

where T_n and U_n are the Chebyshev polynomials of first and second kind, respectively.

The roots of the Chebyshev polynomials are given as follow

$$s_j=\cos\left(\frac{\pi\,(2j-1)}{2\,N}\right) \qquad j=1,.....,N \tag{D8}$$

Consider a function $f_1(t)$

$$f_1(t)=G_1(t)\,(t-d)^{\alpha}\,(c-t)^{\beta}\,, \qquad d<t<c\,,\quad -1<\alpha<0\,,\quad -1<\beta<0 \tag{D9}$$

Cauchy Integral is defined as,

$$\frac{1}{\pi}\int_{d}^{c}\frac{f_1(t)}{t-y}\,dt \tag{D10}$$

Asymptotic behavior of the Cauchy integral near the end points can be expressed as follows

$$\frac{1}{\pi}\int_{d}^{c}\frac{f_1(t)}{t-y}\,dt \cong G_1(d)\,(d-y)^{\alpha}\,(c-d)^{\beta} - G_1(c)\,(c-d)^{\alpha}\,(y-c)^{\beta} \tag{D11}$$