ENERGY BOUNDS FOR SOME NONSTANDARD PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

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ENERGY BOUNDS FOR SOME NONSTANDARD PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

ENERGY BOUNDS FOR SOME NONSTANDARD PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

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This thesis is a survey of the studies of Ames, Payne and Schaefer about the partial differential equations with nonstandard auxiliary conditions; this is where the values of the solution are prescribed as a combination of initial time $t = 0$ and at a later time $t = T$.

The first chapter is introductory and contains some historical background of the problem, basic definitions and theorems. In Chapter 2 energy bounds and pointwise bounds for the solutions of the nonstandard hyperbolic problems have been investigated and by means of energy bound the uniqueness of solutions is examined. Similar discussions for the nonstandard parabolic problems have been presented in Chapter 3. Lastly in Chapter 4 a new continuous dependence result has been derived for the nonstandard problem.

Keywords: Nonstandard problem, energy bound, pointwise bound, continuous dependence, structural stability.

ÖZ

KISMİ TÜREVLİ DENKLEMLERDE BAZI STANDART OLMAYAN PROBLEMLER İÇİN ENERJİ SINIRLARI

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Bu tez, Ames, Payne ve Schaefer'ın standart olmayan koşullar içeren kısmi türevli denklemlerle ilgili çalışmalarının bir derlemesini içermektedir. Burada koşullar, $t = 0$ başlangıç zamanı ve daha sonraki bir $t = T$ zamanının bir kombinasyonu olarak verilmiştir.

İlk bölüm problemin tarihçesini, bazı tanım ve teoremleri içermektedir. İkinci bölümde standart olmayan hiperbolik problemlerin çözümleri için enerji sınırları ve noktasal sınırlar incelenmiş ve enerji sınırından yararlanarak problemin çözümünün tekliği araştırılmıştır. Standart olmayan parabolik problemler için benzer incelemeler üçüncü bölümde sunulmuştur. Son olarak dördüncü bölümde standart olmayan problemle ilgili yeni bir sürekli bağımlılık sonucu elde edilmiştir.

Anahtar Kelimeler: Standart olmayan problem, enerji sınırı, noktasal sınır, sürekli bağımlılık, yapısal kararlılık.

To my family

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In this study we shall be concerned with the nonstandard problems in partial differential equations.

Recently, Ames, Payne and Schaefer [1, 16] considered a class of initial–boundary value problems in which the initial data are not given at time $t = 0$, but instead as a linear combination of data at time $t = 0$ and at a later time $t = T$. We call this type of initial data as the *nonstandard auxiliary conditions.* As stated in [22], such problems are introduced, for example, to give estimates of solution behaviour in an improperly posed problem when the data is given at time $t = T$ and one wishes to compute the solution backward in time.

Payne and Schaefer [16] primarily study a nonstandard problem for differential operators that involve a second order time–derivative term and have application to hyperbolic partial differential equations. The nonstandard auxiliary conditions they employed are

$$\begin{aligned} \alpha u(x,0) + u(x,T) &= g(x), \\ \beta u_t(x,0) + u_t(x,T) &= h(x), \end{aligned} \tag{1.1}$$

where α and β are nonzero constants. These conditions have been also considered by Quintanilla and Straughan [22] to establish a priori solution bounds for a class of problem in thermoelasticity.

In [1], Ames, Payne and Schaefer investigate similar type of problems for differential operators that involve a first order time–derivative term and have application to parabolic equations. The non–standard condition they have considered,

$$\alpha u(x,0) + u(x,T) = g(x), \tag{1.2}$$

for α non–zero constant, has been also discussed in several papers (see [2], [5], [15]). Clark and Oppenheimer [5] studied to find a solution $u : [0,T] \to H$ for the problem

$$\begin{aligned} &u'(t) + Au(t) = 0, \quad 0 < t < T \\ &u(T) = f \end{aligned} \tag{1.3}$$

for some prescribed final value f in a Hilbert space H where A is a self–adjoint operator on H such that $-A$ generates a compact contraction semi–group on H. Such final value problems (1.3) are not well posed. One of the methods for constructing solutions to such problems is the quasireversibility method introduced by Lattes and Lions [11]. The idea in this method is to perturb the ill-posed problem into a well–posed one and then use the solution of this well–posed problem to construct an approximate solution for the ill–posed one. Clark and Oppenheimer approximate the final value problem by perturbing the final value condition. This yields the nonstandard problem,

$$\begin{aligned} &u'(t) + Au(t) = 0, \quad 0 < t < T \\ &\alpha u(0) + u(T) = f. \end{aligned} \tag{1.4}$$

They showed that the approximate problem (1.4) is well posed for each $\alpha > 0$ and that the solutions u_α convergence on $[0,T]$ iff the original problem has a classical solution.

Ames and Payne [2] determined a continuous dependence result for the backward heat equation under the nonstandard condition. They showed that the solution of

$$\begin{aligned} &u_t(x,t) + \Delta u(x,t) = 0 \quad \text{in } \Omega \times (0,T), \\ &u = 0, \quad \text{on } \partial\Omega \times [0,T], \\ &u(x,0) + \epsilon u(x,T) = f(x), \quad x \in \Omega \end{aligned} \tag{1.5}$$

depends continuously on the perturbation parameter ϵ. Showalter [23] calls this problem a "quasi-boundary-value" approximation to the initial value problem for the backward heat equation.

In recent years, Payne, Schaefer and Song [15] derived energy bounds for Dirichlet type boundary value problems for the Navier-Stokes and Stokes equations when data are given as a combination of the initial time $t = 0$ and a later time $t = T$. They considered the following problem for the velocity components $u_i(x, t), i = 1, 2, 3$, of an incompressible fluid,

$$\begin{aligned} u_{i,t} + u_j u_{i,j} - \Delta u_i &= -p_{,i} \quad \text{in } \Omega \times (0, T) \\ u_{j,j} &= 0 \quad \text{in } \Omega \times (0, T) \\ u_i(x, t) &= 0 \quad \text{on } \Omega \times [0, T] \\ \alpha_i u_i(x, 0) + u_i(x, T) &= g_i(x) \quad \text{in } \Omega \end{aligned} \tag{1.6}$$

where Ω is a bounded domain in R^3 with sufficiently smooth boundary $\partial\Omega$. Here, the α_i are nonzero constants, p is an unknown pressure term and g_i are the prescribed data. They determined the range of values for the parameters α_i for which a bound for the energy expression

$$E(t) = \int_\Omega u_j(x, t) u_j(x, t)\, dx$$

can be established.

In this thesis we will examine works of Ames, Payne and Schaefer [1, 16] in detail. In Chapter 2, we will study the hyperbolic partial differential equations with the prescribed nonstandard auxiliary conditions. Assuming a solution exists, we will derive energy bounds by means of differential inequalities and determine ranges of values of the parameters in nonstandard conditions for which it is possible to obtain such a bound. Afterwards, we will discuss the uniqueness of solution and formal representation of the unique solution.

In chapter 3, we will investigate the energy bounds and pointwise bounds for the nonstandard parabolic problems. We will derive a bound for the energy expression and study the consequences of such a bound. We also examine the pointwise bounds for the solution and its gradient with the help of the theorems established by Payne and Philippin. [20]

The last chapter is devoted to the structural stability. In 1999, Payne and Straughan [17] examined some continuous dependence results for the linearized equations for the flow of a Maxwell viscoelastic fluid. They studied how solutions to

$$\begin{gathered}\lambda\frac{\partial u_i}{\partial t^2}+\frac{\partial u_i}{\partial t}=\Delta u_i-\frac{\partial p}{\partial x_i},\quad (x,t)\in\Omega\times(t>0)\\ u_i=f_i(x),\quad x\in\partial\Omega \\ u_i(x,0)=g_i(x),\quad \frac{\partial u_i}{\partial t}(x,0)=h_i(x),\end{gathered}\tag{1.7}$$

behaves under changes in the relaxation parameter λ on a bounded domain Ω. In the last chapter we make use of the work of Payne and Straughan and try to show a new continuous dependence result for the nonstandard problem.

1.2 Preliminaries

Elementary Inequalities:

Following is a collection of elementary, but fundamental inequalities that will be used in this study. [7]

1. **Cauchy's Inequality with ϵ:**

$$a.b\le\epsilon a^2+\frac{b^2}{4\epsilon}\quad(a,b>0;\epsilon>0)$$

   In particular, for $\epsilon=1/2$ we have Cauchy's Inequality.

2. **Cauchy–Schwarz Inequality:**
   Let X be an inner product space and let $x,y\in X$. Then

$$|\langle x,y\rangle|\le\|x\|\,\|y\|$$

   with equality holding iff x and y are linearly dependent, where $\|.\|$ is the norm defined on X.

Basic Definitions and Theorems

Definition 1.2.1. *Let $A : D(A) \to H$ be a linear operator which is densely defined in a Hilbert space H. Then,*

- *A is called* symmetric *if*

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

for all $x, y \in D(A)$.

- *A is called* positive *if*

$$\langle Ax, x\rangle \geq 0$$

for all $x, y \in D(A)$. If $\langle Ax, x\rangle > 0$ for all $x \neq 0$, then A is called positive–definite.

Definition 1.2.2.

- *An orthonormal system $\{\phi_k\}$ in a Hilbert space H is* complete *if there is no nonzero element $x \in H$ which is orthogonal to every element of $\{\phi_k\}$.* (In other words, a complete orthonormal system cannot be extended to a larger orthonormal system by adding new elements. [12])

- *A complete orthonormal system is called an* orthonormal basis *for a Hilbert space.*

Theorem 1.2.1. *Let $\{\phi_k\}$ be a complete orthonormal set in a Hilbert space H. Then every element x of H can be written as,*

$$x = \sum_{k=1}^{\infty} \langle x, \phi_k\rangle \, \phi_k$$

where $\langle x, \phi_k\rangle$ are the Fourier coefficients of x.

Theorem 1.2.2. *(Divergence Theorem) Let $\Omega \subset R^N$ be a bounded domain with C^1 boundary surface $\partial\Omega$. Let n be the unit outward normal vector on $\partial\Omega$ and F be a C^1 vector field on $\bar{\Omega}$. Then ,*

$$\int_\Omega \nabla \cdot F \, dV = \int_{\partial\Omega} F \cdot n \, dS$$

If we define F as the gradient of scalar field ϕ , we can substitute $\nabla\phi$ for F in the above formula to give

$$\int_\Omega \nabla^2\phi \, dV = \int_{\partial\Omega} \nabla\phi \cdot n \, dS$$

Theorem 1.2.3. *(Green's first Identity) [10] The Laplace operator acting on a function $u(x) = u(x_1, \cdots, x_N)$ of class C^2 in a region Ω is defined by*

$$\Delta = \sum_{k=1}^{N} D_k^2.$$

For $u, v \in C^2(\bar{\Omega})$, we have

$$\int_\Omega v \, \Delta u = -\int_\Omega \sum_i v_{x_i} u_{x_i} \, dx + \int_{\partial\Omega} v \, \frac{du}{dn} \, dS$$

where $\frac{du}{dn}$ indicates differentiation in the direction of the exterior normal to $\partial\Omega$.

Theorem 1.2.4. *Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$. The integral representation of the solution of initial–boundary value problem*

$$\begin{aligned} &u_t = \Delta u + Q(x,t), \quad x \in \Omega, t > 0, \\ &u(x,0) = f(x), \quad x \in \Omega, \\ &u(x,t) = 0, \quad x \in \partial\Omega \end{aligned} \tag{1.8}$$

can be written in the form,

$$u(x,t) = \int_\Omega f(x_0) G(x,t;x_0,0) \, dx_0 + \int_\Omega \int_0^t Q(x_0,t_0) G(x,t;x_0,t_0) \, dt_o \, dx_0$$

where $G(x,t;x_0,t_0)$ is the Green's function for the heat operator.

The Green's function at $t = 0$, $G(x,t;x_0,0)$, expresses the influence of the initial temperature at x_0 on the temperature at position x and time t. Moreover, $G(x,t;x_0,t_0)$ shows the influence on the temperature at the position x and time t of the forcing term $Q(x_0,t_0)$ at position x_0 and time t_0. [9]

Theorem 1.2.5. *[26] Let A be a positive operator on a Hilbert space H. There exists a unique positive square root $A^{1/2}$ of A; that is, there exists a unique positive operator B on H such that $B^2 = A$.*

CHAPTER 2

NONSTANDARD HYPERBOLIC PROBLEMS

2.1 Introduction

In this chapter we will consider problems of the form

$$\frac{d^2u}{dt^2} + Au = F, \quad \alpha u(0) + u(T) = g, \quad \beta\frac{du}{dt}(0) + \frac{du}{dt}(T) = h$$

for $t \in (0, T)$, where A is a time independent, positive definite symmetric operator from a dense linear subspace D of a real Hilbert space H into H; F, g and h are sufficiently regular prescribed functions and α and β are nonzero constants.

Throughout this chapter, we assume A is a differential operator acting on functions which satisfy appropriate homogeneous Dirichlet boundary conditions. In Section 2.2, by means of differential inequalities, we will establish energy bounds for inhomogeneous differential equation where the inhomogeneous term is time dependent and time independent, respectively. We restrict our attention for some special values of α and β, and in Section 2.4 we will show that the unique representation of the solution for homogeneous equation can be found in this restricted interval when A has a complete eigenspace. In Section 2.5 we will obtain pointwise bounds for the special case of the wave equation in $N \leq 3$ space dimensions. In the last section we use the technique developed in previous sections for finding energy bounds for the damped equation and for the non linear Kirchoff string.

2.2 Inhomogeneous Equation

2.2.1 Inhomogeneous Equation with time–dependent source term:

Assume that Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$ and let x=$(x_1, \cdots, x_N)$. We consider the nonstandard problem

$$u_{tt} + Au = F(x,t), \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\alpha u(x,0) + u(x,T) = g(x), \quad \text{in } \Omega, \tag{2.2}$$

$$\beta u_t(x,0) + u_t(x,T) = h(x), \quad \text{in } \Omega, \tag{2.3}$$

where A is a linear, time independent, symmetric, positive–definite differential operator acting on sufficiently differentiable functions that satisfy homogeneous Dirichlet boundary conditions. F, g, h are sufficiently regular prescribed functions, α and β are nonzero constants and the subscript denotes the partial differentiation with respect to t. We will use L_2 inner product and norm notation in the form

$$\langle g,h\rangle = \int_\Omega g(x)h(x)\,dx, \qquad \langle u(t),v(t)\rangle = \int_\Omega u(x,t)v(x,t)\,dx,$$
$$\|u(t)\|^2 = \langle u(t),u(t)\rangle .$$

Theorem 2.2.1. *Let u be a solution of (2.1)–(2.3) and α and β are nonzero constants satisfying either $|\alpha| > 1$, $|\beta| > 1$ or $|\alpha| < 1$, $|\beta| < 1$. Then the energy function given by,*

$$E(t) = \|u_t(t)\|^2 + \langle Au(t), u(t)\rangle \tag{2.4}$$

satisfies an a priori bound of the form,

$$E(t) \leq C_1 \int_0^T \|F(\eta)\|^2 \, d\eta + C_2 \int_0^T e^{\gamma\eta} \int_\eta^T \|F(\xi)\|^2 \, d\xi \, d\eta + C_3 \langle Ag, g\rangle + C_4 \|h\|^2 \tag{2.5}$$

for computable constants C_1, C_2, C_3 and C_4 depending on T and a constant γ.

Proof: We shall discuss the cases $|\alpha| > 1$, $|\beta| > 1$ and $|\alpha| < 1$, $|\beta| < 1$ seperately. (The reason for not considering the other choices of α and β will be explained in Section 3)

Case 1: $|\alpha| > 1$, $|\beta| > 1$

We begin with the homogeneous equation

$$u_{tt} + Au = 0; \tag{2.6}$$

multiply (2.6) by $2u_t$ and integrate over $\Omega \times (0, t)$, $0 \le t \le T$, to have

$$\int_0^t \int_\Omega (2u_\eta u_{\eta\eta} + 2u_\eta \, A \, u) \; dx \; d\eta = 0. \tag{2.7}$$

Since A is a symmetric operator, we can write

$$\int_\Omega 2u_\eta Au \; d\eta = \int_\Omega (u_\eta Au + uAu_\eta) \; d\eta.$$

Thus from (2.7) we get,

$$\int_0^t \int_\Omega (u_\eta^2 + uAu)_\eta \, dx \, d\eta = 0. \tag{2.8}$$

Using the energy function given by (2.4) we can rewrite (2.8) as

$$\int_0^t E'(\eta) \, d\eta = 0$$

from which we deduce the conservation of energy principle

$$E(t) = E(0), \qquad 0 \le t \le T. \tag{2.9}$$

Let us consider the inhomogeneous equation (2.1). Again multiplying (2.1) by $2u_t$ and integrating over $\Omega \times (0, t)$, we get

$$\int_0^t \int_\Omega (2u_\eta u_{\eta\eta} + 2u_\eta Au) \, dx \, d\eta = 2 \int_0^t \int_\Omega u_\eta F(x, \eta) \, dx \, d\eta.$$

From the above discussion the left hand side of the above equation is $E(t)-E(0)$. Thus, we have

$$E(t) - E(0) = 2\int_0^t \langle F, u_\eta \rangle \; d\eta$$

i.e.,

$$E(t) = 2\int_0^t \langle F, u_\eta \rangle \; d\eta + E(0), \qquad 0 \le t \le T. \tag{2.10}$$

Now using Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, respectively, we get

$$\begin{aligned} E(t) &\le 2\int_0^t \|F\| \, \|u_\eta\| \; d\eta + E(0) \\ &\le 2\int_0^t (\epsilon \, \|F(\eta)\|^2 + \frac{1}{4\epsilon} \, \|u_\eta(\eta)\|^2) \, d\eta + E(0) \\ &= \frac{1}{\gamma}\int_0^t \|F(\eta)\|^2 \; d\eta + \gamma \int_0^t \|u_\eta(\eta)\|^2 \; d\eta + E(0) \end{aligned} \tag{2.11}$$

where $\gamma = \dfrac{1}{2\epsilon} > 0$. From (2.4) we can write

$$\|u_t(t)\|^2 \le E(t). \tag{2.12}$$

Thus, substitution of (2.12) into (2.11) yields,

$$E(t) \le \; \frac{1}{\gamma}\int_0^t \|F(\eta)\|^2 \; d\eta + \gamma \int_0^t E(\eta) \, d\eta + E(0). \tag{2.13}$$

Now setting

$$P(t) = \int_0^t E(\eta) \, d\eta$$

and rewriting (2.13) in terms of $P(t)$ gives us

$$P'(t) \le \; \frac{1}{\gamma} S^2(t) + \gamma P(t) + E(0), \quad S^2(t) = \int_0^t \|F(\eta)\|^2 \; d\eta. \tag{2.14}$$

We now solve (2.14) for $P(t)$. Indeed, this is a linear ordinary differential inequality with integrating factor $e^{-\gamma t}$. Solving this inequality over the interval $[0, t]$ we obtain

$$P(t) \leq \frac{1}{\gamma} \int_0^t e^{\gamma(t-\eta)}\, S^2(\eta)\, d\eta + \frac{E(0)}{\gamma}(e^{\gamma t} - 1), \tag{2.15}$$

and hence inserting (2.15) into (2.14) we get

$$E(t) \leq \frac{1}{\gamma} S^2(t) + \int_0^t e^{\gamma(t-\eta)}\, S^2(\eta)\, d\eta + E(0) e^{\gamma t}. \tag{2.16}$$

Now we have to find a bound for $E(0)$ in terms of the data. To this end, we evaluate (2.16) at $t = T$ and then use the auxiliary conditions (2.2)–(2.3) to write the terms of $E(T)$ in terms of $u(0)$ and $u_t(0)$.
That is, we substitute

$$\begin{aligned} E(T) &= \|u_t(T)\|^2 + \langle Au(T), u(T)\rangle \\ &= \|h - \beta u_t(0)\|^2 + \langle A\left(g - \alpha u(0)\right), g - \alpha u(0)\rangle \end{aligned}$$

so that we have

$$\begin{aligned} &\|h - \beta u_t(0)\|^2 + \langle A\left(g - \alpha u(0)\right), g - \alpha u(0)\rangle \\ &\qquad\qquad \leq \frac{1}{\gamma} S^2(T) + \int_0^T e^{\gamma(T-\eta)}\, S^2(\eta)\, d\eta + E(0) e^{\gamma T}. \end{aligned}$$

Writing the terms explicitly and arranging them, we get

$$\begin{aligned} &\|h\|^2 + \beta^2 \|u_t(0)\|^2 + \langle Ag, g\rangle + \alpha^2 \langle Au(0), u(0)\rangle \\ &\qquad \leq 2\beta \langle h, u_t(0)\rangle + 2\alpha \langle Ag, u(0)\rangle + \frac{1}{\gamma} S^2(T) + \int_0^T e^{\gamma(T-\eta)} S^2(\eta)\, d\eta \\ &\qquad\quad + (\|u_t(0)\|^2 + \big(\langle Au(0), u(0)\rangle\big) e^{\gamma T} \\ &\qquad \leq 2\,|\,\beta|\,|\langle h, u_t(0)\rangle| + 2\,|\,\alpha|\,|\langle Ag, u(0)\rangle| + \frac{1}{\gamma} S^2(T) \\ &\qquad\quad + \int_0^T e^{\gamma(T-\eta)} S^2(\eta)\, d\eta + \big(\|u_t(0)\|^2 + \langle Au(0), u(0)\rangle\big) e^{\gamma T}. \end{aligned}$$

Since A is symmetric positive definite operator, from Theorem 1.2.5 we can write

$$\langle Ag, u(0)\rangle = \left\langle A^{1/2}g, A^{1/2}u(0)\right\rangle.$$

Substituting this property into the above inequality and then using Cauchy–Schwarz Inequality we obtain,

$$\begin{aligned}
&\|h\|^2 + \beta^2 \|u_t(0)\|^2 + \langle Ag, g\rangle + \alpha^2 \langle Au(0), u(0)\rangle \\
&\qquad \le 2\,|\beta|\,\|h\|\,\|u_t(0)\| + 2\,|\alpha|\,\left\|A^{1/2}g\right\|\left\|A^{1/2}u(0)\right\| + \frac{1}{\gamma}S^2(T) \\
&\qquad\qquad + \int_0^T e^{\gamma(T-\eta)}\, S^2(\eta)\, d\eta + (\|u_t(0)\|^2 + \langle Au(0), u(0)\rangle)e^{\gamma T}.
\end{aligned}$$

We now apply Cauchy's Inequality with ϵ to have,

$$\begin{aligned}
&\|h\|^2 + \beta^2 \|u_t(0)\|^2 + \langle Ag, g\rangle + \alpha^2 \langle Au(0), u(0)\rangle \\
&\qquad \le 2\,|\beta|\,(\epsilon\,\|h\|^2 + \frac{1}{4\epsilon}\,\|u_t(0)\|^2) + 2\,|\alpha|\,(\bar{\epsilon}\left\|A^{1/2}g\right\|^2 + \frac{1}{4\bar{\epsilon}}\left\|A^{1/2}u(0)\right\|^2) \\
&\qquad\qquad + \frac{1}{\gamma}S^2(T) + \int_0^T e^{\gamma(T-\eta)}\, S^2(\eta)\, d\eta + (\|u_t(0)\|^2 + \langle Au(0), u(0)\rangle)e^{\gamma T}.
\end{aligned}$$

Since

$$\left\|A^{1/2}u(0)\right\|^2 = \left\langle A^{1/2}u(0), A^{1/2}u(0)\right\rangle = \langle Au(0), u(0)\rangle$$

and

$$\left\|A^{1/2}g\right\|^2 = \left\langle A^{1/2}g, A^{1/2}g\right\rangle = \langle Ag, g\rangle,$$

rewriting the above inequality and arranging terms we get

$$\begin{aligned}
&\left(\beta^2 - e^{\gamma T} - |\beta|\,\sigma_1\right)\|u_t(0)\|^2 + \left(\alpha^2 - e^{\gamma T} - |\alpha|\,\sigma_2\right)\langle Au(0), u(0)\rangle \le \\
&\left(\frac{|\alpha|}{\sigma_2} - 1\right)\langle Ag, g\rangle + \left(\frac{|\beta|}{\sigma_1} - 1\right)\|h\|^2 + \frac{1}{\gamma}S^2(T) + \int_0^T e^{\gamma(T-\eta)}\, S^2(\eta)\, d\eta, \quad (2.17)
\end{aligned}$$

where $\sigma_1 = \dfrac{1}{2\epsilon} > 0$ and $\sigma_2 = \dfrac{1}{2\bar{\epsilon}} > 0$. Thus if $|\alpha| > 1, |\beta| > 1$, for

$$0 < \sigma_1 < \frac{\beta^2 - 1}{|\beta|} \quad , \quad 0 < \sigma_2 < \frac{\alpha^2 - 1}{|\alpha|}$$

we can choose γ,

$$0 < \gamma < \frac{1}{T} \ln \left[\min \left\{ \alpha^2 - |\alpha| \, \sigma_2 \ \ , \ \ \beta^2 - |\beta| \, \sigma_1 \right\} \right] \tag{2.18}$$

so that

$$\beta^2 - e^{\gamma T} - |\beta| \, \sigma_1 > 0 \quad , \quad \alpha^2 - e^{\gamma T} - |\alpha| \, \sigma_2 > 0$$

and

$$\left(\frac{|\alpha|}{\sigma_2} - 1 \right) > 0 \quad , \quad \left(\frac{|\beta|}{\sigma_1} - 1 \right) > 0.$$

Letting

$$\min \left\{ \beta^2 - e^{\gamma T} - |\beta| \, \sigma_1 \ \ , \ \ \alpha^2 - e^{\gamma T} - |\alpha| \, \sigma_2 \right\} = \mu \ \ \text{(constant)},$$

from (2.17) we can write

$$\mu \left(\|u_t(0)\|^2 + \langle Au(0), u(0) \rangle \right) \leq \left(\frac{|\, \alpha \,|}{\sigma_2} - 1 \right) \langle Ag, g \rangle + \left(\frac{|\, \beta \,|}{\sigma_1} - 1 \right) \|h\|^2 + \frac{1}{\gamma} S^2(T) + \int_0^T e^{\gamma (T - \eta)} \, S^2(\eta) \, d\eta$$

from which we deduce

$$E(0) \leq K_1 \, \langle Ag, g \rangle + K_2 \, \|h\|^2 + K_3 S^2(T) + K_4 \int_0^T e^{\gamma (T - \eta)} \, S^2(\eta) \, d\eta \tag{2.19}$$

for computable constants K_1, K_2, K_3 and K_4. Hence a bound for $E(0)$ is obtained in terms of data.

Finally, substituting (2.19) into (2.16) we obtain the energy bound

$$E(t) \leq K_1 e^{\gamma T} \langle Ag, g \rangle \; + \; K_2 e^{\gamma T} \|h\|^2 \; + \; \left(\frac{1}{\gamma} \; + \; K_3 e^{\gamma T} \right) S^2(T)$$
$$+ \left(1 + K_4 e^{\gamma T}\right) \int_0^T e^{\gamma(T-\eta)} \, S^2(\eta) \, d\eta, \quad 0 \leq t \leq T \tag{2.20}$$

when $|\alpha| > 1, |\beta| > 1$.

Case 2: $|\boldsymbol{\alpha}| < \mathbf{1}$, $|\boldsymbol{\beta}| < \mathbf{1}$

In the present case we again begin by multiplying (2.1) by $2u_t$ and then integrating over $\Omega \times (t, T)$. As done in the previous case, one can easily obtain that this calculation yields,

$$E(t) = E(T) - 2 \int_t^T \langle F, u_\eta \rangle \; d\eta, \quad 0 \leq t \leq T. \tag{2.21}$$

Now using Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, respectively, we have

$$\begin{aligned} E(t) &\leq E(T) + 2 \int_t^T \|F\| \; \|u_\eta\| \, d\eta \\ &\leq E(T) + \frac{1}{\tilde{\gamma}} \int_t^T \|F(\eta)\|^2 \; d\eta + \tilde{\gamma} \int_t^T \|u_\eta\|^2 \; d\eta, \end{aligned}$$

for $\tilde{\gamma} > 0$. Substituting (2.12) into the above inequality we obtain,

$$E(t) \leq E(T) + \frac{1}{\tilde{\gamma}} \int_t^T \|F(\eta)\|^2 \; d\eta + \tilde{\gamma} \int_t^T E(\eta) \; d\eta. \tag{2.22}$$

We set

$$\tilde{P}(t) = \int_t^T E(\eta) \; d\eta$$

and rewrite (2.22) as,

$$-\tilde{P}'(t) \leq E(T) + \frac{1}{\tilde{\gamma}} \tilde{S}^2(t) + \tilde{\gamma} \tilde{P}(t), \quad \tilde{S}^2(t) = \int_t^T \|F(\eta)\|^2 \; d\eta. \tag{2.23}$$

Now solving (2.23) for $\tilde{P}(t)$ over the interval (t, T), we get

$$\tilde{P}(t) \leq \frac{E(T)}{\tilde{\gamma}} \left[e^{\tilde{\gamma}(T-t)} - 1\right] + \frac{1}{\tilde{\gamma}} \int_t^T e^{\tilde{\gamma}(\eta-t)} \tilde{S}^2(\eta)\, d\eta. \tag{2.24}$$

Substituting (2.24) into (2.23) and writing $-\tilde{P}'(t) = E(t)$ yields,

$$E(t) \leq \frac{1}{\tilde{\gamma}} \tilde{S}^2(t) + E(T) e^{\tilde{\gamma}(T-t)} + \int_t^T e^{\tilde{\gamma}(\eta-t)} \tilde{S}^2(\eta)\, d\eta, \quad 0 \leq t \leq T. \tag{2.25}$$

In this case we have to find a bound for $E(T)$ in terms of data. Similar to the first case, we evaluate (2.25) at $t = 0$ and then use the auxiliary conditions (2.2)–(2.3) to write the terms of $E(0)$ in terms of $u(T)$ and $u_t(T)$. That is, we substitute

$$\begin{aligned} E(0) &= \|u_t(0)\|^2 + \langle Au(0), u(0) \rangle \\ &= \frac{1}{\beta^2} \|h - u_t(T)\|^2 + \frac{1}{\alpha^2} \langle A(g - u(T)), g - u(T) \rangle \end{aligned}$$

to get,

$$\begin{aligned} &\frac{1}{\beta^2} \|h - u_t(T)\|^2 + \frac{1}{\alpha^2} \langle A(g - u(T)), g - u(T) \rangle \\ &\qquad\qquad \leq \frac{1}{\tilde{\gamma}} \tilde{S}^2(0) + E(T) e^{\tilde{\gamma} T} + \int_0^T e^{\tilde{\gamma}\eta} \tilde{S}^2(\eta)\, d\eta. \end{aligned}$$

Writing the terms explicitly we have,

$$\begin{aligned} &\frac{1}{\beta^2} \|u_t(T)\|^2 + \frac{1}{\alpha^2} \langle Au(T), u(T) \rangle + \frac{1}{\beta^2} \|h\|^2 + \frac{1}{\alpha^2} \langle Ag, g \rangle \\ &\quad \leq \frac{1}{\tilde{\gamma}} \tilde{S}^2(0) + \left(\|u_t(T)\|^2 + \langle Au(T), u(T) \rangle \right) e^{\tilde{\gamma} T} + \int_0^T e^{\tilde{\gamma}\eta} \tilde{S}^2(\eta)\, d\eta \\ &\qquad + \frac{2}{\beta^2} \langle h, u_t(T) \rangle + \frac{2}{\alpha^2} \langle Ag, u(T) \rangle . \end{aligned}$$

After applying Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, respectively, and then arranging terms, we obtain

$$(\frac{1}{\beta^2} - e^{\tilde{\gamma}T} - \frac{\sigma_3}{\beta^2}) \|u_t(T)\|^2 + (\frac{1}{\alpha^2} - e^{\tilde{\gamma}T} - \frac{\sigma_4}{\alpha^2}) \langle Au(T), u(T)\rangle$$
$$\leq \frac{1}{\tilde{\gamma}}\tilde{S}^2(0) + \int_0^T e^{\tilde{\gamma}\eta}\tilde{S}^2(\eta)\, d\eta + \frac{1}{\beta^2}\left(\frac{1}{\sigma_3} - 1\right)\|h\|^2 + \frac{1}{\alpha^2}\left(\frac{1}{\sigma_4} - 1\right)\langle Ag, g\rangle \tag{2.26}$$

for $\sigma_3, \sigma_4 > 0$. Thus if $|\alpha| < 1, |\beta| < 1$ for,

$$\sigma_3 < 1 - \beta^2 \quad , \quad \sigma_4 < 1 - \alpha^2$$

we choose $\tilde{\gamma}$

$$0 < \tilde{\gamma} < \frac{1}{T}\ln\left[\min\left\{\frac{1-\sigma_3}{\beta^2}\ ,\ \frac{1-\sigma_4}{\alpha^2}\right\}\right], \tag{2.27}$$

so that,

$$\frac{1}{\beta^2} - e^{\tilde{\gamma}T} - \frac{\sigma_3}{\beta^2} > 0 \quad , \quad \frac{1}{\alpha^2} - e^{\tilde{\gamma}T} - \frac{\sigma_4}{\alpha^2} > 0$$

and

$$\frac{1}{\beta^2}\left(\frac{1}{\sigma_3} - 1\right) > 0 \quad , \quad \frac{1}{\alpha^2}\left(\frac{1}{\sigma_4} - 1\right) > 0.$$

Hence, proceeding as in the first case, from (2.26) we can write,

$$E(T) \leq K_5\tilde{S}^2(0) + K_6\int_0^T e^{\tilde{\gamma}\eta}\tilde{S}^2(\eta)\, d\eta + K_7\|h\|^2 + K_8\langle Ag, g\rangle \tag{2.28}$$

for computable constants K_5, K_6, K_7 and K_8. Substituting this inequality into (2.25), it follows that,

$$E(t) \leq \left(\frac{1}{\tilde{\gamma}} + K_5e^{\tilde{\gamma}T}\right)\tilde{S}^2(0) + \left(1 + K_6e^{\tilde{\gamma}}(T)\right)\int_0^T e^{\tilde{\gamma}\eta}\tilde{S}^2(\eta)\, d\eta$$
$$+ K_7e^{\tilde{\gamma}T}\|h\|^2 + K_8e^{\tilde{\gamma}T}\langle Ag, g\rangle, \quad 0 \leq t \leq T, \tag{2.29}$$

when $|\alpha| < 1, |\beta| < 1$. Finally, writing $\tilde{S}^2(t)$ explicitly gives the bound (2.5), as desired.

Thus we see that the constants C_1, C_2, C_3 and C_4 in (2.5) depend on T and

γ that satisfies either (2.18) or (2.27) for $|\alpha| > 1, |\beta| > 1$ and $|\alpha| < 1, |\beta| < 1$, respectively.

We remark that, in fact, sharper estimates than (2.5) are possible in each case. For instance, in the first case $|\alpha| > 1, |\beta| > 1$ we could use (2.19) to bound $E(0)$ in (2.16) and not extend t to T in the other terms of (2.16). A similar procedure applies in the second case $|\alpha| < 1, |\beta| < 1$ to (2.25) and (2.28) where we extended the integrals in (2.25) to the interval $[0, T]$ to obtain the common estimate (2.5).

2.2.2 Inhomogeneous Equation with time–independent source term:

In this section we will consider the nonstandard problem (2.1) where the inhomogeneous term is time independent. That is we have,

$$u_{tt} + Au = f(x), \quad \text{in } \Omega \times (0, T). \tag{2.30}$$

In this case we have the following theorem:

Theorem 2.2.2. *If u is a solution of the problem (2.30), (2.2), (2.3), with α and β satisfying either $|\alpha| > 1$, $|\beta| > 1$ or $|\alpha| < 1$, $|\beta| < 1$, then the energy function given by (2.4) satisfies an a priori inequality of the form,*

$$E(t) \leq c_1 \|h\|^2 + c_2 \langle Ag, g \rangle + c_3 \|f\|^2 + c_4 \|g\|^2, \quad 0 \leq t \leq T, \tag{2.31}$$

for computable constants c_1, c_2, c_3 and c_4.

Proof: We make some modifications in the previous argument and we will obtain an energy bound which is independent of T. We again examine two cases seperately.

Case 1: $|\boldsymbol{\alpha}| > \mathbf{1}$, $|\boldsymbol{\beta}| > \mathbf{1}$

As done previously, we begin with multiplying (2.30) by $2u_t$ and integrating over

$\Omega \times (0,t)$. This calculation yields

$$\begin{aligned} E(t) - E(0) &= 2\int_0^t \langle f, u_\eta \rangle \ d\eta \\ &= 2\int_0^t (\, \langle f, u\rangle \,)_\eta \ d\eta \end{aligned}$$

that is,

$$E(t) = 2\left[\langle f, u(t)\rangle - \langle f, u(0)\rangle\right] + E(0). \tag{2.32}$$

Writing this equation for $t = T$ and using the auxiliary conditions, we get

$$\begin{aligned} &\|h - \beta u_t(0)\|^2 + \langle A(g - \alpha u(0)), g - \alpha u(0)\rangle \\ &\qquad = 2\left[\langle f, g - \alpha u(0)\rangle - \langle f, u(0)\rangle\right] + \|u_t(0)\|^2 + \langle Au(0), u(0)\rangle \,. \end{aligned}$$

So we have,

$$\begin{aligned} &\|h\|^2 + \left(\beta^2 - 1\right)\|u_t(0)\|^2 + \langle Ag, g\rangle + \left(\alpha^2 - 1\right)\langle Au(0), u(0)\rangle \\ &\quad = 2\beta\,\langle u_t(0), h\rangle + 2\alpha\,\langle Ag, u(0)\rangle + 2\,\langle f, g\rangle - 2(\alpha+1)\,\langle f, u(0)\rangle \\ &\quad \le 2\,|\beta|\,|\langle u_t(0), h\rangle| + 2\,|\alpha|\,|\langle Ag, u(0)\rangle| + 2\,|\langle f, g\rangle| + 2\,|\alpha+1|\,|\langle f, u(0)\rangle| \\ &\quad \le |\beta|\left(\epsilon_1\,\|u_t(0)\|^2 + \frac{1}{\epsilon_1}\,\|h\|^2\right) + |\alpha|\left(\epsilon_2\,\langle Au(0), u(0)\rangle + \frac{1}{\epsilon_2}\,\langle Ag, g\rangle\right) \\ &\qquad\quad + \|f\|^2 + \|g\|^2 + |\alpha+1|\left(\frac{1}{\epsilon_3}\,\|f\|^2 + \epsilon_3\,\|u(0)\|^2\right), \end{aligned}$$

where we have used Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, respectively; and ϵ_1, ϵ_2, ϵ_3 are positive constants. Hence, on collecting terms we obtain,

$$\left(\beta^2 - 1 - |\beta|\,\epsilon_1\right) \|u_t(0)\|^2 + \left(\alpha^2 - 1 - |\alpha|\,\epsilon_2\right) \langle Au(0), u(0)\rangle$$
$$\leq \left(\frac{|\beta|}{\epsilon_1} - 1\right) \|h\|^2 + \left(\frac{|\alpha|}{\epsilon_2} - 1\right) \langle Ag, g\rangle + \left(\frac{|\alpha+1|}{\epsilon_3} + 1\right) \|f\|^2$$
$$+ |\alpha+1|\,\epsilon_3 \|u(0)\|^2 + \|g\|^2 . \quad (2.33)$$

Now we have to find a bound for $\|u(0)\|^2$. Since A is assumed to be positive definite we can write,

$$\langle Au(t), u(t)\rangle \geq \lambda \|u(t)\|^2 , \qquad 0 \leq t \leq T$$

for some positive constant λ. In particular

$$\|u(0)\|^2 \leq \frac{1}{\lambda} \langle Au(0), u(0)\rangle .$$

Substitution of this inequality into (2.33) yields

$$\left(\beta^2 - 1 - |\beta|\,\epsilon_1\right) \|u_t(0)\|^2 + \left(\alpha^2 - |\alpha|\,\epsilon_2 - 1 - \frac{|\alpha+1|}{\lambda}\epsilon_3\right) \langle Au(0), u(0)\rangle$$
$$\leq \left(\frac{\beta}{\epsilon_1} - 1\right) \|h\|^2 + \left(\frac{\alpha}{\epsilon_2} - 1\right) \langle Ag, g\rangle + \left(\frac{|\alpha+1|}{\epsilon_3} + 1\right) \|f\|^2 + \|g\|^2 . \quad (2.34)$$

We consider the case $|\alpha| > 1, |\beta| > 1$. Since ϵ_1, ϵ_2, ϵ_3 are arbitrary positive constants we can choose them sufficiently small so that the coefficients of the above inequality become positive. Thus, using similar discussions in the previous section, it follows that,

$$E(0) \leq Q^2 := k_1 \|h\|^2 + k_2 \langle Ag, g\rangle + k_3 \|f\|^2 + k_4 \|g\|^2 , \qquad (2.35)$$

for computable constants k_1, k_2, k_3, k_4. Inserting (2.35) into (2.32) gives the bound

$$E(t) \leq Q^2 + 2\left[\langle f, u(t)\rangle - \langle f, u(0)\rangle\right]$$

from which we deduce

$$E(t) \leq Q^2 + \left(1 + \frac{1}{\epsilon_4}\right) \|f\|^2 + \epsilon_4 \|u(t)\|^2 + \|u(0)\|^2$$

using Cauchy–Schwarz Inequalities, for $\epsilon_4 > 0$. Now writing $E(t)$ explicitly and using the positive definiteness of A, we have

$$\|u_t(t)\|^2 + \langle Au(t), u(t)\rangle \leq Q^2 + \left(1 + \frac{1}{\epsilon_4}\right) \|f\|^2 + \frac{\epsilon_4}{\lambda} \langle Au(t), u(t)\rangle + \frac{1}{\lambda} \langle Au(0), u(0)\rangle$$

or,

$$\|u_t(t)\|^2 + \left(1 - \frac{\epsilon_4}{\lambda}\right) \langle Au(t), u(t)\rangle \leq Q^2 + \left(1 + \frac{1}{\epsilon_4}\right) \|f\|^2 + \frac{1}{\lambda} \langle Au(0), u(0)\rangle \quad (2.36)$$

for $\lambda > 0$.

Now we are looking for a bound for the last term in (2.36). From (2.4) we can write

$$\langle Au(0), u(0)\rangle \leq E(0),$$

so that taking (2.35) into account, we have

$$\frac{1}{\lambda} \langle Au(0), u(0)\rangle \leq \frac{1}{\lambda} Q^2. \quad (2.37)$$

Substituting (2.37) into (2.36) gives the bound,

$$\|u_t(t)\|^2 + \left(1 - \frac{\epsilon_4}{\lambda}\right) \langle Au(t), u(t)\rangle \leq \left(1 + \frac{1}{\lambda}\right) Q^2 + \left(1 + \frac{1}{\epsilon_4}\right) \|f\|^2.$$

Finally choosing ϵ_4 appropriately we derive the energy bound

$$E(t) \leq k_5 \|h\|^2 + k_6 \langle Ag, g\rangle + k_7 \|f\|^2 + k_8 \|g\|^2, \quad 0 \leq t \leq T \quad (2.38)$$

for computable constants k_5, k_6, k_7, k_8 when $|\alpha| > 1$, $|\beta| > 1$, as desired.

<u>Case 2:</u> $\boldsymbol{|\alpha| < 1, |\beta| < 1}$

We begin by evaluating (2.32) for $t = T$. That is,

$$E(T) = 2\left[\langle f, u(T)\rangle - \langle f, u(0)\rangle\right] + E(0). \tag{2.39}$$

Substituting

$$E(0) = \frac{1}{\beta^2}\left\|h - u_t(T)\right\|^2 + \frac{1}{\alpha^2}\left\langle A(g - u(T)), g - u(T)\right\rangle$$

we obtain,

$$\begin{aligned}\left\|u_t(T)\right\|^2 + \langle Au(T), u(T)\rangle = \frac{1}{\beta^2}\left\|h - u_t(T)\right\|^2 + \frac{1}{\alpha^2}\left\langle A(g - u(T)), (g - u(T))\right\rangle \\ + 2\left[\langle f, u(T)\rangle - \left\langle f, \frac{1}{\alpha}(g - u(T))\right\rangle\right].\end{aligned}$$

On collecting terms we get

$$\begin{aligned}&\left(\frac{1}{\beta^2} - 1\right)\left\|u_t(T)\right\|^2 + \left(\frac{1}{\alpha^2} - 1\right)\langle Au(T), u(T)\rangle + \frac{1}{\beta^2}\|h\|^2 + \frac{1}{\alpha^2}\langle Ag, g\rangle \\ &\quad = \frac{2}{\beta^2}\langle h, u_t(T)\rangle + \frac{2}{\alpha^2}\langle Ag, u(T)\rangle - 2\left[\langle f, u(T)\rangle - \left\langle f, \frac{1}{\alpha}(g - u(T))\right\rangle\right].\end{aligned}$$

Since we assumed $|\alpha| < 1, |\beta| < 1$, the left hand side of the above inequality becomes positive. Taking absolute value of both sides and applying Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, we obtain

$$\begin{aligned}&\left(\frac{1}{\beta^2} - 1\right)\left\|u_t(T)\right\|^2 + \left(\frac{1}{\alpha^2} - 1\right)\langle Au(T), u(T)\rangle + \frac{1}{\beta^2}\|h\|^2 + \frac{1}{\alpha^2}\langle Ag, g\rangle \\ &\quad \le \frac{1}{\beta^2}\left(\frac{1}{\epsilon_5}\|h\|^2 + \epsilon_5\left\|u_t(T)\right\|^2\right) + \frac{1}{\alpha^2}\left(\frac{1}{\epsilon_6}\langle Ag, g\rangle + \epsilon_6\langle Au(T), u(T)\rangle\right) \\ &\qquad + 2\left|\left[\langle f, u(T)\rangle - \left\langle f, \frac{1}{\alpha}(g - u(T))\right\rangle\right]\right|.\end{aligned} \tag{2.40}$$

Since

$$2\left|\left[\langle f,u(T)\rangle-\left\langle f,\frac{1}{\alpha}(g-u(T))\right\rangle\right]\right| = 2\left|\left(1+\frac{1}{\alpha}\right)\langle f,u(T)\rangle-\frac{1}{\alpha}\langle f,g\rangle\right|$$
$$\leq 2\left[\left|1+\frac{1}{\alpha}\right|\left|\langle f,u(T)\rangle\right|+\frac{1}{|\alpha|}\left|\langle f,g\rangle\right|\right]$$
$$\leq \left|1+\frac{1}{\alpha}\right|\left(\frac{1}{\epsilon_7}\|f\|^2+\epsilon_7\|u(T)\|^2\right)$$
$$+\frac{1}{|\alpha|}\left(\|f\|^2+\|g\|^2\right),$$

we finally have

$$2\left[\langle f,u(T)\rangle-\left\langle f,\frac{1}{\alpha}(g-u(T))\right\rangle\right]$$
$$\leq\left|1+\frac{1}{\alpha}\right|\left(\frac{1}{\epsilon_7}\|f\|^2+\frac{\epsilon_7}{\lambda}\langle Au(T),u(T)\rangle\right)+\frac{1}{|\alpha|}\left(\|f\|^2+\|g\|^2\right) \quad (2.41)$$

from the positive definiteness of A, for $\epsilon_7>0$.

Thus, on combining (2.40) and (2.41) we derive

$$\left(\frac{1}{\beta^2}-1-\frac{\epsilon_5}{\beta^2}\right)\|u_t(T)\|^2+\left(\frac{1}{\alpha^2}-1-\frac{\epsilon_6}{\alpha^2}-\left|1+\frac{1}{\alpha}\right|\frac{\epsilon_7}{\lambda}\right)\langle Au(T),u(T)\rangle$$
$$\leq\frac{1}{\beta^2}\left(\frac{1}{\epsilon_5}-1\right)\|h\|^2+\frac{1}{\alpha^2}\left(\frac{1}{\epsilon_6}-1\right)\langle Ag,g\rangle+\left[\left|1+\frac{1}{\alpha}\right|\frac{1}{\epsilon_7}+\frac{1}{|\alpha|}\right]\|f\|^2+\frac{1}{|\alpha|}\|g\|^2. \quad (2.42)$$

For sufficiently small ϵ_5, ϵ_6 and ϵ_7, it follows that,

$$E(T)\leq P^2:=k_9\|h\|^2+k_{10}\langle Ag,g\rangle+k_{11}\|f\|^2+k_{12}\|g\|^2 \quad (2.43)$$

for computable constants k_9, k_{10}, k_{11}, k_{12}. We now use the equation

$$E(t)=E(T)-2\int_t^T\langle f,u_\eta\rangle\,d\eta$$

(which is obtained by multiplying (2.30) by $2u_t$ and integrating over $\Omega\times(t,T)$)

and (2.43) to write

$$\|u_t(t)\|^2 + \langle Au(t), u(t)\rangle \leq P^2 - 2\left[\langle f, u(T)\rangle - \langle f, u(t)\rangle\right]. \tag{2.44}$$

Since

1. $2\langle f, u(T)\rangle \leq \|f\|^2 + \|u(T)\|^2 \leq \|f\|^2 + \dfrac{1}{\lambda}\langle Au(T), u(T)\rangle$,

2. $2\langle f, u(t)\rangle \leq \dfrac{1}{\epsilon_8}\|f\|^2 + \dfrac{\epsilon_8}{\lambda}\langle Au(t), u(t)\rangle$,

for $\epsilon_8 > 0$, we have

$$\begin{aligned}\|u_t(t)\|^2 + \langle Au(t), u(t)\rangle &\leq P^2 + \left(1 + \frac{1}{\epsilon_8}\right)\|f\|^2 + \frac{\epsilon_8}{\lambda}\langle Au(t), u(t)\rangle \\ &\quad + \frac{1}{\lambda}\langle Au(T), u(T)\rangle \\ &\leq P^2 + \left(1 + \frac{1}{\epsilon_8}\right)\|f\|^2 + \frac{\epsilon_8}{\lambda}\langle Au(t), u(t)\rangle + \frac{1}{\lambda}P^2,\end{aligned} \tag{2.45}$$

that is,

$$\|u_t(t)\|^2 + \left(1 - \frac{\epsilon_8}{\lambda}\right)\langle Au(t), u(t)\rangle \leq \left(1 + \frac{1}{\lambda}\right)P^2 + \left(1 + \frac{1}{\epsilon_8}\right)\|f\|^2. \tag{2.46}$$

Choosing ϵ_8 and λ appropriately, we finally obtain,

$$E(t) \leq k_{13}\|h\|^2 + k_{14}\langle Ag, g\rangle + k_{15}\|f\|^2 + k_{16}\|g\|^2, \quad 0 \leq t \leq T \tag{2.47}$$

for computable constants $k_{13}, k_{14}, k_{15}, k_{16}$ when $|\alpha| < 1, |\beta| < 1$, which is of the form (2.31).

2.3 Uniqueness

In the previous two sections we obtained energy bounds for the nonstandard problem when the nonzero constants α and β satisfy either $|\alpha| > 1, |\beta| > 1$ or $|\alpha| < 1, |\beta| < 1$. Thus we have the following theorem.

Theorem 2.3.1. *If a solution exists to problem (2.1)–(2.3) with either F(x,t) or f(x) in (2.1), then it is unique, provided* $|\alpha| > 1, |\beta| > 1$ *or* $|\alpha| < 1, |\beta| < 1$.

Proof: We will show the uniqueness for $F(x,t)$. Similar development is possible when the inhomogeneous term is time independent.

Let u and v be two different solutions of the problem (2.1)–(2.3) and let $u - v = w$. Then w solves

$$\begin{aligned} & w_{tt} + Aw = 0, \quad \text{in } \Omega \times (0,T), \\ & \alpha w(0) + w(T) = 0, \quad \text{in } \Omega, \\ & \beta w_t(0) + w_t(T) = 0, \quad \text{in } \Omega. \end{aligned} \tag{2.48}$$

We will use the energy function to show the uniqueness. In Section 2.1, we have shown that the energy function $E(t)$ satisfies a bound of the form

$$E(t) \leq C_1 \int_0^T \|F(\eta)\|^2 \, d\eta + C_2 \int_0^T e^{\gamma\eta} \int_\eta^T \|F(\xi)\|^2 \, d\xi \, d\eta + C_3 \langle Ag, g \rangle + C_4 \|h\|^2$$

when $|\alpha| > 1, |\beta| > 1$ or $|\alpha| < 1, |\beta| < 1$. Since the problem (2.48) involves homogeneous differential equation and homogeneous auxiliary conditions, we get

$$E(t) \leq 0.$$

But the definition of the energy function implies,

$$E(t) = \|w_t(t)\|^2 + \langle Aw(t), w(t) \rangle \geq 0.$$

So we have

$$E(t) = \|w_t(t)\|^2 + \langle Aw(t), w(t) \rangle = 0,$$

from which we get $w = 0$ implying the uniqueness of the solution.

We now assume that A is such that there is a complete eigenspace and consider problem (2.1)–(2.3) with $F(x,t) = 0$. Then we can determine the formal representation of the unique solution and a condition for the existence of such.

Let λ_n and ϕ_n denote the eigenvalues and eigenfunctions, respectively, of the eigenvalue problem

$$\begin{aligned} A\phi - \lambda\phi = 0 \quad \text{in } \Omega, \\ \phi = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.49}$$

(We assume that the eigenfunctions have been made orthonormal by the Gram–Schmidt process.) Since there is a complete eigenspace, the set of eigenfunctions $\{\phi_n\}$ forms a complete orthonormal set. This leads to an expansion

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x) \tag{2.50}$$

for the solution $u(x,t)$ of the problem (2.1)–(2.3) with $F(x,t) = 0$. Substituting (2.50) into the equation (2.1) with $F(x,t) = 0$, one finds that the $c_n(t)$ are solutions of the ordinary differential equation

$$c_n''(t) + \lambda_n c_n(t) = 0.$$

Hence, the series representation of the solution of (2.1)–(2.3) with $F(x,t) = 0$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin\sqrt{\lambda_n}t + b_n \cos\sqrt{\lambda_n}t\right)\phi_n(x), \tag{2.51}$$

where the coefficients a_n and b_n are determined by the auxiliary conditions. Now using (2.51) in the first auxiliary condition (2.2) we get,

$$\sum_{n=1}^{\infty} \left[a_n\left(\alpha + \cos\sqrt{\lambda_n}T\right) + b_n \sin\sqrt{\lambda_n}T\right]\phi_n(x) = g(x).$$

Using orthonormality of eigenfunctions, we have

$$a_m\left(\alpha + \cos\sqrt{\lambda_m}T\right) + b_m \sin\sqrt{\lambda_m}T = \langle g(x), \phi_m(x)\rangle.$$

Similarly, from the second auxiliary condition (2.3) we get,

$$\sqrt{\lambda_m}T\left[a_m \sin\sqrt{\lambda_m}T + b_m\left(\beta + \cos\sqrt{\lambda_m}T\right)\right] = \langle h(x), \phi_m(x)\rangle\,.$$

Therefore, the constants a_n and b_n satisfy the system of equations

$$\begin{aligned} a_n\left(\alpha + \cos\sqrt{\lambda_n}T\right) + b_n \sin\sqrt{\lambda_n}T &= g_n, \\ -a_n \sin\sqrt{\lambda_n}T + b_n\left(\beta + \cos\sqrt{\lambda_n}T\right) &= h_n/\sqrt{\lambda_n}, \end{aligned} \tag{2.52}$$

where $\langle g(x), \phi_n(x)\rangle = g_n$ and $\langle h(x), \phi_n(x)\rangle = h_n$ are the Fourier coefficients of g and h, respectively. Writing (2.52) in the matrix form, we get,

$$\begin{pmatrix} \alpha + \cos\sqrt{\lambda_n}T & \sin\sqrt{\lambda_n}T \\ -\sin\sqrt{\lambda_n}T & \beta + \cos\sqrt{\lambda_n}T \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} g_n \\ h_n/\sqrt{\lambda_n} \end{pmatrix}$$

We denote the coefficient determinant of this system by D_n, i.e.,

$$D_n = \alpha\beta + (\alpha+\beta)\cos\sqrt{\lambda_n}T + 1. \tag{2.53}$$

Thus if $D_n \neq 0$ for each n, then the system (2.52) has a unique solution for each n and we obtain a unique series representation of (2.1)–(2.3) with $F(x,t) = 0$ from (2.51). Let us examine the condition $D_n \neq 0$. That is,

$$\cos\sqrt{\lambda_n}T \neq -\frac{1+\alpha\beta}{\alpha+\beta}\,. \tag{2.54}$$

Indeed this is possible if

$$\left|\frac{1+\alpha\beta}{\alpha+\beta}\right| > 1. \tag{2.55}$$

We now show that (2.55) is satisfied either $|\alpha| > 1, |\beta| > 1$ or $|\alpha| < 1, |\beta| < 1$. (That is why we only discussed two cases in the previous sections.)

Let us consider the case $\boldsymbol{|\alpha| > 1}$, $\boldsymbol{|\beta| > 1}$. Similar development is possible for $|\boldsymbol{\alpha}| < \mathbf{1}$, $|\boldsymbol{\beta}| < \mathbf{1}$. $|\alpha| > 1, |\beta| > 1$ implies $\alpha^2 > 1$, $\beta^2 > 1$. So we can write,

$$\alpha^2(\beta^2 - 1) > \beta^2 - 1$$

i.e.,

$$\alpha^2\beta^2 + 1 > \alpha^2 + \beta^2.$$

Adding $2\alpha\beta$ on both sides of the above inequality gives us

$$\alpha^2\beta^2 + 2\alpha\beta + 1 > (\alpha + \beta)^2$$

which implies

$$\frac{(\alpha\beta)^2 + 2\alpha\beta + 1}{(\alpha + \beta)^2} > 1$$

i.e.

$$\frac{(\alpha\beta + 1)^2}{(\alpha + \beta)^2} > 1$$

from which it follows that,

$$\left|\frac{1 + \alpha\beta}{\alpha + \beta}\right| > 1,$$

as desired.
Thus, (2.54) holds for $|\alpha| > 1, |\beta| > 1$. Therefore, the solution is unique and the series representation of the unique solution can be obtained.

Now if $D_n = 0$ for some $n = k$, from the Cramer's rule, either the solution of (2.52) does not exist or it may not be unique depending upon whether G_k and/or H_k do not vanish or do vanish, where

$$G_k = \begin{vmatrix} g_k & \sin\sqrt{\lambda_k}T \\ h_k/\sqrt{\lambda_k} & \beta + \cos\sqrt{\lambda_k}T \end{vmatrix}, \qquad H_k = \begin{vmatrix} \alpha + \cos\sqrt{\lambda_k}T & g_k \\ -\sin\sqrt{\lambda_k}T & h_k/\sqrt{\lambda_k}T \end{vmatrix}$$

As a simple example for the nonexistence of the solution when $0 < \alpha < 1$, $\beta > 1$, we take $g(x) = h(x)$ and suppose $D_n = 0$, for some n.
This implies,

$$\cos\sqrt{\lambda_n}T = -\frac{1 + \alpha\beta}{\alpha + \beta} \quad \text{for some } n.$$

Then we have

$$\begin{aligned} G_n &= g_n\left(\beta - \frac{1+\alpha\beta}{\alpha+\beta}\right) - \frac{h_n}{\sqrt{\lambda_n}}\left(\frac{\sqrt{(\beta^2-1)(1-\alpha^2)}}{\alpha+\beta}\right) \\ &= \frac{g_n\sqrt{\beta^2-1}}{\alpha+\beta}\left[\sqrt{\beta^2-1} - \frac{1}{\sqrt{\lambda_n}}\sqrt{1-\alpha^2}\right] \end{aligned}$$

and it is easily seen that G_n doesn't vanish when

$$\beta > \left[1 + \frac{1}{\lambda_n}(1-\alpha^2)\right]^{1/2}$$

(that is, $\beta > 1$), which yields the nonexistence of the solution.

2.4 Pointwise bounds for the wave equation

In this section we again consider problem (2.1)–(2.3) with $F(x,t) = 0$ and show how pointwise bounds may be obtained at some time t_1, $0 \leq t_1 \leq T$, assuming a solution exists. We proceed generally at first, then we shall fix the operator A as the negative of Laplacian and we shall restrict the dimension. In this way, we derive pointwise bounds for the special case of the wave equation in $N \leq 3$ dimensions.

We assume $u \in C^3$ and start with

$$u_{ttt} + Au_t = 0.$$

As done before, multiplying the above equation by $2u_{tt}$ and integrating over $\Omega \times (0,t)$ yields,

$$\int_0^t \int_\Omega \left[u_{\eta\eta}^2 + u_\eta A u_\eta\right]_\eta \, dx \, d\eta = 0.$$

Thus it follows

$$\|u_{tt}(t)\|^2 + \langle Au_t(t), u_t(t)\rangle = \|u_{tt}(0)\|^2 + \langle Au_t(0), u_t(0)\rangle . \qquad (2.56)$$

Now, from the homogeneous form of (2.1) we have

$$u_{tt}(x,t) = -Au(x,t).$$

In particular

$$u_{tt}(0) = -Au(0) \text{ and } u_{tt}(T) = -Au(T),$$

where we suppress the spatial argument. Hence, from (2.2) we can write,

$$\alpha u_{tt}(0) + u_{tt}(T) = -A(\alpha u(0) + u(T)) = -Ag. \tag{2.57}$$

Evaluating (2.56) at $t = T$ then using (2.57) and (2.3), we get

$$\|Ag + \alpha u_{tt}(0)\|^2 + \langle A(h - \beta u_t(0), h - \beta u_t(0))\rangle = \|u_{tt}(0)\|^2 + \langle Au_t(0), u_t(0)\rangle .$$

On combining terms,

$$\begin{aligned}(\alpha^2 - 1)\|u_{tt}(0)\|^2 + \|Ag\|^2 + (\beta^2 - 1)\langle Au_t(0), u_t(0)\rangle + \langle Ah, h\rangle \\ = -2\alpha\langle Ag, u_{tt}(0)\rangle + 2\beta\langle h, Au_t(0)\rangle .\end{aligned}$$

Since we assumed solution exists, particularly we can take $|\alpha| > 1$ and $|\beta| > 1$. Taking absolute value of both sides and using Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, it follows that,

$$\begin{aligned}(\alpha^2-1)\|u_{tt}(0)\|^2 &+ \|Ag\|^2 + (\beta^2 - 1)\langle Au_t(0), u_t(0)\rangle + \langle Ah, h\rangle \\ &\leq 2|\alpha||\langle Ag, u_{tt}(0)\rangle| + 2|\beta||\langle h, Au_t(0)\rangle| \\ &\leq |\alpha|\left(\frac{1}{\theta_1}\|Ag\|^2 + \theta_1\|u_{tt}(0)\|^2\right) + |\beta|\left(\frac{1}{\theta_2}\langle Ah, h\rangle + \theta_2\langle Au_t(0), u_t(0)\rangle\right).\end{aligned}$$

That is,

$$\begin{aligned}(\alpha^2 - |\alpha|\theta_1 - 1)\|u_{tt}(0)\|^2 + (\beta^2 - |\beta|\theta_2 - 1)\langle Au_t(0), u_t(0)\rangle \\ \leq \left(\frac{|\alpha|}{\theta_1} - 1\right)\|Ag\|^2 + \left(\frac{|\beta|}{\theta_2} - 1\right)\langle Ah, h\rangle\end{aligned}$$

for positive constants θ_1 and $\theta_2 > 0$. Choosing θ_1 and θ_2 sufficiently small, we have

$$\|u_{tt}(0)\|^2 + \langle Au_t(0), u_t(0)\rangle \leq \kappa_1 \|Ag\|^2 + \kappa_2 \langle Ah, h\rangle \tag{2.58}$$

for computable constants κ_1 and κ_2. Note that, from (2.56) and (2.58) we get

$$\|u_{tt}(t)\|^2 \leq \kappa_1 \|Ag\|^2 + \kappa_2 \langle Ah, h\rangle .$$

We now consider the homogeneous form of (2.1) and choose the operator A as the negative of Laplacian with homogeneous Dirichlet boundary conditions. Thus, taking the above inequality into account and using Green's first identity we can write

$$\begin{aligned} \|u_{tt}(t_1)\|^2 = \|\Delta u(t_1)\|^2 &\leq \kappa_1 \|\Delta g\|^2 - \kappa_2 \int_\Omega h.\Delta h \, dx \\ &= \kappa_1 \|\Delta g\|^2 + \kappa_2 \int_\Omega \sum_{i=1}^{N} h_{x_i}^2 \, dx \\ &= \kappa_1 \|\Delta g\|^2 + \kappa_2 \|\nabla h\|^2 \end{aligned} \tag{2.59}$$

where ∇ denotes the gradient operator, $0 \leq t_1 \leq T$. Hence we obtain a bound for $\|\Delta u(t_1)\|$.

By Green's formula we can write,

$$u(x, t_1) = -\int_\Omega G(x, t_1; \xi, t_1)\Delta u(\xi, t_1) \, d\xi \; ,$$

where G denotes the Green's function for the operator $-\Delta$ under Dirichlet boundary conditions. Applying Schwarz Inequality we have,

$$|u(x, t_1)| \leq \left(\int_\Omega G^2 \, d\xi\right)^{1/2} \left(\int_\Omega [\Delta u(t_1)]^2 \, d\xi\right)^{1/2}$$

or,

$$|u(x, t_1)| \leq \|G\| \, \|\Delta u(t_1)\| . \tag{2.60}$$

Weinberger [25] determined a bound for $\|G\|_{p'}$ in a bounded N–dimensional domain. He showed

$$\|G\|_{p'} \leq \lambda^{-1} V^{2/N-1/p} K_{p,N}$$

where p is any number greater than $\frac{N}{2}$, $(1/p+1/p'=1)$, λ is a lower bound for the eigenvalues of the symmetric matrix $a_{ij}(x)$ for the self adjoint elliptic operator

$$L \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_i}\right]$$

and V is the volume of Ω. The best possible constant $K_{p,N}$ is given explicitly in [25].

Since we use L_2 norm we have $p = 2$ here, and so we are restricted to $N = 3$ since $p > N/2$. Thus using (2.59) to bound $\|\Delta u(t_1)\|$ and Weinberger bound on $\|G\|$, we can obtain pointwise bounds for u at the point (x, t_1) when $N \leq 3$, by means of (2.60).

2.5 Some extensions

In this section we will extend the technique developed in previous sections to a damped equation and to a nonlinear Kirchoff string, respectively.

Firstly, we investigate the derivation of energy bound when a damping term is present in (2.1). That is, we consider the problem

$$\begin{aligned} &u_{tt} + au_t + Au = 0, \quad \text{in } \Omega \times (0,T) \\ &\alpha u(0) + u(T) = f(x), \quad \text{in } \Omega \\ &\beta u_t(0) + u_t(T) = g(x), \quad \text{in } \Omega \end{aligned} \tag{2.61}$$

where a is a positive constant. (A similar development is also possible when a is negative.) We start with $|\alpha| > 1$, $|\beta| > 1$. Multiplying (2.61) by $2u_t$ and integrating over $\Omega \times (0,t)$ yields,

$$\|u_t(t)\|^2 + 2a\int_0^t \|u_\eta\|^2 \, d\eta + \langle Au(t), u(t)\rangle = \|u_t(0)\|^2 + \langle Au(0), u(0)\rangle \tag{2.62}$$

for $0 \leq t \leq T$. We evaluate (2.62) at $t = T$, drop the integral term and use auxiliary conditions to have

$$\|h - \beta u_t(0)\|^2 + \langle A(g - \alpha u(0)), g - \alpha u(0)\rangle \leq \|u_t(0)\|^2 + \langle Au(0), u(0)\rangle .$$

Proceeding as in Section 2.3, one can easily determine,

$$E(0) = \|u_t(0)\|^2 + \langle Au(0), u(0)\rangle \leq D_1 \|h\|^2 + D_2 \langle Ag, g\rangle$$

for computable constants D_1 and D_2. Consequently, from (2.62) we get,

$$\bar{E}(t) \leq D_1 \|h\|^2 + D_2 \langle Ag, g\rangle \tag{2.63}$$

where $\bar{E}(t)$ denotes the left hand side of (2.62).

In the case of $|\alpha| < 1$, $|\beta| < 1$ we again multiply (2.61) by $2u_t$ and integrate over $\Omega \times (t, T)$ to have

$$\|u_t(t)\|^2 + \langle Au(t), u(t)\rangle = 2a \int_t^T \|u_\eta\|^2 \; d\eta + \|u_t(T)\|^2 + \langle Au(T), u(T)\rangle . \tag{2.64}$$

Using (2.4) we obtain from (2.64) that

$$E(t) \leq 2a \int_t^T E(\eta)\; d\eta + E(T), \quad 0 \leq t \leq T. \tag{2.65}$$

Letting

$$S(t) = \int_t^T E(\eta)\; d\eta$$

we can rewrite (2.65) as

$$-S'(t) \leq 2a\; S(t) + E(T). \tag{2.66}$$

Solving (2.66) for $S(t)$, over the interval (t, T), we obtain,

$$S(t) \leq \frac{E(T)}{2a}\left[e^{2a(T-t)} - 1\right] . \tag{2.67}$$

Now substituting (2.67) into (2.66) yields,

$$E(t) = S'(t) \leq E(T)e^{2a(T-t)}, \quad 0 \leq t \leq T. \tag{2.68}$$

In particular we have

$$E(0) \leq E(T)e^{2aT}.$$

Writing $E(0) = \frac{1}{\beta^2}\|h - u_t(T)\|^2 + \frac{1}{\alpha^2}\langle A(g - u(T)), g - u(T)\rangle$ and applying Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, it follows that

$$\begin{aligned}\left(\frac{1}{\beta^2} - e^{2aT} - \frac{\delta_1}{\beta^2}\right)\|u_t(T)\|^2 + \left(\frac{1}{\alpha^2} - e^{2aT} - \frac{\delta_2}{\alpha^2}\right)\langle Au(T), u(T)\rangle \\ \leq \frac{1}{\beta^2}\left(\frac{1}{\delta_1} - 1\right)\|h\|^2 + \frac{1}{\alpha^2}\left(\frac{1}{\delta_2} - 1\right)\langle Ag, g\rangle\end{aligned}$$

for positive constants δ_1 and δ_2. If we have the further restriction

$$|\alpha|, |\beta| < e^{-aT}$$

we can make the coefficients of the above inequality positive for sufficiently small δ_1 and δ_2. Thus we obtain the bound

$$E(T) \leq D_3\|h\|^2 + D_4\langle Ag, g\rangle$$

for computable constants D_3 and D_4. Consequently it follows from (2.68) that

$$E(t) \leq \left[D_3\|h\|^2 + D_4\langle Ag, g\rangle\right]e^{2aT}, \quad 0 \leq t \leq T, \tag{2.69}$$

provided $|\alpha|, |\beta| < e^{-aT}$.

We now try to find an energy bound for the generalized Kirchhoff string. In [27], generalized Kirchhoff string is defined as the differential equation,

$$u_{tt} - m\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{q-2}u_t = f(x,t) \quad \text{in } [0,\infty) \times \Omega$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, m is a nonnegative function, $|u_t|^{q-2} u_t$ is damping and $f(x,t)$ is an external force.

For simplicity, we will discuss the homogeneous generalized Kirchoff string with no damping term and we will take $m = \rho^{'}$, for some nonnegative differentiable function ρ. The prime here denotes the ordinary differentiation. Thus, we consider the differential equation

$$u_{tt} - \rho^{'} \left(\int_{\Omega} |\nabla u(t)|^2 \, dx \right) \Delta u = 0, \quad \text{in } \Omega \times (0,T), \tag{2.70}$$

and determine a bound for the energy identity

$$E(t) = \|u_t(t)\|^2 + \rho \left(\|\nabla u(t)\|^2 \right). \tag{2.71}$$

Multiplying (2.70) by $2u_t$ and integrating over $\Omega \times (0,t)$, we get

$$\|u_t(t)\|^2 - \|u_t(0)\|^2 = 2 \int_0^t \int_{\Omega} \left[\rho^{'} \left(\|\nabla u\|^2 \right) u_\eta \Delta u \right] \, dx \, d\eta. \tag{2.72}$$

We now examine the right hand side of the above equation. Since, $\nabla \left(u_\eta \nabla u \right) = u_\eta \Delta u + \nabla u_\eta \nabla u$, we have

$$2 \int_0^t \int_{\Omega} \left[\rho^{'} \left(\|\nabla u\|^2 \right) u_\eta \, \Delta u \right] \, dx \, d\eta$$
$$= 2 \int_0^t \int_{\Omega} \left[\rho^{'} \left(\|\nabla u\|^2 \right) \left(\nabla \left(u_\eta \nabla u \right) - \nabla u_\eta \nabla u \right) \right] \, dx \, d\eta.$$

Using the fact that, $\nabla u_\eta \nabla u = \frac{1}{2} \frac{\partial}{\partial \eta} |\nabla u|^2$ and applying divergence theorem we rewrite the right hand side of the above equation as,

$$2 \int_0^t \int_{\Omega} \left[\rho^{'} \left(\|\nabla u\|^2 \right) \left(\nabla \left(u_\eta \nabla u \right) - \nabla u_\eta \nabla u \right) \right] \, dx \, d\eta$$
$$= \int_0^t \rho^{'} \left(\|\nabla u\|^2 \right) \left\{ 2 \int_{\partial\Omega} \left(u_\eta \nabla u \right) . \vec{n} \, dS_x - \int_{\Omega} \frac{\partial}{\partial \eta} |\nabla u|^2 \, dx \right\} \, d\eta$$

where $\vec{n}$ denotes the exterior unit normal vector to the boundary. Since we consider problems whose solutions satisfy Dirichlet boundary conditions, the integral taken on the boundary vanishes and we finally have

$$2\int_0^t \int_\Omega \left[\rho^{'}\left(\|\nabla u\|^2\right) u_\eta\, \Delta u\right]\, dx\, d\eta = -\int_0^t \rho^{'}\left(\|\nabla u\|^2\right) \frac{d}{d\eta}\, \|\nabla u\|^2\, d\eta.$$

Hence (2.72) becomes,

$$\|u_t(t)\|^2 + \int_0^t \rho^{'}\left(\|\nabla u\|^2\right) \frac{d}{d\eta} \|\nabla u\|^2\, d\eta = \|u_t(0)\|^2. \tag{2.73}$$

Now letting $\|\nabla u\|^2 = z$ we rewrite (2.73) as,

$$\|u_t(t)\|^2 + \int_{\|\nabla u(0)\|^2}^{\|\nabla u(t)\|^2} \rho^{'}(z)\, dz = \|u_t(0)\|^2$$

and finally obtain,

$$\|u_t(t)\|^2 + \rho\left(\|\nabla u(t)\|^2\right) = \|u_t(0)\|^2 + \rho\left(\|\nabla u(0)\|^2\right), \quad 0 \le t \le T. \tag{2.74}$$

Now we have to bound the right hand side of (2.74). To this end, we evaluate (2.74) at $t = T$ and use (2.2), (2.3) so that,

$$\|h - \beta u_t(0)\|^2 + \rho\left(\|\nabla\left(g - \alpha u(0)\right)\|^2\right) = \|u_t(0)\|^2 + \rho\left(\|\nabla u(0)\|^2\right).$$

On collecting terms, we have

$$\|h\|^2 + \left(\beta^2 - 1\right)\|u_t(0)\|^2 + \rho\left(\|\nabla g - \alpha \nabla u(0)\|^2\right) = 2\beta\left\langle h, u_t(0)\right\rangle + \rho\left(\|\nabla u(0)\|^2\right). \tag{2.75}$$

We now assume that ρ satisfies an inequality of the form,

$$\rho\left(\|\nabla g - \alpha\nabla u(0)\|^2\right) \ge k_1(\alpha,\delta)\rho\left(\|\nabla u(0)\|^2\right) - k_2(\alpha,\delta)\rho\left(\|\nabla g\|^2\right) \tag{2.76}$$

where δ is a positive constant, $k_1(\alpha,\delta) > 1$ and $k_2(\alpha,\delta) > 0$. This is possible, for example, ρ is of the form $\rho(s) = s^p$. Let us check that (2.76) is satisfied, for instance, when we choose $\rho(s) = s^p$ with $p = 2$ and $|\alpha| > 1$.

$$\begin{aligned}
\rho\left(\|\nabla g - \alpha \nabla u(0)\|^2\right) &= \left(\|\nabla g - \alpha \nabla u(0)\|^2\right)^2 \\
&= \left(\|\nabla g\|^2 - 2\alpha \langle \nabla g, \nabla u(0)\rangle + \alpha^2 \|\nabla u(0)\|^2\right)^2 \\
&= \|\nabla g\|^4 + 4\alpha^2 \langle \nabla g, \nabla u(0)\rangle^2 + \alpha^4 \|\nabla u(0)\|^4 \\
&\quad - 4\alpha \|\nabla g\|^2 \langle \nabla g, \nabla u(0)\rangle + 2\alpha^2 \|\nabla g\|^2 \|\nabla u(0)\|^2 \\
&\quad - 4\alpha^3 \|\nabla u(0)\|^2 \langle \nabla g, \nabla u(0)\rangle . \qquad (2.77)
\end{aligned}$$

We will examine the terms with negative coefficients in (2.77). Since for positive a and b,

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

we can write

$$\begin{aligned}
4\alpha \|\nabla g\|^2 \langle \nabla g, \nabla u(0)\rangle &\leq 4\,|\alpha|\, \|\nabla g\|^2 \,|\langle \nabla g, \nabla u(0)\rangle| \\
&\leq 4\,|\alpha|\, \|\nabla g\|^3 \|\nabla u(0)\| \\
&= 4 \left(\frac{\|\nabla g\|^4}{\sigma_1}\right)^{3/4} \left(\sigma_1^3 |\alpha|^4 \|\nabla u(0)\|^4\right)^{1/4} \\
&\leq 4 \left(\frac{\|\nabla g\|^4}{\sigma_1 4/3} + \frac{\sigma_1^3 |\alpha|^4 \|\nabla u(0)\|^4}{4}\right) \\
&= \frac{3}{\sigma_1} \|\nabla g\|^4 + \sigma_1^3 |\alpha|^4 \|\nabla u(0)\|^4 .
\end{aligned}$$

Similarly,

$$\begin{aligned}
4\alpha^3 \|\nabla u(0)\|^2 \langle \nabla g, \nabla u(0)\rangle &\leq 4 \left(\|\nabla g\|^4\right)^{1/4} \left(\alpha^4 \|\nabla u(0)\|^4\right)^{3/4} \\
&= 4 \left(\frac{\|\nabla g\|^4}{\sigma_2^3}\right)^{1/4} \left(\sigma_2 \alpha^4 \|\nabla u(0)\|^4\right)^{3/4} \\
&\leq 4 \left(\frac{\|\nabla g\|^4}{4\sigma_2^3}\right) + \sigma_2 \alpha^4 \frac{\|\nabla u(0)\|^4}{4/3} \\
&= \frac{1}{\sigma_2^3} \|\nabla g\|^4 + 3\sigma_2 \alpha^4 \|\nabla u(0)\|^4
\end{aligned}$$

for positive constants σ_1 and σ_2. Substituting these inequalities into (2.77) we get,

$$\begin{aligned}\rho\left(\|\nabla g-\alpha\nabla u(0)\|^2\right) \geq \|\nabla g\|^4 + 4\alpha^2\left\langle\nabla g,\nabla u(0)\right\rangle^2 + \alpha^4\|\nabla u(0)\|^4 \\ + 2\alpha^2\|\nabla g\|^2\|\nabla u(0)\|^2 - \frac{3}{\sigma_1}\|\nabla g\|^4 - \sigma_1^3|\alpha|^4\|\nabla u(0)\|^4 \\ - \frac{1}{\sigma_2^3}\|\nabla g\|^4 - 3\sigma_2\alpha^4\|\nabla u(0)\|^4 .\end{aligned}$$

Consequently we have,

$$\begin{aligned}\rho\left(\|\nabla g-\alpha\nabla u(0)\|^2\right) \geq \alpha^4\left(1-\sigma_1^3-3\sigma_2\right)\rho\left(\|\nabla u(0)\|^2\right) \\ - \left(\frac{3}{\sigma_1}+\frac{1}{\sigma_2^3}-1\right)\rho\left(\|\nabla g\|^2\right)\end{aligned}$$

which is of the form (2.76).

We now return back to equation (2.75) and use the assumption (2.76) to write,

$$\begin{aligned}\|h\|^2 + \left(\beta^2-1\right)\|u_t(0)\|^2 + k_1\,\rho\left(\|\nabla u(0)\|^2\right) - k_2\,\rho\left(\|\nabla g\|^2\right) \\ \leq 2\beta\left\langle h,u_t(0)\right\rangle + \rho\left(\|\nabla u(0)\|^2\right). \quad (2.78)\end{aligned}$$

Using Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, we can write,

$$\begin{aligned}\|h\|^2 + \left(\beta^2-1\right)\|u_t(0)\|^2 + (k_1-1)\,\rho\left(\|\nabla u(0)\|^2\right) \\ \leq k_2\,\rho\left(\|\nabla g\|^2\right) + |\beta|\,\tau\,\|u_t(0)\|^2 + \frac{|\beta|}{\tau}\|h\|^2 \quad (2.79)\end{aligned}$$

for $\tau > 0$. That is, we obtain

$$\begin{aligned}\left(\beta^2-|\beta|\,\tau-1\right)\|u_t(0)\|^2 + (k_1(\alpha,\delta)-1)\,\rho\left(\|\nabla u(0)\|^2\right) \\ \leq \left(\frac{|\beta|}{\tau}-1\right)\|h\|^2 + k_2(\alpha,\delta)\rho\left(\|\nabla g\|^2\right).\end{aligned}$$

Thus, for $|\beta| > 1$ we can choose τ sufficiently small so that the coefficients of the above inequality become positive. Hence, as done in the previous sections we can write

$$\|u_t(0)\|^2 + \rho\left(\|\nabla u(0)\|^2\right) \leq \tilde{c}_1 \|h\|^2 + \tilde{c}_2\rho\left(\|\nabla g\|^2\right) \tag{2.80}$$

for computable constants $\tilde{c}_1$ and $\tilde{c}_2$.

Consequently, from (2.74), we find the energy bound as,

$$\|u_t(t)\|^2 + \rho\left(\|\nabla u(t)\|^2\right) \leq \tilde{c}_1 \|h\|^2 + \tilde{c}_2\rho\left(\|\nabla g\|^2\right) \tag{2.81}$$

for $0 \leq t \leq T$.

CHAPTER 3

NONSTANDARD PARABOLIC PROBLEMS

3.1 Introduction

In this chapter we study similar type of problems for differential operators that involve a first order time–derivative term. We consider problems of the form,

$$\frac{du}{dt} + Au = f, \quad \alpha u(0) + u(T) = g$$

for $t \in (0, T)$, where A is densely defined, linear, time independent positive definite symmetric operator and α is a nonzero constant. We note that A is a differential operator acting on functions that satisfy appropriate homogeneous boundary conditions.

In Section 3.2 we will obtain energy bounds for the given nonstandard problem and we will discuss the uniqueness of the solution. In Section 3.3 we will consider the special case of the nonstandard problem when A is the negative of the Laplace operator and determine pointwise bounds for the solution and its gradient by means of maximum principle.

3.2 Energy Bound

Let Ω be a bounded domain in R^N with smooth boundary $\partial\Omega$ and let $x = (x_1, \cdots, x_N)$. We consider the nonstandard problem

$$u_t + Au = f(x,t), \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\alpha u(x,0) + u(x,T) = g(x), \quad \text{in } \Omega, \tag{3.2}$$

where A is a linear, time independent, symmetric positive definite differential operator acting on sufficiently differentiable functions that satisfy appropriate homogeneous boundary conditions. We also assume that g satisfies an appropriate compatibility condition on $\partial\Omega$ and that a solution to (3.1)–(3.2) exists. As in Chapter 2, we will use the L_2 inner product. We shall determine a bound for the energy expression given by,

$$E(t) = \|u(t)\|^2 = \int_\Omega u^2(x,t)\,dx \tag{3.3}$$

from which the uniqueness of the solution and continuous dependence on the data can be deduced. We begin with differentiating (3.3). This yields,

$$\frac{dE}{dt} = 2\int_\Omega u\,u_t\;dx = 2\,\langle u, f - Au\rangle\,.$$

Since A is positive definite, we can write

$$\langle Au, u\rangle \geq \lambda\,\|u\|^2\,,$$

for $\lambda > 0$. Thus, we have

$$\begin{aligned}\frac{dE}{dt} &\leq 2\,\langle u, f\rangle - 2\lambda\,\|u\|^2 \\ &= 2\,\langle u, f\rangle - 2\lambda E.\end{aligned} \tag{3.4}$$

Applying Cauchy–Schwarz Inequality and Cauchy's Inequality with ϵ, we obtain,

$$\frac{dE}{dt} + 2\lambda E \leq 2\left(\epsilon \|u\|^2 + \frac{1}{4\epsilon}\|f\|^2\right),$$

$$\frac{dE}{dt} \leq -2(\lambda - \epsilon)E + \frac{1}{2\epsilon}\|f\|^2 \tag{3.5}$$

where ϵ is an arbitrarily chosen constant satisfying $0 < \epsilon < \lambda$. Solving this differential inequality over the interval $[t_1, t]$, we get

$$E(t) \leq e^{-2(\lambda-\epsilon)(t-t_1)}E(t_1) + \frac{1}{2\epsilon}\int_{t_1}^{t}\|f\|^2 e^{-2(\lambda-\epsilon)(t-\eta)}\, d\eta$$

that is,

$$\|u(t)\|^2 \leq \|u(t_1)\|^2 e^{-2(\lambda-\epsilon)(t-t_1)} + \frac{1}{2\epsilon}\int_{t_1}^{t}\|f\|^2 e^{-2(\lambda-\epsilon)(t-\eta)}\, d\eta \tag{3.6}$$

where $0 \leq t_1 \leq t \leq T$. We note that if

$$\Phi(t) = \|u(t)\| = \left[\int_\Omega u^2(x,t)\, dx\right]^{1/2},$$

then

$$\frac{d\Phi}{dt} = \frac{\int_\Omega u\, u_t\, dx}{\left[\int_\Omega u^2(x,t)\, dx\right]^{1/2}} = \frac{\langle u, f\rangle - \langle u, Au\rangle}{\Phi}$$

$$\leq \frac{\langle u, f\rangle}{\Phi} - \lambda\frac{\|u\|^2}{\Phi}.$$

Thus we have

$$\frac{d\Phi}{dt} + \lambda\Phi \leq \frac{\langle u, f\rangle}{\Phi}$$

and hence we obtain

$$\frac{d\Phi}{dt} + \lambda\Phi \leq \left[\int_\Omega f^2\, dx\right]^{1/2}$$

by using Cauchy–Schwarz Inequality. The solution here is,

$$\Phi(t) \leq e^{-\lambda t}\Phi(0) + \int_0^t e^{-\lambda(t-\eta)} \|f\| \, d\eta$$

or

$$\|u(t)\| \leq e^{-\lambda t} \|u(0)\| + \int_0^t e^{-\lambda(t-\eta)} \|f\| \, d\eta \tag{3.7}$$

for $0 \leq t \leq T$. We now set

$$v(\eta) = u(2t - \eta), \qquad 0 < t < T/2 \tag{3.8}$$

where we suppress the spatial variable. Using this in (3.1) we get,

$$v_\eta - Av = -f(2t - \eta).$$

Now consider the integral,

$$\int_0^t [\langle v, u_\eta + Au\rangle + \langle u, v_\eta - Av\rangle] \, d\eta$$

Using (3.8) we can write,

$$\int_0^t [\langle v, u_\eta + Au\rangle + \langle u, v_\eta - Av\rangle] \, d\eta = \int_0^t [\langle u(2t-\eta), f(\eta)\rangle - \langle u(\eta), f(2t-\eta)\rangle] \, d\eta. \tag{3.9}$$

One can easily see that the integral on the left hand side of (3.9) is $\langle v, u\rangle \,|_0^t$. Letting $2t - \eta = \sigma$, we can rewrite (3.9) as,

$$\langle v, u\rangle \,|_0^t = \int_t^{2t} \langle u(\sigma), f(2t-\sigma)\rangle \, d\sigma - \int_0^t \langle u(\eta), f(2t-\eta)\rangle \, d\eta$$

or

$$\langle v(t), u(t)\rangle = \langle v(0), u(0)\rangle + \int_t^{2t} \langle u(\sigma), f(2t-\sigma)\rangle \, d\sigma - \int_0^t \langle u(\eta), f(2t-\eta)\rangle \, d\eta.$$

From (3.8) we have $v(t) = u(t)$ and $v(0) = u(2t)$. Substituting these into above equation and using Cauchy–Schwarz Inequalities, we find

$$\|u(t)\|^2 \leq \langle u(2t), u(0)\rangle + \int_t^{2t} \|u\| \, \|f\| \ d\eta + \int_0^t \|u\| \, \|f\| \ d\eta$$

and

$$\|u(t)\|^2 \leq \langle u(2t), u(0)\rangle + \int_0^{2t} \|u\| \, \|f\| \ d\eta \tag{3.10}$$

for $0 \leq t \leq T/2$. Evaluating (3.10) at $t = T/2$ yields

$$\|u(T/2)\|^2 \leq \langle u(T), u(0)\rangle + \int_0^{T} \|u\| \, \|f\| \ d\eta. \tag{3.11}$$

We now apply (3.6) on the interval $[T/2, T]$, so that

$$\|u(T)\|^2 \leq \|u(T/2)\|^2 \, e^{-(\lambda-\epsilon)T} + F_1(\|f\|^2), \tag{3.12}$$

where

$$F_1(\|f\|^2) = \frac{1}{2\epsilon} \int_{T/2}^{T} \|f\|^2 \, e^{-2(\lambda-\epsilon)(T-\eta)} \ d\eta. \tag{3.13}$$

Now substitute (3.11) into (3.12) so that

$$\|u(T)\|^2 \leq \left(\langle u(T), u(0)\rangle + \int_0^{T} \|u\| \, \|f\| \ d\eta \right) e^{-(\lambda-\epsilon)T} + F_1(\|f\|^2).$$

Using (3.7) in the above inequality, we get

$$\begin{aligned}
\|u(T)\|^2 &\leq (\langle u(T), u(0)\rangle) \, e^{-(\lambda-\epsilon)T} \\
&\quad + \int_0^{T} \|f\| \left\{ \|u(0)\| \, e^{-\lambda\sigma} + \int_0^{\sigma} e^{-\lambda(\sigma-\eta)} \, \|f\| \ d\eta \right\} \ d\sigma \, e^{-(\lambda-\epsilon)T} + F_1(\|f\|^2) \\
&\leq (\langle u(T), u(0)\rangle) \, e^{-(\lambda-\epsilon)T} \\
&\quad + \|u(0)\| \int_0^{T} \|f\| \, e^{-\lambda\sigma} \ d\sigma \, e^{-(\lambda-\epsilon)T} + F_1(\|f\|^2) + F_2(\|f\|^2)
\end{aligned} \tag{3.14}$$

where

$$F_2(\|f\|^2) = \int_0^T \|f(\sigma)\| \int_0^\sigma \|f(\eta)\| \, e^{-\lambda(\sigma-\eta)} \, d\eta \, d\sigma \, e^{-(\lambda-\epsilon)T}. \tag{3.15}$$

In order to complete our bound for $\|u(t)\|^2$, we seek a bound for $\|u(0)\|$ in (3.6) when $t_1 = 0$. To establish this, we use the auxiliary condition (3.2) in (3.14). That is, we replace $u(T)$ in (3.14) by $g - \alpha u(0)$ to have

$$\|g - \alpha u(0)\|^2 \leq \langle g - \alpha u(0), u(0)\rangle \, e^{-(\lambda-\epsilon)T} + \|u(0)\| \int_0^T \|f\| \, e^{-\lambda\sigma} \, d\sigma e^{-(\lambda-\epsilon)T} + F_1(\|f\|^2) + F_2(\|f\|^2).$$

On collecting terms we get

$$\|g\|^2 + \left(\alpha^2 + \alpha e^{-(\lambda-\epsilon)T}\right) \|u(0)\|^2 \leq \left(2\alpha + e^{-(\lambda-\epsilon)T}\right) \langle g, u(0)\rangle + \|u(0)\| \int_0^T \|f\| \, e^{-\lambda\sigma} \, d\sigma e^{-(\lambda-\epsilon)T} + F_1(\|f\|^2) + F_2(\|f\|^2),$$

and finally using Cauchy–Schwarz Inequality we obtain,

$$\left\{\alpha^2 + \alpha e^{-(\lambda-\epsilon)T}\right\} \|u(0)\|^2 + \|g\|^2 \leq \left\{ \left|2\alpha + e^{-(\lambda-\epsilon)T}\right| \|g\| + \int_0^T \|f\| \, e^{-\lambda\sigma} \, d\sigma e^{-(\lambda-\epsilon)T} \right\} \|u(0)\| + F_1(\|f\|^2) + F_2(\|f\|^2).$$

Hence we get a quadratic inequality. We can solve this quadratic inequality to find a bound for $\|u(0)\|$ in terms of data. In fact, we can write,

$$\|u(0)\| \leq \frac{B + \sqrt{B^2 + 4DC}}{2D}$$

where

$$
\begin{aligned}
B &= \left|2\alpha + e^{-(\lambda-\epsilon)T}\right| \|g\| + \int_0^T \|f\| \, e^{-\lambda\sigma} \, d\sigma e^{-(\lambda-\epsilon)T}, \\
D &= \alpha^2 + \alpha e^{-(\lambda-\epsilon)T}, \\
C &= F_1(\|f\|^2) + F_2(\|f\|^2) - \|g\|^2 .
\end{aligned}
$$

Consequently from (3.6) we have the energy bound as,

$$
E(t) = \|u(t)\|^2 \leq \left[\frac{B + \sqrt{B^2 + 4DC}}{2D}\right]^2 e^{-2(\lambda-\epsilon)t} + \frac{1}{2\epsilon} \int_0^t \|f\|^2 \, e^{-2(\lambda-\epsilon)(t-\eta)} \, d\eta \tag{3.16}
$$

for $0 \leq t \leq T$. We note that the energy inequality we obtain is valid for all $\alpha > 0$ and for $\alpha < 0$ provided $|\alpha| > e^{-(\lambda-\epsilon)T}$.

The interval $-e^{-\lambda T} \leq \alpha \leq 0$ is not considered in the above analysis since (3.1) may not have a solution or the solution may not be unique in this restricted interval. To investigate this, we consider the homegeneous form of problem (3.1), i.e. $f(x,t) = 0$, and assume A is such that there is a complete eigenspace. Let γ_n and w_n denote the eigenvalues and eigenfunctions, respectively, of the eigenvalue problem

$$
\begin{aligned}
Aw - \gamma w &= 0 \quad \text{in}\, \Omega, \\
w &= 0 \quad \text{on}\, \partial\Omega.
\end{aligned}
$$

We can easily obtain that the series solution of (3.1) with $f(x,t) = 0$, is

$$
u(x,t) = \sum_{n=1}^{\infty} c_n e^{\gamma_n t} w_n(x).
$$

Putting $t = 0$ and $t = T$ in this series, we find that the auxiliary condition becomes

$$
\alpha \sum_{n=1}^{\infty} c_n w_n(x) + \sum_{n=1}^{\infty} c_n e^{-\gamma_n T} w_n(x) = g(x)
$$

or

$$\sum_{n=1}^{\infty} c_n(\alpha + e^{-\gamma_n T}) w_n(x) = g(x).$$

Using orthonormality of the eigenfunctions and letting $\langle g(x), w_n(x) \rangle = g_n$, the Fourier coefficient of $g(x)$, we determine the constants c_n from the equation,

$$c_n(\alpha + e^{-\gamma_n T}) = g_n.$$

Thus, a unique solution exists if $\alpha + e^{-\gamma_n T} \neq 0$ for each n, but there is no solution if $g_n \neq 0$ and $\alpha + e^{-\gamma_n T} = 0$ for some n, and a non–unique solution if $g_n = 0$ and $\alpha + e^{-\gamma_n T} = 0$ for some n.

In the case of the homogeneous form of (3.1), the computation of the energy bound is much more simpler. In this case, in place of (3.4) we have

$$\frac{dE}{dt} \leq -2\lambda E,$$

so that the solution is of the form,

$$E(t) \leq E(t_1) e^{-2\lambda(t-t_1)}.$$

Thus, in place of (3.6), (3.7), (3.11), (3.12) we have,

$$\|u(t)\|^2 \leq \|u(t_1)\|^2 e^{-2\lambda(t-t_1)} \quad 0 \leq t_1 \leq t \leq T \tag{3.17}$$

$$\|u(t)\| \leq e^{-\lambda t} \|u(0)\| \quad 0 \leq t \leq T \tag{3.18}$$

$$\|u(T/2)\|^2 \leq \langle u(T), u(0) \rangle \tag{3.19}$$

$$\|u(T)\|^2 \leq \left\| u(\frac{1}{2}T) \right\|^2 e^{-\lambda T} \tag{3.20}$$

respectively. Thus, (3.19) and (3.20) implies

$$\|u(T)\|^2 \leq \langle u(T), u(0) \rangle e^{-\lambda T}.$$

Writing $u(T) = g - \alpha u(0)$ in the above equation, we get

$$\begin{aligned}\|g\|^2 - 2\alpha \langle g, u(0)\rangle + \alpha^2 \|u(0)\|^2 &\le \langle (g - \alpha u(0)), u(0)\rangle e^{-\lambda T} \\ &= \langle g, u(0)\rangle e^{-\lambda T} - \alpha e^{-\lambda T} \|u(0)\|^2\end{aligned}$$

Consequently, on combining terms and using Cauchy–Schwarz Inequality we have

$$\left(\alpha^2 + \alpha e^{-\lambda T}\right) \|u(0)\|^2 + \|g\|^2 \le \left|2\alpha + e^{-\lambda T}\right| \|g\| \|u(0)\| . \tag{3.21}$$

Hence, we again have a quadratic inequality. Solving this inequality for $\|u(0)\|$, we obtain

$$\|u(0)\| \le \left[\frac{\left|2\alpha + e^{-\lambda T}\right| + e^{-\lambda T}}{2\left(\alpha^2 + \alpha e^{-\lambda T}\right)}\right] \|g\| ,$$

and finally from (3.17) we get the energy bound as

$$\|u(t)\|^2 \le \left[\frac{\left|2\alpha + e^{-\lambda T}\right| + e^{-\lambda T}}{2\left(\alpha^2 + \alpha e^{-\lambda T}\right)}\right]^2 \|g\|^2 e^{-2\lambda t} \tag{3.22}$$

for $0 \le t \le T$ when $\alpha^2 + \alpha e^{-\lambda T} > 0$.

When $\alpha > 0$, the term in brackets in (3.22) becomes $\dfrac{1}{\alpha^2}$, and we get a simpler bound:

$$\|u(t)\|^2 \le \frac{1}{\alpha^2} \|g\|^2 e^{-2\lambda t}, \quad 0 \le t \le T. \tag{3.23}$$

Similarly, if $\alpha < -e^{-\lambda T}$, from (3.22) we obtain,

$$\|u(t)\|^2 \le \frac{1}{\left[\alpha + e^{-\lambda T}\right]^2} \|g\|^2 e^{-2\lambda t}, \quad 0 \le t \le T. \tag{3.24}$$

3.3 Pointwise Bounds

In this section we will consider the nonstandard problem (3.1)–(3.2) when the operator A is taken as the negative of the Laplace operator Δ.

That is, we have

$$\begin{aligned} u_t - \Delta u = 0 \quad &\text{in } \ \Omega \times (0,T), \\ u = 0 \quad &\text{on } \ \partial\Omega \times (0,T), \\ \alpha u(x,0) + u(x,T) = g(x) \quad &\text{in } \ \Omega, \end{aligned} \tag{3.25}$$

where Ω is a convex domain in R^N with $C^{2+\beta}$ boundary and $g(x) \neq 0$. Assuming a solution exists, we will derive pointwise bounds for the solution and its gradient by using maximum principle argument. To this end, we set,

$$\Psi(x,t) = \left[|\nabla u|^2 + au^2\right] e^{2at} \tag{3.26}$$

where a is a positive constant and ∇u denotes the gradient of u. In [20] Payne and Philippin derived a maximum principle for $\Psi(x,t)$. We shall give theorems that are discussed in [20] and needed for the determination of the pointwise bound for the solution and its gradient.

Theorem 3.3.1. *Let $u(x,t)$ be a classical solution of*

$$u_t - \Delta u = 0 \quad x \in \Omega, \ t > 0$$

in a bounded region $\Omega \subset R^N$ for some interval $(0,T)$. Then $\Psi(x,t)$ defined in (3.26) takes its maximum value either at a boundary point $(\hat{x},\hat{t})$ with $\hat{x} \in \partial\Omega$, $0 \leq \hat{t} \leq T$, or at an interior critical point $(\bar{x},\bar{t})$ of u with $\bar{x} \in \Omega$, $0 \leq \bar{t} \leq T$ or initially, at a point $(\tilde{x},0)$ with $\tilde{x} \in \Omega$.
i.e., we have

$$\Psi(x,t) \leq \max \begin{cases} \Psi(\hat{x},\hat{t}), & \hat{x} \in \partial\Omega, \ 0 \leq \hat{t} \leq T, \\ \Psi(\bar{x},\bar{t}), & \text{with } \nabla u(\bar{x},\bar{t}) = 0, \\ \max\limits_{x\in\Omega} \left\{|\nabla h|^2 + ah^2\right\}, \end{cases} \tag{3.27}$$

with $u(x,0) = h(x)$.

To establish (3.27) they showed that Ψ satisfies the parabolic differential inequality

$$\Delta\Psi + |\nabla u|^{-2} W_k \Psi_{,k} - \Psi_t \geq 0, \quad \text{in } \Omega \times (0,T)$$

where W_k is the k^{th} component of a vector field which is regular throughout $\Omega \times (0,T)$. $\Psi_{,k}$ denotes the spatial differentiation with respect to x_k, $(k \neq t)$, and the repeated index denotes summation from 1 to N. It follows then from the standard maximum principle for parabolic equations that (3.27) holds.

Theorem 3.3.2. *Let $u(x,t)$ be a classical solution of*

$$u_t - \Delta u = 0 \quad x \in \Omega, t > 0$$

and let Ψ defined by (3.26). If Ω is a convex domain with $C^{2+\beta}$ boundary $\partial\Omega$, and if $u(x,t)$ satisfies either

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{3.28}$$

or

$$\frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, t > 0, \tag{3.29}$$

where $\frac{\partial u}{\partial n}$ is the normal derivative directed outward from Ω, then Ψ cannot take its maximum value at a point of $\partial\Omega$, unless $u(x,0) = h \equiv 0$.

For the proof of the above theorem they assumed in contradiction to the conclusion of the theorem that Ψ takes its maximum value at a point (x^*, t^*), with $x^* \in \partial\Omega$, and $0 < t^* \leq T$. Then they showed by the use of the maximum principle on the boundary [18] that this is impossible.

Theorem 3.3.3. *Let $u(x,t)$ be a classical solution of*

$$u_t - \Delta u = 0 \quad x \in \Omega, t > 0$$

with

$$u(x,0) = h(x), \quad x \in \Omega$$
$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0$$

defined on a bounded domain $\Omega \subset R^N$. *Then the function* Ψ *defined in (3.26) cannot take its maximum value at a critical point of* u *provided that,*

$$0 \leq a \leq \frac{\pi^2}{4d^2}$$

where d *is the radius of the largest ball contained in* Ω. *Furthermore if* Ω *is convex with* $C^{2+\beta}$ *boundary, we then have*

$$|\nabla u|^2 + \frac{\pi^2}{4d^2}u^2 \leq G^2\, e^{\frac{-\pi^2 t}{2d^2}}$$

with

$$G^2 := \max_{\Omega}\left\{|\nabla h|^2 + \frac{\pi^2}{4d^2}h^2\right\}$$

Now making use of the above theorems we will examine the function Ψ defined in (3.26).

From Theorem 3.3.1, Ψ takes its maximum value either,

1. at a point P on $\partial\Omega$ when $t = t^*$ for $0 \leq t^* \leq T$.

2. at an interior point Q where $\nabla u = 0$, or

3. in $\bar{\Omega}$ when $t = 0$.

By Theorem 3.3.2, Ψ cannot take its maximum value at a point of $\partial\Omega$, since $g(x) \neq 0$. (In fact, $g(x) \neq 0$ implies $u(x,0) \neq 0$. On the contrary, if we assume $u(x,0) = 0$, then the nonstandard problem will be converted into the standard final value problem, which we don't deal with.)

By Theorem 3.3.3, Ψ cannot take its maximum value at an interior point Q where $\nabla u = 0$ when $0 \leq a \leq \frac{\Pi^2}{4d^2}$. Consequently Ψ takes its maximum initially when a

is so chosen, i.e.,

$$|\nabla u(x,t)|^2 + au^2(x,t) \le \left[|\nabla u(x,0)|^2 + au^2(x,0)\right]_M e^{-2at} \tag{3.30}$$

where $[.]_M$ denotes the maximum over $\bar{\Omega}$. We now evaluate (3.30) at $t = T$ so that

$$\left[|\nabla u(T)|^2 + au^2(T)\right]_M \le \left[|\nabla u(0)|^2 + au^2(0)\right]_M e^{-2aT}. \tag{3.31}$$

From the auxiliary condition in (3.32) we can write

$$\begin{aligned} |\alpha u(0)|^2 &\le (|u(T)| + |g|)^2 \\ &\le 2\left(|u(T)|^2 + |g|^2\right) \end{aligned}$$

by means of Cauchy Schwarz Inequality. Similarly, we have

$$|\alpha \nabla u(0)|^2 \le 2\left(|\nabla u(T)|^2 + |\nabla g|^2\right).$$

Now, combining these inequalities

$$|\alpha|^2 \left[|\nabla u(0)|^2 + au(0)^2\right]_M \quad \le \quad 2\left[\left(|\nabla u(T)|^2 + au^2(T)\right) + \left(|\nabla g|^2 + ag^2\right)\right]_M$$

and taking the square root of both sides, we obtain

$$|\alpha| \left[|\nabla u(0)|^2 + au(0)^2\right]_M^{1/2} \le \sqrt{2}\left[\left(|\nabla u(T)|^2 + au^2(T)\right)_M^{1/2} + \left(|\nabla g|^2 + ag^2\right)_M^{1/2}\right]. \tag{3.32}$$

From (3.31) and (3.32) we get

$$\begin{aligned} |\alpha| \left[|\nabla u(0)|^2 + au^2(0)\right]_M^{1/2} - \sqrt{2}\left[|\nabla g|^2 + ag^2\right]_M^{1/2} \\ \le \sqrt{2}\left[|\nabla u(0)|^2 + au^2(0)\right]_M^{1/2} e^{-aT} \end{aligned}$$

which leads to the bound

$$\left[|\nabla u(0)|^2 + au^2(0)\right] \le \frac{2\left[|\nabla g|^2 + ag^2\right]_M}{\left[|\alpha| - \sqrt{2}e^{-aT}\right]^2}$$

provided $|\alpha| > \sqrt{2}e^{-aT}$. Finally from (3.30), we have

$$|\nabla u(x,t)|^2 + au^2(x,t) \leq \frac{2\left[|\nabla g|^2 + ag^2\right]_M}{\left[|\alpha| - \sqrt{2}e^{-aT}\right]^2} e^{-2at},$$

and letting $a \to \pi^2/4d^2$, we obtain the pointwise bound

$$|\nabla u(x,t)|^2 + (\pi^2/4d^2)u^2(x,t) \leq \frac{2\left[|\nabla g|^2 + (\pi^2/4d^2)g^2\right]_M}{\left[|\alpha| - \sqrt{2}e^{-\pi^2 T/4d^2}\right]^2} e^{-(\pi^2/2d^2)t}. \tag{3.33}$$

The above result can be summarized in the following theorem.

Theorem 3.3.4. *If u is a solution of (3.25), where Ω is a convex domain with $C^{2+\beta}$ boundary, $g(x) \neq 0$ and $|\alpha| > \sqrt{2}\exp(-\pi^2 T/4d^2)$, with d the radius of the largest ball contained in Ω, then u and its gradient ∇u satisfy the bound (3.33).*

We can also obtain a pointwise bound for the solution $u(x,t)$ of (3.25) using the Green's function. Let G be the Green's function for the heat operator under Dirichlet boundary conditions. From the integral representation (Theorem 1.2.4), we have

$$u(x,t) = \int_\Omega G(x,t;\xi,0)u(\xi,0)\,d\xi \tag{3.34}$$

from which we can write

$$u(x,t) = \int_\Omega \left\{\frac{u(\xi,0)}{w}\right\} wG(x,t;\xi,0)\,d\xi, \tag{3.35}$$

where w is the first eigenfunction of the fixed membrane problem

$$\begin{aligned} \Delta w + \lambda w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \\ w &> 0 \quad \text{in } \Omega. \end{aligned} \tag{3.36}$$

From (3.35) we may write,

$$u(x,t) \leq \sup_\Omega \left\{\frac{u(x,0)}{w}\right\} \int_\Omega wG(x,t;\xi,0)\,d\xi.$$

We observe that the last integral in the above inequality is the solution of (3.25) with $u(x,0) = w$, i.e.,

$$\int_{\Omega} wG(x,t;\xi,0)\,d\xi = w\,e^{-\lambda t}.$$

Thus we have,

$$|u(x,t)| \leq \sup_{\Omega}\left\{\frac{|u(x,0)|}{w}\right\} w\,e^{-\lambda t} \tag{3.37}$$

and it follows for $t = T$ that,

$$\sup_{\Omega}\frac{|u(x,T)|}{w} \leq \sup_{\Omega}\frac{|u(x,0)|}{w}\,e^{-\lambda T}. \tag{3.38}$$

From the auxiliary condition in (3.25), we have

$$|\alpha|\sup_{\Omega}\frac{|u(x,0)|}{w} \leq \sup_{\Omega}\frac{|u(x,T)|}{w} + \sup_{\Omega}\frac{|g(x)|}{w}. \tag{3.39}$$

Substituting (3.38) into (3.39), we get

$$|\alpha|\sup_{\Omega}\frac{|u(x,0)|}{w} \leq \sup_{\Omega}\frac{|u(x,0)|}{w}\,e^{-\lambda T} + \sup_{\Omega}\frac{|g(x)|}{w}$$

which yields the bound

$$\sup_{\Omega}\frac{|u(x,0)|}{w} \leq \frac{1}{(|\alpha| - e^{-\lambda T})}\,\sup_{\Omega}\frac{|g(x)|}{w}$$

provided $|\alpha| > e^{-\lambda T}$. Consequently, from (3.37) we obtain the pointwise bound

$$|u(x,t)| \leq \frac{1}{(|\alpha| - e^{-\lambda T})}\,\sup_{\Omega}\left\{\frac{|g(x)|}{w}\right\} we^{-\lambda t} \tag{3.40}$$

for $0 \leq t \leq T$ when $|\alpha| > e^{-\lambda T}$.

CHAPTER 4

STRUCTURAL STABILITY

4.1 Introduction

Given a problem, the concept of structural stability is the continuous dependence of solutions on changes in the model itself rather than on the initial data. In other words, small perturbations in the equation itself or in the coefficients of partial differential equation causes a small change in the solution. For example, Payne and Straughan [17] have shown how a solution for the linearized equations for the flow of a Maxwell viscoelastic fluid behaves under changes in the relaxation parameter (see also [18] and [19]). Similar to their work, we investigate the structural stability for a nonstandard problem.

4.2 Continuous Dependence on the Parameter

In this section we shall again consider the damped equation

$$u_{tt} + \lambda u_t + Au = 0 \text{ in } \Omega \times (0, t) \tag{4.1}$$

and we will try to derive the continuous dependence result on λ when the nonstandard auxiliary conditions

$$\alpha u(0) + u(T) = g \text{ in } \Omega \tag{4.2}$$

$$\beta u_t(0) + u_t(T) = h \text{ in } \Omega \tag{4.3}$$

are given.

Now let u and v be solutions to (4.1)–(4.3) for different values λ_1 and λ_2, respectively. Thus u and v satisfy the problems

$$\begin{aligned} u_{tt} + \lambda_1 u_t + Au = 0 \quad & \text{in } \Omega \times (0, t), \\ \alpha u(0) + u(T) = g \quad & \text{in } \Omega, \\ \beta u_t(0) + u_t(T) = h \quad & \text{in } \Omega. \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} v_{tt} + \lambda_2 v_t + Av = 0 \quad & \text{in } \Omega \times (0, t), \\ \alpha v(0) + v(T) = g \quad & \text{in } \Omega, \\ \beta v_t(0) + v_t(T) = h \quad & \text{in } \Omega. \end{aligned} \tag{4.5}$$

for prescribed data functions g and h.

To study continuous dependence on λ, we define $w = u - v$. Then w satisfies the problem

$$\begin{aligned} w_{tt} + \lambda_1 w_t + (\lambda_1 - \lambda_2) v_t + Aw = 0, \\ \alpha w(0) + w(T) = 0, \\ \beta w_t(0) + w_t(T) = 0, \end{aligned} \tag{4.6}$$

or equivalently it maybe convenient to rewrite (4.6) in the form,

$$\begin{aligned} w_{tt} + (\lambda_1 - \lambda_2) u_t + \lambda_2 w_t + Aw = 0, \\ \alpha w(0) + w(T) = 0, \\ \beta w_t(0) + w_t(T) = 0. \end{aligned} \tag{4.7}$$

To derive the continuous dependence estimate, we multiply the differential equation in (4.6) by $2w_t$ and integrate over $\Omega \times (0, t)$.

$$\|w_t(t)\|^2 + 2\lambda_1 \int_0^t \|w_\eta\|^2 \, d\eta + \langle Aw(t), w(t) \rangle =$$
$$\|w_t(0)\|^2 + \langle Aw(0), w(0) \rangle - 2(\lambda_1 - \lambda_2) \int_\Omega \int_0^t w_\eta v_\eta \, d\eta \, dx.$$

Writing

$$E(t) = \|w_t(t)\|^2 + \langle Aw(t), w(t)\rangle$$

and taking the absolute value of both side we get

$$E(t) + 2\lambda_1 \int_0^t \|w_\eta\|^2 \leq E(0) + 2\int_0^t |\langle w_\eta, (\lambda_1 - \lambda_2)v_\eta\rangle| \; d\eta.$$

Now using Cauchy Schwarz Inequality and Cauchy's Inequality with ϵ, respectively, we have

$$\begin{aligned} E(t) + 2\lambda_1 \int_0^t \|w_\eta\|^2 \; d\eta &\leq E(0) + 2\int_0^t \|w_\eta\| \, \|(\lambda_1 - \lambda_2)v_\eta\| \; d\eta \\ &\leq E(0) + \theta_1 \int_0^t \|w_\eta\|^2 + \frac{(\lambda_1 - \lambda_2)^2}{\theta_1} \int_0^t \|v_\eta\|^2 \; d\eta \end{aligned}$$

from which we finally obtain

$$E(t) + (2\lambda_1 - \theta_1) \int_0^t \|w_\eta\|^2 \; d\eta \leq E(0) + \frac{(\lambda_1 - \lambda_2)^2}{\theta_1} \int_0^t \|v_\eta\|^2 \; d\eta \qquad (4.8)$$

for positive constant θ_1. To estimate the integral on the right hand side of (4.8), we multiply (4.5) by $2v_t$ and integrate over $\Omega \times (0, t)$. So we have,

$$\|v_t(t)\|^2 + 2\lambda_2 \int_0^t \|v_\eta\|^2 \; d\eta + \langle Av(t), v(t)\rangle = \|v_t(0)\|^2 + \langle Av(0), v(0)\rangle$$

from which we can write

$$2\lambda_2 \int_0^t \|v_\eta\|^2 \; d\eta \leq \|v_t(0)\|^2 + \langle Av(0), v(0)\rangle \,. \qquad (4.9)$$

Now we have to bound the right hand side of (4.9). But in Section 2.5 we have already deduced

$$\|v_t(0)\|^2 + \langle Av(0), v(0)\rangle \leq D_1 \|h\|^2 + D_2 \langle Ag, g\rangle$$

for computable constants D_1 and D_2. Hence, using this bound in (4.9) we get

$$\int_0^t \|v_\eta\|^2 \, d\eta \leq \frac{1}{2\lambda_2}\left(D_1 \|h\|^2 + D_2 \langle Ag, g\rangle\right). \tag{4.10}$$

Substituting (4.10) into (4.8) we obtain,

$$E(t)+(2\lambda_1-\theta_1)\int_0^t \|w_\eta\|^2 \, d\eta \leq E(0)+\frac{(\lambda_1-\lambda_2)^2}{2\lambda_2}\left(D_3 \|h\|^2 + D_4 \langle Ag, g\rangle\right) \tag{4.11}$$

for $D_3 = \frac{D_1}{\theta_1}, D_4 = \frac{D_2}{\theta_1} > 0$. This time, we seek a bound for $E(0)$. To accomplish this, we evaluate (4.11) at $t = T$ and by means of the auxiliary condition in (4.6) we substitute

$$E(T) = \beta^2 \|w_t(0)\|^2 + \alpha^2 \langle Aw(0), w(0)\rangle$$

so that

$$\begin{aligned}\left(\beta^2 - 1\right)\|w_t(0)\|^2 + (2\lambda_1 - \theta_1)\int_0^T \|w_\eta\|^2 \, d\eta + \left(\alpha^2 - 1\right)\langle Aw(0), w(0)\rangle \\ \leq \frac{(\lambda_1-\lambda_2)^2}{2\lambda_2}\left(D_3 \|h\|^2 + D_4 \langle Ag, g\rangle\right).\end{aligned}$$

Choosing θ_1 sufficiently small, dropping the integral term and letting

$$\min\left\{\beta^2 - 1, \alpha^2 - 1\right\} = \gamma, \quad \gamma > 0$$

we can write

$$\begin{aligned}E(0) = \|w_t(0)\|^2 + \langle Aw(0), w(0)\rangle &\leq \frac{1}{\gamma}\frac{(\lambda_1-\lambda_2)^2}{2\lambda_2}\left(D_3 \|h\|^2 + D_4 \langle Ag, g\rangle\right) \\ &= \frac{(\lambda_1-\lambda_2)^2}{2\lambda_2}\left(D_5 \|h\|^2 + D_6 \langle Ag, g\rangle\right), \quad (4.12)\end{aligned}$$

for $D_5 = \frac{D_3}{\gamma} > 0$ and $D_6 = \frac{D_4}{\gamma} > 0$. Finally substituting (4.12) into (4.11) we obtain

$$E(t) + (2\lambda_1 - \theta_1) \int_0^t \|w_\eta\|^2 \, d\eta \leq$$
$$\frac{(\lambda_1 - \lambda_2)^2}{2\lambda_2} \left(D_3 \|h\|^2 + D_4 \langle Ag, g \rangle + D_5 \|h\|^2 + D_6 \langle Ag, g \rangle \right)$$

i.e.,

$$E(t) + (2\lambda_1 - \theta_1) \int_0^t \|w_\eta\|^2 \, d\eta \leq \frac{(\lambda_1 - \lambda_2)^2}{2\lambda_2} \left[D_7 \|h\|^2 + D_8 \langle Ag, g \rangle \right] \quad (4.13)$$

for positive constants D_7 and D_8. A similar argument commencing with (4.7) leads to (4.13) with λ_1 and λ_2 are interchanged. That is,

$$E(t) + (2\lambda_2 - \theta_2) \int_0^t \|w_\eta\|^2 \, d\eta \leq \frac{(\lambda_1 - \lambda_2)^2}{2\lambda_1} \left[D_9 \|h\|^2 + D_{10} \langle Ag, g \rangle \right] \quad (4.14)$$

Thus adding (4.13) and (4.14), we may easily see that

$$2 \|w_t(t)\|^2 + [2(\lambda_1 + \lambda_2) - \theta] \int_0^t \|w_\eta\|^2 \, d\eta + 2 \langle Aw(t), w(t) \rangle$$
$$\leq \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \left(D_{11} \|h\|^2 + D_{12} \langle Ag, g \rangle \right), \quad (4.15)$$

for computable constants D_{11} and D_{12} and $\theta = \theta_1 + \theta_2$. Consequently, estimate (4.15) establishes continuous dependence on λ provided θ is chosen so that $2(\lambda_1 + \lambda_2) - \theta > 0$.

We can summarize the above results in the following theorem:

Theorem 4.2.1. *The solution of the problem (4.1)–(4.3) depends continuously on the coefficient λ.*

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