FRACTURE OF A THREE LAYER ELASTIC PANEL

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ABSTRACT

FRACTURE OF A THREE LAYER ELASTIC PANEL

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The panel is symmetrical about both x- and y- axes. The central strip (strip1) of width $2h_1$ contains a central transverse crack of width 2a on x-axis. The two strips (strip2) contain transverse cracks of width c-b also on x-axis. The panel is subjected to axial loads with uniform intensities p_1 and p_2 in strip1 and strip2 , respectively at $y = \pm\infty$. Materials of all strips are assumed to be linearly elastic and isotropic. Due to double symmetry, only one quarter of the problem $(0 \le x \le \infty$ and $0 \le y \le \infty)$ will be considered.

The solutions are obtained by using Fourier transforms both in x and y-directions. Summing several solutions is due to the necessity for sufficient number of unknowns in general expressions in order to be able to satisfy all boundary conditions of the problem. The conditions at the edges of the strips and at the interfaces are satisfied and the general expressions for a three layer panel become expressions for the panel with free edges. Use of remaining boundary conditions leads the formulation to a system of two singular integral equations. These equations are converted to a system of linear algebraic equations which is solved numerically

Keywords :Crack, Fracture , Stress Intensity Factor, Singular Integral Equations

ÖZ

ÜÇ KATMANLI ELASTİK PANELİN ÇATLAMASI

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Tez yöneticisi : Prof. Dr. M. Ruşen Geçit

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Üç katmanlı , x ve y eksenlerine göre simetrik olan bir panelin ortasında bulunan şerit1 adlı şeritte x ekseni üzerinde ara yüze dik olan bir çatlak ve yanlardaki şerit2 olarak adlandırılan iki yandaki şeritlerde de yine x ekseni üzerinde bulunan ve arayüze dik olan 2 çatlak daha bulunmaktadır. Şeritler y $\pm\infty$ da p_1 ve p_2 değerinde üniform yüklere maruz kalmaktadır. Şerit malzemeleri lineer elastik ve izotrop kabul edilmektedir.

Çözümler hem x hem de y ekseni yönünde Fourier Dönüşümü kullanılarak bulunmaktadır. Yeterli sayıda bilinmeyi sağlamak amacıyla çok sayıda farklı çözümler toplanmaktadır. Önce şeritlerin kenarlarındaki ve ara yüzlerindeki sınır koşulları sağlanmaktadır. Kalan sınır koşullarının kullanımı da iki tekil integral denklemden oluşan bir sistem vermektedir. Bu denklem sistemi sayısal olarak çözülecek olan lineer cebrik denklem sistemine dönüştürülmektedir.

Anahtar Kelimeler : Çatlak, Kırılma, Gerilme Şiddeti Katsayısı, Tekil İntegral Denklemler

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TABLE OF CONTENTS

PLAGIARISM.....iii
ABSTRACT.....iv
ÖZ v
ACKNOWLEDGEMENTS vi
TABLE OF CONTENTS vii
LIST OF TABLES ix
LIST OF FIGURES x
NOMENCLATURE xviii

CHAPTER

1.INTRODUCTION.....1
1.1 LiteratureReview.....1
1.2 A Short Introduction on Methods of Solution of the Problem.....6
2. FORMULATION OF THE PROBLEM.....8
2.1 Introduction.....8
2.2 Perturbation Solution.....12
2.2.1 An Infinte Medium Having Crack.....12
2.2.2 An Infinte Medium Subjected to Arbitrary Symmetric Loads.....14
2.2.3 General Expressions of Perturbation Problem for Strip 1.....15
2.2.4 General Expressions of Perturbation Problem for

Strip 2..16
2.2.5 Boundary and Continuity Conditions..................18
2.3 Integral Equations...21
2.3.1 The Case of Embedded Cracks
($a<h_1$, $0<b$, $c<h_2$)..24
2.3.2 The Case of Broken Middle Strip
($a=h_1$, $0<b$, $c<h_2$)..25
2.3.3 The Case of Crack in Strip 2 Touching the Interface
($a<h_1$, $0=b$,$c<h_2$)...26
2.4 Stress Intensity Factors..27
3. NUMERICAL SOLUTION...31
3.1 Embedded Cracks ($a<h_1$, $0<b$, $c<h_2$)...............................33
3.2 Broken Strip 1 ($a=h_1$, $0<b$, $c<h_2$)....................................34
3.3 Crack in Strip 2 Touching the Interface
($a<h_1$, $0=b$,$c<h_2$)..35
3.4 Calculation of Stress Intensity Factors..................................36
4. RESULTS AND CONCLUSIONS...38
4.1 Results...38
4.2 Suggesions for Further Studies.......................................45

REFERENCES...138
APPENDIX...142
VITA..179

LIST OF TABLES

TABLE

4.1 Comparison of the Results of This Study with
the Results of G.D.Gupta and F.Erdoğan (1974) 38

LIST OF FIGURES

FIGURES

2.1.	Three Layer Panel…………………………………..………………	8
2.2.	Superposition…………………………..……………………...	9
2.3.	Schematic Representation of General Formulation…………..…….	10
2.4	Schematic Representation of Infinite Medium Having a Crack….…	12
4.1.	Comparison of the Results for k_a for an Infinite Strip with a Transverse Central Crack with a/h_1 for 1: aluminum 2: aluminum $\lambda=\mu_1/\mu_2 = 1$, $h_2=h_1$ Plane Strain……………………………………..	46
4.2.	Comparison of the Results of Variation of k_b with b/h_2 for 1:epoxy 2: aluminum $\mu_2/\mu_1 = 23.077$, $h_2=0.35h_1$, $a=0.01h_1$, $c=0.9h_2$, Plain Strain…………………………………………………………	47
4.3.	Comparison of the Results of Variation of k_c with b/h_2 for 1:epoxy 2: aluminum $\mu_2/\mu_1 = 23.077$, $h_2=0.35h_1$, $a=0.01h_1$, $c=0.9h_2$., Plain Strain…………………………………………………………	48
4.4	Comparison of the Results of Variation of k_b with b/h_2 for 1:steel 2: aluminum $\lambda=\mu_1/\mu_2 = 3$, $h_2=0.35h_1$, $a=0.01h_1$, $c=0.9h_2$., Plain Strain…	49
4.5.	Comparison of the Results of Variation of k_c with b/h_2 for 1:steel 2: aluminum $\lambda=\mu_1/\mu_2 = 3$, $h_2=0.35h_1$, $a=0.01h_1$, $c=0.9h_2$., Plain Strain…	50
4.6.	Variation of k_a with a/h_1 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress…………………………………….	51
4.7	Variation of k_b with a/h_1 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress…………………………………………	52
4.8	Variation of k_c with a/h_1 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………	53
4.9.	Variation of k_a with b/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………….	54

4.10 Variation of k_b with b/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………….. 55
4.11. Variation of k_c with b/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………….. 56

4.12. Variation of k_a with c/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $b=0.1h_2$, Plane Stress……………………………………….. 57
4.13. Variation of k_b with c/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $b=0.1h_2$, Plane Stress……………………………………….. 58
4.14. Variation of k_c with c/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $b=0.1h_2$, Plane Stress……………………………………….. 59
4.15. Variation of k_a with a/h_1 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 60
4.16. Variation of k_b with a/h_1 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 61
4.17. Variation of k_c with a/h_1 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 62
4.18. Variation of k_a with b/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 63
4.19. Variation of k_b with b/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 64
4.20. Variation of k_c with b/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 65
4.21. Variation of k_a with c/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 66
4.22. Variation of k_b with c/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 67

4.23. Variation of k_c with c/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 68
4.24. Variation of k_a with a/h_1 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………………… 69
4.25. Variation of k_b with a/h_1 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 70
4.26. Variation of k_c with a/h_1 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………………… 71

4.27. Variation of k_a with b/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 72
4.28. Variation of k_b with b/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 73
4.29. Variation of k_c with b/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 74
4.30. Variation of k_a with c/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 75
4.31. Variation of k_b with c/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 76
4.32. Variation of k_c with c/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 77
4.33. Variation of k_a with a/h_1 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 78
4.34. Variation of k_b with a/h_1 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 79
4.35. Variation of k_c with a/h_1 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 80

4.36. Variation of k_a with b/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 81

4.37. Variation of k_b with b/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 82

4.38. Variation of k_c with b/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, Plane Stress………………………………………………. 83

4.39. Variation of k_a with c/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 84

4.40. Variation of k_b with c/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 85

4.41. Variation of k_c with c/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………………. 86

4.42. Variation of k_a with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………. 87

4.43. Variation of k_b with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………. 88

4.44. Variation of k_c with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………. 89

4.45. Variation of k_a with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………. 90

4.46. Variation of k_b with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………. 91

4.47. Variation of k_c with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, Plane Stress……………………………………. 92

4.48. Variation of k_a with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $b=0.1h_2$, Plane Stress……………………………………. 93

4.49. Variation of k_b with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………. 94
4.50. Variation of k_c with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $b=0.1h_2$, Plane Stress………………………………………. 95
4.51. Variation of k_a with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $c=0.9h_2$, Plane Stress…………………………………… 96
4.52. Variation of k_a with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $c=0.9h_2$, Plane Stress……………………………………………… 97
4.53. Variation of k_a with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $c=0.9h_2$, Plane Stress……………………………………………… 98
4.54. Variation of k_b with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $c=0.9h_2$, Plane Stress…………………………………… 99
4.55. Variation of k_b with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $c=0.9h_2$, Plane Stress……………………………………………… 100
4.56. Variation of k_b with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $c=0.9h_2$, Plane Stress……………………………………………… 101

4.57. Variation of k_c with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $c=0.9h_2$, Plane Stress…………………………………… 102
4.58. Variation of k_c with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $c=0.9h_2$, Plane Stress……………………………………………… 103
4.59. Variation of k_c with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $c=0.9h_2$, Plane Stress……………………………………………… 104
4.60. Variation of k_a with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $b=0.1h_2$, Plane Stress…………………………………… 105
4.61. Variation of k_a with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $b=0.1h_2$, Plane Stress……………………………………………… 106

4.62. Variation of k_a with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $b=0.1h_2$, Plane Stress…… 107

4.63. Variation of k_b with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $b=0.1h_2$, Plane Stress…… 108

4.64. Variation of k_b with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $b=0.1h_2$, Plane Stress…… 109

4.65. Variation of k_b with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $b=0.1h_2$, Plane Stress…… 110

4.66. Variation of k_c with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $b=0.1h_2$, Plane Stress…… 111

4.67. Variation of k_c with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $b=0.1h_2$, Plane Stress…… 112

4.68. Variation of k_c with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $b=0.1h_2$, Plane Stress…… 113

4.69. Variation of k_a with a/h_1 with $h_2=h_1$, $b=0.1h_2$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$…… 114

4.70. Variation of k_b with a/h_1 with $h_2=h_1$, $b=0.1h_2$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$ …… 115

4.71. Variation of k_c with a/h_1 with $h_2=h_1$, $b=0.1h_2$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$…… 116

4.72. Variation of k_a with b/h_2 with $h_2=h_1$, $a=0.1h_1$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$…… 117

4.73. Variation of k_b with b/h_2 with $h_2=h_1$, $a=0.1h_1$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$…… 118

4.74. Variation of k_c with b/h_2 with $h_2=h_1$, $a=0.1h_1$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$…… 119

4.75. Variation of k_a with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.1h_1$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 120

4.76. Variation of k_b with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.1h_1$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 121

4.77. Variation of k_c with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.1h_1$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 122

4.78. Variation of k_a with a/h_1 with $h_2=h_1$, $b=0.89h_2$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 123

4.79. Variation of k_b with a/h_1 with $h_2=h_1$, $b=0.89h_2$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 124

4.80. Variation of k_c with a/h_1 with $h_2=h_1$, $b=0.89h_2$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 125

4.81. Variation of k_a with b/h_2 with $h_2=h_1$, $a=0.98h_1$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 126

4.82. Variation of k_b with b/h_2 with $h_2=h_1$, $a=0.98h_1$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 127

4.83. Variation of k_c with b/h_2 with $h_2=h_1$, $a=0.98h_1$, $c=0.9h_2$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 128

4.84. Variation of k_a with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.98h_1$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 129

4.85. Variation of k_b with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.98h_1$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 130

4.86. Variation of k_c with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.98h_1$, Plane Stress, $\lambda=\mu_1/\mu_2$………… 131

4.87. Variation of k_b with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2=23.077$, $h_1=h_2$, $c=0.9h_2$, Plane Stress………… 132

4.88. Variation of k_b with b/h_2 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, Plane Stress………………………………………… 133
4.89. Variation of k_b with c/h_2 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, Plane Stress………………………………………… 134
4.90 Variation of k_c with a/h_1 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, Plane Stress………………………………………… 135
4.91. Variation of k_c with b/h_2 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, Plane Stress………………………………………… 136
4.92. Variation of k_c with c/h_2 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, Plane Stress………………………………………… 137

NOMENCLATURE

2a	Width of the crack in strip1
(c-b)	Width of the crack in strip2
$2h_1$	Width of strip1
h_2	Width of strip2
p_1	Uniform intensity of the axial load in strip1
p_2	Uniform intensity of the axial load in strip2
x,y,z	Rectangular coordinates
u,v	Displacement components in x- and y-directions
σ , τ	Normal and shearing stresses
κ	($3-4\nu$) for plane strain and $(3-\nu) / (1+\nu)$ for plane stress
E	Young 's Modulus
μ	Modulus of rigidity
ν	Poisson's ratio
α, β	Fourier transform variables
m(t)	Crack surface displacement derivative
α, Ψ, γ	Powers of singularity at the edges of the cracks
ξ, η,ε, ρ	Non-dimensional coordinates on the crack in strip1
$m^{**}(t)$	Hölder-continuous function on the crack
K	Mode I stress intensity factor at the edge of the crack
k	Normilized Mode I stress intensity factor at the edge of the crack

CHAPTER I

INTRODUCTION

Despite the early works on roots of cause of fracture, quantitative relation between fracture stress and flaw size was made obvious by the work of Griffith, which was published in 1920. With his work on the brittle fracture of glass, he applied a stress analysis on an elliptical hole to the unstable propagation of a crack.

Because of their singularity nature and their related problems, strip based problems constitute a significant portion of the field of fracture mechanics. In general, these types of problems can be simplified and represented by boundary value problems, which are solved by both analytical and numerical methods.

In this manner, although problems concerning the single crack and collinear cracks in a material have been studied before, three layers of infinite panel with collinear cracks on the x-axis symmetric with respect to the y- axis have not been solved yet. This problem will be investigated by studying partial differential equations of the linear plane elasticity theory and by using singular integral equations.

1.1 Literature Review

Sneddon (1945) considered the distribution of stress produced in the interior of an elastic solid by the opening of an internal crack under the action of pressure applied to its surface. The analysis is given for the “Griffith’s cracks” and for circular cracks.

In his work, G.R. Irwin(1957) has pointed out that for somewhat brittle tensile fractures in situations such that a generalized plane-stress and a plane-strain analysis is appropriate , the influence of the test configuration, loads, and crack length upon the stresses near an end of the crack may be expressed in terms of two parameters. One of these is an adjustable uniform stress parallel to the direction of a crack extension. Also in his work, it is shown that the other parameter, called stress-intensity factor, proportional to the square root of the crack length and the intensity of force tending to cause crack extension.

G.C. Sih , P.C. Paris and F. Erdogan (1962) studied a complex variable method for evaluating the strength of stress singularities at crack tips in plane problems and plate bending problems. The results suggest the possibility of extension of the Griffith-Irwin fracture theory to arbitrary plane extensional and/or bending problems in plates.

F. Erdogan and G.C. Sih (1963) examined the crack extension in a large plate subjected to general plane loading theoretically and experimentally. In this work, it is found that under skew-symmetric plane loading of brittle materials the “sliding” or the crack extension in its own plane does not take place, instead crack grows in the direction approximately 70 degrees from the plane of the crack. In their work, it is also shown that, in general plane loading, the fracture criterion in terms of stress intensity factor is an ellipse.

F. Erdogan (1965) considered the problem of two bonded dissimilar semi-infinite planes containing cracks along the bond. The external loads considered include the tractions on the crack surfaces, in-plane moments, residual stresses due to temperature changes, concentrated load and couple acting at an arbitrary location in the plane, and one sided wedge loading of crack.

M.P. Stallybrass (1970) found an exact solution for an elastic half plane containing a crack perpendicular to the free surface, when the faces of the crack are subjected to a particular, but rather general, distribution of pressure. The solution is based on an application of the Weiener-Hopf technique to the governing integral equation. This integral equation can also be used as a basis for computing the stress and displacement fields for an arbitrary distribution of crack pressure.

P.D. Hilton and G.C. Sih (1971) studied the redistribution of stresses in a laminate composite due to the presence of a crack or flaw situated normal to the bond lines. The many-layered composite is idealized to the case of a single layer of dissimilar material containing a crack, which is sandwiched between two other layers of infinite height. Using the integral transform technique, the problem is formulated in terms of integral equations and solved for the singular stress field near the crack tip. The effects of crack size, layer height and the material properties of the composite on the stress intensity factor are illustrated graphically

I.N. Sneddon and R.P. Srivastav (1971) studied the problem of determining the stress field in an elastic strip of finite width when pressure is applied to the faces of a Griffith crack situated symmetrically within it. Stress intensity factor is computed by obtaining the numerical solution of the Fredholm integral equation.

I.N. Sneddon and S.C. Das (1971) solved the problem of determining the stress and displacement fields in an elastic half–plane containing an edge crack normal to the free surface when the crack faces are subjected to normal pressure by using the idea of reducing it to a mixed boundary value problem for the quarter plane. The theory of dual integral equations is used to reduce the boundary value problem to that of solving a pair of simultaneous integral equations of Fredholm type.

F. Erdogan and G.D. Gupta (1971) studied the plane strain problem for a bonded medium composed of three different materials. It is assumed that the medium contains a flaw on one of the interfaces, which may be idealized as a crack. The integral equations for the general problem are obtained, which turn out to be a system of singular integral equations of the second kind. The singularity of the system is removed and the equations are solved by taking advantage of the fact that the fundamental function of the integral equation is the weight function of Jacobi polynomials.

G. D. Gupta (1973) considered the problem of a laminate composite in presence of a crack located normal to the bond lines. Stress analysis of the limiting case when the crack extends to the bond lines is carried out. Integral transforms technique is used to formulate the problem in terms of a singular integral equation from which the power of stress singularity around the crack tip terminating at the interface is obtained.

D.B. Bogy (1973) considered the plane elastostatic problem for a crack in a strip composite loaded with normal or shearing traction is reduced to a single integral equation. The integral equation is solved numerically and the dependence of the stress intensity factors on the material parameters is displayed graphically.

G.D. Gupta and F. Erdogan (1974) studied the problem of edge cracks in an infinite strip. The elastostatic plane problem of an infinite strip containing two symmetrically located internal cracks perpendicular to the boundary is formulated in terms of a singular integral equation with the derivative of the crack surface displacement as the density function. The limiting case of the edge cracks is then considered in some detail.

S. Krenk (1975) studied a method to deal with an inclined crack in an elastic strip. No assumptions of symmetry are made. The method involves the solution for a cracked plane and uncracked strip and results in two coupled singular integral equations with finite interval of integration.

M.R. Geçit (1979a) studied the plane problem of a cracked elastic surface layer bonded to an elastic half space. The surface layer is assumed to contain a transverse crack whose surface is subjected to uniform compression. The problem is formulated in terms of a singular integral equation, the derivative of the crack surface displacement being the density function. By using appropriate quadrature formulas, the integral equation reduces to a system of linear algebraic equations.

M.R. Geçit (1979b) considered the elastostatic plane problem of an infinite strip containing two non-symmetrically located collinear cracks perpendicular to the sides. The strip is assumed to be isotropic and subjected to uniaxial tension. General expressions for field quantities are obtained by using the Fourier transform technique: these expressions, together with relevant boundary conditions, give singular integral equations in terms of the derivative of the crack surface displacements.

M.B. Civelek and F. Erdogan (1982) solved the problem of the general plane problem for an infinite strip containing multiple cracks perpendicular to its boundaries. The problem is reduced to a system of singular integral equations

A.F.H. Blaibel and M.R. Gecit (1989) studied the elastostatic plane problem of a semi-infinite strip bonded to an infinite strip along its short end and subjected to a bending moment at its far end. The infinite strip is bonded to a rigid substrate along its entire lower side. Formulation of the problem is reduced to a system of three singular integral equations of the second kind. These integral equations are solved numerically and the interface stress distributions and the stress intensity factors at the corners are calculated for various geometries and material combinations.

Ryvkin (1998) studied fracture behavior of an infinite periodically layered composite body, with a Mode 3 crack parallel to the layering. Upon deriving the Green function for a dislocation in a layered space, the problem is reduced to a singular integral equation of the first kind.

Shen, Kuang and Hu (1999) considered the interface crack problems of a multilayered anisotropic medium under a state of generalized plane deformation. The problem is reduced to the solution of a system of singular integral equations by means of Fourier transform method.

Khraishi and Demir (2002) made comments on some of the different numerical techniques commonly employed in evaluating Cauchy singular integrals of the first kind; e.g. as pertaining to 2D through cracks in a brittle material undergoing Mode I loading.

1.2 A Short Introduction and Methods of Solution of the Problem

The panel is symmetrical about both x- and y- axes. The central strip (Strip 1) of width $2h_1$ contains a central transverse crack of width 2a on x-axis. The two strips (Strip 2) contain transverse cracks of width c-b also on x-axis. The panel is subjected to axial loads with uniform intensities p_1 and p_2 in Strip 1 and Strip 2, respectively at $y = \pm\infty$. Materials of all strips are assumed to be linearly elastic and isotropic. Due to double symmetry, only one quarter of the problem $(0 \le x \le \infty$ and $0 \le y \le \infty)$ will be considered.

Solution for the infinite panel loaded at infinity having cracks with traction-free surfaces is obtained by superposition of the following two problems: (i) an infinite panel loaded at infinity with no cracks (uniform solution), (ii) an infinite panel with cracks whose surfaces are subjected to the negative of the stresses at the location of these cracks obtained from problem (i) (perturbation problem). These solutions are obtained by using Fourier transforms both in x- and y-directions. Summing several solutions is due to the necessity for sufficient number of unknowns in general expressions in order to be able to satisfy all boundary conditions of the problem. The conditions at the edges of the strips and at the interfaces are satisfied and the general expressions for the panel become expressions for the three-layer panel with free edges. Use of remaining boundary conditions leads the formulation to a system of two singular integral equations. These equations are converted to a system of linear algebraic equations, which is solved numerically.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Introduction

In this work, a three-layer panel problem containing a symmetrical central strip and two other strips on each side will be solved.

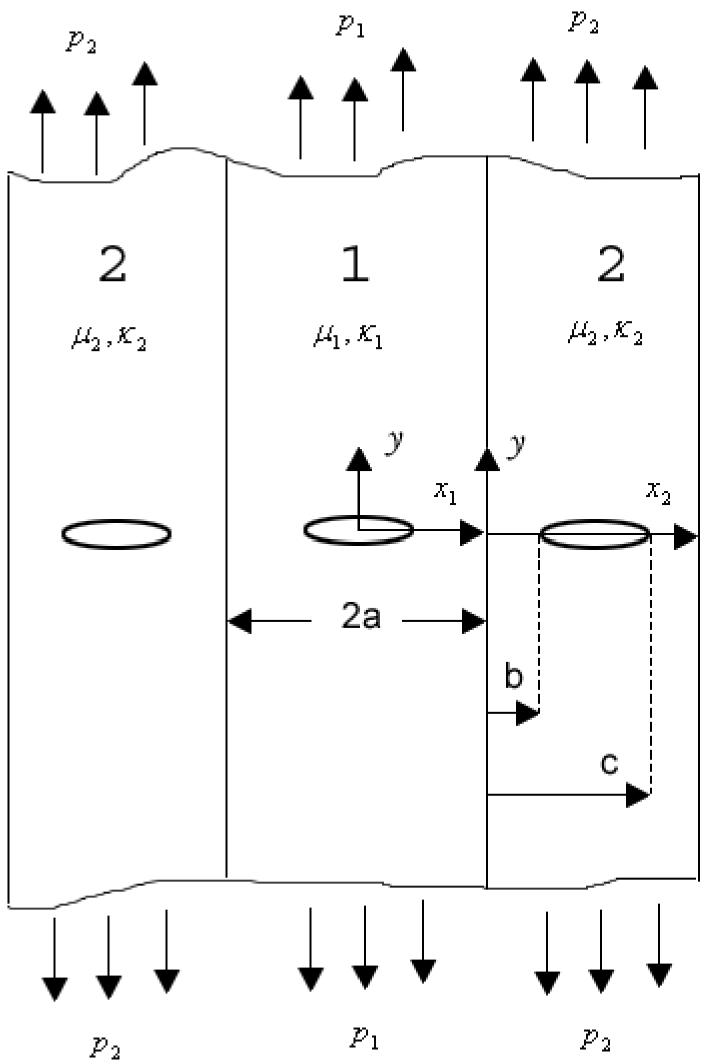

Figure 2.1 Three Layer Panel

Solution for the infinite panel loaded at infinity having cracks with traction free surfaces is obtained by superposition of the following two problems: (i) an infinite panel loaded at infinity with no cracks (uniform solution), (ii) an infinite panel with cracks whose surfaces are subjected to the negative of the stresses at the location of these cracks obtained from problem (i) (perturbation problem) as shown in Figure 2.2.

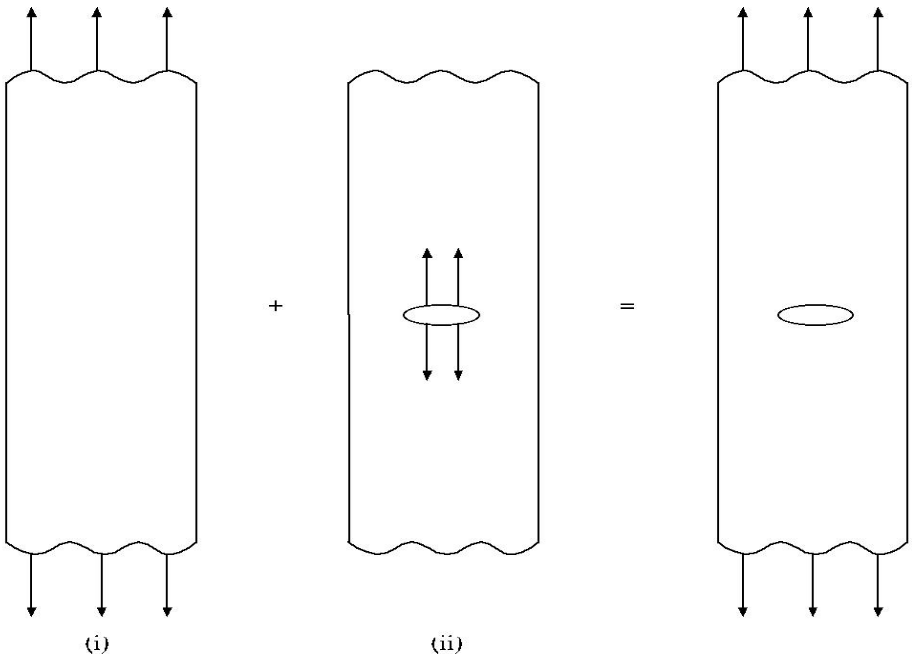

Figure 2.2 Superposition

General expressions for the perturbation problem (ii) can be obtained by adding the general solutions for (a) an infinite medium subjected to arbitrary symmetric loads, and (b) an infinite medium having cracks as shown in Figure 2.3.

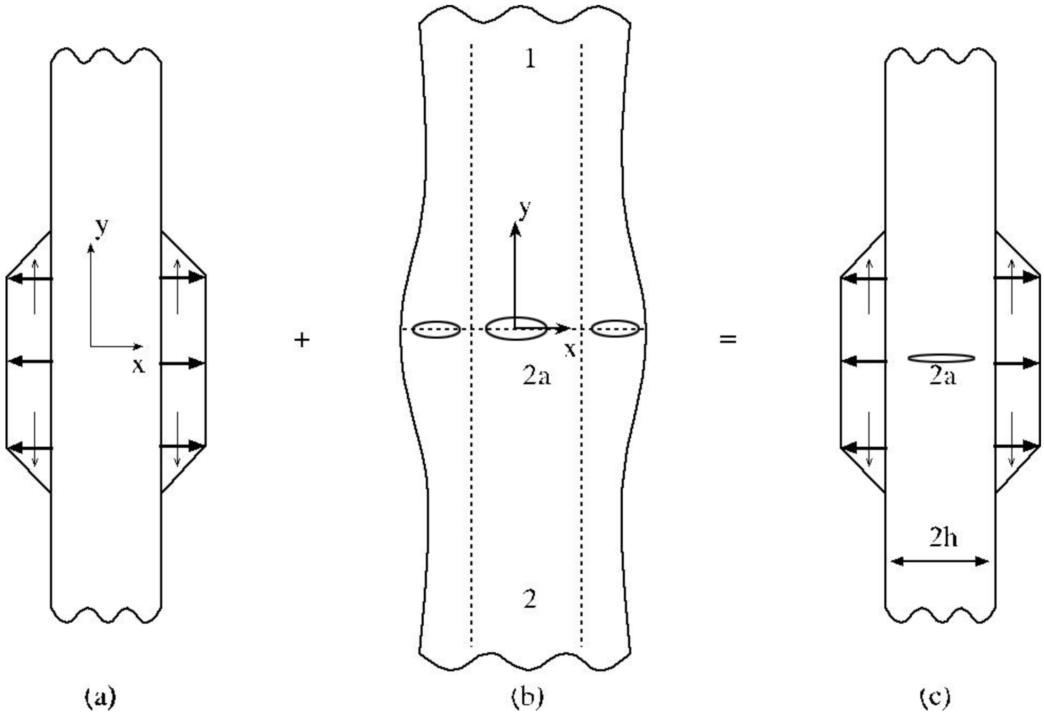

Figure 2.3 Schematic Representation of General Formulation

For linearly elastic, isotropic and two dimensional problems, the field equations can be listed as follows:

Navier Equations:

$$(\kappa+1)\frac{\partial^2 u}{\partial x^2}+(\kappa-1)\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 v}{\partial x \partial y}=0 \qquad (2.1.a\text{-}b)$$

$$2\frac{\partial^2 u}{\partial x \partial y}+(\kappa-1)\frac{\partial^2 v}{\partial x^2}+(\kappa+1)\frac{\partial^2 v}{\partial y^2}=0$$

where u and v are the x- and y- components of the displacement vector; $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, ν being the Poisson's ratio .

Stress-Displacement Relations:

$$\frac{1}{2\mu}\sigma_x = \frac{\kappa+1}{2(\kappa-1)}\frac{\partial u}{\partial x} + \frac{3-\kappa}{2(\kappa-1)}\frac{\partial v}{\partial y}$$

$$\frac{1}{2\mu}\sigma_y = \frac{3-\kappa}{2(\kappa-1)}\frac{\partial u}{\partial x} + \frac{\kappa+1}{2(\kappa-1)}\frac{\partial v}{\partial y} \qquad (2.2.a\text{-}c)$$

$$\frac{1}{2\mu}\tau_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

where σ and τ denote normal and shearing stresses, μ is the shear modulus.

These equations must be solved with boundary conditions given below:

$\sigma_{x2}(h_2, y) = 0 \qquad (0 \le y < \infty)$

$\tau_{xy2}(h2, y) = 0 \qquad (0 \le y < \infty)$

$u_1(h_1, y) = u_2(0, y) \qquad (0 \le y < \infty)$

$v_1(h1, y) = v_2(0, y) \qquad (0 \le y < \infty)$

$\sigma_{x1}(h_1, y) = \sigma_{x2}(0, y) \qquad (0 \le y < \infty)$

$\tau_{xy1}(h_1, y) = \tau_{xy2}(0, y) \qquad (0 \le y < \infty)$

$\sigma_{y1}(x_1, \infty) = p_1 \qquad (0 \le x_1 \le h_1) \qquad$ (2.3. a-l)

$\sigma_{y2}(x_2, \infty) = p_2 \qquad (0 \le x_2 \le h_2)$

$\sigma_{y1}(x_1, 0) = 0 \qquad (0 \le x_1 \le a)$

$\sigma_{y2}(x_2, 0) = 0 \qquad (b < x_1 < c)$

$v_1(x_1, 0) = 0 \qquad (a < x_1 < h_1)$

$v_2(x_2, 0) = 0 \qquad (0 < x_2 < b \text{ and } c < x_2 < h_2)$

in which the subscripts 1 and 2 indicate the central strip (Strip 1) and the strips on the sides (Strip 2), respectively.

2.2 Perturbation Solution

2.2.1 An Infinite Medium Having Crack

Solution for Strip 1 will be obtained first. Navier Equations, written again in the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\left(\frac{2}{\kappa-1}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y \partial x}\right)=0$$

$$\left(\frac{2}{\kappa-1}\right)\left(\frac{\partial^2 u}{\partial y \partial x}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad (2.4.a\text{-}b)$$

will be solved by using Fourier transforms in x-direction. In order to realize the crack at y = 0, one can consider two half planes 0 ≤ y < 0 and -∞ < y ≤ 0, solved Navier equations for these regions separately, and then match the solutions at y = 0 such that

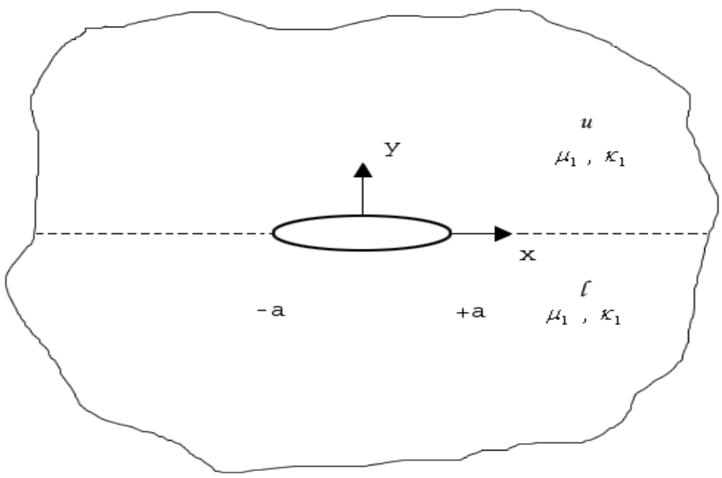

Figure 2.4.: Schematic Representation of Infinite Medium Having a Crack

$\sigma_y^u(x,0^+) = \sigma_y^l(x,0^-)$, $\tau_{xy}(x,0) = 0$, $u^u(x,0^+) = u^l(x,0^-)$ $(0 < x < \infty)$

$\sigma_y(x,0) = known$ $(0 \le x < a)$

$v^u(x,0^+) = v^l(x,0^-)$ $(a < x < \infty)$. (2.5 a-e)

In these conditions superscripts u and ℓ refer to the upper and lower half planes, respectively. One may introduce

$$m_1(x) = \frac{\partial}{\partial x} v_1(x,0) \quad \text{such that} \qquad (2.6\text{ a, b})$$

$m_1(x) = 0$ $(a < x < \infty)$

Hence the condition (2.5e) is satisfied. Then, by taking Fourier sine transform of (2.4a) and the cosine transform of (2.4b) in x-direction, solving the resulting ordinary differential equations, and making use of conditions (2.5a-c), (2.6a,b) one can obtain the following expressions for the upper half plane

$$u_1^1(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} M_1(\alpha)\left(\frac{1-\kappa_1}{2} + \alpha y\right)e^{-\alpha y}\sin(\alpha x)d\alpha \qquad (2.7\text{ a, b})$$

$$v_1^1(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} M_1(\alpha)\left(\frac{\kappa_1+1}{2} + \alpha y\right)e^{-\alpha y}\cos(\alpha x)d\alpha$$

in which the subscript 1 denotes the symmetric middle strip (Strip 1), superscript 1 indicates that this is the solution for the cracked medium, μ_1 is the shear modulus for the middle strip and

$$M_1(\alpha) = -\frac{2}{\kappa_1+1}\int_0^a m_1(x)\sin(\alpha x)dx \qquad (2.8)$$

Substitution of (2.7) in (2.2) gives

$$\sigma_{1x}^{1}(x,y) = -\frac{4\mu_1}{\pi}\int_0^\infty M_1(\alpha)(1-\alpha y)e^{-\alpha y}\cos(\alpha x)d\alpha$$

$$\sigma_{1y}^{1}(x,y) = -\frac{4\mu_1}{\pi}\int_0^\infty M_1(\alpha)(1+\alpha y)e^{-\alpha y}\cos(\alpha x)d\alpha \qquad (2.9a\text{-}c)$$

$$\tau_{1yx}^{1}(x,y) = -\frac{4\mu_1}{\pi}\int_0^\infty M_1(\alpha)(\alpha y)e^{-\alpha y}\sin(\alpha x)d\alpha$$

2.2.2 An Infinite Medium Subjected to Arbitrary Symmetric Loads

Now, consider infinite medium exhibiting double symmetry with respect to x- and y-axis. Imagine that this medium is subjected to arbitrary but symmetric loads applied not at y = ∞. Taking the Fourier cosine transform of (2.4a), sine transform of (2.4b), solving the resulting ordinary differential equations and then taking the inverse Fourier sine/cosine transforms, one can obtain the following expressions;

$$u_1^2(x,y) = -\frac{2}{\pi}\int_0^\infty \frac{1}{\beta}\left[\left(A_1 - \frac{(\kappa_1 - 1)}{2}B_1\right)\sinh(\beta x) + \beta B_1 x\cosh(\beta x)\right]\cos(\beta y)d\beta$$

(2.10a)

$$v_1^2(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\beta}\left[\left(A_1 + \frac{(\kappa_1 + 1)}{2}B_1\right)\cosh(\beta x) + \beta B_1 x\sinh(\beta x)\right]\sin(\beta y)d\beta$$

(2.10b)

for the displacement. Here, superscript 2 indicates that this is the solution for the medium with no crack; A_1 and B_1 are yet unknowns. Substitution of (2.10) in (2.2) gives

$$\sigma_{1x}^{2}(x,y) = \frac{-4\mu_1}{\pi}\int_0^\infty \left[A_1 \cosh(\beta x) + B_1 \beta x \sinh(\beta x)\right]\cos(\beta y)d\beta$$

$$\sigma_{1y}^{2}(x,y) = \frac{4\mu_1}{\pi}\int_0^\infty \left[(A_1 + 2B_1)\cosh(\beta x) + \beta x B_1 \sinh(\beta x)\right]\cos(\beta y)d\beta \qquad (2.11a\text{-}c)$$

$$\tau_{1xy}^{2}(x,y) = \frac{4\mu_1}{\pi}\int_0^\infty \left[(A_1 + B_1)\sinh(\beta x) + \beta x B_1 \cosh(\beta x)\right]\sin(\beta y)d\beta$$

for the stresses.

2.2.3 General Expressions of Perturbation Problem for Strip 1

When the expressions obtained in the last two sections are collected, the general expressions for Strip 1 are obtained in such a manner that these expressions contain sufficient number of unknown quantities to be determined from the boundary conditions.

They can be written explicitly as follows,

$$u_1(x,y) = u_1^1(x,y) + u_1^2(x,y)$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} M_1(\alpha)\left(\frac{\kappa_1 - 1}{2} - \alpha y\right)e^{-\alpha y}\sin(\alpha x)d\alpha$$

$$-\frac{2}{\pi}\int_0^\infty \frac{1}{\beta}\left[\left(A_1 - \frac{(\kappa_1 - 1)}{2}B_1\right)\sinh(\beta x) + \beta B_1 x \cosh(\beta x)\right]\cos(\beta y)d\beta$$

$$v_1(x,y) = v_1^1(x,y) + v_1^2(x,y)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} M_1(\alpha)\left(\frac{\kappa_1 + 1}{2} + \alpha y\right)e^{-\alpha y}\cos(\alpha x)d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \frac{1}{\beta}\left[\left(A_1 + \frac{(\kappa_1 + 1)}{2}B_1\right)\cosh(\beta x) + \beta B_1 x \sinh(\beta x)\right]\sin(\beta y)d\beta$$

$$\sigma_{1x}(x,y) = \sigma_{1x}^{1}(x,y) + \sigma_{1x}^{2}(x,y)$$

$$= -\frac{4\mu_1}{\pi}\int_0^\infty M_1(\alpha)(1-\alpha y)e^{-\alpha y}\cos(\alpha x)\,d\alpha$$

$$-\frac{4\mu_1}{\pi}\int_0^\infty \left[A_1\cosh(\beta x) + B_1\beta x\sinh(\beta x)\right]\cos(\beta y)\,d\beta$$

$$\sigma_{1y}(x,y) = \sigma_{1y}^{1}(x,y) + \sigma_{1y}^{2}(x,y)$$

$$= -\frac{4\mu_1}{\pi}\int_0^\infty M_1(\alpha)(1+\alpha y)e^{-\alpha y}\cos(\alpha x)\,d\alpha$$

$$+\frac{4\mu_1}{\pi}\int_0^\infty \left[(A_1 + 2B_1)\cosh(\beta x) + \beta x B_1\sinh(\beta x)\right]\cos(\beta y)\,d\beta$$

$$\tau_{1xy}(x,y) = \tau_{1yx}^{1}(x,y) + \tau_{2yx}^{2}(x,y)$$

$$= -\frac{4\mu_1}{\pi}\int_0^\infty M_1(\alpha)(\alpha y)e^{-\alpha y}\sin(\alpha x)\,d\alpha$$

$$+\frac{4\mu_1}{\pi}\int_0^\infty \left[(A_1 + B_1)\sinh(\beta x) + \beta x B_1\cosh(\beta x)\right]\sin(\beta y)\,d\beta$$

(2.12a-e)

2.2.4 General Expression of Perturbation Problem for Strip 2

General expressions for strips (or layers) on the sides (Strip 2) can be obtained by following a procedure similar to that described in the previous sections for Strip 1. The only difference is that Strip 2 has single symmetry about x-axis. Hence, one can obtain the following expressions for Strip 2:

$$u_2(x,y) = -\frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} M_2(\alpha)\left(\frac{\kappa_2 - 1}{2} - \alpha y\right) e^{-\alpha y}\sin(\alpha x)\,d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \left[\left(-A_2 + \frac{\kappa_2}{\beta}C_2 - xB_2\right)\sinh(\beta x)\right.$$

$$\left.+\left(D_2 + \frac{\kappa_2}{\beta}B_2 - xC_2\right)\cosh(\beta x)\right]\cos(\beta y)\,d\beta\ ,$$

$$v_2(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha} M_2(\alpha)\left(\frac{\kappa_2 + 1}{2} + \alpha y\right) e^{-\alpha y}\cos(\alpha x)\,d\alpha$$

$$+\frac{2}{\pi}\int_0^\infty \left[(A_2 + B_2 x)\cosh(\beta x) + (-D_2 + C_2 x)\sinh(\beta x)\right]\sin(\beta y)\,d\beta\ ,$$

$$\sigma_{2x} = -\frac{4\mu_2}{\pi}\int_0^\infty M_2(\alpha)(1-\alpha y)e^{-\alpha y}\cos(\alpha x)\,d\alpha$$

$$+\frac{4\mu_2}{\pi}\int_0^\infty \left[\left(-A_2\beta + \frac{(\kappa_2+1)}{2}C_2 - B_2\beta x\right)\cosh(\beta x)\right.$$

$$\left.+\left(D_2\beta + \frac{(\kappa_2+1)}{2}B_2 - C_2\beta x\right)\sinh(\beta x)\right]\cos(\beta y)\,d\beta\ ,$$

$$\sigma_{2y} = -\frac{4\mu_2}{\pi}\int_0^\infty M_2(\alpha)(1+\alpha y)e^{-\alpha y}\cos(\alpha x)\,d\alpha$$

$$+\frac{4\mu_2}{\pi}\int_0^\infty \left[\left(A_2\beta + \frac{(3-\kappa_2)}{2}C_2 + B_2\beta x\right)\cosh(\beta x)\right.$$

$$\left.+\left(-D_2\beta + \frac{(3-\kappa_2)}{2}B_2 + C_2\beta x\right)\sinh(\beta x)\right]\cos(\beta y)\,d\beta\ ,$$

$$\tau_{2xy} = -\frac{4\mu_2}{\pi}\int_0^\infty M_2(\alpha)(\alpha y)e^{-\alpha y}\sin(\alpha x)\,d\alpha$$

$$+\frac{4\mu_2}{\pi}\int_0^\infty \left[\left(-D_2\beta + \frac{(1-\kappa_2)}{2}B_2 + C_2\right)\cosh(\beta x)\right.$$

$$\left.+\left(A_2\beta + \frac{(1-\kappa_2)}{2}C_2 + B_2\beta x\right)\sinh(\beta x)\right]\sin(\beta y)\,d\beta$$

(2.13a-e)

In which the subscript 2 denotes Strip 2, A_2, B_2, C_2, D_2 are yet unknowns to be determined from the boundary and continuity conditions at the interface,

$$M_2(\alpha) = -\frac{2}{\kappa_2 + 1}\int_b^c m_2(x)\sin(\alpha x)dx \quad ,$$

$$m_2(x) = \frac{\partial}{\partial x} v_2(x,0) \qquad \text{(2.14a-c)}$$

$$m_2(x) = 0 \qquad (0 < x < b,\ c < x < h_2)$$

2.2.5. Boundary and Continuity Conditions

General expressions for Strip 1 and Strip 2 contain a total number of eight unknowns; A_1, B_1, A_2, B_2, C_2, D_2, M_1, M_2. Six of these, namely, A_1, B_1, A_2, B_2, C_2 and D_2 can be related to the remaining two, M_1 and M_2, by the use of the conditions on the free side of Strip 2 and on the interface between Strip 1 and Strip 2, Eq.(2.3a-f). The resulting system of six equations can be written in the form:

$$\left(-\beta\ \cosh(\beta h_2)\right)A_2 + \left(-\beta\ h_2\cosh(\beta h_2) + \frac{(\kappa_2+1)}{2}\sinh(\beta h_2)\right)B_2$$

$$+\left(\frac{\kappa_2+1}{2}\cosh(\beta h_2) - \beta h_2\sinh(\beta h_2)\right)C_2 + \beta\sinh(\beta h_2)D_2 =$$

$$\frac{4\beta^2}{\pi}\int_0^\infty \frac{\alpha}{\left(\alpha^2+\beta^2\right)^2} M_2(\alpha)\cos(\alpha h_2)d\alpha$$

(2.15a)

$$(\beta \sinh(\beta h_2))A_2 + \left(\beta h_2 \sinh(\beta h_2) + \frac{(1-\kappa_2)}{2}\cosh(\beta h_2)\right)B_2$$

$$+\left(\frac{(1-\kappa_2)}{2}\sinh(\beta h_2) + \beta h_2 \cosh(\beta h_2)\right)C_2 + (-\beta \cosh(\beta h_2))D_2 =$$

$$\frac{4\beta}{\pi}\int_0^\infty \frac{\alpha^2}{(\alpha^2+\beta^2)^2} M_2(\alpha)\sin(\alpha h_2)d\alpha$$

(2.15b)

$$\left(\frac{1}{\beta}\sinh(\beta h_1)\right)A_1 + \left(\frac{(1-\kappa_1)}{2\beta}\sinh(\beta h_1) + h_1\cosh(\beta h_1)\right)B_1$$

$$+\left(\frac{\kappa_2}{\beta}\right)B_2 + D_2 = \frac{-1}{\pi}\int_0^\infty\left(\frac{\kappa_1 - 1}{(\alpha^2+\beta^2)} - \frac{2(\alpha^2-\beta^2)}{(\alpha^2+\beta^2)^2}\right)M_1(\alpha)\sin(\alpha h_1)d\alpha$$

(2.15c)

$$\left(\frac{1}{\beta}\cosh(\beta h_1)\right)A_1 + \left(\frac{\kappa_1+1}{2\beta}\cosh(\beta h_1) + h_1\sinh(\beta h_1)\right)B_1 - A_2 =$$

$$\frac{\beta}{\pi}\int_0^\infty \frac{1}{\alpha}\left[M_2(\alpha)\left[\frac{(\kappa_2+1)}{(\alpha^2+\beta^2)} + \frac{4\alpha^2}{(\alpha^2+\beta^2)^2}\right] - M_1(\alpha)\cos(\alpha h_1)\left[\frac{(\kappa_1+1)}{(\alpha^2+\beta^2)} + \frac{4\alpha^2}{(\alpha^2+\beta^2)^2}\right]\right]d\alpha$$

(2.15d)

$$(2\mu_1\cosh(\beta h_1))A_1 + (2\mu_1\beta h_1\sinh(\beta h_1))B_1 - (2\mu_2\beta)A_2 + (\mu_2(\kappa_2+1))C_2 =$$

$$\frac{8\beta^2}{\pi}\int_0^\infty (\mu_2 M_2(\alpha) - \mu_1 M_1(\alpha)\cos(\alpha h_1))\frac{\alpha}{(\alpha^2+\beta^2)^2}d\alpha$$

(2.15e)

$$(2\mu_1 \sinh(\beta h_1))A_1 + (2\mu_1(\sinh(\beta h_1) + \beta h_1 \cosh(\beta h_1)))B_1$$

$$+((\kappa_2 - 1)\mu_2)B_2 + (2\mu_2\beta)D_2 = \frac{8\mu_1\beta}{\pi}\int_0^\infty M_1(\alpha)\sin(\alpha h_1)\frac{\alpha^2}{(\alpha^2+\beta^2)^2}d\alpha$$

(2.15f)

These equations are solved simultaneously and A1, B_1, A_2, B_2, C_2, D_2 are determined in terms of M_1 and M_2

$$A_1 = \frac{1}{\Delta}\sum_{i=1}^{6} a_{1i} Int_i(\alpha)$$

$$B_1 = \frac{1}{\Delta}\sum_{i=1}^{6} b_{1i} Int_i(\alpha)$$

$$A_2 = \frac{1}{\Delta}\sum_{i=1}^{6} a_{2i} Int_i(\alpha)$$

$$B_2 = \frac{1}{\Delta}\sum_{i=1}^{6} b_{2i} Int_i(\alpha)$$

$$C_2 = \frac{1}{\Delta}\sum_{i=1}^{6} c_{2i} Int_i(\alpha)$$

$$D_2 = \frac{1}{\Delta}\sum_{i=1}^{6} d_{2i} Int_i(\alpha)$$

(2.16.a-f)

where a_{1i}, b_{1i}, a_{2i}, b_{2i}, c_{2i}, d_{2i}, Δ, $Int_i(\alpha)$ (i = 1, 6) are given in Appendix A.

Substituting Eq.(2.16) in Eqs.(2.3.g) and (2.3.h) and rearranging the resulting expressions, one can write the following expressions for the perturbation problem in $0 < y < \infty$:

$$\sigma_{iy}(x,y)=\frac{1}{\pi}\int_{-a}^{a}\widetilde{f}_{i1}(x,y,t)m_1(t)dt+\frac{1}{\pi}\int_{b}^{c}\widetilde{f}_{i2}(x,y,t)m_2(t)dt \qquad (\text{i} = 1, 2)$$

(2.17a, b)

where, $\widetilde{f}_{ij}(x,y,t)$ (i,j = 1,2) are given in Appendix B. Note here that the integrals from 0 to a are converted into integrals from –a to a by considering the fact that m_1 is an odd function.

2.3. Integral Equations

Now if one substitutes Eq.(2.17) in the boundary conditions on the crack surfaces, Eqs.(2.3g-h), the following singular integral equation are obtained from Eq.(2.3g) ;

$$\frac{1}{\pi}\int_{-a}^{a}\frac{4\mu_1 m_1(t)}{(\kappa_1+1)}\left[\frac{1}{(t-x)}+k_{11}(x,t)\right]dt+\frac{1}{\pi}\int_{b}^{c}\frac{\mu_1 m_2(t)}{(\kappa_2+1)p_1}\left[k_{12}(x,t)\right]dt=-p_1$$

(-a < x < a) (2.18a)

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\mu_2 m_1(t)}{(\kappa_1+1)}\left[k_{21}(x,t)\right]dt+\frac{1}{\pi}\int_{b}^{c}\frac{\mu_2 m_2(t)}{(\kappa_2+1)p_2}\left[\frac{4}{t-x}+\frac{4}{t+x}+k_{22}(x,t)\right]dt=-p_2$$

(b < x < c) (2.18b)

where

$$k_{ij}(x,t)=\int_{0}^{\infty}K_{ij}(x,t,\beta)d\beta \text{ ,(i,j=1,2)} \qquad (2.19)$$

and $K_{ij}(x, t, \beta)$; (i, j =1, 2) are given in Appendix C. These singular integral equations, Eqs.(2.18a-b), must be solved in such a way that the single-valuedness conditions for the cracks

$$\int_{-a}^{a} m_1(t)dt = 0$$

(2.20a, b)

$$\int_{b}^{c} m_2(t)dt = 0$$

are also satisfied.

When the cracks are embedded, kernels $k_{ij}(x, t)$, (i, j=1,2) are all bounded and there is the simple Cauchy Kernel alone. However, when either (or both) crack touches the interface, $k_{ij}(x, t)$, (i, j=1, 2) contain unbounded parts due to behavior of $K_{ij}(x, t, \beta)$ (i, j=1, 2) as $\beta \to \infty$. These unbounded parts together with the simple Cauchy Kernel $(t-x)^{-1}$ constitute a set of generalized Cauchy Kernels.

Non-vanishing portions of $K_{ij}(x,t,\beta)$ as $\beta \to \infty$;

$$K_{ij\infty} = \lim_{s\to\infty} K_{ij}(x,t,\beta) \quad \text{(i, j=1, 2)} \tag{2.21a}$$

$$K_{ijb}(x,t,\beta) = K_{ij}(x,t,\beta) - K_{ij\infty}(x,t,\beta) \quad \text{(i, j=1, 2)} \tag{2.21b}$$

are given in Appendix D. Integrating $K_{ij\infty}$;

$$k_{ijs}(x,t) = \int_{0}^{\infty} K_{ij\infty} d\beta \qquad \text{(i, j=1, 2)}$$

(2.22)

are obtained. These are also given in Appendix D. with the definition given in Eqs.(2.21a-b) and Eq.(2.22), the singular integral equation (2.18) can be written in the form

$$\frac{4}{\pi}\int_{-a}^{a}\left[\frac{1}{t-x_1}+k_{11s}(x_1,t)+\int_0^\infty K_{11b}(x_1,t,\beta)d\beta\right]m_1(t)dt$$

$$+\frac{\kappa_1+1}{\kappa_2+1}\frac{1}{\pi}\int_b^c\left[k_{12s}(x_1,t)+\int_0^\infty K_{12b}(x_1,t,\beta)d\beta\right]m_2(t)dt=-\frac{\kappa_1+1}{\mu_1}P_1$$

$(-a < x_1 < a)$ (2.23. a)

$$\frac{\kappa_2+1}{\kappa_1+1}\frac{1}{\pi}\int_{-a}^{a}\left[k_{21s}(x_2,t)+\int_0^\infty K_{21b}(x_2,t,\beta)d\beta\right]m_1(t)dt$$

$$+\frac{1}{\pi}\int_b^c\left[\frac{4}{t-x_2}+\frac{4}{t+x_2}+k_{22s}(x_2,t)+\int_0^\infty K_{22b}(x_2,t,\beta)d\beta\right]m_2(t)dt=-\frac{\kappa_2+1}{\mu_2}p_2$$

$(b < x_2 < c)$ (2.23. b)

The unknown function $m_1(t)$ is singular at $x_1 = \mp a$ and $m_2(t)$ is singular at $x_2 = b$ and $x_2 = c$. Their singular behavior can be examined and determined by the complex function technique given in Muskhelishvili (1953). Their singular behavior can be investigated by first writing;

$$m_1(x_1)=\frac{m_1^*(x_1)}{\left(a^2-x_1^2\right)^\alpha} \quad , \quad -a \le x_1 \le a, \quad 0 < \mathrm{Re}(\alpha) < 1$$

(2.24a,b)

$$m_2(x_2)=\frac{m_2^*(x_2)}{\left(c-x_2\right)^\gamma (x_2-b)^\psi} \quad , \quad b \le x_2 \le c \ , \quad 0 < \mathrm{Re}(\gamma, \Psi) < 1$$

where $m_1^*(t)$ and $m_2^*(t)$ are Hölder-continuous functions in the respective intervals [-a , a] and [b , c] and α , Ψ and γ are unknown constants.

2.3.1. The Case of Embedded Cracks (a < h$_1$, 0 < b , c < h$_2$)

Calculating the integrals containing simple Cauchy kernels by the formulas given in Muskhelisvili (1953),

$$\frac{1}{\pi}\int_{-a}^{a}\frac{m_1(t)dt}{(t-x_1)}=\frac{m_1^*(-a)\cot(\pi\alpha)}{(2a)^{\alpha}(a+x_1)^{\alpha}}-\frac{m_1^*(a)\cot(\pi\alpha)}{(2a)^{\alpha}(a-x_1)^{\alpha}}+F_1(x_1)\,,$$

$$\frac{1}{\pi}\int_{-a}^{a}\frac{m_2(t)dt}{(t-x_2)}=\frac{m_2^*(b)\cot(\pi\psi)}{(c-b)^{\gamma}(x_2-b)^{\psi}}-\frac{m_2^*(c)\cot(\pi\gamma)}{(c-b)^{\psi}(c-x_2)^{\gamma}}+F_2(x_2)$$

(2.25a,b)

are obtained.

$F_1(x_1)$ and $F_2(x_2)$ are all bounded everywhere except at the end points $\pm a$, b and c. If Eq.(2.23a) is multiplied by $(a-x_1)^{\alpha}$ and then the limiting case $x_1 \to a$ is considered with the help of Eqs.(2.24a) and (2.25a), Eq.(2.23a) can be reduced to

$$\cot(\pi\alpha)=0 \tag{2.26}$$

Similarly, if Eq.(2.23b) is multiplied by $(x_2-b)^{\psi}$ then considering the limiting case $x_2 \to b$, by the use of Eqs. (2.24b) and (2.25b) , Eqs. (2.23b) can be reduced to

$$\cot(\pi\psi)=0 \tag{2.27}$$

Finally, if Eq.(2.23b) is multiplied by $(c-x_2)^{\gamma}$, then considering the limiting case $x_2 \to c$, by the use of Eqs.(2.24b) and (2.25b), Eq.(2.23b) can be reduced to

$$\cot(\pi\gamma)=0 \tag{2.28}$$

Eqs.(2.26-2.28) are in exact agreement with those given in previous works, e.g., Bogy (1973), Cook and Erdogan (1972), Erdogan (1965), Gecit (1979a), Gecit (1979b), Gupta and Erdogan (1974), Isida (1966) and Krenk (1975). From Eqs. (2.26-28), $\alpha = \psi = \gamma = 1/2$ is obtained as the acceptable power of stress singularity at the edges of cracks embedded in homogenous media.

2.3.2. The Case of Broken Middle Strip (a=h_1, 0<b, c<h_2)

When the edges of the crack in Strip 1 approach the interfaces at $x_1 = \pm h_1$, the middle Strip will break completely.

The integrals containing generalized Cauchy Kernels are calculated again using the formulas given in Muskhelishvili (1953):

$$\frac{1}{\pi}\int_{-h_1}^{+h_1} \frac{m_1(t)dt}{t-(2h_1 \pm x_1)} = \frac{-m_1^*(h_1)}{(2h_1)^\alpha \sin(\pi\alpha)} \frac{1}{(h_1 \pm x_1)^\alpha} + F_3(\pm x_1) \qquad (2.29)$$

where F_3 is bounded everywhere except at the end points $x_1 = \pm h_1$. Now, noting that a=h1 and employing Eqs.(2.24a), (2.25a) and (2.29) in Eq.(2.23a), multiplying Eq.(2.23a) by $(h_1 - x_1)^\alpha$, and then considering the limiting case $x_1 \to h_1$ in the resulting equation, following characteristic equation is obtained for determination of the power of singularity, α:

$$2\cos(\pi\alpha) + \frac{1-\lambda}{\kappa_1 + \lambda}[4\alpha(\alpha-2)+3] - \frac{1-\kappa_1\kappa_2}{1+\lambda\kappa_2} = 0 \qquad (2.30)$$

in which

$$\lambda = \mu_1/\mu_2 \qquad (2.31)$$

Eq.(2.31) is in perfect agreement with the previous results, e.g., Cook and Erdogan (1972), Erdogan (1965), Erdogan and Biricikoglu (1973), Gecit (1979a), Gecit (1979b). Note that the cracks in the other strips are still embedded cracks. Therefore, $\psi = 1/2$ in this case too.

2.3.3 The Case of Crack in Strip2 Touching the Interface ($a<h_1$, $0=b$, $c<h_2$)

In this case the edges of the crack in Strip1 and the edge at the $x_2=c$ of the crack in Strip 2 are embedded in homogeneous media while the edge at $x_2=b$ of the crack in Strip 2 is touching the interface at $x_2=0$. The generalized Cauchy Kernels of this case are also calculated by using the formulas given in Muskhelishvili (1953):

$$\frac{1}{\pi}\int_0^c \frac{m_2(t)dt}{t} = \frac{m_2^*(0)}{c^{\psi}\sin(\pi\beta)}\frac{1}{x_2^{\psi}} + F_4(x_2), \tag{2.32}$$

where F_4 is bounded everywhere except at the end point $x_2=0$. Now, employing Eqs.(2.24b), (2.25b) and (2.32) in Eq.(2.23b), multiplying Eq.(2.23b) by x_2^{ψ}, and then considering the limiting case $x_2 \to 0$ in the resulting equation, following characteristic equation is obtained for β:

$$2\cos(\pi\psi) + \frac{\lambda - 1}{1 + \lambda\kappa_2}\left[4\psi(\psi - 2) + 3\right] - \frac{1 - \kappa_1\kappa_2}{\lambda + \kappa_1} = 0 \tag{2.33}$$

this equation is similar to Eq.(2.30) and one can be converted to the other by simply interchanging the subscripts.

2.4. Stress Intensity Factors

Stresses become infinite at the edges of the cracks $(x_1 = \pm a, x_2 = b, c; y = 0)$. Consequently, the stress state near the crack edges is conveniently expressed by means of the so called stress intensity factors, Erdogan (1985) here, Mode-I (opening mode) stress intensity factors will be evaluated only. For this purpose, Eqs.(2.17) will be evaluated at y $=0^+$ first. Then, the integrals containing singular terms will be calculated in front of the crack edges. This requires close examination of those integrals again using the relevant integral formulas giving Muskhelishvili(1953) outside the cuts $-a \le x_1 \le a$, $b \le x_2 \le c$. This consideration requires revision in the definitions given in Eqs.(2.24a,b):

Embedded Crack in Strip1 ($a<h_1$);

$$m_1(x_1) = \frac{m_1^*(x_1)}{\sqrt{a^2 - x_1^2}} = \frac{m_1^*(x_1)(a + x_1)^{-1/2}}{(a - x_1)^{1/2}}, \ (x_1 \to a_1), \tag{2.34}$$

Broken Strip1 ($a=h_1$)

$$m_1(x_1) = \frac{m_1^*(x_1)}{(h_1^2 - x_1^2)^{\alpha}} = \frac{m_1^*(x_1)(h_1 + x_1)^{-\alpha}}{(h_1 - x_1)^{\alpha}}, \ (x_1 \to h_1), \tag{2.35}$$

Embedded Cracks in Strip2 ($0<b$, $c<h_2$)

$$m_2(x_2) = \frac{m_2^*(x_2)}{\sqrt{x_2 - b}\sqrt{c - x_2}} = \frac{m_2^*(x_2)(c - x_2)^{-1/2}}{(x_2 - b)^{1/2}}, \ (x_2 \to b), \tag{2.36a,b}$$

$$= \frac{m_2^*(x_2)(x_2 - b)^{-1/2}}{(c - x_2)^{1/2}}, \ (x_2 \to c)$$

Crack in Strip2 is touching the interface ($0=b$);

$$m_2(x_2)=\frac{m_2^*(x_2)}{x_2^{\psi}\sqrt{c-x_2}}=\frac{m_2^*(x_2)(c-x_2)^{-1/2}}{x_2^{\psi}},\ (x_2\to 0)$$

$$=\frac{m_2^*(x_2)x_2^{-\psi}}{(c-x_2)^{1/2}},\ (x_2\to c), \tag{2.37a, b}$$

The Mode-I stress intensity factors are defined in the form

$$K_a=\lim_{x_1\to a}\sqrt{2(x_1-a)}\sigma_{1y}(x_1,0),\ (a<h_1)$$

$$=\lim_{x_2\to 0}\sqrt{2}x_2^{\alpha}\sigma_{2y}(x_2,0),\ (a=h_1), \tag{2.38a, b}$$

$$K_b=\lim_{x_2\to b}\sqrt{2(b-x_2)}\sigma_{2y}(x_2,0),\ (0<b)$$

$$=\lim_{x_1\to h_1}\sqrt{2}(h_1-x_1)^{\psi}\sigma_{1y}(x_1,0),\ (0=b), \tag{2.39a, b}$$

$$K_c=\lim_{x_2\to c}\sqrt{2(x_2-c)}\sigma_{2y}(x_2,0),\ (c<h_2) \tag{2.40}$$

Expressions for the necessary stress components at y=0 can conveniently be expressed as

$$\sigma_{1y}(x_1,0)=\frac{4\mu_1}{1+\kappa_1}\frac{1}{\pi}\int_{-a}^{a}\frac{m_1(t)}{t-x_1}dt+\frac{2\mu_1}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\frac{1}{\pi}\int_{b}^{c}\{[3-\kappa_1-(1-3\kappa_2)\lambda]$$

$$+2[1-\kappa_1-(1-\kappa_2)\lambda]\frac{x_1-h_1}{t-x_1+h_1}\ \}\frac{m_2(t)dt}{t-x_1+h_1}+\sigma_{1yb}(x_1,0),$$

(2.41a, b)

$$\sigma_{2y}(x_2,0)=\frac{2\mu_1}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\frac{1}{\pi}\int_{-a}^{a}\{[3\kappa_1-1+(1-\kappa_2)\lambda]$$

$$+2[\kappa_1-1+(1-\kappa_2)\lambda]\frac{x_2}{t-h_1-x_2}\ \}\frac{m_1(t)dt}{t-h_1-x_2}+\frac{4\mu_2}{1+\kappa_2}\frac{1}{\pi}\int_{b}^{c}\frac{m_2(t)dt}{t-x_2}+\sigma_{2yb}(x_2,0)$$

where σ_{1yb} and σ_{2yb} contain all bounded terms.

The singular integrals in these expressions are calculated by using the formulas given in Muskhelishvili (1953).

When $a < h_1$, near the edge at $x_1 = a$:

$$\frac{1}{\pi}\int_{-a}^{a}\frac{m_1(t)}{t-x_1}dt=-\frac{m_1^*(a)}{\sqrt{2a}}\frac{1}{\sqrt{x_1 a}}+\phi_1(x_1), \tag{2.42}$$

when $0<b$, $c<h_2$, near the edges at $x_2=b$, c:

$$\frac{1}{\pi}\int_{b}^{c}\frac{m_2(t)}{t-x_2}dt=\frac{im_2^*(b)}{\sqrt{c-b}}\frac{1}{\sqrt{x_2-b}}-\frac{m_2^*(c)}{\sqrt{c-b}}\frac{1}{\sqrt{x_2-c}}+\phi_2(x_2), \tag{2.43}$$

when $a=h_1$, $0<b$, near the edge at $x_1=h_1$:

$$\frac{1}{\pi}\int_{-h_1}^{h_1}\frac{m_1(t)}{t-h_1-x_2}dt=-\frac{m_1^*(h_1)}{(2h_1)^{\alpha}\sin(\pi\alpha)}\frac{1}{x_2^{\alpha}}+\phi_3(x_2), \tag{2.44}$$

when $a<h_1$, $0=b$, $c<h_2$, near the edge at $x_2=0$:

$$\frac{1}{\pi}\int_{0}^{c}\frac{m_2(t)}{t-x_1+h_1}dt=\frac{m_2^*(0)e^{i\pi\psi}}{\sqrt{c}\sin(\pi\psi)}\frac{1}{(x_1-h_1)^{\psi}}+\phi_4(x_1), \tag{2.45}$$

when $a<h_1$, $0=b$, $c<h_2$, near the edge at $x_2=c$:

$$\frac{1}{\pi}\int_{0}^{c}\frac{m_2(t)}{t-x_2}dt=-\frac{m_2^*(c)}{c^{\psi}}\frac{1}{\sqrt{x_2-c}}+\phi_5(x_2), \tag{2.46}$$

where ϕ_i, (i=1-5), are bounded functions.

Using the appropriate formulas from Eqs.(2.42)-(2.45) and also Eqs.(2.41) in Eqs.(2.38)-(2.40), the expressions for the stress intensity factors are obtained in the form:

$$K_a = -\frac{4\mu_1}{1+\kappa_1}\frac{m_1^*(a)}{\sqrt{a}}, \text{ (a<h}_1\text{)},$$

$$= \frac{1-3\kappa_1+2(\kappa_1-1)\alpha+(1-\kappa_2)\lambda(\alpha-1)}{(\lambda+\kappa_1)(1+\lambda\kappa_2)2^{1/2+\alpha}\sin(\pi\alpha)}4\mu_1\frac{m_1^*(h_1)}{h_1^\alpha}, \ (a=h_1), \qquad (2.47a, b)$$

$$K_b = \frac{4\mu_2}{1+\kappa_2}\frac{m_2^*(b)}{\sqrt{\frac{c-b}{2}}}, \text{ (0<b, c<h}_2\text{)},$$

$$= \frac{3-\kappa_1-2(1-\kappa_1)\psi-(1-3\kappa_2)\lambda+2(1-\kappa_2)\lambda\psi}{(\lambda+\kappa_1)(1+\lambda\kappa_2)\sqrt{2}\sin(\pi\psi)}4\mu_1\frac{m_2^*(0)}{\sqrt{c}}, \text{ (a<h}_1\text{, 0=b, c<h}_2\text{)},$$

(2.48a, b)

$$K_c = -\frac{4\mu_2}{1+\kappa_2}\frac{m_2^*(c)}{\sqrt{\frac{c-b}{2}}}, \text{ (0<b, c<h}_2\text{)}$$

$$= -\sqrt{2}\frac{4\mu_2}{1+\kappa_2}\frac{m_2^*(c)}{c^\psi}, \text{ (0=b, c<h}_2\text{)}, \qquad (2.49a, b)$$

CHAPTER III

NUMERICAL SOLUTION

First, dimensionless variables will be introduced on the cracks.

$x_1 = a\xi$, $t = a\eta$, (-a < (x_1, t) < a), (3.1a, b)

$x_2 = \frac{c-b}{2}\varepsilon + \frac{c+b}{2}$, $t = \frac{c-b}{2}\rho + \frac{c+b}{2}$ (b < (x_2, t) < c) (3.2a, b)

so that the singular integral equations, Eqs.(2.23a, b), and the single valuedness conditions, Eqs.(2.20a, b), can be rewritten in the form:

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\eta-\xi} + \bar{k}_{11s}(\xi,\eta) + \bar{k}_{11b}(\xi,\eta)\right]\bar{m}_1(\eta)d\eta$$

$$+\frac{1}{\pi}\int_{-1}^{1}\left[\bar{k}_{12s}(\xi,\rho) + \bar{k}_{12b}(\xi,\rho)\right]\bar{m}_2(\rho)d\rho = -1 \text{, (-1<ξ<1),} \quad (3.3a, b)$$

$$\frac{1}{\pi}\int_{-1}^{1}\left[\bar{k}_{21s}(\varepsilon,\eta) + \bar{k}_{21b}(\varepsilon,\eta)\right]\bar{m}_1(\eta)d\eta$$

$$+\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\rho-\varepsilon} + \frac{1}{\rho+\varepsilon+2\frac{c+b}{c-b}} + \bar{k}_{22s}(\varepsilon,\rho) + \bar{k}_{22b}(\varepsilon,\rho)\right]\bar{m}_2(\rho)d\rho = -1$$

(-1<ε<1), (3.4a, b)

$$\int_{-1}^{1}\bar{m}_1(\eta)d\eta = 0, \quad \int_{-1}^{1}\bar{m}_2(\rho)d\rho = 0$$

where

$$\bar{k}_{11s}(\xi,\eta) = ak_{11s}(a\xi,a\eta),$$

$$\bar{k}_{11b}(\xi,\eta) = \frac{a}{h_1}\int_{0}^{\infty}K_{11b}\left(a\xi,a\eta,\frac{\omega}{h_1}\right)d\omega \text{, (ω=β}h_1\text{),}$$

$$\bar{k}_{12s}(\xi,\rho)=\frac{\kappa_1+1}{\kappa_2+1}\frac{c-b}{8}\bar{k}_{12s}\left(a\xi,\frac{c-b}{2}\rho,\frac{c+b}{2}\right),$$

$$\bar{k}_{12b}(\xi,\rho)=\frac{\kappa_1+1}{\kappa_2+1}\frac{c-b}{8h_2}\int_0^\infty K_{12b}\left(a\xi,\frac{c-b}{2}\rho+\frac{c+b}{2},\frac{\omega}{h_2}\right)d\omega,\ (\omega=\beta h_2),$$

$$\bar{k}_{21s}(\varepsilon,\eta)=\frac{\kappa_2+1}{\kappa_1+1}\frac{a}{4}k_{21s}\left(\frac{c-b}{2}\varepsilon+\frac{c+b}{2},a\eta\right),$$

$$\bar{k}_{21b}(\varepsilon,\eta)=\frac{\kappa_2+1}{\kappa_1+1}\frac{a}{4h_1}\int_0^\infty K_{21b}\left(\frac{c-b}{2}\varepsilon+\frac{c+b}{2},a\eta,\frac{\omega}{h_1}\right)d\omega,\ (\omega=\beta h_1),$$

$$\bar{k}_{22s}(\varepsilon,\rho)=\frac{c-b}{8}k_{22s}\left(\frac{c-b}{2}\varepsilon+\frac{c+b}{2},\frac{c-b}{2}\rho+\frac{c+b}{2}\right),$$

$$\bar{k}_{22b}=\frac{c-b}{8h_2}\int_0^\infty K_{22b}\left(\frac{c-b}{2}\varepsilon+\frac{c+b}{2},\frac{c-b}{2}\rho+\frac{c+b}{2},\frac{\omega}{h_2}\right)d\omega,\ (\omega=\beta h_2)$$

(3.5a-h)

and

$$\bar{m}_1(\eta)=\frac{4\mu_1 m_1(a\eta)}{(\kappa_1+1)p_1},$$

$$\bar{m}_2(\rho)=\frac{4\mu_2 m2\left(\frac{c-b}{2}\rho+\frac{c+b}{2}\right)}{(\kappa_2+1)p_2} \tag{3.6a, b}$$

Note that the relation $\dfrac{(\kappa_1+1)p_1}{\mu_1}=\dfrac{(\kappa_2+1)p_2}{\mu_2}$ (3.7)

As a result of the requirement that the strips should not separate at $y=\pm\infty$ has been used. The integrals in Eqs.(3.3) and (3.4) will be evaluated by means of Gauss quadrature formulas. The infinite integral will be calculated by using the Laguerre Integration formula, Abramowitz and Stegun (1965).

3.1. Embedded Cracks (a < h_1, 0 < b, c < h_2)

In this case, for all integrals in both η and ρ, Gauss-Lobatto quadrature formula (Krenk (1975), will be used. Then, the following linear algebraic equations are obtained:

$$\sum_{i=1}^{n} C_i \left\{ \left[\frac{1}{\eta_i - \xi_j} + \bar{k}_{11s}(\xi_j, \eta_i) + \bar{k}_{11b}(\xi_j, \eta_i) \right] m_1^{**}(\eta_i) \right.$$

$$\left. + \left[\bar{k}_{12s}(\xi_j, \rho_i) + \bar{k}_{12b}(\xi_j, \rho_i) \right] m_2^{**}(\rho_i) \right\} = -1 \text{, (j=1,.....,n-1),} \tag{3.8}$$

$$\sum_{i=1}^{n} C_i \left\{ \left[\bar{k}_{21s}(\varepsilon_j, \eta_i) + \bar{k}_{21b}(\varepsilon_j, \eta_i) \right] m_1^{**}(\eta_i) \right.$$

$$\left. + \left[\frac{1}{\rho_i - \varepsilon_j} + \frac{1}{\rho_i + \varepsilon_j + 2\frac{c+b}{c-b}} + \bar{k}_{22s}(\varepsilon_j, \rho_i) + \bar{k}_{22b}(\varepsilon_j, \rho_i) \right] m_2^{**}(\rho_i) \right\} = -1$$

(j=1,.....,n-1)

$$\sum_{i=1}^{n} C_i m_1^{**}(\eta_i) = 0 \text{ , } \sum_{i=1}^{n} C_i m_2^{**}(\rho_i) = 0$$

where

$$\eta_i = \rho_i = \cos\left(\frac{i-1}{n-1} \pi \right), \text{ (i=1,......,n) ,}$$

$$\xi_j = \varepsilon_j = \cos\left(\frac{2j-1}{2n-2} \pi \right), \text{ (j=1,........,n-1),}$$

$$C_i = \frac{1}{(n-1)}, \text{ (i=2,.....,n-1);} \quad C_1 = C_n = \frac{1}{2(n-1)} \tag{3.9a-c}$$

$m_1^{**}(\eta)$ and $m_2^{**}(\rho)$ are defined by

$$\overline{m}_1(\eta)=\frac{m_1^{**}(\eta)}{\sqrt{1-\eta^2}},$$

$$\overline{m}_2(\rho)=\frac{m_2^{**}(\rho)}{\sqrt{1-\rho^2}}, \qquad (3.10a, b)$$

3.2. Broken Strip1 $(a=h_1,\ 0<b,\ c<h_2)$

In this case, the integrals on ρ will again be evaluated by Gauss-Lobatto quadrature formula, while the integrals in η will be evaluated by Gauss-Jacobi integration formula. The resulting linear algebraic equations will be in the form:

$$\sum_{i=1}^{n}\left\{W_i\left[\frac{1}{\eta_\iota-\xi_j}+\overline{k}_{11s}(\xi_j,\eta_i)+\overline{k}_{11b}(\xi_j,\eta_i)\right]m_1^{***}(\eta_i)\right.$$
$$\left.+C_i\left[\overline{k}_{12s}(\xi_j,\rho_i)+\overline{k}_{12b}(\xi_j,\rho_i)\right]m_2^{**}(\rho_i)\ \right\}=-1, \qquad \text{(j=1,……,n-1)}$$

$$\sum_{i=1}^{n}\left\{W_i\left[\overline{k}_{21s}(\varepsilon_j,\eta_i)+\overline{k}_{21b}(\varepsilon_j,\eta_i)\right]m_1^{***}(\eta_i)\right.$$
$$\left.+C_i\left[\frac{1}{\rho_i-\varepsilon_j}+\frac{1}{\rho_i+\varepsilon_j+2\dfrac{c+b}{c-b}}+\overline{k}_{22s}(\varepsilon_j,\rho_i)+\overline{k}_{22b}(\varepsilon_j,\rho_i)\right]m_2^{**}(\rho_i)\ \right\}=-1,$$

(j=1,……,n-1),

$$\sum_{i=1}^{n}W_i m_1^{***}(\eta_i)=0,\ \sum_{i=1}^{n}C_i m_2^{**}(\rho_i)=0 \qquad (3.11)$$

where ρ_i, C_i (i=1,…..,n), and ε_j, (j=1,…..,n-1), are still given by Eqs.(3.9a-c). However, ξ_j, (j=1, … ,n-1), and η_i, W_i , (i=1, … ,n), are the roots and the weights of the Jacobi polynomials.

$$P_{n-1}^{(1-\alpha,1-\alpha)}(\xi_j) = 0, \text{ (j=1,…..,n-1)},$$

$$P_n^{(-\alpha,-\alpha)}(\eta_j) = 0, \text{ (i=1,….,n)}, \qquad (3.12a, b)$$

and

$$m_1^{***}(\eta) = \overline{m}_1(\eta)(1-\eta^2)^\alpha, \qquad (3.13)$$

3.3. Crack in Strip 2 Touching the Interface(a<h_1, 0=b, c<h_2)

In this case, in contrary to the case in Section 3.2, the integrals on η will be evaluated by Gauss-Lobatto quadrature formula, while the integrals on ρ will be evaluated by Gauss-Jacobi integration formula. The system of linear algebraic equations will look like

$$\sum_{i=1}^{n}\Bigg\{C_i\left[\frac{1}{\eta_i-\xi_j}+\bar{k}_{11s}(\xi_j,\eta_i)+\bar{k}_{11b}(\xi_j,\eta_i)\right]m_1^{**}(\eta_i)$$
$$+W_i\left[\bar{k}_{12s}(\xi_j,\rho_i)+\bar{k}_{12b}(\xi_j,\rho_i)\right]m_2^{***}(\rho_i)\ \Bigg\}=-1, \qquad \text{(j=1, … , n-1)},$$

$$\sum_{i=1}^{n}\Bigg\{C_i\left[\bar{k}_{21s}(\varepsilon_j,\eta_i)+\bar{k}_{21b}(\varepsilon_j,\eta_i)\right]m_1^{**}(\eta_i)$$
$$+W_i\left[\frac{1}{\rho_i-\varepsilon_j}+\frac{1}{\rho_i+\varepsilon_j+2\frac{c+b}{c-b}}+\bar{k}_{22s}(\varepsilon_j,\rho_i)+\bar{k}_{22b}(\varepsilon_j,\rho_i)\right]m_2^{***}(\rho_i)\ \Bigg\}=-1,$$

(j=1, … , n-1),

$$\sum_{\imath=1}^{n} C_i m_1^{**}(\eta_i) = 0\,,\ \sum_{\imath=1}^{n} W_i m_2^{***}(\rho_i) = 0 \tag{3.14}$$

where η_i, C_i, (i=1,…,n), and ξ_j, (j=1,….,n-1) are given by Eqs.(3.9a-c). on the other hand, ε_j, (j=1, … , n-1), and ρ_i, W_i, (i=1, … , n), are the roots and the weights of Jacobi polynomials.

$$P_{n-1}^{(1-\psi,0.5)}(\varepsilon_j) = 0\,,\ \text{(j=1,…..,n-1)},$$

$$P_{n}^{(-\psi,-0.5)}(\rho_j) = 0\,,\ \text{(i=1,….,n)}, \tag{3.15a, b}$$

and

$$m_2^{***}(\rho) = \overline{m}_1(\rho)(1-\eta)^{1/2}(1+\eta)^{\psi}\,, \tag{3.16}$$

3.4. Calculation of Stress Intensity Factors

Once the system of linear algebraic equations is solved and the unknown functions are calculated at discrete collocation points, every relevant quantity can be calculated. For example, the stress intensity factors

$$K_a = -m_1^{**}(1)p_1\sqrt{a}\,,\ (a<h_1) \tag{3.17a, b}$$

$$= \frac{1-3\kappa_1 + 2(\kappa_1-1)\alpha + (1-\kappa_2)\lambda(\alpha-1)}{(\lambda+\kappa_1)(1+\lambda\kappa_2)2^{1/2+\alpha}\sin(\pi\alpha)}(1+\kappa_1)m_1^{**}(1)p_1h_1^{\alpha}\,,\qquad (a=h_1,\ 0<b),$$

$$K_b = m_2^{**}(-1)p_2\sqrt{\frac{c-b}{2}}\,,\ (0<b,\ c<h_2) \tag{3.18a, b}$$

$$= \frac{3-\kappa_1 - 2(1-\kappa_1)\psi - (1-3\kappa_2)\lambda + 2(1-\kappa_2)\lambda\psi}{(\lambda+\kappa_1)(1+\lambda\kappa_2)\sin(\pi\psi)}\frac{\lambda(1+\kappa_2)}{\sqrt{2}}m_2^{***}(-1)p_2\left(\frac{c}{2}\right)^{\psi}$$

($a<h_1$, $0=b$, $c<h_2$),

$$K_c = -m_2^{**}(1) p_2 \sqrt{\frac{c-b}{2}}\text{, (0<b, c<h}_2\text{),}$$

$$= -2^{1/2-\psi} m_2^{***}(1) p_2 \sqrt{\frac{c}{2}}\text{, (a<h}_1\text{, 0=b, c<h}_2\text{)} \qquad (3.19a, b)$$

It would obviously be advantageous to present the stress intensity factors in normalized form:

$$k_a = \frac{K_a}{p_1\sqrt{a}} = -m_1^{**}(1)\text{, (a<h}_1\text{),}$$

$$= \frac{K_a}{p_1 h_1^{\alpha}} = \frac{1-3\kappa_1+2(\kappa_1-1)\alpha+(1-\kappa_2)\lambda(\alpha-1)}{(\lambda+\kappa_1)(1+\lambda\kappa_2)2^{1/2+\alpha}\sin(\pi\alpha)}(1+\kappa_1)m_1^{**}(1), \qquad (3.20a, b)$$

(a=h$_1$, 0<b)

$$k_b = \frac{K_b}{p_2\sqrt{\frac{c-b}{2}}} = m_2^{**}(-1), \qquad \text{(0<b, c<h}_2\text{)}$$

$$= \frac{K_b}{p_2\left(\frac{c}{2}\right)^{\psi}} = \frac{3-\kappa_1-2(1-\kappa_1)\psi-(1-3\kappa_2)\lambda+2(1-\kappa_2)\lambda\psi}{(\lambda+\kappa_1)(1+\lambda\kappa_2)\sin(\pi\psi)}\frac{\lambda(1+\kappa_2)}{\sqrt{2}}m_2^{***}(-1),$$

(a<h$_1$, 0=b, c<h$_2$),

(3.21a, b)

$$k_c = \frac{K_c}{p_2\sqrt{\frac{c-b}{2}}} = -m_2^{**}(1)\text{, (0<b, c<h}_2\text{)}$$

$$= \frac{K_c}{p_2\sqrt{\frac{c}{2}}} = -2^{1/2-\psi} m_2^{***}(1)\text{, (a<h}_1\text{, 0=b, c<h}_2\text{),} \qquad (3.22a, b)$$

CHAPTER IV

RESULTS AND CONCLUSIONS

4.1. Results

a/h Gupta(1974)	b/h Gupta(1974)	k_b Gupta(1974)	k_b Present Study	k_c Gupta(1974)	k_c Present Study
0.1	0.5	1.1746	1.1755	1.1169	1.1144
0.2	0.6	1.1102	1.1086	1.0961	1.0936
0.4	0.8	1.0984	1.0963	1.1250	1.1217
0.5	0.9	1.1290	1.1272	1.2278	1.2210
0.6	1.0	1.6080	1.6020	$\rightarrow \infty$	$\rightarrow \infty$

Table 4.1 : Comparison of the Results of This Study With the Results of G.D. Gupta and F. Erdogan (1974)

In this study, 6 examples are considered and five of these examples are obtained for plane stress conditions only with fixed ratio of h_1/h_2 =1. The last example is considered with different geometric ratios such as h_1/h_2 = 0.5, 1, 2 for different widths of the both Strip 1 and Strip 2 with plane stress case.

The values given in Table4.1 and Figs. (4.1 –4.5) show the results of this study are in good agreement with those given in G.D.Gupta and F.Erdogan(1974), Gecit(1979-a) and I.N.Sneddon and R.P.Srivastav(1971).

Figs. 4.6-4.14 show the variations of the normalized stress intensity factors k_a, k_b and k_c for various crack length combinations in Strip 1 and Strip 2 with the softer material at the middle. ($\lambda = \mu_1/ \mu_2$=1/23.077)

Figure 4.6 shows when $c = 0.9h_2$ is fixed as constant, k_a decreases when the crack in Strip 1 approaches to the interface, whereas Figs 4.7 and 4.8 show that k_b and k_c are increasing slightly.

Figures 4.9–4.11 show that k_a, k_b and k_c have decreasing tendencies all together with different rates as b/h_2 ratio approaches to the value 0.9, when $c = 0.9h_2$ is kept as constant. For $a = 0.98h_1$, k_a and k_b has the strongest decreasing tendency, as compared to that of k_c.

Figures 4.12–4.13 show that k_a, k_b and k_c increase with growing size of the crack in Strip2 when $b = 0.1h_2$ is kept as constant. It is noted that k_c has the strongest variation rate when compared to k_a and k_b depending on the length of the central crack.

Figures 4.15–4.23 show the variations of k_a, k_b and k_c for various crack length combinations in Strip 1 and Strip 2 with softer material at the middle ($\lambda = 1/3$).

Figure 4.15 shows that when $c = 0.9h_2$ is fixed; k_a has a decreasing behavior when the cracks in the outer materials are not closer than $b = 0.3\ h_2$ to the interface. If b/h_2 ratio gets closer to zero, k_a increases up to $a/h_1 = 0.9$ then decreases to zero. However, in Figures 4.16 and 4.17, k_b and k_c are increasing as a/h_1 ratio goes to 1 in Strip 1.

Figures 4.18–4.20 show the same characteristics as seen in Figures 4.9–4.11 but in the latter case, k_a, k_b and k_c have lower values when compared to the former.

Figures 4.21–4.23 exhibit the same increasing behavior as Figures 4.12–4.14. After $c = 0.7\ h_2$; k_c increases dramatically.

Figures 4.24 – 4.32 show the variations of k_a, k_b and k_c for various crack length combinations of cracks in Strip 1 and Strip 2 with the same material constants (λ = 1). In this case the problem reduces to the case of a strip made of one type of material containing three cracks on x-axis with different lengths.

Figures 4.24 – 4.26 show that when c = 0.9 h_2 is fixed as constant, k_a, k_b and k_c are increasing, unlike the behavior seen in Figures 4.6 – 4.11. It is noted that the values of k_a has the strongest rate of increase as a/ h_l approaches to 1.

Figures 4.27–4.29 show that k_a, k_b and k_c decrease as b gets closer to c. Also, k_b has the biggest and k_c has the smallest decreasing rate for a = 0.98 h_l and c = 0.9 h_2.

Figures 4.30–4.32 show that k_a values start from the interval of [1, 1.2] and constantly increase as c gets away from the interface when b=0.1 h_2 is kept as constant. However, for a/h_l $\geq$ 0.7, k_b and especially k_c values start decreasing, and after around c = 0.7 h_2 , they start to increase again and pass the initial values and stop at some relatively higher values.

Figures 4.33–4.41 show the variations of k_a, k_b and k_c for various crack length combinations in Strip1 and Strip 2 with stiffer material at the middle (λ = 3), unlikely the previous cases.

Figures 4.33–4.35 show k_a, k_b and k_c are increasing as a/h_l approaches to 1 when c = 0.9 h_2 is kept as constant, and the values of k_b for the b = 0.3 h_2 have the strongest increasing rate because of the interaction with the crack in Strip 1

Figures 4.36–4.38 show that k_a, k_b and k_c increase as a/h_l increases. The largest values occur as b/h_2 goes to zero, it is also noted that k_b is the largest of the three stress intensity factors. This is due to greater degree of interaction between the cracks in Strip 1 and Strip 2 when a/h_1 is greater. For smaller values of a/h_1, k_b starts with relatively smaller values when b/h_2 is very close to zero in Strip 2, for which inner edge of the crack is in close vicinity of the central crack so that it is in a highly disturb stress field. As b/h_2 increases, k_b also increases since this effect decreases as the inner edge of the crack gets away from the interface. With further increase in b/h_2, k_b decreases due to the fact that the crack length decreases.

Figures 4.39–4.41 show that k_a values start from the interval of [1.2 ; 2] and increase slightly with increasing values of a/h_1 as c/h_2 approaches to 1. However, for the high values of a/h_1, k_b and especially k_c values first decrease and after passing $c = 0.7\ h_2$, they start to increase again.

Figures 4.42–4.50 show the variations of k_a, k_b and k_c for various crack length combinations in Strip1 and Strip2 with relatively stiffer panel at the middle of two softer panels. ($\lambda = 23.077$)

Figures 4.42–4.44 show that as k_a increases rapidly and is not affected too much by the length of the cracks in the strips at the two sides since the material of Strip 1 is considerably stiffer. However, k_b and k_c both increase as a/h_1 approaches to 1.

Figures 4.45–4.47 show that the values of k_a are not affected regardless of the size of the crack in Strip 1, the only dominant variable, which determines the value of k_a, is the size of the crack in Strip 1. However, as b approaches to c, k_b and k_c are both affected by the size of the crack in Strip1 and by the size of the crack in Strip 2. But in general, both k_b and k_c decrease when b/h_2 gets closer to the value of 0.9.

Figures 4.48–4.50 show that as c/h_2 approaches to 1, k_a is not affected by the increasing size of the crack in Strip 2. However, with increasing crack length in Strip 2, the values of k_b decrease continuously. On the other hand, the values of k_c show a decreasing behavior until $c/h_2 \approx 0.75$. Then k_c values start to increase again because of being in close proximity to the outer edge of the strip.

Figures 4.51–4.68 show the variations of k_a, k_b and k_c for various widths of Strip 1 with various crack length combinations in Strip 1 and Strip 2 with stronger material at the middle and relatively softer (weaker) materials at the sides ($\lambda = 69.231$).

Figures 4.51–4.53 show the variations of k_a , k_b and k_c as function of b/h_2 for different a/h_1 ratios when the width of the Strip 2 is decreased from $h_1 = 0.5\ h_2$ and to $h_1 = h_2$ then to $h_1 = 2\ h_2$. Strip 2 is a considerably softer material then Strip 1 at the middle; hence decreasing the width of the cracks in Strip 2 (softer material) has no important effect on the values of k_a .

Figures 4.54–4.56 show an interesting behavior of k_b when $c = 0.9\ h_2$ is kept as constant (close to the edge of Strip 2). As b/h_2 ratio increases, the values of k_b increases up to some level then after some critical distance from the interface, the values start to decrease. Also the values of k_b are at the highest level when $h_1 = 2\ h_2$ (namely when width of the softer material is considerably smaller) and $a = 0.95\ h_1$ for all 3 different types of geometries.

Figures 4.57–4.59 show a general behavior of k_c when $c = 0.9\ h_2$ is kept as constant, as the b/h_2 ratio increases from zero to 0.9. It is observed that k_c values decrease slowly as with decreasing crack length in Strip 2. tendency. Also, as the width of the Strip 2 gets smaller, k_c attains higher values.

Figures 4.60–4.62 show that k_a values are at the highest level when the width of Strip 2 is at the smallest value $h_1 = 0.5\ h_2$ compared with other two cases, namely $h_1 = 2\ h_2$ and $h_1 = h_2$. Furthermore, for all the cases, the effect of c/h_2 ratio on k_a is not discernible at all.

Figures 4.63–4.65 show that k_b values are at the highest level when the width of the Strip 2 is smallest, namely when $h_1 = 2\ h_2$, k_b is around 7.4 for $a = 0.95\ h_1$; which occurs when c/h_2 is very close to 0.1. The values of k_b show a decreasing behavior when the crack in Strip 2 approaches to the outer edge of the Strip 2.

Figures 4.66 – 4.68 show k_c values are at the highest level when the width of Strip 2 is smallest, namely $h_1 = 2h_2$, where $b = 0.1\ h_2$ is kept as constant. In all the three different geometries the k_c values decrease up to $c = 0.8\ h_2$ and then increase for higher values of c/h_2.

Figures 4.69-4.71 show that k_a values have an interesting behavior depending on different values of λ. In the cases of λ values smaller than one; for which Strip 1 is weaker than Strip 2, k_a has a decreasing behavior as the ratio a/h_1 approaches to 1, namely, the crack in the Strip 1 approaches to the interface. On the other hand, for the cases of $\lambda \geq 1$; where Strip 1 is stronger than Strip 2, k_a values are increasing dramatically proportional to the increasing values of λ. On the other hand, k_b and k_c show similar behaviors to each other. When $\lambda \leq 1$, k_b and k_c are not effected too much, on the contrary for $\lambda \geq 1$, both values increase dramatically.

In figures 4.72-4.74 the variation of k_a, k_b, k_c are given as functions of b/h_2 for different values of λ. In these figures $h_1 = h_2$, $a = 0.1\ h_1$, $c = 0.9\ h_2$ are fixed. As can be seen from Fig. 4.72 k_a takes its maximum value for smallest λ, when b/h_2 goes to zero. For all λ values, k_a decreases to unity with increasing b/h_2 ratio. As b/h_2 goes to zero, only for $\lambda \leq 1$ values, k_b

increases but for $\lambda \geq 1$ values, k_b decreases. The same behavior can be seen for k_c. In these situations, k_b is always greater than k_c values.

Figures 4.75–4.77 indicate that when $\lambda \leq 1$, k_a, k_b and k_c values are increasing dramatically, and for $\lambda \geq 1$ values, k_a, k_b and k_c values are not increasing as much as the previous cases.

Figures 4.78-4.80 show that k_a values are increasing when $\lambda \geq 1$ as a/h_1 goes to1. Note that, (c-b) is very small and away from the interface. On the other hand, for $\lambda \leq 1$, k_a values decrease as a/h_1 goes to 1. However, k_b and k_c values increase as a/h_1 gets larger for all values of λ except for $\lambda = 1/3$ and 1/23.077. Moreover, k_b and k_c values are considerably close to each other for corresponding λ values, since the length of the crack in Strip 2 is very small.

Figures 4.81-4.83 show that k_a values increase dramatically as b/h_2 ratio approaches to zero. The increasing behavior is getting stronger as the values of λ decrease. The same decreasing tendency proportional to λ, except for λ = 23.077, can be seen also in k_b and k_c values. For $\lambda = 23.077$, k_b and k_c increase at the close vicinity of zero.

Figures 4.84-4.86 indicate that there is a different behavior in k_b and k_c compared to k_a depending on λ values such that k_a values are increasing independent from the values of λ. On the other hand, k_b values are increasing when c/h_2 ratio approaches to1 with $\lambda < 1$. However, for the values of $\lambda \geq 1$, the values of k_b first decrease and then starts to increase. For the k_c values, the most interesting thing is, when $c=0.8h_2$, the values show increasing tendency independent of the previous behavior.

Figures 4.87 and 4.90 show that both k_b and k_c values are increasing as a/h_1 approaches to 1 and they increase to very high values when the Strip 1 breaks completely.

4.2. Suggestions for Further Studies

In this thesis, solutions for the cases 1-3 given below are obtained and the remaining cases from 4 to 7 are yet to be found.

In addition to these cases, some more possible problem suggestions are numbered from 8 to 12.

1. $a < h_1$, $0 < b$, $c < h_2$
2. $a = h_1$, $0 < b$, $c < h_2$
3. $a < h_1$, $b = 0$, $c < h_2$
4. $a < h_1$, $b = 0$, $c = h_2$
5. $a = h_1$, $0 < b$, $c = h_2$
6. $a = h_1$, $0 = b$, $c < h_2$
7. $a = h_1$, $0 = b$, $c = h_2$
8. Two symmetrical inclusions can be inserted into Strip1
9. Two symmetrical inclusions can be inserted into Strip2
10. Case 8 and case 9 can be applied at the same time
11. The materials of the panel can be assumed as anisotropic
12. Some package programs like ANSYS or MARC can be used to verify the results found in this study.

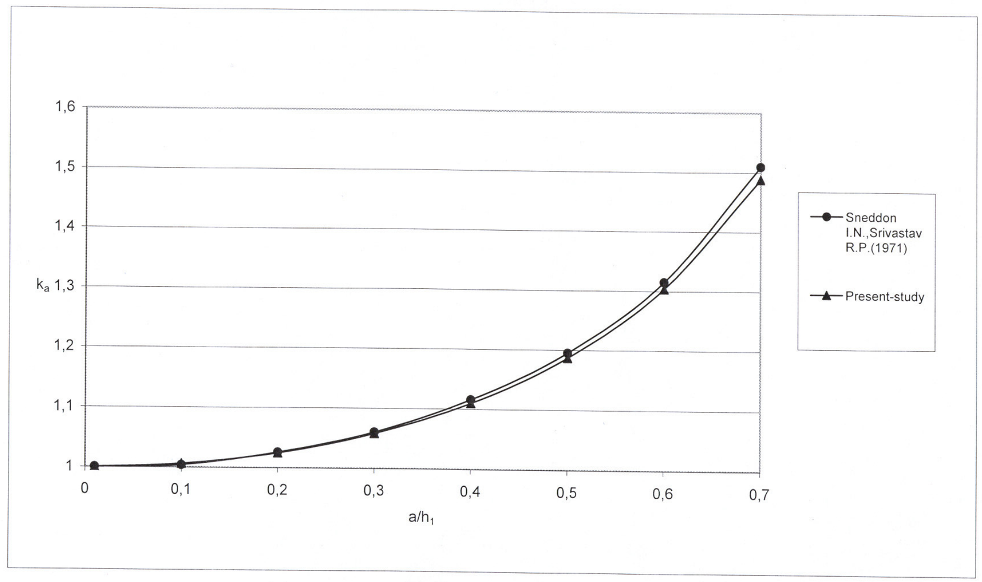

Figure 4.1. Comparison of the Results for an Infinite Strip with a Transverse Central Crack

$\nu_1 = \nu_2 = 0.3$ (plane strain) r1=0.4

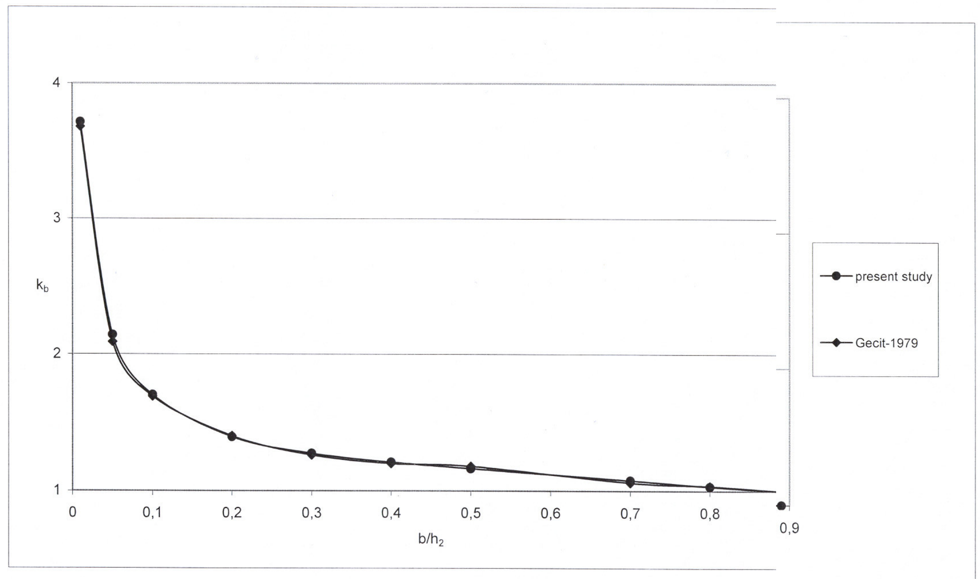

Figure 4.2. Comparison of the Variaion of k_b for ν_1=0.35 ν_2=0.3 r1=h_2/h_1=0.35 r2=a/h_1=0.001 r4=0.9 h_2 (pl. strain)

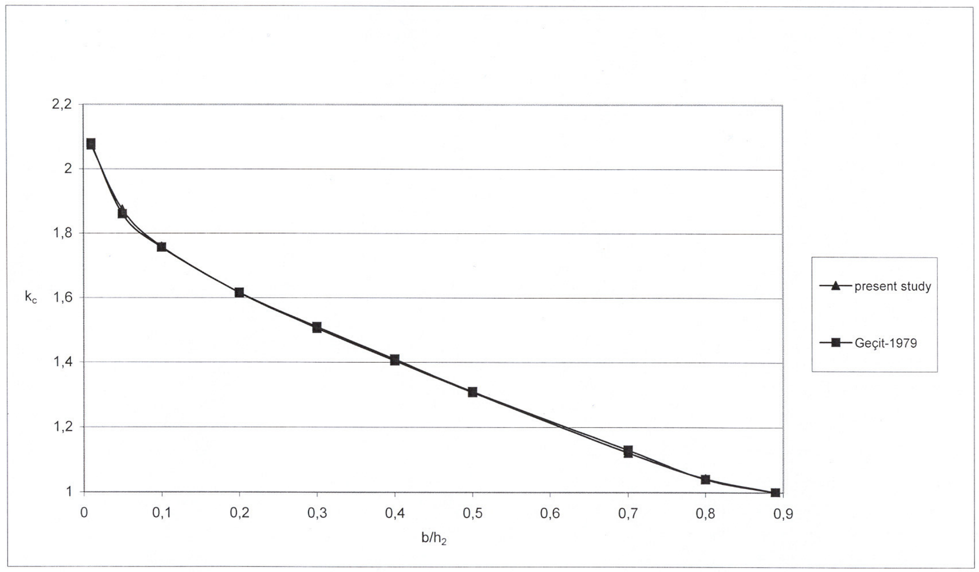

Figure 4.3. Comparison of the Variation for k_c for ν_1=0.35 ν_2=0.3 r1=h_2/h_1=0.35 r2=a/h_1=0.001 c=0.9 h_2 (pl. strain)

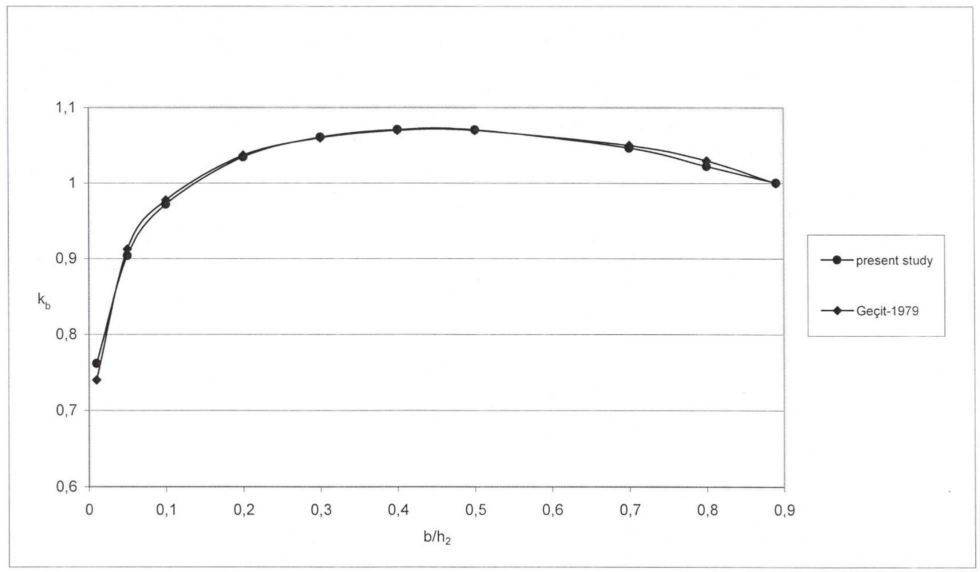

Figure 4.4. Comparison of the Variation for k_b for $\nu_1 = \nu_2 = 0.3$ $\lambda = 3$ r1=0.35 r2=a/h_1=0.001 r4=0.9 (Pl.Strain)

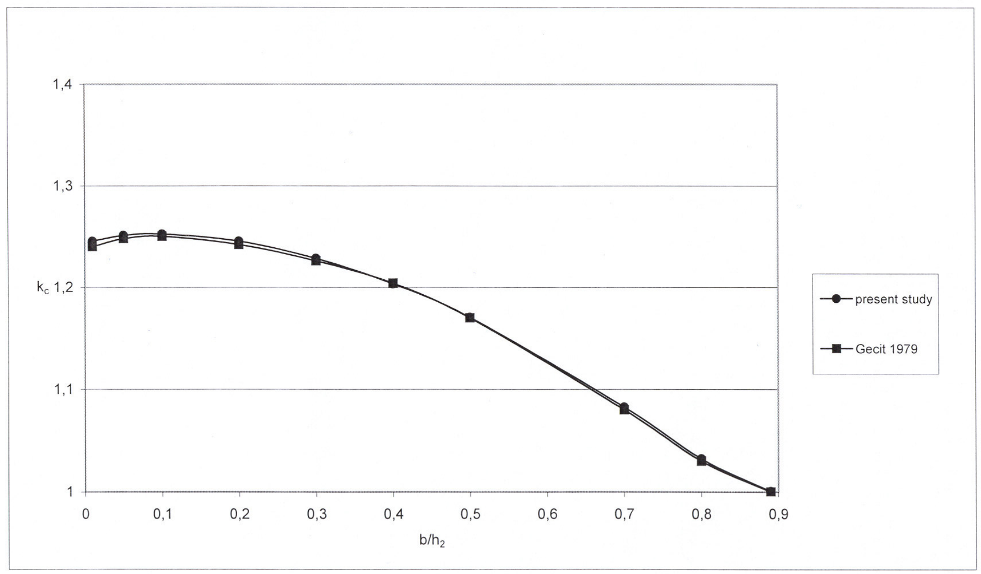

Figure 4.5. Comparison of the Variation for k_c for $\nu_1 = \nu_2 = 0.3$ $\lambda = 3.0$ r1=0.35 r2=a/h_1=0.001 c=0.9 h_2 (Pl.Strain)

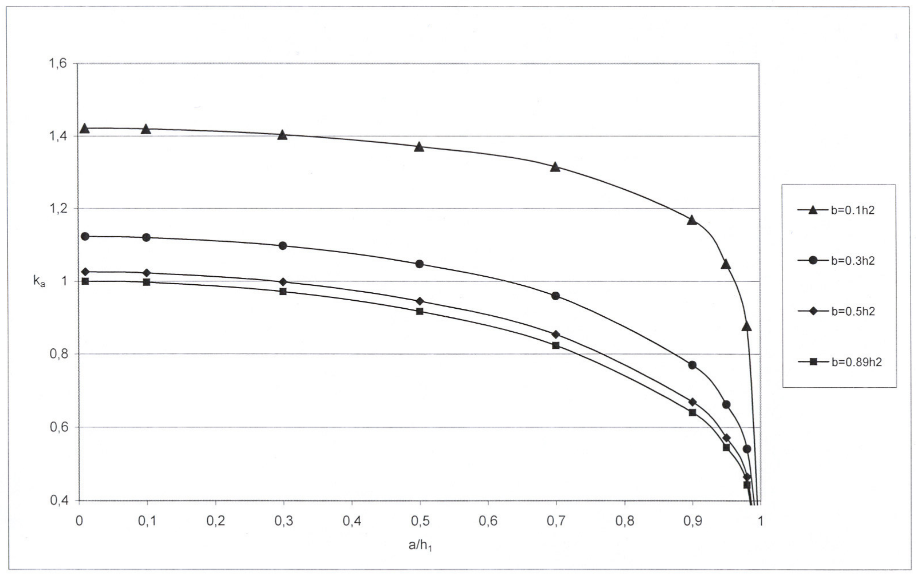

Figure 4.6. Variation of k_a with a/h_1 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1=23.077$, $h_2=h_1$, $c=0.9h_2$ (Plane Stress)

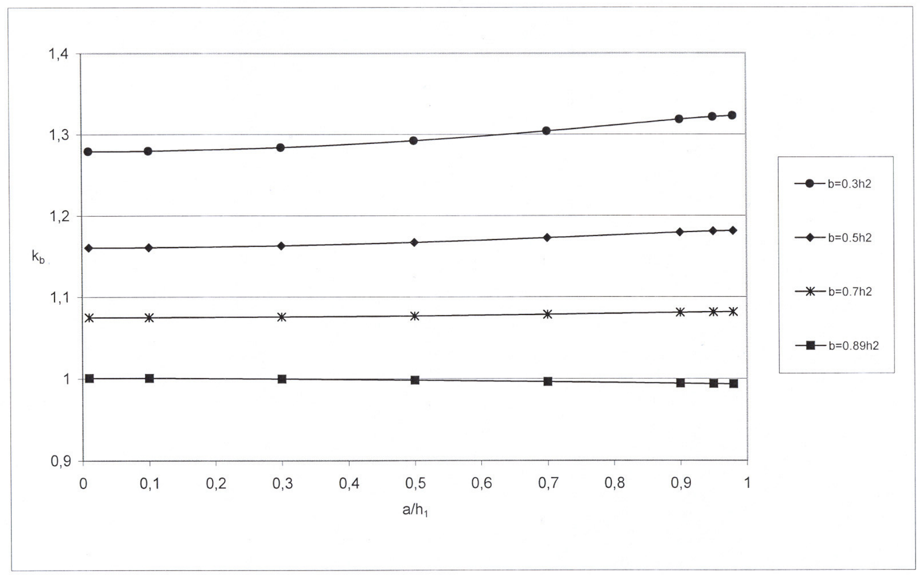

Figure 4.7. Variation of k_b with a/h_1 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

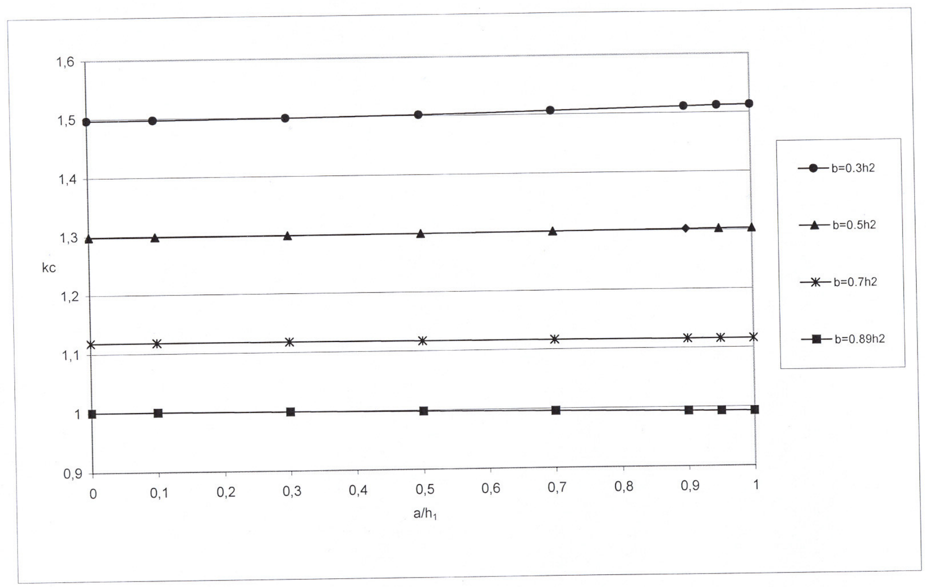

Figure 4.8. Variation of k_c with a/h_1 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

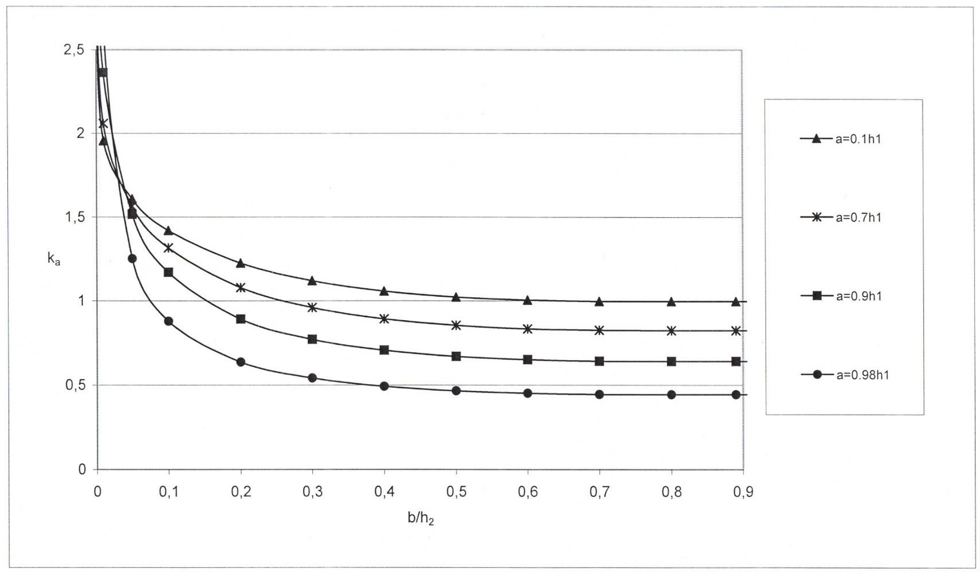

Figure 4.9. Variation of k_a with b/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

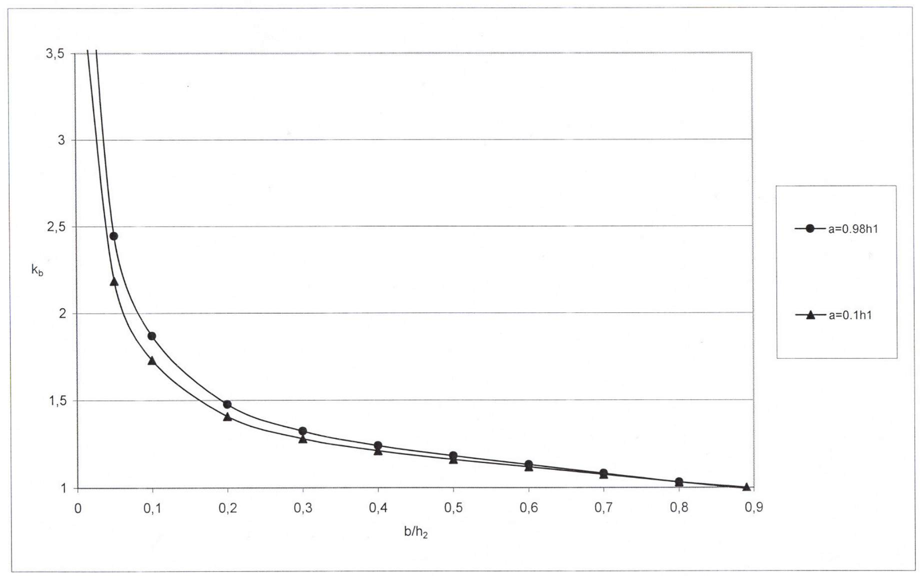

Figure 4.10. Variation of k_b with b/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

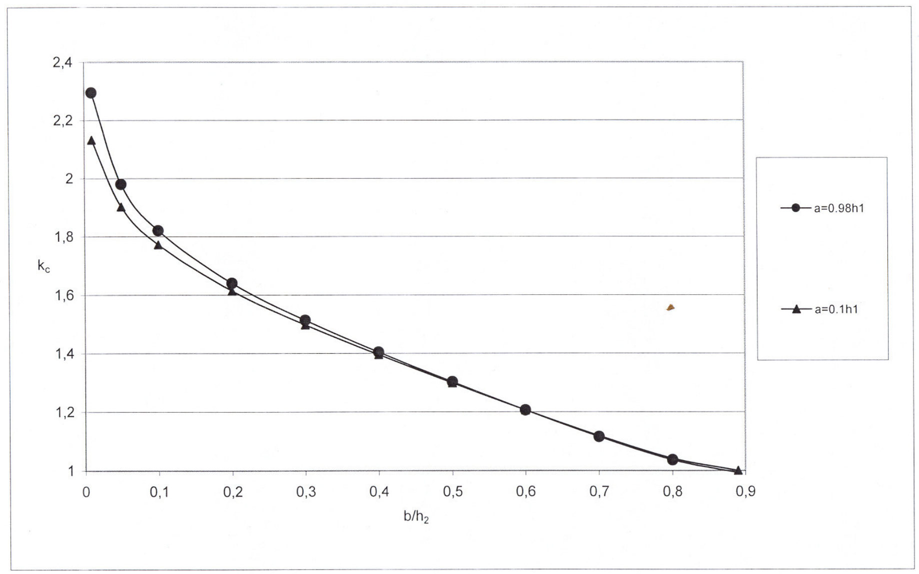

Figure 4.11. Variation of k_c with b/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

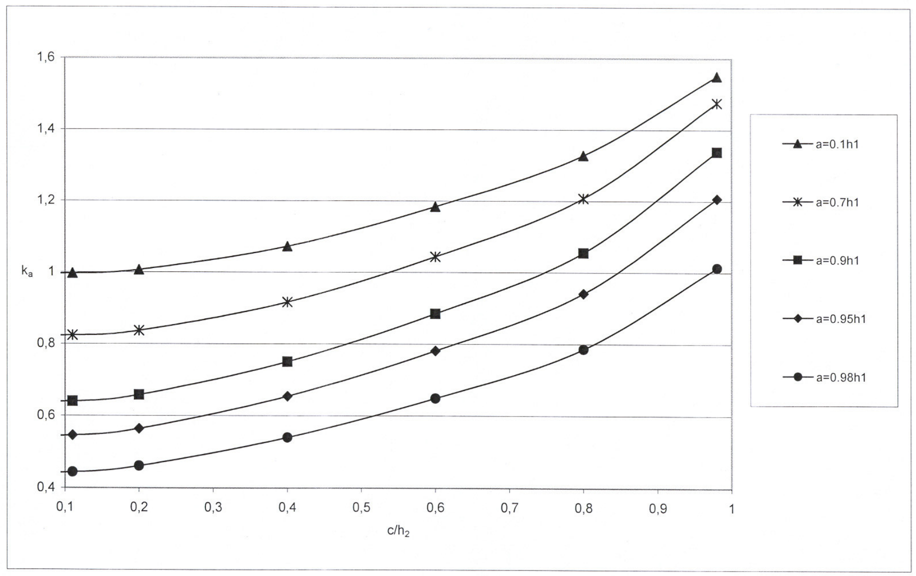

Figure 4.12. Variation of k_a with c/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

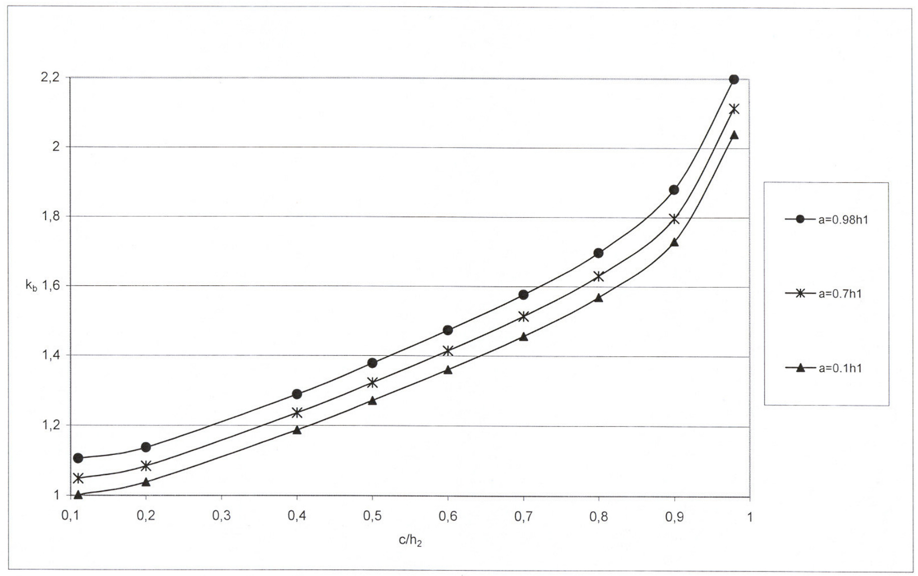

Figure 4.13. Variation of k_b with c/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $b=0.1h_2$,(Plane Stress)

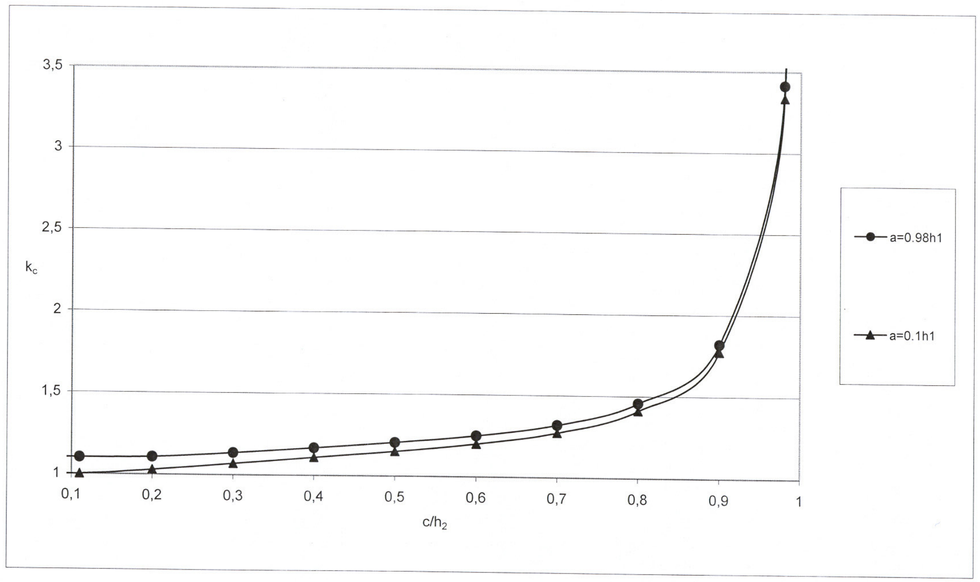

Figure 4.14. Variation of k_c with c/h_2 for 1:epoxy, 2:aluminum, $\mu_2/\mu_1 = 23.077$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

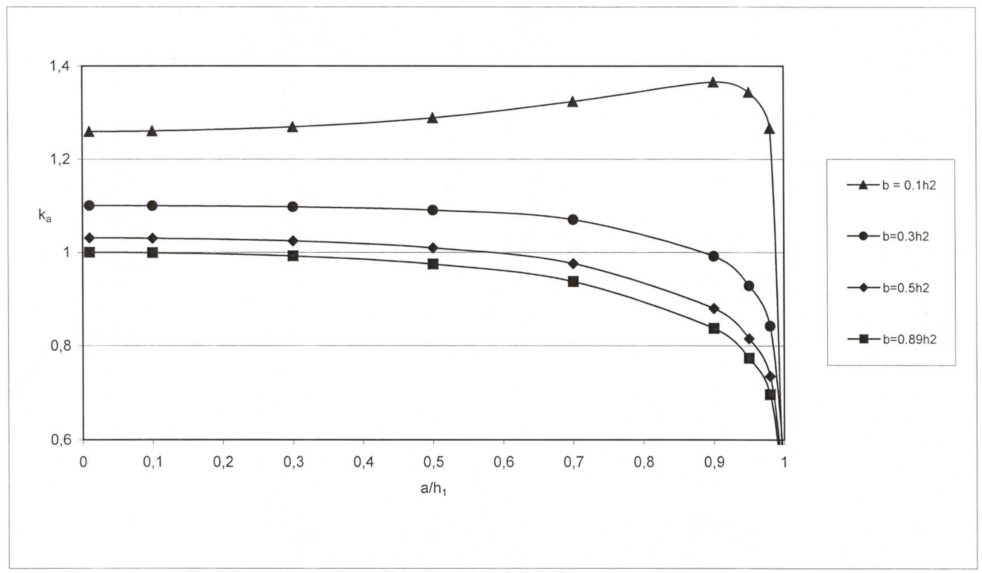

Figure 4.15. Variation of k_a with a/h_1 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

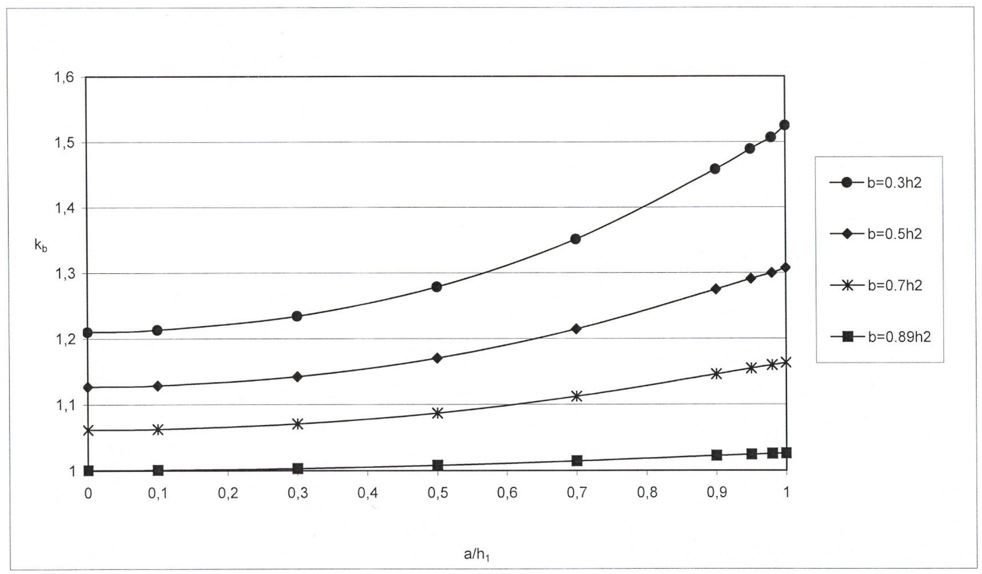

Figure 4.16. Variation of k_b with a/h_1 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

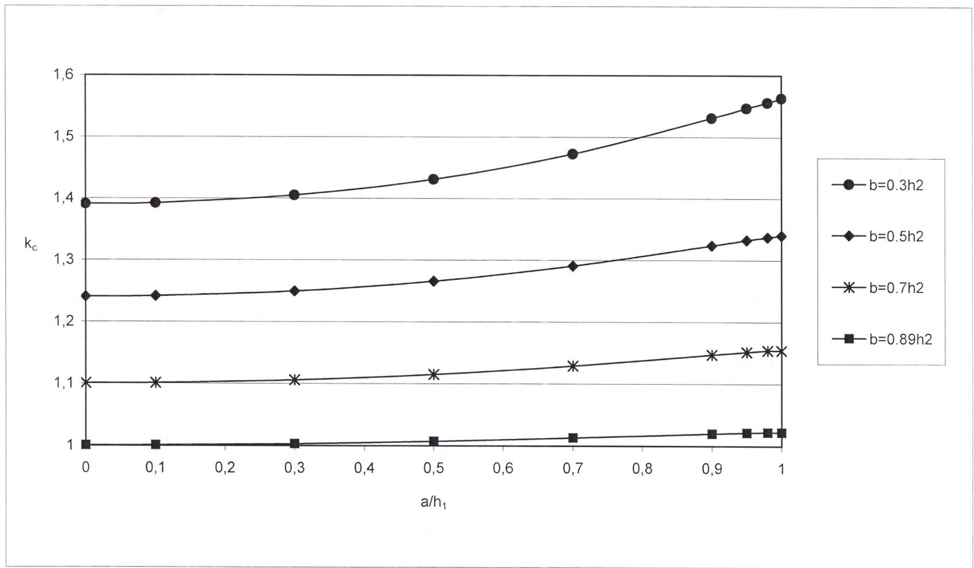

Figure 4.17. Variation of k_c with a/h_1 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

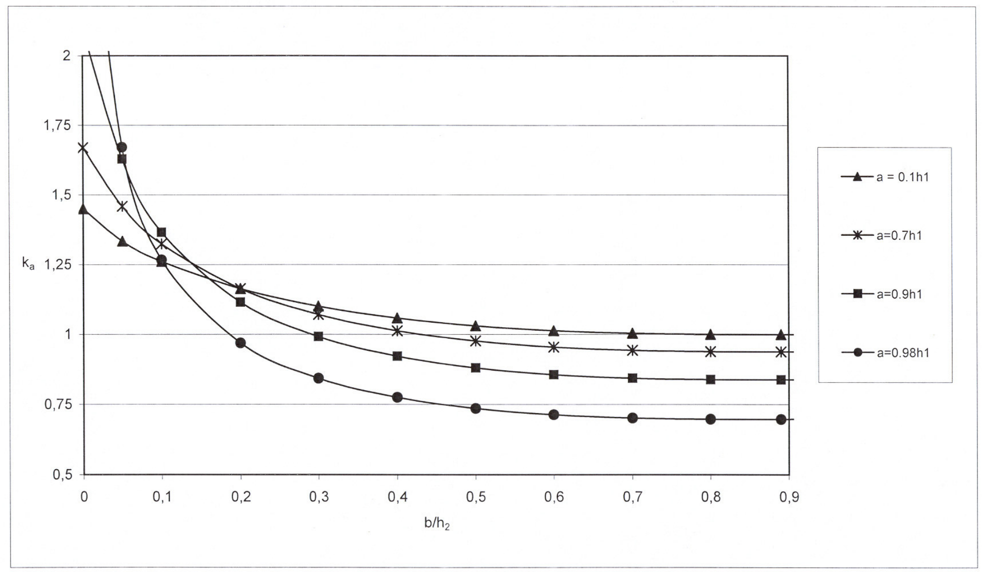

Figure 4.18. Variation of k_a with b/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

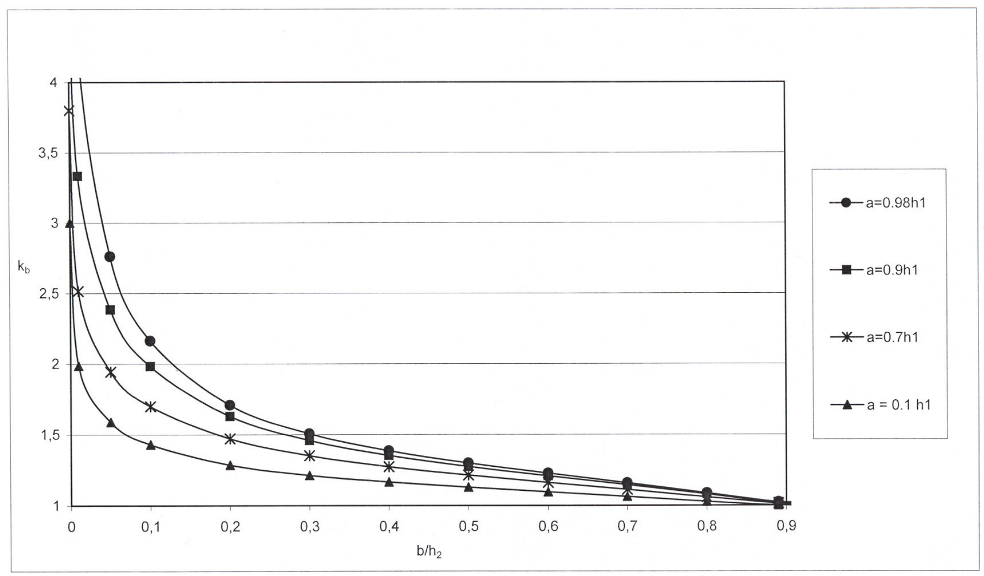

Figure 4.19. Variation of k_b with b/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

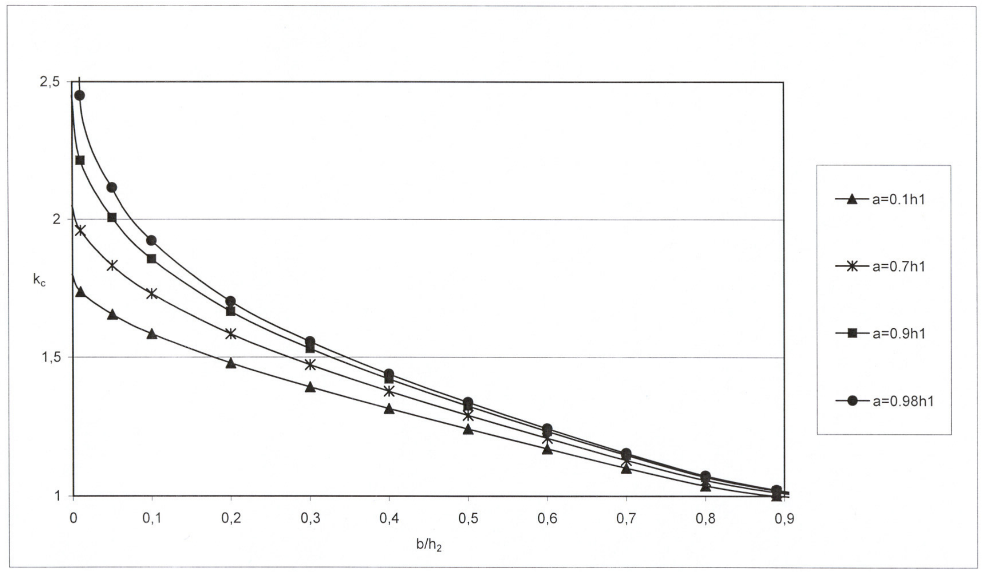

Figure 4.20. Variation of k_c with b/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

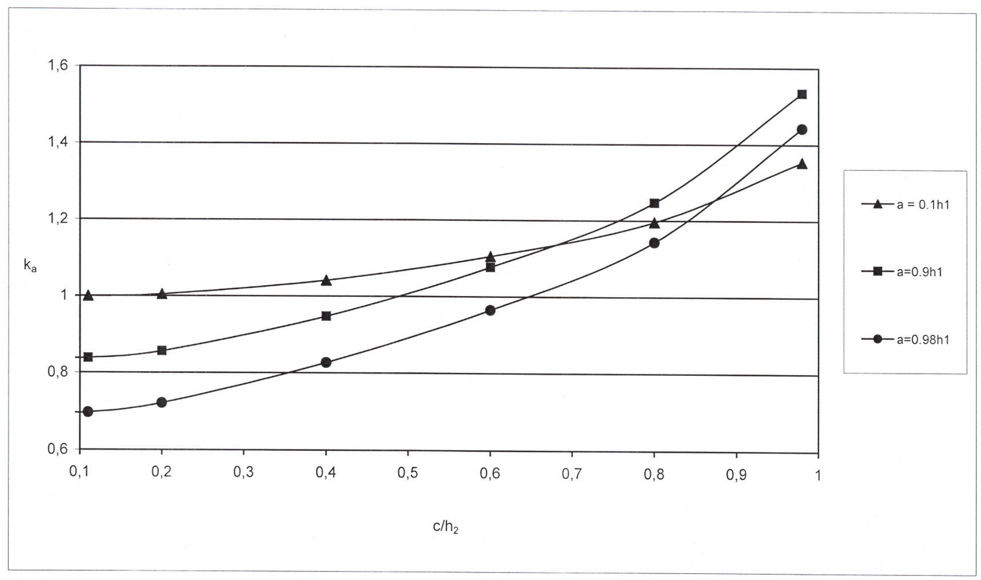

Figure 4.21. Variation of k_a with c/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

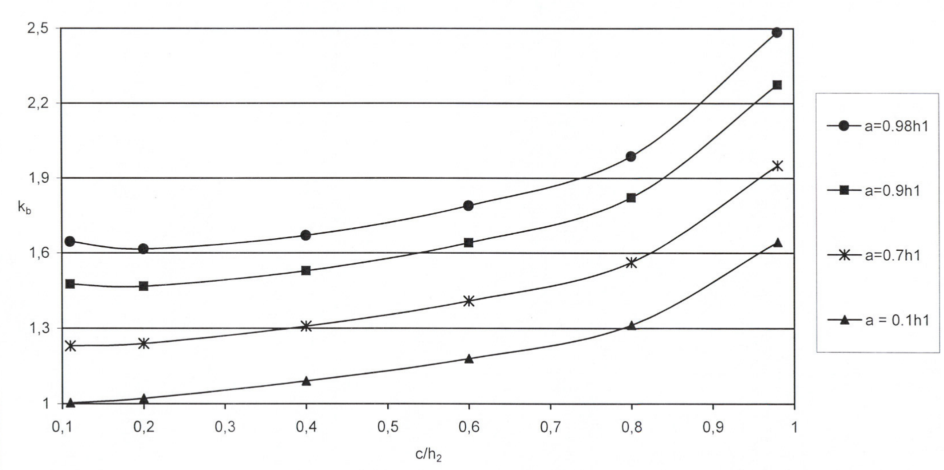

Figure 4.22. Variation of k_b with c/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

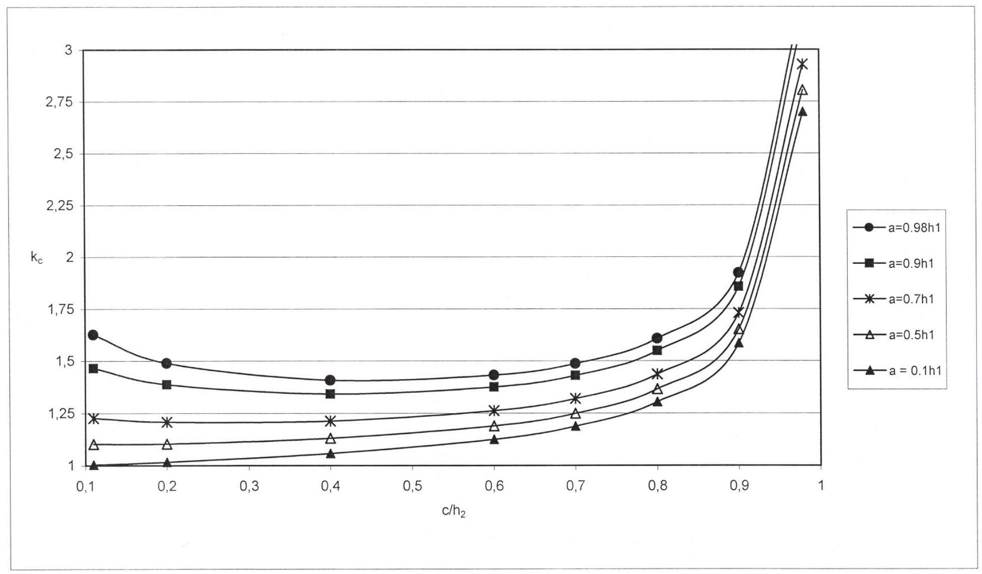

Figure 4.23. Variation of k_c with c/h_2 for 1:aliminum, 2:steel, $\mu_2/\mu_1 = 3$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

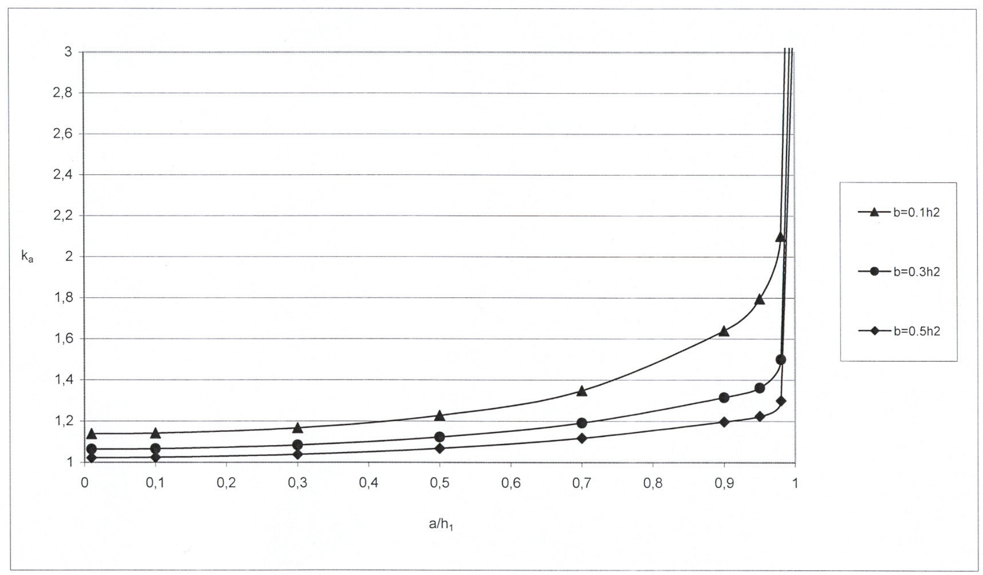

Figure 4.24. Variation of k_a with a/h_1 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

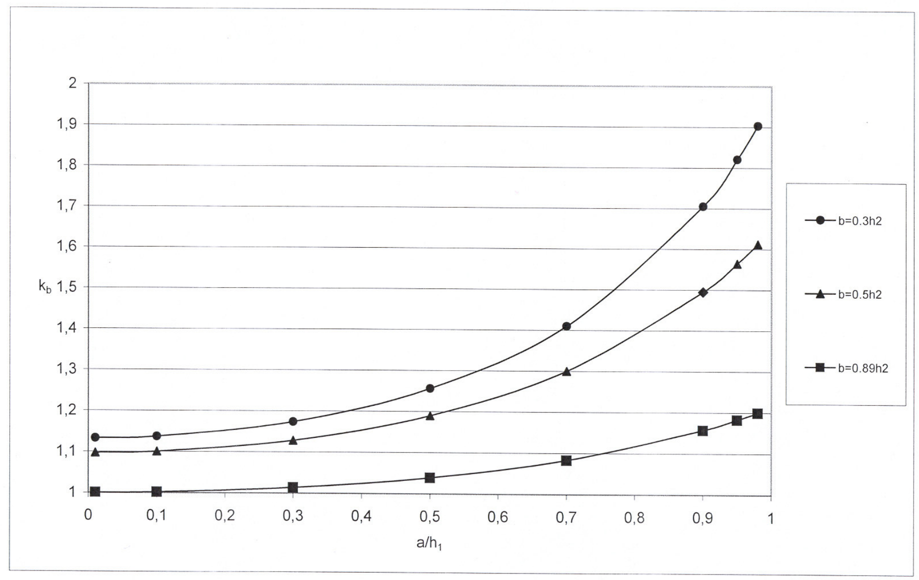

Figure 4.25. Variation of k_b with a/h_1 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

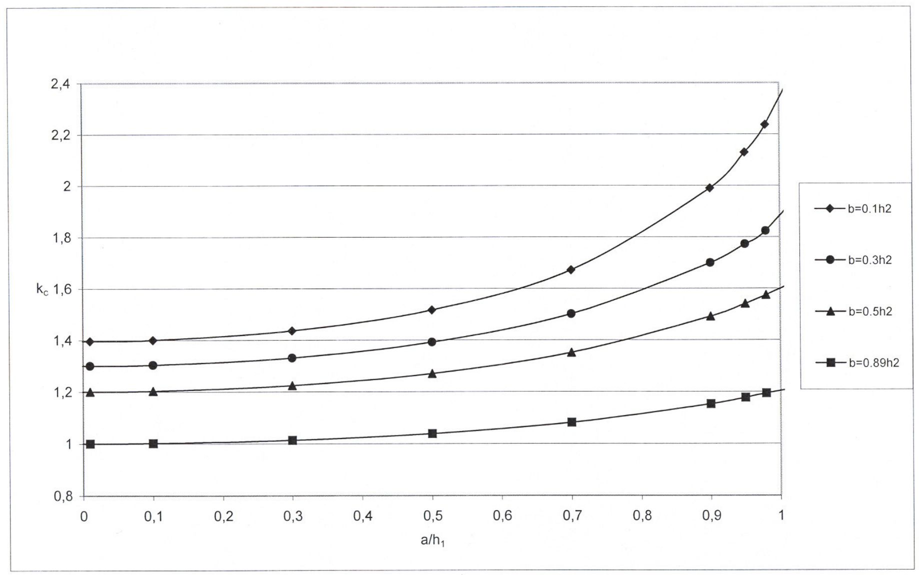

Figure 4.26. Variation of k_c with a/h_1 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, h_2=h_1, c=0.9h_2, (Plane Stress)

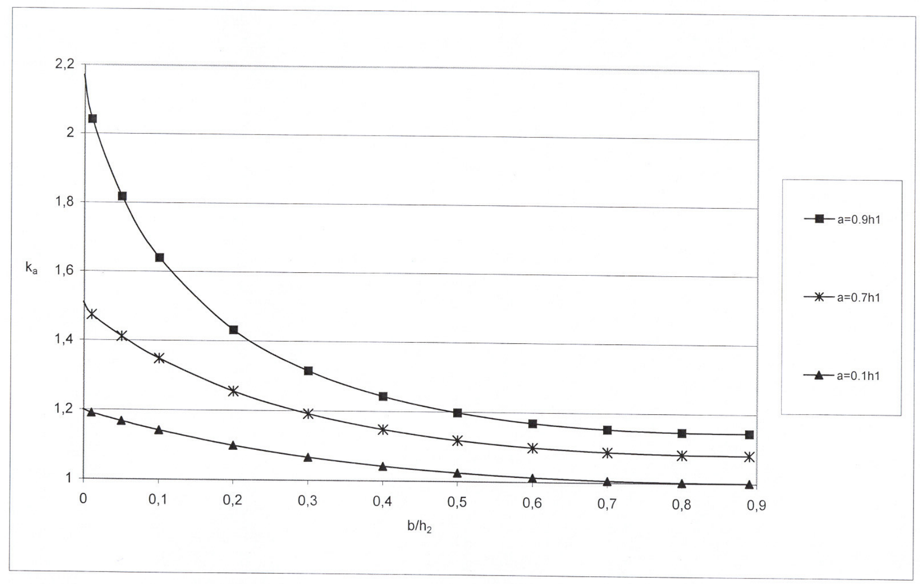

Figure 4.27. Variation of k_a with b/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

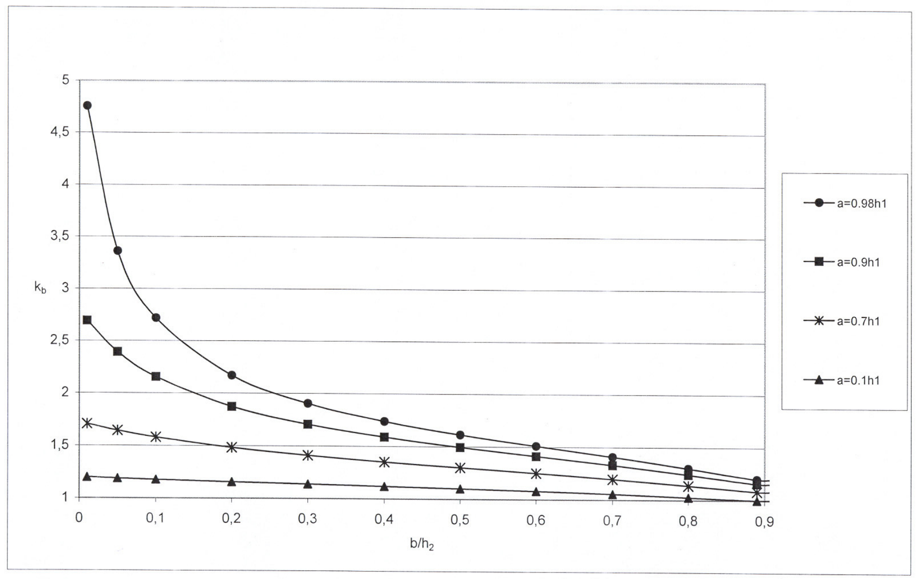

Figure 4.28. Variation of k_b with b/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

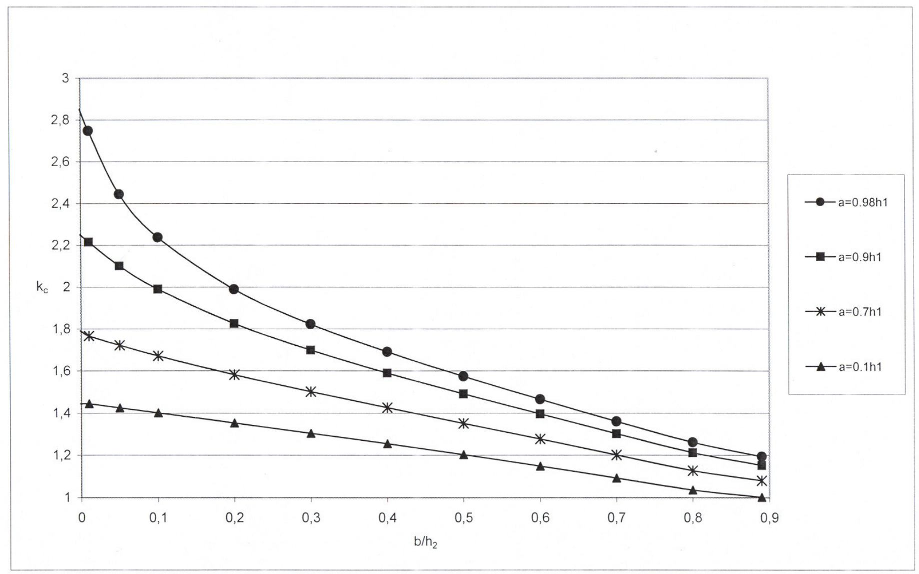

Figure 4.29. Variation of k_c with b/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, h_2=h_1, c=0.9h_2, (Plane Stress)

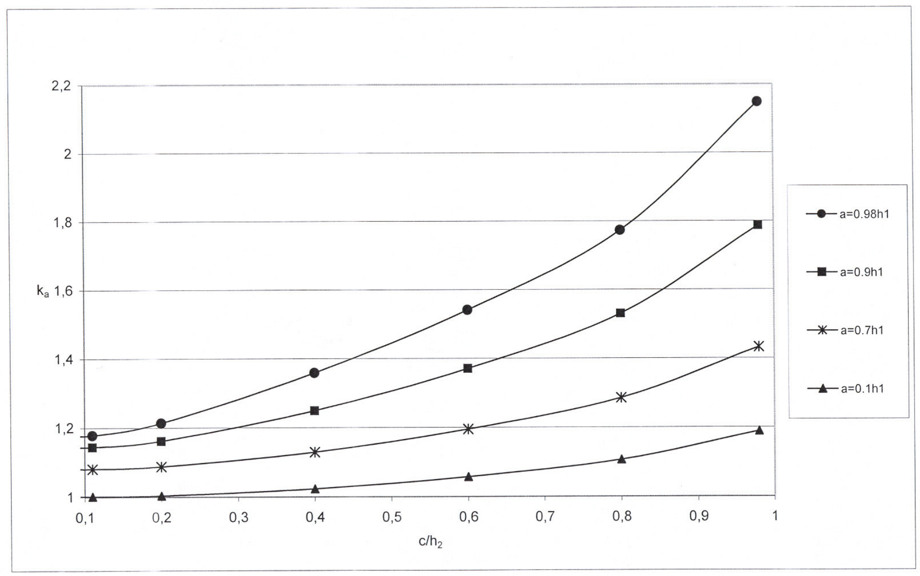

Figure 4.30. Variation of k_a with c/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

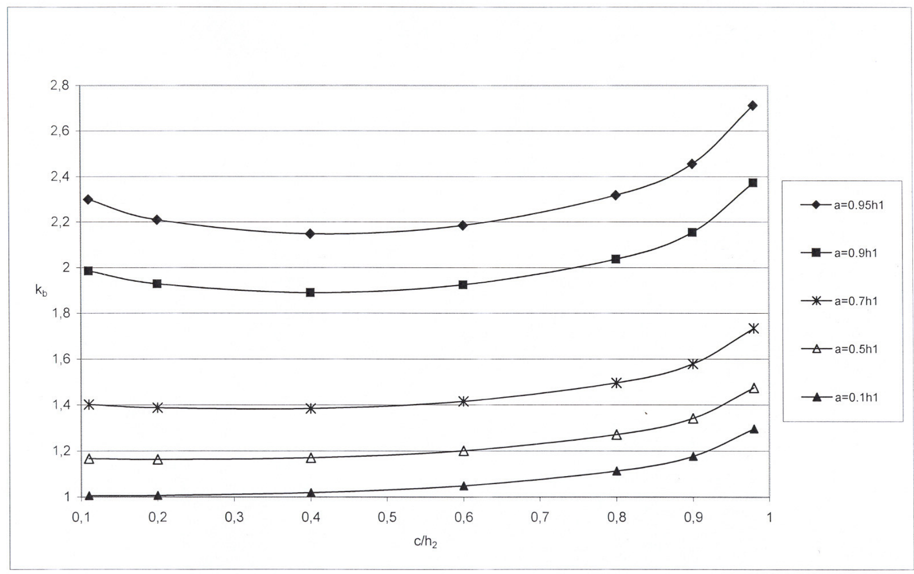

Figure 4.31. Variation of k_b with c/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

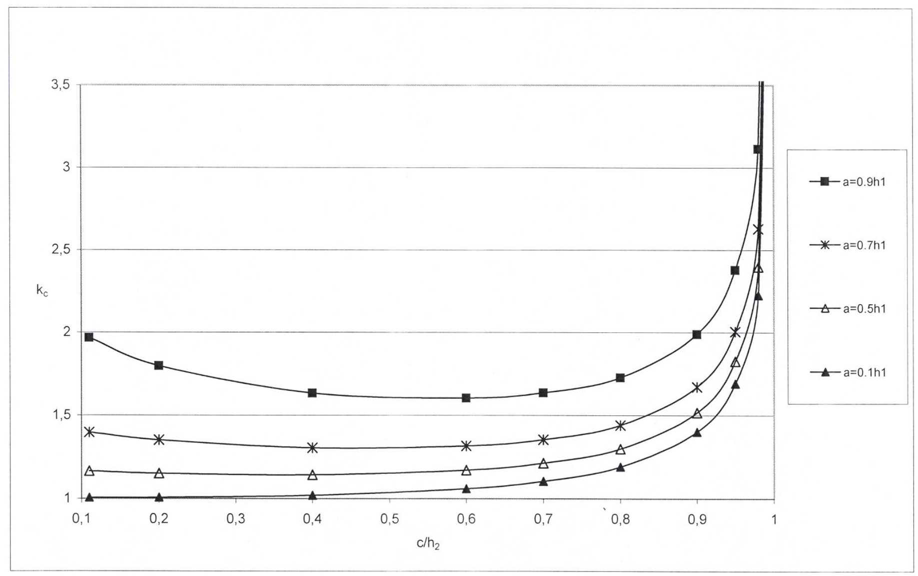

Figure 4.32. Variation of k_c with c/h_2 for 1:steel, 2:steel, $\mu_1/\mu_2 = 1$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

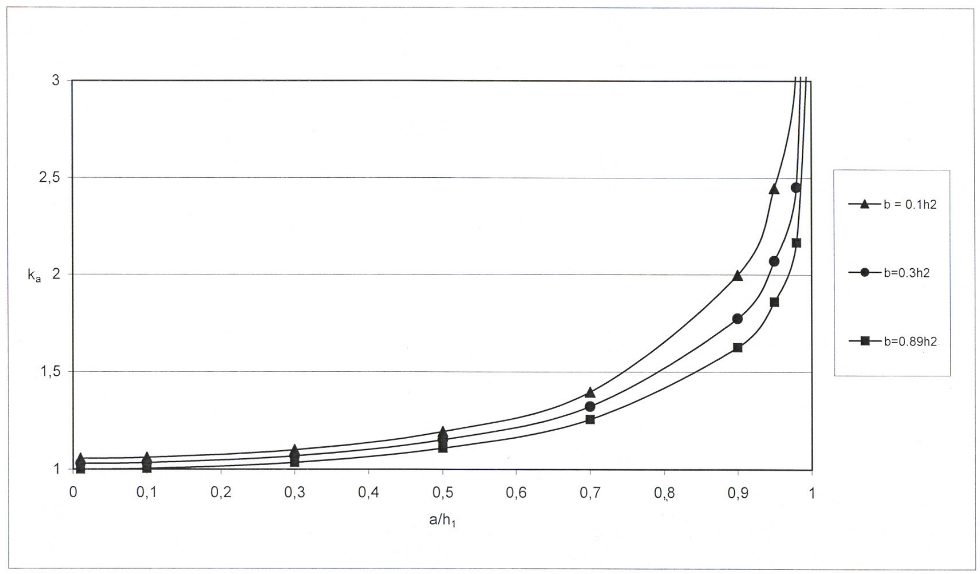

Figure 4.33. Variation of k_a with a/h_1 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

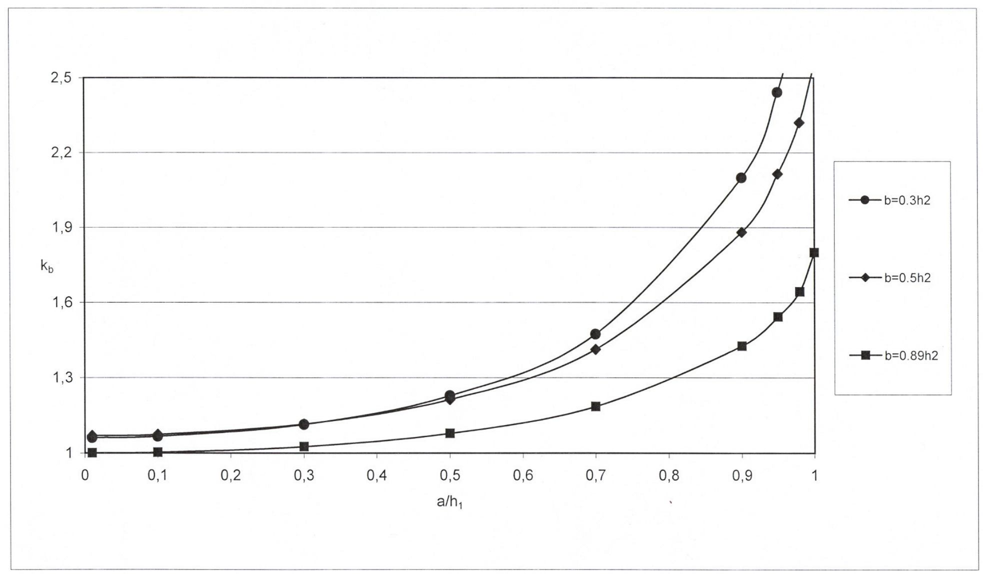

Figure 4.34. Variation of k_b with a/h_1 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

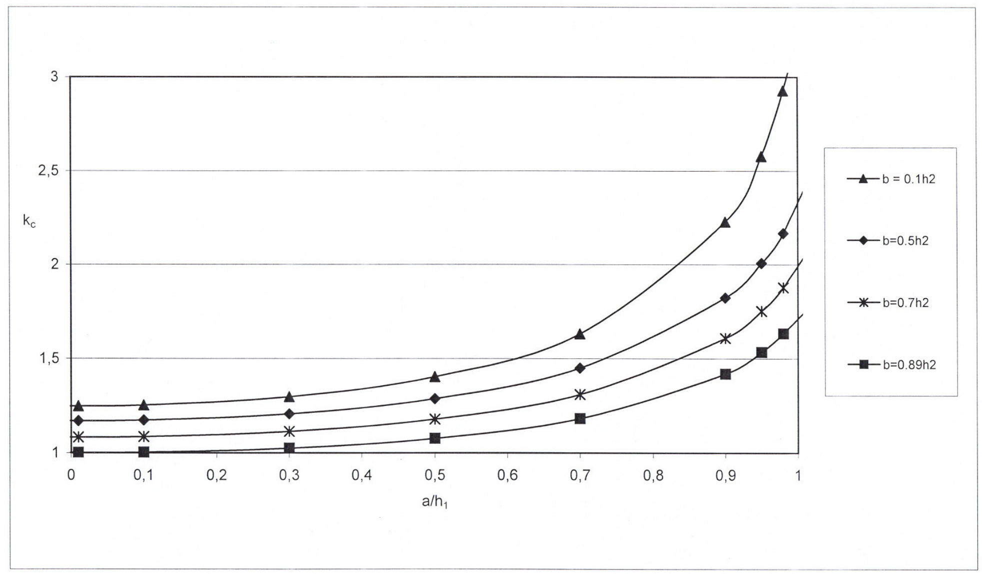

Figure 4.35. Variation of k_c with a/h_1 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

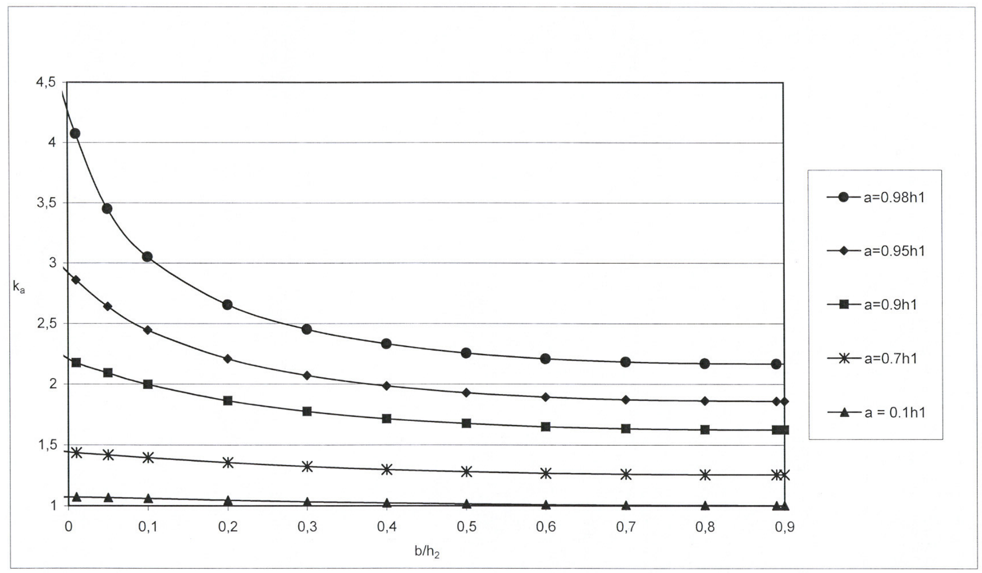

Figure 4.36. Variation of k_a with b/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, h_2=h_1, c=0.9h_2, (Plane Stress)

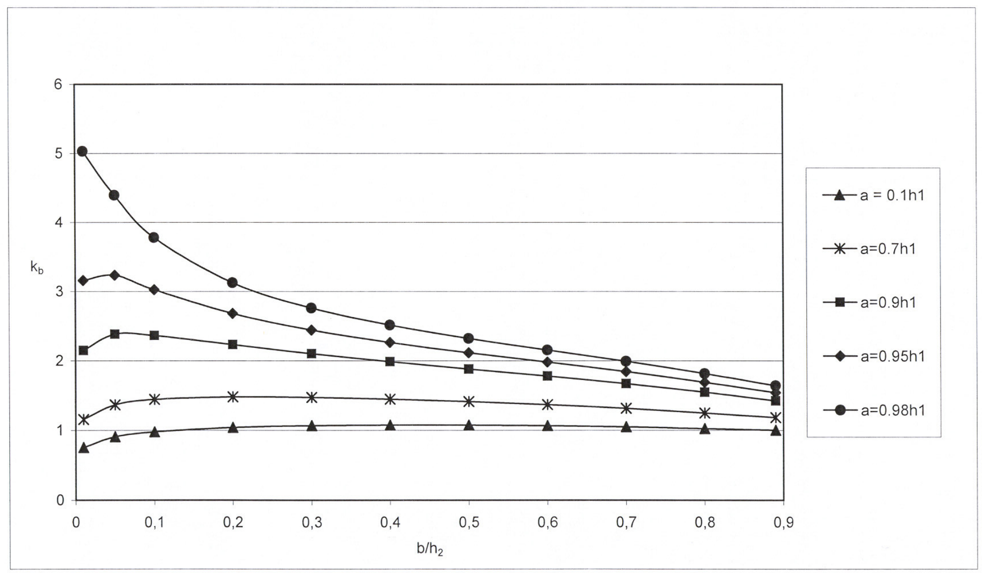

Figure 4.37. Variation of k_b with b/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

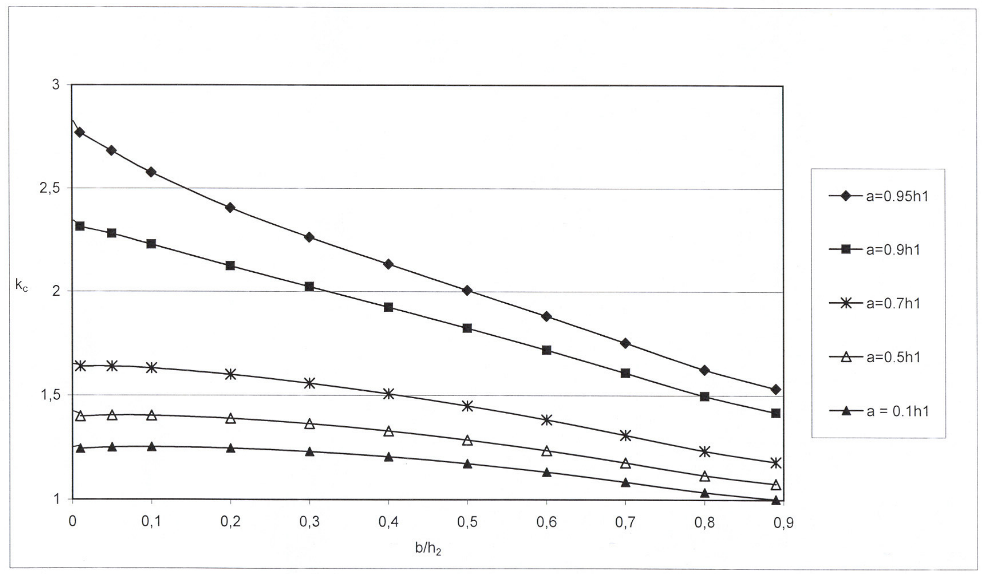

Figure 4.38. Variation of k_c with b/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

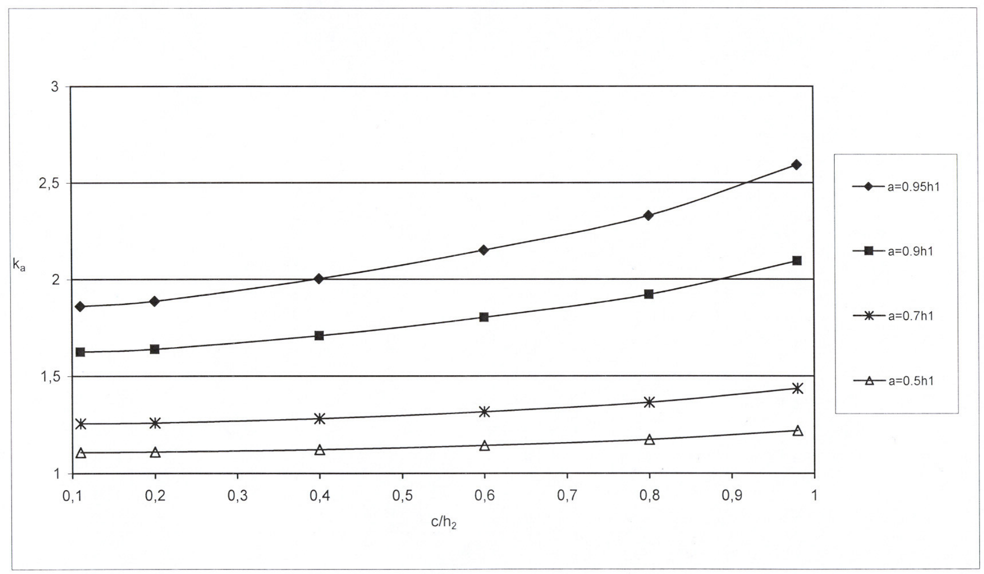

Figure 4.39. Variation of k_a with c/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

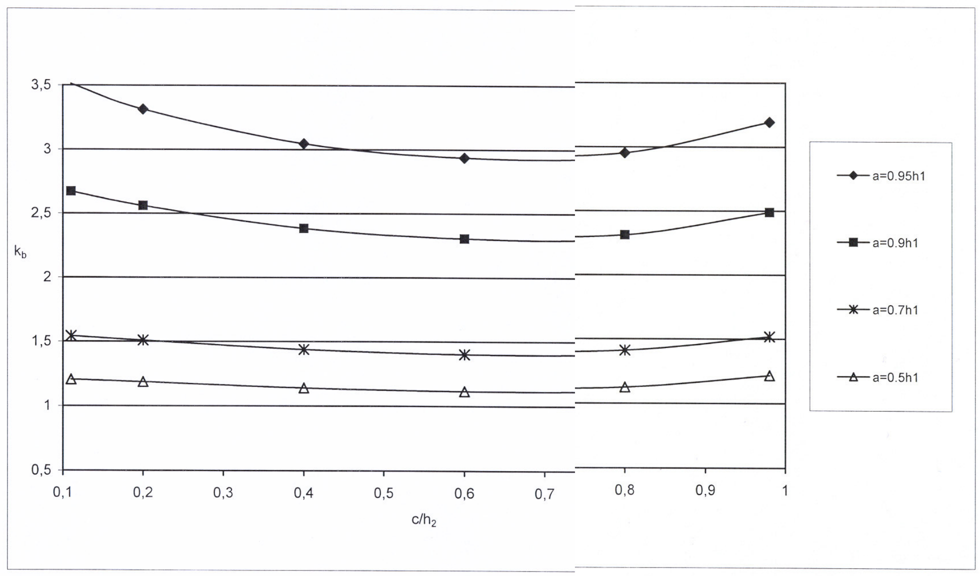

Figure 4.40. Variation of k_b with c/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

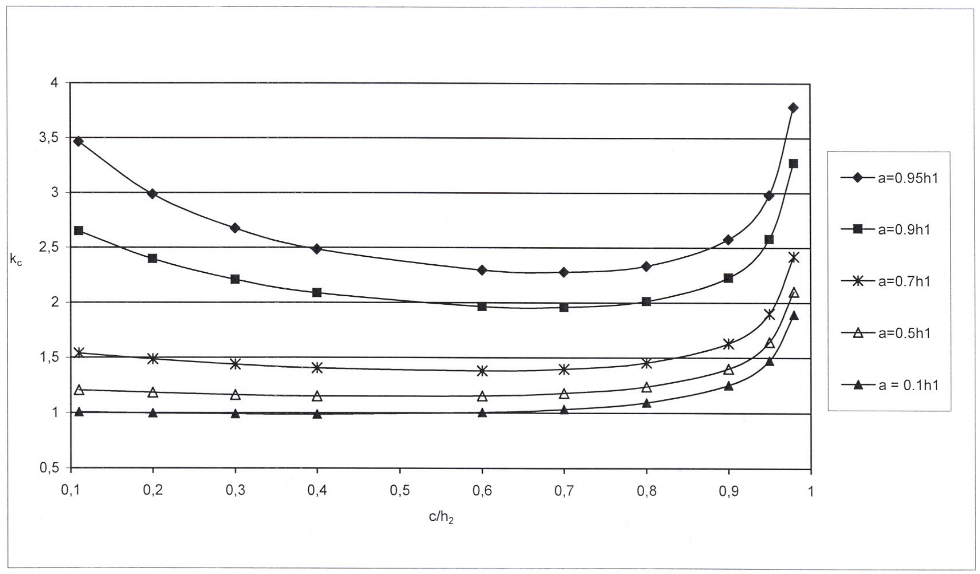

Figure 4.41. Variation of k_c with c/h_2 for 1:steel, 2:aluminum, $\mu_1/\mu_2 = 3$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

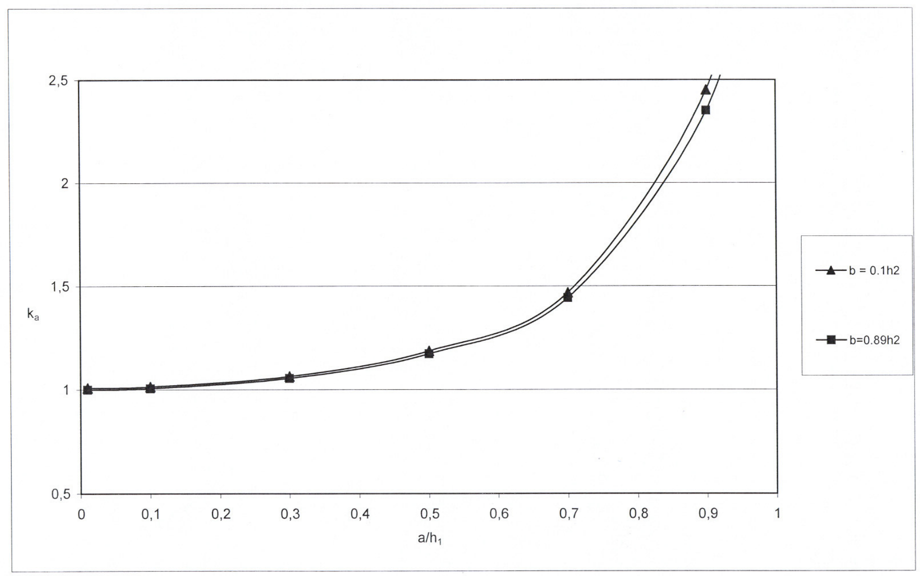

Figure 4.42. Variation of k_a with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

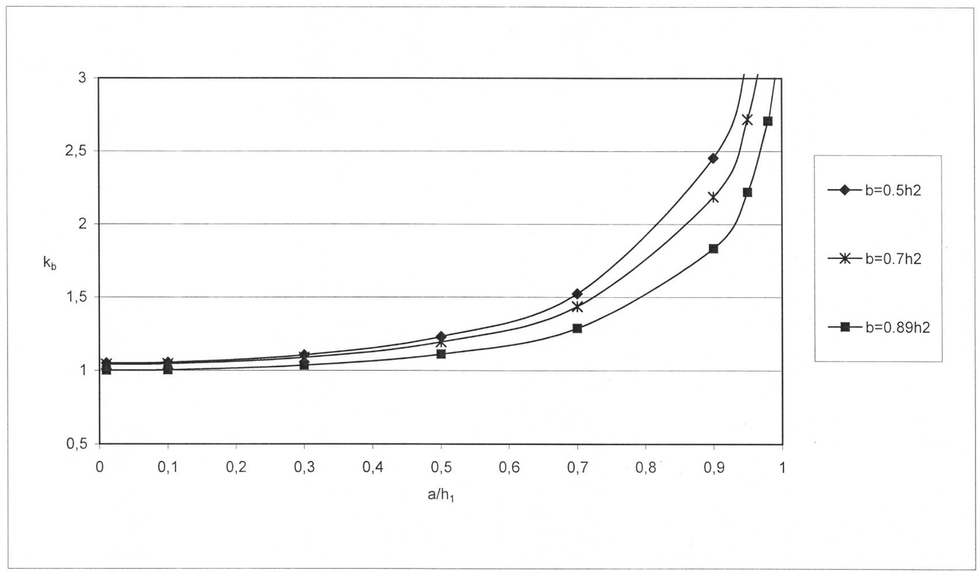

Figure 4.43. Variation of k_b with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

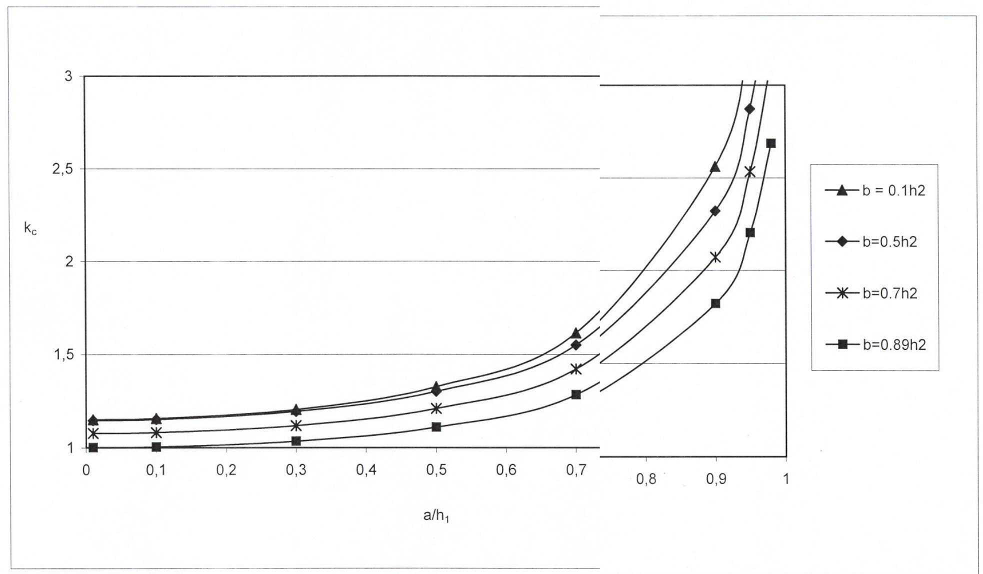

Figure 4.44. Variation of k_c with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$,(Plane Stress)

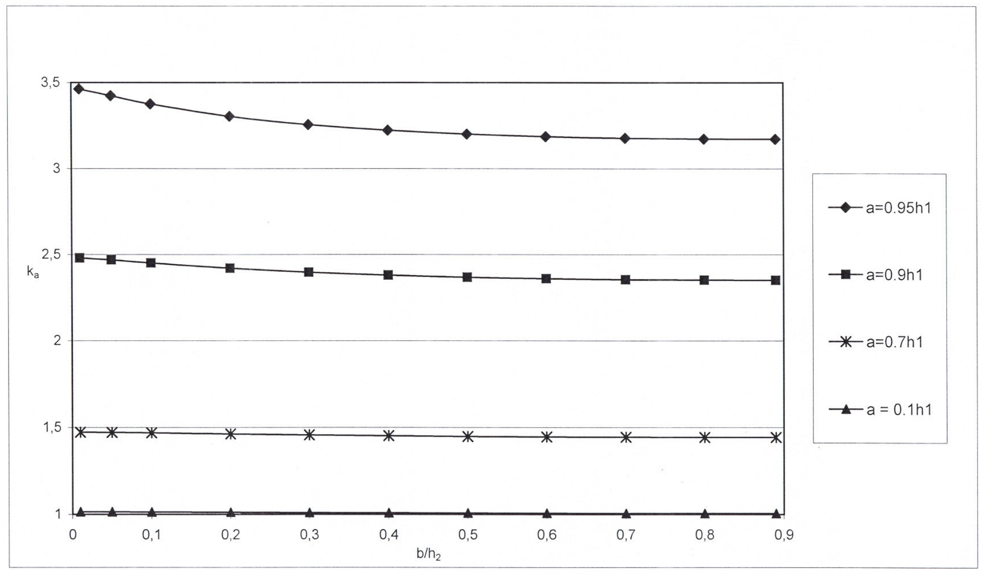

Figure 4.45. Variation of k_a with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

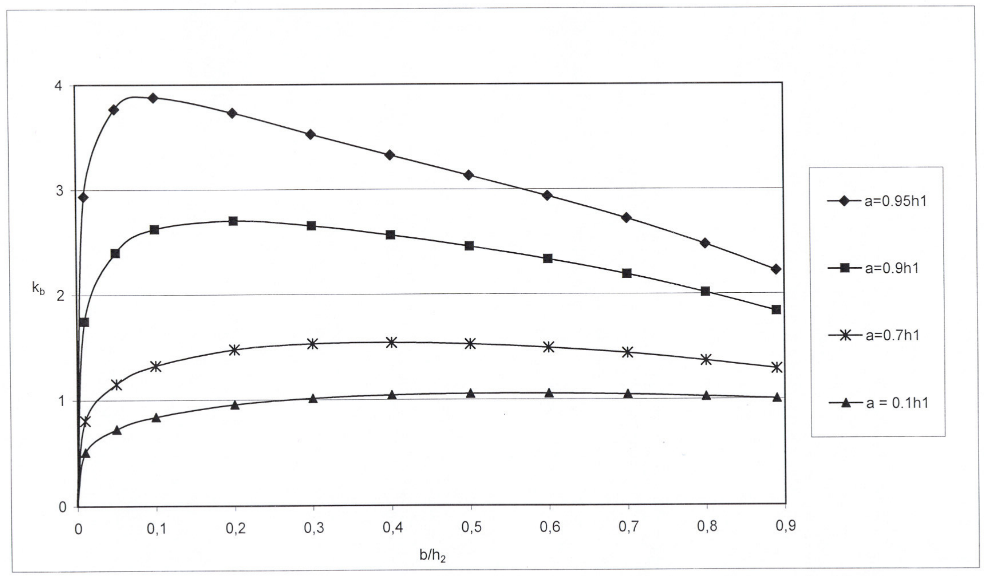

Figure 4.46. Variation of k_b with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

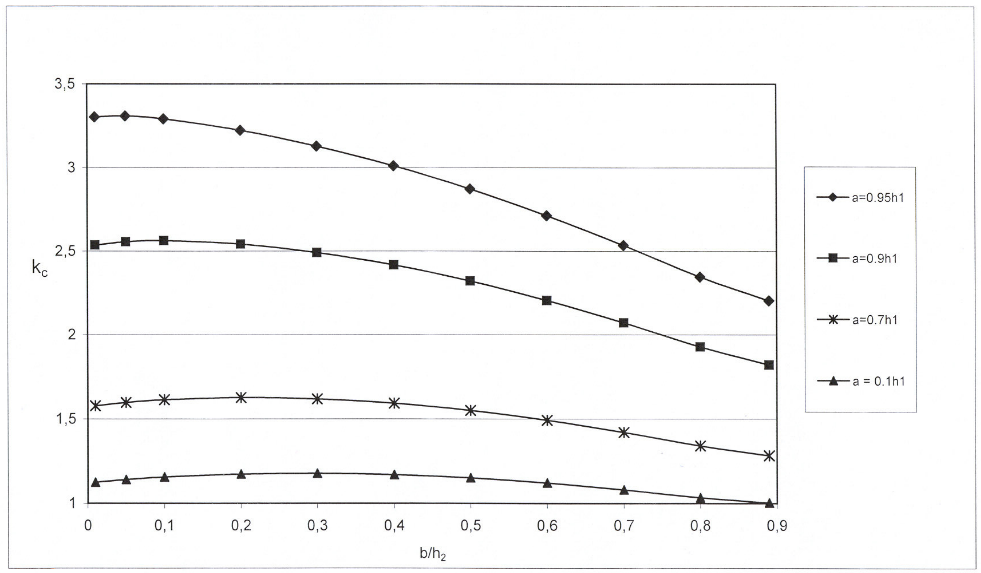

Figure 4.47. Variation of k_c with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $c=0.9h_2$, (Plane Stress)

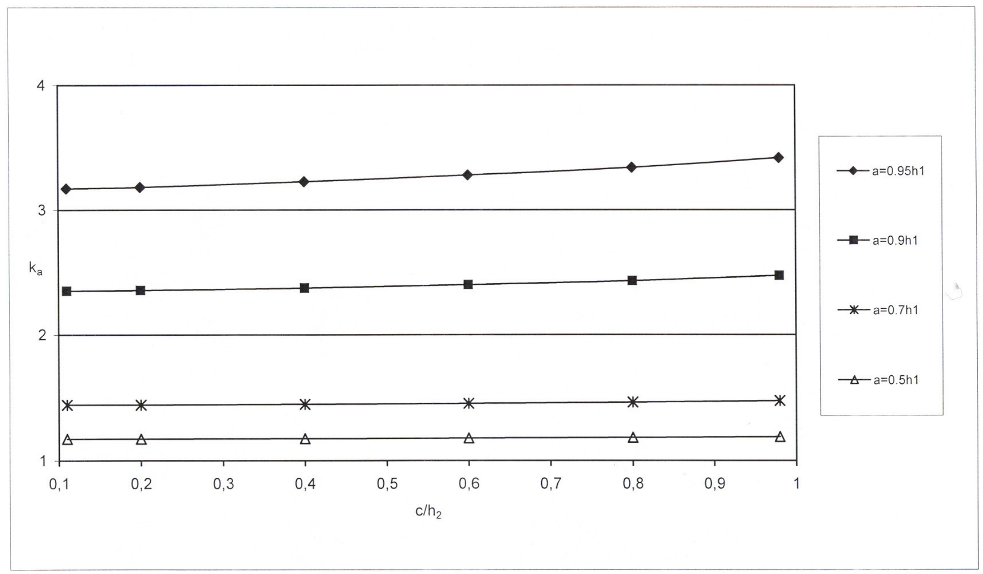

Figure 4.48. Variation of k_a with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

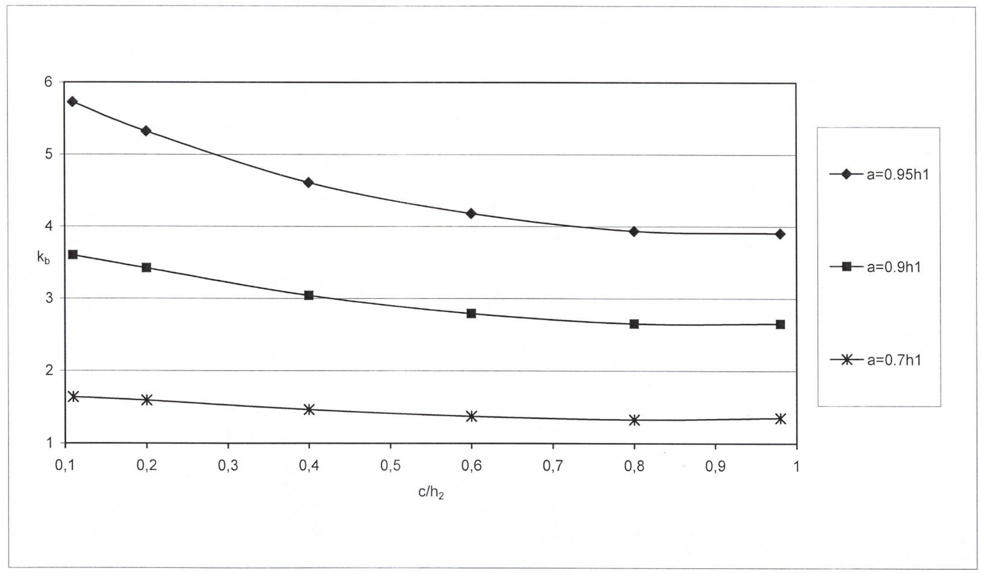

Figure 4.49. Variation of k_b with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

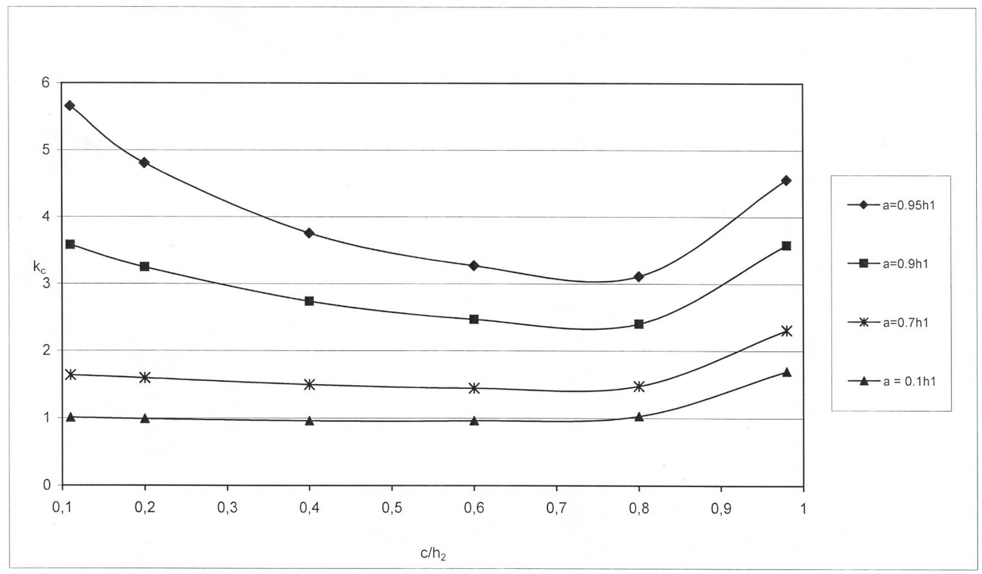

Figure 4.50. Variation of k_c with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_2=h_1$, $b=0.1h_2$, (Plane Stress)

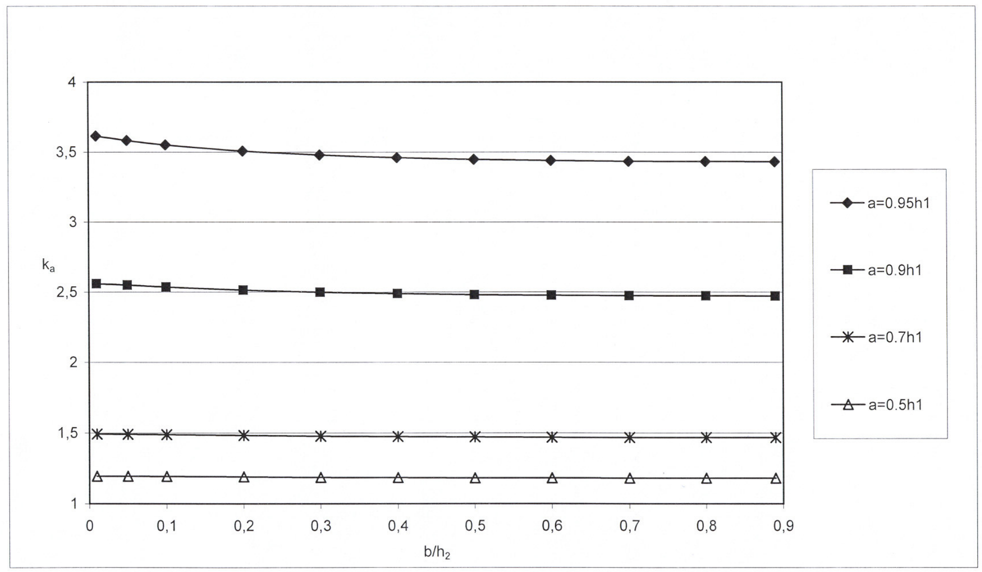

Figure 4.51. Variation of k_a with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $c=0.9h_2$, (Plane Stress)

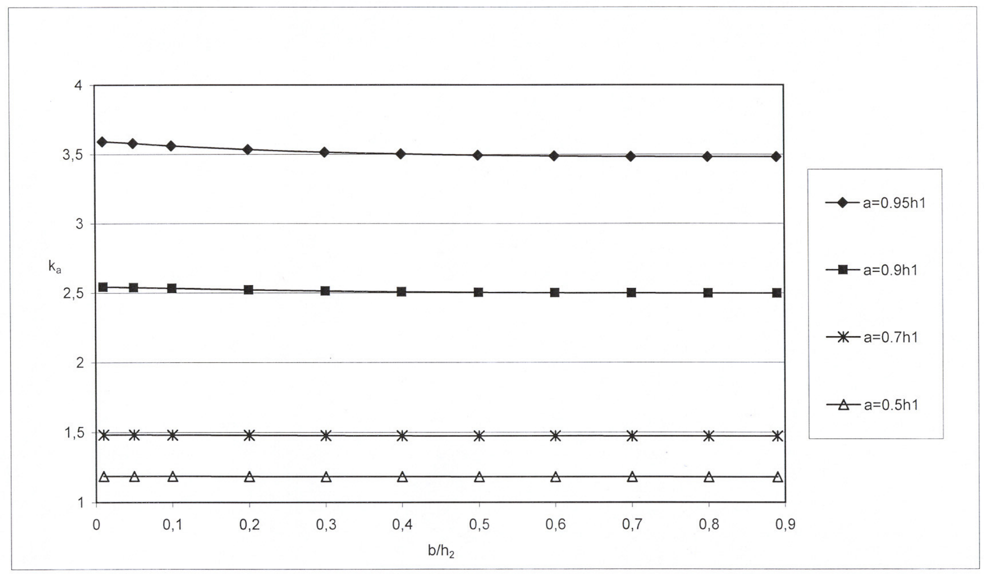

Figure 4.52. Variation of k_a with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $c=0.9h_2$, (Plane Stress)

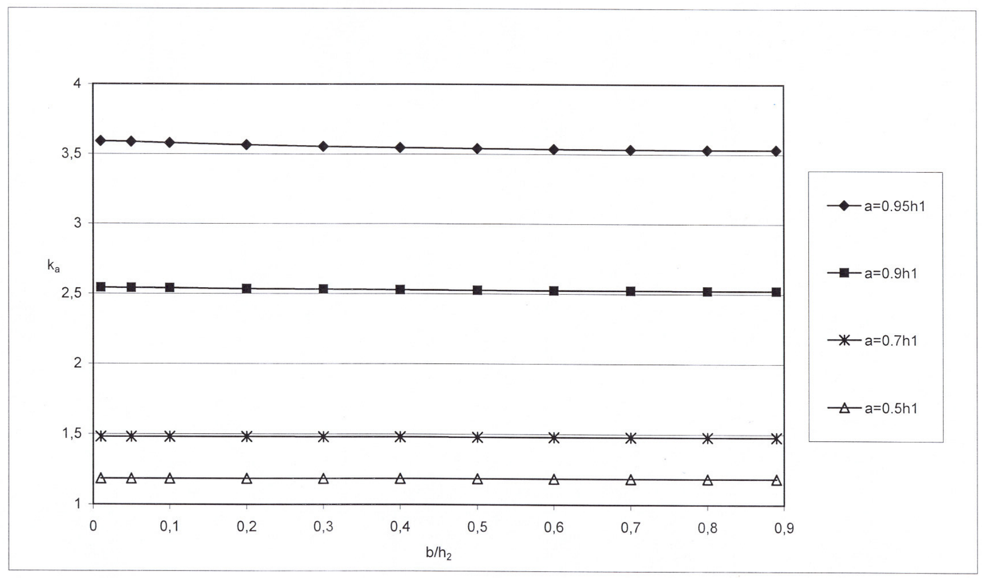

Figure 4.53. Variation of k_a with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $c=0.9h_2$, (Plane Stress)

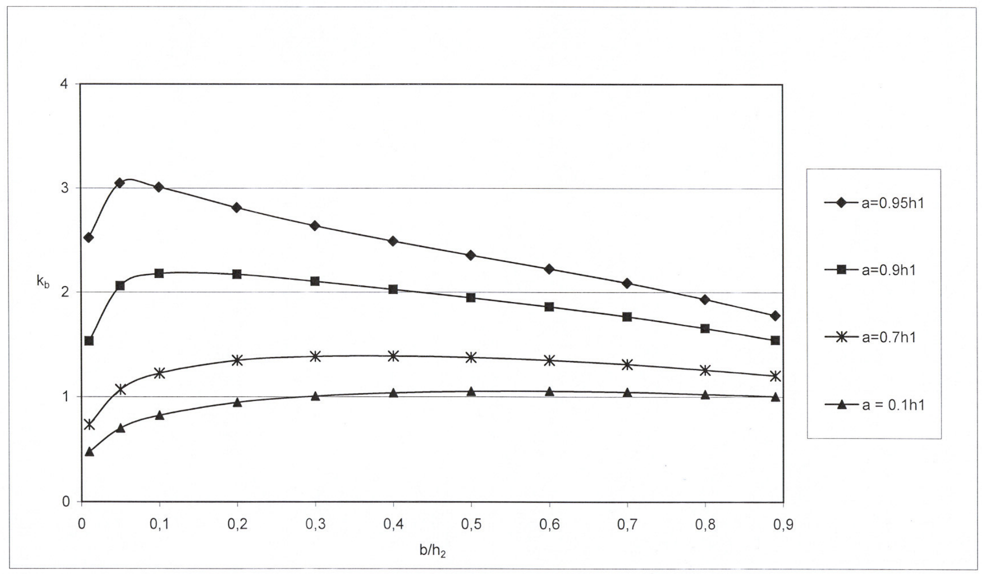

Figure 4.54. Variation of k_b with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $c=0.9h_2$, (Plane Stress)

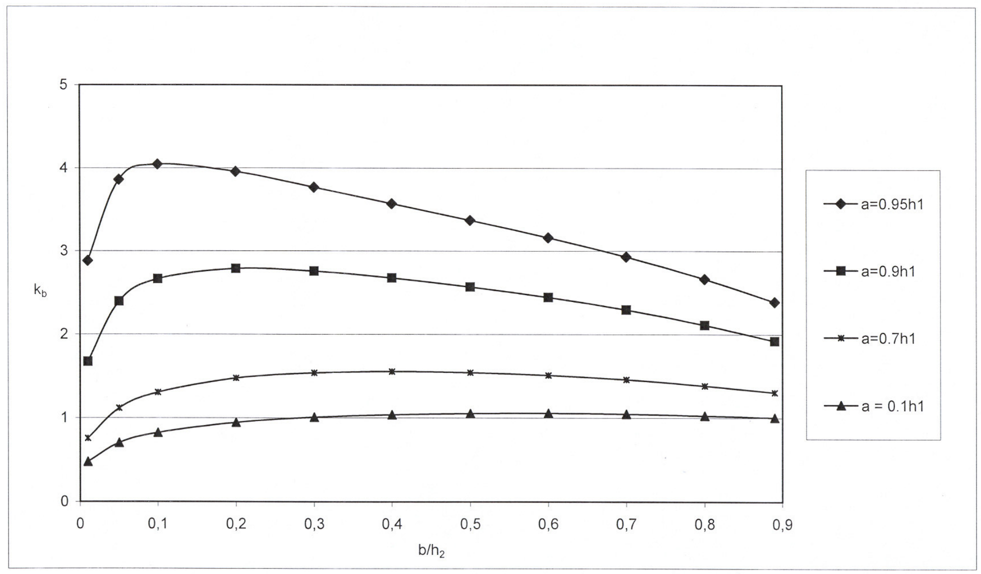

Figure 4.55. Variation of k_b with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $c=0.9h_2$, (Plane Stress)

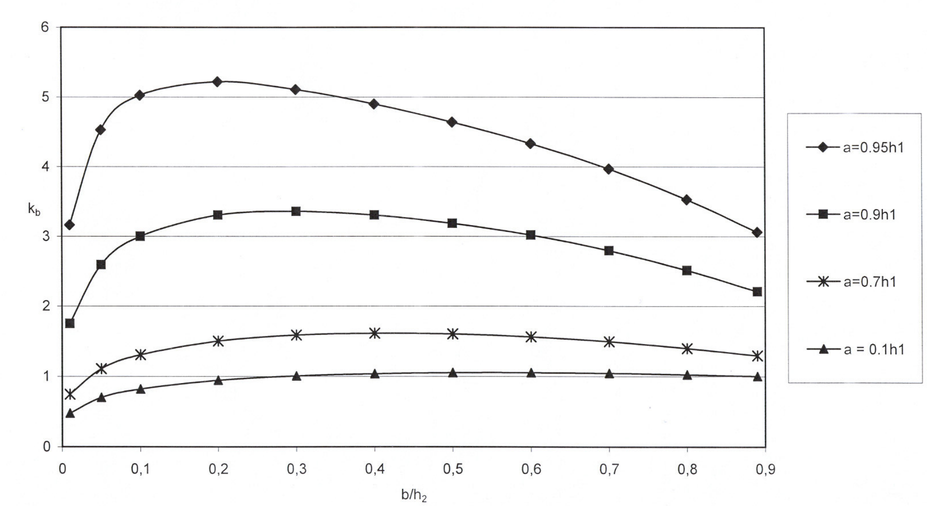

Figure 4.56. Variation of k_b with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $c=0.9h_2$,(Plane Stress)

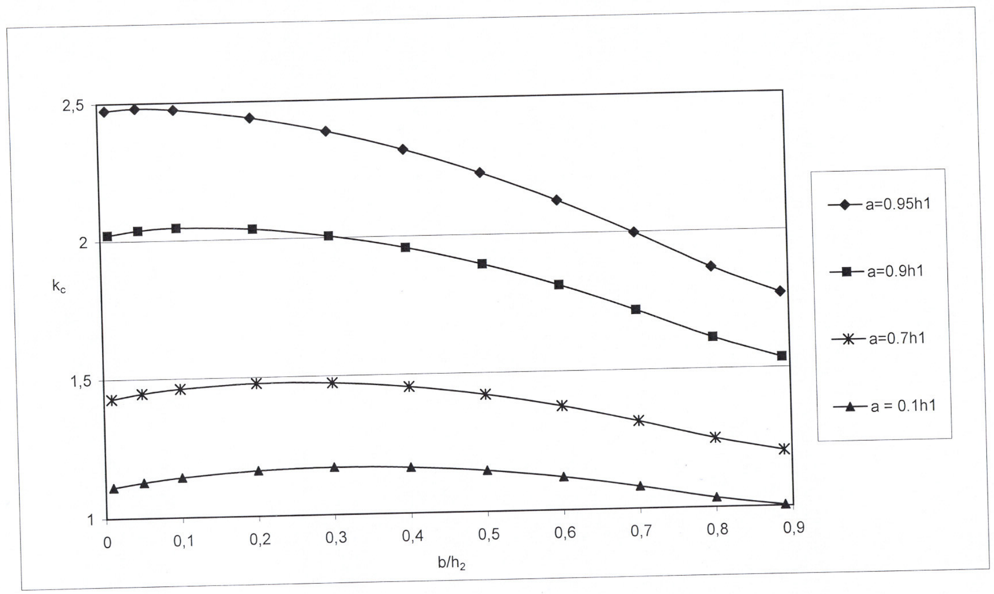

Figure 4.57. Variation of k_c with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $c=0.9h_2$, (Plane Stress)

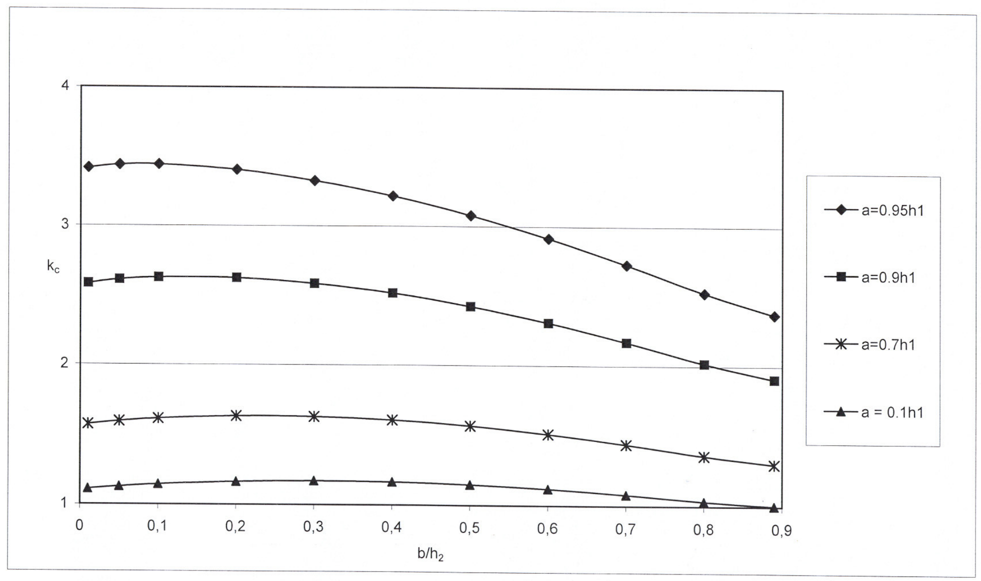

Figure 4.58. Variation of k_c with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $c=0.9h_2$, (Plane Stress)

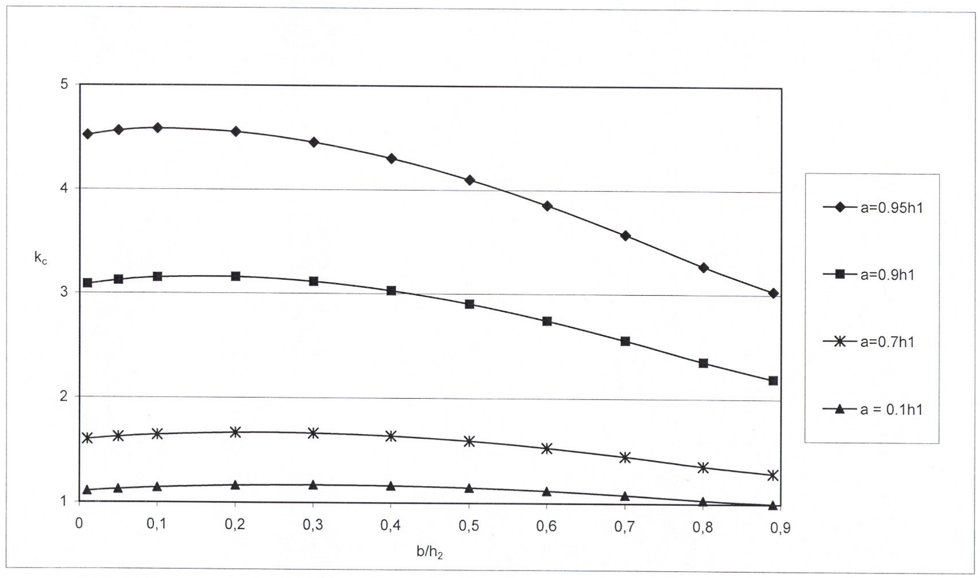

Figure 4.59. Variation of k_c with b/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $c=0.9h_2$, (Plane Stress)

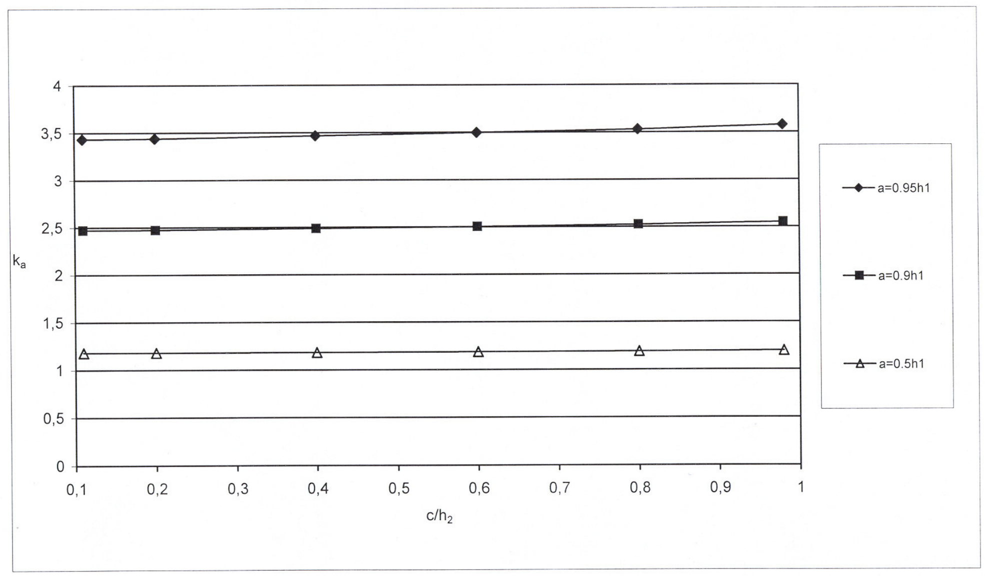

Figure 4.60. Variation of k_a with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $b=0.1h_2$,(Plane Stress)

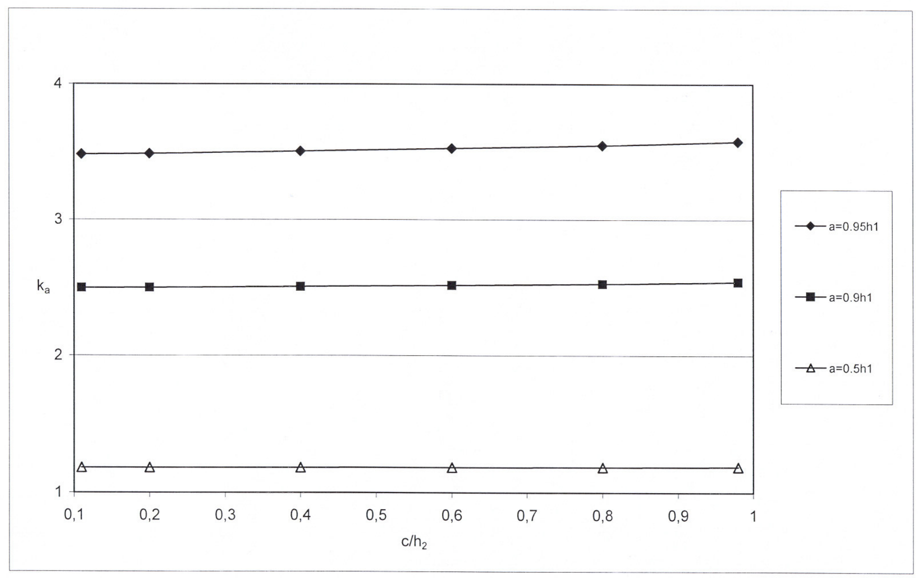

Figure 4.61. Variation of k_a with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $b=0.1h_2$, (Plane Stress)

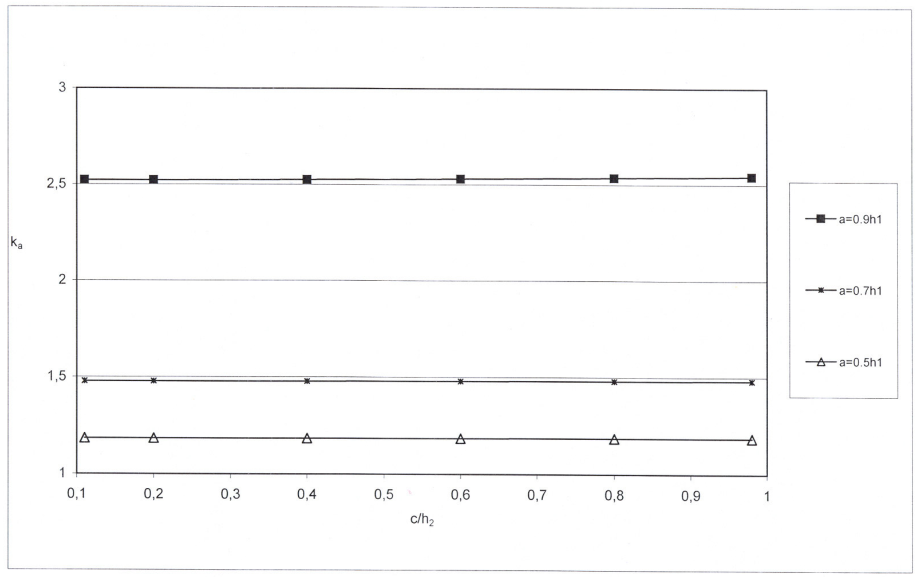

Figure 4.62. Variation of k_a with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $b=0.1h_2$, (Plane Stress)

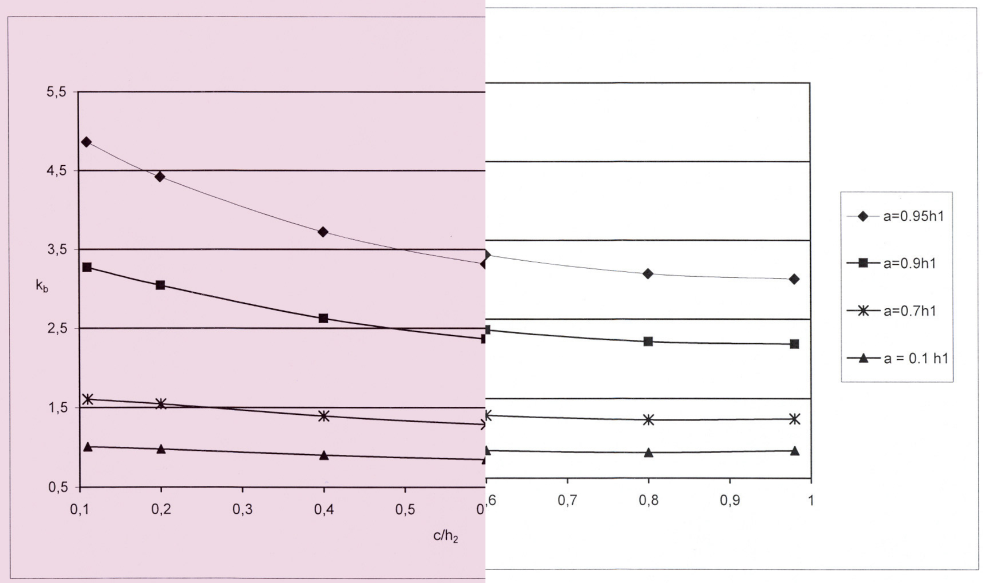

Figure 4.63. Variation of k_b with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=0.5h_2$, $b=0.1h_2$, (Plane Stress)

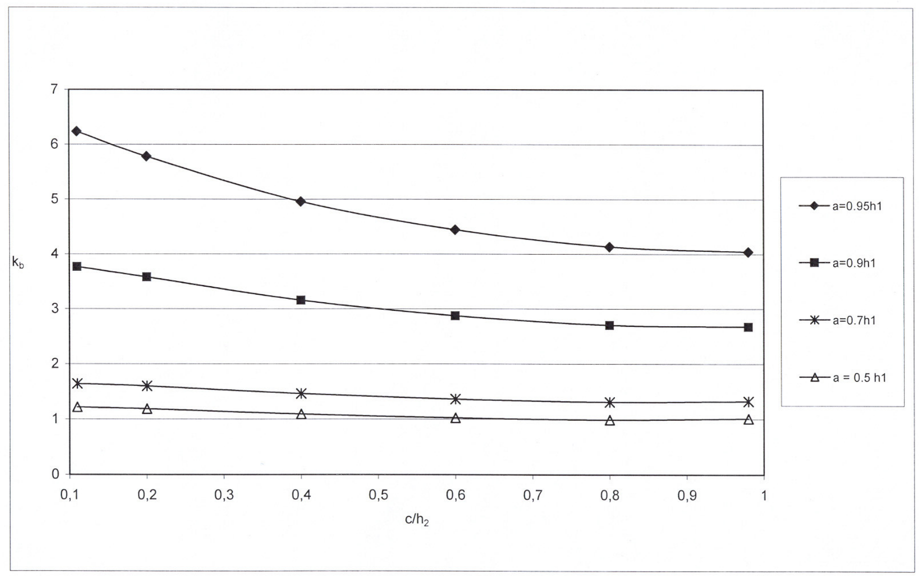

Figure 4.64. Variation of k_b with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $b=0.1h_2$, (Plane Stress)

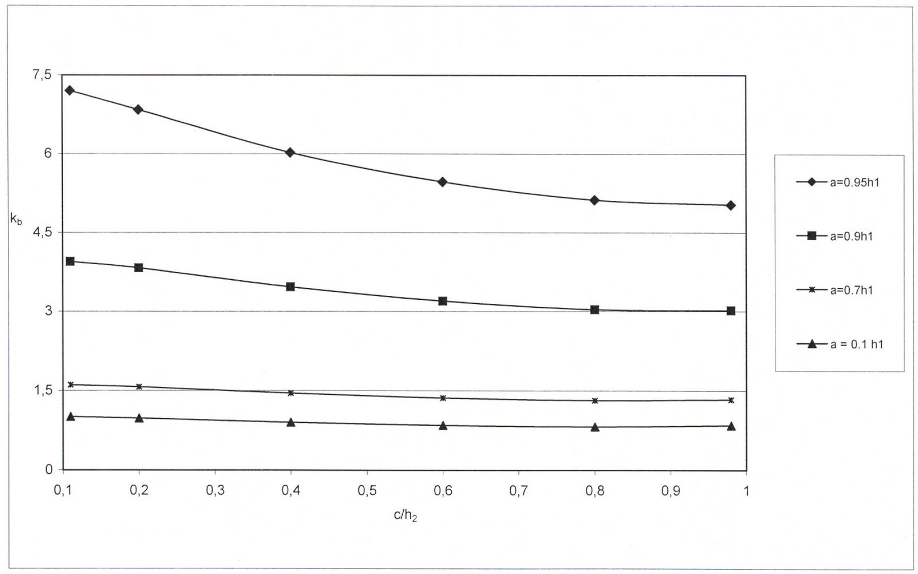

Figure 4.65. Variation of k_b with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $b=0.1h_2$, (Plane Stress)

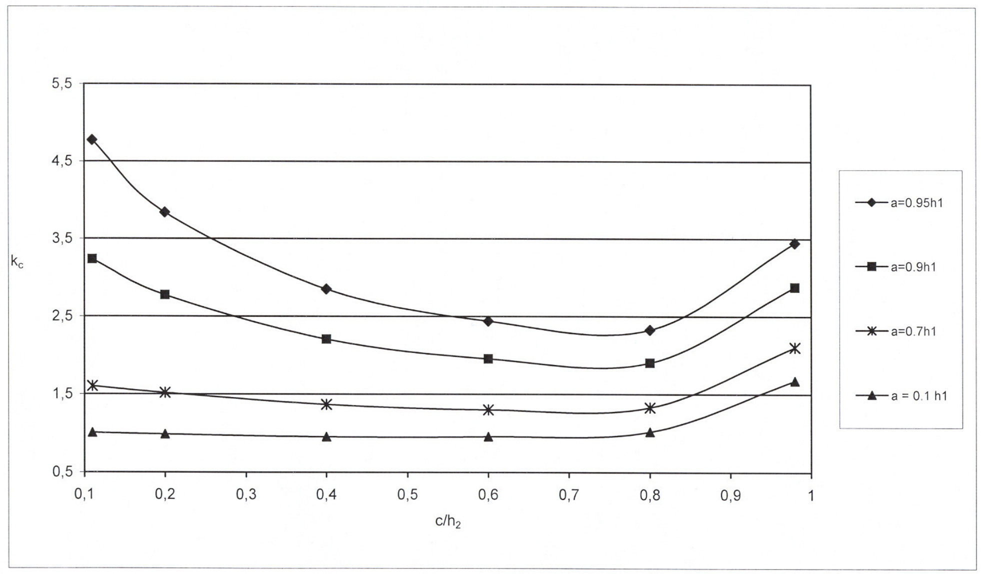

Figure 4.66. Variation of k_c with c/h_2 for 1:steel, 2:epoxy, μ_1/μ_2 = 69.231, h_1=0.5h_2, b=0.1h_2, (Plane Stress)

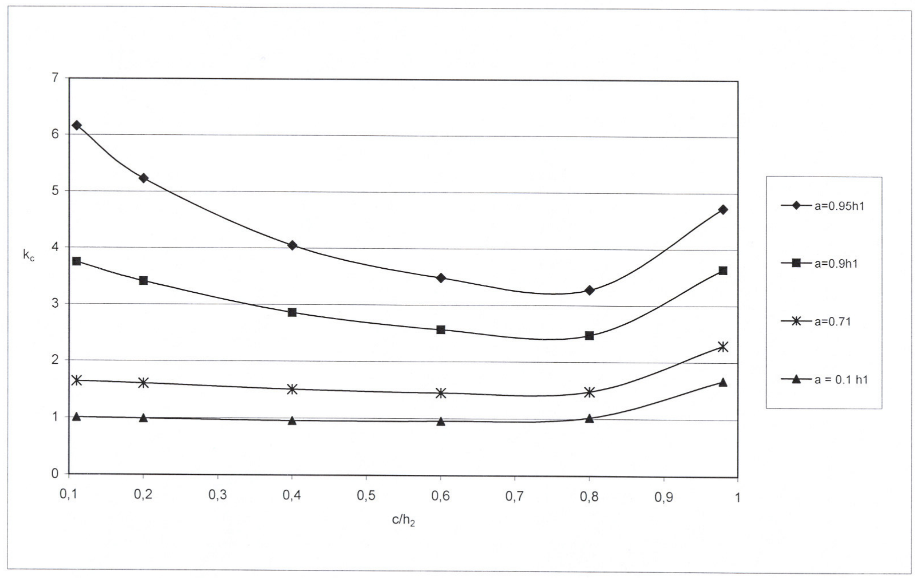

Figure 4.67. Variation of k_c with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=h_2$, $b=0.1h_2$, (Plane Stress)

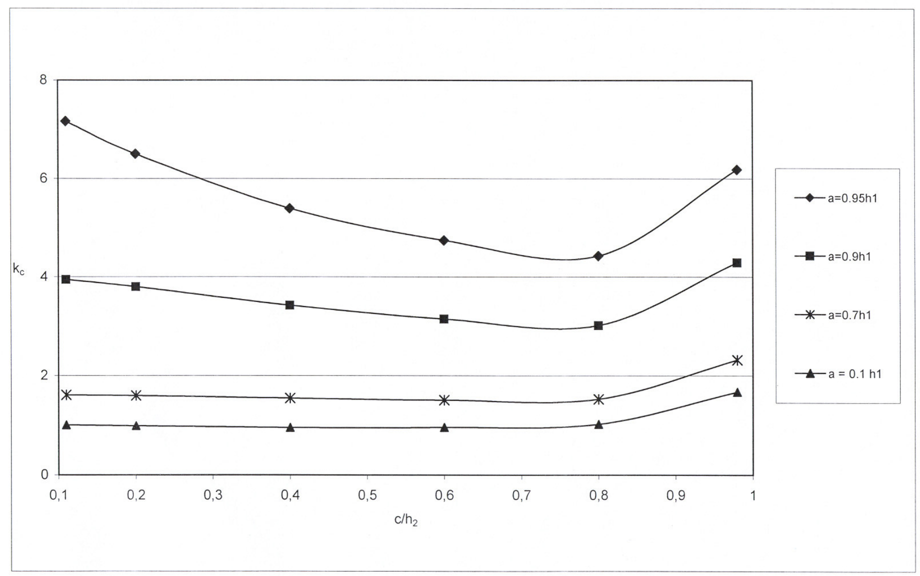

Figure 4.68. Variation of k_c with c/h_2 for 1:steel, 2:epoxy, $\mu_1/\mu_2 = 69.231$, $h_1=2h_2$, $b=0.1h_2$, (Plane Stress)

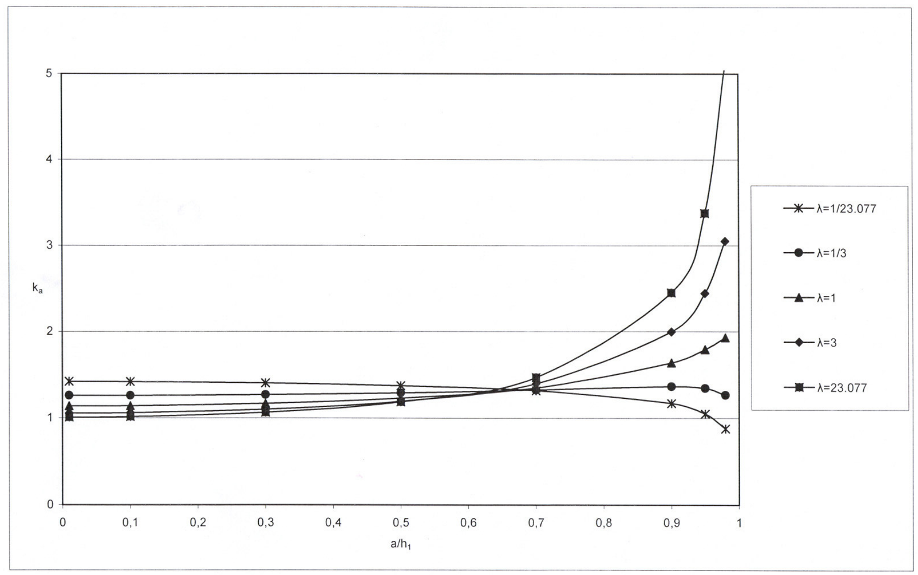

Figure 4.69. Variation of k_a with a/h_1 with $h_2=h_1$, $b=0.1h_2$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

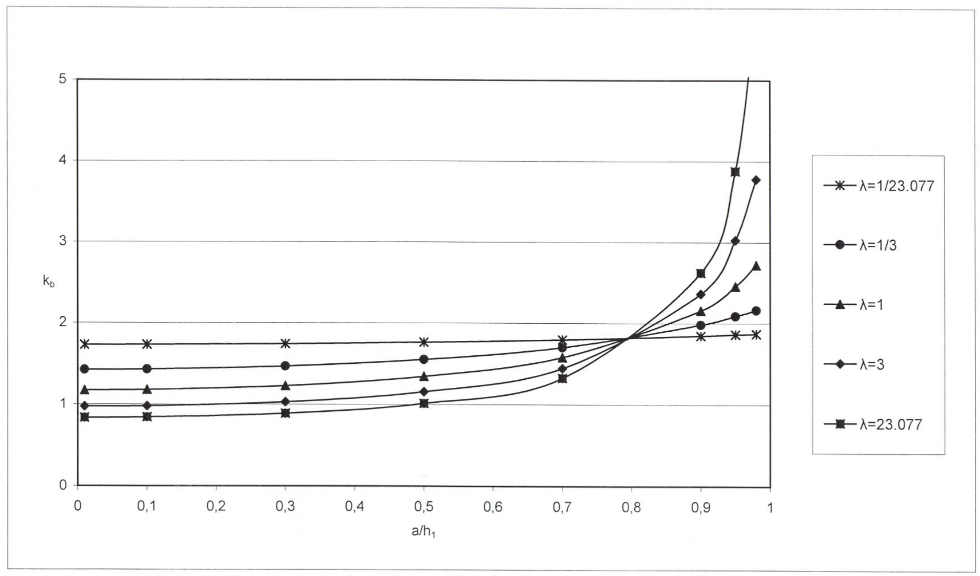

Figure 4.70. Variation of k_b with a/h_1 with $h_2=h_1$, $b=0.1h_2$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

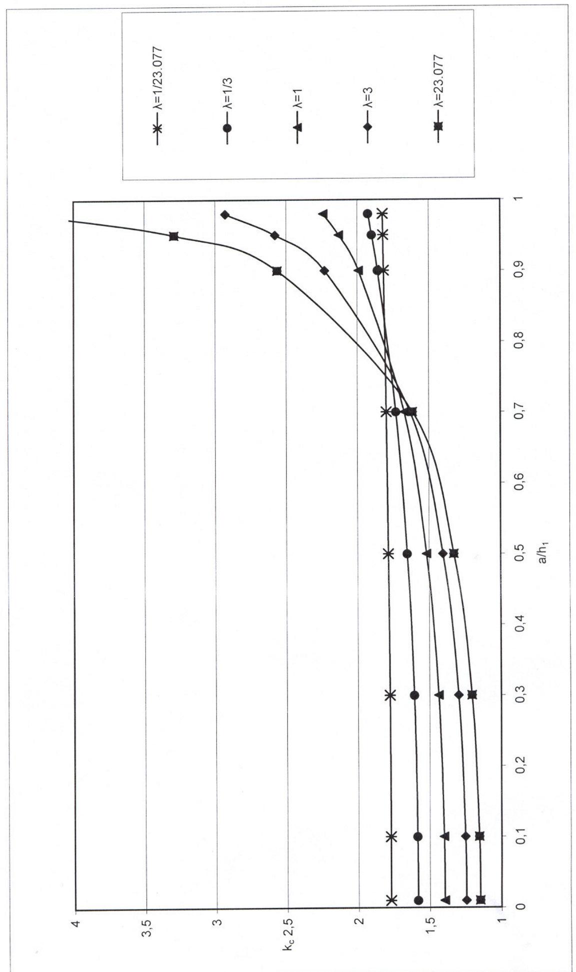

Figure 4.71. Variation of k_c with a/h_1 with $h_2=h_1$, $b=0.1h_2$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

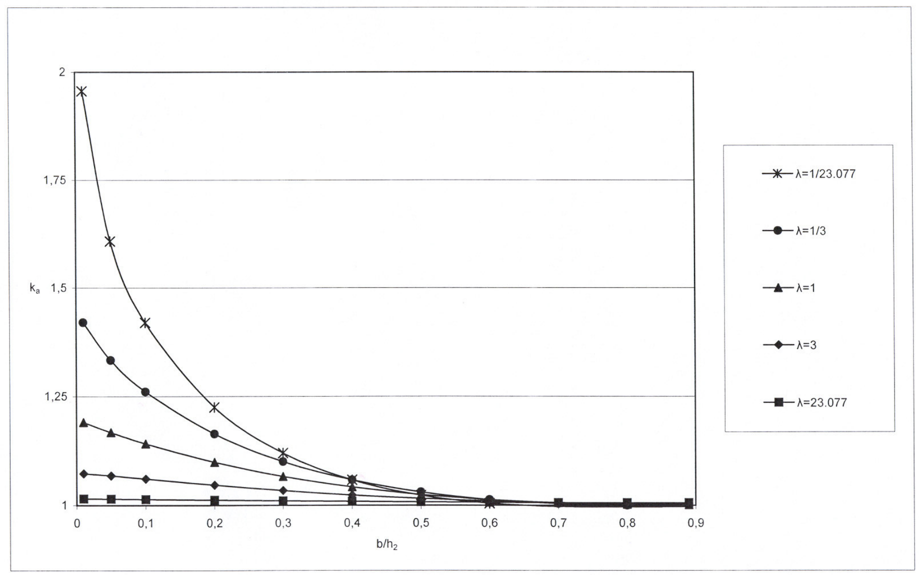

Figure 4.72. Variation of k_a with b/h_2 with $h_2=h_1$, $a=0.1h_1$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

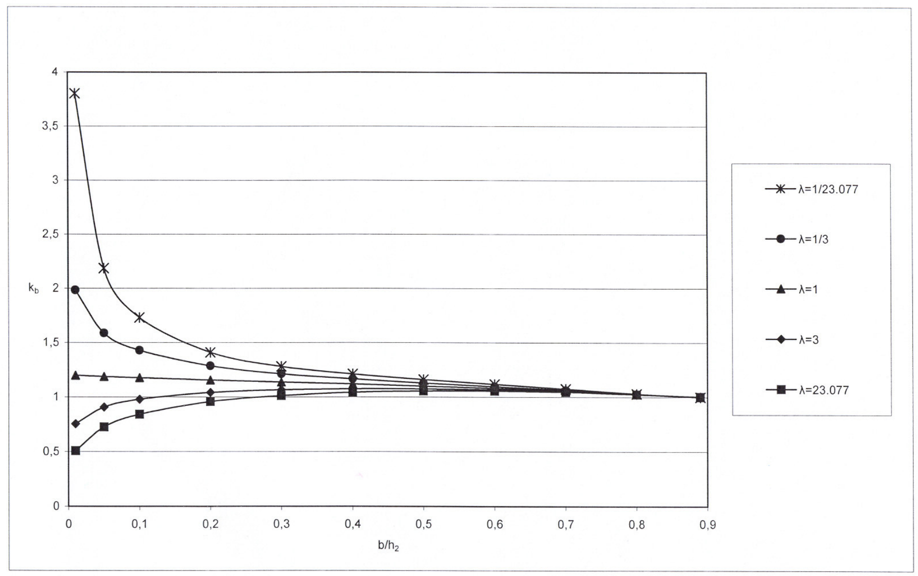

Figure 4.73. Variation of k_b with b/h_2 with $h_2=h_1$, $a=0.1h_1$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

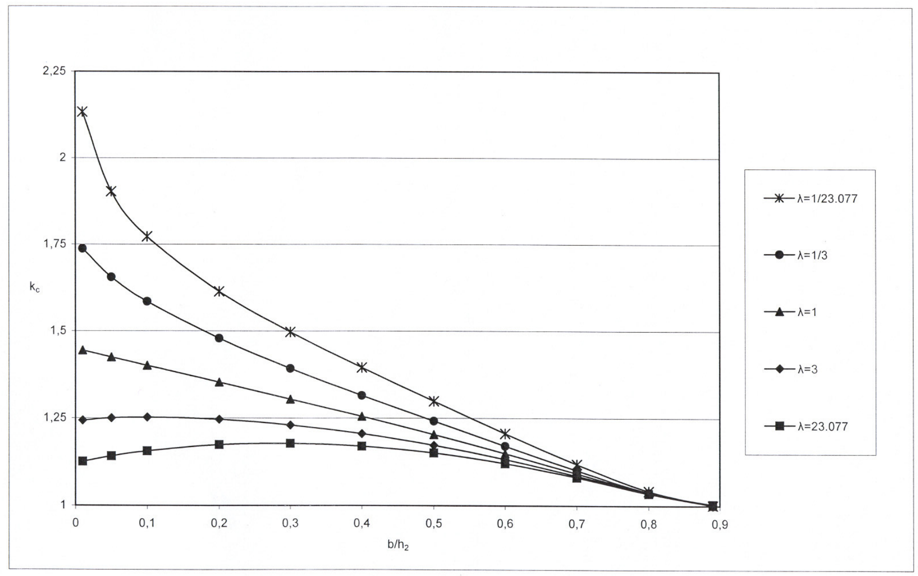

Figure 4.74. Variation of k_c with b/h_2 with $h_2=h_1$, $a=0.1h_1$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

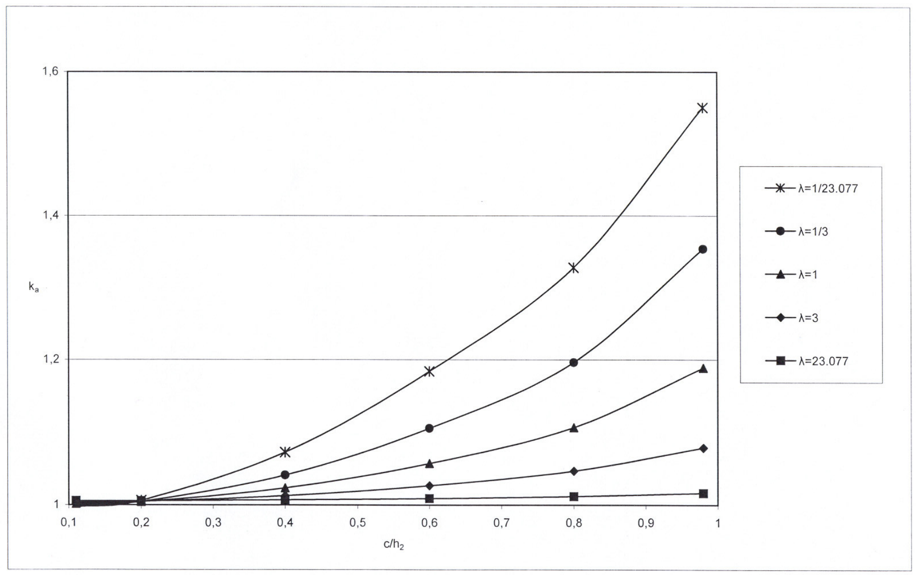

Figure 4.75. Variation of k_a with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.1h_1$, (Plane Stress), $\lambda=\mu_1/\mu_2$

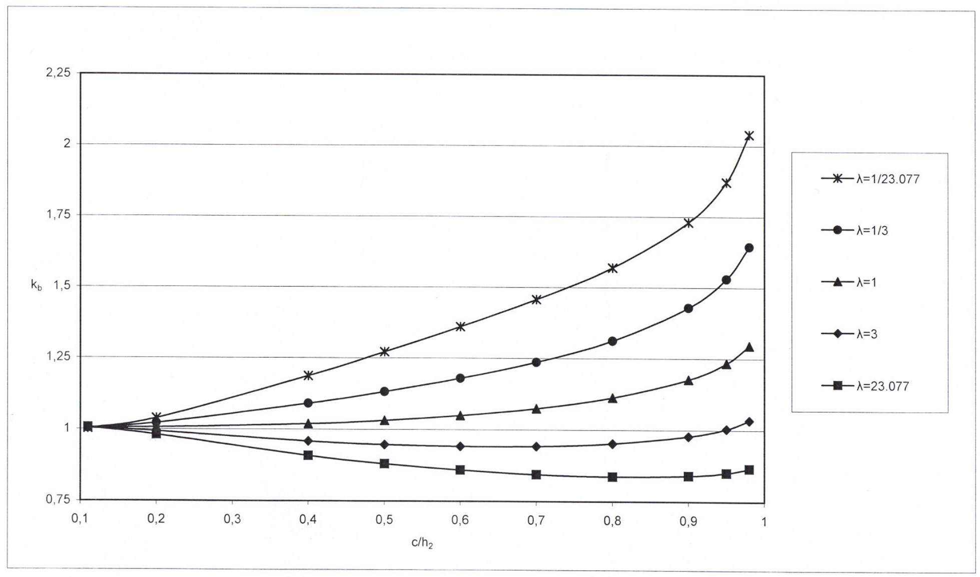

Figure 4.76. Variation of k_b with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.1h_1$,(Plane Stress), $\lambda=\mu_1/\mu_2$

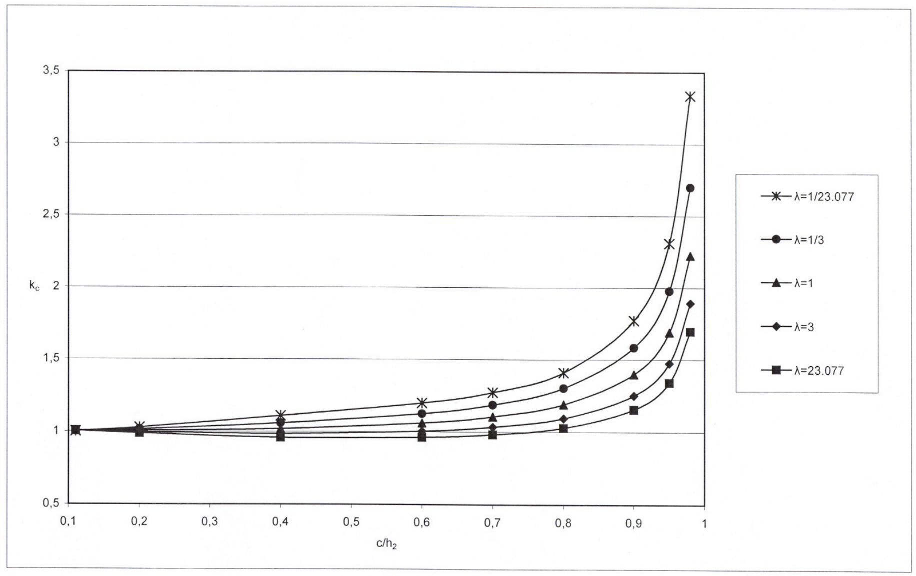

Figure 4.77. Variation of k_c with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.1h_1$, (Plane Stress), $\lambda=\mu_1/\mu_2$

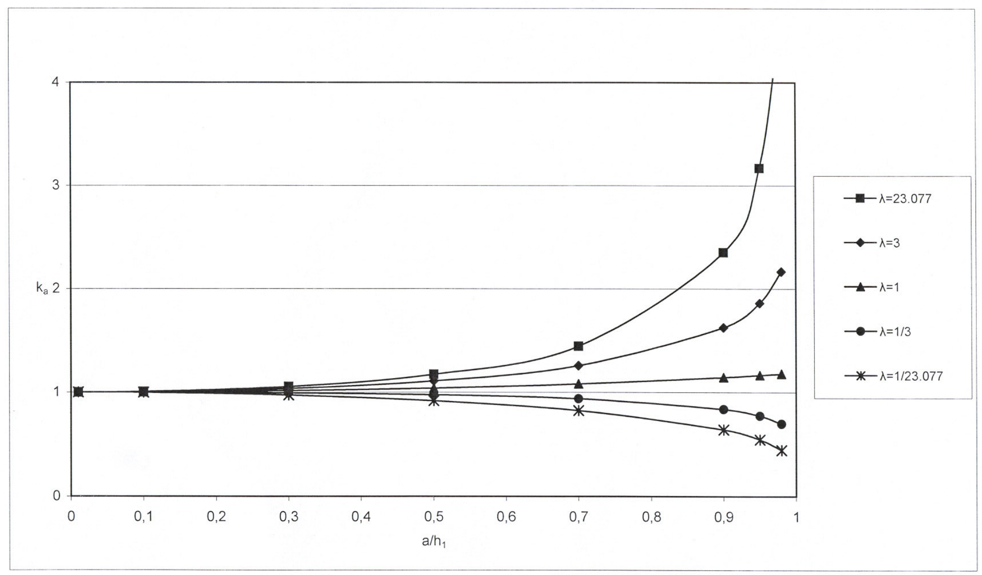

Figure 4.78. Variation of k_a with a/h_1 with $h_2=h_1$, $b=0.89h_2$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

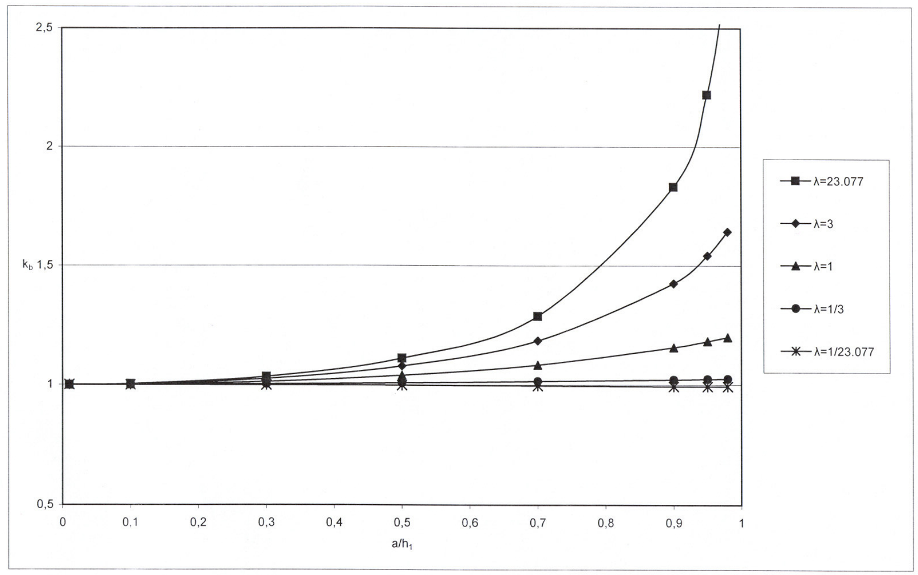

Figure 4.79 Variation of k_b with a/h_1 with $h_2=h_1$, $b=0.89h_2$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

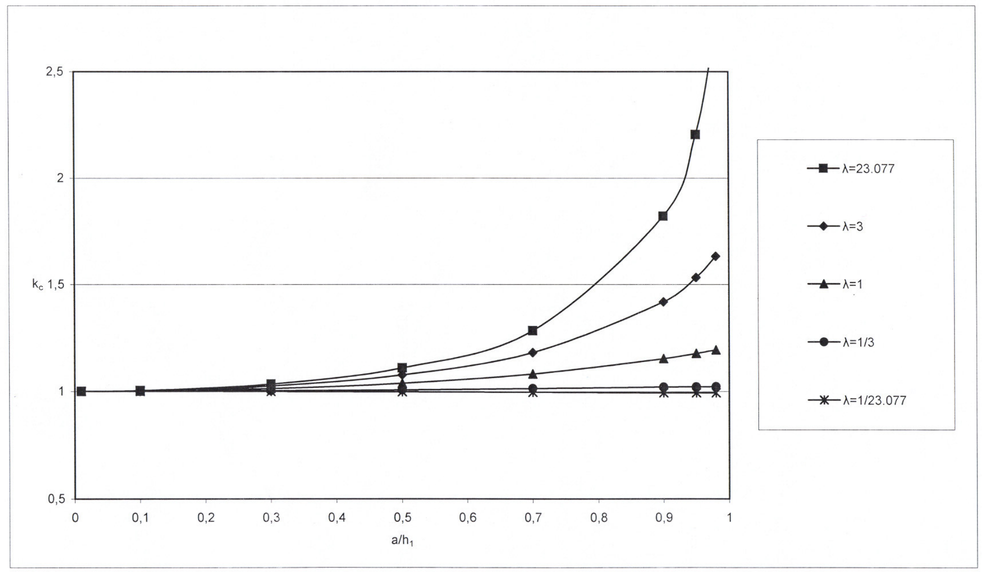

Figure 4.80. Variation of k_c with a/h_1 with $h_2=h_1$, $b=0.89h_2$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

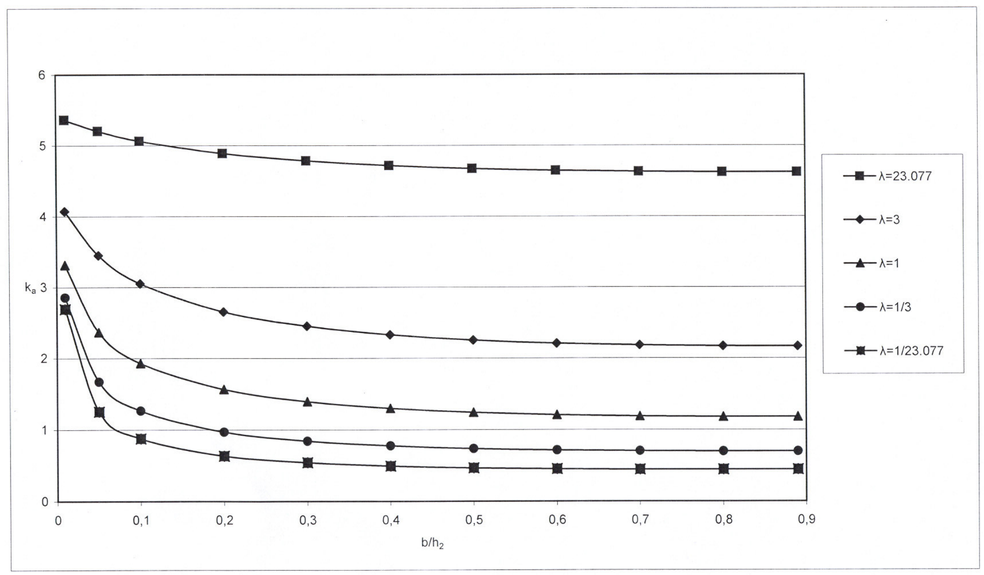

Figure 4.81. Variation of k_a with b/h_2 with $h_2=h_1$, $a=0.98h_1$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

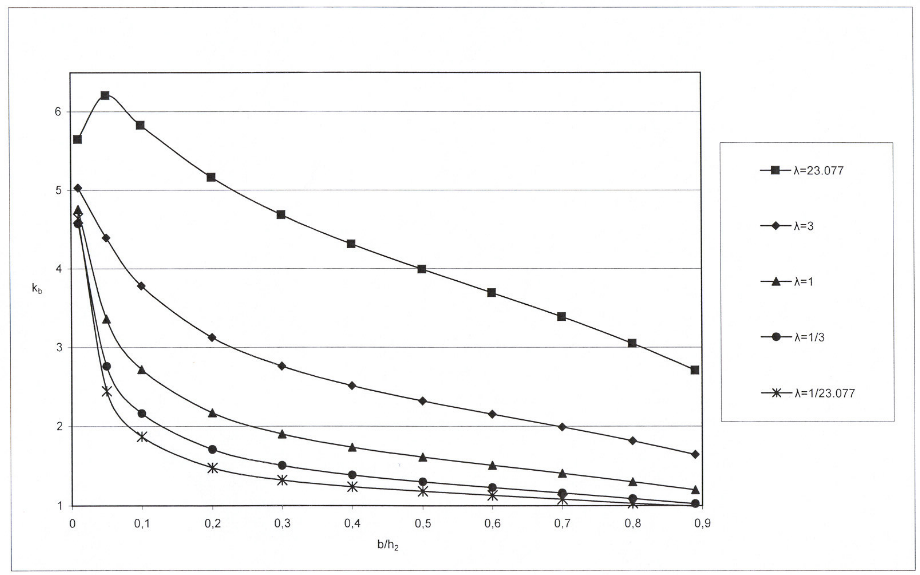

Figure 4.82. Variation of k_b with b/h_2 with $h_2=h_1$, $a=0.98h_1$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

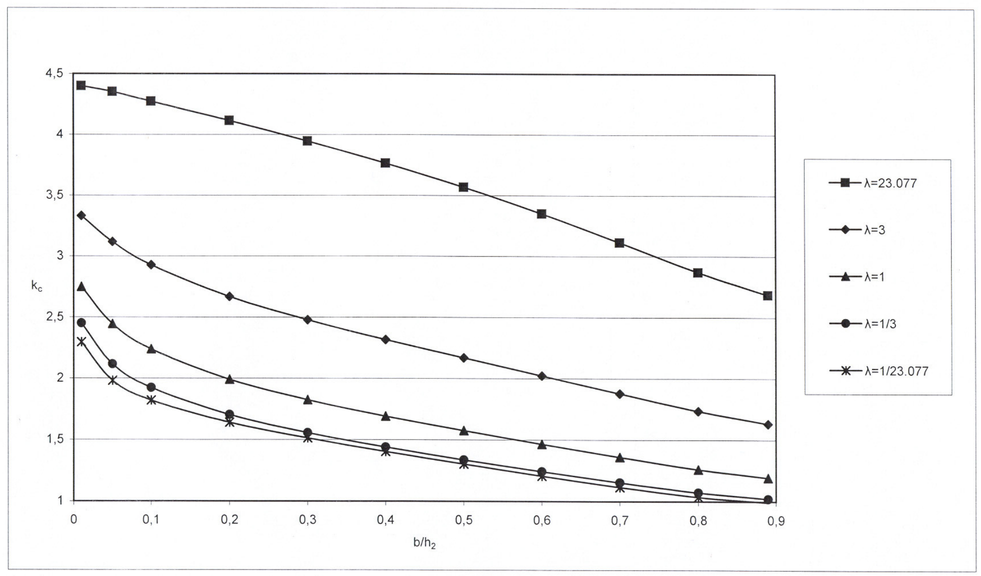

Figure 4.83. Variation of k_c with b/h_2 with $h_2=h_1$, $a=0.98h_1$, $c=0.9h_2$, (Plane Stress), $\lambda=\mu_1/\mu_2$

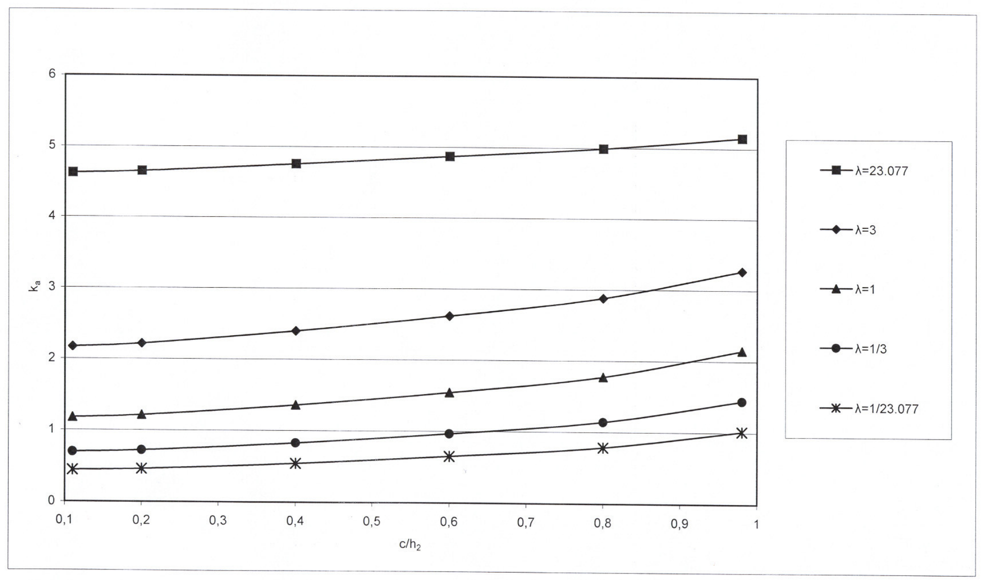

Figure 4.84. Variation of k_a with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.98h_1$, (Plane Stress), $\lambda=\mu_1/\mu_2$

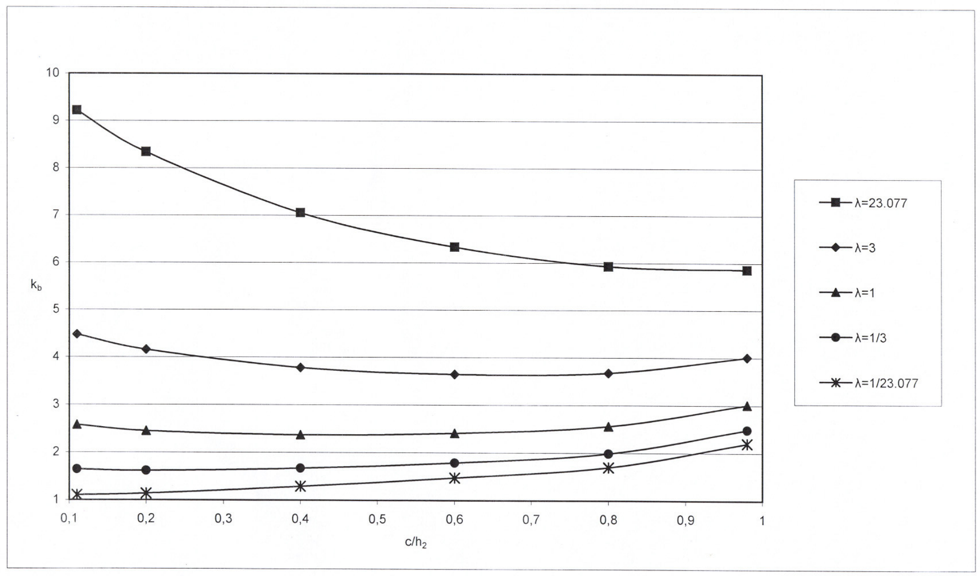

Figure 4.85. Variation of k_b with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.98h_1$, (Plane Stress), $\lambda=\mu_1/\mu_2$

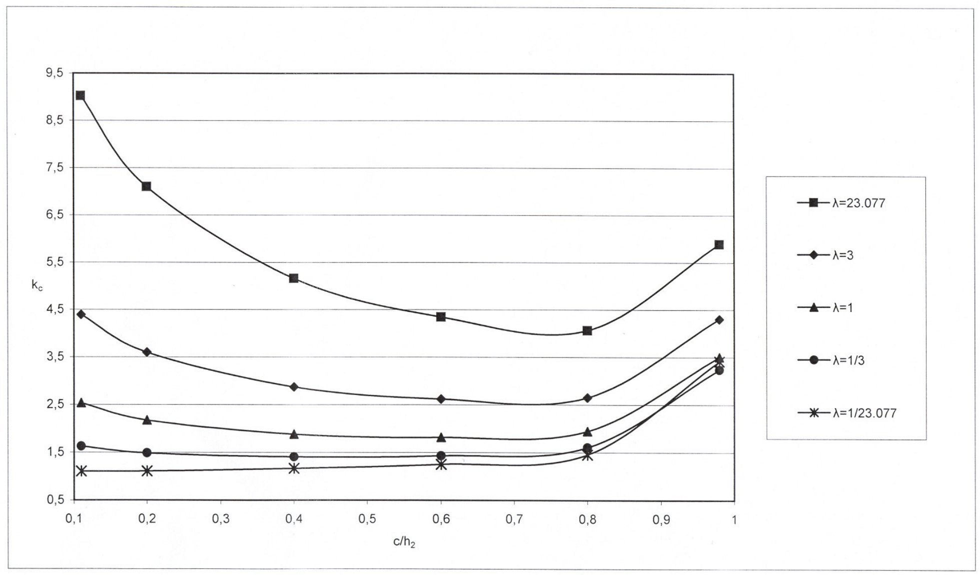

Figure 4.86. Variation of k_c with c/h_2 with $h_2=h_1$, $b=0.1h_2$, $a=0.98h_1$,(Plane Stress), $\lambda=\mu_1/\mu_2$

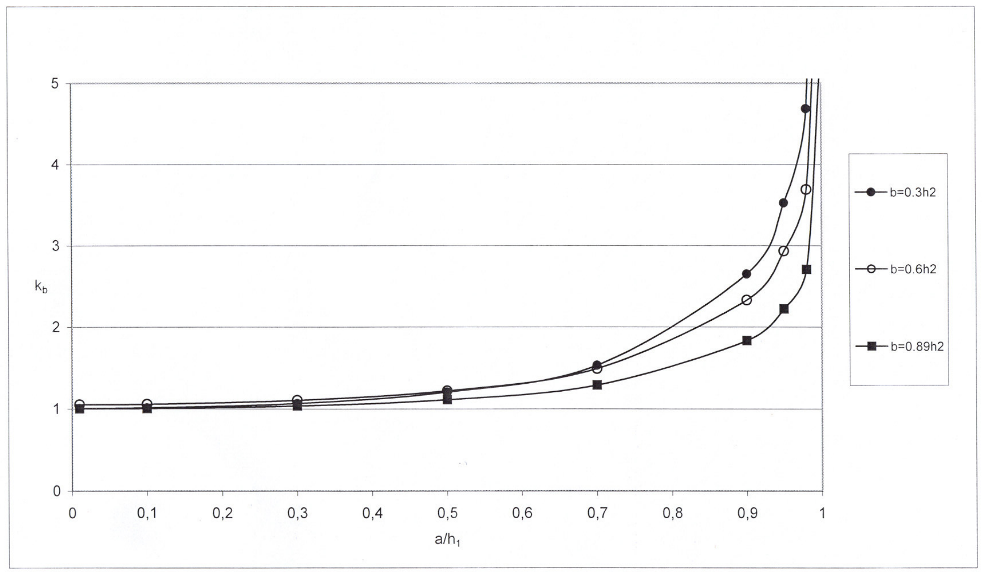

Figure 4.87. Variation of k_b with a/h_1 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, a=h_1, (Plane Stress)

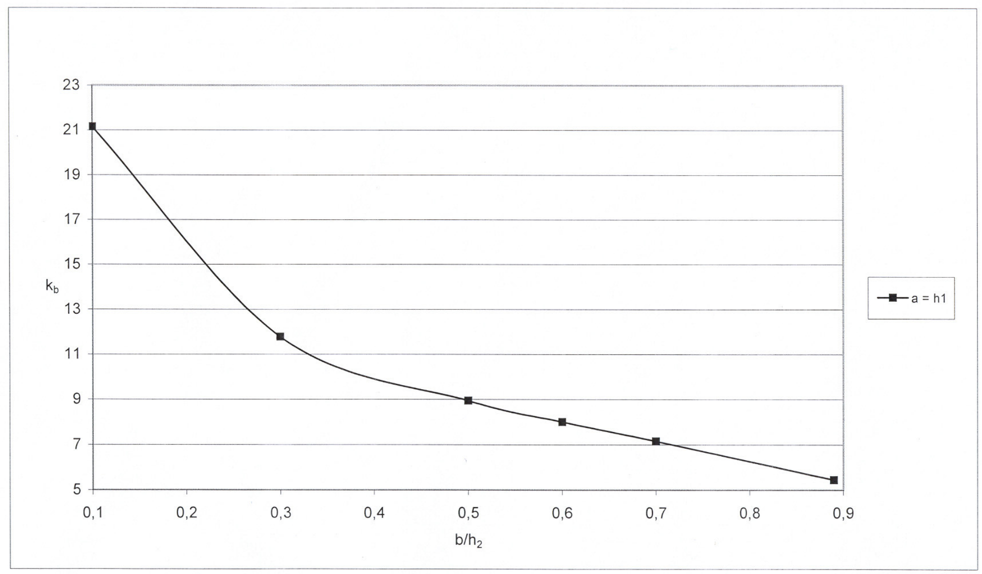

Figure 4.88. Variation of k_b with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_1 = h_2$, $c = 0.9h_2$, $a = h_1$, (Plane Stress)

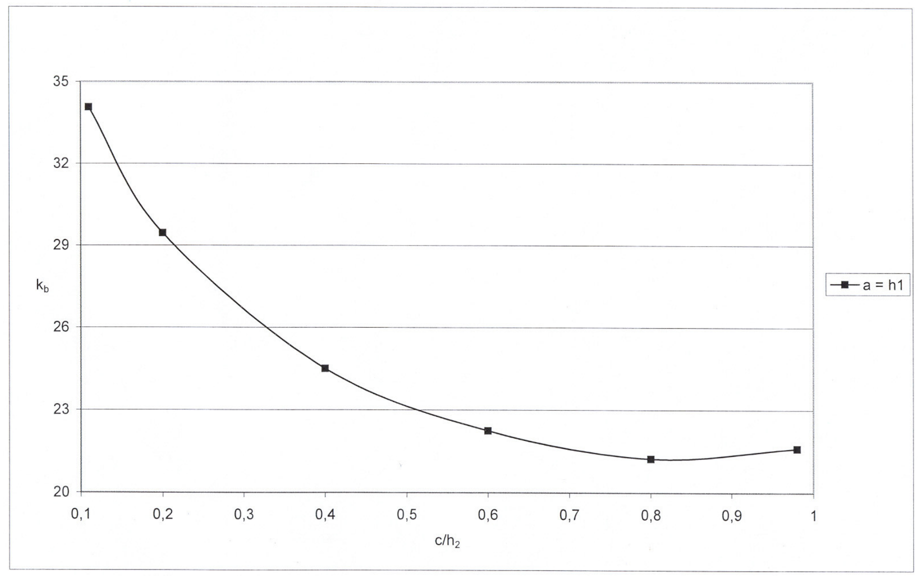

Figure 4.89. Variation of k_b with c/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_1 = h_2$, $c = 0.9h_2$, $a = h_1$, (Plane Stress)

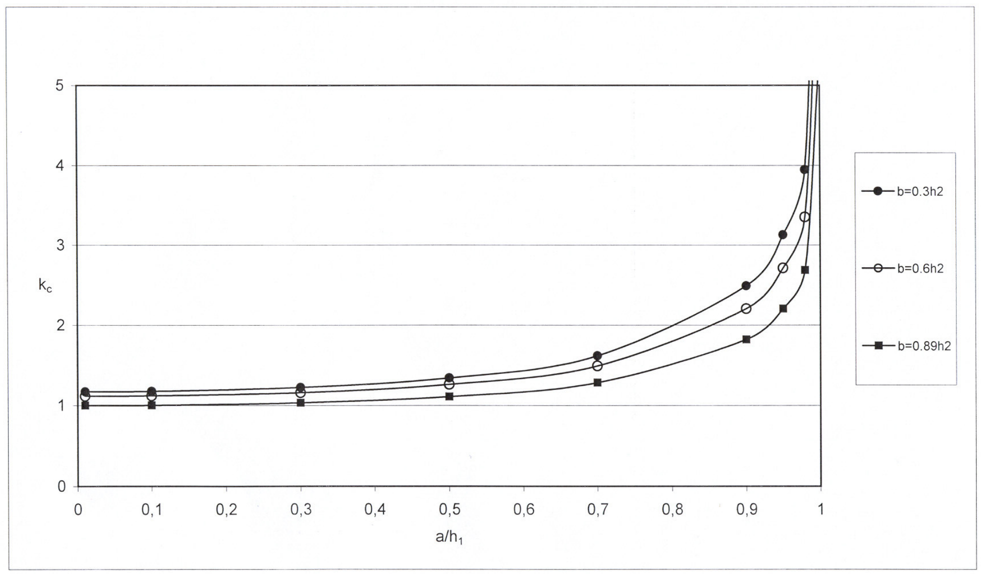

Figure 4.90. Variation of k_c with a/h_1 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_1=h_2$, $c=0.9h_2$,(Plane Stress)

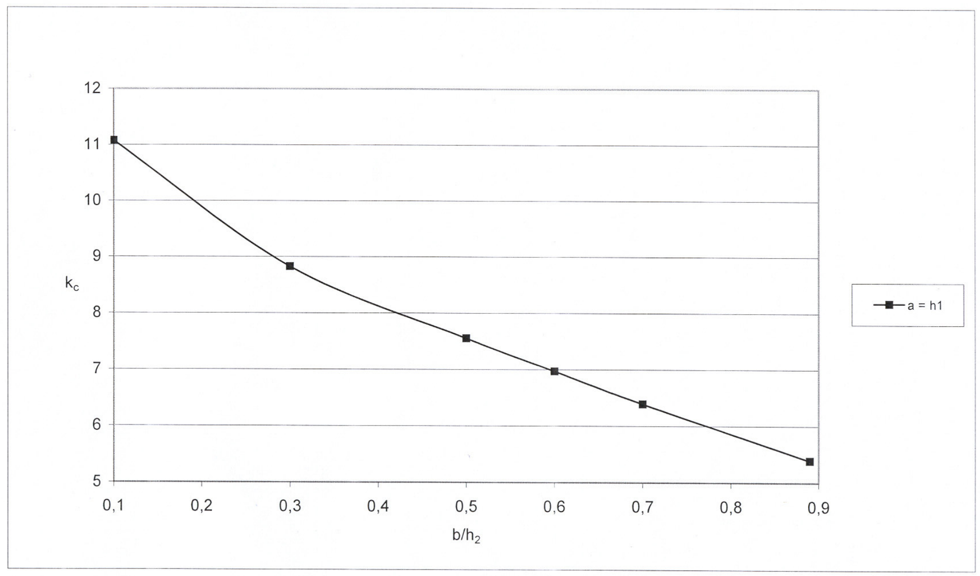

Figure 4.91. Variation of k_c with b/h_2 for 1:aluminum, 2:epoxy, $\mu_1/\mu_2 = 23.077$, $h_1 = h_2$, $c = 0.9h_2$, (Plane Stress)

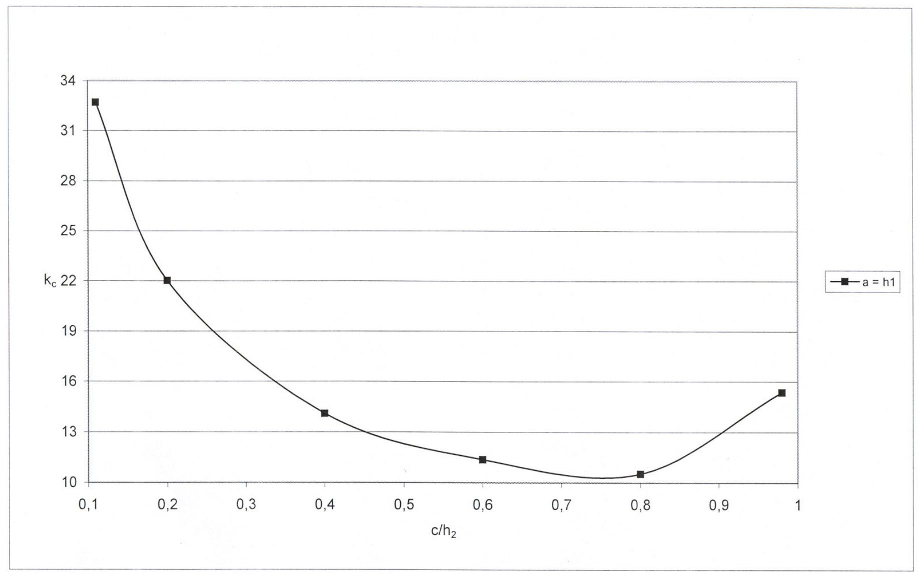

Figure 4.92. Variation of k_c with c/h_2 for 1:aluminum, 2:epoxy, μ_1/μ_2 =23.077, h_1=h_2, c=0.9h_2, (Plane Stres)

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APPENDIX A

Expressions for quantities appearing in Eqs.(2.16) are given in the form

$$a_{11} - 2\beta^2 a_3 f_{19}\mu_2(f_{37}\mu_1 + f_{38}\mu_2) \qquad \text{(A.1)}$$

$$a_{12} = -8\beta a_3\mu_2\left(\frac{1}{4} f_{19} f_{39}\mu_1 + f_{40}\mu_2\right)$$

$$a_{13} = \beta\mu_2\left(-\frac{1}{2} f_{32}\mu_1 + f_{33}\mu_2\right)$$

$$a_{14} = \beta^2\mu_2(-2 f_{30}\mu_1 + f_{31}\mu_2)$$

$$a_{15} = -\beta^2 f_{16}(- f_{42} + f_{45} + f_{41}\mu_2)$$

$$a_{16} = -4\beta\mu_1\left(b_1 f_{34}\mu_1 + \frac{1}{4} a_3 f_{35}\mu_2 - \frac{f_{36}\mu_2}{4}\right)$$

$$b_{11} = 4\beta^2 a_3 f_{19}\mu_2(- f_{53}\mu_1 + f_{54}\mu_2)$$

$$b_{12} = 4\beta a_3 f_{19}\mu_2(f_{58}\mu_1 + f_{59}\mu_2)$$

$$b_{13} = \beta\mu_2(f_{57}\mu_1 + f_1 f_{16} f_{21}\mu_2)$$

$$b_{14} = \beta^2\mu_2(f_{55}\mu_1 + f_{56}\mu_2)$$

$$b_{15} = -4\beta^2 f_{16}(f_2 f_{50}\mu_1 + f_{51}\mu_2)$$

$$b_{16} = 4\beta f_{16}\mu_1(f_1 f_{50}\mu_1 - f_{52}\mu_2)$$

$$a_{21} = 2\beta a_3\mu_2\left(-\frac{1}{2} f_{76}\mu_1 + 2b_2(e_3 - e_4) f_{63}\mu_2\right)$$

$$a_{22} = 2a_3\mu_2\left(\frac{f_{79}\mu_1}{2} + f_{80}\mu_2\right)$$

$$a_{23} = \frac{1}{4}(-4 f_{73} + f_{74})\mu_1\mu_2$$

$$a_{24} = -\frac{1}{2}\beta\mu_1\left(f_{66}\mu_1 + \frac{f_{69}\mu_2}{2} \right)$$

$$a_{25} = 2\beta\left(-\frac{1}{2} f_{81}\mu_1 + f_{83}\mu_2 \right)$$

$$a_{26} = 2\mu_1\left(\frac{f_{71}\mu_1}{2} - a_3 f_{72}\mu_2 \right)$$

$$b_{21} = -2\beta^2 (f_{101} - 2a_3 f_{103}\mu_2)$$

$$b_{22} = -2\beta(f_{106} - 2a_3 f_{105}\mu_2)$$

$$b_{23} = -\frac{1}{2}\beta\mu_1(f_{87} f_{90} f_{95}\mu_1 + f_{97}\mu_2)$$

$$b_{24} = \frac{1}{2}\beta^2\mu_1(f_{92}\mu_1 - f_{94}\mu_2)$$

$$b_{25} = 4\beta^2\left(\frac{f_{108}\mu_1}{2} + f_{63} f_{86}\mu_2 \right)$$

$$b_{26} = -2\beta\mu_1(f_{100}\mu_1 + f_4^2 f_{88} f_{90}\mu_2)$$

$$c_{21} = 4\beta^2\left(a_6 f_{129} + \frac{1}{2} f_{130}\mu_1^2 + 2 f_{131}\mu_2^2 \right)$$

$$c_{22} = 2\beta(a_6 f_{134} + f_{135}\mu_1^2 + 2 f_{80}\mu_2^2)$$

$$c_{23} = -\frac{1}{2}\beta\mu_1(- f_{125}\mu_1 + f_{126}\mu_2)$$

$$c_{24} = -\frac{1}{2}\beta^2\mu_1(f_{124}\mu_1 - f_{122}\mu_2)$$

$$c_{25} = 2\beta^2(f_{119}\mu_1 + f_{120}\mu_2)$$

$$c_{26} = 4\beta\mu_1\left(\frac{f_{117}\mu_1}{2} - f_{72}\mu_2 \right)$$

$$d_{21} = 2\beta(2 f_{162}\kappa_2 + f_{159}\kappa_2^2\mu_1 + f_{158}\mu_2)$$

$$d_{22} = 2(f_{170}\kappa_2 + f_{167}\kappa_2^2\mu_1 + f_{166}\mu_2)$$

$$d_{23} = -\frac{1}{2}\mu_1\left(f_{149}\mu_1 + \frac{f_{151}\mu_2}{2}\right)$$

$$d_{24} = -\frac{1}{2}\beta\mu_1\left(f_{147}\kappa_2 + \frac{f_{148}\mu_2}{2}\right)$$

$$d_{25} = -2\beta\left(\frac{f_{173}\mu_1}{2} + f_{174}\mu_2\right)$$

$$d_{26} = \mu_1(f_{155}\mu_1 + f_{156}\mu_2)$$

$$INT_1 = \int_0^\infty \frac{\alpha Cos[\alpha h_2]M_2[\alpha]}{(\alpha^2+\beta^2)^2} dl\alpha \qquad \text{(A.2)}$$

$$INT_2 = \int_0^\infty \frac{\alpha^2 Sin[\alpha h_2]M_2[\alpha]}{(\alpha^2+\beta^2)^2} dl\alpha$$

$$INT_3 = \int_0^\infty \frac{Sin[\alpha h_1](-3\alpha^2+\beta^2+(\alpha^2+\beta^2)\kappa_1)M_1[\alpha]}{(\alpha^2+\beta^2)^2} dl\alpha$$

$$INT_4 = \int_0^\infty \frac{-Cos[\alpha h_1](5\alpha^2+\beta^2+(\alpha^2+\beta^2)\kappa_1)M_1[\alpha]+(5\alpha^2+\beta^2+(\alpha^2+\beta^2)\kappa_2)M_2[\alpha]}{\alpha(\alpha^2+\beta^2)^2} dl\alpha$$

$$INT_5 = \int_0^\infty \frac{\alpha(-Cos[\alpha h_1]\mu_1 M_1[\alpha]+\mu_2 M_2[\alpha])}{(\alpha^2+\beta^2)^2} dl\alpha$$

$$INT_6 = \int_0^\infty \frac{\alpha^2 Sin[\alpha h_1]M_2[\alpha]}{(\alpha^2+\beta^2)^2} dl\alpha$$

$$a_1 = 1 + \kappa_1 \tag{A.3}$$
$$a_2 = -1 + \kappa_1$$
$$a_3 = 1 + \kappa_2$$
$$a_4 = -1 + \kappa_2$$
$$a_5 = \mu_1 - \mu_2$$
$$a_6 = \mu_1 \mu_2$$
$$a_7 = \lambda + \kappa_1$$
$$a_8 = -1 + \lambda$$
$$a_9 = 1 + \lambda \kappa_2$$
$$a_{10} = -\kappa_1 + \lambda \kappa_2$$
$$-a_{10} = \kappa_1 - \lambda \kappa_2$$
$$a_{11} = \lambda - \lambda \kappa_2$$
$$a_{12} = a_8 a_9$$
$$-3a_8 = 3 - 3\lambda$$

$$b_1 = \beta h_1 \tag{A.4}$$
$$b_1 = \beta h_1$$
$$b_2 = \beta h_2$$
$$b_2 = \beta h_2$$
$$b_2^2 = \beta h_2^2$$
$$b_1^2 = \beta^2 h_1^2$$

$$d_1 = t\beta \tag{A.5}$$
$$d_2 = tx$$
$$d_3 = x\beta$$
$$d_4 = \beta^2 \lambda$$
$$\lambda d_4 = \beta^2 \lambda^2$$

$$e_1 = e^{b_1} \tag{A.6}$$
$$e_2 = e^{-b_1}$$
$$e_3 = e^{b_2}$$
$$e_4 = e^{-b_2}$$
$$e_5 = e^{2b_1}$$
$$e_6 = e^{-2b_1}$$

$$e_7 = e^{2b_2}$$
$$e_8 = e^{-2b_2}$$
$$e_9 = e^{4b_1}$$
$$e_{10} = e^{4b_2}$$
$$e_{11} = e^{-4b_1}$$
$$e_{12} = e^{-4b_2}$$
$$e_{13} = e^{t\beta}$$
$$e_{14} = e^{x\beta}$$
$$e_{15} = e^{-x\beta}$$
$$e_{16} = e^{-2b_1-4b_2+d_1+d_3}$$
$$e_{17} = e^{-6b_1+d_1+d_3}$$
$$e_{18} = e^{-4b_1-2b_2+d_1-d_3}$$
$$e_{19} = e^{-2b_1-4b_2+d_1-d_3}$$
$$e_{20} = e^{-6b_1-4b_2+d_1-d_3}$$
$$e_{21} = e^{-6b_1+d_1-d_3}$$
$$e_{22} = e^{-6b_1-4b_2+d_1+d_3}$$
$$e_{23} = e^{-6b_1-2b_2+d_1-d_3}$$
$$e_{24} = e^{-6b_1-2b_2+d_1+d_3}$$
$$e_{25} = e^{-2b_1-2b_2+d_1+d_3}$$
$$e_{26} = e^{-2b_1-2b_2+d_1-d_3}$$
$$e_{27} = e^{-4b_1-2b_2+d_1+d_3}$$
$$e_{28} = e^{-4b_1+d_1+d_3}$$
$$e_{29} = e^{-4b_1+d_1-d_3}$$
$$e_{30} = e^{-4b_1-4b_2+d_1-d_3}$$
$$e_{31} = e^{-4b_1-4b_2+d_1+d_3}$$
$$e_{32} = e^{-5b_1-4b_2-d_3}$$
$$e_{33} = e^{-5b_1-2b_2-d_3}$$
$$e_{34} = e^{-5b_1-4b_2-d_1-d_3}$$
$$e_{35} = e^{-3b_1-4b_2-d_1-d_3}$$
$$e_{36} = e^{-5b_1-d_1-d_3}$$
$$e_{37} = e^{-b_1-4b_2-d_1-d_3}$$
$$e_{38} = e^{-5b_1-2b_2-d_1-d_3}$$
$$e_{39} = e^{-5b_1-4b_2+d_3}$$

$$e_{40} = e^{-5b_1+4b_2+d_3}$$
$$e_{41} = e^{-5b_1+d_3}$$
$$e_{42} = e^{-5b_1-2b_2+d_3}$$
$$e_{43} = e^{-5b_1-4b_2-d_1+d_3}$$
$$e_{44} = e^{-3b_1-4b_2-d_1+d_3}$$
$$e_{45} = e^{-5b_1-d_1+d_3}$$
$$e_{46} = e^{-b_1-4b_2-d_1+d_3}$$
$$e_{47} = e^{-5b_1-2b_2-d_1+d_3}$$
$$e_{48} = e^{-b_1-4b_2+d_1+d_3}$$
$$e_{49} = e^{-b_1-4b_2+d_1-d_3}$$
$$e_{50} = e^{-3b_1-2b_2+d_1+d_3}$$
$$e_{51} = e^{-3b_1-4b_2+d_1+d_3}$$
$$e_{52} = e^{-5b_1-d_3}$$
$$e_{53} = e^{-3b_1-4b_2+d_1-d_3}$$
$$e_{54} = e^{-b_1-4d_2-d_3}$$
$$e_{55} = e^{-b_1-4d_2+d_3}$$
$$e_{56} = e^{-b_1-2d_2-d_3}$$
$$e_{57} = e^{-b_1-2d_2+d_3}$$
$$e_{58} = e^{-3b_1-2b_2+d_1-d_3}$$
$$e_{59} = e^{-3b_1-d_3}$$
$$e_{60} = e^{-3b_1-d_1-d_3}$$
$$e_{61} = e^{-3b_1+d_3}$$
$$e_{62} = e^{-3b_1-d_1+d_3}$$
$$e_{63} = e^{-b_1-2b_2+d_1-d_3}$$
$$e_{64} = e^{-b_1-2b_2+d_1+d_3}$$
$$e_{65} = e^{-3b_1-4b_2+d_3}$$
$$e_{66} = e^{-3b_1-4b_2-d_3}$$
$$e_{67} = e^{-b_1-2b_2-d_1-d_3}$$
$$e_{68} = e^{-b_1-2b_2-d_1+d_3}$$
$$e_{69} = e^{-3b_1-2b_2-d_3}$$
$$e_{70} = e^{-3b_1-2b_2+d_3}$$
$$e_{71} = e^{-3b_1-2b_2-d_1+d_3}$$
$$e_{72} = e^{-3b_1-2b_2-d_1-d_3}$$

$$e_{73} = e^{-5b_1-4b_2+d_1-d_3}$$
$$e_{74} = e^{-5b_1+d_1-d_3}$$
$$e_{75} = e^{-5b_1-2b_2+d_1-d_3}$$
$$e_{76} = e^{-3b_1+d_1-d_3}$$
$$e_{77} = e^{-4b_1-2b_2-d_1-d_3}$$
$$e_{78} = e^{-2b_1-2b_2-d_1-d_3}$$
$$e_{79} = e^{-4b_2+d_1-d_3}$$
$$e_{80} = e^{-4b_1-4b_2-d_1+d_3}$$
$$e_{81} = e^{-4b_2-d_1+d_3}$$
$$e_{82} = e^{-2b_1-4b_2-d_1+d_3}$$
$$e_{83} = e^{-4b_1-d_3}$$
$$e_{84} = e^{-4b_2+d_3}$$
$$e_{85} = e^{-4b_1-6b_2+d_1+d_3}$$
$$e_{86} = e^{-6b_2+d_1+d_3}$$
$$e_{87} = e^{-2b_1-6b_2+d_1+d_3}$$
$$e_{88} = e^{-2b_1-d_1-d_3}$$
$$e_{89} = e^{-2b_2-d_3}$$
$$e_{90} = e^{-4b_1-4b_2+d_3}$$
$$e_{91} = e^{-d_3}$$
$$e_{92} = e^{-4b_1-2b_2+d_3}$$
$$e_{93} = e^{-4b_1-2b_2-d_3}$$
$$e_{94} = e^{-4b_1-d_1-d_3}$$
$$e_{95} = e^{-4b_1-4b_2-d_1-d_3}$$
$$e_{96} = e^{-4b_2-d_1-d_3}$$
$$e_{97} = e^{-2b_1-4b_2-d_1-d_3}$$
$$e_{98} = e^{-2b_1-d_3}$$
$$e_{99} = e^{-2b_1-4b_2+d_3}$$
$$e_{100} = e^{-2b_2+d_3}$$
$$e_{101} = e^{-2b_2-d_1-d_3}$$
$$e_{102} = e^{-2b_1-2b_2-d_3}$$
$$e_{103} = e^{-2b_1-2b_2+d_3}$$
$$e_{104} = e^{-2b_2+d_1-d_3}$$
$$e_{105} = e^{-2b_2-d_1+d_3}$$

$$e_{106} = e^{-4b_1 - 2b_2 - d_1 + d_3}$$
$$e_{107} = e^{-4b_2 + d_1 + d_3}$$
$$e_{108} = e^{-2b_1 - 2b_2 - d_1 + d_3}$$

$$e_2 e_8 = e^{-\beta(h_1 + 2h_2)} \quad \text{(A.7)}$$
$$e_5 e_7 = e^{2\beta(h_1 + h_2)}$$
$$e_2 e_7 = e^{-\beta(h_1 - 2h_2)}$$
$$e_1 e_8 = e^{\beta(h_1 - 2h_2)}$$
$$e_1 e_7 = e^{\beta(h_1 + 2h_2)}$$
$$e_1 e_4 = e^{\beta(h_1 - h_2)}$$
$$e_2 e_3 = e^{\beta(-h_1 + h_2)}$$
$$e_2 e_4 = e^{-\beta(h_1 + h_2)}$$
$$e_6 e_8 = e^{-2\beta(h_1 + h_2)}$$
$$e_6 e_4 = e^{-\beta(2h_1 + h_2)}$$
$$e_5 e_{10} = e^{2\beta(h_1 + 2h_2)}$$
$$e_{11} e_4 = e^{-\beta(4h_1 + h_2)}$$
$$e_6 e_7 = e^{-2\beta(h_1 - h_2)}$$
$$e_5 e_8 = e^{2\beta(h_1 - h_2)}$$
$$e_5 e_4 = e^{\beta(2h_1 - h_2)}$$
$$e_6 e_3 = e^{\beta(-2h_1 + h_2)}$$
$$e_5 e_3 = e^{\beta(2h_1 + h_2)}$$
$$e_6 e_{12} = e^{-2\beta(h_1 + 2h_2)}$$

$$f_1 = 1 + e_5 \quad \text{(A.8)}$$
$$f_2 = -1 + e_5$$
$$f_3 = 1 + e_7$$
$$f_4 = -1 + e_7$$
$$f_5 = e_1 + e_2$$
$$f_6 = e_1 - e_2$$
$$f_7 = e_5 e_7$$
$$f_8 = 1 + f_7$$
$$f_9 = -1 + f_7$$

$$f_{10} = b_1 f_1$$
$$f_{11} = f_2 + f_{10}$$
$$f_{12} = -a_2 f_2 + 2 f_{10}$$
$$f_{13} = e_1 e_7$$
$$f_{14} = e_2 e_7$$
$$f_{15} = e_1 e_8$$
$$f_{16} = e_2 e_8$$
$$f_{17} = e_2 e_3$$
$$f_{18} = e_1 e_4$$
$$f_{19} = e_2 e_4$$
$$f_{20} = 2e_1 + 2e_2 + 4b_2^2 f_5 - f_{13} - f_{14} - f_{15} - f_{16}$$
$$f_{21} = -4b_2^2 e_7 + f_4^2$$
$$f_{22} = e_2 f_2$$
$$f_{23} = f_4 f_{16}$$
$$f_{24} = -1 - 4b_2 e_7 + e_{10}$$
$$f_{25} = f_2 f_{16}$$
$$f_{26} = e_5 + e_7 + f_8 \kappa_2$$
$$f_{27} = b_2 f_4$$
$$f_{28} = a_4 f_4$$
$$f_4 f_{28} = a_4 f_4^2$$
$$f_{29} = b_2 f_3$$
$$f_{30} = f_6\left(2b_2^2 + \frac{1}{4} e_8 f_4 f_{28}\right) + \frac{1}{2} b_1 (4b_2^2 e_2 f_1 + 2a_3 b_2 f_{22} + f_{23}(e_5 - e_7 + f_9 \kappa_2))$$
$$f_{31} = b_1 f_{20} + \frac{1}{2} a_2 f_{21} f_{25}$$
$$f_{32} = a_3 f_{24} f_{25} + 2b_1(-2a_3 b_2 e_2 f_1 + 4b_2^2 f_{22} f_{23} f_{26} \kappa_2)$$
$$f_{33} = \frac{a_1 f_{20}}{2} - b_1 f_{21} f_{25}$$
$$f_{34} = f_6(1 + 4b_2^2 + (e_7 + e_8)\kappa_2 + \kappa_2^2)$$
$$f_{35} = f_{12} f_{16} f_{24}$$
$$f_{36} = (a_1 f_1 + 2b_1 f_2) f_{16} (8b_2^2 e_7 + f_4 f_{28})$$
$$f_{37} = f_2(a_3 f_3 + 2 f_{27}) - 2b_1(-e_5 + 2b_2(e_5 - e_7) - e_7 - f_8 \kappa_2)$$
$$f_{38} = a_1 f_1 f_4 + 2b_1(2b_2(e_5 - e_7) + f_2 f_4) + 2b_2(e_5 + e_7 + f_8 \kappa_1)$$
$$f_{39} = (f_2(f_{28} + 2 f_{29}) + 2b_1(e_5 - e_7 + 2b_2(e_5 - e_7) + f_9 \kappa_2))$$
$$f_{40} = -\frac{1}{4} a_2 f_2 f_4 f_{19} + \frac{1}{2} b_1(-2b_2(f_{17} + f_{18}) + f_1 f_4 f_{19}) + \frac{1}{2} b_2 f_{19}(-e_5 + e_7 + f_9 \kappa_1)$$

$$f_{41} = -a_3\left(-1+4b_2e_7e_{10}\right)(a_1 f_1 + 2b_1 f_2)$$
$$f_{42} = (a_3 f_3 - 2 f_{27})(f_{11}(a_3 f_{31} + 2 f_{27})\mu_1 - f_{12} f_{27}\mu_2)$$
$$f_{43} = -f_{11}(f_{28} + 2 f_{29})$$
$$f_{44} = f_{12}(1 - e_7 + 2 f_{29})$$
$$f_{45} = (-f_{28} + 2 f_{29})(f_{43}\mu_1 + f_{44}\mu_2)$$
$$f_{46} = e_5 + e_7$$
$$f_{47} = e_5 - e_7$$
$$f_{48} = b_2 f_{47}$$
$$f_{49} = a_3 b_2$$
$$f_{50} = e_7 + 4b_2^2 e_7 + (1+e_{10})\kappa_2 + e_7\kappa_2^2$$
$$f_{51} = -4b_2^2 e_7 f_2 + 2e_7 f_1 f_{49} + f_4(f_8 + f_{46}\kappa_2)$$
$$f_{52} = 4b_2^2 e_7 f_1 + 2e_7 f_2 f_{49} - f_4(f_9 + f_{47}\kappa_2)$$
$$f_{53} = -e_5 + e_7 + 2b_2 f_{46} + \kappa_2 - f_7\kappa_2$$
$$f_{54} = f_1 f_4 + 2b_2 f_{46}$$
$$f_{55} = 4b_2^2 f_{22} + f_{23} f_{26} + 2e_2 f_1 f_{49}$$
$$f_{56} = -2e_1 + 2e_2 + f_{13} - f_{14} + f_{15} - f_{16} - 4b_2^2 f_{22}$$
$$f_{57} = 4b_2^2 e_2 f_1 - 2 f_{22} f_{49} + f_{23}(f_{47} + f_9\kappa_2)$$
$$f_{58} = f_{26} + 2 f_{48}$$
$$f_{59} = f_2 f_4 - 2 f_{48}$$
$$f_{60} = -1 + e_9$$
$$f_{61} = -1 + f_7$$
$$f_{62} = 1 + f_7$$
$$f_{63} = 2b_1 - \frac{1}{2} e_6 f_{60}\kappa_1$$
$$f_{64} = -e_5 + e_7 - f_{61}\kappa_2$$
$$f_{65} = f_2 f_{49}$$
$$f_{66} = e_6 e_8 f_{50}(4b_1 e_5 + f_{60})$$
$$f_{67} = f_9 + f_{47}\kappa_2 + \kappa_1(f_{47} + f_{61}\kappa_2)$$
$$f_{68} = 4a_2 b_2^2 e_7 f_1 - 2a_1 e_7 f_{65} + f_4 f_{67}$$
$$f_{69} = -4b_1 e_8(8b_2^2 e_7 + f_4 f_{28}) + e_6 e_8 f_2 f_{68}$$
$$f_{70} = 4a_1 b_2^2 e_7 f_1 - 2a_2 e_7 f_{65} - f_4(f_9 + f_{64}\kappa_1 + f_{47}\kappa_2)$$
$$f_{71} = a_1 e_6 e_8 f_1^2 f_{50}$$
$$f_{72} = (-4b_2 + e_7 - e_8) f_{63}$$
$$f_{73} = a_3 b_1 e_8 f_{24}$$
$$f_{74} = e_6 e_8 f_1 f_{70}$$

$$f_{75} = f_9 + 2b_2(f_8 + f_{46}\kappa_1) + e_5\kappa_2 - e_7\kappa_2 + \kappa_1(-e_5 + e_7 + \kappa_2 - f_7\kappa_2)$$
$$f_{76} = e_4e_6(4b_1e_5(a_3f_3 + 2f_{27}) + f_1f_{75})$$
$$f_{77} = f_9 + (-e_5 + e_7)\kappa_1$$
$$f_{78} = f_8 - 2b_2f_{77} + e_5\kappa_2 + e_7\kappa_2 + \kappa_1(f_{46} + f_{62}\kappa_2)$$
$$f_{79} = -4b_1e_4(f_{28} + 2f_{29}) + e_4e_6f_1f_{78}$$
$$f_{80} = e_4e_6(1 - e_7 + f_{29})(4b_1e_5 - f_{60}\kappa_1)$$
$$f_{81} = e_6e_8f_{50}(-8b_1e_5 + a_2f_{60})$$
$$f_{82} = 8b_2^2e_6 + e_6e_8f_4f_{28}$$
$$f_{83} = -8b_1(2b_2^2 + \frac{1}{4}e_8f_4f_{28}) + \frac{1}{2}f_{60}f_{82}\kappa_1$$
$$f_{84} = e_3 + e_4$$
$$f_{85} = e_3 - e_4$$
$$f_{86} = 4b_2 + e_7 - e_8$$
$$-1f_{86} = -4b_2 - e_7 + e_8$$
$$f_{87} = 4b_1e_5 + f_{60}$$
$$f_{88} = -4b_1e_5 + f_{60}\kappa_1$$
$$f_{89} = a_2b_2$$
$$f_{90} = e_6e_8$$
$$f_{91} = a_1b_2$$
$$f_{92} = (4b_1 + e_5 - e_6)f_{86}$$
$$f_{93} = 2e_7f_1f_{89} + f_4(-e_5 - e_7 + f_8\kappa_1)$$
$$f_{94} = 4b_1f_{86} - f_2f_{90}f_{93}$$
$$f_{95} = 1 + e_{10} + 2e_7\kappa_2$$
$$f_{96} = 2e_7f_1f_{91} + f_4(f_{46} + f_8\kappa_1)$$
$$f_{97} = -4b_1e_8f_4^2 + f_1f_{90}f_{96}$$
$$f_{98} = -1 + e_5e_{10} + 2e_7f_1f_{91} - e_7\kappa_2 + f_7\kappa_2 + \kappa_1(-e_5 + e_{10} - e_7f_2\kappa_2)$$
$$f_{99} = e_7 + e_8 + 2\kappa_2$$
$$f_{100} = f_1f_{90}f_{98} + 4b_1f_{99}$$
$$f_{101} = e_5e_4e_6(-f_{28} + 2f_{29})(f_{87}\mu_1 + f_{88}\mu_2)$$
$$f_{102} = 4b_1e_5f_4 + f_2(f_9 + f_{47}\kappa_1)$$
$$f_{103} = \frac{1}{2}e_4e_6f_{102}\mu_1 + f_{85}(-2b_1 + \frac{1}{2}e_6f_{60}\kappa_1)\mu_2$$
$$f_{104} = 4b_1e_4f_3 + e_4e_6f_2(f_8 + f_{46}\kappa_1)$$
$$f_{105} = \frac{f_{104}\mu_1}{2} - f_{63}f_{84}\mu_2$$
$$f_{106} = a_5e_4e_6(-a_3f_3 + 2f_{27})(f_{87}\mu_1 + f_{88}\mu_2)$$

$$f_{107} = 1 - e_5 e_{10} + 2e_7 f_1 f_{89} + e_7\kappa_2 - f_7\kappa_2 + \kappa_1(-e_5 + e_{10} - e_7 f_2\kappa_2)$$
$$f_{108} = -4b_1 f_{86} + f_2 f_{90} f_{107}$$
$$f_{109} = e_4 e_6$$
$$f_{110} = -4b_2 + e_7 - e_8$$
$$f_{111} = f_2 f_{91}$$
$$f_{112} = e_5 + e_{10} + e_7 f_1\kappa_2$$
$$f_{113} = e_7\kappa_2 + f_7\kappa_2\ )$$
$$f_{114} = 1 + e_5 e_{10}$$
$$f_{115} = f_{113} + f_{114}$$
$$f_{116} = -e_5 + e_7$$
$$f_{117} = 4b_1 f_{110} + f_1 f_{90}(2e_7 f_2 f_{89} + f_{115} + f_{112}\kappa_1)$$
$$f_{118} = -2e_7 f_{111} + f_{115} - f_{112}\kappa_1$$
$$f_{119} = 4b_1 f_{99} + f_2 f_{90} f_{118}$$
$$f_{120} = f_4^2 f_{88} f_{90}$$
$$f_{121} = f_2 f_{90}(2e_7 f_{111} + f_4(f_{116} - f_9\kappa_1))$$
$$f_{122} = 4b_1 e_8 f_4^2 + f_{121}$$
$$f_{123} = 2e_7 f_2 f_{89} - f_4(f_{116} + f_9\kappa_1)$$
$$f_{124} = f_{87} f_{90} f_{95}$$
$$f_{125} = (4b_1 + e_5 - e_6) f_{110}$$
$$f_{126} = 4b_1 f_{110} + f_1 f_{90} f_{123}$$
$$f_{127} = -2b_1 e_4(a_3 f_3 + 4 f_{27})$$
$$f_{128} = a_3 f_1 f_{109}(f_{47} - f_9\kappa_1)$$
$$f_{129} = a_2 f_{27} f_{60} f_{109} + f_{127} - \frac{f_{128}}{2}$$
$$f_{130} = (a_3 f_3 + 2 f_{27}) f_{87} f_{109}$$
$$f_{131} = b_2 f_{63} f_{85}$$
$$f_{132} = 4 f_{29} + f_4(-3 + \kappa_2)$$
$$f_{133} = 2 - e_5 - e_7 + 2 f_7 + 2a_2 f_2 f_{29} + e_5\kappa_2 + e_7\kappa_2 + \kappa_1(-1 + 2e_5 + 2e_7 - f_7 + f_8\kappa_2)$$
$$f_{134} = -4b_1 e_4 f_{132} + e_4(e_5 + e_9)e_{11} f_{133}$$
$$f_{135} = (f_{28} + 2 f_{29}) f_{87} f_{109}$$
$$f_{136} = e_5 + e_6$$
$$f_{137} = e_5 - e_6$$
$$f_{138} = e_7 - e_8$$
$$f_{139} = e_5 + e_6$$
$$\beta b_2^2 = \beta^3 h_2^2$$

$$f_{140} = 4b_1 e_5 - f_{60}\kappa_1$$
$$f_{141} = (f_4 + f_{29}) f_{109} f_{140}$$
$$f_{142} = -8b_2^2 e_7 + a_3 f_4^2$$
$$b_2 f_{89} = a_2 b_2^2$$
$$f_{143} = f_{87} f_{109}$$
$$f_{144} = -1 - e_7 + 2 f_{27}$$
$$f_{145} = f_4 + 2 f_{29}$$
$$f_{146} = -4a_1 b_2^2 e_7 f_2 - 2e_7 f_1 f_{89} + f_4 (f_8 - f_{46}\kappa_1)$$
$$f_{147} = (-1 + 4b_2 e_7 + e_{10}) f_{87} f_{90}\mu_1 - \frac{f_{94}\mu_2}{2}$$
$$f_{148} = 4b_1 f_{86} + f_2 f_{90} f_{146}$$
$$f_{149} = (4b_1 + f_{137})(1 + 4b_2^2 - \kappa_2^2)$$
$$f_{150} = -1 - f_7 + f_{46}\kappa_2 + \kappa_1(-e_5 - e_7 + f_8\kappa_2)$$
$$f_{151} = 4b_1 e_8 f_{142} - f_1 f_{90}(-4b_2 e_7 f_2 f_{89} + 2a_4 e_7 f_1 f_{91} + f_4 f_{150})$$
$$f_{152} = -8h_1(\beta + 4\beta b_2^2 - \beta\kappa_2^2)$$
$$f_{153} = e_7 f_2 + (-1 + e_{10}) f_1\kappa_2 - e_7 f_2\kappa_2^2$$
$$f_{154} = e_7 - f_7 + 4b_2 e_7 f_2 f_{89} + f_{153}\kappa_1 - \kappa_2 - e_5\kappa_2 + e_{10}\kappa_2 + e_5 e_{10}\kappa_2 + 4e_7 f_1 f_{91}\kappa_2 - e_7\kappa_2^2 + f_7\kappa_2^2$$
$$f_{155} = f_{152} + f_1 f_{90} f_{154}$$
$$f_{156} = f_{88} f_{90} f_{142}$$
$$f_{157} = 4b_1 e_4 f_{145} + f_2 f_{109}(f_9 + f_{47}\kappa_1 + 2b_2(f_8 - f_{46}\kappa_1))$$
$$f_{158} = \frac{f_{157}\mu_1}{2} - f_{141}\mu_2$$
$$f_{159} = -f_4 f_{143}\mu_1 + \frac{1}{2}(4b_1 e_4 f_4 - f_2 f_{109}(f_{47} + f_9\kappa_1))\mu_2$$
$$f_{160} = -e_3 + e_4 - e_3 e_5 - 2e_4 e_5 + 2e_3 e_6 + f_{109} + f_2(2 + 2f_7 + f_{46}) f_{109}\kappa_1$$
$$f_{161} = -6b_1 e_4 (f_4 f_{29}) + \frac{3}{4} a_2 f_4 f_{60} f_{109} + \frac{b_2 f_{160}}{2}$$
$$f_{162} = a_6 f_{161} + \frac{1}{2} f_{143} f_{145}\mu_1^2 + \frac{1}{2} f_{141}\mu_2^2$$
$$f_{163} = -1 - f_7 + f_{46}\kappa_1 + 2b_2(f_9 + f_{47}\kappa_1)$$
$$f_{164} = 4b_1 e_4 f_{144} + f_2 f_{109} f_{163}$$
$$f_{165} = b_2 f_{85}\left(-2b_1 + \frac{1}{2} e_6 f_{60}\kappa_1\right)$$
$$f_{166} = \frac{f_{164}\mu_1}{2} + 2 f_{165}\mu_2$$

$$f_{167} = -f_{84}(4b_1 + f_{137})\mu_1 + \frac{1}{2}(4b_1 e_4 f_3 + f_2 f_{109}(f_{46} - f_8\kappa_1))\mu_2$$
$$f_{168} = -2e_5 + 2e_7 + f_9 + (2 - 2f_7 + f_{47})\kappa_1$$
$$f_{169} = -\frac{1}{2}a_1 f_2^2 f_4 f_{109} - b_2(12b_1 e_4 f_4 + f_2 f_{109} f_{168})$$
$$f_{170} = a_6 f_{169} + f_{143} f_{144}\mu_1^2 + 2f_{131}\mu_2^2$$
$$f_{171} = -2 + 4a_1 b_2^2 e_6 f_2^2 + f_{136} + f_2 f_{90} f_{153}\kappa_1 + (e_6 e_7 + e_5 e_8)\kappa_2$$
$$f_{172} = f_{171} - 8b_1 e_8(-1 + e_{10})\kappa_2 - 8b_2\left(4b_1 - \frac{1}{2}a_2 e_6 f_{60}\right)\kappa_2$$
$$f_{173} = f_{172} - (f_7 + f_{90})\kappa_2 + 2\kappa_2^2 - f_{139}\kappa_2^2$$
$$f_{174} = a_4 f_{63} f_{86}$$
$$f_{175} = (8b_2^2 e_7 + f_4 f_{28})h_1 - a_1 a_3 e_7 h_2$$
$$f_{176} = \kappa_1 + \kappa_2 + e_7(1 + \kappa_1\kappa_2)$$
$$f_{177} = 4b_2 e_7 f_{60} f_{89} + f_4(1 + e_9 f_{176} + e_7\kappa_1 + e_7\kappa_2 + \kappa_1\kappa_2)$$
$$f_{178} = \kappa_2 + e_{10}\kappa_2 + e_7(1 + 4b_2^2 + \kappa_2^2)$$
$$f_{179} = a_6 f_{177} + f_{87} f_{178}\mu_1^2 - f_{21} f_{140}\mu_2^2$$
$$f_{180} = -4\beta a_6 f_7 f_{175} - e_7 f_{179}$$
$$\Delta = \frac{1}{2}\pi e_6 e_{12} f_{180}$$

APPENDIX B

Expressions for quantities appearing in Eqs.(2.17) are given in the form

(B.1)

$$\int_0^\infty \frac{\alpha}{\left(\alpha^2+\beta^2\right)^2}\sin(\alpha t)\cos(\alpha h_1)d\alpha = \frac{\pi}{8\beta}\left(-\left(h_1-t\right)e^{-(h_1-t)\beta}+\left(h_1+t\right)e^{-(h_1+t)\beta}\right)$$

$$\int_0^\infty \frac{1}{\alpha\left(\alpha^2+\beta^2\right)^2}\sin(\alpha t)\cos(\alpha h_1)d\alpha = \frac{\pi}{8\beta^4}\left\{\left(2+\beta\left(h_1-t\right)\right)e^{-\beta(h_1-t)}-\left(2+\beta\left(h_1+t\right)\right)e^{-\beta(h_1+t)}\right\}$$

$$\int_0^\infty \frac{\alpha^2}{\left(\alpha^2+\beta^2\right)^2}\sin(\alpha t)\sin(\alpha h_1)d\alpha = \frac{\pi}{8\beta}\left\{\left(1-\beta\left(h_1-t\right)\right)e^{-\beta(h_1-t)}-\left(1-\beta\left(h_1+t\right)\right)e^{-\beta(h_1+t)}\right\}$$

$$\int_0^\infty \frac{1}{\left(\alpha^2+\beta^2\right)^2}\sin(\alpha t)\sin(\alpha h_1)d\alpha = \frac{\pi}{8\beta^3}\left\{\left(1+\beta\left(h_1-t\right)\right)e^{-\beta(h_1-t)}-\left(1+\beta\left(h_1+t\right)\right)e^{-\beta(h_1+t)}\right\}$$

$$\int_0^\infty \frac{\alpha}{\left(\alpha^2+\beta^2\right)^2}\sin(\alpha t)d\alpha = \frac{\pi}{4\beta}\left(te^{-t\beta}\right)$$

$$\int_0^\infty \frac{1}{\alpha\left(\alpha^2+\beta^2\right)^2}\sin(\alpha t)d\alpha = \frac{\pi}{4\beta^4}\left(2-\left(2+\beta t\right)e^{-t\beta}\right)$$

$$f_{181} = 1+e^{2x\beta} \qquad \text{(B.2)}$$

$$f_{182} = 2+x\beta$$

$$f_{183} = 2-x\beta+e^{2x\beta}\,f_{182}$$

$$f_{184} = e^{-x\beta}\beta$$

$$\beta f_{184} = e^{-x\beta}\beta^2$$

$$f_{185} = f_{37}\,f_{181}\mu_1+2\,f_{53}\,f_{183}\mu_1+(\,f_{38}\,f_{181}-2\,f_{54}\,f_{183})\mu_2$$

$$f_{186} = -4\,f_{40}\,f_{181}\mu_2+\,f_{19}(-\,f_{39}\,f_{181}\mu_1+2\,f_{183}(\,f_{58}\mu_1+\,f_{59}\mu_2))$$

$$f_{187} = -\,f_{32}\,f_{181}\mu_1+2(\,f_{57}\,f_{183}\mu_1+(\,f_{33}\,f_{181}+\,f_1\,f_{16}\,f_{21}\,f_{183})\mu_2)$$

$$f_{188} = -2 f_{30} f_{181}\mu_1 + f_{55} f_{183}\mu_1 + (f_{31} f_{181} + f_{56} f_{183})\mu_2$$
$$f_{189} = -f_{42} f_{181} + f_{45} f_{181} + 8 f_2 f_{50}\mu_1$$
$$f_{190} = 8e^{2x\beta} f_2 f_{50}\mu_1 - 4x\beta f_2 f_{50}\mu_1$$
$$f_{191} = 4e^{2x\beta} x\beta f_2 f_{50}\mu_1 + f_{41}\mu_2$$
$$f_{192} = e^{2x\beta} f_{41}\mu_2 + 8 f_{51}\mu_2$$
$$f_{193} = 8e^{2x\beta} f_{51}\mu_2 - 4x\beta f_{51}\mu_2$$
$$f_{194} = 4e^{2x\beta} x\beta f_{51}\mu_2$$
$$f_{195} = f_{189} + f_{190} + f_{191} + f_{192} + f_{193} + f_{194}$$
$$f_{196} = -4b_1 f_{34} f_{181}\mu_1 + 4 f_1 f_{16} f_{50} f_{183}\mu_1$$
$$f_{197} = a_3 f_{35} f_{181} - f_{36} f_{181} + 4 f_{16} f_{52} f_{183}$$
$$f_{198} = -8\beta a_3 f_{19} f_{184} f_{185}\mu_1\mu_2$$
$$f_{199} = 8a_3 f_{184} f_{186}\mu_1\mu_2$$
$$f_{200} = 2 f_{184} f_{187}\mu_1\mu_2$$
$$f_{201} = 4\beta f_{184} f_{188}\mu_1\mu_2$$
$$f_{202} = -4\beta f_{16} f_{184} f_{195}\mu_1$$
$$f_{203} = 4 f_{184}\mu_1^2 (f_{196} - f_{197}\mu_2)$$
$$f_{204} = \frac{1}{\pi e_6 e_{12} f_{180}}$$
$$f_{205} = f_{198} f_{204}$$
$$f_{206} = f_{199} f_{204}$$
$$f_{207} = f_{200} f_{204}$$
$$f_{208} = f_{201} f_{204}$$
$$f_{209} = f_{202} f_{204}$$
$$f_{210} = f_{203} f_{204}$$
$$f_{211} = 3 + 2x\beta$$
$$f_{212} = -3 + 2x\beta$$
$$f_{213} = -1 + e^{2x\beta}$$
$$f_{214} = 3 - 2x\beta$$
$$f_{215} = e^{2x\beta}$$
$$f_{216} = f_{211} f_{215}$$
$$f_{217} = f_{212} + f_{216} - f_{213}\kappa_2$$
$$f_{218} = f_{214} + f_{216} - f_{181}\kappa_2$$
$$a_6 = \mu_1\mu_2$$
$$f_{219} = a_3 a_6$$
$$a_7 = \mu_1^2$$

$$a_8 = \mu_2^2$$
$$f_{220} = a_3 a_8$$
$$f_{221} = f_{184} f_{204}$$
$$f_{222} = 3a_7 f_{130}$$
$$f_{223} = 2x\beta a_7 f_{130}$$
$$-1 f_{223} = -2x\beta a_7 f_{130}$$
$$f_{224} = a_8 f_{131}$$
$$f_{225} = f_{63} f_{220}$$
$$f_{226} = -f_{101} f_{217} + 2a_6 f_{129} f_{218} - f_{76} f_{219} - f_{76} f_{215} f_{219} + f_{222}$$
$$f_{227} = f_{215} f_{222} - f_{223} + f_{215} f_{223} + 12 f_{224} - 8x\beta f_{224}$$
$$f_{228} = 12 f_{215} f_{224} + 8x\beta f_{215} f_{224} + 4b_2 e_3 f_{225} - 4b_2 e_4 f_{225}$$
$$f_{229} = 4b_2 e_3 f_{215} f_{225} - 4b_2 e_4 f_{215} f_{225}$$
$$f_{230} = f_{229} - a_7 f_{130} \kappa_2$$
$$f_{231} = 4 f_{162} \kappa_2 - a_7 f_{130} f_{215} \kappa_2 - 4 f_{162} f_{215} \kappa_2 - 4 f_{224} \kappa_2 - 4 f_{215} f_{224} \kappa_2$$
$$f_{232} = 2 f_{159} \kappa_2^2 \mu_1 - 2 f_{159} f_{215} \kappa_2^2 \mu_1 - 6a_3 f_{103} \mu_2 + 4x\beta a_3 f_{103} \mu_2$$
$$f_{233} = 2 f_{158} \mu_2 + 6a_3 f_{103} f_{215} \mu_2 + 4x\beta a_3 f_{103} f_{215} \mu_2 - 2 f_{158} f_{215} \mu_2$$
$$f_{234} = 2a_3 f_{103} \kappa_2 \mu_2 - 2a_3 f_{103} f_{215} \kappa_2 \mu_2$$
$$f_{235} = 4\beta f_{221} (f_{226} + f_{227} + f_{228} + f_{230} + f_{231} + f_{232} + f_{233} + f_{234}) \mu_2$$
$$f_{236} = 6a_8 f_{80} - 4x\beta a_8 f_{80} + 3a_7 f_{135} - 2x\beta a_7 f_{135} + 6a_8 f_{80} f_{215}$$
$$f_{237} = 4x\beta a_8 f_{80} f_{215} + 3a_7 f_{135} f_{215} + 2x\beta a_7 f_{135} f_{215} - f_{106} f_{217}$$
$$f_{238} = a_6 f_{134} f_{218} + f_{79} f_{219} + f_{79} f_{215} f_{219} + 2 f_{80} f_{220} + 2 f_{80} f_{215} f_{220}$$
$$f_{239} = -2a_8 f_{80} \kappa_2 - a_7 f_{135} \kappa_2 + 2 f_{170} \kappa_2 - 2a_8 f_{80} f_{215} \kappa_2 - a_7 f_{135} f_{215} \kappa_2 - 2 f_{170} f_{215} \kappa_2$$
$$f_{240} = 2 f_{167} \kappa_2^2 \mu_1 - 2 f_{167} f_{215} \kappa_2^2 \mu_1 - 6a_3 f_{105} \mu_2 + 4x\beta a_3 f_{105} \mu_2 + 2 f_{166} \mu_2$$
$$f_{241} = 6a_3 f_{105} f_{215} \mu_2 + 4x\beta a_3 f_{105} f_{215} \mu_2 - 2 f_{166} f_{215} \mu_2$$
$$f_{242} = 2a_3 f_{105} \kappa_2 \mu_2 - 2a_3 f_{105} f_{215} \kappa_2 \mu_2$$
$$f_{243} = 4 f_{221} (f_{236} + f_{237} + f_{238} + f_{239} + f_{240} + f_{241} + f_{242}) \mu_2$$
$$f_{244} = 2 f_{149} \mu_1 - 2 f_{149} f_{215} \mu_1 + f_{87} f_{90} f_{95} f_{217} \mu_1 - f_{125} f_{218} \mu_1$$
$$f_{245} = 4 f_{73} \mu_2 + f_{74} \mu_2 - 3 f_{97} \mu_2 + 2x\beta f_{97} \mu_2 + 3 f_{126} \mu_2 - 2x\beta f_{126} \mu_2$$
$$f_{246} = f_{151} \mu_2 + 4 f_{73} f_{215} \mu_2 - f_{74} f_{215} \mu_2 + 3 f_{97} f_{215} \mu_2$$
$$f_{247} = 2x\beta f_{97} f_{215} \mu_2 + 3 f_{126} f_{215} \mu_2 + 2x\beta f_{126} f_{215} \mu_2 - f_{151} f_{215} \mu_2$$
$$f_{248} = f_{97} \kappa_2 \mu_2 - f_{126} \kappa_2 \mu_2 - f_{97} f_{215} \kappa_2 \mu_2 - f_{126} f_{215} \kappa_2 \mu_2$$
$$f_{249} = -a_6 f_{221} (f_{244} + f_{245} + f_{246} + f_{247} + f_{248})$$
$$f_{250} = 2 f_{147} f_{213} \kappa_2 - 3 f_{92} \mu_1 + 2x\beta f_{92} \mu_1 - 3 f_{124} \mu_1$$
$$f_{251} = 2x\beta f_{124} \mu_1 - 2 f_{66} f_{181} \mu_1 + 3 f_{92} f_{215} \mu_1$$
$$f_{252} = 2x\beta f_{92} f_{215} \mu_1 - 3 f_{124} f_{215} \mu_1 - 2x\beta f_{124} f_{215} \mu_1 + f_{92} \kappa_2 \mu_1$$

$$f_{253} = f_{124}\kappa_2\mu_1 - f_{92} f_{215}\kappa_2\mu_1 + f_{124} f_{215}\kappa_2\mu_1 - f_{69}\mu_2$$
$$f_{254} = 3f_{94}\mu_2 - 2x\beta f_{94}\mu_2 + 3f_{122}\mu_2 - 2x\beta f_{122}\mu_2 - f_{148}\mu_2 - f_{69} f_{215}\mu_2 - 3f_{94} f_{215}\mu_2 - 2x\beta f_{94} f_{215}\mu_2$$
$$f_{255} = 3 f_{122} f_{215}\mu_2 + 2x\beta f_{122} f_{215}\mu_2$$
$$f_{256} = f_{148} f_{215}\mu_2 - f_{94}\kappa_2\mu_2 - f_{122}\kappa_2\mu_2 + f_{94} f_{215}\kappa_2\mu_2 - f_{122} f_{215}\kappa_2\mu_2$$
$$f_{257} = \beta a_6 f_{221}(f_{250} + f_{251} + f_{252} + f_{253} + f_{254} + f_{255} + f_{256})$$
$$f_{258} = 3 f_{119}\mu_1 - 2x\beta f_{119}\mu_1 - f_{173}\mu_1 - f_{81} f_{181}\mu_1 + 3 f_{119} f_{215}\mu_1$$
$$f_{259} = 2x\beta f_{119} f_{215}\mu_1 + f_{173} f_{215}\mu_1$$
$$f_{260} = f_{108} f_{217}\mu_1 - f_{119}\kappa_2\mu_1 - f_{119} f_{215}\kappa_2\mu_1 + 2 f_{83}\mu_2 - 6 f_{63} f_{86}\mu_2$$
$$f_{261} = 4x\beta f_{63} f_{86}\mu_2 + 3 f_{120}\mu_2 - 2x\beta f_{120}\mu_2 - 2 f_{174}\mu_2 + 2 f_{83} f_{215}\mu_2$$
$$f_{262} = 6 f_{63} f_{86} f_{215}\mu_2 + 4x\beta f_{63} f_{86} f_{215}\mu_2 + 3 f_{120} f_{215}\mu_2$$
$$f_{263} = 2x\beta f_{120} f_{215}\mu_2 + 2 f_{174} f_{215}\mu_2$$
$$f_{264} = 2 f_{63} f_{86}\kappa_2\mu_2 - f_{120}\kappa_2\mu_2 - 2 f_{63} f_{86} f_{215}\kappa_2\mu_2 - f_{120} f_{215}\kappa_2\mu_2$$
$$f_{265} = 4\beta f_{221}(f_{258} + f_{259} + f_{260} + f_{261} + f_{262} + f_{263} + f_{264})\mu_2$$
$$f_{266} = -3 f_{117}\mu_1 + 2x\beta f_{117}\mu_1 - f_{155}\mu_1 - f_{71} f_{181}\mu_1 - 3 f_{117} f_{215}\mu_1 - 2x\beta f_{117} f_{215}\mu_1$$
$$f_{267} = f_{155} f_{215}\mu_1 + f_{100} f_{217}\mu_1 + f_{117}\kappa_2\mu_1 + f_{117} f_{215}\kappa_2\mu_1$$
$$f_{268} = 6 f_{72}\mu_2 - 4x\beta f_{72}\mu_2 + 2a_3 f_{72}\mu_2 - 3 f_4^2 f_{88} f_{90}\mu_2 + 2x\beta f_4^2 f_{88} f_{90}\mu_2 - f_{156}\mu_2$$
$$f_{269} = 6 f_{72} f_{215}\mu_2 + 4x\beta f_{72} f_{215}\mu_2 + 2a_3 f_{72} f_{215}\mu_2$$
$$f_{270} = 3 f_4^2 f_{88} f_{90} f_{215}\mu_2 + 2x\beta f_4^2 f_{88} f_{90} f_{215}\mu_2 + f_{156} f_{215}\mu_2 - 2 f_{72}\kappa_2\mu_2$$
$$f_{271} = f_4^2 f_{88} f_{90}\kappa_2\mu_2 - 2 f_{72} f_{215}\kappa_2\mu_2 - f_4^2 f_{88} f_{90} f_{215}\kappa_2\mu_2$$
$$f_{272} = -4a_6 f_{221}(f_{266} + f_{267} + f_{268} + f_{269} + f_{270} + f_{271})$$

$$\phi_1 = f_{205}\left(\frac{-\pi}{4\beta^2(\kappa_2 + 1)}\right) \qquad \text{(B.3)}$$

$$\phi_2 = f_{206}\left(\frac{-\pi}{4\beta(\kappa_2 + 1)}\right)$$

$$\phi_3 = f_{207}\left(\frac{-\pi(\kappa_1 - 3)}{4\beta(\kappa_1 + 1)}\right)$$

$$\phi_4 = f_{207}\left(\frac{-\pi}{4\beta}\right)$$

$$\phi_5 = f_{208}\left(\frac{\pi(\kappa_1+5)}{4\beta^2(\kappa_1+1)}\right)$$

$$\phi_6 = f_{208}\left(\frac{\pi}{4\beta^2}\right)$$

$$\phi_7 = f_{208}\left(\frac{-\pi(\kappa_2+5)}{2\beta(\kappa_2+1)}\right)$$

$$\phi_8 = f_{208}\left(\frac{-\pi}{2\beta^2}\right)$$

$$\phi_9 = f_{209}\left(\frac{\pi\mu_1}{4\beta^2(\kappa_1+1)}\right)$$

$$\phi_{10} = f_{209}\left(\frac{-\pi\mu_2}{2\beta(\kappa_2+1)}\right)$$

$$\phi_{11} = f_{210}\left(\frac{-\pi}{4\beta(\kappa_1+1)}\right)$$

$$\varphi\phi_1 = f_{235}\left(\frac{-\pi}{4\beta^2(\kappa_2+1)}\right) \tag{B.4}$$

$$\phi\phi_2 = f_{243}\left(\frac{-\pi}{4\beta(\kappa_2+1)}\right)$$

$$\phi\phi_3 = f_{249}\left(\frac{-\pi(\kappa_1-3)}{4\beta(\kappa_1+1)}\right)$$

$$\phi\phi_4 = f_{249}\left(\frac{-\pi}{4\beta}\right)$$

$$\phi\phi_5 = f_{257}\left(\frac{\pi(\kappa_1+5)}{4\beta^2(\kappa_1+1)}\right)$$

$$\phi\phi_6 = f_{257}\left(\frac{-\pi}{2\beta^2}\right)$$

$$\phi\phi_7 = f_{257}\left(\frac{-\pi(\kappa_2+5)}{2\beta(\kappa_2+1)}\right)$$

$$\phi\phi_8 = f_{257}\left(\frac{-\pi}{2\beta^2}\right)$$

$$\phi\phi_9 = f_{265}\left(\frac{\pi\mu_1}{4\beta^2(\kappa_1+1)}\right)$$

$$\phi\phi_{10} = f_{265}\left(\frac{-\pi\mu_2}{2\beta(\kappa_2+1)}\right)$$

$$\phi\phi_{11} = f_{272}\left(\frac{-\pi}{4\beta(\kappa_1+1)}\right)$$

$$s_{11} = -\phi_3 + \phi_4 - \phi_5 + \phi_6 - \phi_9 - \phi_{11} \qquad \text{(B.5)}$$

$$s_{12} = \phi_3 + \phi_4 + 2\phi_6 + \phi_{11}$$

$$s_{21} = \phi_7 t + \phi_{10} t - \phi_8(2 + t\beta)$$

$$s_{22} = (-\phi_1 - \phi_2)\beta(h_2 - t) + \phi_2$$

$$s_{23} = (-\phi_1 - \phi_2)\beta(h_2 + t) + \phi_2$$

$$ss_{11} = -\phi\phi_3 + \phi\phi_4 - \phi\phi_5 + \phi\phi_6 - \phi\phi_9 - \phi\phi_{11} \qquad \text{(B.6)}$$

$$ss_{12} = \phi\phi_3 + \phi\phi_4 + 2\phi\phi_6 + \phi\phi_{11}$$

$$ss_{21} = \phi\phi_7 t + \phi\phi_{10} t - \phi\phi_8(2 + t\beta)$$

$$ss_{22} = \left(-\phi\phi_1 - \phi\phi_2\right)\beta\left(h_2 - t\right) + \phi\phi_2$$

$$ss_{23} = \left(-\phi\phi_1 - \phi\phi_2\right)\beta\left(h_2 + t\right) + \phi\phi_2$$

(B.7)

$$\widetilde{f}_{11}(x,y,t) = \frac{4\mu_1}{(\kappa_1+1)(t-x)} + \int_0^\infty \left\{ s_{11}\beta(h_1 - t)e^{-\beta(h_1 - t)} + s_{12}e^{-\beta(h_1-t)} \right\}\cos(\beta y)d\beta$$

$$\widetilde{f}_{12}(x,y,t) = \int_0^\infty \left\{ 2\phi_8 + s_{21}e^{-t\beta} + s_{22}e^{-\beta(h_2 - t)} - s_{23}e^{-\beta(h_2+t)} \right\}\cos(\beta y)d\beta$$

(B.8)

$$\widetilde{f}_{21}(x,y,t) = \int_0^\infty \left\{ ss_{11}\beta(h_1 - t)e^{-\beta(h_1 - t)} + ss_{12}e^{-\beta(h_1-t)} \right\}\cos(\beta y)d\beta$$

$$\widetilde{f}_{22}(x,y,t) = \frac{\mu_2}{(\kappa_2+1)}\left(\frac{4}{t-x} + \frac{4}{t+x}\right) + 4\int_0^\infty \left\{ 2\phi\phi_8 + ss_{21}e^{-t\beta} + ss_{22}e^{-\beta(h_2 - t)} - ss_{23}e^{-\beta(h_2+t)} \right\}\cos(\beta y)d\beta$$

APPENDIX C

Expressions for quantities appearing in Eqs.(2.18) are given in the form when $y \rightarrow 0$,

$$k_{11}(x,t) = \frac{(\kappa_1 + 1)}{4\mu_1}\tilde{f}_{11}(x,0,t) - \frac{1}{t-x} \tag{C-1}$$

$$k_{12}(x,t) = \frac{(\kappa_2 + 1)}{\mu_1}\tilde{f}_{12}(x,0,t) \tag{C-2}$$

$$k_{21}(x,t) = \frac{(\kappa_1 + 1)}{\mu_2}\tilde{f}_{21}(x,0,t) \tag{C-3}$$

$$k_{22}(x,t) = \frac{(\kappa_2 + 1)}{4\mu_2}\tilde{f}_{22}(x,0,t) - \left(\frac{1}{t-x} + \frac{1}{t-x}\right) \tag{C-4}$$

APPENDIX D

$$f_{273} = (2 + d_1 + d_3) \tag{D.1}$$
$$f_{274} = (2 + d_1 - d_3)$$
$$f_{275} = (-3 + 2d_3)$$
$$f_{276} = (3 + 2d_3)$$
$$f_{277} = (3 - 4\lambda + 2a_8 d_3 + 2a_8 d_1 f_{275})$$
$$f_{278} = (-3 + 4\lambda + 2a_8 d_3 + 2a_8 d_1 f_{276})$$
$$f_{279} = (1 + 2d_1)$$
$$f_{280} = (a_8 f_{275} f_{279} + a_{11}\kappa_1 + \kappa_1^2 + \lambda f_{277}\kappa_2)$$
$$f_{281} = (a_8 f_{276} f_{279} + \lambda a_4 \kappa_1 - \kappa_1^2 + \lambda f_{278}\kappa_2)$$
$$f_{282} = (-2 + a_{11})$$
$$f_{283} = -3a_8 - 4a_{12}b_1^2 + 6d_1 - 6\lambda d_1 + 4\beta^2 d_2 + 2d_3 - 2\lambda d_3$$
$$f_{284} = -4d_2 d_4 + 4a_{12}b_1 f_{273} + f_{283} + a_{11}\kappa_1 + \kappa_1^2$$
$$f_{285} = f_{284} + 3\lambda\kappa_2 - 4\lambda^2\kappa_2 + 6\lambda d_1\kappa_2 - 6\lambda^2 d_1\kappa_2$$
$$f_{286} = f_{285} + 2\lambda d_3\kappa_2 - 2\lambda^2 d_3\kappa_2 + 4d_2 d_4\kappa_2 - 4\lambda d_2 d_4\kappa_2$$
$$f_{287} = -3a_8 - 4a_{12}b_1^2 + 6d_1 - 6\lambda d_1 - 4\beta^2 d_2 - 2d_3 + 2\lambda d_3$$
$$f_{288} = 4d_2 d_4 + 4a_{12}b_1 f_{274} + f_{287} + a_{11}\kappa_1 + \kappa_1^2 + 3\lambda\kappa_2 - 4\lambda^2\kappa_2$$
$$f_{289} = 6\lambda d_1\kappa_2 - 6\lambda^2 d_1\kappa_2 - 2\lambda d_3\kappa_2 + 2\lambda^2 d_3\kappa_2 - 4d_2 d_4\kappa_2 + 4\lambda d_2 d_4\kappa_2$$
$$f_{290} = f_{288} + f_{289}$$
$$f_{291} = -a_7 a_{10} f_{274} - 8a_8 a_9 b_1^2 f_{274} - 2b_1 f_{280} + 8\beta^3 a_8 a_9 h_1^3$$
$$f_{292} = -a_7 a_{10} f_{273} - 8a_8 a_9 b_1^2 f_{273} + 2b_1 f_{281} + 8\beta^3 a_8 a_9 h_1^3$$
$$f_{293} = a_7^2 a_9 f_{274} + 8a_8^2 a_9 b_1^2 f_{274} + 2a_8 b_1 f_{280} - 8\beta^3 a_8^2 a_9 h_1^3$$
$$f_{294} = -a_7^2 a_9 f_{273} - 8a_8^2 a_9 b_1^2 f_{273} + 2a_8 b_1 f_{281} + 8\beta^3 a_8^2 a_9 h_1^3$$
$$f_{295} = \lambda^2 + 4a_7^2 b_2^2 + 6d_1$$
$$f_{296} = \beta + \beta\lambda\kappa_2$$
$$f_{297} = (\lambda + a_4 + \lambda\kappa_2^2)$$
$$f_{298} = 4a_7 a_8 b_2^2 + \lambda f_{297} + f_{282}\kappa_1$$
$$f_{299} = -3 + 3\lambda + 4a_{12}b_1^2 - 6d_1 + 6\lambda d_1 - 4\beta^2 d_2 - 2d_3 + 2\lambda d_3$$
$$f_{300} = 4d_2 d_4 - 4a_{12}b_1 f_{273} + f_{299} + \lambda a_4\kappa_1 - \kappa_1^2 - 3\lambda\kappa_2$$
$$f_{301} = 4\lambda^2\kappa_2 - 6\lambda d_1\kappa_2 + 6\lambda^2 d_1\kappa_2 - 2\lambda d_3\kappa_2 + 2\lambda^2 d_3\kappa_2$$
$$f_{302} = f_{300} + f_{301} - 4d_2 d_4\kappa_2 + 4\lambda d_2 d_4\kappa_2$$

$$f_{303} = 3\lambda^2\kappa_2^2 + 6\lambda^2 d_1\kappa_2^2 - 2\lambda^2 d_3\kappa_2^2 - 4\lambda d_2 d_4\kappa_2^2$$
$$f_{304} = 4 f_{296}^2 h_1^2 + 2\lambda\kappa_1 + \kappa_1^2 + 6\lambda\kappa_2 + 12\lambda d_1\kappa_2 - 4\lambda d_3\kappa_2 - 8d_2 d_4\kappa_2$$
$$f_{305} = f_{295} + f_{303} + f_{304}$$
$$f_{306} = 3 - 4\beta^2 d_2 - 2d_3 - 4a_7 a_9 b_2 f_{274} - 4a_9 b_1(-2a_7 b_2 + a_9 f_{274}) + f_{305}$$
$$f_{307} = 8d_2 d_4\kappa_2 + 3\lambda^2\kappa_2^2 + 6\lambda^2 d_1\kappa_2^2 + 2\lambda^2 d_3\kappa_2^2 + 4\lambda d_2 d_4\kappa_2^2$$
$$f_{308} = f_{307} + 4 f_{296}^2 h_1^2 + 2\lambda\kappa_1 + \kappa_1^2 + 6\lambda\kappa_2 + 12\lambda d_1\kappa_2 + 4\lambda d_3\kappa_2$$
$$f_{309} = 4\beta^2 d_2 - 4a_7 a_9 b_2 f_{273} - 4a_9 b_1(-2a_7 b_2 + a_9 f_{273}) + f_{276} + f_{295} + f_{308}$$
$$f_{310} = (-3 + 2\lambda - a_8\kappa_2 + \lambda\kappa_2^2)$$
$$f_{311} = (2 - 2\lambda + \lambda^2 + 2\lambda\kappa_2 + \lambda^2\kappa_2^2)$$
$$f_{312} = (\lambda^2 f_{297} + \lambda f_{310}\kappa_1 + f_{282}\kappa_1^2)$$
$$f_{313} = (4a_8^2 b_2^2 + f_{311})$$
$$f_{314} = a_7^2 a_{12} b_2^2$$
$$f_{315} = (-\lambda a_1 f_{49} + 4a_8^2 b_2^2 f_{274} + f_{274} + f_{311})$$
$$f_{316} = (4\lambda a_1 a_{12} f_{49} f_{274} + 4a_8^2 b_2^2 f_{280} + f_{280} + f_{311})$$
$$f_{317} = 8a_{12} b_1^2 f_{315} + 2b_1 f_{316} - 8\beta^3 a_{12} f_{313} h_1^3$$
$$f_{318} = 2\lambda a_1 f_{49} f_{280} + a_9 f_{274} f_{312} + 4 f_{274} f_{314} + f_{317}$$
$$f_{319} = (3 - 4\lambda - 2a_8 d_3 - 2a_8 d_1 f_{276})$$
$$f_{320} = (-\lambda a_1 f_{49} + 4a_8^2 b_2^2 f_{273} + f_{273} + f_{311})$$
$$f_{321} = (-4\lambda a_1 a_{12} f_{49} f_{273} + 4a_8^2 b_2^2 f_{281} + f_{281} + f_{311})$$
$$f_{322} = -a_9 f_{273} f_{312} - 4 f_{273} f_{314} - 8a_{12} b_1^2 f_{320} + 2b_1 f_{321}$$
$$f_{323} = (a_8(-1 - 2d_1) f_{276} + a_{11}\kappa_1 + \kappa_1^2 + \lambda f_{319}\kappa_2)$$
$$f_{324} = (f_{322} - 2\lambda a_1 f_{49} f_{323} + 8\beta^3 a_{12} f_{313} h_1^3)$$
$$f_{325} = 2\lambda a_1 a_3 f_{286}$$
$$f_{326} = 2a_8 a_{10} f_{286}$$
$$f_{327} = 2\lambda a_1 a_3 f_{290}$$
$$f_{328} = 2a_7 a_9 f_{290}$$
$$f_{329} = 2a_8 a_{10} f_{290}$$
$$f_{330} = 4a_{12} f_{291}$$
$$f_{331} = 4a_{12} f_{292}$$
$$f_{332} = -4a_9 f_{293}$$
$$f_{333} = 4a_9 f_{294}$$

$$f_{334} = 2f_{290}f_{298}$$
$$f_{335} = -2a_7a_9f_{302}$$
$$f_{336} = -2f_{298}f_{302}$$
$$f_{337} = -2\lambda a_1a_3f_{306}$$
$$f_{338} = -2\lambda a_1a_3f_{309}$$
$$f_{339} = -4f_{318}$$
$$f_{340} = 4f_{324}$$
$$f_{341} = a_7^2a_9^2 + 4a_7a_8a_9^2b_1e_6 - a_7a_{10}a_{12}e_{11} + a_7a_{10}a_{12}e_{12}$$
$$f_{342} = 4a_7a_8a_9^2b_1e_6e_{12} - a_7^2a_9^2e_{11}e_{12} + a_7a_9e_8f_{298} - a_7a_9e_8e_{11}f_{298}$$
$$f_{343} = f_{341} + f_{342} + 4\beta a_7a_9f_{90}(f_{313}h_1 + \lambda a_1a_3h_2)$$
$$f_{344} = -\lambda + f_{275} + 2f_{296}h_1 - \kappa_1 - 3\lambda\kappa_2 + 2\lambda d_3\kappa_2$$
$$f_{345} = -\lambda + 2\lambda d_1 + f_{275} + 2f_{296}h_1 + (-1+2d_1)\kappa_1 - 3\lambda\kappa_2 + 2\lambda d_3\kappa_2$$
$$f_{346} = a_7 + f_{276} - 2f_{296}h_1 + 3\lambda\kappa_2 + 2\lambda d_3\kappa_2$$
$$f_{347} = \lambda - 2\lambda d_1 + f_{276} - 2f_{296}h_1 + (1-2d_1)\kappa_1 + 3\lambda\kappa_2 + 2\lambda d_3\kappa_2$$
$$f_{348} = -1 + 2d_1$$
$$f_{349} = -2d_3 + 2\lambda d_3 + 4d_2d_4$$
$$f_{350} = -3 + 3\lambda + a_{10} - 6d_1 + 6\lambda d_1 - 4\beta^2d_2 - 2a_8b_1f_{279} + f_{349}$$
$$f_{351} = -3a_8 - a_{10} + 6d_1 - 6\lambda d_1 - 4\beta^2d_2 + 2a_8b_1f_{279} + f_{349}$$
$$f_{352} = 1 + 4b_2^2 - 2b_2f_{279}$$
$$f_{353} = a_{10} + a_8f_{276}f_{279}$$
$$f_{354} = -3a_8 - a_{10} + 2d_3 - 2\lambda d_3 - 4a_8b_2^2f_{276} - 2a_8b_1f_{352} + 2b_2f_{353} + 2d_1\kappa_1 - 2\lambda d_1\kappa_2$$
$$f_{355} = 3\lambda\kappa_2 - 2\lambda d_3\kappa_2$$
$$f_{356} = -2f_{296}h_1 - f_{279}\kappa_1 - 3\lambda\kappa_2$$
$$f_{357} = \lambda + 2\lambda d_1 + f_{276} - f_{356} + 2\lambda d_3\kappa_2$$
$$f_{358} = 3 + a_7 - 2d_3 + f_{355} - 2f_{296}h_1$$
$$f_{359} = -\lambda - 2\lambda d_1 + f_{275} + f_{356} + 2\lambda d_3\kappa_2$$
$$f_{360} = -1 + 2a_7b_2 - \lambda\kappa_2$$
$$f_{361} = 3 + \lambda + 2\lambda d_1$$
$$f_{362} = -2 + d_3$$
$$f_{363} = 2 + d_3$$
$$f_{364} = 3 - 2d_3$$
$$f_{365} = 3 + 2d_3$$

$$f_{366} = \lambda^2 + 4a_7^2 b_2^2$$
$$f_{367} = 2\lambda\kappa_1 + \kappa_1^2 + 6\lambda\kappa_2$$
$$f_{368} = -3a_8 + 2a_{12}b_1 - 2d_3 + 2\lambda d_3 + f_{355} + a_{11}\kappa_1 + \kappa_1^2 - 4\lambda^2\kappa_2 + 2\lambda^2 d_3\kappa_2$$
$$f_{369} = -3 + 3\lambda - 2a_{12}b_1 - 2d_3 + 2\lambda d_3 + \lambda a_4\kappa_1 - \kappa_1^2 - 3\lambda\kappa_2 + 4\lambda^2\kappa_2 - 2\lambda d_3\kappa_2 + 2\lambda^2 d_3\kappa_2$$
$$f_{370} = -2b_2 + f_{279}$$
$$f_{371} = 2a_9 b_1 f_{360}$$
$$f_{372} = 4a_7 a_9 b_2 f_{362} + f_{364} + f_{366} + f_{367} + f_{371} - 4\lambda d_3\kappa_2 + 3\lambda^2\kappa_2^2 - 2\lambda^2 d_3\kappa_2^2$$
$$f_{373} = f_{276} - 4a_7 a_9 b_2 f_{363} + f_{366} + f_{367} + f_{371} + 4\lambda d_3\kappa_2 + 3\lambda^2\kappa_2^2 + 2\lambda^2 d_3\kappa_2^2$$
$$f_{374} = a_8 f_{275} f_{279} + \kappa_1 - \lambda\kappa_2$$
$$f_{375} = -3a_8 - a_{10} - 2d_3 + 2\lambda d_3 + 4a_8 b_2^2 f_{275} - 2a_8 b_1 f_{352} - 2b_2 f_{374} + 2d_1\kappa_1 - 2\lambda d_1\kappa_2$$
$$f_{376} = -6 - \lambda + 4d_3 - \kappa_1 + 2\lambda f_{275}\kappa_2$$
$$f_{377} = 8a_{12}b_1^2 - a_7(-a_{10} + a_8 f_{275}) + 2a_8 b_1 f_{376}$$
$$f_{378} = -6 - \lambda + 2\lambda d_1 + 4d_3 + f_{348}\kappa_1 + 2\lambda f_{275}\kappa_2$$
$$f_{379} = 8a_{12}b_1^2 + a_7(a_{10} + a_8 f_{275} f_{348}) + 2a_8 b_1 f_{378}$$
$$f_{380} = 6 + a_7 + 4d_3 + 2\lambda f_{276}\kappa_2$$
$$f_{381} = -8a_{12}b_1^2 - a_7(a_{10} + a_8 f_{276}) + 2a_8 b_1 f_{380}$$
$$f_{382} = 1 - 2d_1$$
$$f_{383} = 6 + \lambda - 2\lambda d_1 + 4d_3 + f_{382}\kappa_1 + 2\lambda f_{276}\kappa_2$$
$$f_{384} = -8a_{12}b_1^2 + a_7(-a_{10} + a_8 f_{276} f_{348}) + 2a_8 b_1 f_{383}$$
$$f_{385} = -2d_3 + f_{361} + f_{279}\kappa_1 + \lambda f_{364}\kappa_2$$
$$f_{386} = 2b_2 f_{385} - \kappa_1 - 3\lambda\kappa_2 - 6\lambda d_1\kappa_2 + 2\lambda d_3\kappa_2 + 4d_2 d_4\kappa_2$$
$$f_{387} = -\lambda - 4a_7 b_2^2 - 6d_1 + 4\beta^2 d_2 + f_{275} + 2a_9 b_1 f_{370} + f_{386}$$
$$f_{388} = 2d_3 + f_{361} + f_{279}\kappa_1 + \lambda f_{365}\kappa_2$$
$$f_{389} = -3 + 2d_3$$
$$f_{390} = 2b_1 + f_{275}$$
$$f_{391} = -2b_1 + f_{276}$$
$$f_{392} = -1 + 2\lambda + \lambda\kappa_2$$
$$f_{393} = 3\lambda\kappa_2 + 6\lambda d_1\kappa_2 + 2\lambda d_3\kappa_2 + 4d_2 d_4\kappa_2$$
$$f_{394} = a_7 + 4a_{13}b_2^2 + 6d_1 + 4\beta^2 d_2 + f_{276} - 2a_9 b_1 f_{370} - 2b_2 f_{388} + f_{393}$$
$$f_{395} = a_7 + f_{276} + \lambda f_{365}\kappa_2$$
$$f_{396} = -6 + 5\lambda + 2\lambda^2 - 4d_3 + 4\lambda d_3 + f_{392}\kappa_1 + \lambda(-6 + 7\lambda + 4a_8 d_3)\kappa_2$$

$$f_{397} = -8a_{12}b_1^2 + a_7 f_{395} + 2b_1 f_{396}$$
$$f_{398} = -\lambda + f_{275} - \kappa_1 + f_{389}\kappa_2$$
$$f_{399} = 6 - 5\lambda - 2\lambda^2 - 4d_3 + 4\lambda d_3 - f_{392}\kappa_1 + \lambda(6 - 7\lambda + 4a_8 d_3)\kappa_2$$
$$f_{400} = 8a_{12}b_1^2 + a_7 f_{398} + 2b_1 f_{399}$$
$$f_{401} = \lambda a_8 f_{348} + f_{390}\kappa_2 + 2\lambda f_{390}\kappa_2^2 + \lambda^2 f_{390}\kappa_2^3$$
$$f_{402} = -2\lambda + 2\lambda^2 + 2a_9^2 b_1 + 4\lambda d_1 - 4\lambda^2 d_1 + f_{275} + 2\lambda f_{389}\kappa_2 + \lambda^2 f_{389}\kappa_2^2$$
$$f_{403} = 3\lambda + a_{10} + 6d_1 - 6\lambda d_1 - 4\beta^2 d_2 - 2\lambda d_3 + 4d_2 d_4 + f_{275} + 2a_8 b_1 f_{348}$$
$$f_{404} = 4a_7^2 a_8 b_2^2 f_{348} + \lambda f_{401} + 2a_7 a_9 b_2 f_{403} - f_{402}\kappa_1 + a_8 f_{348}\kappa_1^2$$
$$f_{405} = -3a_8 - a_{10} - 6d_1 + 6\lambda d_1 - 4\beta^2 d_2 + 2d_3 - 2\lambda d_3 + 4d_2 d_4 - 2a_8 b_1 f_{348}$$
$$f_{406} = -\lambda a_8 f_{348} + f_{391}\kappa_2 + 2\lambda f_{391}\kappa_2^2 + \lambda^2 f_{391}\kappa_2^3$$
$$f_{407} = 2\lambda - 2\lambda^2 - 2a_9^2 b_1 - 4\lambda d_1 + 4\lambda^2 d_1 + f_{276} + 2\lambda f_{365}\kappa_2 + \lambda^2 f_{365}\kappa_2^2$$
$$f_{408} = -4a_7^2 a_8 b_2^2 f_{348} + 2a_7 a_9 b_2 f_{405} + \lambda f_{406} - f_{407}\kappa_1 - a_8 f_{348}\kappa_1^2$$
$$f_{409} = -1 + d_3$$
$$f_{410} = \lambda - \lambda d_3$$
$$f_{411} = \lambda + \lambda d_3$$
$$f_{412} = a_{15} - d_3 + f_{410}\kappa_2$$
$$f_{413} = 3 - 2\lambda + a_{14}d_3$$
$$f_{414} = 3 - 2\lambda + 2a_8 d_3 + \kappa_1$$
$$f_{415} = f_{275} + f_{410} - \kappa_1 + \lambda f_{409}\kappa_2$$
$$f_{416} = a_{15} + 3\kappa_1$$
$$f_{417} = -\lambda + \lambda d_3 + f_{364} + f_{412}\kappa_1 + \kappa_1^2 + \lambda f_{413}\kappa_2$$
$$f_{418} = 4a_9 b_1^2 f_{313} + 2a_7 a_9 b_2^2 f_{414} - a_7 a_9 f_{415} - 2\lambda f_{49} f_{417}$$
$$f_{419} = -12 + 10\lambda - 3\lambda^2 - 2\lambda^3 + 8d_3 - 8\lambda d_3 + 4\lambda^2 d_3$$
$$f_{420} = -2\kappa_1 + 3\lambda\kappa_1 - 2\lambda^2\kappa_1 - 24\lambda\kappa_2 + 11\lambda^2\kappa_2 - 7\lambda^3\kappa_2$$
$$f_{421} = 16\lambda d_3\kappa_2 - 8\lambda^2 d_3\kappa_2 + 4\lambda^3 d_3\kappa_2 - \lambda\kappa_1\kappa_2 - \lambda^2\kappa_1\kappa_2 - 18\lambda^2\kappa_2^2 - \lambda^3\kappa_2^2$$
$$f_{422} = 12\lambda^2 d_3\kappa_2^2 - \lambda^2\kappa_1\kappa_2^2 - 6\lambda^3\kappa_2^3 + 4\lambda^3 d_3\kappa_2^3$$
$$f_{423} = f_{418} + b_1(4a_8 b_2^2 f_{399} + 2\lambda a_9 f_{49} f_{416} + f_{419} + f_{420} + f_{421} + f_{422})$$
$$f_{424} = -3 + 2\lambda + 2a_8 d_3 - \kappa_1$$
$$f_{425} = -\lambda - \lambda d_3 + f_{276} + \kappa_1 + f_{411}\kappa_2$$
$$f_{426} = a_{15} + d_3 + f_{411}\kappa_2$$
$$f_{427} = -3 + 2\lambda + a_{14}d_3$$

$$f_{428} = -3 - 2d_3 + f_{411} - f_{426}\kappa_1 - \kappa_1^2 + \lambda f_{427}\kappa_2$$
$$f_{429} = -4a_9 b_1^2 f_{313} + 2a_7 a_9 b_2^2 f_{424} - a_7 a_9 f_{425} - 2\lambda f_{49} f_{428}$$
$$f_{430} = 12 - 10\lambda + 3\lambda^2 + 2\lambda^3 + 8d_3 - 8\lambda d_3 + 4\lambda^2 d_3 + 4a_8 b_2^2 f_{396} - 2\lambda a_9 f_{49} f_{416}$$
$$f_{431} = 2\kappa_1 - 3\lambda\kappa_1 + 2\lambda^2\kappa_1 + 24\lambda\kappa_2 - 11\lambda^2\kappa_2 + 7\lambda^3\kappa_2 + 16\lambda d_3\kappa_2 - 8\lambda^2 d_3\kappa_2$$
$$f_{432} = 4\lambda^3 d_3\kappa_2 + \lambda\kappa_1\kappa_2 + \lambda^2\kappa_1\kappa_2 + 18\lambda^2\kappa_2^2 + \lambda^3\kappa_2^2$$
$$f_{433} = 12\lambda^2 d_3\kappa_2^2 + \lambda^2\kappa_1\kappa_2^2 + 6\lambda^3\kappa_2^3 + 4\lambda^3 d_3\kappa_2^2$$
$$f_{434} = f_{429} + b_1(f_{430} + f_{431} + f_{432} + f_{433})$$
$$f_{435} = -3 - \lambda + 6d_1 + 4\beta^2 d_2 - 2d_3 - \kappa_1 + \lambda f_{276} f_{348}\kappa_2$$
$$f_{436} = \lambda - 2\lambda d_1 + f_{276} + f_{382}\kappa_1 + \lambda f_{365}\kappa_2$$
$$f_{437} = -1 + a_{14}\kappa_2$$
$$f_{438} = \lambda f_{437} - f_{392}\kappa_1$$
$$f_{439} = 3 + a_{15}\kappa_2$$
$$f_{440} = 1 + 2\lambda + 3\lambda\kappa_2$$
$$f_{441} = 3 - 4\lambda + 2\lambda^2 + 2\lambda\kappa_2 + \lambda^2\kappa_2^2$$
$$f_{442} = \lambda f_{439} + f_{440}\kappa_1$$
$$f_{443} = 12 - 9\lambda + 2\lambda^2 + 2\lambda^3 - 6\lambda d_1 + 8\lambda^2 d_1 - 4\lambda^3 d_1 + 8d_3 - 8\lambda d_3$$
$$f_{444} = 4\lambda^2 d_3 + 8a_8^2 b_2^2 f_{436} - 2a_9 b_2 f_{442} + f_{443} - f_{348} f_{441}\kappa_1$$
$$f_{445} = 24\lambda\kappa_2 - 10\lambda^2\kappa_2 + 6\lambda^3\kappa_2 - 4\lambda^2 d_1\kappa_2$$
$$f_{446} = 16\lambda d_3\kappa_2 - 8\lambda^2 d_3\kappa_2 + 4\lambda^3 d_3\kappa_2 + 18\lambda^2\kappa_2^2 + \lambda^3\kappa_2^2 - 2\lambda^3 d_1\kappa_2^2$$
$$f_{447} = f_{444} + f_{445} + f_{446} + 12\lambda^2 d_3\kappa_2^2 + 6\lambda^3\kappa_2^3 + 4\lambda^3 d_3\kappa_2^3$$
$$f_{448} = 4a_7^2 a_9 b_2^2 + 8a_9 b_1^2 f_{313} - a_7 a_9 f_{435} + 2b_2 f_{436} f_{438} - 2b_1 f_{447}$$
$$f_{449} = -\lambda + 2\lambda d_1 + f_{275} + f_{348}\kappa_1 + \lambda f_{389}\kappa_2$$
$$f_{450} = a_7 - 6d_1 + 4\beta^2 d_2 + f_{364} + \lambda f_{275} f_{348}\kappa_2$$
$$f_{451} = 4a_7^2 a_9 b_2^2 + 8a_9 b_1^2 f_{313} - 2b_2 f_{438} f_{449} + a_7 a_9 f_{450}$$
$$f_{452} = -12 + 9\lambda - 2\lambda^2 - 2\lambda^3 + 6\lambda d_1 - 8\lambda^2 d_1 + 4\lambda^3 d_1$$
$$f_{453} = f_{348} f_{441}\kappa_1 - 24\lambda\kappa_2 + 10\lambda^2\kappa_2 - 6\lambda^3\kappa_2 + 4\lambda^2 d_1\kappa_2 + 16\lambda d_3\kappa_2 - 8\lambda^2 d_3\kappa_2$$
$$f_{454} = 4\lambda^3 d_3\kappa_2 - 18\lambda^2\kappa_2^2 - \lambda^3\kappa_2^2 + 2\lambda^3 d_1\kappa_2^2 + 12\lambda^2 d_3\kappa_2^2 - 6\lambda^3\kappa_2^3 + 4\lambda^3 d_3\kappa_2^3$$
$$f_{455} = f_{451} + 2b_1(8d_3 - 8\lambda d_3 + 4\lambda^2 d_3 + 2a_9 b_2 f_{442} + 8a_8^2 b_2^2 f_{449} + f_{452} + f_{453} + f_{454})$$
$$f_{456} = 2a_3 a_7 a_9 f_{344}$$
$$f_{457} = 2a_3 f_{298} f_{344}$$
$$f_{458} = 2a_3 a_7 a_9 f_{345}$$

$$f_{459} = -2a_3 a_7 a_9 f_{345}$$
$$f_{460} = 8a_3 a_{12} b_1 f_{345}$$
$$f_{461} = -2a_{10} f_{220} f_{345}$$
$$f_{462} = 2a_{10} f_{220} f_{345}$$
$$f_{463} = -2a_3 f_{298} f_{345}$$
$$f_{464} = -2a_3 a_7 a_9 f_{346}$$
$$f_{465} = -2a_{10} f_{220} f_{346}$$
$$f_{466} = -2a_3 f_{298} f_{346}$$
$$f_{467} = 2a_3 a_7 a_9 f_{347}$$
$$f_{468} = -8a_3 a_{12} b_1 f_{347}$$
$$f_{469} = 2a_{10} f_{220} f_{347}$$
$$f_{470} = -2a_{10} f_{220} f_{347}$$
$$f_{471} = 2a_3 f_{298} f_{347}$$
$$f_{472} = -2a_3 a_7 a_9 f_{350}$$
$$f_{473} = 2a_3 a_7 a_9 f_{351}$$
$$f_{474} = 2a_3 a_7 a_9 f_{354}$$
$$f_{475} = -2a_3 a_7 a_9 f_{357}$$
$$f_{476} = -2a_{10} f_{220} f_{358}$$
$$f_{477} = 2a_3 a_7 a_9 f_{359}$$
$$f_{478} = -2\lambda a_3^2 f_{368}$$
$$f_{479} = 2\lambda a_3^2 f_{369}$$
$$f_{480} = 2\lambda a_3^2 f_{372}$$
$$f_{481} = 2\lambda a_3^2 f_{373}$$
$$f_{482} = 2a_3 a_7 a_9 f_{375}$$
$$f_{483} = -2a_3 a_9 f_{377}$$
$$f_{484} = 2a_3 a_9 f_{379}$$
$$f_{485} = 2a_3 a_9 f_{381}$$
$$f_{486} = -2a_3 a_9 f_{384}$$
$$f_{487} = 2a_3 a_7 a_9 f_{387}$$
$$f_{488} = -2a_3 a_7 a_9 f_{394}$$
$$f_{489} = 2a_3 a_9 f_{397}$$
$$f_{490} = -2a_3 a_9 f_{400}$$

$$f_{491} = 2a_3 f_{404}$$
$$f_{492} = -2a_3 f_{408}$$
$$f_{493} = -4a_3 f_{423}$$
$$f_{494} = 4a_3 f_{434}$$
$$f_{495} = 2a_3 f_{448}$$
$$f_{496} = 2a_3 f_{455}$$
$$f_{497} = -3 + 3\lambda + a_{10} - 6d_1 + 6\lambda d_1 - 4\beta^2 d_2 - 2a_8 b_1 f_{276} + f_{349}$$
$$f_{498} = 3\lambda + a_9 + 2d_1 + 2\lambda d_3 + 2 f_{296} h_1 + f_{276}\kappa_1 + 2\lambda d_1 \kappa_2$$
$$f_{499} = 3\lambda + a_9 + 2d_1 - 2\lambda d_3 - 2 f_{296} h_1 + f_{364}\kappa_1 + 2\lambda d_1 \kappa_2$$
$$f_{500} = 1 + 4b_2^2 - 2b_2 f_{276}$$
$$f_{501} = a_{10} + a_8 f_{276} f_{279}$$
$$f_{502} = \lambda + 2\lambda d_1$$
$$f_{503} = -2b_2 + f_{365}$$
$$f_{504} = a_8 - 2d_1 + 2\lambda d_1 + 4a_8 b_2^2 f_{279}$$
$$f_{505} = 2a_8 b_1 f_{500} - 2b_2 f_{501} + f_{504} - 3\kappa_1 - 2d_3\kappa_1 + 3\lambda\kappa_2 + 2\lambda d_3 \kappa_2$$
$$f_{506} = 3\lambda + 2\lambda d_3 + f_{279} + f_{365}\kappa_1 + f_{502}\kappa_2$$
$$f_{507} = a_7 + 4a_{13} b_2^2 + 6d_1 + 4\beta^2 d_2 + f_{276} + f_{393} - 2a_9 b_1 f_{503} - 2b_2 f_{506}$$
$$f_{508} = -a_{10} + a_8 f_{275} f_{279}$$
$$f_{509} = 2 + 3\lambda + 4d_1 - 2\lambda d_3 + f_{364}\kappa_1 + 2 f_{502}\kappa_2$$
$$f_{510} = -8a_{12} b_1^2 + a_7 f_{508} + 2a_8 b_1 f_{509}$$
$$f_{511} = 1 + 2d_1$$
$$f_{512} = -3a_8 - a_{10} + 6d_1 - 6\lambda d_1 - 4\beta^2 d_2 - 2a_8 b_1 f_{275} + f_{349}$$
$$f_{513} = -2b_1 + f_{279}$$
$$f_{514} = -1 + 2b_1 - 2d_1$$
$$f_{515} = \lambda a_8 f_{275} + f_{514}\kappa_2 - 2\lambda f_{513}\kappa_2^2 - \lambda^2 f_{513}\kappa_2^3$$
$$f_{516} = 4a_7^2 a_8 b_2^2 f_{275} - 2a_7 a_9 b_2 f_{512} + \lambda f_{515} + a_8 f_{275}\kappa_1^2$$
$$f_{517} = 6\lambda - 6\lambda^2 - 2a_9^2 b_1 - 4\lambda d_3 + 4\lambda^2 d_3 + f_{279} + 2 f_{502}\kappa_2 + \lambda^2 f_{511}\kappa_2^2$$
$$f_{518} = f_{516} + f_{517}\kappa_1$$
$$f_{519} = -\lambda - 6d_1 + 4\beta^2 d_2 + f_{275} - \kappa_1 + \lambda f_{279} f_{389}\kappa_2$$
$$f_{520} = 3\lambda - 2\lambda d_3 + f_{279} + f_{364}\kappa_1 + f_{502}\kappa_2$$
$$f_{521} = 4a_7^2 a_9 b_2^2 + 8a_9 b_1^2 f_{313} - a_7 a_9 f_{519}$$

$$f_{522} = 4 + 5\lambda - 10\lambda^2 + 6\lambda^3 + 8d_1 - 8\lambda d_1 + 4\lambda^2 d_1 - 6\lambda d_3$$
$$f_{523} = 8\lambda^2 d_3 - 4\lambda^3 d_3 - 2a_9 b_2 f_{442} + 8a_8^2 b_2^2 f_{520} + f_{522} - f_{275} f_{441}\kappa_1$$
$$f_{524} = 8\lambda\kappa_2 + 2\lambda^2\kappa_2 + 2\lambda^3\kappa_2 + 16\lambda d_1\kappa_2 - 8\lambda^2 d_1\kappa_2$$
$$f_{525} = f_{523} + f_{524} + 4\lambda^3 d_1\kappa_2 - 4\lambda^2 d_3\kappa_2 + 6\lambda^2\kappa_2^2 + 3\lambda^3\kappa_2^2$$
$$f_{526} = f_{525} + 12\lambda^2 d_1\kappa_2^2 - 2\lambda^3 d_3\kappa_2^2 + 2\lambda^3\kappa_2^3 + 4\lambda^3 d_1\kappa_2^3$$
$$f_{527} = 2b_2 f_{438} f_{520} + f_{521} - 2b_1 f_{526}$$
$$f_{528} = -2\lambda a_1 a_7 a_9 f_{497}$$
$$f_{529} = -2\lambda a_1 a_7 a_9 f_{498}$$
$$f_{530} = 2\lambda a_1 a_8 a_{10} f_{499}$$
$$f_{531} = 8\lambda a_1 a_{12} b_1 f_{499}$$
$$f_{532} = -2\lambda a_1 a_7 a_9 f_{499}$$
$$f_{533} = -2\lambda a_1 a_8 a_{10} f_{499}$$
$$f_{534} = -2\lambda a_1 f_{298} f_{499}$$
$$f_{535} = -2\lambda a_1 a_7 a_9 f_{505}$$
$$f_{536} = -2\lambda a_1 a_7 a_9 f_{507}$$
$$f_{537} = 2\lambda a_1 a_9 f_{510}$$
$$f_{538} = -2\lambda a_1 f_{518}$$
$$f_{539} = -2\lambda a_1 f_{527}$$
$$f_{540} = -2 + d_1 - d_3$$
$$f_{541} = -a_{10} + a_8 f_{275}$$
$$f_{542} = a_{10} + a_8 f_{276}$$
$$f_{543} = 2b_2^2 - 2b_2 f_{273} + d_1 f_{276} + f_{363}$$
$$f_{544} = d_1 + f_{362}$$
$$f_{545} = a_7 f_{544} + 2 f_{296} h_1$$
$$f_{546} = 3\lambda + a_9$$
$$f_{547} = a_8 + 4a_8 b_2^2 + f_{355} - 2b_2 f_{541} + f_{275}\kappa_1$$
$$f_{548} = 2\lambda d_3 + f_{546} + f_{276}\kappa_1$$
$$f_{549} = -4 + 3\lambda + 2d_3 - 2\lambda d_3 + 2a_8 d_1 f_{275} - \lambda\kappa_2$$
$$f_{550} = -2\lambda d_3 + f_{546} + f_{364}\kappa_1$$
$$f_{551} = -1 - 3\lambda + 2\lambda d_3 + f_{389}\kappa_1 - \lambda\kappa_2$$
$$f_{552} = 1 - \lambda - 4a_8 b_2^2 + 2b_2 f_{542} + f_{276}\kappa_1 - 3\lambda\kappa_2 - 2\lambda d_3\kappa_2$$
$$f_{553} = -\lambda - 4a_7 b_2^2 + f_{275} + 2b_2 f_{551} - \kappa_1 - 3\lambda\kappa_2 + 2\lambda d_3\kappa_2$$

$$f_{554} = a_8 f_{275} f_{348} + \kappa_2 + \lambda\kappa_2^2$$
$$f_{555} = \lambda f_{554} + f_{549}\kappa_1$$
$$f_{556} = a_1 f_{362} + 2b_1(-2 + 3\lambda - 2a_8 d_3 + \lambda\kappa_2)$$
$$f_{557} = a_1 f_{363} + 2b_1(-2 + 3\lambda + 2a_8 d_3 + \lambda\kappa_2)$$
$$f_{558} = 2\lambda d_3 + f_{546} + f_{365}\kappa_1$$
$$f_{559} = a_7 + 4a_{13}b_2^2 + f_{276} - 2b_2 f_{558} + 3\lambda\kappa_2 + 2\lambda d_3\kappa_2$$
$$f_{560} = 4 - 6\lambda + 3\lambda^2 - 2d_3 + 4\lambda d_3 - 2\lambda^2 d_3 + 2a_8^2 d_1 f_{275} + 2\lambda\kappa_2 + \lambda^2\kappa_2^2$$
$$f_{561} = a_8^2 f_{275} f_{348} - \kappa_2 - 2\lambda\kappa_2^2 - \lambda^2\kappa_2^3$$
$$f_{562} = a_8^2 f_{275} f_{348} + \kappa_2^2 + 2\lambda\kappa_2^3 + \lambda^2\kappa_2^4$$
$$f_{563} = 4a_7^2 a_8^2 b_2^2 f_{275} f_{348} + 4a_7 a_{10} a_{12} b_2 f_{544}$$
$$f_{564} = \lambda^2 f_{562} + f_{563} + 2\lambda f_{561}\kappa_1 + f_{560}\kappa_1^2$$
$$f_{565} = -2b_2^2 + f_{362} + 2b_2 f_{362}$$
$$f_{566} = 2 - 3\lambda + 2a_8 d_3 - \lambda\kappa_2$$
$$f_{567} = -2 - \lambda - 4a_8 b_2^2$$
$$f_{568} = 2d_3 + 2b_2 f_{566} + f_{567} + \lambda f_{275}\kappa_2$$
$$f_{569} = a_1 f_{565} + 2b_1 f_{568}$$
$$f_{570} = 2b_2(-2 + 3\lambda + 2a_8 d_3 + \lambda\kappa_2)$$
$$f_{571} = -2d_3 + f_{567} + f_{570} - 3\lambda\kappa_2 - 2\lambda d_3\kappa_2$$
$$f_{572} = -a_1(-2 - 2b_2^2 - d_3 + 2b_2 f_{363}) + 2b_1 f_{571}$$
$$f_{573} = 4 - 3\lambda + 2d_1 - 2d_3 + 2\lambda d_3 + f_{502}\kappa_2$$
$$f_{574} = a_8 f_{275} - f_{279}\kappa_2 - f_{502}\kappa_2^2$$
$$f_{575} = a_8 f_{275} f_{279} - \kappa_2 - \lambda\kappa_2^2$$
$$f_{576} = 4 - 3\lambda - 2d_3 + 2\lambda d_3 + 2a_8 d_1 f_{275} + \lambda\kappa_2$$
$$f_{577} = \lambda f_{575} + f_{576}\kappa_1$$
$$f_{578} = 4a_7 a_8 b_2^2 f_{275} + \lambda f_{574} - 2b_2 f_{577} + f_{573}\kappa_1$$
$$f_{579} = 4 - \lambda + 2a_8 d_1 + 2d_3 + \lambda f_{276}\kappa_2$$
$$f_{580} = 4a_7 a_8 b_2^2 f_{348} + f_{579}\kappa_1 + \lambda(a_8 f_{348} - f_{276}\kappa_2 - \lambda f_{276}\kappa_2^2)$$
$$f_{581} = 4 - 3\lambda + 2d_3 - 2\lambda d_3 + 2a_8 d_1 f_{276} + \lambda\kappa_2$$
$$f_{582} = a_8 f_{276} f_{348} - \kappa_2 - \lambda\kappa_2^2$$
$$f_{583} = f_{580} - 2b_2(\lambda f_{582} + f_{581}\kappa_1)$$
$$f_{584} = \lambda f_{297} + f_{282}\kappa_1$$

$$f_{585} = 4 - \lambda + 2d_3 + 2d_1 f_{276} + \lambda f_{276} f_{279} \kappa_2$$
$$f_{586} = a_8 + f_{276} f_{279} \kappa_2 + \lambda f_{276} f_{279} \kappa_2^2$$
$$f_{587} = -6 + 5\lambda - 2d_3 + 2\lambda d_3 + 2a_8 d_1 f_{276} - \lambda \kappa_2$$
$$f_{588} = 5 + 2d_3 + 2d_1 f_{276}$$
$$f_{589} = a_8 f_{588} + \kappa_2 + \lambda \kappa_2^2$$
$$f_{590} = \lambda f_{589} + f_{587} \kappa_1$$
$$f_{591} = -4b_2 f_{273} f_{584} + \lambda f_{586} + 4b_2^2 f_{590} - 16\beta^3 a_7 a_8 f_{273} h_2^3$$
$$f_{592} = f_{591} + 16\beta^4 a_7 a_8 h_2^4 - f_{585} \kappa_1$$
$$f_{593} = f_{592} + 16\beta^4 a_7 a_8 h_2^4 - f_{585} \kappa_1$$
$$f_{594} = -2b_2^2 + 2b_2 f_{274} + d_1 f_{275} + f_{362}$$
$$f_{595} = 4 - 6\lambda + 3\lambda^2 + 2d_1 - 2d_3 + 4\lambda d_3 - 2\lambda^2 d_3 - 4a_8^2 b_2^2 f_{275}$$
$$f_{596} = f_{595} + 2\lambda \kappa_2 + 4\lambda d_1 \kappa_2 + \lambda^2 \kappa_2^2 + 2\lambda^2 d_1 \kappa_2^2$$
$$f_{597} = -4 + 6\lambda - 3\lambda^2 + 2d_3 - 4\lambda d_3 + 2\lambda^2 d_3$$
$$f_{598} = -2 - b_2^2 - d_3 + d_1 f_{276} + 2b_2(-d_1 + f_{363})$$
$$f_{599} = -4 + 6\lambda - 3\lambda^2 - 2d_3 + 4\lambda d_3 - 2\lambda^2 d_3$$
$$f_{600} = 2a_8^2 d_1 f_{276} + f_{599} - 2\lambda \kappa_2 - \lambda^2 \kappa_2^2$$
$$f_{601} = 4 - 2\lambda + \lambda^2 - 2d_1 + 4\lambda_1 - 2\lambda^2 d_1 + 2d_3 - 4a_8^2 b_2^2 f_{348}$$
$$f_{602} = 2b_2 f_{600} + f_{601} + 6\lambda \kappa_2 + 4\lambda d_3 \kappa_2 + 3\lambda^2 \kappa_2^2 + 2\lambda^2 d_3 \kappa_2^2$$
$$f_{603} = \lambda a_1 a_3 f_{598} + 2b_1 f_{602}$$
$$f_{604} = 2a_8^2 d_1 f_{275} + f_{597} - 2\lambda \kappa_2 - \lambda^2 \kappa_2^2$$
$$f_{605} = \lambda a_1 a_3 f_{594} + 2b_1 (f_{596} + 2b_2 f_{604})$$
$$f_{606} = 1 - 3\lambda + \lambda^2$$
$$f_{607} = 1 - 5\lambda + 2\lambda^2$$
$$f_{608} = 2 - d_3 + d_1 f_{275}$$
$$f_{609} = 1 + 2\lambda - a_{16} d_3 + a_{16} d_1 f_{275}$$
$$f_{610} = 3 + 4\lambda + 2\lambda^2 - a_{16}^2 d_3 + a_{16}^2 d_1 f_{275}$$
$$f_{611} = 2a_{15} - a_{16} d_3 + a_{16} d_1 f_{275}$$
$$f_{612} = 1 + \lambda(-4 + 2d_3 + 2d_1 f_{364})$$
$$f_{613} = 4 - 10\lambda + 3\lambda^2 - 2d_3 + 6\lambda d_3 - 2\lambda^2 d_3 + 2d_1 f_{275} f_{606} - 2\lambda f_{609} \kappa_2 - \lambda^2 f_{275} f_{348} \kappa_2^2$$
$$f_{614} = 8\lambda - 3\lambda^2 - 5\lambda d_3 + 2\lambda^2 d_3 + f_{409} - d_1 f_{275} f_{607} + f_{610} \kappa_2 + \lambda f_{611} \kappa_2^2 + \lambda^2 \kappa_2^3$$
$$f_{615} = 2 - 6\lambda + 3\lambda^2 - 2d_3 + 4\lambda d_3 - 2\lambda^2 d_3 + 2a_8^2 d_1 f_{275} - a_{16} \kappa_2 - \lambda \kappa_2^2$$

$$f_{616} = -1 - 2\lambda f_{275} f_{348} + \lambda^2 f_{275} f_{348} - 2a_{16} f_{608}\kappa_2 + f_{612}\kappa_2^2 + 2\lambda\kappa_2^3 + \lambda^2\kappa_2^4$$
$$f_{617} = -7\lambda + 3\lambda^2 + 4\lambda d_3 - 2\lambda^2 d_3 + 2a_8^2 d_1 f_{275} + f_{364} - \lambda a_{16}\kappa_2 - \lambda^2\kappa_2^2$$
$$f_{618} = \lambda f_{615} + f_{617}\kappa_1$$
$$f_{619} = -2(4a_7^2 a_9^2 b_2 f_{544} + \lambda^2 f_{616} + 4a_7 b_2^2 f_{618} - 2\lambda f_{614}\kappa_1 + f_{613}\kappa_1^2)$$
$$f_{620} = -2\lambda + 8\lambda^2 - 9\lambda^3 + 3\lambda^4 + 6\lambda d_1 - 18\lambda^2 d_1 + 18\lambda^3 d_1 - 6\lambda^4 d_1 - 12\beta^2\lambda^3 d_2$$
$$f_{621} = 4\beta^2\lambda^4 d_2 + 2\lambda d_3 - 6\lambda^2 d_3 + 6\lambda^3 d_3 - 2\lambda^4 d_3 - 4d_2 d_4 + 12\lambda d_2 d_4$$
$$f_{622} = 4a_7 a_8 a_9^2 b_2 f_{544} + 4a_8 b_2^2 f_{618} + f_{620} + f_{621} - 4\kappa_1 + 10\lambda\kappa_1 - 10\lambda^2\kappa_1 + 3\lambda^3\kappa_1$$
$$f_{623} = 4\lambda^2\kappa_2^2 - \lambda^3\kappa_2^2 - 2\lambda^2\kappa_1\kappa_2^2 - \lambda^3\kappa_1\kappa_2^2 + 3\lambda^3\kappa_2^3 - \lambda^3\kappa_1\kappa_2^3 + \lambda^4\kappa_2^4$$
$$f_{624} = 12d_2 d_4\kappa_1 - 12\lambda d_2 d_4\kappa_1 + 2\lambda\kappa_2 - \lambda^3\kappa_2 - 2\lambda\kappa_1\kappa_2 - \lambda^3\kappa_1\kappa_2$$
$$f_{625} = 6d_1\kappa_1 - 18\lambda d_1\kappa_1 + 18\lambda^2 d_1\kappa_1 - 6\lambda^3 d_1\kappa_1 - 4\beta^2 d_2\kappa_1 + 4\beta^2\lambda^3 d_2\kappa_1$$
$$f_{626} = f_{622} + f_{623} + f_{624} + f_{625} + 2d_3\kappa_1 - 6\lambda d_3\kappa_1 + 6\lambda^2 d_3\kappa_1 - 2\lambda^3 d_3\kappa_1$$
$$f_{627} = 8\beta(f_{626} h_1 + \lambda a_1 a_3 f_{555} h_2)$$
$$f_{628} = d_1 f_{276} + f_{363}$$
$$f_{629} = -2 - d_1 - d_3 - 8b_2^2 f_{273} + 4b_2 f_{628} + 8\beta^3 h_2^3$$
$$f_{630} = 6 - 10\lambda + 5\lambda^2 + 2d_3 - 4\lambda d_3 + 2\lambda^2 d_3$$
$$f_{631} = 2a_8^2 d_1 f_{276} + f_{630} + 2\lambda\kappa_2 + \lambda^2\kappa_2^2$$
$$f_{632} = 4 - 2\lambda + \lambda^2 + 6d_1 + 4\beta^2 d_2 + 2d_3 + f_{307} - 4b_2 f_{273} f_{311}$$
$$f_{633} = 4b_2^2 f_{631} + f_{632} - 16\beta^3 a_8^2 f_{273} h_2^3 + 16\beta^4 a_8^2 h_2^4$$
$$f_{634} = f_{633} + 6\lambda\kappa_2 + 12\lambda d_1\kappa_2 + 4\lambda d_3\kappa_2$$
$$f_{635} = 4a_7 a_9(\lambda a_1 a_3 f_{629} + 2b_1 f_{634})$$
$$f_{636} = 4a_7 a_{10} a_{12} f_{274}$$
$$f_{637} = 16a_7 a_8 a_9^2 b_1 f_{274}$$
$$f_{638} = -4a_7^2 a_9^2 f_{274}$$
$$f_{639} = -4a_7^2 a_9^2 f_{540}$$
$$f_{640} = 4a_7 a_{10} a_{12} f_{540}$$
$$f_{641} = 16a_7 a_8 a_9^2 b_1 f_{540}$$
$$f_{642} = 2\lambda a_3 a_7 a_9 f_{541}$$
$$f_{643} = 2\lambda a_3 a_7 a_9 f_{542}$$
$$f_{644} = -4a_7^2 a_9^2 f_{543}$$
$$f_{645} = 4a_7 a_{10} a_{12} f_{543}$$
$$f_{646} = 16a_7 a_8 a_9^2 b_1 f_{543}$$

$$f_{647} = -4\lambda a_1 a_3 a_9 f_{545}$$
$$f_{648} = 2\lambda a_3 a_7 a_9 f_{547}$$
$$f_{649} = -2\lambda a_3 a_7 a_9 f_{548}$$
$$f_{650} = 2\lambda a_3 a_7 a_9 f_{550}$$
$$f_{651} = 2\lambda a_3 a_7 a_9 f_{552}$$
$$f_{652} = 2\lambda a_3 a_7 a_9 f_{553}$$
$$f_{653} = 2\lambda a_1 a_3 f_{555}$$
$$f_{654} = -2a_7 a_9 f_{555}$$
$$f_{655} = 2a_8 a_{10} f_{555}$$
$$f_{656} = 8a_{12} b_1 f_{555}$$
$$f_{657} = 4\lambda a_3 a_7 a_9 f_{556}$$
$$f_{658} = 4\lambda a_3 a_7 a_9 f_{557}$$
$$f_{659} = 2\lambda a_3 a_7 a_9 f_{559}$$
$$f_{660} = 2 f_{564}$$
$$f_{661} = -4\lambda a_3 a_7 a_9 f_{569}$$
$$f_{662} = -4\lambda a_3 a_7 a_9 f_{572}$$
$$f_{663} = -2a_7 a_9 f_{578}$$
$$f_{664} = 2a_7 a_9 f_{578}$$
$$f_{665} = 2a_7 a_9 f_{583}$$
$$f_{666} = -2a_7 a_9 f_{583}$$
$$f_{667} = -2a_7 a_9 f_{592}$$
$$f_{668} = 2a_7 a_9 f_{592}$$
$$f_{669} = -2a_7 a_9 f_{603}$$
$$f_{670} = -4a_7 a_9 f_{603}$$
$$f_{671} = 4a_7 a_9 f_{605}$$

(D.2)

$$K_{11b} = \frac{1}{f_{343}}\Big(e_{16} f_{325} + e_{17} f_{326} + e_{19} f_{327} + e_{20} f_{328} + e_{21} f_{329} + e_{29} f_{330} + e_{28} f_{331} + e_{30} f_{332} + e_{31} f_{333} + e_{23} f_{334} + e_{22} f_{335} + e_{24} f_{336} + e_{26} f_{337} + e_{25} f_{338} + e_{18} f_{339} + e_{27} f_{340}\Big)$$

$$K_{12b} = \frac{1}{f_{343}}\big(e_{32} f_{456} + e_{33} f_{457} + e_{34} f_{459} + e_{35} f_{460} + e_{36} f_{461} + e_{37} f_{462} + e_{38} f_{463} + e_{39} f_{464} +$$

$$e_{41} f_{465} + e_{42} f_{466} + e_{43} f_{467} + e_{44} f_{468} + e_{45} f_{469} + e_{46} f_{470} + e_{47} f_{471} + e_{48} f_{472} +$$

$$e_{49} f_{473} + e_{50} f_{474} + e_{51} f_{475} + e_{52} f_{476} + e_{53} f_{477} + e_{54} f_{478} + e_{55} f_{479} + e_{56} f_{480} +$$

$$e_{57} f_{481} + e_{58} f_{482} + e_{59} f_{483} + e_{60} f_{484} + e_{61} f_{485} + e_{62} f_{486} + e_{63} f_{487} + e_{64} f_{488} +$$

$$e_{65} f_{489} + e_{66} f_{490} + e_{67} f_{491} + e_{68} f_{492} + e_{69} f_{493} + e_{70} f_{494} + e_{71} f_{495} + e_{72} f_{496} \big)$$

$$K_{21b} = \frac{1}{f_{343}}\big(e_{48} f_{528} + e_{51} f_{529} + e_{49} f_{530} + e_{53} f_{531} + e_{73} f_{532} + e_{74} f_{533} + e_{75} f_{534} + e_{50} f_{535} +$$

$$e_{64} f_{536} + e_{76} f_{537} + e_{63} f_{538} + e_{58} f_{539} \big)$$

$$K_{22b} = \frac{1}{f_{343}}\big(e_{77} f_{619} + e_{78} f_{627} + e_{16} f_{635} + e_{79} f_{636} + e_{19} f_{637} + e_{30} f_{638} + e_{80} f_{639} + e_{81} f_{640} +$$

$$e_{82} f_{641} + e_{83} f_{642} + e_{84} f_{643} + e_{85} f_{644} + e_{86} f_{645} + e_{87} f_{646} + e_{88} f_{647} + e_{89} f_{648} +$$
$$e_{90} f_{649} + e_{91} f_{650} + e_{92} f_{651} + e_{93} f_{652} + e_{94} f_{653} + e_{95} f_{654} + e_{96} f_{655} + e_{97} f_{656} +$$
$$e_{98} f_{657} + e_{99} f_{658} + e_{100} f_{659} + e_{101} f_{660} + e_{102} f_{661} + e_{103} f_{662} + e_{104} f_{663} + e_{18} f_{664} +$$
$$e_{105} f_{665} + e_{106} f_{667} + e_{107} f_{668} + e_{31} f_{669} + e_{108} f_{670} + e_{26} f_{671} \big)$$

$$k_{11s}=\frac{-8\kappa_2\mu_1^2-2(\kappa_2-1)(\kappa_1-3)\mu_1\mu_2+2(\kappa_1^2+3)\mu_2^2}{(2h_1-t-x)(\lambda+\kappa_1)(1+\lambda\kappa_2)}+\frac{(h_1-x)4(\mu_1-\mu_2)(\kappa_2\mu_1+\mu_2)}{(2h_1-t-x)^2(\lambda+\kappa_1)(1+\lambda\kappa_2)}$$

$$+\frac{-8\kappa_2\mu_1^2-2(\kappa_2-1)(\kappa_1-3)\mu_1\mu_2+2(\kappa_1^2+3)\mu_2^2}{(2h_1-t+x)(\lambda+\kappa_1)(1+\lambda\kappa_2)}+\frac{(h_1+x)4(\mu_1-\mu_2)(\kappa_2\mu_1+\mu_2)}{(2h_1-t+x)^2(\lambda+\kappa_1)(1+\lambda\kappa_2)}$$

$$+\left\{\frac{\kappa_2\mu_1^2-(\kappa_2-1)\mu_1\mu_2-\mu_2^2}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\right\}\left(\frac{12}{(2h_1-t-x)}-\frac{28(h_1-x)}{(2h_1-t-x)^2}+\frac{16(h_1-x)^2}{(2h_1-t-x)^3}\right)$$

$$+\left\{\frac{\kappa_2\mu_1^2-(\kappa_2-1)\mu_1\mu_2-\mu_2^2}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\right\}\left(\frac{-12}{(2h_1-t+x)}+\frac{28(h_1+x)}{(2h_1-t+x)^2}-\frac{16(h_1+x)^2}{(2h_1-t+x)^3}\right)$$

$$k_{12s}=2(1+\kappa_2)\left\{\frac{1}{(1+\lambda\kappa_2)}+\frac{1}{(\lambda+\kappa_1)}\right\}\frac{1}{(x-h_1)}-2(1+\kappa_2)\left\{\frac{1}{(1+\lambda\kappa_2)}+\frac{1}{(\lambda+\kappa_1)}\right\}\frac{1}{(x+h_1)}$$

$$+\frac{2(1+\kappa_2)\{3(1+\lambda\kappa_2)-(\lambda+\kappa_1)\}}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\frac{1}{(t-x+h_1)}+\frac{4(1+\kappa_2)\{(1+\lambda\kappa_2)-(\lambda+\kappa_1)\}}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\frac{(x-h_1)}{(t-x+h_1)^2}$$

$$+\frac{2(1+\kappa_2)\{3(1+\lambda\kappa_2)-(\lambda+\kappa_1)\}}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\frac{1}{(t+x+h_1)}+\frac{4(1+\kappa_2)\{-(1+\lambda\kappa_2)+(\lambda+\kappa_1)\}}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\frac{(x+h_1)}{(t+x+h_1)^2}$$

$$k_{21s}=\left\{\frac{-2\lambda(1+\kappa_1)}{(\lambda+\kappa_1)(1+\lambda\kappa_2)}\right\}\left\{\frac{3(\kappa_1+\lambda)-(1+\lambda\kappa_2)}{(x+h_1-t)}+\frac{2(\lambda(\kappa_2-1)-(\kappa_1-1))x}{(x+h_1-t)^2}\right\}$$

$$k_{22s}=\frac{4}{(2h_2-t-x)}+\frac{-24(h_2-x)}{(2h_2-t-x)^2}+\frac{16(h_2-x)^2}{(2h_2-t-x)^3}$$

$$+\frac{(-2\lambda\kappa_2(\kappa_1+1+\lambda(\kappa_2+2)))+((\kappa_1+\lambda)(6\lambda-8)-2\lambda)}{(\lambda+\kappa_1)(1+\lambda\kappa_2)(t+x)}+\frac{-24(\lambda-1)}{(1+\lambda\kappa_2)}*\frac{x}{(t+x)^2}+\frac{16(\lambda-1)}{(1+\lambda\kappa_2)}*\frac{x^2}{(t+x)^3}$$

VITA

Mehmet Tarık Atay, son of Hatice and İhsan who were both teachers, was born in Akşehir on November 13, 1969. He attended Hamdullah Suphi Primary School and Bahçelievler Secondary Schoool and Bahçelievler Deneme Highschool, all are in Ankara. He obtained his B.Sc. degree from the Mathematics Department in Middle East Technical University in 1993 and his M.Sc. degree on Applied Mathematics from Illinios Institute of Technology, Chicago in 1996. He has been in Engineering Sciences Department since September 1998.