STURM COMPARISON THEORY FOR IMPULSIVE DIFFERENTIAL EQUATIONS

ABDULLAH ÖZBEKLER

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Prof. Dr. Canan ÖZGEN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Şafak ALPAY
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Ağacık ZAFER
Supervisor

Examining Committee Members

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Prof. Dr. Ağacık ZAFER	(METU)	
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Name, Last name	:	Abdullah ÖZBEKLER
Signature	:	

ABSTRACT

STURM COMPARISON THEORY FOR IMPULSIVE DIFFERENTIAL EQUATIONS

ÖZBEKLER, Abdullah
Ph.D., Department of Mathematics
Supervisor: Prof. Dr. Ağacık ZAFER

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In this thesis, we investigate Sturmian comparison theory and oscillation for second order impulsive differential equations with fixed moments of impulse actions. It is shown that impulse actions may greatly alter the oscillation behavior of solutions.

In chapter two, besides Sturmian type comparison results, we give Leightonian type comparison theorems and obtain Wirtinger type inequalities for linear, half-linear and non-selfadjoint equations. We present analogous results for forced super linear and super half-linear equations with damping.

In chapter three, we derive sufficient conditions for oscillation of nonlinear equations. Integral averaging, function averaging techniques as well as interval criteria for oscillation are discussed. Oscillation criteria for solutions of impulsive Hill's equation with damping and forced linear equations with damping are established.

Keywords: Sturm, Leighton, Wirtinger, Damping, Hill's Equation, Impulse.

ÖZ

İMPALSİF DİFERANSİYEL DENKLEMLERDE STURM KARŞILAŞTIRMA TEORİLERİ

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Bu tezde, impals etkisi sabit zamanlı impalsif diferansiyel denklemler için Sturm tipi karşılaştırma teorisi ve salınımını araştırdık. İmpals etkilerinin, çözümlerin davranışını önemli ölçüde değiştirebileceği gösterildi.

İkinci bölümde, Sturm tipi karşılaştırma sonuçlarıyla birlikte, lineer, yarı-lineer ve kendine eşlenik olmayan denklemler için Leighton tipi karşılaştırma teoremleri verdik ve Wirtinger tipi eşitsizlikler elde ettik. Damping terimli kuvvetlendirilmiş süper lineer ve süper yarı-lineer denklemler için benzer sonuçlar sunduk.

Üçüncü bölümde, lineer olmayan denklemlerin salınımı için yeterli koşulları elde ettik. Salınım için aralık kritelerinin yanısıra integral ortalama, fonksiyon ortalama metodları ele alındı. Damping terimli İmpalsif Hill denklemi ve kuvvetlendirilmiş lineer denklemler için salınım kriterleri kanıtlandı.

Anahtar Kelimeler: Sturm, Leighton, Wirtinger, Damping Terim, Hill Denklemi, İmpals.

To the memory of my grandfather and father,
Tahir AYDIN
and
Vahit ÖZBEKLER

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Many evolution process are characterized by the fact that they are subject to short-time perturbation whose duration is negligible in comparison with the duration of the process. This results in a sudden change of the state of the process. For example, when a hammer hits a string which is already oscillating, it experiences a rapid change of velocity; a pendulum of a clock, meanwhile, undergoes a sudden change of momentum when it crosses its equilibrium position; and so on.

For the description of the continuous change of such processes, ordinary differential equations are used, while the short-time perturbations of those processes are described by sudden changes of their states at certain times. It becomes, therefore, necessary to study dynamical systems with discontinuous trajectories, or with impulse effect, shortly as they are called, impulsive differential equations, or sometimes, differential equations with impulse actions.

In the last a few decades the theory of impulsive differential equations has been developed very rapidly due to the fact that such equations find a wide range of applications modelling adequately many real processes observed in modern technology, engineering, physics and biology, etc. [2, 49, 56, 67, 68, 69, 72, 74, 86]. Moreover, impulsive differential equations is richer in applications compared to the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations. Numerous aspects of qualitative theory and the existence and uniqueness theorems of solutions of impulsive differential equations subject the initial conditions has been investigated in the monographs of Samoilenko and Perestjuk [62], Bainov, Lakshmikantham and Simeonov [37], Bainov and Simeonov [4, 5, 6].

The oscillation theory is one of the directions which initiated the investigations on the qualitative properties of the differential equations. Its occurrence started with the classical works of Sturm [66] and Kneser [28, 29], and still attracts attention of many mathematicians as they find various applications.

The attractiveness of the oscillation theory links rather strongly the occurrence of new objects to be investigated. Such fast development can be observed in studying the oscillatory properties of the impulsive differential equations. The paper of K. Gopalsamy and B. G. Zhang [12] is the first investigation on oscillatory properties of impulsive differential equations. In the last decade D.D. Bainov, M. B. Dimitrova, Yu. I. Domshlak, E. I. Minchev, J. Yan and P. S. Simeonov have studied the oscillatory properties of various classes of impulsive differential equations. The book by Bainov and Simeonov [7] is the only source dealing with the subject.

The classical Sturmian comparison theory of second order ordinary differential equations is known to be the basis for study of numerous important properties of their solutions and, especially, of their oscillatory properties. The principal improvement in this direction was achieved due to the results of Sturmian theory (Sturm comparison theorem, oscillation and nonoscillation theorem, zeros separation theorem, dichotomy theorem) although many of the more recent investigations (especially for nonlinear equations) are no more based on this theory. The first investigation on Sturmian theory for second order impulsive differential equations was published in 1996, the paper of Bainov, Domshlak and Simeonov [3](see also [7]).

In this thesis, we investigate Sturmian comparison theory and oscillation of solutions for second-order impulsive differential equations with fixed moments of impulse actions. It is shown that impulse actions may greatly change oscillatory behavior of solutions.

The comparison and oscillation property of solutions of second order equations is of special interest, and therefore, it has been the subject of many investigations. The interest in this subject is due to the fact that many physical systems are modelled by such equations.

The thesis is organized as follows: In chapter 2, besides Sturmian type comparison results, we also give Leigtonian type comparison theorems and obtain Wirtinger type inequalities for linear, half-linear and non self-adjoint equations. We present analogous results for forced super linear and super half-linear equations with damping. In chapter 3, we work on the oscillation theory for nonlinear equations. Integral averaging, function averaging techniques as well as interval criteria for oscillation are also discussed. Several criteria for oscillation of impulsive Hill's equation with periodic damping and forced linear equations with damping are established.

1.2 Impulsive Differential Equations

The impulsive differential equations are adequate mathematical models of processes and phenomena characterized by as continuous as jumpwise changes of the phase variables describing the processes. The continuous change is prescribed by the differential equation which can be ordinary one or partial. The jumpwise change is prescribed by jump conditions which determine the moments and magnitudes of the jumpwise (impulse) change of some of the phase variables.

In this thesis, it is assumed that

$$0 < \theta_1 < \theta_2 < \dots \qquad \text{and} \qquad \lim_{i\to\infty} \theta_i = \infty.$$

Let $n \in \mathbb{N}$, $\mathcal{I} \subset \mathbb{R}$, and let the sequence $\{\theta_i\}$ be fixed. We denote by $\mathcal{PLC}(\mathcal{I})$ the space of all functions $\psi : \mathcal{I} \to \mathbb{R}$ such that ψ is continuous for all $t \neq \theta_i$ at which $\psi(t)$ is continuous from left and has discontinuity of the first kind. As usual by $\mathcal{PLC}^n(\mathcal{I})$ we mean the space of functions $\psi : \mathcal{I} \to \mathbb{R}$ such that $\psi^{(k)} \in \mathcal{PLC}(\mathcal{I})$, $k = 0, 1, 2, ..., n$.

For $\psi \in \mathcal{PLC}(\mathcal{I})$, $\Delta\psi(t)|_{t=\theta_i}$ denotes the jump at $t = \theta_i \in \mathcal{I}$, i.e.

$$\Delta\psi(\theta_i) = \psi(\theta_i^+) - \psi(\theta_i^-),$$

where

$$\psi(\theta_i^\pm) = \lim_{h\to 0^+} \psi(\theta_i \pm h).$$

Note that if $\psi \in \mathcal{PLC}(\mathcal{I})$ and $\Delta\psi(\theta_i) = 0$ for all $i \in \mathbb{N}$, then ψ becomes continuous and vice versa.

The mathematical model of a real process which experiences certain impulses at fixed moments $\{\theta_i\}$ could be given by an impulsive differential equation

$$\begin{aligned} x' &= f(t, x), \qquad && t \neq \theta_i; \\ \Delta x &= I_i(x), && t = \theta_i, \qquad i \in \mathbb{N} \end{aligned} \tag{1.1}$$

where $x' = dx/dt$. The function $x = \psi(t)$ is said to be a solution of the equation (1.1) on an interval $\mathcal{J} = (a, b)$ if $\psi \in \mathcal{PLC}^1(\mathcal{J})$ satisfies

$$\psi'(t) = f(t, \psi(t)), \qquad t \neq \theta_i$$

and

$$\psi(\theta_i^+) - \psi(\theta_i^-) = I_i(\psi(\theta_i^-)), \qquad \theta_i \in \mathcal{J}.$$

An initial condition

$$x(t_0) = x_0 \quad \text{or} \quad x(t_0^+) = x_0, \tag{1.2}$$

can be associated with equation (1.1). For basic theory of initial value problems (1.1) and (1.2), we refer to [5, 6, 62].

CHAPTER 2

STURMIAN COMPARISON THEORY

2.1 Introduction

Although numerous aspects of qualitative theory are contained in the monographs [37, 62], there appears to be less known about the oscillation theory, especially the Sturmian theory, of impulsive differential equations when compared to equations without impulses. Therefore, our objective is to make a contribution to the impulsive differential equations in this direction. Specifically, we are interested in a Picone's formula so as to obtain comparison theorems of Leighton and Sturm-Picone types for second order impulsive differential equations.

Sturmian type comparison theorems for linear equations without impulse effect are very classical and well known [16, 18, 70]. However, there is hardly any result for impulsive equations.

Consider the second order linear ordinary differential equations

$$l[x] = (k(t)x')' + p(t)x = 0, \tag{2.1}$$

$$L[y] = (m(t)y')' + q(t)y = 0 \tag{2.2}$$

where $k, p, m, q \in C(J)$, $J \subset \mathbb{R}$. The classical Sturmian comparison theorem asserts that, if equation (2.1) has a nontrivial solution $x(t)$ with two zeros t_1 and t_2 in J, $t_1 < t_2$, under the assumption that $k(t) \geq m(t)$ and $q(t) \geq p(t)$ for $t \in [t_1, t_2]$, then every nontrivial solution $y(t)$ of equation (2.2) has a zero in (t_1, t_2) unless $y(t)$ is a constant multiple of $x(t)$.

The proof of the well-known Sturm-Picone comparison theorem given by Picone [59] in 1909 (see also [30, 31, 70, 71]) was based on employing the

Picone's formula

$$\frac{x}{y}\,(ykx' - xmy')\Big|_a^b = \int_a^b \Big[(k-m)(x')^2 + (q-p)x^2 + m(x' - \frac{x}{y}\,y')^2 + \frac{x}{y}\,\{yl[x] - xL[y]\}\Big]\,dt \tag{2.3}$$

which holds for all real valued functions x and y defined on an interval $[a, b]$ such that x, y, kx' and my' are differentiable on $[a, b]$ and $y \neq 0$ for $t \in [a, b]$. The formula (2.3) has also been used for establishing Wirtinger type inequalities for solutions of ordinary differential equations [30, 70], and generalized to linear non self-adjoint equations [30, p. 11].

Recently, Jaroš and Kusano [22] have shown that Picone's identity (2.3) can be generalized to the half-linear equations

$$l_\alpha[x] = (k(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x) = 0, \tag{2.4}$$

$$L_\alpha[y] = (m(t)\varphi_\alpha(y'))' + q(t)\varphi_\alpha(y) = 0, \tag{2.5}$$

where $\varphi_\alpha(s) = |s|^{\alpha-1}s$ and α is a positive constant. The generalized Picone's identity is written as follows:

$$\begin{aligned} \frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(y)}\Big[\varphi_\alpha(y)k(t)\varphi_\alpha(x') - \varphi_\alpha(x)m(t)\varphi_\alpha(y')\Big]\right\} \\ = [k(t) - p(t)]|x'|^{\alpha+1} + [q(t) - p(t)]|x|^{\alpha+1} + m(t)\,\Phi_\alpha\big(x', xy'/y\big) \\ + \frac{x}{\varphi_\alpha(y)}\Big\{\varphi_\alpha(y)\,l_\alpha[x] - \varphi_\alpha(x)\,L_\alpha[y]\Big\} \end{aligned} \tag{2.6}$$

where

$$\Phi_\alpha(u, v) := u\,\varphi_\alpha(u) + \alpha\,v\,\varphi_\alpha(v) - (\alpha+1)\,u\,\varphi_\alpha(v). \tag{2.7}$$

There were several attempts to extend Picone's formula to nonlinear equations (see, for instance, [14]). Jaroš, Kusano and Yoshida [23, 24] showed how Picone's formula can be used, rather surprising but simple way, to extend the classical Sturm theory to forced super-linear and super half-linear equations. In [24], they compared the solutions of (2.4) with those of

$$(m(t)\varphi_\alpha(y'))' + q(t)\varphi_\beta(y) = f(t), \qquad \beta \geq \alpha > 0 \tag{2.8}$$

by employing

$$\frac{d}{dt}\left\{\frac{x}{\varphi_\alpha(y)}\left[\varphi_\alpha(y)\,k\,\varphi_\alpha(x')-\varphi_\alpha(x)\,m\,\varphi_\alpha(y')\right]\right\}=(k-m)|x'|^{\alpha+1}$$
$$+\left\{q|y|^{\beta-\alpha}-\frac{f}{\varphi_\alpha(y)}-p\right\}|x|^{\alpha+1}+m\,\Phi_\alpha(x',xy'/y). \qquad (2.9)$$

The first investigation on oscillatory properties of impulsive differential equations is due by Gopalsamy and Zhang [12]. Later, several investigations have been done for various classes of impulsive differential equations, see [3, 8, 12, 17, 50, 63] and references cited therein. As far as the Sturmian theory is concerned, to the best of our knowledge, the first work has appeared in the literature in 1996, in which Bainov, Domshlak and Simeonov [3] studied the Sturmian comparison theory for second order linear impulsive differential equations of the form

$$\begin{aligned} &x''+p(t)x=0, && t\neq\theta_i;\\ &\Delta x'+p_i\,x=0, && t=\theta_i. \end{aligned} \qquad (2.10)$$

They proved theorems on linear dependence, zeros-separation, dichotomy, oscillation, and nonoscillation of solutions of linear impulsive equations.

In this chapter, we obtain some analogous results in [3]. In Section 2.2, we deal with linear and half-linear impulsive equations. In Section 2.3, we obtain some analogous results for non self-adjoint impulsive equations and in the last section we extend the previous results to forced super half-linear impulsive equations with damping. Examples are also provided to illustrate the results.

2.2 Linear and Half-Linear Equations

It is well-known that the Sturmian theory for linear and half-linear differential equations plays an important role in the study of qualitative behavior of solutions of both linear and nonlinear equations.

Consider half-linear equations of the form

$$\begin{aligned} (k(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x) &= 0, && t \neq \theta_i; \\ \Delta(k(t)\varphi_\alpha(x')) + p_i\varphi_\alpha(x) &= 0, && t = \theta_i. \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} (m(t)\varphi_\alpha(y'))' + q(t)\varphi_\alpha(y) &= 0, && t \neq \theta_i; \\ \Delta(m(t)\varphi_\alpha(y')) + q_i\varphi_\alpha(y) &= 0, && t = \theta_i \end{aligned} \tag{2.12}$$

where α is a positive constant, $\{p_i\}$, $\{q_i\}$ and $\{\theta_i\}$ are real sequences with $\theta_1 > t_0$ for some fixed $t_0 \in \mathbb{R}$, and $k, m, p, q \in \mathcal{PLC}[t_0, \infty)$ with $k(t) > 0$ and $m(t) > 0$.

Note that the above equations become linear if $\alpha = 1$.

By a solution $x(t)$ of (2.11) on an interval $J \subset [t_0, \infty)$ we mean a nontrivial continuous function $x(t)$ defined on J such that $k(t)\varphi_\alpha(x') \in \mathcal{PLC}^1(J)$ and $x(t)$ satisfies (2.11). A solution $y(t)$ of (2.12) is defined in a similar manner.

The existence and uniqueness of the solutions of (2.4) subject the initial condition has been investigated by Elbert [11], Kusano and Kitano [32].

The purpose of this section is to obtain some Sturmian type comparison theorems for both linear and half-linear impulsive differential equations. By applying the results, several oscillation criteria are also established.

The pioneering works of Elbert [11] and Mirzov [54] showed that there is a striking similarity between linear and half-linear equations without impulse, showing that many results in the Sturmian comparison and oscillation theory for linear equations can be carried over almost literatim and verbatim to half-linear equation, see e.g. [22, 44]. Motivated by this we attempt to obtain analogous comparison results for second order half-linear impulsive differential equations.

In order to prove our results, we need the following well-known inequality.

Lemma 2.2.1. *[15] Let $A, B \in \mathbb{R}$ and $\beta > 0$ be a constant, then $\Phi_\beta(A, B)$ defined by (2.7) satisfies*

$$\Phi_\beta(A, B) \geq 0, \tag{2.13}$$

where equality holds if and only if $A = B$.

Our first result is the following Sturm-Picone type comparison theorem.

Theorem 2.2.2 (Sturm-Picone type comparison). *Let* $x(t)$ *be a solution of (2.11) having two consecutive zeros* a *and* b *in* J*. Suppose that* $p(t) \leq q(t)$ *and* $m(t) \leq k(t)$ *are satisfied for all* $t \in [a, b]$*, and that* $p_i \leq q_i$ *for all* $i \in \mathbb{N}$ *for which* $\theta_i \in [a, b]$*. If either* $p(t) \not\equiv q(t)$ *or* $k(t) \not\equiv m(t)$ *or* $p_i \not\equiv q_i$*, then any solution* $y(t)$ *of (2.12) must have at least one zero in* (a, b)*.*

Proof. Assume that $y(t)$ never vanishes on (a, b). Define

$$u(t) := \frac{x}{\varphi_\alpha(y)} \Big[\varphi_\alpha(y) k(t) \varphi_\alpha(x') - \varphi_\alpha(x) m(t) \varphi_\alpha(y') \Big], \tag{2.14}$$

where the dependence on t of x and y are suppressed. It is not difficult to see that

$$\begin{aligned} u'(t) &= [k(t) - p(t)]|x'|^{\alpha+1} + [q(t) - p(t)]|x|^{\alpha+1} \\ &\quad + m(t)\, \Phi_\alpha\big(x', xy'/y\big), \quad t \neq \theta_i \end{aligned} \tag{2.15}$$

$$\Delta u(t) = (q_i - p_i)|x|^{\alpha+1}, \qquad t = \theta_i. \tag{2.16}$$

The last term of (2.15) is integrable over (a, b) if $y(a) \neq 0$ and $y(b) \neq 0$. Moreover, $u(a^+) = u(b^-) = 0$ in this case. Suppose that $y(a^+) = 0$. The case $y(b^-) = 0$ is similar. Since $y'(a^+) \neq 0$ and

$$\lim_{t \to a^+} \frac{x(t)}{y(t)} = \lim_{t \to a^+} \frac{x'(t)}{y'(t)} < \infty,$$

we get

$$\lim_{t \to a^+} \varphi_\alpha\left(\frac{x(t)}{y(t)}\right) < \infty,$$

and so

$$\begin{aligned} \lim_{t \to a^+} \Phi_\alpha\big(x', xy'/y\big) = \lim_{t \to a^+} \Bigg[& x'\varphi_\alpha(x') + \alpha\left(\frac{x}{y}\right)\varphi_\alpha\left(\frac{x}{y}\right) y'\varphi_\alpha(y') \\ & -(\alpha+1)x'\varphi_\alpha(y')\varphi_\alpha\left(\frac{x}{y}\right)\Bigg] < \infty. \end{aligned}$$

Moreover,

$$\lim_{t\to a^+} u(t) = \lim_{t\to a^+} x\left[k(t)\varphi_\alpha(x') - m(t)\varphi_\alpha\left(\frac{x}{y}\right)\varphi_\alpha(y')\right] = 0.$$

Integrating (2.15) from a to b and using (2.16), we see that

$$\begin{aligned} 0 = &\int_a^b \Big\{[k(t) - m(t)]|x'|^{\alpha+1} + [q(t) - p(t)]|x|^{\alpha+1}\Big\}dt \\ &+ \int_a^b m(t)\Phi_\alpha\big(x', xy'/y\big)dt + \sum_{a\le\theta_i<b}(q_i - p_i)|x(\theta_i)|^{\alpha+1}. \end{aligned} \tag{2.17}$$

It is clear that (2.17) is not possible under our assumptions and Lemma 2.2.1 with $u = x'$ and $v = xy'/y$, and hence $y(t)$ must have a zero in (a, b). □

Corollary 2.2.3. *The zeros of two linearly independent solutions $x(t)$ and $y(t)$ of (2.11) separate each other.*

Proof. Let a and b be two consecutive zeros of $x(t)$. Assume that $y(t)$ never vanishes on (a, b). Then, in view of (2.17), we see that

$$0 = \int_a^b k(t)\,\Phi_\alpha\big(x', xy'/y\big)\,dt. \tag{2.18}$$

Since $x(t)$ and $y(t)$ are linearly independent, (2.18) leads to a contradiction due to Lemma 2.2.1. Therefore $y(t)$ must have a zero in (a, b). Moreover, $y(t)$ cannot have more than one zero in (a, b) as a and b are consecutive zeros of $x(t)$. □

Definition 2.2.4. *A nontrivial function $\xi(t)$ is called oscillatory if it has arbitrarily large zeros. Otherwise, $\xi(t)$ is said to be nonoscillatory. A nonoscillatory function is either eventually positive or eventually negative, i.e. there exists a $t^* \in \mathbb{R}$ such that $\xi(t) \neq 0$ for all $t > t^*$. A differential equation is called oscillatory if every solution of the equation is oscillatory and nonoscillatory if it has at least one nonoscillatory solution.*

Corollary 2.2.5. *Suppose that $p(t) \le q(t)$ and $m(t) \le k(t)$ are satisfied for all $t \in [t_*, \infty)$ for some $t_* \ge t_0$, and that $p_i \le q_i$ for all $i \in \mathbb{N}$ for which $\theta_i \ge t_*$. If either $p(t) \not\equiv q(t)$ or $k(t) \not\equiv m(t)$ or $p_i \not\equiv q_i$, then every solution $y(t)$ of (2.12) is oscillatory whenever a solution $x(t)$ of (2.11) is oscillatory.*

Corollary 2.2.6. *The solutions of (2.11) are either all oscillatory or all nonoscillatory.*

Theorem 2.2.7 (Leighton-type comparison). *Let $x(t)$ be a solution of (2.11) having two consecutive zeros a and b in J. Suppose that*

$$\begin{aligned} V_\alpha[x] := &\int_a^b \Big\{ [k(t) - m(t)]|x'(t)|^{\alpha+1} + [q(t) - p(t)]|x(t)|^{\alpha+1} \Big\} dt \\ &+ \sum_{a \le \theta_i < b} (q_i - p_i)|x(\theta_i)|^{\alpha+1} > 0. \end{aligned}$$

Then any nontrivial solution $y(t)$ of (2.12) must have at least one zero in (a, b).

Proof. Assume that $y(t)$ has no zero in (a, b). Define the function $u(t)$ as in (2.14). Clearly, (2.15) and (2.16) hold. It follows that

$$\begin{aligned} 0 &= u(b^-) - u(a^+) \\ &= \int_a^b \Big\{ [k(t) - m(t)]|x'(t)|^{\alpha+1} + [q(t) - p(t)]|x(t)|^{\alpha+1} \Big\} dt \\ &\quad + \int_a^b m(t)\, \Phi_\alpha\big(x', xy'/y\big) dt + \sum_{a \le \theta_i < b} (q_i - p_i)|x(\theta_i)|^{\alpha+1}, \end{aligned}$$

and that

$$V_\alpha[x] = -\int_a^b m(t)\, \Phi_\alpha\big(x', xy'/y\big)\, dt \le 0,$$

which is a contradiction. Therefore, $y(t)$ must have a zero on (a, b). □

If $V_\alpha[x] \ge 0$ then we may conclude that either $y(t)$ has a zero in (a, b) or $y(t)$ is a constant multiple of $x(t)$. As a consequence of Theorem 2.2.2 and Theorem 2.2.7, we have the following oscillation result.

Corollary 2.2.8. *Suppose for a given $T \geq t_*$ there exists an interval $(a, b) \subset [T, \infty)$ for which either the conditions of Theorem 2.2.2 or Theorem 2.2.7 are satisfied, then every solution $y(t)$ of (2.12) is oscillatory.*

Example 2.2.9. Consider

$$\begin{aligned} &x'' - a^2 x = 0, && t \neq i, \quad (i \in \mathbb{N}) \\ &\Delta x' + 2a \coth(a/2)\, x = 0, && t = i \end{aligned} \tag{2.19}$$

where $a > 0$ is a fixed real number. It is not difficult to see that $x(t) = x_i(t)$,

$$x_i(t) = \frac{(-1)^{i+1}}{e^a - 1}\left\{e^{a(t-i+1)} - e^{a(i-t)}\right\}, \qquad t \in (i-1, i],$$

is a solution defined on $[1/2, \infty)$. Clearly, this solution is oscillatory with zeros at $t_i = (2i-1)/2$, $i \in \mathbb{N}$. From Corollary 2.2.6, we may conclude that all solutions of (2.19) are oscillatory. Applying Theorem 2.2.2 we deduce that if there exists an $n_0 \in \mathbb{N}$ such that

$$0 < m(t) \leq 1, \qquad q(t) \geq -a^2, \qquad q_i \geq 2a \coth(a/2)$$

for all $t \geq n_0$, and all $i \geq n_0$, where a is any positive real number, then (2.12) with $\alpha = 1$ and $\theta_i = i$ is oscillatory.

The lemma below, cf. [3, Lemma 1], provides more choices of test equations which can be used for comparison purposes.

Lemma 2.2.10. *Let ψ be a positive and continuous function for $t \geq a$ with $\psi' \in \mathcal{PLC}^1[a, \infty)$, where a is a fixed real number, and $k \in \mathcal{PLC}^2[a, \infty)$. Then the function $x(t) = \dfrac{1}{\sqrt{k(t)\psi(t)}} \sin\left(\displaystyle\int_a^t \psi(s)ds\right)$ is a solution of*

$$\begin{aligned} (k(t)x')' + p(t)x = 0, && t \neq \theta_i, \quad (i \in \mathbb{N}) \\ \Delta k(t)x' + p_i x = 0, && t = \theta_i \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} p(t) &= \frac{1}{2}\, k''(t) - \frac{(k'(t))^2}{4k(t)} + k(t)\left[\frac{\psi''(t)}{2\psi(t)} + \psi^2(t) - \frac{3}{4}\left(\frac{\psi'(t)}{\psi(t)}\right)^2\right] \\ p_i &= \frac{1}{2\psi(\theta_i)}\Big[\psi(\theta_i)\Delta k'(\theta_i) + k(\theta_i)\Delta\psi'(\theta_i)\Big]. \end{aligned}$$

It is obvious that if $\displaystyle\int_a^\infty \psi(t)\,dt = \infty$ then $x(t)$ is oscillatory.

By choosing specific functions, we may obtain several oscillation criteria for equation (2.12) with $\alpha = 1$.

Example 2.2.11. Let $k(t) = t^2/4$, $\psi(t) = \dfrac{2i - t}{i(i+1)}$, $\; i - 1 < t \le i$, $\;(i \in \mathbb{N})$. We see that

$$x(t) = \frac{2}{t\sqrt{\psi(t)}} \sin\left(\int_a^t \psi(s)ds\right)$$

is an oscillatory solution of

$$\begin{aligned}
&(t^2x')' + t^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right]x = 0, && t \ne i;\\
&\Delta(t^2x') + \frac{i}{i+2}\,x = 0, && t = i.
\end{aligned}$$

In view of Theorem 2.2.2 we easily see that every solution of (2.12) with $\alpha = 1$ and $\theta_i = i$ is oscillatory if there exists an $n_0 \in \mathbb{N}$ such that

$$m(t) \le t^2, \qquad q(t) \ge t^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right], \qquad q_i \ge \frac{i}{i+2}$$

for all $t \in (i-1, i]$ and $i \ge n_0$.

The generalized sine function $S(t)$ is defined [11] as the unique solution of

$$(|u'|^{\alpha-1}u')' + \alpha|u|^{\alpha-1}u = 0, \qquad u(0) = 0, \quad u'(0) = 1,$$

where $\alpha > 0$ is a fixed real number. We note that the generalized cosine function $C(t)$ is then defined by $C(t) = S'(t)$, and the generalized tangent function $T(t)$ becomes

$$T(t) = \frac{S(t)}{C(t)}, \quad t \ne \frac{\pi_\alpha}{2} \pmod{\pi_\alpha}, \qquad \pi_\alpha = \frac{2\pi}{\alpha+1}\Big/ \sin\frac{\pi}{\alpha+1}.$$

Lemma 2.2.12. *Let ψ be a positive and continuous function defined for $t \ge a$ with $\psi' \in \mathcal{PLC}^1[a,\infty)$, where a is a fixed real number. If $\lim_{t\to\infty}\psi(t) = \infty$,* *then* $x(t) = S\left\{\psi^\beta(t)\right\}$, *with* $\beta = \alpha/(\alpha+1)$, *is an oscillatory solution of (2.11) where*

$$\begin{aligned}
&k(t) = \psi^\beta(t),\\
&p(t) = \alpha\,\beta^{\alpha+1}\,\psi^{-\beta/\alpha}(t)\,|\psi'(t)|^{\alpha+1} - \alpha\,\beta^\alpha\,\frac{|\psi'(t)|^{\alpha-1}\psi''(t)}{\varphi_\alpha(T(\psi^\beta(t)))},\\
&p_i = -\beta^\alpha\,\frac{\Delta\varphi_\alpha(\psi'(\theta_i))}{\varphi_\alpha(T(\psi^\beta(\theta_i)))}.
\end{aligned}$$

The proof of the above Lemma can be accomplished by a direct substitution. Using Lemma 2.2.12, we obtain the following particular case.

Example 2.2.13. Let $\xi_i = 2^{2i-2}(i-1)!^2/(2i-1)!$ for $i \in \mathbb{N}$. Consider

$$\begin{aligned}
&((t+i)^\beta\,\xi_i^\beta\,\varphi_\alpha(x'))' + \alpha\,\beta^{\alpha+1}\,(t+i)^{-\beta/\alpha}\,\xi_i^{\beta(\alpha+2)}\,\varphi_\alpha(x) = 0, && t \neq i;\\
&\Delta[(t+i)\,\xi_i]^\beta\,\varphi_\alpha(x') + (\beta\,\xi_i)^\alpha\,\frac{(2i+1)^\alpha - (2i)^\alpha}{(2i+1)^\alpha\,\varphi_\alpha(T((2i\,\xi_i)^\beta))}\,\varphi_\alpha(x) = 0, && t = i
\end{aligned}$$

where $\beta = \alpha(\alpha+1)^{-1}$. Clearly $\psi(t) = (t+i)\,\xi_i$ and so by Lemma 2.2.12, $x(t) = S\left\{(t+i)^\beta\,\xi_i^\beta\right\}$ is an oscillatory solution.

Applying Theorem 2.2.2 we easily see that every solution of (2.12) with $\theta_i = i$ is oscillatory if there exists an $n_0 \in \mathbb{N}$ such that for all $i \geq n_0$,

$$\begin{aligned}
&m(t) \leq (t+i)^\beta\,\xi_i^\beta, && t \in (i-1, i],\\
&q(t) \geq \alpha\,\beta^{\alpha+1}\,(t+i)^{-\beta/\alpha}\,\xi_i^{\beta(\alpha+2)}, && t \in (i-1, i],\\
&q_i \geq (\beta\,\xi_i)^\alpha\,\frac{(2i+1)^\alpha - (2i)^\alpha}{(2i+1)^\alpha\,\varphi_\alpha(T((2i\,\xi_i)^\beta))}.
\end{aligned}$$

As in the classical case we may employ the Sturmian comparison theory to establish sufficient conditions for oscillation of second order nonlinear impulsive equations of the form

$$\begin{aligned}
&(m(t)\varphi_\alpha(x'))' + f(t, x, x') = 0, && t \neq \theta_i;\\
&\Delta(m(t)\varphi_\alpha(x')) + f_i(x, x') = 0, && t = \theta_i
\end{aligned} \tag{2.21}$$

where $f(t, u, v)$ and $f_i(u, v)$, $i \in \mathbb{N}$, are real valued continuous functions defined for all $t \geq t_0 \geq 0$ and for all for all $(u, v) \in \mathbb{R}^2$, m, φ_α, and $\{\theta_i\}$ are as previously defined.

Theorem 2.2.14. *Suppose that*

$$\begin{aligned} &(k(t)\varphi_\alpha(y'))' + q(t)\varphi_\alpha(y) = 0, && t \neq \theta_i;\\ &\Delta(k(t)\varphi_\alpha(y')) + q_i\,\varphi_\alpha(y) = 0, && t = \theta_i \end{aligned} \tag{2.22}$$

is oscillatory. If $k(t) \geq m(t)$ *and*

$$\varphi_\alpha(u)\, f(t,u,v) \geq q(t)\,\varphi_\alpha^2(u), \qquad \varphi_\alpha(u)\, f_i(u,v) \geq q_i\,\varphi_\alpha^2(u) \tag{2.23}$$

for all $t \geq t_0$ *and for all* $(u,v) \in \mathbb{R}^2$*, then every solution of (2.21) is also oscillatory.*

Proof. Let us assume on the contrary that there exists a nonoscillatory solution $w(t)$ of (2.21) while every solution of (2.22) is oscillatory. Consider the impulsive system

$$\begin{aligned} &(m(t)\varphi_\alpha(y'))' + p(t)\varphi_\alpha(y) = 0, && t \neq \theta_i;\\ &\Delta(m(t)\varphi_\alpha(y')) + p_i\,\varphi_\alpha(y) = 0, && t = \theta_i \end{aligned} \tag{2.24}$$

where

$$p(t) = \frac{f(t, w(t), w'(t))}{\varphi_\alpha(w(t))}, \qquad p_i = \frac{f_i(w(\theta_i), w'(\theta_i))}{\varphi_\alpha(w(\theta_i))}.$$

Clearly, $w(t)$ is also solution of (2.24). Let $x(t)$ be a solution of (2.22) such that $x(a) = x(b) = 0$ and $x(t) > 0$ for all $t \in (a,b)$, where $a \geq t_0$ is sufficiently large. Since $m(t) \leq k(t)$ by our hypothesis and $q(t) \leq p(t)$ for $t \geq t_0$, and $q_i \leq p_i$ for all $i \in \mathbb{N}$ for which $\theta_i \geq t_0$ by (2.23), we may apply Theorem 2.2.2 to deduce that $w(t)$ must have a zero in (a,b), which is a contradiction. □

If $\alpha = 1$ and $k(t) \equiv 1$ then the above result reduces to Theorem 13 in [3].

2.3 Non-Selfadjoint Equations

Consider the second order linear impulsive differential equations of the form

$$\begin{aligned} &l[x] = (k(t)x')' + r(t)x' + p(t)x = 0, && t \neq \theta_i;\\ &l_0[x] = \Delta(k(t)x') + p_i\,x = 0, && t = \theta_i \end{aligned} \tag{2.25}$$

and

$$
\begin{aligned}
&L[y] = (m(t)y')' + s(t)y' + q(t)y = 0, && t \neq \theta_i; \\
&L_0[y] = \Delta(m(t)y') + q_i\, y = 0, && t = \theta_i,
\end{aligned}
\tag{2.26}
$$

where $\{p_i\}$, $\{q_i\}$ and $\{\theta_i\}$ are real sequences with $\theta_1 > t_0$ for some fixed $t_0 \in \mathbb{R}$, and that k, m, r, s, p, $q \in \mathcal{PLC}(I)$ with $k(t) > 0$ and $m(t) > 0$ for all $t \in I \subset [t_0, \infty)$.

By a solution of (2.25) on an interval I we mean a nontrivial continuous function $x(t)$ defined on I such that $x' \in \mathcal{PLC}(I)$, $kx' \in \mathcal{PLC}^1(I)$, and $x(t)$ satisfies (2.25). It is not difficult to see that such solutions exist.

In this section, our purpose is to modify (2.3) and thereby extend the results in [30] to linear impulsive differential equations with damping and also generalize some of the results given in [3]. In particular, we establish a Wirtinger type inequality and a Leighton type comparison theorem together with some oscillation criteria for linear non-selfadjoint equations.

Let I_0 be a nondegenerate subinterval of I. In what follows we shall make use of the following condition:

$$
k(t) \neq m(t) \text{ whenever } r(t) \neq s(t), \qquad t \in I_0. \tag{H}
$$

It is well known that condition (H) is crucial in obtaining a Picone's formula in the case when impulses are absent. If (H) fails to hold then Wirtinger, Leighton, and Sturm-Picone type results require employing a so called "device of Picard". We will show how this is possible for impulsive differential equations as well.

Let (H) be satisfied. Suppose that x and y are continuous functions defined on I_0 such that $x', y' \in \mathcal{PLC}(I_0)$ and $kx', my' \in \mathcal{PLC}^1(I_0)$. These simply mean that x and y are in the domain of l, l_0 and L, L_0, respectively. If $y(t) \neq 0$ for any $t \in I_0$, then we may define

$$
w(t) = \frac{x(t)}{y(t)}\, [y(t)k(t)x'(t) - x(t)m(t)y'(t)] \quad \text{for } t \in I_0.
$$

For clarity we suppress the variable t. Clearly,

$$w' = (k-m)(x')^2 + (q-p)x^2 + m(x' - \frac{x}{y}\, y')^2 + x^2\, \frac{sy'}{y} - rxx'$$
$$+\frac{x}{y}\, \{yl[x] - xL[y]\}, \qquad t \neq \theta_i; \tag{2.27}$$

$$\Delta w = x\, \{l_0[x] - p_i x\} - \frac{x^2}{y}\, \{L_0[y] - q_i y\}, \qquad t = \theta_i. \tag{2.28}$$

In view of (2.25) and (2.26) it is not difficult to see, cf.[30], from (2.27) and (2.28) that

$$\begin{aligned} w' &= (k-m)(x')^2 + (q-p)x^2 + m(x' - \frac{xy'}{y})^2 - sx(x' - \frac{xy'}{y}) \\ &\quad + (s-r)xx' + \frac{x}{y}\, \{yl[x] - xL[y]\}, \quad t \neq \theta_i \\ &= \left\{q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right\}x^2 + (k-m)\left\{x' + \frac{(s-r)}{2(k-m)}\, x\right\}^2 \\ &\quad + \frac{m}{y^2}\left(x'y - xy' - \frac{s}{2m}\, xy\right)^2 + \frac{x}{y}\, \{yl[x] - xL[y]\}, \quad t \neq \theta_i \end{aligned} \tag{2.29}$$

and

$$\Delta w = (q_i - p_i)\, x^2 + \frac{x}{y}\, \{yl_0[x] - xL_0[y]\}, \quad t = \theta_i. \tag{2.30}$$

Employing the identity

$$w(\beta) - w(\alpha) = \int_\alpha^\beta w'(t)\, dt + \sum_{\alpha \le \theta_i < \beta} \Delta w(\theta_i),$$

we easily obtain the following Picone's formula.

Theorem 2.3.1 (Picone's formula). *Let* (H) *be satisfied. Suppose that* x *and* y *are continuous functions defined on* I_0 *such that* $x', y' \in \mathcal{PLC}(I_0)$ *and*

$kx', my' \in \mathcal{PLC}^1(I_0)$. *If* $y(t) \neq 0$ *for any* $t \in I_0$, *and* $[\alpha, \beta] \subseteq I_0$ *then*

$$
\begin{aligned}
\frac{x}{y}\,(ykx' - xmy')\Big|_{\alpha}^{\beta} &= \int_{\alpha}^{\beta} \Bigg\{\Big[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\Big]x^2 \\
&+(k-m)\Big\{x' + \frac{(s-r)}{2(k-m)}\,x\Big\}^2 + \frac{m}{y^2}\Big(x'y - xy' - \frac{s}{2m}\,xy\Big)^2 \qquad (2.31)\\
&+\frac{x}{y}\,\{yl[x] - xL[y]\}\Bigg\}\,dt + \sum_{\alpha \le \theta_i < \beta} \Big[(q_i - p_i)\,x^2 + \frac{x}{y}\,\{yl_0[x] - xL_0[y]\}\Big].
\end{aligned}
$$

In a similar manner we derive a Wirtinger type inequality.

Theorem 2.3.2 (Wirtinger type inequality). *If there exists a solution* x *of (2.25) such that* $x \neq 0$ *on* (a, b), *then*

$$
W[\eta] := \int_a^b \Big\{p\,\eta^2 - k\Big(\eta' - \frac{r}{2k}\,\eta\Big)^2\Big\}dt + \sum_{a \le \theta_i < b} p_i\,\eta^2 \le 0, \qquad \eta \in \Omega_{rk}, \quad (2.32)
$$

where

$$
\Omega_{rk} = \big\{\eta \in C[a,b] : r\eta' \in \mathcal{PLC}[a,b],\ k\eta' \in \mathcal{PLC}^1[a,b],\ \eta(a) = \eta(b) = 0\big\}.
$$

Proof. Let x be a solution of (2.25) such that $x(t) \neq 0$ for any $t \in (a, b)$. Setting $m \equiv k$, $q \equiv p$, $s \equiv r$, and $q_i = p_i$, replacing x by η and y by x in (2.27) and (2.28) we see that

$$
\begin{aligned}
w' &= k(\eta' - \frac{\eta}{x}\,x')^2 + \eta^2\,\frac{rx'}{x} - r\eta\eta' + \eta l[\eta], && t \neq \theta_i \\
&= \eta(k\eta')' + \big(p - \frac{r^2}{4k}\big)\eta^2 + r\eta\eta' + \frac{k}{x^2}(\eta' x - \eta x' - \frac{r\eta x}{2k})^2, && t \neq \theta_i \quad (2.33)
\end{aligned}
$$

and

$$
\Delta w = \eta\{\Delta(k\eta') + p_i\eta\}, \qquad t = \theta_i, \qquad (2.34)
$$

It is clear that if $x(a^+) \neq 0$ and $x(b^-) \neq 0$, then the last term in (2.33) is integrable over (a, b). If $x(a^+) = 0$, then since $x'(a^+) \neq 0$ (otherwise, we have only the trivial solution) it follows that

$$
\lim_{t \to a^+} \Big\{\frac{\eta'(t)x(t) - \eta(t)x'(t)}{x(t)} - \frac{r(t)\eta(t)}{2k(t)}\Big\} = \eta'(a^+) - \eta'(a^+) - \frac{r(a^+)\eta(a^+)}{2k(a^+)} = 0.
$$

The same argument applies if $x(b^-) = 0$. Thus, the last term in (2.33) is integrable on (a, b).

We now claim that $w(a^+) = w(b^-) = 0$. Let us consider $w(a^+) = 0$. The case $w(b^-) = 0$ is similar. If $x(a^+) \neq 0$, then we certainly have $w(a^+) = 0$. In case $x(a^+) = 0$, it follows from

$$\lim_{t \to a^+} \frac{\eta(t)}{x(t)} = \lim_{t \to a^+} \frac{\eta'(t)}{x'(t)} < \infty$$

that

$$w(a^+) = \lim_{t \to a^+} \frac{\eta(t)}{x(t)} \Big\{ k(t)\eta'(t)x(t) - k(t)\eta(t)x'(t) \Big\} = 0.$$

Integrating (2.33) over (a, b) and using (2.34) we see that

$$\int_a^b \eta(k\eta')'\,dt + \int_a^b \Big\{ \big(p - \frac{r^2}{4k}\big)\,\eta^2 + r\eta\eta' \Big\} dt$$
$$+ \int_a^b \frac{k}{x^2} \Big\{ \eta' x - \eta x' - \frac{r}{2k}\,\eta\, x \Big\}^2 dt + \sum_{a \le \theta_i < b} \eta\{\Delta(k\eta') + p_i\eta\} = 0.$$

Applying the integration by parts formula to the first integral leads to

$$W[\eta] = -\int_a^b \frac{k}{x^2} \Big\{ \eta' x - \eta x' - \frac{r}{2k}\,\eta\, x \Big\}^2 dt \le 0.$$

□

As a corollary we have the following criterion on the existence of a zero of a solution of (2.25). This result may be considered as an extension of Lemma 1.3 in [70].

Corollary 2.3.3. *If there exists an $\eta \in \Omega_{rk}$ such that $W[\eta] > 0$ then every solution x of (2.25) has a zero in (a, b).*

As an immediate consequence of Corollary 2.3.3, we have the following oscillation result.

Corollary 2.3.4. *Suppose for any given $t_1 \geq t_0$ there exists an interval $(a,b) \subset [t_1,\infty)$ and a function $\eta \in \Omega_{rk}$ for which $W[\eta] > 0$, then (2.25) is oscillatory.*

Next, we provide a Leighton type comparison result between nontrivial solutions of (2.25) and (2.26), which may be considered as an extension of the classical comparison theorem of Leighton [38, Corollary 1].

Theorem 2.3.5 (Leighton type comparison). *Suppose that there exists a solution $x \in \Omega_{rk}$ of (2.25). If* (H) *is satisfied with $(a,b) \subset I_0$ and*

$$L[x] := \int_a^b \left\{ \left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right] x^2 + (k-m) \left[x' + \frac{s-r}{2(k-m)}\, x \right]^2 \right\} dt + \sum_{a \leq \theta_i < b} (q_i - p_i)\, x^2 > 0, \tag{2.35}$$

then every solution y of (2.26) must have at least one zero in (a,b).

Proof. Let $\alpha = a+\epsilon$ and $\beta = b-\epsilon \in I_0$. Since x and y are solutions of (2.25) and (2.26) respectively, we have $l[x] \equiv l_0[x] \equiv L[y] \equiv L_0[y] \equiv 0$. Employing Picone's formula (2.31) we see that

$$\frac{x}{y}\,(ykx' - xmy')\Big|_{a+\epsilon}^{b-\epsilon} = \int_{a+\epsilon}^{b-\epsilon} \left[\left\{ q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right\} x^2 + (k-m) \left\{ x' + \frac{(s-r)}{2(k-m)}\, x \right\}^2 + \frac{m}{y^2} \left\{ x'y - xy' - \frac{s}{2m}\, xy \right\}^2 \right] dt + \sum_{a+\epsilon \leq \theta_i < b-\epsilon} (q_i - p_i)\, x^2. \tag{2.36}$$

As in the proof of Theorem 2.3.2, the functions under integral sign are all integrable and regardless of the values of $y(a)$ or $y(b)$, left-hand side of (2.36) tends to zero as $\epsilon \to 0^+$. Clearly (2.36) results in

$$L[x] \leq 0,$$

a contradiction to (2.35). □

Corollary 2.3.6 (Sturm-Picone type comparison). *Let x be a solution of (2.25) having two consecutive zeros $a, b \in I_0$. Suppose* (H) *holds, and*

$$k \geq m, \tag{2.37}$$
$$q \geq p + \frac{(s-r)^2}{4(k-m)} + \frac{s^2}{4m} \tag{2.38}$$

for all $t \in [a, b]$, and

$$q_i \geq p_i \tag{2.39}$$

for all $i \in \mathbb{N}$ for which $\theta_i \in [a, b]$.

If either (2.37) or (2.38) is strict in a subinterval of $[a, b]$ or (2.39) is strict for some $i \in \mathbb{N}$, then every solution y of (2.26) must have at least one zero on (a, b).

We note that, if there is no impulse then we recover Theorem 2.1 in [30].

Corollary 2.3.7. *Suppose that conditions (2.37)-(2.38) are satisfied for all $t \in [t_*, \infty)$ for some integer $t_* \geq t_0$, and that (2.39) is satisfied for all $i \in \mathbb{N}$ for which $\theta_i \geq t_*$. If one of the inequalities (2.37)–(2.39) is strict then (2.26) is oscillatory whenever any solution x of (2.25) is oscillatory.*

As a consequence of Theorem 2.3.5 and Corollary 2.3.6, we have the following oscillation result.

Corollary 2.3.8. *Suppose for any given $t_1 \geq t_0$ there exists an interval $(a, b) \subset [t_1, \infty)$ for which either the conditions of Theorem 2.3.5 or Corollary 2.3.6 are satisfied, then (2.26) is oscillatory.*

If (H) does not hold, we introduce a setting which is based on a device of Picard [58] (see also [30, p. 12]) and leads to different versions of Corollary 2.3.6. Indeed, for any $h \in \mathcal{PLC}^1(I)$ we have

$$\frac{d}{dt}\,(x^2 h) = 2xx'h + x^2 h', \quad t \neq \theta_i.$$

Let

$$v := \frac{x}{y}\,(ykx' - xmy') + x^2 h, \quad t \in I.$$

It follows that

$$\begin{aligned}
v' &= \left\{q - p + h' - \frac{(s-r+2h)^2}{4(k-m)} - \frac{s^2}{4m}\right\}x^2 + (k-m)\left\{x' + \frac{s-r+2h}{2(k-m)}\,x\right\}^2 \\
&\quad + \frac{m}{y^2}\left\{x'y - xy' - \frac{s}{2m}\,xy\right\}^2, \qquad t \neq \theta_i; \\
\Delta v &= (q_i - p_i)\,x^2 + x^2\,\Delta h, \qquad t = \theta_i.
\end{aligned}$$

Assuming that $r, s \in \mathcal{PLC}^1(I)$, the choice of $h = (r-s)/2$ yields

$$\begin{aligned}
v' &= (k-m)(x')^2 + \left\{q - p - \frac{s'-r'}{2} - \frac{s^2}{4m}\right\}x^2 \\
&\quad + \frac{m}{y^2}\left\{x'y - xy' - \frac{s}{2m}\,xy\right\}^2, && t \neq \theta_i; \\
\Delta v &= \left\{q_i - p_i - \frac{1}{2}\,\big(\Delta s - \Delta r\big)\right\}\,x^2, && t = \theta_i.
\end{aligned}$$

Then, we have the following result.

Theorem 2.3.9 (A Device of Picard). *Let $r, s \in \mathcal{PLC}^1(I)$ and x be a solution of (2.25) having two consecutive zeros a and b in I. Suppose that*

$$k \geq m, \tag{2.40}$$

$$q \geq p + \frac{1}{2}\,(s' - r') + \frac{s^2}{4m} \tag{2.41}$$

are satisfied for all $t \in [a, b]$, and that

$$q_i \geq p_i + \frac{1}{2}\,\big(\Delta s - \Delta r\big) \tag{2.42}$$

for all $i \in \mathbb{N}$ for which $\theta_i \in [a, b]$.

If either (2.40) or (2.41) is strict in a subinterval of $[a, b]$ or (2.42) is strict for some i, then any solution y of (2.26) must have at least one zero in (a, b).

Corollary 2.3.10. *Suppose that (2.40)-(2.41) are satisfied for all* $t \in [t_*, \infty)$ *for some integer* $t_* \geq t_0$*, and that (2.42) is satisfied for all* $i \in \mathbb{N}$ *for which* $\theta_i \geq t_*$*. If* $r, s \in \mathcal{PLC}^1[t_*, \infty)$ *and one of the inequalities (2.40)-(2.42) is strict, then (2.26) is oscillatory whenever any solution* x *of (2.25) is oscillatory.*

As a consequence of Theorem 2.3.9, we have the following Leighton type comparison result which is analogous to Theorem 2.3.5.

Theorem 2.3.11 (Leighton type comparison). *Let* $r, s \in \mathcal{PLC}^1[a, b]$*. If there exists a solution* $x \in \Omega_{rk}$ *of (2.25) such that*

$$\mathcal{L}[x] := \int_a^b \left\{ \left(k - m\right) \left(x'\right)^2 + \left[q - p - \frac{1}{2}\left(s' - r'\right) - \frac{s^2}{4m}\right] x^2 \right\} dt + \sum_{a \leq \theta_i < b} \left\{ q_i - p_i - \frac{1}{2}\left(\Delta s - \Delta r\right) \right\} x^2 > 0,$$

then every solution y *of (2.26) must have at least one zero in* (a, b)*.*

As a consequence of Theorem 2.3.9 and Theorem 2.3.11, we have the following oscillation result.

Corollary 2.3.12. *Suppose for any given* $t_1 \geq t_0$ *there exists an interval* $(a, b) \subset [t_1, \infty)$ *for which either the conditions of Theorem 2.3.9 or Theorem 2.3.11 are satisfied, then (2.26) is oscillatory.*

Moreover, it is possible to obtain results for (2.26) analogous to Theorem 2.3.2 and Corollary 2.3.3.

Theorem 2.3.13 (Wirtinger type inequality). *If there exists a solution* y *of (2.26) such that* $y \neq 0$ *on* (a, b)*, then for* $s \in \mathcal{PLC}^1[a, b]$ *and for all* $\eta \in \Omega_{sm}$

$$\mathcal{W}[\eta] := \int_a^b \left\{ \left(q - \frac{s^2}{2m} - \frac{s'}{2}\right)\eta^2 - m(\eta')^2 \right\} dt + \sum_{a \leq \theta_i < b} \left(q_i - \frac{1}{2}\,\Delta s\right) \eta^2 \leq 0.$$

Corollary 2.3.14. *If there exists an* $\eta \in \Omega_{sm}$ *with* $s \in \mathcal{PLC}^1[a,b]$ *such that* $\mathcal{W}[\eta] > 0$ *then every solution* y *of (2.26) must have at least one zero in* (a,b)*.*

As an immediate consequence of Corollary 2.3.14, we have the following oscillation result.

Corollary 2.3.15. *Suppose for any given* $t_1 \geq t_0$ *there exists an interval* $(a,b) \subset [t_1, \infty)$ *and a function* $\eta \in \Omega_{sm}$ *with* $s \in \mathcal{PLC}^1(I)$ *for which* $\mathcal{W}[\eta] > 0$*, then (2.26) is oscillatory.*

Example 2.3.16. *Consider*

$$\begin{aligned} &x'' - 2cx' + c^2 x = 0, \qquad && t \neq i, \quad (i \in \mathbb{N}) \\ &\Delta x' + 2(1+\coth c)\, x = 0, && t = i \end{aligned} \tag{2.43}$$

where c *is a fixed real number. It is easy to verify that* $x(t) = x_i(t)$*, where*

$$x_i(t) = (-1)^i \, e^{c\,(t-i)+1} \left\{ (e^c+1)(i-t) - 1 \right\}, \qquad t \in (i-1, i], \quad (i \in \mathbb{N})$$

is a continuous solution of (2.43). Clearly, this solution is oscillatory with zeros at $t_i = i - (e^c+1)^{-1}$*,* $i \in \mathbb{N}$*.*

Note that if the impulse conditions are dropped then the equation has no oscillatory solution.

Applying Corollary 2.3.7 and Corollary 2.3.10 we get the following oscillation criteria (a) and (b), respectively.

(a) If there exists an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &k(t) \leq 1 \\ &k(t) < 1 \quad \text{whenever} \quad r(t) \neq -2c, \\ &p(t) \geq c^2 + \frac{\{r(t)+2c\}^2}{4\{1-k(t)\}} + \frac{r^2(t)}{4k(t)} \\ &p_i \geq 2(1+\coth c) \end{aligned}$$

for all $t \geq n_0$, and for all $i \geq n_0$, then (2.25) with $\theta_i = i$ is oscillatory.

(b) If there exists an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &k(t) \leq 1 \\ &p(t) \geq c^2 + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)} \\ &p_i \geq 2(1 + \coth c) + \frac{1}{2}\,\Delta r(i) \end{aligned}$$

for all $t \geq n_0$, and for all $i \geq n_0$, then (2.25) with $\theta_i = i$ is oscillatory.

The lemma below, cf. [3, Lemma 1.] can be proved directly.

Lemma 2.3.17. *Let ψ be a positive and continuous function for $t \geq a$ with $\psi' \in \mathcal{PLC}^1[a,\infty)$, where a is a fixed real number. Suppose that $k \in \mathcal{PLC}^2[a,\infty)$ and $r \in \mathcal{PLC}^1[a,\infty)$. Then the function*

$$x(t) = \frac{1}{\sqrt{k(t)\psi(t)}}\ \exp\left(-\frac{1}{2}\int_a^t \frac{r(s)}{k(s)}ds\right)\ \sin\left(\int_a^t \psi(s)ds\right), \quad t \geq a \quad (2.44)$$

is a solution of (2.25) where

$$\begin{aligned} p(t) &= \frac{1}{2}\left\{k''(t) + r'(t) + r(t)\,\frac{k'(t)}{k(t)} + \frac{r^2(t)}{k(t)}\right\} - \frac{\{k'(t)+r(t)\}^2}{4k(t)} \\ &\quad + k(t)\left\{\frac{\psi''(t)}{2\psi(t)} + \psi^2(t) - \frac{3}{4}\left(\frac{\psi'(t)}{\psi(t)}\right)^2\right\}, \\ p_i &= \frac{1}{2\psi(\theta_i)}\left[\psi(\theta_i)\Delta k'(\theta_i) + k(\theta_i)\Delta\psi'(\theta_i)\right] + \frac{1}{2}\,\Delta r(\theta_i), \quad \theta_i > a. \end{aligned}$$

It is obvious that if $\displaystyle\int_a^\infty \psi(t)\,dt = \infty$ then $x(t)$ defined in (2.44) is oscillatory.

Clearly, Lemma 2.3.17 can be used to derive general oscillation criteria for (2.25). We prefer, however, to establish more concrete oscillation criteria by making use of the following particular cases of Lemma 2.3.17.

Example 2.3.18. *Let $k(t) = t^2/4$, $r(t) = -t/4$ and $\psi(t) = \dfrac{2i-t}{i(i+1)}$, $i-1 < t \leq i$, $i \in \mathbb{N}$. In view of Lemma 2.3.17, we see that $x(t) = x_i(t)$, where*

$$x_i(t) = \frac{2}{\sqrt{\beta\ t\ \psi(t)}}\ \sin\left(\int_1^t \psi(s)ds\right), \quad t \in (i-1, i], \quad (i \in \mathbb{N})$$

is an oscillatory solution of

$$(t^2x')' - tx' + \left\{t^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right] - \frac{1}{4}\right\} x = 0, \; t \in (i-1, i),$$
$$\Delta(t^2x') + \frac{i}{i+2}\, x = 0, \qquad t = i, \quad (i \in \mathbb{N}).$$

Example 2.3.19. *Let* $k(t) = \xi_i^2(t+i)^2$, $r(t) = -\xi_i^2(t+i)$ *and* $\psi(t) = \dfrac{2i-t}{i(i+1)}$, $i-1 < t \le i$, $i \in \mathbb{N}$, *where*

$$\xi_i = \frac{2^{2i-2}(i-1)!^2}{(2i-1)!} \quad \textit{for} \quad i \in \mathbb{N}.$$

In view of Lemma 2.3.17, we see that $x(t) = x_i(t)$, *where*

$$x_i(t) = \frac{1}{\xi_i(t+i)\sqrt{\psi(t)}}\, \exp\left(\frac{1}{2}\int_\gamma^t \frac{ds}{s+i}\right) \sin\left(\int_1^t \psi(s)ds\right), \quad t \in (i-1, i]$$

is an oscillatory solution of

$$(\xi_i^2(t+i)^2x')' - \xi_i^2(t+i)x' + \left\{(t+i)^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right] - \frac{1}{4}\right\}\xi_i^2\, x = 0, \qquad t \in (i-1, i),$$
$$\Delta(\xi_i^2(t+i)^2x') + \frac{i(7i+2)}{(i+2)(2i+1)}\, \xi_i^2\, x = 0, \qquad t = i.$$

In view of the above examples, by applying Corollary 2.3.7 and Corollary 2.3.10 we easily see that (2.25) with $\theta_i = i$ is oscillatory if there exists an $n_0 \in \mathbb{N}$ such that, for each fixed $i \ge n_0$ and for all $t \in (i-1, i]$, any one of the following conditions (a)–(d) holds:

(a) $k(t) \le t^2$; $\quad k(t) < t^2$ whenever $r(t) \ne -t$;
$$p(t) \ge t^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right] - \frac{1}{4} + \frac{\{r(t)+t\}^2}{4\{t^2-k(t)\}} + \frac{r^2(t)}{4k(t)};$$
$$p_i \ge \frac{i}{i+2}.$$

(b) $k(t) \le t^2;$

$$p(t) \ge t^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right] + \frac{1}{4} + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)};$$

$$p_i \ge \frac{i}{i+2} + \frac{1}{2}\,\Delta r(i).$$

(c) $k(t) \le \xi_i^2(t+i)^2;\quad k(t) < \xi_i^2(t+i)^2$ whenever $r(t) \ne -\xi_i^2(t+i);$

$$p(t) \ge \xi_i^2(t+i)^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right] - \frac{1}{4}\,\xi_i^2 + \frac{\{r(t)+\xi_i^2(t+i)\}^2}{4\{\xi_i^2(t+i)^2 - k(t)\}} + \frac{r^2(t)}{4k(t)};$$

$$p_i \ge \frac{i(7i+2)}{(i+2)(2i+1)}\,\xi_i^2.$$

(d) $k(t) \le \xi_i^2(t+i)^2;$

$$p(t) \ge \xi_i^2(t+i)^2\left[\left(\frac{2i-t}{i(i+1)}\right)^2 - \frac{3}{4}\left(\frac{1}{2i-t}\right)^2\right] + \frac{1}{4}\,\xi_i^2 + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)};$$

$$p_i \ge \frac{6i^2}{(i+2)(2i+1)}\,\xi_i^2 + \frac{1}{2}\,\Delta r(i).$$

Consider the nonlinear impulsive equations of the form

$$\begin{aligned} &(m(t)z')' + s(t)z' + f(t,z,z') = 0, && t \ne \theta_i; \\ &\Delta(m(t)z') + f_i(z,z') = 0, && t = \theta_i \end{aligned} \tag{2.45}$$

where $f(t,u,v)$ and $f_i(u,v)$, $i \in \mathbb{N}$, are real-valued continuous functions defined for all $t \ge t_0 \ge 0$ and for all $(u,v) \in \mathbb{R}^2$, m, s, and $\{\theta_i\}$ are as previously defined. It is tacitly assumed that there exist solutions of (2.45) which are continuous and defined for all $t \ge t_0$ satisfying $\sup\{|z(t)|,\ t \ge T\} > 0$ for all $T \ge t_0$. This last condition simply means that the solutions are nontrivial in the neighborhood of ∞.

The following oscillation criteria can be easily established, cf [3].

Theorem 2.3.20. *Suppose that* (H) *holds,* $k(t) \ge m(t)$, *and*

$$\begin{aligned} u\,f(t,u,v) &\ge \left\{p(t) + \frac{[s(t)-r(t)]^2}{4[k(t)-m(t)]} + \frac{s^2(t)}{4m(t)}\right\}u^2, \\ u\,f_i(u,v) &\ge p_i\,u^2 \end{aligned} \tag{2.46}$$

for all $t \geq t_0$ and for all $(u,v) \in \mathbb{R}^2$. If (2.25) is oscillatory, then so is (2.45).

Proof. Let us assume on the contrary that there exists a nonoscillatory solution $w(t)$ of (2.45) while every solution of (2.25) is oscillatory. Consider the linear impulsive system

$$\begin{aligned} &(m(t)z')' + s(t)z' + q(t)z = 0, \qquad && t \neq \theta_i;\\ &\Delta(m(t)z') + q_i\, z = 0, && t = \theta_i \end{aligned} \tag{2.47}$$

where

$$q(t) = \frac{f(t,w(t),w'(t))}{w(t)}, \qquad q_i = \frac{f_i(w(\theta_i),w'(\theta_i))}{w(\theta_i)}.$$

Clearly, $w(t)$ is also solution of (2.47). Let $x(t)$ be an oscillatory solution of (2.25) such that $x(a) = x(b) = 0$ and $x(t) > 0$ for all $t \in (a,b)$. Since $m(t) \leq k(t)$ by our hypothesis and

$$q(t) \geq p(t) + \frac{[s(t)-r(t)]^2}{4[k(t)-m(t)]} + \frac{s^2(t)}{4m(t)}$$

for $t \geq a$, and $q_i \geq p_i$ for all $i \in \mathbb{N}$ for which $\theta_i \geq a$ by (2.46), we may apply Corollary 2.3.6 to deduce that $w(t)$ must have a zero in (a,b), which is a contradiction. □

Alternatively, if (H) fails but $r, s \in \mathcal{PLC}^1[t_0,\infty)$, then as an application of Theorem 2.3.9 we have the following result.

Theorem 2.3.21. *Suppose that $r, s \in \mathcal{PLC}^1[t_0,\infty)$, $k(t) \geq m(t)$, and*

$$\begin{aligned} u\, f(t,u,v) &\geq \left\{ p(t) + \frac{1}{2}\,[s'(t) - r'(t)] + \frac{s^2(t)}{4m(t)} \right\} u^2,\\ u\, f_i(u,v) &\geq \left\{ p_i + \frac{1}{2}\,[\Delta s(\theta_i) - \Delta r(\theta_i)] \right\} u^2 \end{aligned}$$

for all $t \geq t_0$ and for all $(u,v) \in \mathbb{R}^2$. If (2.25) is oscillatory, then so is (2.45).

2.4 Super Half-Linear Equations

Consider the forced second order super half-linear impulsive differential equation of the form

$$\begin{aligned} &(m(t)\varphi_\alpha(y'))' + s(t)\varphi_\alpha(y') + q(t)\varphi_\beta(y) = f(t), && t \neq \theta_i, \\ &\Delta(m(t)\varphi_\alpha(y')) + q_i\,\varphi_\beta(y) = f_i, && t = \theta_i, \end{aligned} \tag{2.48}$$

where α and β are real constants with $\beta \geq \alpha > 0$. Further we assume that

(i) $\{q_i\}$, $\{f_i\}$ and $\{\theta_i\}$ are real sequences with $\theta_1 > t_0$ for some fixed $t_0 \in \mathbb{R}$;

(ii) $m, s, q, f \in \mathcal{PLC}[t_0, \infty)$; $m(t) > 0$.

By a solution of (2.48), we mean a continuous function $y(t)$ defined on $[t_0, \infty)$ such that $y, m\varphi_\alpha(y') \in \mathcal{PLC}^1[\mathrm{t}_0, \infty)$ and (2.48) is fulfilled for all $t \geq t_0$. Existence of such solutions can be proved in a similar manner performed for equations without impulse effect [11].

In [24], some oscillation criteria about equation (2.8) are given and the results improve and extend those in [23, 55]. In 2004, W. Tong Li [45] obtained several interval oscillation criteria by use of Riccati techniques for the equation

$$(m(t)\varphi_\alpha(y'))' + s(t)\varphi_\alpha(y') + q(t)\varphi_\beta(y) = f(t), \qquad \beta > \alpha > 0. \tag{2.49}$$

The case $\alpha = 1$, $\beta > 1$ and $s(t) \equiv 0$, has been studied by Nasr [55] by using the technique duo to El-Sayed [10]. Recently, Jaroš, Kusano and Yoshida [23] studied the same equation by using Picone's formula which improves the results of Nasr [55].

Theorem 2.4.1. *Suppose that for any given $t_* \geq t_0$, there exist intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $t_* \leq a_1 < b_1 \leq a_2 < b_2$, such that*

(A) *$q(t) \geq 0$ for all $t \in \{I_1 \cup I_2\} \setminus \{\theta_i\}$ and $q_i \geq 0$ for all $i \in \mathbb{N}$ for which $\theta_i \in I_1 \cup I_2$;*

(B) $f(t)\begin{cases} \le 0, & t \in I_1 \setminus \{\theta_i\} \\ \ge 0, & t \in I_2 \setminus \{\theta_i\} \end{cases}$; $\quad f_i \begin{cases} \le 0, & \theta_i \in I_1 \\ \ge 0, & \theta_i \in I_2 \end{cases}$ *for all* $\; i \in \mathbb{N}$.

If there exists $\eta \in \mathcal{D}(a_j, b_j) = \{z \in C^1(I_j) : z(t) \not\equiv 0, z(a_j) = z(b_j) = 0\}$, $j = 1, 2$, *such that*

$$\begin{aligned} W_{\alpha\beta}[\eta; I_j] &:= \int_{a_j}^{b_j} \Big\{ \widetilde{q}\,|\eta|^{\alpha+1} - m\Big|\eta' - \frac{s}{(\alpha+1)m}\,\eta\Big|^{\alpha+1} \Big\} dt \\ &\quad + \sum_{a_j \le \theta_i < b_j} \widetilde{q}_i\,|\eta|^{\alpha+1} \ge 0, \end{aligned} \tag{2.50}$$

for $j = 1, 2$, *where*

$$\begin{aligned} \widetilde{q}(t) &= \beta\,\alpha^{-\alpha/\beta}(\beta-\alpha)^{(\alpha-\beta)/\beta}\,[q(t)]^{\alpha/\beta}\,|f(t)|^{(\beta-\alpha)/\beta}, \\ \widetilde{q}_i &= \beta\,\alpha^{-\alpha/\beta}(\beta-\alpha)^{(\alpha-\beta)/\beta}\,[q_i]^{\alpha/\beta}\,|f_i|^{(\beta-\alpha)/\beta} \end{aligned}$$

with the convention that $0^0 = 1$, *then all solutions of (2.48) are oscillatory.*

Proof. Suppose that y is a nonoscillatory solution of (2.48) which is eventually positive, say $y(t) > 0$ when $t \in [t^*, \infty)$ for some $t^* \ge t_*$ depending on the solution y. By assumption, we can choose $a_1, b_1 \ge t^*$ so that $f(t) \le 0$ on $I_1 \setminus \{\theta_i\}$ and $f_i \le 0$ for $\theta_i \in I_1$ with $a_1 < b_1$.

Define

$$\nu := -\frac{m\,\varphi_\alpha(y')}{\varphi_\alpha(y)}\,|\eta|^{\alpha+1} \quad \text{for} \quad t \in I_1$$

where the dependence of t of the functions are suppressed. It follows from equation (2.48) that $\nu(t)$ satisfies the pair of identities

$$\begin{aligned} \nu' &= \alpha m\Big|\frac{\eta y'}{y}\Big|^{\alpha+1} - (\alpha+1)m\Big(\eta' - \frac{s}{(\alpha+1)m}\,\eta\Big)\varphi_\alpha\Big(\frac{\eta y'}{y}\Big) \\ &\quad + \Big[q|y|^{\beta-\alpha} - \frac{f}{\varphi_\alpha(y)}\Big]\,|\eta|^{\alpha+1}, \quad t \ne \theta_i; \\ &= m\,\Phi_\alpha\Big(\eta' - \frac{s}{(\alpha+1)m}\,\eta, \frac{\eta y'}{y}\Big) + \Big[q|y|^{\beta-\alpha} + \frac{|f|}{|y|^\alpha}\Big]\,|\eta|^{\alpha+1} \\ &\quad - m\Big|\eta' - \frac{s}{(\alpha+1)m}\,\eta\Big|^{\alpha+1}, \qquad t \ne \theta_i; \end{aligned} \tag{2.51}$$

and

$$\Delta\nu = -\frac{|\eta|^{\alpha+1}}{\varphi_\alpha(y)}\,\Delta m\varphi_\alpha(y') = \left[q_i|y|^{\beta-\alpha} - \frac{f_i}{\varphi_\alpha(y)}\right]|\eta|^{\alpha+1}, \qquad t=\theta_i;$$
$$= \left[q_i|y|^{\beta-\alpha} + \frac{|f_i|}{|y|^\alpha}\right]|\eta|^{\alpha+1}, \qquad t=\theta_i. \qquad (2.52)$$

Define the function $G(u):\mathbb{R}^+\to\mathbb{R}^+$

$$G(u) := \lambda_1\, u^{\beta-\alpha} + \frac{\lambda_2}{u^\alpha}, \qquad \lambda_{1,2}\ge 0, \quad \beta\ge\alpha>0,$$

and observe that

$$\min_{u>0} G(u) = \beta\,\alpha^{-\alpha/\beta}(\beta-\alpha)^{(\alpha-\beta)/\beta}\,\lambda_1^{\alpha/\beta}\,\lambda_2^{(\beta-\alpha)/\beta}. \qquad (2.53)$$

Taking the nonnegativity of $q(t)$ and q_i into account, and considering the expressions in brackets on the right-hand sides of (2.51) and (2.52) as the functions of $y(t)$ and $y(\theta_i)$ respectively, (2.53) yields

$$\nu' \ \ge\ \widetilde{q}\,|\eta|^{\alpha+1} - m\Big|\eta' - \frac{s}{(\alpha+1)m}\,\eta\Big|^{\alpha+1}$$
$$+m\,\Phi_\alpha\left(\eta' - \frac{s}{(\alpha+1)m}\,\eta, \frac{\eta y'}{y}\right), \qquad t\ne\theta_i; \qquad (2.54)$$
$$\Delta\nu \ \ge\ \widetilde{q}_i\,|\eta|^{\alpha+1}, \qquad t=\theta_i. \qquad (2.55)$$

Integrating (2.54) over I_1 and using (2.55), we see that

$$0 \ge W_{\alpha\beta}[\eta; I_1] + \int_{a_1}^{b_1} m\,\Phi_\alpha\left(\eta' - \frac{s}{(\alpha+1)m}\,\eta, \frac{\eta y'}{y}\right)dt. \qquad (2.56)$$

Since $W_{\alpha\beta}[\eta; I_1]\ge 0$, (2.56) yields

$$\eta' y - \eta y' - \frac{s}{(\alpha+1)m}\,\eta y = 0 \quad \text{on} \quad I_1.$$

Since $y(t)>0$, it follows that

$$\eta = C_0\, y \exp\left(\frac{1}{\alpha+1}\int^t \frac{s}{m}\,d\tau\right) \quad \text{on} \quad I_1,$$

for some constant C_0. Since $\eta \in \mathcal{D}(a_1, b_1)$ and $\eta \not\equiv 0$, this is incompatible to the fact that $y(t) > 0$ on I_1.

When $y(t)$ is eventually negative, we use $\eta \in \mathcal{D}(a_2, b_2)$ and $f(t) \geq 0$ on $I_2 \setminus \{\theta_i\}$ and $f_i \geq 0$ for $\theta_i \in I_2$ to reach a similar contradiction. This contradiction proves that $y(t)$ must be oscillatory. The proof is complete. $\square$

Note that, if $s(t) \equiv 0$ and $q_i = f_i = 0$ with $\alpha = \beta > 0$, then we recover the results in [46].

Next, we prove the following result.

Theorem 2.4.2. *Let $x(t)$ be an oscillatory solution of (2.11) with zeros at $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$. Suppose that for any given $t_* \geq t_0$, there exist intervals $I_1 = [t_{n_1}, t_{m_1}]$, $I_2 = [t_{n_2}, t_{m_2}] \subset [t_*, \infty)$ on which* (A)-(B) *hold.*

If

$$\int_{t_{n_j}}^{t_{m_j}} \left\{(k-m)|x'|^{\alpha+1} + (\widetilde{q} - p)\,|x|^{\alpha+1}\right\}dt + \sum_{t_{n_j} \leq \theta_i < t_{m_j}} \left(\widetilde{q}_i - p_i\right)|x|^{\alpha+1} > 0, \qquad (2.57)$$

for $j = 1, 2$, then all solutions of (2.48) with $s(t) \equiv 0$ are oscillatory.

Proof. Suppose that y is a nonoscillatory solution of (2.48) which is eventually positive, say $y(t) > 0$ when $t \in [t^*, \infty)$ for some $t^* \geq t_*$ depending on the solution y. By assumption, $I_1 \subset [t^*, \infty)$ so that $f(t) \leq 0$ on $I_1 \setminus \{\theta_i\}$ and $f_i \leq 0$ for $\theta_i \in I_1$.

Define

$$w(t) := \frac{x}{\varphi_\alpha(y)}\left[\varphi_\alpha(y)\, k\, \varphi_\alpha(x') - \varphi_\alpha(x)\, m\, \varphi_\alpha(y')\right] \quad \text{for} \quad t \in I_1.$$

For abbreviation we secrete the variable t. Clearly

$$\begin{aligned} w' &= (k-m)|x'|^{\alpha+1} + \left[q|y|^{\beta-\alpha} + \frac{|f|}{|y|^\alpha}\right]|x|^{\alpha+1} - p\,|x|^{\alpha+1} \\ &+ m\,\Phi_\alpha(x', xy'/y), \qquad t \neq \theta_i; \qquad (2.58) \\ \Delta w &= \left[q_i|y|^{\beta-\alpha} + \frac{|f_i|}{|y|^\alpha}\right]|x|^\alpha - p_i\,|x|^{\alpha+1}, \quad t = \theta_i. \qquad (2.59) \end{aligned}$$

In the view of (2.53), it is not difficult to see from (2.58) and (2.59) that

$$w' \geq (k-m)|x'|^{\alpha+1} + (\widetilde{q}-p)\,|x|^{\alpha+1} + m\,\Phi_\alpha(x', xy'/y), \qquad t \neq \theta_i; \quad (2.60)$$

and

$$\Delta w \geq (\widetilde{q}_i - p_i)\,|x|^{\alpha+1}, \qquad t = \theta_i. \quad (2.61)$$

Integrating (2.60) over I_1 and using (2.61) and (2.57), we get

$$\int_{t_{n_1}}^{t_{m_1}} m\,\Phi_\alpha(x', xy'/y)\,dt \leq 0 \quad (2.62)$$

which yields $\Phi_\alpha(x', xy'/y) = 0$ on I_1. Since $y(t) > 0$, it follows that $x = C_1\,y$ on I_1, for some constant C_1. This is incompatible with the fact that $y(t) > 0$ on I_1.

When $y(t)$ is eventually negative, we choose the interval $I_2 \subset [T, \infty)$ for some $T \geq t_*$ so that $f(t) \geq 0$ on $I_2 \setminus \{\theta_i\}$ and $f_i \geq 0$ for $\theta_i \in I_2$ to reach a similar contradiction. This contradiction proves that $y(t)$ must be oscillatory. The proof is complete. □

Note that if there is no impulse effect, we recover the results in [23] and [24].

Theorem 2.4.2 does not work when $s(t) \neq 0$. However, it is possible to obtain analogous results for the equation (2.48) if $\alpha = 1$. The first one will be obtained by comparing the solutions of equation

$$\begin{aligned} &(m(t)y')' + s(t)y' + q(t)\varphi_\beta(y) = f(t), && t \neq \theta_i, \\ &\Delta(m(t)y') + q_i\,\varphi_\beta(y) = f_i, && t = \theta_i, \end{aligned} \quad (2.63)$$

$\beta > 1$, and those of non self-adjoint equation (2.25).

The following comparison result can be considered as an extension of the results in [1, pp. 358], [3, Corollary 1] , [23, Theorem 2], [30, pp. 12].

Theorem 2.4.3. *Let $x(t)$ be an oscillatory solution of (2.25) with zeros at $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$. Suppose that for any given $t_* \geq t_0$, there exist intervals $I_1 = [t_{n_1}, t_{m_1}]$, $I_2 = [t_{n_2}, t_{m_2}] \subset [t_*, \infty)$ on which* (A)-(B) *hold.*

If (H) is satisfied and

$$\int_{t_{n_j}}^{t_{m_j}} \left\{ \left[\widetilde{q} - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right] x^2 + (k-m)\left[x' + \frac{(s-r)}{2(k-m)}\, x\right]^2 \right\} dt + \sum_{t_{n_j} \leq \theta_i < t_{m_j}} \left(\widetilde{q}_i - p_i\right) x^2 > 0 \tag{2.64}$$

for $j = 1, 2$, then all solutions of (2.63) are oscillatory.

Proof. Suppose that y is a nonoscillatory solution of (2.48) which is eventually positive, say $y(t) > 0$ when $t \in [t^*, \infty)$ for some $t^* \geq t_*$ depending on the solution y. By assumption, we can choose $I_1 \subset [t^*, \infty)$ so that $f(t) \leq 0$ on $I_1 \setminus \{\theta_i\}$ and $f_i \leq 0$ for $\theta_i \in I_1$.

Define

$$\rho(t) := \frac{x(t)}{y(t)} \left\{ y(t)kx'(t) - x(t)my'(t) \right\} \quad \text{for} \quad t \in I_1.$$

For abbreviation we secrete the variable t. Clearly

$$\begin{aligned} \rho' &= \left[q|v|^{\beta-1} + \frac{|f|}{|y|}\right] x^2 - p\, x^2 + (k-m)(x')^2 + m\,(x' - \frac{x}{y}\, y')^2 \\ &+ \frac{sy'}{y}\, x^2 - r\, xx', \qquad\qquad t \neq \theta_i; \end{aligned} \tag{2.65}$$

$$\Delta\rho = \left[q_i|y|^{\beta-1} + \frac{|f_i|}{|y|}\right] x^2 - p_i\, x^2, \qquad t = \theta_i. \tag{2.66}$$

In the view of (2.53), it is not difficult to see, cf.[30], from (2.65) and (2.66) that

$$\begin{aligned} \rho' \geq & \left[\widetilde{q} - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right] x^2 + (k-m)\left[x' + \frac{(s-r)}{2(k-m)}\, x\right]^2 \\ & + \frac{m}{y^2}\left(x'y - xy' - \frac{s}{2m}\, xy\right)^2, \qquad t \neq \theta_i; \end{aligned} \tag{2.67}$$

and

$$\Delta\rho \geq (\widetilde{q}_i - p_i)\, x^2, \qquad t = \theta_i. \tag{2.68}$$

Integrating (2.67) over I_1 and using (2.68) and (2.64), we get

$$\int_{t_{n_1}}^{t_{m_1}} \frac{m}{y^2} \left\{ x'y - xy' - \frac{s}{2m}\, xy \right\}^2 dt \leq 0. \tag{2.69}$$

From inequality (2.69), we conclude that

$$x'y - xy' - \frac{s}{2m}\, xy = 0 \quad \text{on} \quad I_1.$$

As before, it follows that

$$x = C_2\, y \exp\left(\int^t \frac{s}{2m}\, d\tau \right) \quad \text{on} \quad I_1,$$

for some constant C_2. Since $x(t_{n_1}) = x(t_{m_1}) = 0$, this is incompatible with the fact that $y(t) > 0$ on I_1.

When $y(t)$ is eventually negative, we choose the interval $I_2 \subset [T, \infty)$ for some $T \geq t_*$ so that $f(t) \geq 0$ on $I_2 \setminus \{\theta_i\}$ and $f_i \geq 0$ for $\theta_i \in I_2$ to reach a similar contradiction. This contradiction proves that $y(t)$ must be oscillatory. The proof is complete. □

Note that if there is no impulse and $s(t) \equiv 0$, then we recover the results in [23].

If (H) does not hold, we introduce a device of Picard [58](see also [30, p. 12]). Clearly, for any $h \in \mathcal{PLC}^1(I)$ we have

$$\frac{d}{dt}\,(x^2 h) = 2xx'h + x^2 h', \qquad t \neq \theta_i.$$

Let

$$\mu := \frac{x}{y}\,(ykx' - xmy') + x^2 h, \qquad t \in I.$$

It follows that

$$
\begin{aligned}
\mu' &\geq \left\{\widetilde{q} - p + h' - \frac{(s-r+2h)^2}{4(k-m)} - \frac{s^2}{4m}\right\}x^2 \\
&+(k-m)\left\{x' + \frac{s-r+2h}{2(k-m)}\,x\right\}^2 + \frac{m}{y^2}\left\{x'y - xy' - \frac{s}{2m}\,xy\right\}^2, \quad t \neq \theta_i \\
\Delta\mu &\geq (\widetilde{q}_i - p_i + \Delta h)\,x^2, \qquad\qquad t = \theta_i.
\end{aligned}
$$

Assuming that $r, s \in \mathcal{PLC}^1(I)$, the choice of $h = (r-s)/2$ yields

$$
\begin{aligned}
\mu' \;&\geq\; \left\{\widetilde{q} - p - \frac{1}{2}\,(s'-r') - \frac{s^2}{4m}\right\}x^2 + (k-m)(x')^2 \\
&+\; \frac{m}{y^2}\left\{x'y - xy' - \frac{s}{2m}\,xy\right\}^2, \qquad\qquad t \neq \theta_i \\
\Delta\mu \;&\geq\; \left\{\widetilde{q}_i - p_i - \frac{1}{2}\,\big(\Delta s - \Delta r\big)\right\}x^2, \qquad t = \theta_i.
\end{aligned}
$$

Then, we have the following result which is analogous to Theorem 2.4.3.

Theorem 2.4.4 (A Device of Picard). *Let $x(t)$ be an oscillatory solution of (2.25) with zeros at $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$. Suppose that for any given $t_* \geq t_0$, there exist intervals $I_1 = [t_{n_1}, t_{m_1}]$, $I_2 = [t_{n_2}, t_{m_2}] \subset [t_*, \infty)$ on which* (A)-(B) *hold.*

If $r, s \in \mathcal{PLC}^1(I_j)$ for $j = 1, 2$ and

$$
\begin{aligned}
&\int_{t_{n_j}}^{t_{m_j}} \left\{\left[\widetilde{q} - p - \frac{1}{2}\,(s'-r') - \frac{s^2}{4m}\right]x^2 + (k-m)(x')^2\right\}dt \\
&\qquad\qquad + \sum_{t_{n_j} \leq \theta_i < t_{m_j}} \left[\widetilde{q}_i - p_i - \frac{1}{2}\,\big(\Delta s - \Delta r\big)\right]x^2 > 0,
\end{aligned}
$$

for $j = 1, 2$, then all solutions of (2.63) are oscillatory.

CHAPTER 3

OSCILLATION THEOREMS

3.1 Nonlinear Equations

3.1.1 Introduction

In this chapter, we are interested in oscillation of second order nonlinear impulsive differential equations of the form

$$\begin{aligned} &(r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)f(x) = e(t), && t \neq \theta_i;\\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\, f(x) = e_i, && t = \theta_i \end{aligned} \tag{3.1}$$

where $\alpha > 0$ is a constant, $\{q_i\}$, $\{e_i\}$ and $\{\theta_i\}$ are real sequences, for $i \in \mathbb{N}$, with $\theta_1 > t_0$ for a fixed $t_0 \in \mathbb{R}$.

Throughout this chapter, we assume that

(i) r, p, q, $e \in \mathcal{PLC}([t_0, \infty))$; $r(t) > 0$;

(ii) $f \in C(\mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$ and the inequality

$$f'(s)|f(s)|^{(1-\alpha)/\alpha} \geq K_\alpha > 0 \tag{3.2}$$

holds.

By a solution of equation (3.1), we mean a nontrivial continuous function $x(t)$ for $t \geq t_x > t_0$ such that $r\varphi_\alpha(x') \in \mathcal{PLC}^1([t_x, \infty))$ satisfies equation (3.1).

In special cases (3.1) reduces to

$$(r(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{3.3}$$

$$(r(t)\psi(x)\varphi_\alpha(x'))' + q(t)f(x) = 0, \tag{3.4}$$

and

$$(r(t)\psi(x)x')' + p(t)x' + q(t)f(x) = 0. \tag{3.5}$$

These equations have been the object of intensive studies in recent years. (See [1, 20, 33, 34, 35, 39, 40, 41, 42, 43, 48, 51] for (3.3), [19, 20, 52, 77, 76, 83] for (3.4), and [13, 47, 53, 61, 75, 78, 85] for (3.5)).

In Section 3.1.2, we consider equation (3.4) with $\psi(s) \equiv 1$ and impulse effect, and using integral averaging technique, we extend the results of Coles [9] and Wintner [80].

In Section 3.1.3, we consider equation (3.1) with $e(t) \equiv 0$ and $e_i \equiv 0$, and using function averaging technique, we extend some of the results presented in literature to the impulsive case.

In another special case of (3.1) we have

$$(r(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = e(t), \tag{3.6}$$

which includes

$$(r(t)x')' + q(t)x = e(t). \tag{3.7}$$

In 1993, El-Sayed [10] established an interval criterion for (3.7) and in 1999, Wong [82] proved more general oscillation result for the same equation. Numerous oscillation criteria have been obtained for equation (3.7) (See [26, 27, 60, 64, 65, 73]). Recently, Li and Cheng [46] established an interval oscillation criterion for (3.6). In Section 3.1.4, we consider the equation (3.1) and using interval criteria we extend the results of Wong [82] and Li and Cheng [46] to the impulsive case.

3.1.2 Coles Type Oscillation Criteria

In 1968, Coles [9] studied the oscillation problem for

$$(r(t)x')' + q(t)\,x = 0, \tag{3.8}$$

by considering weighted averages of $\int^t q(\tau)d\tau$. In present work, we study equation

$$\begin{aligned} &(r(t)\varphi_\alpha(x'))' + q(t)f(x) = 0, && t \neq \theta_i; \\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\, f(x) = 0, && t = \theta_i \end{aligned} \tag{3.9}$$

in the special case when $p(t) \equiv e(t) \equiv 0$ and $e_i \equiv 0$, by considering weighted averages of

$$\int^t q(\tau)d\tau + \sum_{\theta_i \leq t} q_i.$$

Theorem 3.1.1. *If there exists a nonnegative, locally integrable function $g(t) : \mathbb{R} \to \mathbb{R}$ such that $\int^t g(\tau)\, d\tau \not\equiv 0$ and satisfying*

$$\int_\beta^\infty \left\{ g(t) \Big(\int_0^t g(s)ds \Big)^{k/\alpha} \Big/ \Big(\int_0^t r(s)g^{\alpha+1}(s)ds \Big)^{1/\alpha} \right\} dt = \infty \tag{3.10}$$

for some k, $0 \leq k < 1$, and for $\beta > 0$ and

$$\lim_{t\to\infty} \mathcal{A}(t) = \infty, \tag{3.11}$$

then the equation (3.9) is oscillatory, where

$$\mathcal{A}(t) := \int_0^t g(s) \left\{ \int_0^s q(\tau)\, d\tau + \sum_{0<\theta_i<s} q_i \right\} ds \Big/ \int_0^t g(s)\, ds. \tag{3.12}$$

Proof. We give a proof for $g(t)$ continuous; the proof easily modified for g locally integrable. Also, if convenient we will change the lower limits of the integrals in (3.12) and (3.10), since the asymptotic behavior as $t \to \infty$ is not changed thereby.

Let $x(t)$ be a nonoscillatory solution of the equation (3.9). Without loss of generality, we assume that $x(t) \neq 0$ for $t \geq \beta$, for large enough β. We define

$$z(t) := \frac{r(t)\varphi_\alpha(x')}{f(x)}, \qquad t \in [\beta, \infty)$$

then $z(t)$ satisfies

$$z' + K_\alpha \frac{|z|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)} + q(t) \le 0, \quad t \ne \theta_i; \tag{3.13}$$

$$\Delta z + q_i = 0, \qquad t = \theta_i \tag{3.14}$$

on $[\beta, \infty)$, where K_α as in (3.2). Integrating (3.13) over $[\beta, t)$ and using (3.14), we see that

$$z(t) + K_\alpha \int_\beta^t \frac{|z(s)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)}\, ds \le z(\beta) - \sum_{\beta \le \theta_i < t} q_i - \int_\beta^t q(s) ds. \tag{3.15}$$

Multiplying equation (3.15) by the function $g(s)$ and integrating over $[\beta, t)$, we obtain

$$\int_\beta^t g(s)z(s)ds + K_\alpha \int_\beta^t g(s) \int_\beta^s \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)} d\tau ds \le \Big[z(\beta) - \mathcal{A}(t) \Big] \int_\beta^t g(s) ds. \tag{3.16}$$

By (3.11), the right hand side of (3.16) tends to $-\infty$; hence, for large values of t,

$$\int_\beta^t g(s)z(s)ds + K_\alpha \int_\beta^t g(s) \int_\beta^s \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)}\, d\tau ds < 0. \tag{3.17}$$

Using Hölder's inequality and (3.17), we obtain

$$\begin{aligned} \left(K_\alpha \int_\beta^t g(s) \int_\beta^s \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)}\, d\tau ds \right)^{\alpha+1} & \\ \le \left(\int_\beta^t g(s)|z(s)|ds \right)^{\alpha+1} & \\ \le \left(\int_\beta^t r(s) g^{\alpha+1}(s) ds \right) \left(\int_\beta^t \frac{|z(s)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(s)}\, ds \right)^{\alpha}. & \end{aligned} \tag{3.18}$$

Let

$$R(t) := K_\alpha \int_\beta^t g(s) \int_\beta^s \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)}\, d\tau ds.$$

Since, for $t \ge \gamma > \beta$,

$$R(t) \ge K_\alpha \Big(\int_\gamma^t g(s)ds \Big) \Big(\int_\beta^\gamma \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)} \, d\tau \Big), \tag{3.19}$$

using inequalities (3.18) and (3.19), we see that

$$\begin{aligned} g^\alpha(t) \, \Big(\int_\gamma^t g(s)ds \Big)^k \Big(\int_\beta^\gamma \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)} \, d\tau \Big)^k \Big/ \int_\beta^t r(s) g^{\alpha+1}(s) ds \\ \le \frac{1}{K_\alpha^{\alpha+k}} \, R^{(k-\alpha-1)}(t)(R'(t))^\alpha. \end{aligned} \tag{3.20}$$

For $\gamma > \beta$, integration of the inequality (3.20) gives

$$\int_\gamma^t \Bigg[g(s) \, \Big(\int_\gamma^s g(\tau)d\tau \Big)^{k/\alpha} \Big/ \Big(\int_\beta^s r(\tau) g^{\alpha+1}(\tau)d\tau \Big)^{1/\alpha} \Bigg] ds \le \mathcal{K} \, R^{(k-1)/\alpha}(\gamma) \tag{3.21}$$

where

$$\mathcal{K} = \frac{\alpha}{K_\alpha(1-k)} \Big(K_\alpha \int_\beta^\gamma \frac{|z(\tau)|^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\tau)} \, d\tau \Big)^{-k/\alpha}.$$

Inequality (3.21) implies that condition (3.10) cannot be hold. This contradiction completes the proof of Theorem 3.1.1. □

Note that if $f(s) = s$, $q_i \equiv 0$ and $\alpha = 1$, we obtain the Coles result [9].

In case $r(s) \equiv 1$, $f(s) = s$ and $\alpha = 1$, equation (3.9) reduces to linear impulsive equation (2.10) and as a consequence of Theorem 3.1.1, we have the following result which is the extension of Wintner's [80] oscillation criteria to impulsive equations.

Corollary 3.1.2. *If*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \Big\{ \int_0^s q(\tau)d\tau + \sum_{0<\theta_i<s} q_i \Big\} ds = \infty,$$

then equation (2.10) is oscillatory.

Proof. Take the function $g(t)$ to be 1, let $k = 0$, and apply Theorem 3.1.1. □

3.1.3 Averaging Method

Throughout this section, we consider

$$\begin{aligned} &(r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)f(x) = 0, && t \neq \theta_i; \\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\, f(x) = 0, && t = \theta_i. \end{aligned} \tag{3.22}$$

Many authors have studied the oscillation problem for the less general equations such as the second order linear equation (3.8) or nonlinear equations (3.4) and (3.5) (see the references cited in section 3.1.1). In 1989, Philos [57] proved two oscillation criteria for equation

$$x'' + q(t)\, x = 0, \tag{3.23}$$

which are considered as extension of the results of Kamenev [25] and Yan [84]. Later, some of the extensions of results of Philos were given (see section 3.1.1).

In present section, we extend the Philos theorems [57] to the impulsive equation (3.22) and we give some analogous results, cf. [51] and [85].

The following Theorem is one of the main result of this section.

Theorem 3.1.3. *Let $D_0 = \{(t,s) : t > s > t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. Assume $H(t,s) \in C^1(D : (0,\infty))$, $h(t,s) \in C(D_0, \mathbb{R})$, $\rho \in C^1([t_0,\infty),(0,\infty))$ satisfy the conditions*

(i) $H(t,t) = 0$ *for* $t \geq t_0$ *and* $H(t,s) > 0$ *on* D_0*;*

(ii) *H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable;*

(iii)

$$\frac{\partial H}{\partial s}(t,s) + \left\{ \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)} \right\} H(t,s) = -h(t,s)\, H^{\alpha/(\alpha+1)}(t,s), \quad (t,s) \in D_0.$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\Gamma_\alpha\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\right]ds\right.$$
$$\left.+\sum_{t_0\le\theta_i<t}H(t,\theta_i)\,\rho(\theta_i)\,q_i\right\}=\infty, \tag{3.24}$$

where

$$\Gamma_\alpha=\left(\frac{\alpha}{K_\alpha}\right)^{\alpha}\left(\frac{1}{\alpha+1}\right)^{\alpha+1} \tag{3.25}$$

and K_α as in (3.2), then equation (3.22) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (3.22). We assume that $x(t)\neq 0$ on $[T,\infty)$ for some sufficiently large $T\ge t_0$. Define

$$w(t):=\rho(t)\,\frac{r(t)\varphi_\alpha(x')}{f(x)},\qquad t\ge T. \tag{3.26}$$

Differentiating (3.26) and making use of (3.22) and (3.2), we obtain

$$w'(t)\le\left\{\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right\}w(t)-\rho(t)q(t)-K_\alpha\,\frac{|w(t)|^{(\alpha+1)/\alpha}}{(\rho(t)r(t))^{1/\alpha}},\qquad t\neq\theta_i; \tag{3.27}$$
$$\Delta w(t)=-q_i\,\rho(t),\qquad t=\theta_i. \tag{3.28}$$

Multiplying (3.27), with t replaced by s, by $H(t,s)$ and integrating from T to t, we have

$$\int_T^t H(t,s)\rho(s)q(s)\,ds\le\int_T^t H(t,s)\left\{\frac{\rho'(s)}{\rho(s)}-\frac{p(s)}{r(s)}\right\}w(s)\,ds$$
$$-K_\alpha\int_T^t H(t,s)\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\,ds-\int_T^t H(t,s)w'(s)\,ds. \tag{3.29}$$

Integration by parts and using (3.28), the last integral on the right hand side of inequality (3.29) becomes

$$\int_T^t H(t,s)w'(s)ds=\int_T^t\left[\frac{\partial}{\partial s}\left\{H(t,s)w(s)\right\}-w(s)\frac{\partial H}{\partial s}(t,s)\right]ds,$$
$$=-H(t,T)w(T)-\sum_{T\le\theta_i<t}H(t,\theta_i)\Delta w(\theta_i)-\int_T^t w(s)\frac{\partial H}{\partial s}(t,s)ds,$$
$$\ge -H(t,T)w(T)+\sum_{T\le\theta_i<t}H(t,\theta_i)\rho(\theta_i)q_i-\int_T^t w(s)\frac{\partial H}{\partial s}(t,s)ds. \tag{3.30}$$

Using (3.29) and (3.30), we obtain

$$
\begin{aligned}
&\int_T^t H(t,s)\rho(s)q(s)\,ds + \sum_{T\le\theta_i<t} H(t,\theta_i)\rho(\theta_i)\,q_i \\
&\quad \le H(t,T)w(T) + \int_T^t \left[\frac{\partial H}{\partial s}(t,s) + H(t,s)\left\{\frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right\}\right] w(s)\,ds \\
&\qquad - K_\alpha \int_T^t H(t,s)\,\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\,ds \\
&\quad \le H(t,T)w(T) - \int_T^t \Bigg[K_\alpha\, H(t,s)\,\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}} \\
&\qquad - |h(t,s)|\,H^{\alpha/(\alpha+1)}(t,s)\,|w(s)|\Bigg]\,ds. \qquad (3.31)
\end{aligned}
$$

Using inequality (2.13) with $\beta = 1/\alpha$

$$
A = \frac{(K_\alpha\,H)^{\alpha/(\alpha+1)}|w|}{(\rho\,r)^{1/(\alpha+1)}} \quad \text{and} \quad B = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\left(\frac{\rho\,r}{K_\alpha^\alpha}\right)^{\alpha/(\alpha+1)} |\,h|^\alpha,
$$

we obtain

$$
\begin{aligned}
&K_\alpha\,H(t,s)\,\frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}} - |h(t,s)|\,H^{\alpha/(\alpha+1)}(t,s)\,|w(s)| \\
&\qquad\qquad \ge -\Gamma_\alpha\,\rho(s)r(s)\,|\,h(t,s)|^{\alpha+1}. \qquad (3.32)
\end{aligned}
$$

From (3.31) and (3.32), we obtain

$$
\begin{aligned}
&\int_T^t \left[H(t,s)\rho(s)q(s) - \Gamma_\alpha\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\right] ds + \sum_{T\le\theta_i<t} H(t,\theta_i)\,\rho(\theta_i)\,q_i \\
&\qquad \le H(t,T)\,w(T) \qquad (3.33) \\
&\qquad \le H(t,T)\,|w(T)| \le H(t,t_0)\,|w(T)| \qquad (3.34)
\end{aligned}
$$

for all $t > T \ge t_0$. In the above inequality we choose $T = T_0$, then we have

$$\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\Gamma_\alpha\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\right]ds+\sum_{t_0\le\theta_i<t}H(t,\theta_i)\,\rho(\theta_i)\,q_i$$

$$=\int_{t_0}^{T_0}\left[H(t,s)\rho(s)q(s)-\Gamma_\alpha\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\right]ds+\sum_{t_0\le\theta_i<T_0}H(t,\theta_i)\,\rho(\theta_i)q_i$$

$$+\int_{T_0}^{t}\left[H(t,s)\rho(s)q(s)-\Gamma_\alpha\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\right]ds+\sum_{T_0\le\theta_i<t}H(t,\theta_i)\,\rho(\theta_i)\,q_i$$

$$\le H(t,t_0)\left\{\int_{t_0}^{T_0}\rho(s)|q(s)|\,ds+\sum_{t_0\le\theta_i<T_0}\rho(\theta_i)\,|q_i|\right\}+H(t,t_0)\,|w(T_0)|.$$

It follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t}\left[H(t,s)\rho(s)q(s)-\Gamma_\alpha\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\right]ds\right.$$
$$\left.+\sum_{t_0\le\theta_i<t}H(t,\theta_i)\,\rho(\theta_i)\,q_i\right\}$$
$$\le\int_{t_0}^{T_0}\rho(s)|q(s)|\,ds+\sum_{t_0\le\theta_i<T_0}\rho(\theta_i)\,|q_i|+|w(T_0)|<\infty,$$

which contradicts with (3.24). This completes the proof. □

As a conclusion of the Theorem 3.1.3, we have the following corollary .

Corollary 3.1.4. *Let condition (3.24) in Theorem 3.1.3 be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left\{\int_{t_0}^{t}H(t,s)\rho(s)q(s)\,ds+\sum_{t_0\le\theta_i<t}H(t,\theta_i)\,\rho(\theta_i)\,q_i\right\}=\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)r(s)\,|h(t,s)|^{\alpha+1}\,ds<\infty,$$

then equation (3.22) is oscillatory.

Our second result is the following Theorem which can be considered as an extension of [51, Theorem 2].

Theorem 3.1.5. *Let the functions H, h and ρ be defined as in Theorem 3.1.3. Moreover, Suppose that*

$$0 < \inf_{s \geq t_0} \left\{ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty \tag{3.35}$$

and

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s) r(s) \left| h(t,s) \right|^{\alpha+1} ds < \infty. \tag{3.36}$$

If there exists a function $A \in C([t_0,\infty);\mathbb{R})$ such that

$$\limsup_{t \to \infty} \int_{t_0}^{t} \frac{(A_+(s))^{(\alpha+1)/\alpha}}{(\rho(s) r(s))^{1/\alpha}} ds = \infty, \tag{3.37}$$

and for every $T \geq t_0$

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \left\{ \int_{T}^{t} \left[H(t,s)\rho(s)q(s) - \Gamma_\alpha \rho(s) r(s) \left| h(t,s) \right|^{\alpha+1} \right] ds + \sum_{T \leq \theta_i < t} H(t,\theta_i)\, \rho(\theta_i)\, q_i \right\} \geq A(T), \tag{3.38}$$

where $A_+(s) = \max\{A(s), 0\}$, then equation (3.22) is oscillatory.

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (3.22) such that $x(t) \neq 0$ on $[T_0,\infty)$ for some sufficiently large $T_0 \geq t_0$. Define $w(t)$ as in (3.26). As in the proof of Theorem 3.1.3, we can obtain (3.31) and (3.33). It follows that

$$\limsup_{t \to \infty} \frac{1}{H(t,T)} \left\{ \int_{T}^{t} \left[H(t,s)\rho(s)q(s) - \Gamma_\alpha \rho(s) r(s) \left| h(t,s) \right|^{\alpha+1} \right] ds + \sum_{T \leq \theta_i < t} H(t,\theta_i)\, \rho(\theta_i)\, q_i \right\} \leq w(T)$$

for all $T \geq T_0$. Thus by (3.38) we have

$$A(T) \leq w(T) \quad \text{for all} \quad T \geq T_0 \tag{3.39}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T_0)} \left\{ \int_{T_0}^{t} H(t,s)\rho(s)q(s)\,ds + \sum_{T_0\le\theta_i<t} H(t,\theta_i)\,\rho(\theta_i)\,q_i \right\} \ge A(T_0). \tag{3.40}$$

Let

$$F(t) := \frac{1}{H(t,T_0)} \int_{T_0}^{t} |h(t,s)w(s)|\, H^{\alpha/(\alpha+1)}(t,s)\,ds$$

and

$$G(t) := \frac{K_\alpha}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\, \frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\,ds$$

for all $t > T_0$. Then, by (3.31) and (3.40), we see that

$$\begin{aligned}\liminf_{t\to\infty}[G(t)-F(t)] = w(T_0) - \limsup_{t\to\infty} \frac{1}{H(t,T_0)} & \left\{ \int_{T_0}^{t} H(t,s)\rho(s)q(s)\,ds \right. \\ & \left. + \sum_{T_0\le\theta_i<t} H(t,\theta_i)\,\rho(\theta_i)\,q_i \right\} \\ & \le w(T_0) - A(T_0) < \infty. \end{aligned} \tag{3.41}$$

Now, claim that

$$\int_{T_0}^{\infty} \frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\,ds < \infty. \tag{3.42}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} \frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\,ds = \infty. \tag{3.43}$$

By (3.35), there is a positive constant η satisfying

$$\inf_{s\ge t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} > \eta > 0. \tag{3.44}$$

On the other hand, by (3.43), for any positive number μ there exists a $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\,ds \ge \frac{\mu}{K_\alpha \eta} \quad \text{for all} \quad t \ge T_1 \tag{3.45}$$

so for all $t \geq T_1$

$$\begin{aligned} G(t) &= \frac{K_\alpha}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\, d\left[\int_{T_0}^{s} \frac{|w(\tau)|^{(\alpha+1)/\alpha}}{(\rho(\tau)r(\tau))^{1/\alpha}}\, d\tau\right] \\ &= \frac{K_\alpha}{H(t,T_0)} \int_{T_0}^{t} \left[-\frac{\partial H}{\partial s}(t,s)\right] \left[\int_{T_0}^{s} \frac{|w(\tau)|^{(\alpha+1)/\alpha}}{(\rho(\tau)r(\tau))^{1/\alpha}}\, d\tau\right] ds \\ &\geq \frac{K_\alpha}{H(t,T_0)} \int_{T_1}^{t} \left[-\frac{\partial H}{\partial s}(t,s)\right] \left[\int_{T_0}^{s} \frac{|w(\tau)|^{(\alpha+1)/\alpha}}{(\rho(\tau)r(\tau))^{1/\alpha}}\, d\tau\right] ds \\ &\geq \frac{\mu}{K_\alpha \eta} \frac{K_\alpha}{H(t,T_0)} \int_{T_1}^{t} \left[-\frac{\partial H}{\partial s}(t,s)\right] ds = \frac{\mu}{\eta} \frac{H(t,T_1)}{H(t,T_0)}. \end{aligned} \tag{3.46}$$

From (3.44) we have

$$\liminf_{t\to\infty} \frac{H(t,T_1)}{H(t,t_0)} > \eta > 0, \tag{3.47}$$

there exists $T_2 \geq T_1$ such that $H(t,T_1)/H(t,t_0) \geq \eta$ for all $t \geq T_2$. Therefore by (3.46), $G(t) \geq \mu$ for all $t \geq T_2$, and since μ is arbitrary constant, we conclude

$$\lim_{t\to\infty} G(t) = \infty. \tag{3.48}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0, ∞) with $\lim_{t\to\infty} t_n = \infty$ and such that

$$\lim_{n\to\infty} [G(t_n) - F(t_n)] = \liminf_{t\to\infty} [G(t) - F(t)].$$

In the view of (3.41), there exists a constant M such that

$$G(t_n) - F(t_n) \leq M \text{ for all sufficiently large } n. \tag{3.49}$$

It follows from (3.48) that

$$\lim_{n\to\infty} G(t_n) = \infty. \tag{3.50}$$

This and (3.49) give

$$\lim_{n\to\infty} F(t_n) = \infty. \tag{3.51}$$

Then by (3.49) and (3.50),

$$\frac{F(t_n)}{G(t_n)} - 1 \geq -\frac{M}{G(t_n)} > -\frac{1}{2} \text{ for } n \text{ large enough.}$$

Thus,
$$\frac{F(t_n)}{G(t_n)} > \frac{1}{2} \text{ for all } n \text{ large enough.}$$
This and (3.51) imply that
$$\lim_{n\to\infty} \frac{F^{\alpha+1}(t_n)}{G^{\alpha}(t_n)} = \infty. \tag{3.52}$$
On the other hand, by Hölder's inequality, we have
$$\begin{aligned}
F(t_n) &= \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} |h(t_n,s)w(s)|\, H^{\alpha/(\alpha+1)}(t_n,s)\, ds \\
&\le \left\{ \frac{K_\alpha}{H(t_n,T_0)} \int_{T_0}^{t_n} H(t_n,s)\, \frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\, ds \right\}^{\alpha/(\alpha+1)} \\
&\quad \times \left\{ \frac{1}{K_\alpha^\alpha\, H(t_n,T_0)} \int_{T_0}^{t_n} \rho(s)r(s)|h(t_n,s)|^{\alpha+1}\, ds \right\}^{1/(\alpha+1)} \\
&\le \frac{G^{\alpha/(\alpha+1)}(t_n)}{K_\alpha^{\alpha/(\alpha+1)}} \left\{ \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} \rho(s)r(s)|h(t_n,s)|^{\alpha+1}\, ds \right\}^{1/(\alpha+1)},
\end{aligned}$$
and therefore
$$\begin{aligned}
\frac{F^{\alpha+1}(t_n)}{G^{\alpha}(t_n)} &\le \frac{1}{K_\alpha^\alpha\, H(t_n,T_0)} \int_{T_0}^{t_n} \rho(s)r(s)|h(t_n,s)|^{\alpha+1}\, ds \\
&\le \frac{1}{K_\alpha^\alpha\, \eta\, H(t_n,t_0)} \int_{t_0}^{t_n} \rho(s)r(s)|h(t_n,s)|^{\alpha+1}\, ds
\end{aligned}$$
for a large n. It follows from (3.52) that
$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} \rho(s)r(s)|h(t_n,s)|^{\alpha+1}\, ds = \infty, \tag{3.53}$$
that is,
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s)r(s)|h(t,s)|^{\alpha+1}\, ds = \infty,$$
which contradicts (3.36). Hence (3.42) holds. Then, it follows from (3.39) that
$$\int_{T_0}^{t} \frac{(A_+(s))^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\, ds \le \int_{T_0}^{\infty} \frac{|w(s)|^{(\alpha+1)/\alpha}}{(\rho(s)r(s))^{1/\alpha}}\, ds < \infty$$
which contradicts (3.37). This completes the proof of Theorem 3.1.5. □

Note that if $\rho(t) \equiv 0$, $p(t) \equiv 0$, $f(s) = s$, $q_i \equiv 0$ and $\alpha = 1$, we obtain the results of Philos [57].

3.1.4 Interval Oscillation Criteria

In this section, we obtain the following interval oscillation criteria for (3.1).

Theorem 3.1.6. *Suppose that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that*

$$e(t) \begin{cases} \leq 0, & t \in [s_1, t_1] \\ \geq 0, & t \in [s_2, t_2] \end{cases} \quad \text{and} \quad e_i \begin{cases} \leq 0, & \theta_i \in [s_1, t_1] \\ \geq 0, & \theta_i \in [s_2, t_2] \end{cases} \tag{3.54}$$

for all $i \in \mathbb{N}$. If there exists $u \in \mathcal{D}(s_k, t_k)$ for $k = 1, 2$, and a positive, nondecreasing function $\phi \in C([t_0, \infty))$ such that

$$\int_{s_k}^{t_k} \left\{ \phi\, q\, |u|^{\alpha+1} - \Gamma_\alpha\, r\, \phi \left| (\alpha+1)\, u' + \left(\frac{\phi'}{\phi} - \frac{p}{r} \right) |u| \right|^{\alpha+1} \right\} dt + \sum_{s_k \leq \theta_i < t_k} \phi\, q_i\, |u|^{\alpha+1} > 0$$

for $k = 1, 2$, where Γ_α is defined as in (3.25), then every solution of the equation (3.1) is oscillatory.

Proof. Suppose now that x be a nonoscillatory solution of equation (3.1) which is positive, say $x > 0$ when $t \geq t_*$ for some t_* depending on the solution x. Now, we define

$$v(t) := \phi(t)\, \frac{r(t)\varphi_\alpha(x')}{f(x)}, \qquad t \geq t_*. \tag{3.55}$$

Then, for every $t \geq t_*$, we obtain

$$v' = \left(\frac{\phi'}{\phi} - \frac{p}{r} \right) v - f'(s)|f(s)|^{(1-\alpha)/\alpha} \frac{|v|^{(\alpha+1)/\alpha}}{(r\phi)^{1/\alpha}} + \left(\frac{e}{f(x)} - q \right) \phi, \qquad t \neq \theta_i; \tag{3.56}$$

$$\Delta v = \left(\frac{e_i}{f(x)} - q_i \right) \phi, \qquad t = \theta_i. \tag{3.57}$$

By assumption, we can choose $s_1, t_1 \geq t_*$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$ and $e_i \leq 0$ for all $i \in \mathbb{N}$ for which $\theta_i \in I_1$ with $s_1 < t_1$. On the interval I_1, using (3.2) and (3.56)-(3.57), $v(t)$ satisfies

$$q\,\phi \leq -v' + \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)v - K_\alpha\,\frac{|v|^{(\alpha+1)/\alpha}}{(r\phi)^{1/\alpha}}, \qquad t \neq \theta_i; \tag{3.58}$$

$$\Delta v + q_i\,\phi \leq 0, \qquad t = \theta_i. \tag{3.59}$$

Let $u \in \mathcal{D}(s_1, t_1)$ be given as in the hypothesis. Multiplying $|u|^{\alpha+1}$ through (3.58) and integrating over I_1, we get

$$\int_{s_1}^{t_1} q\,\phi\,|u|^{\alpha+1} dt \leq \int_{s_1}^{t_1} \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|^{\alpha+1}\,v\,dt - K_\alpha \int_{s_1}^{t_1} \frac{|v|^{(\alpha+1)/\alpha}}{(r\phi)^{1/\alpha}}\,|u|^{\alpha+1}\,dt$$
$$- \int_{s_1}^{t_1} |u|^{\alpha+1}\,v'\,dt. \tag{3.60}$$

Integration by parts and using the fact that $u(s_1) = u(t_1) = 0$ and (3.59), we obtain further that

$$\int_{s_1}^{t_1} q\,\phi\,|u|^{\alpha+1} dt + \sum_{s_1 \leq \theta_i < t_1} q_i\,\phi\,|u|^{\alpha+1}$$
$$\leq (\alpha+1)\int_{s_1}^{t_1} \varphi_\alpha(u)\,u'\,v\,dt + \int_{s_1}^{t_1} \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|^{\alpha+1}\,v\,dt$$
$$-K_\alpha \int_{s_1}^{t_1} \frac{|v|^{(\alpha+1)/\alpha}}{(r\phi)^{1/\alpha}}\,|u|^{\alpha+1} dt$$
$$\leq \int_{s_1}^{t_1} \left|(\alpha+1)\varphi_\alpha(u)\,u' + \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|^{\alpha+1}\right|\,|v|\,dt$$
$$-K_\alpha \int_{s_1}^{t_1} \frac{|u|^{\alpha+1}}{(r\phi)^{1/\alpha}}\,|v|^{(\alpha+1)/\alpha}\,dt.$$

By taking $\beta = 1/\alpha$,

$$A = K_\alpha^{\alpha/(\alpha+1)}\,\frac{|u|^\alpha}{(r\phi)^{1/(\alpha+1)}}\,|v|$$

and

$$B = \left(\frac{\alpha\Gamma_\alpha r\phi}{|u|^{\alpha(\alpha+1)}}\right)^{\alpha/(\alpha+1)} \left|(\alpha+1)\,\varphi_\alpha(u)\,u' + \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|^{\alpha+1}\right|^\alpha,$$

the inequality (2.13) implies that, for $t \in [s_1, t_1]$,

$$\left|(\alpha+1)\,\varphi_\alpha(u)\,u' + \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|^{\alpha+1}\right| |v| - K_\alpha \frac{|u|^{\alpha+1}}{(r\phi)^{1/\alpha}}\,|v|^{(\alpha+1)/\alpha}$$
$$\leq \Gamma_\alpha\, r\,\phi \left|(\alpha+1)\,u' + \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|\right|^{\alpha+1},$$

thus,

$$\int_{s_1}^{t_1} \phi\, q\, |u|^{\alpha+1} dt + \sum_{s_1 \leq \theta_i < t_1} \phi\, q_i\, |u|^{\alpha+1}$$
$$\leq \Gamma_\alpha \int_{s_1}^{t_1} r\,\phi \left|(\alpha+1)\,u' + \left(\frac{\phi'}{\phi} - \frac{p}{r}\right)|u|\right|^{\alpha+1} dt,$$

which contradicts with our assumption.

When $x(t)$ eventually negative, we may employ the fact that $e(t) \geq 0$ on $I_2 = [s_2, t_2]$ and $e_i \geq 0$ for all $i \in \mathbb{N}$ for which $\theta_i \in I_2$ to reach a similar contradiction. The proof is complete. □

When $f(s) = \varphi_\alpha(s)$, then equation (3.1) reduces to forced half-linear impulsive equation with damping

$$\begin{aligned} &(r(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x') + q(t)\varphi_\alpha(x) = e(t), \qquad && t \neq \theta_i; \\ &\Delta(r(t)\varphi_\alpha(x')) + q_i\, \varphi_\alpha(x) = e_i, && t = \theta_i. \end{aligned} \tag{3.61}$$

As a conclusion of Theorem 3.1.6, equation (3.61) is oscillatory if the conditions of Theorem 3.1.6 are all satisfied with $\Gamma_\alpha = (\alpha+1)^{-(\alpha+1)}$. Note that by taking $p(t) \equiv 0$ and $q_i \equiv e_i \equiv 0$ in the equation (3.61), we recover the result of Li and Cheng [46].

Taking $\alpha = 1$ in the equation (3.61), we obtain the forced linear impulsive equation with damping

$$\begin{aligned} &(r(t)x')' + p(t)x' + q(t)x = e(t), \qquad && t \neq \theta_i; \\ &\Delta(r(t)x') + q_i\, x = e_i, && t = \theta_i. \end{aligned} \tag{3.62}$$

Taking $\phi \equiv 1$ and applying Theorem 3.1.6 to equation (3.62), we obtain the following oscillation criteria which can be considered as a generalization of the result given by Wong [82].

Corollary 3.1.7. *Suppose that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that (3.54) holds for all $i \in \mathbb{N}$. If there exist $u \in \mathcal{D}(s_k, t_k)$, for $k = 1, 2$ such that*

$$\int_{s_k}^{t_k} \big[q\, u^2 - r\,(u' - \frac{p}{2r}\, u)^2\big]\, dt + \sum_{s_k \leq \theta_i < t_k} q_i\, u^2 > 0 \tag{3.63}$$

for $k = 1, 2$, then every solution of the equation (3.62) is oscillatory.

We will illustrate our oscillation criterion by means of one example.

Example 3.1.8. *Consider the following special case of equation (3.62),*

$$\begin{aligned} & x'' + (\sin t)x' + (\cos t)x = -e^{\cos t} \sin t, \qquad && t \neq \theta_i; \\ & \Delta x' + (\sin^2 t)\, x = -e^{\cos t} \sin t, && t = \theta_i \end{aligned} \tag{3.64}$$

where $\theta_i = i\pi/m$, $m \in \mathbb{N}$. Here the zeros of the forcing term $-e^{\cos t} \sin t$ are $k\pi$, $k \in \mathbb{Z}$.

Let $u = \sin t$. For any $T \geq 0$, choose $n \in \mathbb{N}$ sufficiently large so that $n\pi \geq T$ and set $s_1 = (2n-1)\pi$ and $t_1 = 2n\pi$ in (3.63), then condition (3.54) is satisfied for all n. It is easy to verify that

$$\begin{aligned} & \int_{(2n-1)\pi}^{2n\pi} \left[\cos t \sin^2 t - (\cos t - \frac{1}{2} \sin^2 t)^2\right] dt + \sum_{(2n-1)\pi \leq \theta_i < 2n\pi} \sin^4 \theta_i \\ & = -\frac{19\pi}{32} + \sum_{i=-m}^{-1} \sin^4(\frac{i\pi}{m}) \\ & = \frac{3m}{8} - \frac{19\pi}{32} \end{aligned} \tag{3.65}$$

and similarly, for $s_2 = 2n\pi$ and $t_2 = (2n+1)\pi$, we obtain

$$\begin{aligned} & \int_{2n\pi}^{(2n+1)\pi} \left[\cos t \sin^2 t - (\cos t - \frac{1}{2} \sin^2 t)^2\right] dt + \sum_{2n\pi \leq \theta_i < (2n+1)\pi} \sin^4 \theta_i \\ & = -\frac{19\pi}{32} + \sum_{i=0}^{m-1} \sin^4(\frac{i\pi}{m}) \\ & = \frac{3m}{8} - \frac{19\pi}{32}. \end{aligned} \tag{3.66}$$

It follows from Corollary 3.1.7 that equation (3.64) is oscillatory if $m \geq 5$.

Note that differential part of equation (3.64) is nonoscillatory with a nonoscillatory solution $x(t) = te^{\cos t}$.

3.2 Hill's Equation with Damping

In this section, we are concerned with second order linear impulsive equation of the form

$$\begin{aligned} &x'' + p(t)x' + q(t)x = 0, && t \neq \theta_i; \\ &\Delta x' + q_i x = 0, && t = \theta_i \end{aligned} \tag{3.67}$$

where $p(t)$, $q(t)$ are continuous functions of period T and $\{q_i\}$, $\{\theta_i\}$ are real sequences satisfying $q_{i+r} = q_i$, $\theta_i + T = \theta_{i+r}$ for all r, $i \in \{1, 2, ...\}$ with $\theta_1 > t_0$ for fixed $t_0 \in \mathbb{R}$.

By a solution of equation (3.67), we mean a nontrivial continuous function $x(t)$ for $t \geq t_x$, $t_x > t_0$, such that $x' \in \mathcal{PLC}^1([t_x, \infty))$ and satisfies equation (3.67).

In present section, we extend the results of Kwong and Wong [36] to the impulsive equation (3.67). Before giving the main results, we need the following two Lemmas which are the extension of the results due to Wintner [79, 81].

Lemma 3.2.1. *Equation (3.67) is nonoscillatory on* $[0, \infty)$ *if and only if there exists a* $t_* \in [0, \infty)$ *and a function* $r \in \mathcal{PLC}([t_*, \infty))$ *such that*

$$\begin{aligned} &r'(t) \geq r^2(t) - p(t)r(t) + q(t), && t \neq \theta_i; \\ &\Delta r(t) \geq q_i, && t = \theta_i \end{aligned} \tag{3.68}$$

for all $t \geq t_*$.

Proof. Assume that $x(t)$ be a solution of equation (3.67) such that it has no zero on $[t_*, \infty)$. Define $r(t) := -x'(t)/x(t)$ for $t \geq t_*$, then $r(t)$ satisfies the impulsive equation

$$\begin{aligned} r'(t) &= r^2(t) - p(t)r(t) + q(t), \qquad && t \neq \theta_i; \\ \Delta r(t) &= q_i, && t = \theta_i. \end{aligned} \tag{3.69}$$

Now, let there exists a function $r \in \mathcal{PLC}([t_*, \infty))$ satisfying (3.68). Define

$$\begin{aligned} f(t) &:= r'(t) - r^2(t) + p(t)r(t) - q(t), \qquad && t \neq \theta_i; \\ f_i &:= \Delta r(\theta_i) - q_i, && i \in \mathbb{N}, \end{aligned}$$

then $f(t) \geq 0$ for $t \geq t_*$ and $f_i \geq 0$ for which $\theta_i \geq t_*$, and we have the following Riccati type impulsive equation:

$$\begin{aligned} r'(t) &= r^2(t) - p(t)r(t) + [q(t) + f(t)], \qquad && t \neq \theta_i; \\ \Delta r(t) &= q_i + f_i, && t = \theta_i. \end{aligned} \tag{3.70}$$

The corresponding equation becomes

$$\begin{aligned} &x'' + p(t)x' + [q(t) + f(t)]x = 0, \qquad && t \neq \theta_i; \\ &\Delta x' + [q_i + f_i]x = 0, && t = \theta_i. \end{aligned} \tag{3.71}$$

Since $q(t) + f(t) \geq q(t)$ and $q_i + f_i \geq q_i$, equation (3.71) is a Sturm majoring for (3.67) and has a positive solution $x(t) = \exp(-\int^t r(\tau)d\tau)$. Hence by Sturmian Oscillation Theorem for impulsive equations (see Corollary 2.2.5), equation (3.67) is nonoscillatory. □

In case $q_i \equiv 0$, Lemma 3.2.1 can be found in [81] and [16, p. 362, Theorem 7.2].

Lemma 3.2.2. *Suppose that*

$$\int^{\infty} \exp\left(- \int^{t} p(\tau)d\tau \right) dt = \infty$$

and

$$\lim_{\omega \to \infty} \left[\int^{\omega} \exp\left(\int^{t} p(\tau)d\tau \right) q(t)dt + \sum_{\theta_i < \omega} \exp\left(\int^{\theta_i} p(\tau)d\tau \right) q_i \right] = \infty, \tag{3.72}$$

then equation (3.67) is oscillatory.

Wintner's [79] original result was proved for the case $p(t) \equiv 0$ and $q_i \equiv 0$, but multiplying the equation (3.67) by the function $\exp(\int^t p(\tau)d\tau)$ and applying Theorem 2 in [3], it is easy to verify that condition (3.72) is an oscillation criterion for equation (3.67).

Theorem 3.2.3. *Let there exist a function $Q \in \mathcal{PLC}([0,\infty))$ such that*

$$Q'(t) = q(t), \qquad t \neq \theta_i; \tag{3.73}$$

where $q(t)$ is periodic of mean value zero, i.e., $\int_0^T q(t)dt = 0$ and a r periodic sequence $\{p_i\}$ such that $\sum_{0<\theta_i<T} p_i = 0$ and

$$\Delta Q(t) = p_i, \qquad t = \theta_i. \tag{3.74}$$

If

$$\begin{aligned} p(t) - Q(t)]\, Q(t) \geq 0, &\qquad 0 \leq t \leq T; \\ p_i \geq q_i, &\qquad 0 \leq \theta_i \leq T, \end{aligned} \tag{3.75}$$

then equation (3.67) is nonoscillatory.

Proof. We note that, if $q(t)$ is periodic with mean value zero, we obtain

$$Q(T) - Q(0) = \int_0^T Q'(t)dt + \sum_{0<\theta_i<T} \Delta Q(\theta_i) = \int_0^T q(t)dt + \sum_{0<\theta_i<T} p_i = 0,$$

which yields $Q(T) = Q(0)$. On the other hand, we have

$$\begin{aligned} Q(t+T) - Q(t) &= [Q(t+T) - Q(T)] - [Q(t) - Q(0)] \\ &= \int_T^{t+T} Q'(\tau)d\tau + \sum_{T\leq\theta_i<t+T} \Delta Q(\theta_i) - \int_0^t Q'(\tau)d\tau - \sum_{0<\theta_i<t} \Delta Q(\theta_i) \\ &= \int_T^{t+T} q(\tau)d\tau - \int_0^t q(\tau)d\tau + \sum_{T\leq\theta_i<t+T} p_i - \sum_{T\leq\theta_{i+r}<t+T} p_i \\ &= \int_0^t [q(\tau+T) - q(\tau)]d\tau - \sum_{T\leq\theta_i<t+T} [p_{i+r} - p_i]. \end{aligned}$$

Since the function $q(t)$ is T periodic and the sequence p_i is r periodic, $Q(t)$ is periodic with period T. Observe that condition (3.75) implies

$$\begin{aligned} Q'(t) &\geq Q^2(t) - p(t)Q(t) + q(t), \qquad && t \neq \theta_i; \\ \Delta Q(t) &\geq q_i, && t = \theta_i \end{aligned} \tag{3.76}$$

which becomes (3.68) if we set $Q(t) = r(t)$ in (3.76). Hence by Lemma 3.2.1, equation (3.67) is nonoscillatory. □

Theorem 3.2.4. *In addition to the assumptions in Theorem 3.2.3, let* $q(t) \not\equiv 0$*,* $p(t)$*,* $Q(t)$ *are also periodic with mean value zero and satisfy*

$$\begin{aligned} [p(t) - Q(t)]\, Q(t) &\leq 0, \qquad && 0 \leq t \leq T; \\ p_i &\leq q_i, && 0 \leq \theta_i \leq T. \end{aligned} \tag{3.77}$$

If either

$$\textit{measure}\Big\{t \in [0,T] :\ [p(t) - Q(t)]\, Q(t) < 0\Big\} > 0 \qquad \textit{or} \qquad p_i < q_i \tag{3.78}$$

for some $i \in \mathbb{N}$ *for which* $\theta_i \in [0,T]$*, then (3.67) is oscillatory.*

Proof. Assume on the contrary that equation (3.67) is nonoscillatory, then without loss of generality there exists a positive solution $x(t)$ on $[t_0, \infty)$ where $t_0 \geq 0$ depends on the solution $x(t)$. Let $r(t) := -x'(t)/x(t)$ on $t \geq t_0$. Then $r(t)$ satisfies the Riccati type impulsive equation (3.69). Define $R(t) = r(t) - Q(t)$. It is easy to verify from (3.69) that $R(t)$ satisfies

$$\begin{aligned} R'(t) &= R^2(t) + [2Q(t) - p(t)]R(t) + Q^2(t) - p(t)Q(t), \qquad && t \neq \theta_i; \\ \Delta R(t) &= q_i - p_i, && t = \theta_i. \end{aligned} \tag{3.79}$$

Since $R(t) \in \mathcal{PLC}([t_0, \infty))$ and satisfies (3.79), we can now apply the sufficiency part of Lemma 3.2.1 to deduce that the second-order impulsive equation

$$\begin{aligned} & z''(t) + [p(t) - 2Q(t)]z'(t) + [Q^2(t) - p(t)Q(t)]z(t) = 0, \qquad && t \neq \theta_i; \\ & \Delta z'(t) + [q_i - p_i]z = 0, && t = \theta_i \end{aligned} \tag{3.80}$$

is nonoscillatory. Since $p(t)$, $Q(t)$ are periodic in T with mean value zero, the function

$$E(t) := \exp \int_0^t \big\{p(\tau) - 2Q(\tau)\big\}\, d\tau$$

is bounded below by a positive constant. Using (3.78), we get

$$\int_0^T E(t)\big\{Q^2(t) - p(t)Q(t)\big\}\, dt + \sum_{0<\theta_i<T} E(\theta_i)\{q_i - p_i\} = \lambda > 0,$$

which implies that condition (3.72) is satisfied. Now apply Lemma 3.2.2 to equation (3.80) and conclude that it is oscillatory. This contradiction proves the Theorem 3.2.4. □

Note that if $q_i \equiv 0$, we recover the results of Kwong and Wong [36].

3.3 Forced Linear Equations

In this section, we consider the forced second order linear impulsive equation

$$\begin{aligned} &(p(t)y')' + q(t)y = f(t), && t \neq \theta_i; \\ &\Delta p(t)y' + q_i y = f_i, && t = \theta_i \end{aligned} \tag{3.81}$$

under the assumption that the unforced equation,

$$\begin{aligned} &(p(t)z')' + q(t)z = 0, && t \neq \theta_i; \\ &\Delta p(t)z' + q_i z = 0, && t = \theta_i \end{aligned} \tag{3.82}$$

is nonoscillatory, where $\{q_i\}$, $\{f_i\}$ and $\{\theta_i\}$ are real sequences with $\theta_1 > t_0$ for fixed $t_0 \in \mathbb{R}$. Throughout this work, we assume that the functions p, $q \in \mathcal{PLC}[t_0, \infty)$ with $p(t) > 0$. Our interest is to establish an oscillation criteria for equation (3.81) without assuming that the functions q and f are of definite signs.

By a solution of equation (3.81), we mean a nontrivial continuous function $y(t)$ for $t \geq t_y > t_0$ such that $py' \in \mathcal{PLC}^1([t_y, \infty))$ and satisfies (3.81).

In order to give an oscillation result for equation (3.81), we need to prove the existence of nonprincipal solution of unforced equation (3.82). Therefore, before giving the main results, we need some Lemmas.

Consider

$$\begin{aligned} Lx &= (p(t)x')' + q(t)x = 0, \qquad t \neq \theta_i; \\ Ix &= \Delta p(t)x' + q_i x = 0, \qquad t = \theta_i. \end{aligned} \tag{3.83}$$

Lemma 3.3.1 (Polya Factorization). *If (3.83) has a continuous solution $u(t)$ with no zeros in $[a, \infty)$, then for all $\eta \in \mathcal{S} = \{\eta \in \mathcal{PLC}^1([a,\infty)) : p\eta' \in \mathcal{PLC}^1([a,\infty))\}$*

$$\begin{aligned} L\eta &= \rho_1(\rho_2(\rho_1\eta)')', \qquad t \neq \theta_i; \qquad t \in [a,\infty), \\ I\eta &= \rho_1 \Delta \rho_2 (\rho_1 \eta)', \qquad t = \theta_i \end{aligned} \tag{3.84}$$

where $\rho_1(t) = 1/u(t)$ and $\rho_2(t) = p(t)u^2(t)$.

Proof. Let $u(t)$ be the solution of (3.83) with no zeros in $[a, \infty)$, namely $Lu \equiv 0$ for $t \neq \theta_i$ and $Iu \equiv 0$ for $t = \theta_i$. Using Lagrange identity, we obtain

$$\begin{aligned} \mu L\eta - \eta L\mu &= [p(t)\, W(\eta,\mu)]', \qquad t \neq \theta_i; \\ \mu I\eta - \eta I\mu &= \Delta p(t)\, W(\eta,\mu), \qquad t = \theta_i \end{aligned} \tag{3.85}$$

where $W(\eta, \mu)$ denotes the Wronskian. Taking $\mu(t) = u(t)$ in (3.85), we obtain the equation (3.84). □

Lemma 3.3.2 (Trench Factorization). *If (3.83) has a positive continuous solution on $[a, \infty)$, then for any $\eta \in \mathcal{S}$*

$$\begin{aligned} L\eta &= \gamma_1(\gamma_2(\gamma_1\eta)')', \qquad t \neq \theta_i; \qquad t \in [a,\infty), \\ I\eta &= \gamma_1 \Delta \gamma_2 (\gamma_1 \eta)', \qquad t = \theta_i \end{aligned} \tag{3.86}$$

where $\gamma_1(t), \gamma_2(t) > 0$ on $[a, \infty)$, and $\displaystyle\int_a^\infty \frac{dt}{\gamma_2(t)} = \infty$.

Proof. If $\int_a^\infty \frac{dt}{\rho_2(t)} = \infty$, take $\gamma_2(t) = \rho_2(t)$ and $\gamma_1(t) = \rho_1(t)$. Suppose $\int_a^\infty \frac{dt}{\rho_2(t)} < \infty$, if we take

$$\gamma_1(t) = \rho_1(t)\left(\int_t^\infty \frac{ds}{\rho_2(s)}\right)^{-1} > 0 \quad \text{and} \quad \gamma_2(t) = \rho_2(t)\left(\int_t^\infty \frac{ds}{\rho_2(s)}\right)^{2} > 0,$$

then $\gamma_1(t)$ and $\gamma_2(t)$ satisfies the equation (3.86) and

$$\begin{aligned}
\int_a^\infty \frac{dt}{\gamma_2(t)} &= \int_a^\infty \frac{1}{\rho_2(t)}\left(\int_t^\infty \frac{ds}{\rho_2(s)}\right)^{-2} dt \\
&= \int_a^\infty \frac{d}{dt}\left(\int_t^\infty \frac{ds}{\rho_2(s)}\right)^{-1} dt \\
&= \left(\int_t^\infty \frac{ds}{\rho_2(s)}\right)^{-1}\Bigg|_{t=a}^{t=\infty} - \sum_{a<\theta_i} \Delta\left(\int_{\theta_i}^\infty \frac{ds}{\rho_2(s)}\right)^{-1} \\
&= \infty.
\end{aligned}$$

□

Theorem 3.3.3. *If (3.83) has a positive solution on $[a,\infty)$, then there exist linearly independent solutions u and $v, (v > 0)$ of equation (3.83) such that $\frac{u}{v} \to 0$ as $t \to \infty$ and*

$$\int_a^\infty \frac{dt}{pu^2} = \infty \quad \textit{and} \quad \int_a^\infty \frac{dt}{pv^2} < \infty.$$

Here the solutions $u(t)$ and $v(t)$ are called principal and nonprincipal solutions of equation (3.83), respectively.

Proof. By Lemma 3.3.2, there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfying equation (3.86). Then, take

$$u(t) = \frac{1}{\gamma_1(t)} \quad \text{and} \quad v_0(t) = \frac{1}{\gamma_1(t)}\int_a^t \frac{ds}{\gamma_2(s)}.$$

Since $Lu = Iu = 0$ and $Lv_0 = Iv_0 = 0$, $u(t)$ and $v_0(t)$ are two linearly independent solutions of equation (3.83) and

$$\lim_{t\to\infty} \frac{u(t)}{v_0(t)} = \lim_{t\to\infty}\left(\int_a^t \frac{ds}{\gamma_2(s)}\right)^{-1} = 0. \tag{3.87}$$

Now, substituting $\eta = u$ and $\mu = v_0$ on (3.85), we get

$$[p(t)\,W(u,v_0)]' = 0, \qquad t \neq \theta_i; \tag{3.88}$$

$$\Delta p(t)\,W(u,v_0) = 0, \qquad t = \theta_i. \tag{3.89}$$

Integrating equation (3.88) over $[a,t]$ and using (3.89), we obtain $W(u,v_0)(t) = c/p(t)$ where the constant $c = p(a)W(u,v_0)(a)$. This implies

$$\left(\frac{v_0}{u}\right)'(t) = \frac{W(u,v_0)(t)}{u^2(t)} = \frac{c}{p(t)u^2(t)}. \tag{3.90}$$

Integrating (3.90) over $[a,\infty)$ and using (3.87), we obtain

$$\begin{aligned}
\int_a^\infty \frac{dt}{p(t)u^2(t)} &= \frac{1}{c}\lim_{A\to\infty}\int_a^A \big(\frac{v_0}{u}\big)'(t)dt \\
&= \frac{1}{c}\lim_{A\to\infty}\left[\frac{v_0(t)}{u(t)}\bigg|_{t=a}^{t=A} - \sum_{a\le\theta_i<A}\Delta\big(\frac{v_0}{u}\big)(\theta_i)\right] \\
&= \frac{1}{c}\lim_{A\to\infty}\left[\frac{v_0(A)}{u(A)} - \frac{v_0(a)}{u(a)} - \sum_{a\le\theta_i<A}\int_{\theta_i}^{\theta_i^+}\gamma_2^{-1}(s)ds\right] \\
&= \frac{1}{c}\lim_{A\to\infty}\frac{v_0(A)}{u(A)} - \frac{v_0(a)}{cu(a)} \\
&= \infty.
\end{aligned}$$

Let $v(t)$ be any solution of (3.83). Then $v(t) = c_1u(t) + c_2v_0(t)$ for some constants c_1, c_2 with $c_2 \neq 0$ and using (3.87), we get

$$\lim_{t\to\infty}\frac{u(t)}{v(t)} = \lim_{t\to\infty}\big[c_1 + c_2\frac{v_0(t)}{u(t)}\big]^{-1} = 0. \tag{3.91}$$

Since $Lu = Iu = 0$ and $Lv = Iv = 0$, $u(t)$ and $v(t)$ satisfy the equalities (3.88) and (3.89) with $v_0(t)$ replaced by $v(t)$. In a similar way, we obtain

$$\left(\frac{u}{v}\right)'(t) = \frac{W(v,u)(t)}{v^2(t)} = \frac{\widetilde{c}}{p(t)v^2(t)} \tag{3.92}$$

where the constant $\widetilde{c} = p(a)W(v,u)(a)$, and

$$\Delta\left(\frac{u}{v}\right)(\theta_i) = \Delta\big[c_1 + c_2\,\frac{v_0(\theta_i)}{u(\theta_i)}\big]^{-1} = 0. \tag{3.93}$$

Integrating (3.92) over $[a, \infty)$ and using (3.91) and (3.93), we have

$$\int_a^\infty \frac{dt}{p(t)v^2(t)} = -\frac{u(a)}{\widetilde{c}v(a)} < \infty. \tag{3.94}$$

The proof of Theorem 3.3.3 is completed. $\square$

Let $z(t)$ be the nonprincipal solution of the unforced equation (3.82), i.e., $z(t)$ satisfies

$$\int^\infty \frac{ds}{p(s)z^2(s)} < \infty. \tag{3.95}$$

Define the following function $H(t)$,

$$H(t) := \int^t \frac{1}{p(s)z^2(s)} \left(\int^s z(\tau)f(\tau)d\tau + \sum_{\theta_i < s} z(\theta_i)f_i \right) ds. \tag{3.96}$$

Theorem 3.3.4. *Suppose that (3.82) is nonoscillatory and let $z(t)$ be a non-principal solution. Then equation (3.81) oscillatory if*

$$\overline{\lim_{t\to\infty}}\, H(t) = -\underline{\lim_{t\to\infty}}\, H(t) = +\infty. \tag{3.97}$$

Proof. The change of variable $y = z(t)w(t)$ transforms (3.81) into

$$(p(t)z^2w')' = f(t)z, \qquad t \neq \theta_i; \tag{3.98}$$

$$\Delta p(t)z^2w' = f_iz, \qquad t = \theta_i. \tag{3.99}$$

When $z(t)$ is a solution of (3.82), we can express $w(t)$ by integration of (3.98) and using (3.99) as follows,

$$w(t) = c_1 + c_2 \int_{t_0}^t \frac{ds}{p(s)z^2} + \int_{t_0}^t \frac{1}{p(s)z^2} \left(\int_{t_0}^s zf(\tau)d\tau + \sum_{t_0 \le \theta_i < s} z(\theta_i)f_i \right) ds$$

where c_1 and c_2 are constants depending on the initial conditions $w(t_0)$ and $w'(t_0)$. Note that $z(t)$ nonprincipal solution, so (3.95) and (3.97) imply that $w(t)$ satisfies

$$\overline{\lim_{t\to\infty}}\, w(t) = -\underline{\lim_{t\to\infty}}\, w(t) = +\infty. \tag{3.100}$$

Because $z(t)$ is nonoscillatory (3.100) implies that $w(t)$ is oscillatory. Hence $y = z(t)w$ is also oscillatory. $\square$

Note that if $q_i \equiv f_i \equiv 0$, we recover the result of Wong [82].

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VITA

PERSONAL INFORMATION

Surname, Name: ÖZBEKLER, Abdullah
Nationality: Turkish (T. C.)
Date and Place of Birth: 12 March 1974, Ankara
Marital Status: Single
Phone: +90 542 581 19 95
email: abdullah@atilim.edu.tr

EDUCATION

Degree	Institution	Year of Graduation
MS	METU, Mathematics	1999
BS	METU, Mathematics Education	1995
High School	Yahya Kemal Beyatlı Lisesi	1991

WORK EXPERIENCE

Year	Place	Enrollment
2004 - Present	Atılım University, Mathematics	Instructor
1996 - 2004	METU, Mathematics	Research Assistant

FOREIGN LANGUAGE

Advanced English

FIELD OF STUDY

Major Field: Differential Equations