INVARIANT SUBSPACES OF POSITIVE OPERATORS ON RIESZ SPACES AND OBSERVATIONS ON $CD_0(K)$-SPACES

MERT ÇAĞLAR

AUGUST 2005

INVARIANT SUBSPACES OF POSITIVE OPERATORS ON RIESZ SPACES AND OBSERVATIONS ON $CD_0(K)$-SPACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

MERT ÇAĞLAR

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

AUGUST 2005

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Canan ÖZGEN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Şafak ALPAY
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Zafer ERCAN
Supervisor

Examining Committee Members

Prof. Dr. Şafak ALPAY	(METU, MATH)	
Assoc. Prof. Dr. Zafer ERCAN	(METU, MATH)	
Prof. Dr. Eduard Yu. EMEL'YANOV	(METU, MATH)	
Assist. Prof. Dr. Cevriye TONYALI	(GAZİ UNIV., MATH)	
Assoc. Prof. Dr. Bahri TURAN	(GAZİ UNIV., MATH)	

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last name: Mert ÇAĞLAR
Signature:

Abstract

INVARIANT SUBSPACES OF POSITIVE OPERATORS ON RIESZ SPACES AND OBSERVATIONS ON $CD_0(K)$-SPACES

Çağlar, Mert
Ph.D., Department of Mathematics
Supervisor: Assoc. Prof. Dr. Zafer ERCAN

August 2005, 29 pages

The present work consists of two main parts. In the first part, invariant subspaces of positive operators and operator families on locally convex solid Riesz spaces are examined. The concept of a weakly-quasinilpotent operator on a locally convex solid Riesz space has been introduced and several results that are known for a single operator on Banach lattices have been generalized to families of positive or close-to-them operators on these spaces.

In the second part, the so-called generalized Alexandroff duplicates are studied and $CD_{\Sigma,\Gamma}(K, E)$-type spaces are investigated. It has then been shown that the space $CD_{\Sigma,\Gamma}(K, E)$ can be represented as the space of E-valued continuous functions on the generalized Alexandroff duplicate of K.

Keywords: Riesz space, positive operator, weak quasinilpotence, $CD_0(K)$-space, Alexandroff duplicate.

Öz

RIESZ UZAYLARI ÜZERİNDEKİ POZİTİF OPERATÖRLERİN DEĞİŞMEZ ALT-UZAYLARI, VE $CD_0(K)$-UZAYLARI ÜZERİNE GÖZLEMLER

Çağlar, Mert
Doktora, Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Zafer ERCAN

Ağustos 2005, 29 sayfa

Eldeki çalışma iki ana bölümden oluşmaktadır. İlk bölümde lokal konveks katı Riesz uzayları üzerindeki pozitif operatörler ve operatör ailelerinin değişmez alt-uzayları incelenmiştir. Lokal konveks katı bir Riesz uzayı üzerinde zayıf-hemen hemen-sıfır-güçlü operatör kavramı tanıtılmış ve Banach örgüleri üzerinde tanımlı tek bir operatör için bilinen pek çok sonuç lokal konveks katı Riesz uzayları üzerindeki pozitif ya da pozitife-benzer operatör ailelerine genelleştirilmiştir.

Çalışmanın ikinci bölümünde genelleştirilmiş Alexandroff kopyaları olarak bilinen uzaylar çalışılmış ve $CD_{\Sigma,\Gamma}(K,E)$-tipi uzaylar tanıtılmıştır. Ardından $CD_{\Sigma,\Gamma}(K,E)$-uzayının, K'nın genelleştirilmiş Alexandroff kopyası üzerindeki E-değerli sürekli fonksiyonlar uzayı olarak temsil edilebileceği gösterilmiştir.

Anahtar Kelimeler: Riesz uzayı, pozitif operatör, zayıf-hemen hemen-sıfır-güçlülük, $CD_0(K)$-uzayı, Alexandroff kopyası.

To my family

Acknowledgments

I would like to express my sincere gratitude to my supervisor, Assoc. Prof. Dr. Zafer ERCAN, for his precious guidance and encouragement throughout the research. Without his efforts, this work would have never come into existence. I offer special thanks to Prof. Dr. Şafak ALPAY for many helpful conversations and the motivation he has given.

It is impossible for me to thank properly to my parents, Reşan and Çetin ÇAĞLAR, and my brother, Yiğit ÇAĞLAR, for their precious love and encouragement they have given to me for all my life.

I wish to wholeheartedly express my appreciation to my beloved, Ferhan KILIÇ, for being with me during the very hard times we have had so far.

I am indepted to the members of the Department of Mathematics for the atmosphere they have provided that broadened my horizons during my graduate work. I am, as usual, grateful to Dr. Ömür UĞUR for his help in computer software. I thank my friend and roommate, Dr. Ali ÖZTÜRK, and my friend, Dr. Müjdat KAYA, for the technical support in computer hardware.

Finally, I thank those who care me all.

Table of Contents

Chapter 1

Introduction

However simple it may seem, the concept of invariant subspace is of fundamental importance and ubiqutious. Having its roots in finite-dimensional linear algebra, it became, since the second quarter of the 20th century, one of the main tools to investigate and to understand the structure of operators. The general *invariant subspace problem* concerns bounded linear operators on complex, infinite-dimensional, separable Hilbert spaces, which are, up to isomorphism, the space of all square-summable sequences of complex numbers, and asks whether there exists a subspace that is mapped to itself by such an operator. It should be noted that there are counterexamples to the corresponding problems on Fréchet spaces [10] and on Banach spaces [16]. Despite the ground-breaking results such as that of V.I. Lomonosov [23] and P. Enflo [16], which provided new directions, several reformulations of the invariant subspace problem look quite different from the original problem and make it one extremely difficult to settle properly. Indeed, it is almost impossible today to even mention the most significant results in this area due to the vast literature with oodles of different directions dedicated to it.

The picture that emerges for ordered normed spaces and Banach lattices is still not tenable, but allows one to have clearer insights and thoroughness when compared to the situation revealed in Hilbert and Banach spaces. It is the strong interest and accumulated work of Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw on the problem [1-4] that has brought to the mathematical community's attention the extra information and facilities gained by the natural order properties that the classical spaces of functional analysis have.

The main goal of the present thesis, which consists of two main parts, is twofold:

The first one is about the invariant subspace problem and it is aimed to extend some results, chosen on an ad hoc basis from the work originally obtained in the setting of positive operators on Banach lattices by numerous authors, to those for positive or close-to-them operators or operator families on locally convex solid Riesz spaces. **Chapter two** deals with the presentation of the invariant subspace problem with a brief historical background and contains the classical theorem of Lomonosov along with its basic consequences.

The purpose of **Chapter three** is to give the main results of this part of thesis. Therein, several results that are known for a single operator on Banach lattices have been generalized to families of positive or close-to-them operators or operator families.

The second goal of the thesis is to present the so-called Alexandroff duplicates and to investigate $CD_{\Sigma,\Gamma}(K,E)$-type spaces, and then to develop a representation theorem for the space $CD_{\Sigma,\Gamma}(K,E)$ as the space of E-valued continuous functions on the generalized Alexandroff duplicate of K, which is achieved in **Chapter four**.

Chapter 2

The invariant subspace problem

2.1 Statement of the problem and some historical background

From now on, the term "operator" will always mean a "linear operator" (i.e., $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all x, y in the domain of T and all $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$). For an arbitrary pair of Banach spaces X and Y, the symbol $L(X, Y)$ will denote the vector space of all continuous operators from X into Y. We shall write $L(X)$ instead of $L(X, X)$. In case where $Y = \mathbb{R}$, $L(X, Y)$ is called the *norm dual* of X and will be denoted by X^*.

Let $T : X \to X$ be an operator on a Banach space X. A subspace V of X is called *T-invariant* if $T(V) \subseteq V$. If V is S-invariant under every continuous operator S which commutes with T (i.e., $ST = TS$), then V is called *T-hyperinvariant*. A vector subspace is *non-trivial* if it is different from $\{0\}$ and X. The invariant subspace problem is the following question:

> *Does there exist a non-trivial closed T-invariant subspace $V \subseteq X$ for the continuous linear operator T on X?*

This question arises naturally from the theory of eigenvectors in finite-dimensional spaces. Recall that an *eigenvalue* of an operator T is a number λ such that there exists an element $x_0 \neq 0$ with the property that $Tx_0 = \lambda x_0$. An element x for which the equation $Tx = \lambda x$ holds is called an *eigenvector* corresponding to the given eigenvalue λ. The set

$$N_\lambda = \{x \in X \mid Tx = \lambda x\}$$

is called the *eigenspace* corresponding to the eigenvalue λ. If x is an eigenvector, of course, $V = \{\lambda x \mid \lambda \in \mathbb{C}\}$ is T-invariant. But there exist operators with no eigenvalues in infinite-dimensional spaces [10]. So, some other concept has to be substituted for it, and the concept of a non-trivial invariant subspace is the broadest and the most natural one.

If X is a finite-dimensional complex Banach space of dimension greater than one, provided that T is not a multiple of the identity operator I, N_λ is a non-trivial closed T-hyperinvariant subspace. Indeed, this subspace is clearly closed and non-trivial since $T \neq \lambda I$; and for each $x \in N_\lambda$ and S in the commutant[1] of T, we have $TSx = STx = S(\lambda x) = \lambda Sx$, so that $Sx \in N_\lambda$. Hence, every non-zero operator T on a finite-dimensional complex Banach space X of dimension greater than one has a non-trivial, closed, hyperinvariant subspace.

If X is non-separable, then the subspace

$$V_x = \overline{\text{span}}\left\{x, Tx, T^2x, \ldots\right\}$$

for a fixed point $0 \neq x \in X$ is a non-trivial closed T-invariant subspace. Thus, the invariant subspace problem is of substance only when X is an infinite-dimensional, separable Banach space.

A similar concept occurs for the hyperinvariant subspaces: instead of taking the iterates of T, we take all operators commuting with T, and define

$$G_x = \overline{\text{span}}\left\{Ax \mid A \text{ commutes with } T\right\}.$$

Then, G_x is a non-trivial, closed, T-hyperinvariant subspace.

We will mention now some milestones in the theory of invariant subspaces.

The first result about the existence of an invariant subspace is the one proved in 1935 by J. von Neumann [21]. He proved that every compact operator on a Hilbert space has a non-trivial, closed, invariant subspace. This result of J. von Neumann was published in a paper by N. Aronszajn and K.T. Smith [4].

[1]The commutant of a continuous operator $T : X \to X$ on a Banach space is the set of all continuous operators on X which commute with T.

In 1947, Godement [19] has proved that for certain classes of operators T, there exist invariant subspaces having an additional property: these subspaces are invariant for all operators commuting with T; that is, those which are called hyperinvariant subspaces in today's terminology.

Although the general operator remains a mystery in Hilbert spaces, one can say quite a bit about the invariant subspaces of a handful of specific operators, and results of this nature are often connected with interesting theorems in analysis. This was first realized in 1947 by A. Beurling [26], who gave a complete classification of the invariant subspaces of the unilateral shift operator; that is, the operator on l_2 defined by $(c_0, c_1, c_2, \ldots) \mapsto (0, c_1, c_2, \ldots)$.

J. Wermer [19] was the first who, in 1952, opening the way towards the results about the invariant subspaces of the quasinilpotent operators[2], obtained a theorem about the existence of invariant subspaces for a class of operators T for which $\|T^n\|$ has a special growth.

In 1954, N. Aronszajn and K.T. Smith [4] proved that compact operators on infinite-dimensional Banach spaces have non-trivial invariant subspaces. This result was the generalization of J. von Neumann's theorem to Banach spaces.

An interesting contribution to the invariant subspace problem was made in 1966 by A.R. Bernstein and A. Robinson [11], who used non-standard analysis to prove the following result: if T is a bounded operator on a Banach space and $p(T)$ is compact for some non-zero polynomial p, then T has an invariant subspace.

In 1968, W. Arveson and J. Feldman [19] have proved that if T is a quasinilpotent operator and the closed algebra generated by T and the identity operator I contains a non-zero compact operator, then T has an invariant subspace.

The most powerful contribution to the invariant subspace problem which astounded the mathematical world came in 1973 from V.I. Lomonosov [23], who introduced an elegant technique which enabled him to solve some hard problems in the theory that pertain to compact operators. He proved that a non-zero compact operator on a Banach space has hyperinvariant subspaces. We will deal with Lomonosov's theorem and its basic consequences in detail in the next section.

[2] See Definition 3.1.1.

Until the middle of the 1970's, the invariant subspace problem was phrased more stronger than our formulation above: it asked whether *every* continuous linear operator on a (separable) Banach space has a non-trivial invariant subspace. This question solved negatively in 1976 by P. Enflo [16], who constructed an example of a continuous operator on a Banach space without a non-trivial closed invariant subspace. Due to his counterexample, the invariant subspace problem for operators on Banach spaces has been confined to the search for various classes of operators for which one can guarantee the existence of an invariant subspace.

An important consequence of Lomonosov's theorem states that every operator T which commutes with an operator different than the identity operator which commutes with a non-zero compact operator has invariant subspaces. The question comes naturally to know whether there are operators which are not of this type. In other words, does the Lomonosov's theorem and, of course, that consequence solve the invariant subspace problem? We know now that this cannot be true in general, since P. Enflo solved it negatively. Moreover, it is still not known how large is the class of operators to which Lomonosov's theorem applies. In 1980, D. Hadvin-E. Nordgren-H. Radjavi-P. Rosenthal [10] produced an example of an operator to which Lomonosov's theorem does not apply; that is, which is not in the "bicommutant" (i.e., the commutant of the commutant) of the compact operators. This operator is a weighted shift on l_2.

In 1985, C.J. Read [27] gave an example of a continuous operator on l_1 without a non-trivial, closed, invariant subspace.

A positive linear operator T from a Banach lattice E into itself is said to be ideal irreducible if there exists no non-trivial closed T-invariant ideal. In 1986, de Pagter [25] proved that every compact quasinilpotent positive operator on a Banach lattice has a non-trivial, closed, invariant ideal. As an immediate consequence of this result, de Pagter [24] obtained a remarkable Andô-Krieger type of result which says that every ideal irreducible compact positive operator on a Banach lattice has positive spectral radius.

In 1995, Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [4] presented their survey which described some recent results for positive and close-to-them operators on Banach lattices. Therein, they showed that an extensive use of the theory of operators on Banach lattices and of their order structure is very helpful in dealing with the invariant subspace problem.

A collection $\mathcal{S}$ of bounded operators on a Banach space is said to be a *multiplicative semigroup*, if for each $S, T \in \mathcal{S}$, the operator ST also belongs to $\mathcal{S}$. An algebra of operators in which every operator is compact and quasinilpotent is called a *Volterra algebra*. There are a few major recent generalizations of Lomonosov's theorem to algebras and semigroups. A very important breakthrough was obtained in 1984 by V.S. Shulman [28], who proved that each non-zero Volterra algebra has a non-trivial, closed, hyperinvariant subspace. It took more than a decade to generalize this result to Volterra semigroups of operators. This deep and significant contribution has been done in 1999 by Y.V. Turovskii [31], who proved that each non-zero multiplicative Volterra semigroup has a non-trivial, closed, hyperinvariant subspace.

2.2 Lomonosov's Theorem

Lomonosov's theorem is one of the most important contributions to the invariant subspace problem. We present here its original form. Recall that an operator $T \in L(X, Y)$ between two normed spaces is said to be *compact* if $\overline{T(U)}$ is compact in Y, where U is the open unit ball in X.

Theorem 2.2.1 (Lomonosov [**23**]). *Let T be a non-zero, compact operator on an infinite-dimensional complex Banach space X. Then T has a non-trivial, closed, hyperinvariant subspace.*

Proof. The proof is by contradiction; suppose that the assertion of the theorem is false. In particular, this means that T lacks eigenvectors.

Take a point x_1 such that $Tx_1 \neq 0$, and take $x_0 = \lambda x_1$ for λ large enough, to get

$$\inf \{\|Tx\| \mid x \in B(x_0, 1)\} > 0,$$

where $B(x_0, 1)$ denotes the closed unit ball with center x_0. We write B for $B(x_0, 1)$.

Let $\mathfrak{R}$ be the algebra of all operators which commute with T. We will show that there is a point $y_0 \in X$, $y_0 \neq 0$, such that

$$\|T'y_0 - x_0\| \geq 1, \quad \forall T' \in \mathfrak{R}. \tag{2.1}$$

This proves the theorem: indeed, let $F = \overline{\text{span}}\,\{T'y_0 \mid T' \in \mathfrak{R}\}$. Then, F is an hyperinvariant subspace for T, and since it does not intersect the interior of B, it cannot be the whole space. So, all we have to do is to prove (2.1).

Assume conversely that

$$\forall y \neq 0, \quad \exists T' \in \mathfrak{R}, \quad \text{with} \quad \|T'y - x_0\| < 1.$$

Since $\overline{TB}$ is compact, we can find a finite number of operators $T'_1, \dots, T'_n$ in $\mathfrak{R}$ such that, for all $y \in \overline{TB}$, there exists an i $(1 \leq i \leq n)$ with $\|T'_i y - x_0\| < 1$.

Set $f(t) = 1 - t$ if $0 \leq t \leq 1$, 0 if not, and let $\Phi : \overline{TB} \to E$ be defined by

$$\Phi(y) = \frac{\sum_{i=1}^n f(\|T'_i y - x_0\|) T'_i y}{\sum_{i=1}^n f(\|T'_i y - x_0\|}.$$

This is a continuous function on the compact $\overline{TB}$, so its image is compact. This image is a convex combination of those of the $T'_i y$ which belong to the ball B, therefore it is contained in this ball.

So $\Phi \circ T$ is a continuous function from B onto a relatively compact subset contained in B. Set $K = \text{con}(\overline{\Phi \circ T(B)})$, (where "con" is the convex hull). Then K is convex, compact, and $\Phi \circ T$ is continuous from K into K, and so has a fixed point x by Schauder's fixed point theorem [14]. This implies

$$\sum_{i=1}^n \alpha_i T'_i T x = x,$$

where

$$\alpha_i = \frac{f(\|T'_i T x - x_0\|)}{\sum_{j=1}^n f(\|T'_j T x - x_0\|)}.$$

We now look at the set

$$G = \left\{ z \in E \mid \sum_{i=1}^n \alpha_i T'_i T z = z \right\}.$$

This is a vector subspace, not reduced to $\{0\}$ since it contains x, and of finite-dimension since $\sum_1^n \alpha_i T'_i T$ is compact, and invariant by T. So, T has an eigenvalue, which contradicts our assumption, and proves (2.1). The proof of the theorem is now complete. □

As pointed out by Lomonosov, a slight variation of this technique shows that an even larger class of operators have hyperinvariant subspaces.

Theorem 2.2.2 (Lomonosov [**23**])**.** *If T is not a multiple of the identity on a complex Banach space and if it commutes with a non-zero compact operator S, then T has hyperinvariant subspaces.*

Proof. In the previous argument, just replace $\overline{TB}$ by $\overline{SB}$, $\Phi \circ T$ by $\Phi \circ S$, and G becomes

$$G = \left\{ z \in E \mid \sum_{i=1}^{n} \alpha_i T_i' S z = z \right\},$$

which is still invariant under T since T commutes with S and the operators T_i'. $\square$

Let us point out the following special case of Theorem 2.2.2.

Corollary 2.2.3. *Let S, T be commuting operators on a complex Banach space such that S commutes with a non-zero compact operator, and is not a multiple of the identity. Then T has a non-trivial, closed, invariant subspace.*

As a rather special case of this corollary, one obtains the theorem of N. Aronszajn and K.T. Smith mentioned in the previous section.

Corollary 2.2.4. *Every compact operator on an infinite-dimensional complex Banach space has a non-trivial invariant subspace.*

An extension of this result, first proved by A.R. Bernstein and A. Robinson, needs only a little work.

Corollary 2.2.5. *Let X be an infinite-dimensional complex Banach space and let T be a continuous operator on X such that $p(T)$ is a compact operator on X for some non-zero complex polynomial $p(z)$. Then T has a non-trivial, closed, invariant subspace.*

Proof. Let

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad \text{with} \quad a_n \neq 0.$$

If $p(T) \neq 0$, then, as $Tp(T) = p(T)T$, the assertion follows from Theorem 2.2.1.

If, on the other hand, $p(T) = 0$, then $a_n T^n = -\sum_{k=0}^{n-1} a_k T^k$, and so

$$V_x = \overline{\operatorname{span}} \left\{ x, Tx, T^2 x, \ldots \right\}$$

is a T-invariant, closed subspace for every $x \neq 0$. $\square$

Chapter 3

Invariant subspaces for positive operators on locally solid Riesz spaces

The fundamental theorem of Lomonosov and its basic consequences mentioned in the previous chapter are, alas, valid for compact operators on Banach spaces. It is of special interest to what extend these type of results can be generalized to positive operators on locally convex solid Riesz spaces and it is the purpose of this chapter to extend several known results for a single operator on a Banach lattice to families of positive or close-to-them operators on locally convex solid Riesz spaces. We refer to [1] and [8] for the whole standard terminology and detailed information about locally convex solid Riesz spaces, respectively.

3.1 Operator families

While studying the invariant subspace problem for positive operators extensively, the concept of an x_0-quasinilpotent operator on a Banach lattice, which was introduced in [2], played a primary role in the work of Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [1-3]. This fact reads as follows.

Definition 3.1.1. *An operator T on a Banach space X is said to be quasinilpotent at a non-zero x_0 if* $\lim_{n\to\infty} \|T^n(x_0)\|^{1/n} = 0$.

It has been proved in [18] that if T is a bounded operator on a Banach space X and x_0 is a nonzero element of X, one has $\lim_{n\to\infty} \|T^n(x_0)\|^{1/n} = 0$ if and only if $\lim_{n\to\infty} |f \circ T^n(x_0)|^{1/n} = 0$ for each $f \in X^*$, where X^* denotes the norm dual of X. This result enables one to introduce the following slightly more general concept which is useful in connection with the invariant subspace problem for operators on locally convex solid Riesz spaces.

Definition 3.1.2. *A non-empty set $\mathcal{M}$ of non-zero linear operators on a topological vector space X is called weakly quasinilpotent at $x_0 \in X$ (or, weakly x_0-quasinilpotent) if $\lim_{n\to\infty} |f \circ \mathcal{M}^n(x_0)|^{1/n} = 0$ for each positive functional $f \in X'$, where*

$$|f \circ \mathcal{M}^n(x_0)| := \sup\{|f \circ M_1 \circ M_2 \circ \cdots \circ M_n(x_0)| \mid M_i \in \mathcal{M}, i = 1, 2, \ldots, n\}$$

and X' denotes the topological dual of X, i.e., the set of all continuous functions with respect to the linear topology of X.

Generalizing the main result of [2] using weak quasinilpotence defined above constitutes the subject matter of this section.

We shall write X_+ for the set $\{x \in X \mid 0 \leq x\}$ for a space X ordered with the partial order " $\leq$ ". An element x of an Archimedean Riesz space X is called an *atom* (or, *discrete*) if the vector subspace generated by x is the (order-) ideal generated by the same element. An Archimedean vector lattice X is called *discrete* if the band generated by the atoms of X is X. It is well-known that an Archimedean vector lattice X is discrete if and only if it is Riesz isomorphic to an order dense Riesz subspace of some Riesz space of the form $\mathbb{R}^I$, where I denotes an index set. The commutant $\mathcal{M}'$ of a set $\mathcal{M}$ of operators on a Riesz space X is the set of all continuous operators T on X satisfying $TM = MT$ for all $M \in \mathcal{M}$.

Throughout the rest of the section, X will denote a discrete Archimedean locally convex solid Riesz space with $\dim(X) > 1$, and x_0 will denote a non-zero element of X_+. The proof of the following theorem is based on the proof of the main theorem of [2].

Theorem 3.1.3. *If $\mathcal{M}$ is a weakly x_0-quasinilpotent set of non-zero continuous positive operators on X, then $\mathcal{M}$ has a common non-trivial, closed, invariant ideal.*

Proof. One can suppose that X is an order dense Riesz subspace of $\mathbb{R}^I$ for some non-empty set I. Since X is order dense in $\mathbb{R}^I$, for each $i \in I$, the characteristic function

$\chi_{\{i\}}$ of $\{i\}$ is in X. One may choose $j \in I$ so that $y = k\chi_{\{j\}} \leq x_0$ for some non-zero positive real number k. Moreover, one might suppose that $k = 1$ (otherwise, x_0 can be replaced by $\frac{1}{k}x_0$). Let $P : X \to X$ be defined by $P((x_i)_{i\in I}) = x_j y$. Then, P is continuous. Now, the following two mutually exclusive cases can be identified:

(i) If $S(x_0) \neq 0$ for some $S \in \mathcal{M}$, then for any $T_1, T_2, \ldots, T_m \in \mathcal{M}$ one has $P \circ T_1, T_2 \ldots T_m \circ S(y) = 0$. Indeed, for any combination $T_1T_2 \ldots T_m$, there exist $0 \leq \alpha$ and $f \in X'_+$ such that $P \circ (T_1T_2 \ldots T_m) \circ S(y) = \alpha y$ and $f(y) = 1$. Then, we have

$$0 \leq \alpha^n y = (P \circ (T_1T_2 \ldots T_m) \circ S)^n(y) \leq ((T_1T_2 \ldots T_m) \circ S)^n(y).$$

This gives

$$0 \leq \alpha^n = \alpha^n f(y) \leq f \circ \mathcal{M}^{n(m+1)}(y),$$

which implies that $\alpha \leq \left|f \circ \mathcal{M}^{n(m+1)}(y)\right|^{1/n} \to 0$ as $n \to \infty$. Hence $\alpha = 0$, that is, $P \circ (T_1T_2 \ldots T_m) \circ S(y) = 0$. We claim that, for any combination $T_1T_2 \ldots T_n$, the ideal generated by $\langle\{T_1T_2 \ldots T_n Sy \mid n \in \mathbb{N}, T_1T_2 \ldots T_n \in \mathcal{M}^n\}\rangle$ is the required invariant subspace, where

$$\mathcal{M}^i = \{T_1 \circ T_2 \circ \cdots \circ T_i \mid T_k \in \mathcal{M}, 1 \leq k \leq n\}$$

Since $S(x_0) \in J$, it follows that $J \neq \{0\}$. It remains only to show that $\overline{J} \neq X$. To see this, observe that J lies inside $\mathrm{Ker}(P)$ and $P(x) = 0$ for each $x \in J$, so $f(x) = 0$. Thus $f(\overline{J}) = \{0\}$. As $f \neq 0$, we have that $\overline{J}$ is non-trivial.

(ii) If $S(x_0) = 0$ for each $S \in \mathcal{M}$, then the closure of the ideal generated by x_0 is the required invariant subspace. This completes the proof. □

Theorem 3.1.4. *Let $\mathcal{M}$ be a set of non-zero continuous positive operators on X with*

$$\lim_{n\to\infty} |f \circ \mathcal{M}^n(x_0)|^{1/n} = 0$$

for each $f \in X'$ and T be a continuous operator on X. If $|T| \in \mathcal{M}$ exists and $|T| \in \mathcal{M}'_+$, then T has a non-trivial, closed, invariant ideal.

Proof. If $S(x_0) = 0$ for each $S \in \mathcal{M}$, then the closure of the ideal generated by $\{|T|^i(x_0) \mid i \in \mathbb{N}\}$ is a non-trivial closed invariant ideal for T. If, on the other hand,

$0 < S(x_0)$ for some $S \in \mathcal{M}$, then as $|T| \in \mathcal{M}$, by a similar argument to that of Theorem 3.1.3, the weak x_0-quasinilpotence of $\mathcal{M}$ implies that $|f\,|T|^n\, S^n y|^n \to 0$ as $n \to \infty$. Then $\{|T|^i\, S(x_0) \mid i = 0, 1, \ldots\}$ is the required invariant ideal of T and the proof is complete. □

The following theorem generalizes the main theorem of [2].

Theorem 3.1.5. *Let $\mathcal{M}$ be a non-empty subset of non-zero continuous positive operators on X and T be a continuous operator on X with module $|T|$. If $|T| \in \mathcal{M}'$ and $\mathcal{M}_i := \{|T|^i \circ M \mid M \in \mathcal{M}\}$ is weakly x_0-quasinilpotent for each $i = 0, 1, \ldots,$ then T has a non-trivial, closed, invariant ideal.*

Proof. The non-trivial invariant subspace $\overline{J}$ for T can be chosen as

$$J = \left\{ x \in X \mid |x| \leq \alpha \sum_{i=0}^{n} |T|^i \circ S(x_0) \text{ for some } \alpha \text{ and } n \right\}$$

if $0 < S(x_0)$ for some $S \in \mathcal{M}$, and

$$J = \left\{ x \in X \mid |x| \leq \alpha \sum_{i=1}^{n} |T|^i\,(x_0) \text{ for some } \alpha \text{ and } n \right\}$$

if $S(x_0) = 0$ for each $S \in \mathcal{M}$, and the proof is complete. □

Corollary 3.1.6 (Abramovich-Aliprantis-Burkinshaw [2])**.** *Let X be a discrete Banach lattice with order continuous norm, S and T be non-zero operators on X such that $0 \leq S$, the module $|T|$ exists and S is quasinilpotent at x_0. If $|T|\,S = S\,|T|$, then T has a non-trivial, closed, invariant ideal.*

Proof. The set $\mathcal{M}_i = \{|T|^i \circ S^m \mid m \in \mathbb{N}\}$ is weakly x_0-quasinilpotent for each $i = 0, 1, \ldots$. Then, by the previous theorem, T has a non-trivial, closed, invariant ideal. □

Let E be an Archimedean Riesz space such that the order dual $E^\sim$ separates the points of E. Let us call a subset $\mathcal{M}$ of non-zero order continuous operators on E *order x_0-quasinilpotent* if $\lim_{n\to\infty} |f \circ \mathcal{M}^n(x)|^{1/n}$ for each $f \in E^\sim$, where

$$|f \circ \mathcal{M}^n(x)| := \sup\{|f \circ M_1 \circ M_2 \circ \cdots \circ M_n(x)| \mid M_i \in \mathcal{M}, 1 \leq i \leq n\}.$$

We can now formulate the following result for order continuous operators on E in the light of the above-mentioned concept. Its proof is rather similar to those of Theorem 3.1.3 and Theorem 3.1.4.

Theorem 3.1.7. *Let E be an Archimedean discrete Riesz space, $\mathcal{M}$ be an order x_0-quasinilpotent subset of non-zero positive order continuous operators on E, and T be an order continuous positive operator on E. Then,*

- **(i)** *$\mathcal{M}$ has a common non-trivial, order closed invariant ideal;*
- **(ii)** *If $|T| \in \mathcal{M}$, then T has a non-trivial, order closed invariant ideal if $S|T| = |T|S$ for all $S \in \mathcal{M}$ and*

$$\lim_{n\to\infty} |f \circ \mathcal{M}^n(x_0)|^{1/n} = 0$$

for each $f \in E^{\sim}$.

3.1.1 Spaces with a Markushevich basis

Some of the results on the existence of invariant subspaces of positive operators on Banach lattices can be extended to Banach spaces ordered by the cone generated by a basis and this idea was first used in [3]. Among many kind of bases on a Banach space, the so-called Markushevich bases can be viewed as a generalization of the classical Schauder bases.

A sequence $(x_n, f_n)_{n\in\mathbb{N}}$ in $X \times X'$, where X is a Hausdorff topological vector space with topological dual X', is called a *Markushevich basis*; if the span of $(x_n)_{n\in\mathbb{N}}$ is dense in X, $f_n(x_n) = 1$ and $f_n(x_m) = 0$ for $n \neq m$, and $(f_n)_{n\in\mathbb{N}}$ separates the points of X. It is obvious that any Schauder basis for a Banach space is a Markushevich basis. Also, it is well-known that a Hausdorff topological vector space X has a Markushevich basis whenever $(X, \sigma(X, X^*))$ and $(X^*, \sigma(X^*, X))$ are separable, and that each separable and metrizable locally convex space has a Markushevich basis [20, 29]. The following theorem generalizes the main result of [18] to families of positive operators on locally convex solid Riesz spaces..

Theorem 3.1.8. *Let X be a discrete, Archimedean locally convex solid Riesz space with a Markushevich basis (x_n, f_n), and x_0 be a positive vector of X. If $T \in L(X)$ is a non-zero, continuous, positive operator and $\mathcal{A}$ is a subalgebra generated by a subset*

$\mathfrak{B}$ *of* $L_+(X)$ *consisting of non-zero, continuous, positive operators such that* $T \in \mathcal{A}'$ *(i.e.,* $AT = TA$ *for all* $A \in \mathcal{A}$*) and* $\mathcal{A}T$ *is weakly* x_0*-quasinilpotent, then* T *has a non-trivial, closed, invariant subspace.*

Proof. Since $x_0 > 0$, there exists a j such that $f_j(x_0) > 0$. Assume, by an appropriate scaling, that $f_j(x_0) > 1$. This implies that $x_0 - x_j \geq 0$. Indeed, if $i \neq j$, then $f_i(x_0 - x_j) = f_i(x_0) \geq 0$; and if $i = j$, then $f_j(x_0 - x_j) = f_i(x_0) - 1 \geq 0$, i.e., $f_i(x_0 - x_j) \geq 0$ for each i.

(i) If $\mathfrak{B}(x_j) = 0$, i.e., $A(x_j) = 0$ for each $A \in \mathcal{A}$, then

$$\bigcap_{A \in \mathcal{A}} \{x \in X \mid Ax = 0\}$$

is a non-trivial closed subspace of X which is T-invariant.

(ii) If $\mathfrak{B}(x_j) > 0$, i.e., $A(x_j) > 0$ for some $A \in \mathcal{A}$, then consider the projection P on X defined by $P(x) = f_j(x)x_j$. Clearly, one has $0 \leq P \leq I$. We claim that $PT^m\mathfrak{B}(x_j) = 0$ holds for each $m \in \mathbb{N} \cup \{0\}$. Indeed, assume that $PT^m\mathfrak{B}(x_j) = \alpha x_j$ for some $\alpha \geq 0$. Then, from the inequalities

$$0 \leq \alpha^n x_j = (PT^m\mathfrak{B})^n(x_j) \leq (T^m\mathfrak{B})^n(x_j) = T^{mn}\mathfrak{B}^n(x_j) \leq T^{mn}\mathfrak{B}^n(x_0)$$

it follows that $0 \leq \alpha \leq f_j(T^{mn}\mathfrak{B}^n(x_0))^{1/n} \to 0$ as $n \to \infty$, so, $\alpha = f_jT^m\mathfrak{B}(x_j) = 0$. Let

$$V := \left\langle \bigcup_{A \in \mathcal{A}} \{T^m A(x_j) \mid m \in \mathbb{N} \cup \{0\}\} \right\rangle.$$

Then, one obtains $\mathfrak{B}(x_j) \in V \neq \{0\}$. Since $PT^m\mathfrak{B}(x_j) = 0$ for each $m \in \mathbb{N} \cup \{0\}$ and $f_j(x_j) = 1$, one has $x_j \notin \overline{V}$. As V is T-invariant, $\overline{V}$ is the required T-invariant subspace. $\square$

Corollary 3.1.9. *If* $|T|$ *exists,* $|T| \in \mathcal{A}'$ *and* $\mathcal{A}_i := \{|T|^i \circ A \mid A \in \mathcal{A}\}$ *is weakly* x_0*-quasinilpotent for each* $i \in \mathbb{N} \cup \{0\}$*, then* T *has a non-trivial, closed, invariant ideal.*

3.2 Operators on complexified spaces

Y.A. Abramovich, C.D. Aliprantis, G. Sirotkin and V.G. Troitsky presented in [5] some open problems and conjectures associated with the invariant subspace problem. Therein, they observe that the case of real Banach spaces, when considered for the search for invariant subspaces, has almost no connection with that of the complex case. Their paper deals in detail with three conjectures on the invariant subspaces for operators on Banach spaces, the last of which ("Conjecture 3") is for the complexification operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ defined by $T_{\mathbb{C}}(x + iy) = Tx + iTy$, which is the natural continuous linear extension of the operator $T : X \to X$ on a Banach space X, where $X_{\mathbb{C}}$ denotes the complexification of X via

$$X_{\mathbb{C}} := X \oplus iX = \{x + iy \mid x, y \in X\}$$

equipped with the norm $||x + iy|| = \sup_{\varphi\in[0,2\pi]} ||x \cos\varphi + y \sin\varphi||$. The conjecture asserts that $T_{\mathbb{C}}$ has no non-trivial closed invariant subspaces provided that T is an operator without non-trivial invariant subspaces on a separable, real Banach space X.

3.2.1 Problems related to "Conjecture 3"

An invariant closed subspace V of $X_{\mathbb{C}}$ is said to be *minimal*, if it follows from $U \subseteq V$ and U a $T_{\mathbb{C}}$-invariant subspace of $X_{\mathbb{C}}$ that either $U = \{0\}$ or $U = V$. The authors examine a problem ("Problem I") related to the above-mentioned conjecture, namely, whether $T_{\mathbb{C}}$ has a minimal non-zero closed invariant subspace whenever T has no non-trivial, closed invariant subspaces. Provided that such an invariant subspace exists, some properties of this subspace are given and it is shown that Conjecture 3 fails to be true under some occurrences.

We observe that the properties mentioned for the $T_{\mathbb{C}}$-invariant subspace W of $X_{\mathbb{C}}$ in Problem I remain also true for order bounded operators on a Riesz space E on which a complete linear topology τ having a countable neighborhood base at θ is defined, and that the conclusion on the failure of Conjecture 3 still holds true in that case. Before giving this result, recall that the complexification $E_{\mathbb{C}}$ of a Riesz space

E is the additive group $E \times E$ with the scalar multiplication

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x)$$

for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in E$, where, having identified $x \in E$ with $(x, 0)$ and ix with $(0, x)$ in $E \times E$, $x + iy$ is written instead of (x, y). For detailed information about complex Riesz spaces, we refer to [24].

Lemma 3.2.1. *Let (E, τ) be a τ-complete Riesz space such that the linear topology τ has a countable neighborhood base at θ, $T : E \to E$ be an order bounded operator without a non-trivial order- (or, uniformly-) closed invariant subspace, and W be an invariant subspace of $T_{\mathbb{C}}$. Then, the following are satisfied:*

(1) *The vector subspace W is infinite-dimensional.*

(2) *If $z = x + iy \in W$ and either $x = 0$ or $y = 0$, then $z = 0$.*

(3) *If $x \in E$, then there exists at most one $y \in E$ such that $x + iy \in W$. If $z = x + iy \in W$, then this unique y will be denoted by Sx, that is, $y = Sx$ and $x + iSx \in W$.*

(4) *Define the following vector subspace of E:*

$$\Delta = \{x \in E \mid \exists y \in E \text{ such that } x + iy \in W\}.$$

Then the mapping $S : \Delta \to E$ is a linear operator with range Δ. Moreover, $S^2 = -I_\Delta$ on Δ (and so, the operator $S : \Delta \to \Delta$ is invertible).

(5) *The subspace Δ is T-invariant, and S and T commute on Δ. In particular, Δ is dense in E.*

(6) *The invertible operator $S : \Delta \to \Delta$ is a closed operator.*

(7) *The operator $S : \Delta \to \Delta$ is continuous and $\Delta = E$.*

Proof. **(1)** Assume by way of contradiction that W is finite-dimensional. Pick a basis $\{z_1, z_2, \dots, z_n\}$ for W and let $z_k = x_k + iy_k$ for each $1 \le k \le n$. If F is the finite-dimensional subspace in E generated by $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, then F is a non-zero (and hence non-trivial, since E is infinite-dimensional) closed T-invariant subspace of E, which is a contradiction. Thus, W is infinite-dimensional.

(2) Let $V := \{y \in E \mid 0 + iy \in W\}$. Clearly, V is a closed subspace of E which is also T-invariant. Indeed, notice that we have $0 + iTy = T_{\mathbb{C}}(0 + iy) \in W$ for each $y \in V$, and so $Ty \in V$. To see that $V = \{0\}$, assume, on the contrary, that $V \neq \{0\}$. Then, since T does not have any non-trivial closed invariant subspaces, $V = X$. This implies that for each $x \in X$, we have $x + i0 = -i(0 + ix) \in W$. In particular, for each $z = x + iy \in E_{\mathbb{C}}$, we have $z = (x + i0) + (0 + iy) \in W$, and so $W = E_{\mathbb{C}}$, which is a contradiction. Therefore, $V = \{0\}$, and the assertion follows.

(3) If $x + iy$ and $x + iy_1$ are elements of W, then

$$0 + i(y - y_1) = (x + iy) - (x + iy_1) \in W,$$

and so, by part (2), we must have $y = y_1$.

(4) The linearity of the mapping $S : \Delta \to E$ follows immediately from the definition of addition and scalar multiplication:

$$\begin{aligned}(x_1 + iSx_1) + (x_2 + iSx_2) &= (x_1 + x_2) + i(Sx_1 + Sx_2)\\ \alpha(x_1 + iSx_1) &= \alpha x_1 + i(\alpha Sx_2).\end{aligned}$$

Now, for each $x \in \Delta$, we have $Sx + i(-x) = -i(x + iSx) \in W$. This implies that $Sx \in W$ and that $S^2x = -x$ for each $x \in \Delta$.

(5) If $x \in \Delta$, then $x + iSx \in W$, and from the $T_{\mathbb{C}}$-invariance of W, we get $Tx + iT(Sx) \in W$. This implies that $Tx \in \Delta$ and that $TSx = STx$. Therefore, Δ is T-invariant and S and T commute on Δ. Since $\Delta \neq \{0\}$, and T has no non-trivial invariant subspaces, it follows that Δ is dense in E.

(6) If $((x_n, Sx_n))_{n\in\mathbb{N}}$ is an order- (or, uniformly-) closed sequence in $E \times E$ such that $(x_n, Sx_n) \to (x, y)$, then, $x_n + iSx_n \to x + iy$ in $E_{\mathbb{C}}$, and so, from the closedness of W, we infer that $x + iy \in W$. This implies that $x \in \Delta$ and $y = Sx$. Therefore, the operator $S : \Delta \to \Delta$ is closed.

(7) As (E, τ) is τ-complete, the map $(u, v) \mapsto u \vee v$ is uniformly continuous. So, by [7, Thm. 2.17, p. 55], τ is a locally solid linear topology. Since τ has a countable neighborhood base at θ, it is metrizable [7, Thm. 2.1, p. 50]. Thus, by [15, Cor. 3, p. 94], $S : \Delta \to \Delta$ is continuous. Then S, as an operator from Δ to E, is uniformly continuous, so it has a continuous linear extension [7, Thm. 2.6, p. 52] $S_1 : E \to E$, since $\overline{\Delta} = E$. Pick a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \Delta$ such that $x_n \xrightarrow{\tau} x$, and note that the sequence $(x_n + iSx_n)_{n\in\mathbb{N}} \subseteq W$ satisfies $(x_n + iSx_n) \xrightarrow{\tau} x + iS_1x$ in $E_{\mathbb{C}}$. This implies that $x + iS_1x \in W$, and so $x \in \Delta$. Therefore, $\Delta = E$. $\square$

We are now in a position to state the following theorems.

Theorem 3.2.2. *The operator $T_{\mathbb{C}} : W \to W$ has no non-trivial closed invariant subspaces, that is, W is a minimal closed invariant subspace of $T_{\mathbb{C}}$.*

Proof. Since S is continuous, it follows from Lemma 3.3.1 (7) that

$$W = \{x + iSx \mid x \in E\}.$$

Now, assume that a non-zero closed subspace W_1 of W is $T_{\mathbb{C}}$-invariant. By Lemma 3.3.1 (5), there is a dense vector subspace Δ_1 of E and a linear operator $S_1 : \Delta_1 \to \Delta_1$ such that $W_1 = \{x + iS_1x \mid x \in \Delta_1\}$. It follows that $S_1x = Sx$ for each $x \in \Delta_1$. This implies that $S_1 : \Delta_1 \to \Delta_1$ is continuous, and as in Lemma 3.3.1 (7), we must have $\Delta_1 = E$. Therefore, one has $W_1 = \{x + iSx \mid x \in E\} = W$, and so W is minimal. $\square$

Theorem 3.2.3. *Conjecture 3 is false if and only if there exists a closed operator $S : \Delta \to \Delta$ that commutes*[1] *with T and satisfies $S^2 = -I$.*

Proof. The "only if" part follows from the above discussion. Now, assume the existence of the operator $S : \Delta \to \Delta$ with the above properties. Since S is closed, it follows that $W := \{x + iSx \mid x \in \Delta\}$ is a non-zero closed vector subspace of $E_{\mathbb{C}}$ that is different than $E_{\mathbb{C}}$. Now, note that W is $T_{\mathbb{C}}$-invariant. $\square$

[1]An operator $S : V \to V$, where V is a vector subspace of X, is said to *commute* with T, if V is T-invariant and $ST = TS$ holds true on V.

Chapter 4

Generalized alexandroff duplicates and $CD_0(K)$-type spaces

Te purpose of this chapter, whose main results are presented in [13], is to define and investigate $CD_{\Sigma,\Gamma}(K,E)$-type spaces, which generalize CD_0-type Banach lattices introduced in [6]. We state that the space $CD_{\Sigma,\Gamma}(K,E)$ can be represented as the space of E-valued continuous functions on the generalized Alexandroff duplicate of K. As a corollary, we obtain the main result of [17] and [30].

4.1 Introduction

Throughout this chapter, E will denote a Banach lattice and Ω, Σ and Γ will stand for topologies on K, where Σ is compact, Γ is locally compact with $\Sigma \subset \Gamma$. These spaces will be denoted by K_Ω, K_Σ and K_Γ, respectively. The closure of a subset A of K_Ω will be denoted by $\mathrm{cl}_\Omega(A)$. As usual, the space of E-valued K_Ω-continuous functions will be denoted by $C(K_\Omega, E)$, or by $C(K,E)$ if there is no possibility of ambiguity. $C_0(K_\Omega, E)$ denotes the space of E-valued K_Ω-continuous functions d on K such that for each $\epsilon > 0$, there exists a compact set M with $||d(k)|| \leq \epsilon$ for each $k \in K \backslash M$. We shall write $C(K_\Omega)$ for $C(K_\Omega, \mathbb{R})$ and $C_0(K_\Gamma)$ for $C_0(K_\Omega, \mathbb{R})$. If K_Σ has no isolated points and K_Γ is discrete, then $C(K_\Sigma, E) \cap C_0(K_\Sigma, E) = \{0\}$, and $CD_0(K_\Sigma, E) = C(K_\Sigma, E) \oplus C_0(K_\Sigma, E)$ is a Banach lattice under the pointwise order and supremum norm. We refer to [6], [9] and [17] for more details on these spaces. $CD_{\Sigma,\Gamma}(K,E)$ will denote the vector space $C(K_\Sigma,E)\times C_0(K_\Sigma,E)$, which is equipped with the coordinatewise algebraic operations. It is easy to see that $CD_{\Sigma,\Gamma}(K,E)$ is a Banach lattice with respect to the order

$$0\leq (f,d) \iff 0\leq f(k) \quad \text{and} \quad 0\leq f(k)+d(k) \quad \text{for each} \quad k\in K$$

and the norm

$$||(f,d)|| = \max\{||f||, ||f+d||\},$$

where $||.||$ is the supremum norm. If K_Σ has no isolated points and K_Γ is discrete, then it is easy to see that $CD_0(K_\Sigma,E)$ and $CD_{\Sigma,\Gamma}(K,E)$ are isometrically Riesz isomorphic spaces.

Let $K\times\{0,1\}$ be topologized by the open base $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$, where

$$\mathcal{A}_1=\{H\times\{1\}\mid H \text{ is } \Gamma\text{-open}\}$$

and

$$\mathcal{A}_2=\{G\times\{0,1\}-M\times\{1\}\mid G \text{ is } \Sigma\text{-open}, \ M \text{ is } \Gamma\text{-compact}\}.$$

Let us denote this topological space by $K_{\Sigma,\Gamma}\otimes\{0,1\}$, which is called the *generalized Alexandroff duplicate* (in case Γ has the discrete topology, we will denote this space by $A(K)$). The space $A(K)$ has been constructed by R. Engelking [15] and it has been generalized to arbitrary locally compact Hausdorff spaces in [12]. It is known that $K_{\Sigma,\Gamma}\otimes\{0,1\}$ is a compact Hausdorff space [15, 22]. The space $A([0,1])$, where $[0,1]$ has the usual topology, has been constructed by P. S. Alexandroff and P. S. Urysohn in [7] as an example of a compact Hausdorff space containing a discrete dense subspace; this space is called the *Alexandroff duplicate* [22, p. 1010].

Definition which is similar to the following was introduced in [17].

Definition 4.1.1. *Let $((k_\alpha,r_\alpha))_{\alpha\in I}$ be a net in $K\times\{0,1\}$, where I is an index set, and $(k,r)\in K\times\{0,1\}$. We say that the net $((k_\alpha,r_\alpha))_{\alpha\in I}$ converges to (k,r), and denote it by $(k_\alpha,r_\alpha)\to(k,r)$, if*

$$f(k_\alpha)+r_\alpha d(k_\alpha)\to f(k)+rd(k)$$

for each $f\in C(K_\Sigma)$ and $d\in C_0(K_\Gamma)$. We denote the space $K\times\{0,1\}$ equipped with this convergence by $K_{\Sigma,\Gamma}\odot\{0,1\}$.

The proof of the following theorem is a simple consequence of the above definition.

Theorem 4.1.2. *Whether $\Sigma \subset \Gamma$ or not, $K_{\Sigma,\Gamma} \odot \{0,1\}$ is a Hausdorff topological space.*

4.2 Main results

In [17], it has been proved that $K_{\Sigma,\Gamma} \odot \{0,1\}$ is a compact Hausdorff space under the convergence given in Definition 4.1.1 if K_Σ has no isolated points and K_Γ is discrete. For certain Banach lattices, representations of these spaces have been constructed in [17] with the topology induced by this.

The space of continuos functions on $K_{\Sigma,\Gamma} \otimes \{0,1\}$ can be identified as follows.

Theorem 4.2.1. *$C(K_{\Sigma,\Gamma} \otimes \{0,1\}, E)$ and $CD_{\Sigma,\Gamma}(K,E)$ are isometrically Riesz isomorphic spaces.*

Proof. Let $f : K \to E$ be a map. Then, in order for f to belong to $C(K_{\Sigma,\Gamma}, E)$ it is necessary and sufficent that

(i) $k \mapsto f(k,0)$ is Σ-continuous; and

(ii) the map $k \mapsto f(k,1) - f(k,0)$ belongs to the space $C_0(K_\Sigma, E)$.

Indeed, suppose that (i) and (ii) are satisfied. Then, $k \mapsto f(k,1)$ is Γ-continuous, being the sum of $f(k,1) - f(k,0)$ and $f(k,0)$, the first of which is Γ-continuous by (ii) and the second is Σ-continuous by (i) and hence also Γ-continuous as $\Sigma \subset \Gamma$. It then follows that f is continuous at each point of $K \times \{1\}$.

Let now $k \in K$. We will show that f is continuous at $(k,0)$. Let $\epsilon > 0$. Then,

$$H := \{k \in K \mid ||f(k,1) - f(k,0)|| \geq \epsilon/2\}$$

is Γ-compact by (ii). Moreover, by (i), there is a Σ-open set G containing k such that $||f(k,0) - f(l,0)|| < \epsilon/2$ for $l \in G$. Set $U := (G \times \{0,1\}) \backslash H \times \{1\}$. Then U is a neighborhood of $(k,0)$ in $K_{\Sigma,\Gamma} \otimes \{0,1\}$. Further, if $(l,i) \in U$, then either $i = 0$, which yields $||f(k,0) - f(l,0)|| < \epsilon/2 < \epsilon$; or $i = 1$, yielding $l \notin H$ and hence

$$||f(l,1) - f(k,0)|| \leq ||f(l,1) - f(l,0)|| + ||f(l,0) - f(k,0)|| < \epsilon.$$

This completes the "if" part.

For the "only if" part, suppose that f is continuous. Then, clearly, (i) holds. Moreover, $k \mapsto f(k,1)$ is Γ-continuous, and hence $k \mapsto f(k,1) - f(k,0)$ is Γ-continuous, too. It remains now to show that $\{k \in K \mid ||f(k,1) - f(k,0)|| \geq \epsilon\}$ is Γ-compact for each $\epsilon > 0$.

Suppose that $V := \{k \in K \mid ||f(k,1) - f(k,0)|| \geq \epsilon\}$ is not Γ-compact for some $\epsilon > 0$. By the compactness of (K, Σ), there exists a $k \in K$ such that $\mathrm{cl}_\Gamma(G) \cap V$ is not Γ-compact for any Σ-neighborhood G of k (otherwise, V would be covered by finitely many Γ-compact subsets and hence would itself be Γ-compact).

Let $U := (G \times \{0,1\}) \backslash \{1\}$ be a basic open set in $K_{\Sigma,\Gamma} \otimes \{0,1\}$ containing $(k,0)$ such that for each $(l,i) \in U$, we have $||f(l,i) - f(k,0)|| < \epsilon/2$. As $\mathrm{cl}_\Gamma(G) \cap V$ is not Γ-compact, there is an $l \in V \cap (G \backslash M)$. Then, both (l,i) and $(l,0)$ belong to U, and hence $||f(l,1) - f(l,0)|| < \epsilon$. However, $||f(l,1) - f(l,0)|| \geq \epsilon$ as $l \in V$, which is a contradiction.

From this, we have the map $\pi : CD_{\Sigma,\Gamma}(K,E) \to C(K_{\Sigma,\Gamma} \otimes \{0,1\}, E)$ defined by

$$\pi(f,d)(k,r) = f(k) + rd(k) \quad \text{for each} \quad (k,r) \in K \times \{0,1\}.$$

It is a straightforward observation that π is a bi-positive and one-to-one linear operator. Let $f \in C(K_{\Sigma,\Gamma} \otimes \{0,1\}, E)$ be given. Define the maps $g, d : K \to E$ via

$$g(k) = f(k,0) \quad \text{and} \quad d(k) = f(k,1) - d(k,0).$$

Then, from the above observation, we have that $(g,d) \in C_{\Sigma,\Gamma}(K,E)$ and $\pi(g,d) = h$, that is, π is also onto. It is also clear that $||\pi(f,d)|| = ||f + d||$. This finishes the "only if" part and the proof of the theorem is now complete. $\square$

Remark 4.2.2. *Note that a characterization similar to the one given in Theorem 4.2.1 holds for functions with values in any metric space as follows: Let (M,d) be a metric space and $f : K \times \{0,1\} \to M$ be a map. Then, $f \in C(K_{\Sigma,\Gamma} \otimes \{0,1\}, M)$ if and only if the following conditions are satisfied:*

(i) $k \mapsto f(k,0)$ *is Σ-continuous;*

(ii) $k \mapsto f(k,1)$ *is Γ-continuous;*

(iii) *for each* $\epsilon > 0$*, the set* $\{k \in K \mid d(f(k,0), f(k,1)) \geq \epsilon\}$ *is* Γ*-compact.*

The following result is a surprising and interesting consequence of Theorem 4.2.1.

Theorem 4.2.3. $K_{\Sigma,\Gamma} \otimes \{0,1\}$ *and* $K_{\Sigma,\Gamma} \odot \{0,1\}$ *are homeomorphic spaces.*

Proof. The assertion follows from Theorem 4.2.1 and from the fact that any compact Hausdorff space X is homeomorphic to a subspace of $(C(X)^*, w^*)$, where the topology on X is the weak topology generated by all continuous functions on it. $\square$

As an immediate consequence of this theorem, we have the following result.

Corollary 4.2.4. $C(K_{\Sigma,\Sigma} \otimes \{0,1\})$ *and* $C_{\Sigma,\Sigma}(K)$ *are isomorphic Riesz spaces.*

The proof of the following fact, which is the main result of [17], follows at once from Theorem 4.2.3.

Corollary 4.2.5. *If* K_Σ *has no isolated points, then the spaces* $CD_0(K,E)$ *and* $C(A(K),E)$ *are isometrically Riesz isomorphic.*

Remark 4.2.6. *It follows from Corollary 4.2.5 and the Banach-Stone theorem that the Kakutani-Krein compact space of* $CD_0(K)$ *is the Alexandroff duplicate* $A(K)$ *of* K_Σ*.*

References

[1] Y.A. Abramovich and C.D. Aliprantis, *An Invitation to Operator Theory*, American Mathematical Society, Graduate Studies in Mathematics, Vol. 50, Providence, RI, 2002.

[2] Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw, "Invariant subspaces of operators on l_p spaces," *J. Func. Anal.* **115** (1993), 418-424.

[3] Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw, "Invariant subspaces for positive operators acting on a Banach space with basis," *Proc. Amer. Math. Soc.* **123** (1995), 1773-1777.

[4] Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw, "The invariant subspace problem: some recent advances," in *Workshop on Measure Theory and Real Analysis* (italian) (Grado, 1995); *Rend. Inst. Mat. Univ. Trieste* **29** (1998), suppl., 3-79 (1999).

[5] Y.A. Abramovich, C.D. Aliprantis, G. Sirotkin and V.G. Troitsky, "Some open problems and conjectures associated with the invariant subspace problem," *Positivity*, to appear.

[6] Y.A. Abramovich and A.W. Wickstead, "Remarkable classes of unital *AM*-spaces," *J. Math. Anal. Appl.* **180** (1993), no. 2, 398-411.

[7] P.S. Alexandroff and P.S. Urysohn, *Mémoire sur les Espaces Topologiques Compacts*, Verh. Kon. Akad. Wetensch. Natarkunde, 14, Amsterdam, 1929.

[8] C.D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics,* 2nd ed., American Mathematical Society, Mathematical Surveys and Monographs, Vol. 105, Providence, RI, 2003.

[9] Ş. Alpay and Z. Ercan, "$CD_0(K, E)$ and $CD_w(K, E)$-spaces as Banach lattices," *Positivity* **4** (2000), no. 3, 213-225.

[10] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988.

[11] A.R. Bernstein and A. Robinson, "Solution of an invariant subspace problem of K.T. Smith and P.R. Halmos," *Pacific J. Math.* **16** (1966), 421-431.

[12] R.E. Chandler, G.D. Faulkner, J.P. Guglielmi and M.C. Memory, "Generalizing the Alexandroff-Urysohn double circumference construction," *Proc. Amer. Math. Soc.* **83** (1981), no. 3, 606-608.

[13] M. Çağlar, Z. Ercan and F. Polat, "Generalized Alexandroff duplicates and $CD_0(K)$-spaces," *Proc. Amer. Math. Soc.*, submitted.

[14] N. Dunford and J.T. Schwartz, *Linear Operators I: General Theory*, Wiley (Interscience), New York, 1958.

[15] R. Engelking, "On the double circumference of Alexandroff," *Bull. Acad. Pol. Sci. Ser. Math.* **16** (1968), 629-634.

[16] P. Enflo, "On the invariant subspace problem for Banach spaces," *Acta Math.* **158** (1987), 213-313.

[17] Z. Ercan, "A concrete description of $CD_0(K)$-spaces as $C(X)$-spaces and its applications," *Proc. Amer. Math. Soc.* **132** (2004), no. 6, 1761-1763.

[18] Z. Ercan and S. Önal, "Invariant subspaces for positive operators acting on a Banach space with Markushevich basis," *Positivity* **8** (2004), no. 2, 123-126.

[19] V.I. Istrăţescu, *Introduction to Linear Operator Theory*, M. Dekker, New York, 1981.

[20] H. Jarchow, *Locally Convex Spaces*, B.G. Teubner, Stuttgart, 1981.

[21] L.V. Kantorovič and G.P. Akilov, *Functional Analysis*, trans. by Howard L. Silcock, 2nd ed., Pergamon Press, Oxford, New York, 1982.

[22] K. Kunen and J.E. Vaughan (eds.), *Handbook of Set-Theoretic Topology*, Elsevier Science Pub. Co., North-Holland, Amsterdam, New York, 1984.

[23] V.I. Lomonosov, "Invariant subspaces for the family of operators which commute with a completely continuous operator," *Functional Anal. Appl.* **7** (1973), 213-214.

[24] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, 1991.

[25] B. de Pagter, "Irreducible compact operators," *Math. Z.* **192** (1986), 149-153.

[26] C. Pearcy (ed.), *Topics in Operator Theory*, American Mathematical Society, Mathematical Surveys, No. 13, Providence, RI, 1974.

[27] C.J. Read, "A solution to the invariant subspace problem on the space l_1," *Bull. London Math. Soc.* **17** (1985), 305-317.

[28] V.S. Shulman, "On invariant subspaces of Volterra operators," *Funktsional Anal. i Prilozhen* **18** (1984), 84-85.

[29] I. Singer, *Bases in Banach Spaces I*, Springer-Verlag, Berlin, New York, 1970.

[30] V.G. Troitsky, "On $CD_0(K)$-spaces," *Vladikavkaz Mat. Zh.* **6** (2004), no. 1, 71-73.

[31] Y.V. Turovskii, "Volterra semigroups have invariant subspaces," *J. Funct. Anal.* **162** (1999), 313-322.

Vita

Mert Çağlar was born in İzmir, on 5 August, 1976. After completing his secondary education at Collège Saint Joseph, he graduated from Atatürk Lycée, both in İzmir. He started his undergraduate studies at the Department of Mathematics of Marmara University, İstanbul, in 1994 and took his B.S. degree in 1998 graduating in first place from the Department of Mathematics and the Faculty of Science and Letters. He became research assistant at the Department of Mathematics, M.E.T.U., in 1999, and started his graduate studies. He received his M.Sc. degree in Mathematics in 2001, with a thesis entitled "Invariant Subspaces of Positive Operators on Banach Lattices."