BASIS IN NUCLEAR FRÉCHET SPACES

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ABSTRACT

BASIS IN NUCLEAR FRÉCHET SPACES

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Existence of basis in locally convex space has been an important problem in functional analysis for more than 40 years. In this thesis the conditions for the existence of basis are examined. This thesis consist of three parts. The first part is about the exterior interpolative conditions. The second part deals with the inner interpolative conditions $DN(\mathcal{Q})$, $\Omega(\mathcal{P})$, $\mathcal{T}(\mathcal{P},\mathcal{Q})$ for seminorm systems $\mathcal{P}$ and $\mathcal{Q}$ on a nuclear Fréchet space. These are sufficient conditions on existence of basis. In the last part, it is shown that for a regular nuclear Köthe space the inner interpolative conditions are satisfied and moreover another type of inner interpolative conditions are introduced.

Keywords: Basis, Nuclear Fréchet Space, Interpolative conditions.

ÖZ

NÜCLEER FRÉCHET UZAYLARINDA TABAN

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Fonksiyonel analizde, lokal konveks uzaylar için tabanın varlığı 40 yıldan fazla süredir önemli bir problemdir. Bu tezde tabanın varlığı için gerekli koşullar incelenmiştir. Bu tez 3 bölümden oluşmaktadır. İlk bölüm dışsal interpolasyon koşulları hakkında bir incelemedir. İkinci bölümde nükleer Fréchet uzayları üzerindeki içsel interpolasyon koşulları, $DN(\mathcal{Q})$, $\Omega(\mathcal{P})$, $\mathcal{T}(\mathcal{P},\mathcal{Q})$ ile ilgilenilmiştir. Bu koşullar tabanın varlığı için yeterli koşullardır. Son bölümde, düzenli nükleer Köthe uzayı için içsel interpolasyon koşullarının sağlandığı gösterilmiş ve son olarak da interpolasyon koşullarının farklı bir versiyonundan bahsedilmiştir.

Anahtar Kelimeler: Taban, Nükleer Fréchet Uzayları, İnterpolasyon Koşulları

To My Family

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

In this chapter, we will give some basic definitions, notations and theorems about Fréchet spaces which will be used in this thesis. For undefined standart concepts and well-known results we refer the reader to [7], [11], [15] and [18].

1.1 Introduction

The locally convex spaces appearing in analysis roughly divides into two classes. The first class is normed spaces. Its theory belongs to classical functional analysis. The second class is nuclear locally convex spaces which were introduced by A. Grothendièck in 1951. These two classes have a trivial intersection in the sense that only finite dimensional locally convex spaces are normable and nuclear.

Grothendièck disclosed a theory of nuclear Fréchet spaces and posed some important questions in the 1950's. One of the important questions is the existence of a basis in a nuclear Fréchet space. B. S. Mitiagin and N. M. Zobin in 1975 answered this question negatively; they constructed a nuclear Fréchet space without basis.

Another important problem which was posed in 1970 by Pelczyński [19] is whether complemented subspaces of nuclear Fréchet spaces with a basis always have a basis, i.e., whether they are isomorphic to Köthe spaces.

On the other hand the existence of basis was proved for many concrete spaces [1],[2],[13],[17] and for many others it is still an open problem. Sufficient conditions for existence of basis were obtained by a method called the " dead-end space method " by various authors. Dead-end space refers to a Hilbert space being continuously and densely embedded in a nuclear Fréchet space and was first introduced in 1971 by B.S.Mitiagin and G.M.Khenkin [13]

In this thesis, we will first discuss the interpolative conditions which depend on a given dead-end space (Exterior interpolative conditions) and describe the " dead-end space " method. In the second part we will introduce some inner interpolative conditions (depending only on the seminorm system of the NFS) and show that they are sufficient for the existence of basis.

Finally, we will prove that for a regular nuclear Köthe space, the inner interpolative conditions introduced above are satified.

1.2 Some Definitions and Preliminaries

1.2.1 Locally Convex Spaces

There are basically two ways to describe a locally convex structure on a given vector space. First is with neighbourhooods, second is with seminorms.

Definition 1.1. A vector space $\mathbf{E}$ over the scalar field $\mathbb{K}$ equipped with a Hausdorff topology for which;

$$addition \;\; + : \mathbf{E} \times \mathbf{E} \to \mathbf{E}$$

and

$$scalar\, multiplication \;\; \cdot : \mathbb{K} \times \mathbf{E} \to \mathbf{E}$$

are continuous is called a *Topological Vector Space.* (TVS)

Definition 1.2. *A locally convex space* is a topological vector space in which each point has a neighbourhood basis consisting of convex sets.

Lemma 1.1. *A locally convex space* $\mathbf{E}$ *has a base* $\boldsymbol{U}$ *of neighbourhoods of the origin with the following properties;*

$C1$ *: If* $U \in \boldsymbol{U}$ *and* $V \in \boldsymbol{U}$*, then there exists* $W \in \boldsymbol{U}$ *with* $W \subset U \cap V$*.*

$C2$ *: If* $U \in \boldsymbol{U}$ *and* $\alpha \neq 0$*,* $\alpha U \in \boldsymbol{U}$*.*

$C3$ *: Each* $U \in \boldsymbol{U}$ *is absolutely convex and absorbing, i.e., for each* $x \in \mathbf{E}$*, there is some* $\lambda > 0$ *with* $x \in \lambda U$

Conversely given a nonempty set $\boldsymbol{U}$ *of subsets of a vector space* $\mathbf{E}$ *with the properties* $C1 - C3$*, there is a topology making* $\mathbf{E}$ *a locally convex space admitting* $\boldsymbol{U}$ *as a base of neighbourhoods.*[See 15]

The relation between seminorms and zero neighbourhoods is discussed below.

Definition 1.3. Let $\mathbf{E}$ be a locally convex space. A collection $\boldsymbol{U}$ of zero neighbourhoods in $\mathbf{E}$ is called *a fundamental system of zero neighbourhoods* if for every zero neighbourhood U, there exists $V \in \boldsymbol{U}$ and $\epsilon > 0$ with $\epsilon V \subset U$

A family $(\mathbf{p}_\alpha)_{\alpha \in I}$ of continuous seminorms on $\mathbf{E}$ is called *a fundamental system of seminorms* if the sets

$$U_\alpha = \{x \in \mathbf{E}; (\mathbf{p}_\alpha(x)) < 1\}, \alpha \in I$$

form a fundamental system of zero neighbourhoods.

Lemma 1.2. *Let* $\mathbf{E}$ *be a* $\mathbb{K}$ *vector space and* $(\mathbf{p}_\alpha)_{\alpha\in I}$ *be a family of seminorms on* $\mathbf{E}$ *satisfying*

1. *For every* $x \in \mathbf{E}$ *with* $x \neq 0$*,* $\exists k \in I$ *with* $(\mathbf{p}_k(x)) > 0$
2. *For every* $i, j \in I$*,* $\exists m \in I$ *and* $C > 0$ *with* $\max(\mathbf{p}_i(x)), (\mathbf{p}_j(x)) \leq C.(\mathbf{p}_m(x))\ \forall x \in \mathbf{E}$*.*

Then there exists a unique locally convex space topology on $\mathbf{E}$ *for which* $(\mathbf{p}_\alpha)_{\alpha\in I}$ *is a fundamental system of seminorms and* $\mathbf{E}$ *is a locally convex space.*

Moreover every locally convex space $\mathbf{E}$ *has a fundamental system of seminorms which satisfies* 1 *and* 2*.[*See 11*]*

One can generate now locally convex spaces from given ones. The method mentioned below is one way of doing this.

Definition 1.4. A $\mathbb{K}$ vector space $\mathbf{E}$ with a family of locally convex spaces $(\mathbf{E}_i)_{i\in\mathbf{I}}$ and linear maps $\pi_i : \mathbf{E} \to \mathbf{E_i}$ is called *a projective system,* if for each $x \in E$, $x \neq 0$, there exists an $i \in I$ with $\pi_i(x) \neq 0$. For every projective system $(p_i : \mathbf{E} \to \mathbf{E_i})_{i\in I}$, the seminorm system $\{p : p = \max_{i\in M} p_i \circ \pi_i\,,\ M \in \epsilon(I)$, p_i is continuous seminorm on $\mathbf{E_i}$, $i \in M$, $\epsilon(I)$ is a bounded subset of I$\}$ induces a locally convex topology on $\mathbf{E}$ which is called *the projective topology of system, i.e.,*

the projective topology is the coarsest topology on $\mathbf{E}$ for which all the maps π_i are continuous.

Definition 1.5. *Let* $(\mathbf{E}_k, \|\cdot\|_k)$ *be a family of locally convex spaces. For* $i \leq j$ *where* $i, j \in I$*, let* $\iota_{i,j} : \mathbf{E}_j \to \mathbf{E}_i$ *be a continuous linear map. The subspace of* $\Pi_i \mathbf{E}_i$*,* $\mathbf{E} = \{x = (x_i) \in \Pi_i \mathbf{E}_i : \iota_{i,j}(x_j) = (x_i)$ *whenever* $i \leq j\}$ *is called the* *projective limit of* $(\mathbf{E}_i, \|\cdot\|)$ *with respect to the mappings* $\iota_{i,j}$ *and it is shown by,*

$$\mathbf{E} \cong \underset{\leftarrow}{\mathbf{limproj}}\, \mathbf{E_i}$$

Definition 1.6. Let $\mathbf{E}$ be a $\mathbb{K}$ vector space and p be a continuous seminorm on $\mathbf{E}$. A norm is defined on $\mathbf{E}/N_p$ by $(\|x + N_p\|)_p := p(x)$ where $N_p := \{x \in \mathbf{E} : p(x) = 0)$ is a closed linear subspace of $\mathbf{E}$.

Then $\mathbf{E}_p := (\widehat{\mathbf{E}/N_p, \|\cdot\|_p})$ is called *the local Banach space* for the seminorm p. For the canonical map $\iota^p : \mathbf{E} \to \mathbf{E_p}$, $\iota^p(x) := x + N_p$ we have $\|\iota^p(x)\|_p = p(x)$ for all $x \in \mathbf{E}$.

Remark 1.1. *The following are well-known,*

(a) Every locally convex space $\mathbf{E}$ is the projective limit for a suitable projective system of Banach spaces. Actually $\mathbf{E}$ is isomorphic to a subspace of a suitable product of Banach spaces.

(b) If $(p_\alpha)_{\alpha \in I}$ is any fundamental system of seminorms for $\mathbf{E}$, the projective topology τ on $\mathbf{E}$ is the projective topology of the system $(\iota^\alpha : \mathbf{E} \to \mathbf{E}_\alpha)_{\alpha \in I}$ where $\mathbf{E}_\alpha$ is the local Banach space for the seminorm p_α.

The dual of a locally convex space is defined below.

Definition 1.7. Let $(\mathbf{E}, \mathbf{E}')$ be a dual pair where E' is the topological dual of $\mathbf{E}$. If U is a subset of $\mathbf{E}$, the subset of $\mathbf{E}'$

$$\{y \in \mathbf{E}' : \sup |y(x)| \leq 1, x \in U\}$$

is called *the polar of* U and is denoted by U°.

Proposition 1.1. *For every absolutely convex zero neighbourhood U in a locally convex space* $\mathbf{E}$*,*

$$\|x\|_U = \sup\{|y(x)| : y \in U^\circ\} \ \ for\ all\ x \in \mathbf{E}$$

Hence if $\boldsymbol{U}$ *is a fundamental system of zero neighbourhoods in a locally convex space* $\mathbf{E}$*,* $\{\|x\|_U\}_{U\in\boldsymbol{U}}$ *is a fundamental system of seminorms in* $\mathbf{E}$*.*

Also we can define the dual seminorm of $\|x\|_k$ by $\|x\|_k^*$ on dual of $\mathbf{E}$, that is

$$\|f\|_k^* = \sup\{|f(x)| : \|x\|_k \leq 1\}\, \forall f \in \mathbf{E}'$$

1.2.2 Fréchet Spaces

Definition 1.8. A complete metrizable locally convex space $\mathbf{E}$ is called *a Fréchet space.*

In a Fréchet space, a fundamental system of seminorms is a countable system of continuous seminorms generating the topology.

We can assume without loss of generality that the seminorm systems considered are increasing. Because we can modify the fundamental system of seminorms $(|\cdot|_n)_{n\in\mathbb{N}}$ as $\|\cdot\|_n = (\max_{1\leq j\leq n} |\cdot|_j)_{n\in\mathbb{N}}$ and hence $\|x\|_n \leq \|x\|_{n+1}\ \forall x \in \mathbf{E}, n \in \mathbb{N}$ and clearly $(|\cdot|_n)_n$ and $(\|\cdot\|_n)_n$ generate the same topology.

Definition 1.9. A matrix $A = (a_j^k)_{j,k\in\mathbb{N}}$ of nonnegative numbers is called *a Köthe matrix* if it satisfies the following conditions :

(i) For each $j \in \mathbb{N}$ *there exists a* $k \in \mathbb{N}$ *with* $a_j^k > 0$*.*

(ii) $a_j^k \leq a_j^{k+1}$ *for all* $j, k \in \mathbb{N}$*.*

For $1 \leq p < \infty$ we define

$$\lambda^p(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \|x\|_k^p := \left(\sum_{j=1}^{\infty} \left| x_j a_j^k \right|^p \right)^{1/p} < \infty \ \ for \ \ all \ \ k \in \mathbb{N} \right\}$$

For $p = \infty$ and $p = 0$

$$\lambda^\infty(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \|x\|_k^\infty := \sup_{j \in \mathbb{N}} |x_j| \, a_j^k < \infty \ for \ all \ k \in \mathbb{N} \right\},$$

$$c_0(A) := \left\{ x \in \lambda^\infty(A) : \lim_{j \to \infty} |x_j| \, a_j^k = 0 \ for \ all \ k \in \mathbb{N} \right\}$$

The seminorms $\|\cdot\|_k^p$ are called *canonical seminorms.*

For every Köthe matrix A, the spaces $\lambda^p(A)$, $1 \leq p \leq \infty$ and $c_0(A)$ are Fréchet spaces.

Definition 1.10. A Köthe space with a Köthe matrix $A = (a_n^k)_{n,k \in \mathbb{N}}$ is called regular if,

$$\frac{a_n^k}{a_n^{k+1}} \geq \frac{a_{n+1}^k}{a_{n+1}^{k+1}}, \ \ \forall \ k, n$$

Definition 1.11. A locally convex space $\mathbf{E}$ is said to be *nuclear,* if for each absolutely convex zero neighbourhood $U \in \mathbf{E}$ there exists an absolutely convex zero neighnourhood V and a measure μ on the σ^*-compact set V°, so that

$$\|x\|_U \leq \int_{V^\circ} |y(x)| \, d\mu(y) \ for \ all \ x \in \mathbf{E}$$

Definition 1.12. A seminorm p on $\mathbb{K}$ vector space $\mathbf{E}$ is called *a Hilbert seminorm* if there exists a semiscalar product $\langle \cdot, \cdot \rangle$ on $\mathbf{E}$ with $p(x) = \sqrt{\langle x, x \rangle}$, $\forall x \in \mathbf{E}$.

Remark 1.2.

1. *Every nuclear space* $\mathbf{E}$ *has a fundamental system of Hilbert seminorms.[See 11]*

2. *If* p *is a Hilbert seminorm on* $\mathbf{E}$ *then the local Banach space* $\mathbf{E}_p$ *is a Hilbert space.*

Definition 1.13. Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces and let $T : \mathbf{E} \to \mathbf{F}$ be a linear map. T *is nuclear,* if there exist sequences $(a_j)_{j\in\mathbb{N}}$ in $\mathbf{E}'$ and $(b_j)_{j\in\mathbb{N}}$ in $\mathbf{F}$ such that

$$T(x) = \sum_{j\in\mathbb{N}} a_j(x)b_j, \forall x \in \mathbf{E} \tag{1.1}$$

and

$$\sum_{j\in\mathbb{N}} \|a_j\| \, \|b_j\| < \infty \tag{1.2}$$

The result below combines the concepts of local Banach spaces and nuclear maps.

Proposition 1.2. $\mathbf{E}$ *is nuclear if and only if for each continuous seminorm* p *on* $\mathbf{E}$*, there exists a continuous seminorm* $q \geq p$ *so that the map*

$$\iota_q^p \; : \; \mathbf{E}_q \to \mathbf{E}_p \tag{1.3}$$

$$x + N_q \to \iota_q^p(x + N_q) = x + N_p \tag{1.4}$$

is nuclear.

A practical way to check nuclearity of Köthe spaces is given below.

Theorem 1.1 (Grothendièck-Pietsch Criterion)**.** *A Köthe space is nuclear if and only if for every $k \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{a_n^k}{a_n^p} < \infty$.*

Remark 1.3. *If a Köthe space is nuclear, then all seminorm systems $\{\|\cdot\|_k^p\}_k$ are equivalent for $1 \leq p \leq \infty$ and $p = 0$.*

In the rest of the work, we show $\lambda^1(A)$ where $A = (a_n^k)_{(k,n)}$ by $\Lambda(a_n^k)$ and call it a Köthe space.

$$\Lambda(a_n^k) = \left\{ x = (x_n)_{n \in \mathbb{N}} : \|x\|_k := \sum_{n=1}^{\infty} |x_n| \, a_n^k < \infty, \forall k \in \mathbb{N} \right\} \tag{1.5}$$

Proposition 1.3. *For a nuclear Köthe space $\Lambda(a_n^k)$, there exists a permutation σ and a subsequence (p_k) at $\mathbb{N}$ such that if $b_n^k = a_{\sigma(n)}^{p_k}$ then*

$$\sum_n \sup_{j \geq n} \frac{b_j^k}{b_j^{k+1}} < \infty \tag{1.6}$$

and $\Lambda(a_n^k) = \Lambda(b_n^k)$ [See 5]

1.2.3 Bases in Locally Convex Spaces

Definition 1.14. A sequence $(x_n)_{n \in \mathbb{N}}$ of elements in a locally convex space $\mathbf{E}$ is called *a basis* if for each element $x \in \mathbf{E}$ there is a uniquely determined sequence of scalars $(\alpha_n)_{n \in \mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} \alpha_n x_n$$

The linear forms x_n^* on $\mathbf{E}$ defined by $\alpha_n = \langle x, x_n \rangle = x_n^*(x)$ *are called coordinate functionals relative to the basis* (x_n).

Definition 1.15. A basis $(x_n)_{n \in \mathbb{N}}$ is called *an absolute basis* if for each continuous seminorm p on $\mathbf{E}$ there exists a continuous seminorm q and a positive constant

C satisfying

$$\sum_{n=1}^{\infty} |x_n^*(x)|\, p(x_n) \leq Cq(x) \ \ \forall x \in \mathbf{E}$$

Theorem 1.2. *(Dynin-Mitiagin Theorem)Every basis in a NFS is absolute.*

Remark 1.4. *We get from the Dynin-Mitiagin Theorem that each nuclear Fréchet space $\mathbf{E}$ with basis can be identified with a nuclear Köthe space $\Lambda(a_{n,k})$. Absoluteness of a basis in a nuclear Fréchet space which has an increasing seminorm system implies that any nuclear Fréchet space with a basis $(e_n)_n$ and seminorm system $(\|\cdot\|_k)_k$ is canonically isomorphic to the nuclear Köthe space $\Lambda(a_{n,k})$ where $a_{n,k} = \|e_n\|_k$ for $k = 1, 2, 3...$, via the map $T : \mathbf{E} \to \Lambda(a_{n,k})$ defined by $T(x) = (\alpha_n)_{n\in\mathbb{N}}$ where $x = \sum_n \alpha_n x_n$. The converse is trivially true since a nuclear Köthe space is a nuclear Fréchet space with basis.*

Now we denote the class of power series spaces which is important because many spaces appearing in analysis are isomorphic to power series spaces.

Definition 1.16. Let $(r_k)_{k\in\mathbb{N}}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be increasing such that $\lim_{k\to\infty} r_k = r$ and $\lim_{n\to\infty} \alpha_n = \infty$. Then the Köthe space $\Lambda(A)$ where $A = (\exp(r_k\alpha_n))_{k,n}$ is called *a power series space* and is denoted by $\Lambda_r(\alpha)$*;*

$$\Lambda_r(\alpha) := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \|x\|_k := \sum_{j\in\mathbb{N}} |x_j| \exp(r_k\alpha_j) < \infty, \forall k \right) \tag{1.7}$$

If $r < \infty$, $\Lambda_r(\alpha)$ is called a power series space of finite type and if $r = \infty$, $\Lambda_r(\alpha)$ is called a power series space of infinite type.

It is clear that a power series space is independent of the choice of the sequence $(r_k)_{k\in\mathbb{N}}$ as long as (r_k) converges to r. Hence we can take

- for $r = 0$, $r_k = -\frac{1}{k}$

- for $r = \infty, r_k = k$

It is not difficult to show that for $r < \infty$ $\Lambda_r(\alpha)$ is isomorphic to $\Lambda_0(\alpha)$. So basically there are two types of power series spaces, namely the ones with $r = 0$ and others with $r = \infty$. From the Grothendièck-Pietsch criterion, we can obtain;

- $\Lambda_\infty(\alpha)$ is nuclear if and only if $\sup\limits_{n\in\mathbb{N}} \dfrac{\log n}{\alpha_n} < \infty$
- $\Lambda_0(\alpha)$ is nuclear if and only if $\lim\limits_{n\to\infty} \dfrac{\log n}{\alpha_n} = 0$.

1.3 Spectral Decomposition

In this section a special representation of compact operators between Hilbert spaces is shown.

Let $\mathbf{H}$ be a Hilbert space and T is a linear, compact and selfadjoint operator and let $(\lambda_n)_{n\in\mathbb{N}_0}$ be the sequence of eigenvalues of T. Then $(\lambda_n)_{n\in\mathbb{N}_0}$ is a real null sequence and there exists an orthonormal system $(e_n)_{n\in\mathbb{N}_0}$, so that

$$T = \sum_{n=0}^{\infty} \lambda_n \langle ., e_n \rangle e_n \tag{1.8}$$

where the series converges in norm.[11]

Let's consider two Hilbert spaces $\mathbf{H}$ *and* $\mathbf{G}$ and $T \in K(\mathbf{H}, \mathbf{G})$ be a compact operator.

Now TT^* is a compact and self adjoint operator from $\mathbf{H}$ to $\mathbf{H}$. By above, there exists a decreasing null sequence $(s_n)_{n\in\mathbb{N}_0}$ and an orthonormal system $(e_n)_{n\in\mathbb{N}_0}$ in $\mathbf{H}$ such that

$$TT^* = \sum_{n=0}^{\infty} s_n^2 \langle \cdot, e_n \rangle e_n \tag{1.9}$$

Let's define $f_n = s_n^{-1} T e_n$ for $n \in \mathbb{N}_0$ with $s_n > 0$. Then for $n, m \in \mathbb{N}_0$ with $s_n > 0$, $s_m > 0$;

$$\begin{aligned}
\langle f_n, f_m \rangle &= \frac{1}{s_n} \frac{1}{s_m} \langle T e_n, T e_m \rangle \\
&= \frac{1}{s_n} \frac{1}{s_m} \langle T T^* e_n, e_m \rangle \\
&= \frac{s_n^2}{s_n s_m} \langle e_n, e_m \rangle \\
&= \delta_{nm}
\end{aligned}$$

If $N = \{n \in \mathbb{N}_0 : s_n > 0\}$ is a finite set, then we extend the orthonormal system $(f_n)_{n \in \mathbb{N}}$ to an orthonormal system $(f_n)_{n \in \mathbb{N}_0}$ in $\mathbf{G}$.

For $y \in \mathbf{H}$ with $y \perp e_n$ for all $n \in \mathbb{N}_0$ we have,

$$\|Ty\|^2 = \langle Ty, Ty \rangle = \langle T T^* y, y \rangle = 0 \; by (1.1)$$

Thus from the definition of $(f_n)_{n \in \mathbb{N}_0}$ we have in either case, for each $x \in \mathbf{H}$,

$$\begin{aligned}
Tx &= T\left(x - \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n \right) + T\left(\sum_{n=0}^{\infty} \langle x, e_n \rangle e_n \right) \\
&= \sum_{n=0}^{\infty} \langle x, e_n \rangle T e_n \\
&= \sum_{n=0}^{\infty} s_n \langle x, e_n \rangle f_n
\end{aligned}$$

Hence we obtain that the series $\sum_{n=0}^{\infty} s_n \langle \cdot, e_n \rangle f_n$ converges to T in norm.

There exists a decreasing null sequence $(s_n)_{n \in \mathbb{N}_0}$ in $[0, \infty[$ and orthonormal systems $(e_n)_{n\in\mathbb{N}_0}$ in $\mathbf{H}$ and $(f_n)_{n\in\mathbb{N}_0}$ in $\mathbf{G}$ so that

$$T = \sum_{n=0}^{\infty} s_n \langle \cdot, e_n \rangle f_n \tag{1.10}$$

where the series converges in the operator norm.

This representation of T is referred to as a Schmidt representation of T.

CHAPTER 2

EXTERIOR INTERPOLATIVE CONDITIONS

In this chapter, first we state and prove an interpolation lemma, then we define and examine some exterior interpolative conditions

2.1 Interpolation Lemma

In this secton we introduce the interpolation lemma of Djakov.

Theorem 2.1. *Suppose* $\mathbf{E}$ *and* $\mathbf{F}$ *are linear spaces,* $|\cdot|_0 \leq |\cdot|_1 \leq |\cdot|_2$ *are seminorms on* $\mathbf{E}$*,* $U_2 \subset U_1 \subset U_0$ *are the corresponding unit balls, and* $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2$ *are seminorms on* $\mathbf{F}$*. If there exists null sequences* $(r_n)_{(n\in\mathbb{N})}$*,* $(s_n)_{(n\in\mathbb{N})}$ *and sequences* $(R_n)_{(n\in\mathbb{N})}$*,* $(S_n)_{(n\in\mathbb{N})}$ *which diverge to infinity satisfying*

(1) $U_1 \subset s_n U_0 + S_n U_2, \quad \forall n \in \mathbb{N}$

(2) $\|\cdot\|_1 \leq R_n \|\cdot\|_0 + r_n \|\cdot\|_2, \quad \forall y \in \mathbf{F}, \quad \forall n \in \mathbb{N}$

(3) $\sum_n s_n R_n \leq C$ *and* $\sum_n S_{n+1} r_n \leq C, \quad for\ some\ C > 0$

Then for any $T : \mathbf{E} \to \mathbf{F}$ *linear with* $\|Tx\|_i \leq C_i |x|_i$ *for some* $C_i > 0, i = 0, 2$ *we have;*

$$\|Tx\|_1 \leq 2C(C_0 + C_2)\,|x|_1 \qquad \forall\, x \in \mathbf{E} \tag{2.1}$$

Proof. Let $x \in U_1$, then by (1) there exists $x_n \in s_n U_0$ and $y_n \in S_n U_2$ $\forall n \in \mathbb{N}$ such that $x = x_n + y_n$

For $n = 1$ we may choose $x_1 = x$ and $y_1 = 0$ without loss of generality $s_1 = 1$. Since

$$x = x_n + y_n = x_{n+1} + y_{n+1} \tag{2.2}$$

$$x_{n+1} - x_n = y_n - y_{n+1} \ \ \forall n \in \mathbb{N} \tag{2.3}$$

$$y_n = \sum_{k=1}^{n-1} y_{k+1} - y_k \tag{2.4}$$

$x_{k+1} - x_k \in (s_{k+1} + s_k)U_0$ and $y_k - y_{k+1} \in (S_{k+1} + S_k)U_2$ because U_0 and U_2 are absolutely convex.

Since $S_n \nearrow \infty$ and $s_n \searrow 0$, $x_{k+1} - x_k \in 2s_k U_0$ and $y_k - y_{k+1} \in 2S_{k+1} U_2$

$$\begin{aligned}
\| Ty_{k+1} - Ty_k \|_1 &\leq R_k \| Ty_{k+1} - Ty_k \|_0 + r_k \| Ty_{k+1} - Ty_k \|_2 && (2.5)\\
&\leq R_k C_0 \, | y_{k+1} - y_k |_0 + r_k C_2 \, | y_{k+1} - y_k |_2 && (2.6)\\
&\leq R_k C_0 2 s_k + r_k C_2 2 S_{k+1} && (2.7)\\
&\leq 2C_0 R_k s_k + 2C_2 r_k S_{k+1} && (2.8)
\end{aligned}$$

Thus ;

$$\begin{aligned}
\| Ty_n \|_1 &\leq \sum_{k=1}^{n-1} \| Ty_{k+1} - Ty_k \|_1 && (2.9)\\
&\leq \sum_{k=1}^{n-1} (2C_0 R_k s_k + 2C_2 r_k S_{k+1}) && (2.10)\\
&\leq 2C_0 \sum_{k=1}^{\infty} R_k s_k + 2C_2 \sum_{k=1}^{\infty} r_k S_{k+1} && (2.11)\\
&\leq 2C(C_0 + C_2) \ \ \forall n && (2.12)
\end{aligned}$$

If we prove $\| Tx_n \|_1 \to 0$ as $n \to \infty$, then $\| Tx \|_1 \leq 2C(C_0 + C_2)$.

$$\begin{aligned}
\|Tx_n\|_1 &\leq R_n \|Tx_n\|_0 + r_n \|Tx_n\|_2 &(2.13)\\
&\leq C_0 R_n |x_n)|_0 + C_2 r_n |x_n)|_2 &(2.14)\\
&\leq C_0 R_n s_n + C_2 r_n |x - y_n|_2 &(2.15)\\
&\leq C_0 R_n s_n + C_2 r_n |x|_2 + C_2 r_n |y_n|_2 &(2.16)\\
&\leq C_0 R_n s_n + C_2 r_n S_n + C_2 r_n |x|_2 &(2.17)
\end{aligned}$$

By (3) since series converge then the general terms $r_n S_n$ and $R_n s_n$ go to zero, so

$$\|Tx_n\|_1 \to 0 \;\; as \;\; n \to \infty \tag{2.18}$$

Thus

$$\|Tx\|_1 \leq \|Tx_n\|_1 + \|Ty_n\|_1 \leq 2C_1(C_0 + C_2) \tag{2.19}$$

■

2.2 Exterior Interpolative Condition

An exterior interpolative condition is one which involves a dead-end space. A dead-end space is a Hilbert space $\mathbf{H}_\infty$ which is continuously and densely imbedded in $(\mathbf{E}, \mathcal{P})$. First we will define an exterior interpolative condition involving a dead-end space $\mathbf{H}_\infty$ and then we will prove that if $\mathbf{E}$ satisfies such an exterior interpolative condition, then $\mathbf{E}$ has a basis. Exterior interpolative type conditions are used first by Mitiagin-Khenkin. We will use in this section, an exterior interpolative condition of Djakov [5].

Definition 2.1 $(DN(\mathcal{P},\infty))$**.** We say that $(\mathbf{E},\mathcal{P})$ has the property $DN(\mathcal{P},\infty)$ if the following is satisfied;
There exist $|\cdot|_0 \in \mathcal{P}$ *and sequences* $(r_n(k))_{(n\in\mathbb{N})}$, $(R_n(k))_{(n\in\mathbb{N})}$ *so that for all* k, *there exists* $l > k$ *with*

$$\| x \|_k \leq R_n(k) \| x \|_0 + r_n(k) \| x \|_\infty , \ \forall n \geq l$$

where $x \in \mathbf{H}_\infty$ and the sequences $(r_n(k))_{(n\in\mathbb{N})}$, $(R_n(k))_{(n\in\mathbb{N})}$ converges to zero and diverges to infinity respectively as $n \to \infty$.

Definition 2.2 $(\Omega(\mathcal{P},\infty))$**.** We say $(\mathbf{E},\mathcal{P})$ has the property $\Omega(\mathcal{P},\infty)$ if the following is satisfied;
There exist $|\cdot|_0 \in \mathcal{P}$ *and sequences* $(s_n(k))_{(n\in\mathbb{N})}$, $(S_n(k))_{(n\in\mathbb{N})}$ *so that for all* k, *there exists* $l > k$ *with*

$$U_k \subset s_n U_0 + S_n U_\infty, \ \forall n \geq l$$

where the sequences $(s_n(k))_{(n\in\mathbb{N})}$, $(S_n(k))_{(n\in\mathbb{N})}$ converges to zero and diverges to infinity respectively as $n \to \infty$.

Now we can state the exterior interpolative condition which was mentioned in the beginning of the Section 2.1.

Definition 2.3 $(\mathcal{T}(\mathcal{P},\mathcal{Q},\infty))$**.** We say that $\mathbf{E}$ has the property $\mathcal{T}(\mathcal{P},\mathcal{Q},\infty)$ where $\mathcal{P}$ and $\mathcal{Q}$ are fundamental systems of seminorms for $\mathbf{E}$ if the following conditions hold;

1. *There exists* $|\cdot|_0 \in \mathcal{P}\cap\mathcal{Q}$ *so that* $(\mathbf{E},\mathcal{P})$ *has the properties* $\Omega(\mathcal{P},\infty)$ *and* $DN(\mathcal{Q},\infty)$.

2. For all k there exists q such that

$$\sum_n s_n(q) R_n(k) < \infty \text{ and } \sum\nolimits_n S_{n+1}(q) r_n(k) < \infty$$

where the sequences $(r_n(k))_n$, $(R_n(k))_n$, $(s_n(q))_n$, $(S_n(q))_n$ are coefficients in the properties $DN(\mathcal{Q},\infty)$ and $\Omega(\mathcal{P},\infty)$ respectively.

All of these three conditions together is called an exterior interpolative condition since they depend not only on the fundamental systems of seminorms but also on the dead-end space $\mathbf{H}_\infty$.

The following remark from literature shows how exterior interpolative condition can be applied to find a basis.

Remark 2.1. *$\underline{D}N$ implies $DN(\mathcal{P},\infty)$ and $\overline{\Omega}_B$ implies $\Omega(\mathcal{P},\infty)$. To prove this recall that a NFS satisfies the properties;*

i) $\underline{D}N$ in case,

$\exists p \in \mathbb{N}$, $\forall k \in \mathbb{N}$, *there exist* $n \in \mathbb{N}$, $0 < \theta < 1$ *and* $C > 0$ *with,*

$$\|x\|_k \leq C \|x\|_p^{1-\theta} \|x\|_n^{\theta}, \ \forall x \in \mathbf{E} \tag{2.20}$$

and if $B \subset \mathbf{E}$ is a bounded set, $\mathbf{E}$ is said to have the property;

ii) $\overline{\Omega}_B$, in case;

$\forall k \in \mathbb{N}$ *there exist* $n \in \mathbb{N}$, $\exists C > 0$ *with*

$$\|y\|_n^{*\,2} \leq C \|y\|_k^* \|y\|_B^*, \ \forall y \in \mathbf{E}^* \tag{2.21}$$

[See 16]

The property $\overline{\Omega}_B$ is equivalent to

For each $p \in \mathbb{N}$ and each $0 < \alpha < 1$ there exist $q \in \mathbb{N}$ and $D > 0$ with

$$\|y\|_q^* \leq D\|y\|_p^{*\,1-\alpha} \|y\|_B^{*\,\alpha}, \forall y \in \mathbf{E}^* \tag{2.22}$$

[Lemma 29.16 of Vogt [11]].

On the other hand the property $\underline{D}N$ trivially implies:

$\exists p \in \mathbb{N}$, $\forall k \in \mathbb{N}$ there exist $0 < \theta < 1$ and $C > 0$ with

$$\|x\|_k \leq C \|x\|_p^{1-\theta} \|x\|_B^{\theta}, \ \forall x \in \mathbf{E} \tag{2.23}$$

Since trivially for $a, b, t > 0$ and $s < 1$

$$\min[at + bt^{1-1/s}] = \frac{1}{s}(\frac{1}{s} - 1)^{s-1} a^{1-s} b^s \tag{2.24}$$

We get $\forall k \in \mathbb{N}$, $\exists 0 < \theta < 1$ and $C_1 > 0$ with

$$\|x\|_k \leq C_1 \|x\|_p^{1-\theta} \|x\|_B^{\theta} \tag{2.25}$$

$$\leq t \|x\|_p + t^{1-\frac{1}{\theta}} \|x\|_B \ \ \forall t > 0. \tag{2.26}$$

,i.e. , $(\mathbf{E}, \mathcal{P})$ has the property $DN(\mathcal{P}, \infty)$.

On the other hand for all $0 < \alpha < 1$, $\exists q \in \mathbb{N}$, $D' > 0$ with

$$\|y\|_q^* \leq D'\|y\|_p^{*\,1-\alpha} \|y\|_B^{*\,\alpha} \tag{2.27}$$

$$\leq r \|y\|_p^* + C_2 r^{1-\frac{1}{\alpha}} \|y\|_B^*, \forall y \in \mathbf{E}^* and \forall r > 0 \tag{2.28}$$

Hence

$$U_q \subset rU_p + C_3 r^{1-\frac{1}{\alpha}} B \;\; for\ appropriate\ C_3 > 0\ and\, q\ \forall r > 0 \tag{2.29}$$

So $(\mathbf{E}, \mathcal{P})$ *has the property* $\Omega(\mathcal{P}, \infty)$ *where the dead-end space* $\mathbf{H}_\infty$ *is* $\mathbf{E}_B$.

Moreover if we choose t *to be* $n+1$ *and* r *to be* $1/n^3$ *then we get* $R_n(k) = n+1$, $s_n(q) = 1/n^3$ *and*

$$\blacklozenge \sum_n R_n(k) s_n(q) = \sum_n \frac{n+1}{n^3} < \infty$$

$$\begin{aligned} \blacklozenge \sum_n r_n(k) S_{n+1}(q) &= \sum_n (n+1)^{1-1/\theta} C_2 \left(\frac{1}{(n+1)^3}\right)^{1-1/\alpha} \\ &= \sum_n C_2 (n+1)^{-(2+1/\theta-3/\alpha)} \end{aligned}$$

If $2 + \frac{1}{\theta} - \frac{1}{\alpha} > 1$, *i.e.* $\alpha > 3\theta$ *then the series above are convergent by p-test. Hence we get the property* $\mathcal{T}(\mathcal{P}, \mathcal{Q}, \infty)$.

∎

Remark 2.2. *The conditions* $DN(\mathcal{P}, \infty)$ *and* $\Omega(\mathcal{P}, \infty)$ *are strictly weaker than the conditions* $\underline{D}N$ *and* $\overline{\Omega}_B$. *Because in Chapter 3 Remark* (3.2) *it is shown that every nuclear Köthe space satisfy the conditions* $DN(\mathcal{P}, \infty)$ *and* $\Omega(\mathcal{P}, \infty)$. *But a nuclear Fréchet space satisfying the conditions* $\underline{D}N$ *and* $\overline{\Omega}_B$ *is isomorphic to a power series space of finite type.[[11] page 373]*

∎

The conditions $\underline{D}N$ and $\overline{\Omega}_B$ mentioned in Remark are exactly the exterior interpolative conditions used by Mitiagin-Henkin to prove the existence of basis.

Their argument goes as follows:

The imbedding $\iota : \mathbf{H}_\infty \hookrightarrow \mathbf{E}_0$ is compact and we can choose a common orthonormal system $(f_n)_{(n\in\mathbb{N})}$ and $(x_n)_{(n\in\mathbb{N})}$ in $\mathbf{H}_\infty$ and $\mathbf{E}_0$ respectively such that

$$\iota \quad : \quad \mathbf{H}_\infty \hookrightarrow \mathbf{E}_0 \tag{2.30}$$

$$\iota(x) = x = \sum_{n=0}^{\infty} \alpha_n \langle x, f_n \rangle_\infty x_n \quad \forall x \in \mathbf{H}_\infty \tag{2.31}$$

for the decreasing null sequence $(\alpha_n)_{n\in\mathbb{N}}$[See Introduction]. If it can be shown that $\sum_n \alpha_n \langle x, f_n \rangle_\infty x_n$ converges in each $\|\cdot\|_k$, then this series converges to x in $\mathbf{E}$ and hence $(x_n)_{(n\in\mathbb{N})}$ will be a basis for $\mathbf{E}$.

From $\underline{D}N(\mathcal{P}, \infty)$ property, fix $\|\cdot\|_0 \in \mathcal{P}$. Then $\forall k \in \mathbb{N}$ there exist $0 < \delta_k < 1$ *and* and $C > 0$ such that;

$$\|x_n\|_k \leq C \|x_n\|_0^{1-\delta_k} \|x_n\|_\infty^{\delta_k} \tag{2.32}$$

Since $\|x_n\|_0 = 1$ and $\|x_n\|_\infty = \left\|\frac{f_n}{\alpha_n}\right\|_\infty = \frac{1}{\alpha_n} \|f_n\|_\infty = \frac{1}{\alpha_n}$ then we have,

$$\|x_n\|_k \leq C(\frac{1}{\alpha_n})^{\delta_k} \tag{2.33}$$

On the other hand from $\overline{\Omega}(\mathcal{P}, \infty)$ property, $\forall s$, $\forall \gamma_s$ with $0 < \gamma_s < 1$ and fix $\|\cdot\|_0^*$ such that;

$$\|f_n\|_s^* \leq D\|f_n\|_0^{*1-\gamma_s} \|f_n\|_\infty^{*\ \gamma_s} \tag{2.34}$$

Since $\|f_n\|_\infty^* = 1$ and $\|f_n\|_0^* = \sup_{\|x\|_0 \leq 1} |\langle x, \alpha_n f_n \rangle_0| = \alpha_n \sup_{\|x\|_0 \leq 1} |\langle x, x_n \rangle_0| = \alpha_n$ we have,

$$\|f_n\|_s^* \leq D(\alpha_n)^{1-\gamma_s} \tag{2.35}$$

Thus $\forall k \in \mathbb{N}$, $\exists s \in \mathbb{N}$ such that,

$$\left\| \sum_n \alpha_n \langle x, f_n \rangle_\infty x_n \right\|_k \leq \sum_n \alpha_n \|f_n\|_s^* \|x\|_s \|x_n\|_k \tag{2.36}$$

$$\leq \sum_n \alpha_n C \frac{1}{\alpha^{\delta_k}} D \alpha^{1-\gamma_s} \|x\|_s \tag{2.37}$$

$$\leq C^* \sum_n \alpha^{2-(\delta_k+\gamma_s)} \|x\|_s \tag{2.38}$$

If we can choose suitable δ_k and γ_s such that $0 < \delta_k + \gamma_s < 1$ and since $\mathbf{E}$ is NFS then $\sum \alpha^{2-(\delta_k+\gamma_s)}$ converges and so $\sum_n \alpha_n \langle x, f_n \rangle_\infty x_n$ converges to x for the Hilbert space $\mathbf{E}_0$.

The condition is true for all k, i.e., the series converges to x in any $\mathbf{E}_k$ and so in $\mathbf{E}$. Hence (x_n) is a basis in $\mathbf{E}$.

■

More generally Djakov [5] proved that for a fixed dead-end space $\mathbf{H}_\infty$ and a NFS with a fundamental system of seminorms, the interpolative condition $DN(\mathcal{P}, \infty)$, $\Omega(\mathcal{P}, \infty)$ and $\mathcal{T}(\mathcal{P}, \mathcal{P}, \infty)$ is sufficient for existence of basis. To see this;

Fix a NFS, $\mathbf{E}$, a Hilbert space $\mathbf{H}_\infty$ which is continuously and densely imbedded in $\mathbf{E}$ such that $\mathbf{E}$ satisfies the exterior interpolative condition

The imbedding $P : \mathbf{H}_\infty \hookrightarrow \mathbf{E}_0$ can be given by $Px = \sum_{n=1}^\infty \alpha_n \langle x, e_n \rangle_\infty f_n$ where $(e_n)_n$ and $(f_n)_n$ are orthonormal basis in $\mathbf{H}_\infty$ and $\mathbf{E}_0$ respectively.

Define the family of operators

$$P_N(x) = \sum_{n=1}^N \alpha_n \langle x, e_n \rangle_\infty f_n \tag{2.39}$$

$$= \sum_{n=1}^N \langle Px, f_n \rangle_0 f_n \quad \forall N \in \mathbb{N} \ \mathit{and} \ \ x \in \mathbf{H}_\infty \tag{2.40}$$

Then,

1. $P_N(P^{-1}(U_0)) \subset \left\{\sum_{n=1}^{N} \alpha_n \langle z, f_n\rangle_0 f_n : z \in U_0\right\} \subset U_0$

2. $If\ x = \sum_{n=1}^{\infty} x_n e_n\ in\ \mathbf{H}_\infty,$

$$P\left(\sum_{n=1}^{N} x_n e_n\right) = \sum_{j=1}^{\infty} \alpha_j \left\langle \sum_{n=1}^{N} x_n e_n, e_j \right\rangle_\infty f_j \tag{2.41}$$

$$= \sum_{n=1}^{N} \alpha_n x_n f_n = P_N(x) \tag{2.42}$$

so $P_N(\overline{U_\infty}) = \left\{\sum_{n=1}^{N} \alpha_n x_n f_n : \|x\|_\infty \leq 1\right\} \subset P(\overline{U_\infty})$

Therefore the family of operators $(P_N)_{N\in\mathbb{N}}$ is equicontinuous both in $\mathbf{E}_0$ and $\mathbf{H}_\infty$.

The same family of operators is equicontinuous in $\mathbf{E}$ by the interpolation lemma. Since $P_N(x) \to P(x)$ for $x \in \mathbf{H}_\infty$ and $\mathbf{H}_\infty$ is dense in $\mathbf{E}$, by the Banach-Steinhaus Theorem [, page 98] the convergence is obtained on the whole of $\mathbf{E}$. Hence we have $\lim_{N\to\infty} P_N(x) = P(x) = x$ for any $x \in \mathbf{E}$, i.e. $(f_n)_n$ is a basis in $\mathbf{E}$.

■

CHAPTER 3

INNER INTERPOLATIVE CONDITIONS

In this chapter we will give some definitions and the main theorem of this thesis about the inner interpolative condition implying the existence of basis.

3.1 Definitions

Let $(\mathbf{E}, \mathcal{P})$ be a nuclear Fréchet space. This section introduces the inner interpolative condition, $DN(\mathcal{Q})$, $\Omega(\mathcal{P})$, $\mathcal{T}(\mathcal{P}, \mathcal{Q})$ for the given fundamental systems of seminorms $\mathcal{P}$ and $\mathcal{Q}$ where $DN(\mathcal{Q})$, $\Omega(\mathcal{P})$ are generalization of the well-known DN-type and Ω-type invariants of Vogt and $\mathcal{T}$ is a compability condition. Later we will see that these conditions are sufficient conditions for the existence of a basis in $\mathbf{E}$.

Suppose $\mathcal{P} = \{\|\cdot\|_k\}_{k=0}^{\infty}$ and $\mathcal{Q} = \{|\cdot|_k\}_{k=0}^{\infty}$ are increasing seminorm systems on a vector space $\mathbf{E}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be as above;

Definition 3.1. If $\|\cdot\|_0 = |\cdot|_0$ and there exists k_0 and $\sigma : \mathbb{N} \to \mathbb{N}$ non-decreasing function with $\sigma(k+1) \leq k$ and $|\cdot|_k \leq M_k \|\cdot\|_{\sigma(k)}$ for all $k \geq k_0$ and for some $M_k \geq 1$, then we will call the seminorm systems $\mathcal{P}$ and $\mathcal{Q}$ a $\sigma - \textit{matched system}$ and denote by $(\mathcal{P}, \mathcal{Q}, \sigma)$.

Also, whenever $\mathcal{P}$ and $\mathcal{Q}$ are fundamental systems of seminorms, we will call $(\mathcal{P}, \mathcal{Q}, \sigma)$ an *equivalent* $\sigma - \textit{matched system}$.

Note that in the notation of $\sigma - matched$ system, $(\mathcal{P}, \mathcal{Q}, \sigma)$, *the order is important.*

Remark 3.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be equivalent increasing seminorm systems on $\mathbf{E}$. We shall show that there exists an equivalent increasing seminorm system $\tilde{\mathcal{P}}$ obtained from $\mathcal{P}$ by passing to a subsequence and $\tilde{\sigma}$ so that we have the $\tilde{\sigma}-matched$ system $\left(\tilde{\mathcal{P}}, \mathcal{Q}, \tilde{\sigma}\right)$:

First, we may assume that $\|\cdot\|_0 = |\cdot|_0$. Otherwise we may include the smallest seminorm in the systems to both seminorm systems. Clearly these systems generate the same topology.

For all k, there exists k' such that $U_{k'} \subset C_k V_k$ for some $C_k > 1$ where $U_{k'}$ and V_k are the unit balls for the seminorms $\|\cdot\|$ and $|\cdot|$ respectively.

Define $\sigma(k) = k'$. Without loss of generality we can choose σ strictly increasing.

Construct a new seminorm system $\tilde{\mathcal{P}}$ by

$$\tilde{U}_{k-1} = U_{\sigma(k)} \text{ for all } k \text{ where } \sigma(k) = k'$$

Then as σ is increasing the new seminorm system is also increasing and $\tilde{U}_{k-1} \subset C_k V_k \ \ \forall k$. If we define $\tilde{\sigma} : \mathbb{N} \to \mathbb{N}$ by $\tilde{\sigma}\,(k) = k-1$ then the condition in Definition 3.1 are satisfied, we have $\left(\tilde{\mathcal{P}}, \mathcal{Q}, \tilde{\sigma}\right)$.

Hence for two equivalent seminorm systems $\mathcal{P}, \mathcal{Q}$ one can find a map $\tilde{\sigma} : \mathbb{N} \to \mathbb{N}$ increasing and a refinement of $\tilde{\mathcal{P}}$ of $\mathcal{P}$ so that we have $\left(\tilde{\mathcal{P}}, \mathcal{Q}, \tilde{\sigma}\right)$.

$\blacksquare$

Now, we define the properties $DN(\mathcal{P})$, and $\Omega(\mathcal{P})$ where $\mathcal{P}$ is a fundamental system of seminorms on $\mathbf{E}$.

Definition 3.2. A NFS $\mathbf{E}$ with an increasing system of seminorms $\mathcal{P} = \{\|\cdot\|_k\}_{k=0}^{\infty}$ is said to have:

i) $DN(\mathcal{P})$ case:

For all k there exists $l > k$, $A_l \in \mathbb{R}^+$ and positive real number sequences $(r_n(k,l))_n$ and $(R_n(k,l))_n$ such that

$$\|x\|_k \leq R_n(k,l)\,\|x\|_0 + r_n(k,l)\,\|x\|_l\,, \qquad \forall x \in \mathbf{E}, \forall n \geq A_l,$$

where $r_n(k,l)$ converges to 0 and $R_n(k,l)$ diverges to ∞ as $n \to \infty$

ii) $\Omega(\mathcal{P})$ case:

For all q there exists $p > q$, $B_p \in \mathbb{R}^+$ and positive real number sequences $(s_n(q,p))_n$ and $(S_n(q,p))_n$ such that

$$U_q \subset s_n(q,p)\,U_0 + S_n(q,p)\,U_p, \forall n \geq B_p$$

where $s_n(q,p)$ converges to 0 and $S_n(q,p)$ diverges to ∞ as $n \to \infty$.

Remark 3.2. *Let $\Lambda(a_n^k)$ be a nuclear Köthe space and let $\mathcal{P}$ denote the natural fundamental system of seminorms for $\Lambda(a_n^k)$.[See Introduction]*

Fix q, let $x = (x_n)_{n=1}^{\infty} \in U_q$ so $\|x\|_q = \sum_{n=0}^{\infty} a_n^q\,|x_n| \leq 1$.

Let's take $X_N = (x_1, x_2, \cdots, x_N, 0, 0, \cdots)$ and $Y_N = (0, 0, ..., y_{N+1}, y_{N+2}, ...)$. Then $x = X_N + Y_N$, $\forall n \in \mathbb{N}$.

Choose $p > q$ such that $(\frac{a_n^q}{a_n^p})_{n\in\mathbb{N}} \in \ell_1$. Then,

$$\begin{aligned}
\|X_N\|_p &= \sum_{n=1}^{N} a_n^p \,|x_n| \\
&\leq \sum_{n=1}^{N} \frac{a_n^p}{a_n^q} a_n^q \,|x_n| \\
&\leq \sup_{1\leq n\leq N} \left\{\frac{a_n^p}{a_n^q}\right\} \sum_{n=1}^{N} a_n^q \,|x_n| \\
&\leq \sup_{1\leq n\leq N} \left\{\frac{a_n^p}{a_n^q}\right\} \|x\|_q \\
&\leq \sup_{1\leq n\leq N} \left\{\frac{a_n^p}{a_n^q}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\|Y_N\|_0 &= \sum_{n=N+1}^{\infty} a_n^0 \,|x_n| \\
&\leq \sum_{n=N+1}^{\infty} \frac{a_n^0}{a_n^q} a_n^q \,|x_n| \\
&\leq \sup_{n\leq N+1} \left\{\frac{a_n^0}{a_n^q}\right\} \sum_{n=N}^{\infty} a_n^q \,|x_n| \\
&\leq \sup_{n\geq N+1} \left\{\frac{a_n^0}{a_n^q}\right\} \|x\|_q \\
&\leq \sup_{n\geq N+1} \left\{\frac{a_n^0}{a_n^q}\right\}
\end{aligned}$$

So we obtain, $\forall q\ \exists\ p > q$ such that

$$U_q \subset \sup_{n\geq N+1} \left\{\frac{a_n^0}{a_n^q}\right\} U_0 + \sup_{1\leq n\leq N} \left\{\frac{a_n^p}{a_n^q}\right\} U_p$$

and we may choose

$$s_N(q,p) = \sup\left\{\frac{a_n^0}{a_n^q} : n \geq N+1\right\}, \;\; S_N(q,p) = \sup\left\{\frac{a_n^p}{a_n^q} : 1 \leq n \leq N\right\}$$

Since $\mathbf{E}$ *is a nuclear Köthe space,* $s_N(q,p)$ *converges to zero and* $S_N(q,p)$ *diverges to infinity as* $n \to \infty$. *So we obtain the condition* $\Omega(\mathcal{P})$.

On the other hand let $x \in \mathbf{E}$. *Then* $\forall k$, $\exists\, l > k$ *with* $(\frac{a_n^k}{a_n^l})_{n\in\mathbb{N}} \in \ell_1$

$$\begin{aligned}
\|x\|_k &= \sum_n a_n^k |x_n| \\
&= \sum_{n\leq N} a_n^k |x_n| + \sum_{n\geq N+1} a_n^k |x_n| \\
&= \sum_{n\leq N} \frac{a_n^k}{a_n^0} a_n^0 |x_n| + \sum_{n\geq N+1} \frac{a_n^k}{a_n^l} a_n^l |x_n| \\
&\leq \sup\left\{\frac{a_n^k}{a_n^0} : 1 \leq n \leq N\right\} \sum_{1\leq n\leq N} a_n^0 |x_n| \\
&\quad + \sup\left\{\frac{a_n^k}{a_n^l} : n \geq N+1\right\} \sum_{n\geq N+1} a_n^l |x_n| \\
&\leq \sup\left\{\frac{a_n^k}{a_n^0} : 1 \leq n \leq N\right\} \|x\|_0 + \sup\left\{\frac{a_n^k}{a_n^l} : n \geq N+1\right\} \|x\|_l
\end{aligned}$$

where $R_N(k,l) = \sup\left\{\frac{a_n^k}{a_n^0} : 1 \leq n \leq N\right\}$ *and* $r_N(k,l) = \sup\left\{\frac{a_n^k}{a_n^l} : n \geq N+1\right\}$. *So we arrive at the condition* $\mathcal{DN}(\mathcal{P})$.

□

Definition 3.3. Fix a NFS $\mathbf{E}$ and a fundamental system of seminorms $\mathcal{P}$ and $\mathcal{Q}$ which are $\sigma-$*matched*. Suppose $\mathcal{P}$ and $\mathcal{Q}$ have the properties $DN(\mathcal{Q})$ and $\Omega(\mathcal{P})$, then $\mathbf{E}$ is said to have the property $(\mathcal{T}(\mathcal{P},\mathcal{Q}))$ for $(\mathcal{P},\mathcal{Q},\sigma)$ if the following hold;

For all k, there exists q such that

- $\sum_n \sup_{p\in\mathcal{I}_n} \{s_n(q,p-1)R_n(k,p-1)\} < \infty$ *and*
- $\sum_n \sup_{p\in\mathcal{I}_n} \left\{S_{n+1}(q,p-1)r_n(k,p-1)n^3\right\} < \infty$

where $(r_n(k,p))_n, (R_n(k,p))_n, (s_n(q,p))_n$ and $(S_n(q,p))_n$ are coefficients in the properties $DN(\mathcal{Q})$ and $\Omega(\mathcal{P})$, respectively, and $\mathcal{I}_n = \{p : n \geq \max(A_p, B_p)\}$.

We will refer to this condition together with $DN(\mathcal{Q})$ and $\Omega(\mathcal{P})$ as a compatible inner interpolative condition.

3.2 Main Theorem

Now we will state the main theorem of this thesis.[20]

Theorem 3.1. *Let* $\mathbf{E}$ *be a NFS and let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be fundamental system of seminorms forming a* $\sigma -$*matched system* $(\mathcal{P},\mathcal{Q},\sigma)$ *on* $\mathbf{E}$*. If* $\mathbf{E}$ *satisfies a compatible inner interpolative conditions then* $\mathbf{E}$ *has a basis.*

Proof. Suppose $(\mathcal{P},\mathcal{Q})$ are equivalent $\sigma -$ *matched* systems which satisfy the hypothesis of the Theorem such that $\mathbf{E}$ has the properties $\Omega(\mathcal{P})$, $DN(\mathcal{Q})$ and $\mathcal{T}(\mathcal{P},\mathcal{Q})$. Moreover, there exists $\sigma : \mathbb{N} \to \mathbb{N}$ nondecreasing for all k, there exists k_0 with $\sigma(k+1) \leq k$ and $U_{\sigma(k)} \subset M_k V_k$ for all $k \geq k_0$.

We require a technical lemma to proceed to the proof, wherein we construct a Hilbert space and a dead-end space which is densely embedded in $\mathbf{E}$ with certain properties. The construction method uses the method in [Lemma 1.1 and Lemma 1.2 [16]] for equivalent $\sigma -$ *matched* seminorm system $(\mathcal{P},\mathcal{Q},\sigma)$.

Lemma 3.1. *For arbitrary sequences* $(\phi_k)_k$*,* $(\psi_k)_k$ *of positive real valued functions on* $\mathbb{R}^+$*,* $(A_k)_{k=1}^\infty$*,* $(B_k)_{k=1}^\infty$ *and* $(M_k)_{k=1}^\infty$ *of positive numbers there exists* $(n_k) \subset \mathbb{N}$ *strictly increasing sequence such that for all* $k \geq k_0$*,* *(1)* $n_{k+1} \geq \psi_k(n_k)$*;*

(2) $n_k \geq 1, n_k \geq M_k, A_k, B_k$*;*

Moreover for any positive sequence $(\epsilon_k)_k$ *with* $\sum_k \epsilon_k \leq 1$*, there exists a hilbertian compact ball* B *in* $\mathbf{E}$ *such that for all* $k \geq k_0$

(3) $U_{\sigma(k)} \subset M_k B + U_0/\phi_k(n_k) \subset n_k B + U_0/\phi_k(n_k)$*;*

(4) $B \subset \frac{n_{k-1}}{\epsilon_{k-1}} U_{\sigma(k)}$*;*

(5) The Hilbert space $H_B = \{x \in \mathbf{E} : \|x\|_B < \infty\}$ *is dense in* $\mathbf{E}$ *and* $\|\cdot\|_B$ *is a continuous hilbertian seminorm in* $\mathbf{E}$*.*

Proof. Since $\mathbf{E}$ is separable, there exists a total sequence $\{\xi_n : n \in \mathbb{N}\}$ where $\xi_n \in U_n$ for all n. We proceed by induction on $k > k_0$.

Since $\mathbf{E}$ is nuclear, the inclusion map $\mathbf{E}_{\sigma(k)} \hookrightarrow \mathbf{E}_0$ is a compact map, since $U_{\sigma(k)} \subset M_k U_k$ and $U_{\sigma(k)}$ is U_0-precompact. Thus we can find a finite set $N_k = \{a_j : j \in I_k\} \subset U_k$ which also contains ξ_k so that

$$\begin{aligned} U_{\sigma(k)} &\subset \bigcup_{j \in I_k} \{M_k a_j + U_0/\phi_k(n_k)\}, i.e. \\ U_{\sigma(k)} &\subset M_k N_k + U_0/\phi_k(n_k). \end{aligned}$$

Now let n_k be an integer greater then

$$\max\left\{\|x\|_{\sigma(k+1)}; x \in N_i, 1 \leq i \leq k; \psi_{k-1}(n_{k-1}); n_{k-1}+1; M_k; A_k; B_k; 1\right\}$$

Thus by induction we obtain finite sets $N_{k_0}, \cdots, N_k; \cdots$ and a sequence (n_k) satisfying for all $k = k_0, k_0+1, \cdots$

1. $n_k \geq \psi_{k-1}(n_{k-1})$;

2. $n_k \nearrow$ and $n_k \geq M_k, A_k, B_k$;

3. $\bigcup_{i=k_0}^{k-1} N_i \subseteq n_k U_{\sigma(k+1)}$

If we denote $\bigcup_i N_i$ by $\{x_n\}$, we have for all $k \geq k_0$,

$$U_{\sigma(k)} \subset M_k N_k + U_0/\phi_k(n_k) \subset n_k \{x_n\} + U_0/\phi_k(n_k)$$

Clearly $N_i \subset U_i \subset U_k$, for $i \geq k$. $\bigcup_{i<k} N_i \subset n_k U_{\sigma(k+1)}$ because $\sigma(k+1) \leq k$. Hence $\{x_n\} \subset n_k U_{\sigma(k+1)}$. Therefore we get a total null sequence, $\{x_n\}$ such that $\|x_n\|_{\sigma(k+1)} \leq n_k$ for all n and for all $k \geq k_0$.

We now proceed exactly as in Lemma 1.2 in [16] to obtain a hilbertian compact ball B from the total null sequence $\{x_n\}$ and a sequence (ϵ_n) so that for all $k \geq k_0$, $x_n \subset B \subset \frac{n_{k-1}}{\epsilon_{k-1}} U_{\sigma(k)}$. Thus $\|\cdot\|_B$ is a continuous seminorm in $\mathbf{E}$. E_B is dense in $\mathbf{E}$, since B contains a total sequence. Hence we complete the proof of the lemma.

■

Now to prove the theorem, we define

$$\phi_p(n) = \max_{1 \leq t \leq n} \left\{ \max_{q \leq p-1} \frac{S_{t+1}(q, \sigma(p))}{s_t(q, \sigma(p))} \right\}.$$

Then $\phi_p(n) \nearrow$ with respect to n. Hence for all m,

$$\frac{S_{t+1}(q, \sigma(p))}{\phi_p(m)} \leq s_t(q, \sigma(p)), \forall t \leq m \tag{3.1}$$

Also choose $\psi_p(n) = 2^{n^2} 2^p$. Now by Lemma, $\exists (n_p) \nearrow$ and a compact hilbertian ball B satisfying the properties listed above. Since $n_p > \psi_{p-1}(n_{p-1}) = 2^{n_{p-1}^2} 2^{p-1}$ we get that $n_{p-1} \geq 1$ and without loss of generality we can choose $n_{p+1} \geq A_{p+1}, B_{p+1}, M_{p+1}$ and A_p, B_p are increasing sequences. This modification does not change the proof of lemma. Moreover we may choose $\epsilon_p = 1/n_p$ which satisfies $\sum_{p\geq k_0} \epsilon_p \leq 1$ in Lemma 3.1. Hence we have for all $p \geq k_0$

$$B \subset n_{p-1}^2 U_{\sigma(p)} \tag{3.2}$$

$$U_{\sigma(p)} \subset n_{p-1} B + U_0/\phi_p(n_p) \tag{3.3}$$

Then by $\Omega(\mathcal{P})$ and (3.3), we have for all $n \geq B_p$

$$\begin{aligned}
U_q &\subset s_n(q,\sigma(p))U_0 + S_n(q,\sigma(p))U_{\sigma(p)} \\
&\subset s_n(q,\sigma(p))U_0 + S_n(q,\sigma(p))n_{p-1}B + \frac{S_n(q,\sigma(p))}{\phi_p(n_p)}U_0 \\
&\subset \left[s_n(q,\sigma(p)) + \frac{S_n(q,\sigma(p))}{\phi_p(n_p)}\right]U_0 + S_n(q,\sigma(p))n_{p-1}B
\end{aligned}$$

For $x \in U_q$, we may find sequences (x_n) and (y_n) so that $x = x_n + y_n$ for all $n \geq B_p$ with

- $x_n \in \left[s_n(q,\sigma(p)) + \frac{S_n(q,\sigma(p))}{\phi_p(n_p)}\right]U_0$,
- $y_n \in S_n(q,\sigma(p))n_{p-1}B$.

Hence,

- $y_n - y_{n+1} \in 2S_{n+1}(q,\sigma(p))n_{p-1}B$ because $S_n(q,\sigma(p))$ is increasing with respect to n;
- $x_n - x_{n+1} \in 2\left[s_n(q,\sigma(p)) + \frac{S_{n+1}(q,\sigma(p))}{\phi_p(n_p)}\right]U_0$ because $s_n(q,\sigma(p))$ is decreasing with respect to n.

As B is a hilbertian ball and $\mathbf{E}_B \hookrightarrow \mathbf{E} \hookrightarrow \mathbf{E}_0$ is nuclear and so compact, we can find a common orthonormal basis in $\mathbf{E}_B$ and $\mathbf{E}_0$. Then consider the imbedding

P_m, $m \in \mathbb{N}$ as in interpolation lemma. So by the Banach-Steinhaus Theorem it suffices to show the continuity of these imbeddings on $\mathbf{E}$. Clearly $\|P_m\|_0 < C_0$ and $\|P_m\|_B < C_B$ where $C_0 \leq 1$ and $C_B \leq 1$ because P_m is a projection with the common orthonormal basis. Define $C = \max C_0, C_B$.

By the property $DN(\mathcal{Q})$, $\forall t > A_p, \forall n > B_p$;

$$\begin{aligned} |P_m y_n - P_m y_{n+1}|_k &\leq R_t(k,\sigma(p))\,|P_m y_n - P_m y_{n+1}|_0 \\ &\quad + r_t(k,\sigma(p))\,|P_m y_n - P_m y_{n+1}|_{\sigma(p)} \end{aligned}$$

By applying (3.2),

$$\begin{aligned} |P_m y_n - P_m y_{n+1}|_k &\leq R_t(k,\sigma(p))\,|P_m y_n - P_m y_{n+1}|_0 \\ &\quad + r_t(k,\sigma(p)) n_{p-1}^2\,|P_m y_n - P_m y_{n+1}|_B \\ &\leq R_t(k,\sigma(p)) C\,\|y_n - y_{n+1}\|_0 \\ &\quad + r_t(k,\sigma(p)) n_{p-1}^2 C\,\|y_n - y_{n+1}\|_B \\ &\leq 2CR_t(k,\sigma(p))\left[s_n(q,\sigma(p)) + \frac{S_{n+1}(q,\sigma(p))}{\phi_p(n_p)}\right] \\ &\quad + 2Cr_t(k,\sigma(p)) n_{p-1}^3 S_{n+1}(q,\sigma(p)). \end{aligned}$$

Let $i = 0, \cdots, n_p - n_{p-1}$. In particular, we can take n and t to be $n_p - i$. Since $n_p \geq A_{p+1}, B_{p+1}$, we get ,

$$\begin{aligned} \left|P_m y_{n_p-i} - P_m y_{n_p-i+1}\right|_k &\leq 2CR_{n_p-i}(k,\sigma(p))\left[s_{n_p-i}(q,\sigma(p)) + \frac{S_{n_p-i+1}(q,\sigma(p))}{\phi_p(n_p)}\right] \\ &\quad + 2Cr_{n_p-i}(k,\sigma(p)) n_{p-1}^3 S_{n_p-i+1}(q,\sigma(p)). \end{aligned}$$

By (3.1), we have

$$s_{n_p-i}(q,\sigma(p))+\frac{S_{n_p-i+1}(q,\sigma(p))}{\phi_p(n_p)} \leq s_{n_p-i}(q,\sigma(p))+s_{n_p-i}(q,\sigma(p)) \leq 2s_{n_p-i}(q,\sigma(p)).$$

Then,

$$\begin{aligned}\left|P_m y_{n_p-i} - P_m y_{n_p-i+1}\right|_k &\leq 4CR_{n_p-i}(k,\sigma(p))s_{n_p-i}(q,\sigma(p)) \\ &\quad +2Cr_{n_p-i}(k,\sigma(p))S_{n_p-i+1}(q,\sigma(p))(n_{p-1}-i)^3\end{aligned}$$

Thus,

$$\begin{aligned}\left|P_m y_{n_{p-1}} - P_m y_{n_p}\right|_k &\leq \sum_{i=0}^{n_p-n_{p-1}-1} \left|P_m y_{n_p-i} - P_m y_{n_p-i+1}\right|_k \\ &\leq 4C \sum_{i=0}^{n_p-n_{p-1}-1} \left\{R_{n_p-i}(k,\sigma(p))s_{n_p-i}(q,\sigma(p))\right. \\ &\quad \left. +r_{n_p-i}(k,\sigma(p))S_{n_p-i+1}(q,\sigma(p))(n_{p-1}-i)^3\right\}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\sum_{p=k_0}^{\infty} \left|P_m y_{n_{p-1}} - P_m y_{n_p}\right|_k &\leq 4C \sum_{p=k_0}^{\infty} \sum\nolimits_{i=0}^{n_p-n_{p-1}-1} \left\{R_{n_p-i}(k,\sigma(p))s_{n_p-i}(q,\sigma(p))\right. \\ &\quad \left. +r_{n_p-i}(k,\sigma(p))S_{n_p-i+1}(q,\sigma(p))(n_p-i)^3\right\} \\ &\leq 4C\sum\nolimits_{m=n_{k_0-1}-1}^{\infty} \left\{R_m(k,\sigma(p))s_m(q,\sigma(p))\right. \\ &\quad \left. +r_m(k,\sigma(p))S_{m+1}(q,\sigma(p))m^3\right\}.\end{aligned}$$

To prove the sum is finite, we must show that $\sigma(p) \in \mathcal{I}_m$.

By the construction of n_p,

$$m \geq n_{p-1} \geq \max(A_p, B_p) \geq \max(A_{p-1}, B_{p-1}) \geq \max(A_{\sigma(p)}, B_{\sigma(p)})$$

Therefore $\sigma(p) \in \mathcal{I}_m = \{t : m \geq \max(A_t, B_t)\}$.

Hence $\sum_{p=k_0}^{\infty} \left|P_m y_{n_{p-1}} - P_m y_{n_p}\right|_k < \infty$. We can choose $x_{n_{k_0}} = x$ and $y_{n_{k_0}} = 0$. Since $y_{n_p} = \sum_{l=k_0}^{p-1} y_{n_{l+1}} - y_{n_l}$,

$$\begin{aligned} \left|P_m y_{n_p}\right|_k &\leq \sum_{l=k_0}^{p-1} \left|P_m y_{n_l} - P_m y_{n_{l+1}}\right|_k \\ &\leq \sum_{l=k_0}^{\infty} \left|P_m y_{n_l} - P_m y_{n_{l+1}}\right|_k < 4C < \infty, \forall p. \end{aligned}$$

Therefore, if we can show that $\left\|P_m x_{n_p}\right\|_k \to 0$ as $p \to \infty$, we get $\|P_m x\|_k < C$ for $x \in U_q$. Now again by the property $DN(\mathcal{Q})$, for all $t > A_p$,

$$\begin{aligned} \left|P_m x_{n_p}\right|_k &\leq R_t(k, \sigma(p)) \left|P_m x_{n_p}\right|_0 + r_t(k, \sigma(p)) \left|P_m x_{n_p}\right|_{\sigma(p)} \\ &\leq R_t(k, \sigma(p)) \left|P_m x_{n_p}\right|_0 + r_t(k, \sigma(p)) n_{p-1}^2 \left|P_m x_{n_p}\right|_B \\ &\leq C R_t(k, \sigma(p)) \left\|x_{n_p}\right\|_0 + C r_t(k, \sigma(p)) n_{p-1}^2 \left\|x_{n_p}\right\|_B. \end{aligned}$$

Recall that $x_{n_p} \in \left[s_{n_p}(q, \sigma(p)) + \frac{S_{n_p+1}(q,\sigma(p))}{\Phi_p(n_p)}\right] U_0$, by (3.1) $x_{n_p} \in 2 s_{n_p}(q, \sigma(p)) U_0$. Moreover, $x_{n_p} = x - y_{n_p}$ and $y_{n_p} \in S_{n_p}(q, \sigma(p)) n_{p-1} B$.

Hence, in particular choosing $t = n_p$, we get,

$$
\begin{aligned}
\left|P_m x_{n_p}\right|_k &\leq 2CR_{n_p}(k,\sigma(p))s_{n_p}(q,\sigma(p)) \\
&\quad + Cr_{n_p}(k,\sigma(p))n_{p-1}^2 \left\|x - y_{n_p}\right\|_B \\
&\leq 2CR_{n_p}(k,\sigma(p))s_{n_p}(q,\sigma(p)) + Cn_{p-1}^2 r_{n_p}(k,\sigma(p)) \left\|x\right\|_B \\
&\quad + 2Cn_{p-1}^3 r_{n_p}(k,\sigma(p))S_{n_p}(q,\sigma(p))
\end{aligned}
$$

As $p \to \infty$, $\left|P_m x_{n_p}\right|_k \to 0$.

Therefore,

$$
\begin{aligned}
\left|P_m x\right|_k &= \left|P_m x_{n_p} + P_m y_{n_p}\right|_k \\
&\leq \left|P_m x_{n_p}\right|_k + \left|P_m y_{n_p}\right|_k \\
&\leq \infty, \forall p.
\end{aligned}
$$

Thus $\left|P_m x\right|_k \leq \infty$ for $x \in U_q$.

■

CHAPTER 4

SOME REMARKS

4.1 Definition and Main Theorem

Sometimes we can work with only one fundamental system of seminorms, $\mathcal{P}$ instead of a $\sigma - matched$ system. So we can rephrase the compability condition as follows:

Definition 4.1. *Suppose a NFS* $\mathbf{E}$ *with a fundamental system of seminorms* $\mathcal{P}$ *satisfies the properties* $DN(\mathcal{P})$ *and* $\Omega(\mathcal{P})$. $\mathbf{E}$ *is said to have the property* $\mathcal{T}(\mathcal{P})$ *if the following holds;*
For all k*, there exists* q *such that*

$$\sum_n \sup_{p\in\mathcal{I}_n} \{s_n(q,p-1)R_n(k,p-1)\} < \infty$$

and

$$\sum_n \sup_{p\in\mathcal{I}_n} \left\{S_{n+1}(q,p-1)r_n(k,p-1)n^3\right\} < \infty$$

where $(r_n(k,p))_n, (R_n(k,p))_n, (s_n(q,p))_n$ *and* $(S_n(q,p))_n$ *are coefficients in the properties* $\mathcal{DN}(\mathcal{P})$ *and* $\Omega(\mathcal{P})$*, respectively, and* $\mathcal{I}_n = \{p : n \geq \max(A_p, B_p)\}$.

We can rephrase the main theorem for one fundamental system of seminorms as;

Corollary 4.1. *Let* $(\mathbf{E}, \mathcal{P})$ *be a NFS. If* $\mathbf{E}$ *satisfies the properties* $DN(\mathcal{P})$, $\Omega(\mathcal{P})$ *and* $\mathcal{T}(\mathcal{P})$ *then* $\mathbf{E}$ *has a basis.*

To prove this corollary, we can use similar steps as in the proof of main theorem in Chapter 3.

Remark 4.1. *In Chapter 3, we showed that every nuclear Köthe space satifies the properties* $\Omega(\mathcal{P})$ *and* $\mathcal{DN}(\mathcal{P})$*. If a nuclear Köthe space is regular then it satisfies the property* $\mathcal{T}(\mathcal{P})$*.*

Proof. *We have* $\forall q\ \exists p > q$ *such that* $(\frac{a_n^q}{a_n^p})_{n\in\mathbb{N}} \in \ell_1$

$$U_q \subset \sup_{n\geq N+1}\left\{\frac{a_n^0}{a_n^q}\right\} U_0 + \sup_{1\leq n\leq N}\left\{\frac{a_n^p}{a_n^q}\right\} U_p$$

and $\forall k$, $\exists\ p > k$ *with* $(\frac{a_n^k}{a_n^p})_{n\in\mathbb{N}} \in \ell_1$

$$\|x\|_k = \sup\left\{\frac{a_n^k}{a_n^0} : 1 \leq n \leq N\right\}\|x\|_0 + \sup\left\{\frac{a_n^k}{a_n^p} : n \geq N+1\right\}\|x\|_p$$

where $s_N(q,p) = \sup\left\{\frac{a_n^0}{a_n^q} : n \geq N+1\right\}$, $S_N(q,p) = \sup\left\{\frac{a_n^p}{a_n^q} : 1 \leq n \leq N\right\}$, $R_N(k,p) = \sup\left\{\frac{a_n^k}{a_n^0} : 1 \leq n \leq N\right\}$ *and* $r_N(k,p) = \sup\left\{\frac{a_n^k}{a_n^p} : n \geq N+1\right\}$.

for all k, $\exists q$ *with* $(\frac{a_n^k}{a_n^q})_{n\in\mathbb{N}} \in \ell_1$

$$\begin{aligned}
i)\sum_n &\sup_{p\in\mathcal{I}_n}\{s_n(q,p-1)R_n(k,p-1)\}\\
\leq \sum_N &\sup_{p\in\mathcal{I}_n}\left\{\sup\left\{\frac{a_n^0}{a_n^q} : n \geq N+1\right\}\sup\left\{\frac{a_n^k}{a_n^0} : 1 \leq n \leq N\right\}\right\}\\
\leq \sum_N &\sup_{p\in\mathcal{I}_n}\left\{\frac{a_{N+1}^0}{a_{N+1}^q}\frac{a_N^k}{a_N^0}\right\}\\
\leq \sum_N &\frac{a_{N+1}^k}{a_{N+1}^q} < \infty
\end{aligned}$$

$$ii)\sum_n \sup_{p\in\mathcal{I}_n}\left\{S_{n+1}(q,p-1)r_n(k,p-1)n^3\right\}$$

$$\leq \sum_N \sup_{p\in\mathcal{I}_n}\left\{\sup\left\{\frac{a_n^{p-1}}{a_n^q}:1\leq n\leq N+1\right\}\sup\left\{\frac{a_n^k}{a_n^{p-1}}:n\geq N+1\right\}N^3\right\}$$

$$\leq \sum_N \sup_{p\in\mathcal{I}_n}\left\{\frac{a_{N+1}^{p-1}}{a_{N+1}^q}\frac{a_{N+1}^k}{a_{N+1}^{p-1}}N^3\right\}$$

$$\leq \sum_n \left\{\frac{a_{N+1}^k}{a_{N+1}^q}N^3\right\}<\infty$$

Hence every regular nuclear Köthe space satisfies the inner interpolative condition.

■

4.2 Another Inner Interpolative Condition

In this section we will give another inner interpolative condition $\mathcal{T}'(\mathcal{P})$.

In the rest of this section we use the notation ;

$$\mathcal{F}=\{\beta:\mathbb{N}\to\mathbb{N}:\beta\ \ is\ \ increasing,\ \ onto\ \ and\ \ \beta(n)\leq n,\ \ \forall n\}$$

Suppose a NFS $\mathbf{E}$ with the increasing seminorm system $\mathcal{P}=\{\|\cdot\|_k\}_{k=0}^{\infty}$ satisfies the properties $DN(\mathcal{P})$ and $\Omega(\mathcal{P})$.

Definition 4.2. $\mathbf{E}$ *is said to satisfy the property* $\mathcal{T}'_\beta(\mathcal{P})$ *for function* $\beta\in\mathcal{F}$ *if the following hold;*
For all k*, there exists* q *and for all* $\kappa\in\mathcal{F}$ *with* $\kappa(n)\leq\beta(n)$ $\forall n\in\mathbb{N}$*;*

$$\sum_n s_n(k,\kappa(n))R_n(q,\kappa(n))<\infty$$

and

$$\sum_n S_{n+1}(k,\kappa(n))r_n(q,\kappa(n))n^3 < \infty$$

where the sequences $(r_n(q,p))_n$*,* $(R_n(q,p))_n$*,* $(s_n(k,l))_n$ *and* $(S_n(k,l))_n$ *are coefficients in the properties* $DN(\mathcal{P})$ *and* $\Omega(\mathcal{P})$*.*

Theorem 4.1. *Let* $(\mathbf{E},\mathcal{P})$ *be a nuclear Fréchet space. If there exists* $\beta \in \mathcal{F}$ *such that the properties* $DN(\mathcal{P})$*,* $\Omega(\mathcal{P})$ *and* $\mathcal{T}'_\beta(\mathcal{P})$ *hold then* $\mathbf{E}$ *has a basis.*

Proof. Proof is similar to the proof of the Main Theorem in Chapter 3,

Fix $\beta \in \mathcal{F}$. Let us consider the sequences $B_k, A_k^\beta = max\{n : \beta(n) = k\}$. Choose $\phi_k(n) = \max_{1\le t\le n}\left\{\max_{q\le k}\frac{S_{t+1}(q,k)}{s_t(q,k)}\right\}$ and $\psi_k(n) = 2^{n^2}2^k$ of positive real valued functions on $\mathbb{R}^+$. Then there exists a strictly increasing sequence $(n_k)_k$ satisfying

$$\begin{aligned} n_{k+1} &\geq \psi(n_k) && (4.1)\\ n_{k-1} &> A_k^\beta && (4.2)\end{aligned}$$

We choose $\epsilon_k = \frac{1}{n_k}$. Then $\sum_k \epsilon_k \leq 1$ since $\epsilon_k = \frac{1}{n_k} \leq \frac{1}{2^{n^2}2^k}$. Then by *Lemma* 3.1 there exists a compact hilbertian ball B so that $\forall\ k$.

$$\begin{aligned} U_k &\subset n_{k-1}B + U_0/\phi_k(n_k) && (4.3)\\ B &\subset n_{k-1}^2 && (4.4)\end{aligned}$$

Then by $\Omega(\mathcal{P})$ and (3.3), we have for all $n \geq B_p$

$$\begin{aligned} U_q &\subset s_n(q,p)U_0 + S_n(q,p)U_p \\ &\subset s_n(q,p)U_0 + S_n(q,p)n_{p-1}B + \frac{S_n(q,p)}{\phi_p(n_p)}U_0 \\ &\subset \left[s_n(q,p) + \frac{S_n(q,p)}{\phi_p(n_p)}\right]U_0 + S_n(q,p)n_{p-1}B \end{aligned}$$

and since B is a hilbertian ball and $\mathbf{E}_B \hookrightarrow \mathbf{E} \hookrightarrow \mathbf{E}_0$ is nuclear and so compact, we can find a common orthonormal basis in $\mathbf{E}_B$ and $\mathbf{E}_0$. Then we consider the projection P_m, $m \in \mathbb{N}$ and by the property $DN(\mathcal{Q})$, $\forall t > A_p, \forall n > B_p$;

$$\begin{aligned} \|P_m y_n - P_m y_{n+1}\|_k &\leq R_t(k,p)\,\|P_m y_n - P_m y_{n+1}\|_0 \\ &\quad + r_t(k,p)\,\|P_m y_n - P_m y_{n+1}\|_p \end{aligned}$$

Let $i = 0, \cdots, n_p - n_{p-1}$. In particular, we can take n and t to be $n_p - i$. Since $n_{p-1} \geq A_p, B_p$, we get ,

$$\begin{aligned} \left\|P_m y_{n_{p-1}} - P_m y_{n_p}\right\|_k &\leq \sum_{i=0}^{n_p-n_{p-1}-1} \left\|P_m y_{n_p-i} - P_m y_{n_p-i+1}\right\|_k \\ &\leq 4C \sum_{i=0}^{n_p-n_{p-1}-1} \big\{R_{n_p-i}(k,p)s_{n_p-i}(q,p) \\ &\quad + r_{n_p-i}(k,p)S_{n_p-i+1}(q,p)(n_{p-1}-i)^3\big\}. \end{aligned}$$

Define $\kappa : \mathbb{N} \to \mathbb{N}$ by $\kappa(n) = p$ for $n \in [n_{p-1}, n_p)$. We have,

$$\begin{aligned} \sum_{p=k_0}^{\infty} \left\|P_m y_{n_{p-1}} - P_m y_{n_p}\right\|_k &\leq 4C\sum\nolimits_{m=1}^{\infty} \{R_m(k,p)s_m(q,p) \\ &\quad + r_m(k,p)S_{m+1}(q,p)m^3\}. \end{aligned}$$

Since for $m \in [n_{p-1}, n_p),\ \kappa(m) = p$, we get

$$\begin{aligned}\sum_{p=k_0}^{\infty} \left\| P_m y_{n_{p-1}} - P_m y_{n_p} \right\|_k &\leq 4C \sum\nolimits_{m=1}^{\infty} \{ R_m(k, \kappa(m)) s_m(q, \kappa(m)) \\ &\quad + r_m(k, \kappa(m)) S_{m+1}(q, \kappa(m)) m^3 \} .\end{aligned}$$

Since $\kappa(n) \leq \beta(n)\ \ \forall n \in \mathbb{N}$ the sum is finite by the $\mathcal{T}'_\beta(\mathcal{P})$ condition. The rest of follows as in the proof of the Main Theorem 3.1

$\blacksquare$

REFERENCES

[1] A. Aytuna, *On Stein manifolds M for which $O(M)$ is isomorphic to $O(\Delta^n)$ as Fréchet spaces,* Manuscripta Math. 62 (1988), no. 3, 297–315.

[2] A. Aytuna, J. Krone, and T. Terzioglu, *Complemented infinite type power series subspaces of nuclear Fréchet spaces,* Math. Ann. 283 (1989), no. 2, 193–202.

[3] N. Bourbaki, *Elements of the History of Mathematics,* Springer-Verlag, Berlin-New York, 1994.

[4] J. Dieudonne, *History of Functional Analysis,* Elsevier Science Publishers B.V., Amsterdam, 1983.

[5] P. B. Djakov, *A criterium for the existence of bases in Nuclear Fréchet spaces,* Doga-Tr. Journal of Mathematics 17 (1993), 171–178

[6] P. Djakov, B. Mitiagin, *Modified construction of nuclear Fréchet spaces without basis,*J. Functional Analysis 23 (1976), no. 4, 415–433.

[7] E. Dubinsky, *The Structure of Nuclear Fréchet spaces,* Springer-Verlag, Berlin-New York, 1979.

[8] J. Krone, *On projections in power series spaces and the existence of bases,* Proc. Amer. Math. Soc. 105 (1989), no. 2, 350–355.

[9] J. Krone, *On Pelczynski's problem,* Advances in the theory of Frchet spaces (Istanbul, 1988), 297–304, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 287, Kluwer Acad. Publ., Dordrecht, 1989.

[10] J. Krone, *Existence of bases and the dual splitting relation for Fréchet spaces* , Studia Math. 92 (1989), no. 1, 37–48.

[11] R. Meise, D. Vogt, *Introduction to Functional Analysis,* Clarendon Press, Oxford, 1997.

[12] A. F. Monna, *Functional Analysis in Historical Perspective,* John Wiley and Sons, New York-Toronto, 1973.

[13] V. S. Mitjagin, G. M. Khenkin, *Linear problems of complex analysis,*(Russian) Uspehi Mat. Nauk 26, 1971, no. 4, (160), 93–152.

[14] A. Pietsch, *Nuclear Locally Convex Spaces,* Springer-Verlag, Berlin-New York, 1972.

[15] A. P. Robertson, W. Robertson *Topological Vector Spaces,* Cambridge at the University Press, Cambridge, 1964

[16] D. Vogt, *Power series space representations of nuclear Fréchet spaces,* Trans. Amer. Math. Soc. , Vol.319 (1990), no. 1, 191–208.

[17] D. Vogt, *Sequence space representations of spaces of test functions and distributions,* Functional analysis, holomorphy, and approximation theory (Rio de Janeiro, 1979), pp. 405–443, Lecture Notes in Pure and Appl. Math., 83, Dekker, New York, 1983.

[18] W. Rudin, *Functional Analysis,* McGraw-Hill Book Company, 1973

[19] A. Pelczyński, *Problem 37,* pp.476, Studia Math. 38 (1970)

[20] A. Ayalp Ross, *Ph.D. Thesis, A Criteria for the Existence of a Basis in Nuclear Fréchet spaces and Applications,* September, 2000