A DYNAMIC THEORY FOR LAMINATED COMPOSITES CONSISTING OF ANISOTROPIC LAYERS

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ABSTRACT

A DYNAMIC THEORY FOR LAMINATED COMPOSITES CONSISTING OF ANISOTROPIC LAYERS

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In this thesis, first a higher order dynamic theory for anisotropic thermoelastic plates is developed. Then, based on this plate theory, two dynamic models, discrete and continuum models (DM and CM), are proposed for layered composites consisting of anisotropic thermoelastic layers. Of the two models, CM is more important, which is established in the study of periodic layered composites using smoothing operations. CM has the properties: it contains inherently the interface and Floquet conditions and facilitates the analysis of the composite, in particular, when the number of laminae in the composite is large; it contains all kinds of deformation modes of the layered composite; its validity range for frequencies and wave numbers may be enlarged by increasing, respectively, the orders of the theory and interface conditions. CM is assessed by comparing its prediction with the exact for the spectra of harmonic waves propagating in various directions of a two-phase periodic layered composite, as well as, for transient dynamic response of a composite slab induced by waves propagating perpendicular to layering. A good comparison is observed in the results and it is found that the model predicts very well the periodic structure of spectra with passing and stopping bands for harmonic waves propagating perpendicular to layering. In view of the results, the physical significance of Floquet wave number is also discussed in the study.

Keywords: layered composites, continuum and discrete models, Floquet periodicity condition, Floquet wave number, spectra.

ÖZ

ANİZOTROP KATMANLARDAN OLUŞAN BİLEŞİK CİSİMLER İÇİN BİR DİNAMİK TEORİ

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Bu tezde ilk olarak anizotrop termoelastik plakalar için yüksek mertebeli bir dinamik teori geliştirilmiştir. Daha sonra, bu plaka teorisine dayanarak, anizotrop termoelastik katmanlardan oluşan tabakalı cisimler için ayrık ve sürekli model (AM ve SM) olmak üzere iki dinamik model sunulmuştur. Bu modellerden, periyodik tabakalı bileşik cisimler için düzleştirme operasyonları uygulanarak geliştirilen SM daha önemlidir. SM'nin dayandığı özellikler sayılacak olursa: arayüzey ve Floquet şartlarını doğal olarak içermekte ve özellikle bileşik cisimdeki tabaka sayısı arttıkça bileşik cismin analizini kolaylaştırmaktadır; tabakalı bileşik cismin tüm şekil değiştirme modlarını içermektedir; frekanslar ve dalga sayıları için geçerlilik alanı, sırasıyla teorinin ve arayüzey şartlarının mertebelerinin artırılmasıyla sağlanabilir. SM'nin geçerliliği, iki fazlı periyodik tabakalı bileşik cisimlerde değişik yönlerde yayılan harmonik dalgaların tafyları (spektrumları) için ve aynı zamanda bileşik bir levhada tabakalara dik yayılan dalgaların neden olduğu geçici dinamik tepki için verdiği çözümlerin kesin çözümlerle karşılaştırılması ile yapılmıştır. Sonuçlarda iyi bir uyuşma gözlenmiş ve modelin tabakalara dik yönde yayılan harmonik dalgaların geçiş ve durduruş bantlarını da içeren tayflarının periyodik yapısını çok iyi tahmin ettiği bulunmuştur. Elde edilen sonuçların ışığında, Floquet dalga sayısının fiziksel önemi de çalışma kapsamında tartışılmıştır.

Anahtar Kelimeler: tabakasal bileşik cisimler, sürekli ve ayrık modeller, Floquet periyodiklik şartı, Floquet dalga sayısı, tayf

to my wife, Latife

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x_i : Cartesian coordinates (i=1…3)

h : half thickness of a layer

τ_{ij} : stresses (i,j=1…3)

ε_{ij} : strains (i,j=1…3)

b_i : body forces (i=1…3)

ρ : mass density

u_i : displacements (i=1…3)

v_i : particle velocities (i=1…3)

C_{ijkl} : elastic coefficients in global frame (i,j,k,l=1…3)

$\hat{C}_{ijkl}$: elastic coefficients in material frame(i,j,k,l=1…3)

C_{ij} : elastic coefficients in global frame in contracted notation (i,j=1…6)

$\hat{C}_{ij}$: elastic coefficients in material frame in contracted notation (i,j=1…6)

β_{ij} : thermal coefficients related to thermal expansion coefficients α_{ij} by $\beta_{ij} = C_{ijkl}\,\alpha_{kl}$ (i,j,k,l=1…3)

θ : ($=T-T_0$) temperature deviation from reference temperature

T : absolute temperature of the current configuration

T_0 : absolute temperature of the reference configuration

q_i : heat flux (i=1…3)

r : rate of heat generation per unit volume

c_v : specific heat per unit volume at constant deformation

τ : retardation time for heat flux

t : time

k_{ij} : coefficients of heat conduction (i,j=1…3)

∂_i : $\partial / \partial x_i$

$\dot{( )}$: time derivative

$\bar{( )}$: nondimensional quantity

ϕ : distribution function

m : order of the theory

Δ : distance between midplanes of two subsequent layers

u_i^n, τ_{ij}^n, b_i^n, q_i^n, θ^n : generalized variables (n=0…m)

$S_i^{\mp}$, $R_i^{\mp}$, $Q^{\mp}$, $\Psi^{\mp}$: face variables

GV : generalized variables

FV : face variables

γ_j, $\gamma^{\pm}$: coefficients (j=0…m)

M_i^n : equations of motion in symbolic form (i=1…3; n=0…m)

C_{ij}^n : constitutive equations in symbolic form (i,j=1…3; n=0…m)

E^n : energy equation in symbolic form (n=0…m)

T_i^n : modified Fourier's law in symbolic form (i=1…3; n=0…m)

$Z_i^{\mp}$, $P^{\mp}$: constitutive equations for FV's in symbolic form (i=1…3)

ω : circular frequency

k : wave number

n	: outer unit normal
κ	: Floquet wave number (FWN)
φ	: inclination angle of a wave (propagating in x_1x_2-plane) from x_1-axis
SHAE	: system of homogeneous algebraic equations
T	: transfer matrix
I	: identity matrix
l	: left-hand eigenvector

CHAPTER 1

INTRODUCTION

Due to their importance in many fields of engineering, such as aerospace, automotive, structural engineering etc., various continuum models (CM) are proposed in literature for the analysis of periodic layered composites. CM's treat the composite material as homogeneous and simplify the analysis when the number of laminae in the composite is large, whereas the exact treatment involves writing the field equations in each lamina and considering the continuity conditions at interfaces.

Among these CM's, effective modulus theories, effective stiffness theories, mixture theories, etc. may be cited. CM's proposed in literature were assessed by using a criterion, which involves the comparison of CM prediction for dispersion spectra of various waves propagating in layered composites with the exact or with the ones obtained by other methods, such as, by finite element method. The interaction between the layers of the composite is induced by reflection and refraction of waves, giving rise to geometric dispersion. Geometric dispersion depends upon the mechanical properties, geometric arrangements and nature of the material interfaces. Validity of an approximate theory for elastodynamic modeling can be decided by comparing the approximate dispersion curves with the exact or experimental ones.

In this study, a higher order CM is developed for periodic layered composites. The formulation of the CM is based on a higher order plate theory for anisotropic thermoelastic plates which is developed as the first stage of this study. The technique employed in developing the approximate theory is a more systematic and improved version of that used in conjunction with isotropic plates and shells [1-4]. This plate theory has an important property: it contains, as field variables, not only generalized variables representing the weighted averages of displacements and stresses, but also face variables (FV) denoting displacements and tractions defined on the faces of the plate. The appearance of FV's in the plate theory played, for the reasons stated below, a crucial role in establishing a consistent model for layered composites: it permits satisfying correctly the interface conditions, thus, accommodating the dispersive properties of layered composites. This property of the approximate theory eliminates the discrepancies which may exist between the displacement (or temperature) distributions assumed over the thickness of the plate and its lateral boundary conditions; thus, it improves, without using any correction factors, the geometric dispersion characteristics of the waves propagating in a plate. This is a superior feature of the proposed theory to the existing theories in literature (see for example, Refs.[5-7]), where some correction coefficients are introduced to remedy the discrepancies which may exist between the assumed displacement distributions over the thickness of the plate and the lateral boundary conditions. Appearance of FV's, further, facilitates performing the smoothing operations in a systematic manner in the formulation of CM.

The CM model developed in this study for periodic layered composites may be viewed as a mixture theory with higher order microstructure. Its formulation involves the use of the following steps: writing the equations of the above mentioned plate theory for each lamina; expressing the interface conditions in terms of FV's of the plate theory; finally, using smoothing operations for the field variables which appear in the equations of the composite body. The originality of the proposed CM and its contribution to existing literature lie along the following lines:

1) Triclinic anisotropy (most general anisotropy with no elastic symmetry) is assumed for the layer material so that the model may be used in the analyses of variety of layered composites, such as, fiber-reinforced and particulate composites.
2) The number of laminae in the unit cell of periodic layered composite is arbitrary. The model contains two-phase composite as a special case.

3) The orders of the theory and continuity conditions in the model are kept arbitrary. The region of validity of the model may be enlarged in wave number-frequency space by increasing these orders. Prediction of the model approaches that of exact as the orders get larger.
4) The model holds for all kinds of deformation modes, such as, those induced by waves propagating perpendicular or parallel or obliquely to layering.
5) The model contains inherently the Floquet wave conditions for the waves propagating in periodic layered composites. This property of the model enables it to predict correctly the filtering behavior of layered composites for the waves normal to layering.
6) Thermal effects are included in the formulation of the model.

The CM is assessed by using the spectral criterion stated previously, which involves the comparison of model prediction with the exact for harmonic waves propagating in an infinite layered composite. The comparison is given in the study for two-phase angle-ply composite laminates and a good match with the exact is observed for the waves propagating in various directions in the composite. The number of modes accommodated by the model increases with the order of theory; its prediction in wave number direction improves with the order of continuity conditions.

It is proved in the study that the CM contains asymptotically the Floquet periodicity conditions as the order of continuity conditions get larger.

The spectral criterion is tested in the study by considering transient shear waves in a composite slab, where the prediction of the model is found to agree quite well with the exact even for lower order theory and continuity conditions.

1.1 Literature Survey

Dynamic behavior of plates

Approximate plate theories are usually derived by expanding the displacement or stress fields of 3D elasticity theory in terms of the thickness coordinate. Among these theories, the first order shear deformation theories of Mindlin [5] and Reissner [8] can be mentioned as pioneering works, where the material is treated as isotropic. Later, various higher order theories [9-12] were proposed to improve their prediction in the dynamic analysis of plates. Reddy's third order shear deformation theory [12] has been widely accepted and it is variationally consistent; however, this theory assumes that transverse shear stresses vanish at the top and bottom of the plate. For extensive review of the approximate plate theories one may refer to Refs.[13-15].

The dynamic behavior of anisotropic plates has been the subject of many researches; but, the emphases in these works were, for some special wave modes, on the investigation of dispersion characteristics of harmonic waves propagating in plates of infinite extent by using mostly the exact theory of elasticity.

Solie and Auld [16] constructed exact plate wave solutions by using the method of partial waves in cubic solids (i.e., materials with three independent elastic constants). Green and Milosavljevic [17] and Rogerson and Kossovitch [18] examined harmonic waves in a transversely isotropic plate (i.e., plate material with five independent elastic constants) propagating in the direction of transverse isotropy axis. In a similar study, Kline et al. [19] considered arbitrary propagation directions in a fiber-reinforced composite (represented by transversely isotropic plate).

In a more detailed study, Nayfeh and Chimenti [20] provided an exact formal solution of the guided waves for the most general case: triclinic plates. From that case, they moved to a special case of monoclinic symmetry (with one plane of symmetry) and determined the dispersion equation of plate waves exactly. Their solution included also higher symmetry materials. Numerical examples for different symmetry cases are presented and dispersion curves for each case are provided. A similar study was performed by Li and Thompson [21] for a monoclinic plate with some additional analytical expressions for the oscillations of lower symmetric and antisymmetric modes. Recently, a theoretical framework for the wave propagation in general anisotropic plates was developed by Shuvalov [22]. In this approach, the propagator matrix method and Stroh's formalism are utilized and real forms for general dispersion equations are derived. Shortly after, Alshits et al. [23] extended the previous study to analyze the asymptotic behavior of the dispersion branches using short-wavelength approximation.

Chimenti [24] reviewed the vast literature on theoretical and experimental work on guided waves in plates. In the books of Nayfeh [25] and Liu and Xi [26], a comprehensive coverage of the wave propagation in anisotropic medium is presented, including coupled wave motion in layered anisotropic plates, elastic waves in fluid loaded solids, piezoelectric effects, transient waves etc., together with some review of the literature on relevant subjects.

Recently appeared works of Verma and Hasebe [27] and Al-Qahtani and Datta [28] incorporate the thermal effects, in the context of generalized thermoelasticity, with the elastic wave propagation in anisotropic plates. In the former, exact methods (for transversely isotropic and triclinic plates) and a semianalytic finite element are presented, and their predictions for thermoelastic waves propagating in a transversely isotropic plate are compared. In the latter, an exact treatment is developed for thermoelastic waves in general anisotropic plates. Dispersion curves for some special cases (monoclinic, orthotropic, transversely isotropic and cubic plates) are given for coupled waves.

Dynamic behavior of layered composites

The CM's proposed in literature may be categorized as effective modulus, effective stiffness and mixture theories. In what follows these models are reviewed briefly.

The simplest of CM's is the effective modulus theory [29,30]: it is proposed for periodic layered composites with isotropic layers and replaces the composite material with homogeneous transversely isotropic or orthotropic material (i.e., material with three orthogonal planes of symmetry) with effective elastic moduli. This model has a major shortcoming: it disregards the dispersive characteristics of the layered composite induced by reflections and/or refractions of waves at layer interfaces. To remedy this drawback of effective modulus theory, various dispersive theories are developed in literature for two-phase periodic elastic or viscoelastic layered composites.

In [31] by Sun et al. a dispersion theory, namely the effective stiffness theory, is proposed for two phase periodic layered composites where the layer properties are assumed to be isotropic. The construction of the theory is based on polynomial expansion of displacements of each layer about the midplanes of the layers. An interface compatibility equation is introduced to satisfy the displacement conditions at the interfaces of adjacent layers. The strain and kinetic energies of the layers are written in terms of displacement expansions. Smoothing the resulting expansions and application of Hamilton's principle yield the displacement equations of the theory. Sve [32] considered waves propagating in an arbitrary angle with the layering through an exact treatment and compared his results with those of the effective stiffness theory. The effective stiffness theory is improved by Drumheller and Bedford [33,34] for laminated composites by including "the evaluation of displacements and stresses, particularly interface stresses". These theories are not capable of predicting the periodic and banded structure of spectra for harmonic waves propagating perpendicular to layering. To remedy this, some matching coefficients are introduced in [35], where the coefficients are determined through matching frequencies with the exact values at the ends of first Brillouin zone. However, it is obvious that this matching procedure requires information about the exact properties of spectra. It is to be noted that the prediction of an approximate theory for the waves perpendicular to layering in layered composites is a good test for its reliability.

Based on classical mixture theories [36,37] several dispersive theories are proposed in literature for layered composites, where the layer material is assumed to be isotropic (see, e.g., Refs.[38-42]). Though formwise they are simple, these theories have some shortcomings: the parameters or constants appearing in them require either the solution of some microstructure boundary value problems or matching with the exact of some spectral properties of waves propagating in layered composites; they accommodate only limited number of modes and not capable to predict the filtering property of the composite for the waves perpendicular to layering.

Higher order models are developed by Mengi et al. [43] and Delph and Hermann [44] for layered composites which accommodates, through matching in the latter, the banded and periodic structure of spectra for waves perpendicular to layering. However, the layer material in these studies are taken as isotropic, which makes the use of these models not suitable for the analyses of layered composites of practical importance, such as, fiber-reinforced and particulate composites.

Through the expansion of displacements and stresses in each lamina, higher order CM's are proposed in [45,46] for some special waves, namely, for compressive waves propagating parallel and perpendicular to layering. In these works, the layer material is assumed to be isotropic. These models are further examined in detail in the works of Hegemier [47] and Hegemier et al. [48].

A multi-scale mixture theory with microstructure is presented in [49,50] for two phase layered composites with orthotropic layer properties. This theory has rather lengthy and complicated equations and is assessed in [50] by considering only the first mode of spectra; no information is given with regard to higher modes and its ability to predict the filtering property of layered composites. In another study , Nayfeh and Gurtman [51] extended the mixture theory of Hegemier et al. [48] to study both transversely and horizontally polarized shear motions in laminated waveguides.

A different approach is used in a two part paper by Mengi [1,43] for two phase periodic thermoelastic layered composites, where the layer property is taken as isotropic. This procedure permits to account for the continuity conditions at layer interfaces; it starts by writing, for each layer, the governing equations of a single layer established in Part 1, and completes the formulation by adding the continuity conditions to these equations and using a smoothing operation. This model can capture the banded and periodic structure of spectra for waves perpendicular to layering. Later, this theory is extended to viscoelastic layered composites by Mengi and Turhan [2] and was appraised by applying it to a transient wave propagation problem in [52]. The same theory is also extended to viscoelastic cylindrical laminated composites by Mengi and Birlik [3] and in [53] it is assessed for axially symmetric elastic waves propagating in a closed circular cylindrical shell. The present thesis work may be considered as a generalization of the work in [43] to the case in which the layer material is anisotropic (with triclinic properties) and the number of laminae in unit cell of the periodic layered composite is arbitrary. This generalization is important since it permits using the theory in the analysis of layered composites of practical importance, such as, fiber-reinforced and particulate composites.

For the sake of completeness, a short literature review will be given for the exact treatment of laminated composites. That treatment involves writing the exact field equations in each layer of the composite and satisfying the continuity conditions at the interfaces. To facilitate this analysis, the matrix transfer method, which is first introduced by Thomson [54], is employed. In this method, a system of equations for the layered media is constructed from the field equations of each layer in view of appropriate interface conditions.

Liu et al. [55] used this technique for the investigation of dispersion relations of Lamb waves in anisotropic laminates. In the works of Nayfeh [25,56,57] matrix transfer method together with linear orthogonal transformations is utilized for the study of harmonic waves in layered anisotropic media.

Braga and Herrmann [58] and Ting and Chadwick [59] applied the matrix transfer method together with Stroh formalism to the analysis of Floquet waves in anisotropic periodically layered media. The former study gives a very detailed analysis of the Floquet wave characteristic equation with stop and pass bands (Brillouin zones). For the composite medium with periodic structures, such as a laminated medium or a fibrous composite, displacements and stresses under harmonic waves can be represented by periodic functions. This representation is given in [59] within the framework of the Floquet (or Bloch) theory, which is commonly being used in crystal lattice studies.

Kohn et al. [60] employed the Floquet theory to the study of the propagation of harmonic elastic waves through periodically layered composite. They developed variational principles in the form of integrals over single cell of the composite and solved the resulting variational equations by the Rayleigh-Ritz procedure. At the end, dispersion relations for some simple illustrative cases were analyzed and satisfactory results for phase velocities and stress profiles were obtained. A detailed discussion of this approach is given in [61,62]. Nemat-Nasser [63] and Minagawa and Nemat-Nasser [64] used a more general variational principle and obtained improved results.

More detailed surveys on the relevant approximate and exact methods can be found in [25,26,35,60,65,66].

1.2 Organization of the Thesis

Contents of this study are presented in seven chapters. A higher order approximate theory for anisotropic plates is developed in Chapter 2. The theory is assessed in Chapter 3 by comparing the dispersion curves of harmonic waves in a fiber-reinforced plate, as well as mode shapes at some cut-off frequencies, with those of exact. Based on this plate theory, two models, namely discrete and continuum models, are constructed in Chapter 4 for two-phase periodic layered composites. In Chapter 5, the two-phase continuum model is extended to a periodic composite with a unit cell having arbitrary number of laminae. Chapter 6 is devoted to the assessment of two-phase continuum model. Within the framework of the study, an interpretation is given for the Floquet wave number. Finally, Chapter 7 contains some conclusions.

CHAPTER 2

A DYNAMIC THEORY FOR TRICLINIC ANISOTROPIC PLATES

In this chapter, an approximate dynamic theory is developed for plates made of a triclinic thermoelastic material by employing a procedure used in [1] in conjunction of isotropic plates. This theory will be used in the subsequent chapters to construct approximate theories for layered composites. A plate which is referred to an x_i-cartesian coordinate system, where $x_1 x_3$-plane coincides with the midplane of the plate, is considered. It is assumed that the plate has a thickness of 2h. The equations governing the linear dynamic thermoelastic behavior of the plate are

equations of motion :

$$\partial_j \tau_{ji} + b_i = \rho \ddot{u}_i \tag{2.1}$$

constitutive equations:

$$\tau_{ij} = C_{ijmn} \partial_m u_n - \beta_{ij} \theta \tag{2.2}$$

energy equation:

$$-\partial_i q_i + r = c_v \dot{\theta} + T_0 \beta_{ij} \partial_i \dot{u}_j \tag{2.3}$$

modified Fourier's law:

$$\tau \dot{q}_i + q_i = -k_{ij} \partial_j \theta \tag{2.4}$$

where

- τ_{ij} : stresses
- b_i : body forces
- ρ : mass density
- u_i : displacements
- C_{ijkl} : elastic coefficients
- β_{ij} : thermal coefficients related to thermal expansion coefficients α_{ij} by $\beta_{ij} = C_{ijkl}\, \alpha_{kl}$

θ : temperature deviation from reference temperature

q_i : heat flux

r : rate of heat generation per unit volume

c_v : specific heat per unit volume at constant deformation

T_0 : absolute temperature of the reference configuration

τ : retardation time for heat flux

k_{ij} : coefficients of heat conduction

and ∂_i stands for $(\partial/\partial x_i)$ and dot denotes time derivative. In writing Eqs.(2.1-2.4), the indicial notation together with summation convention is used where the range of indices is from 1 to 3. The modified Fourier's law in Eq.(2.4) is obtained by adding the term "$\tau \dot{q}_i$" to the left hand side of regular Fourier's equation. This modification permits having a finite speed for thermal waves [67].

For the development of the approximate theory, a modified version of Galerkin's method is used, where a set of distribution functions $\{\phi_n(\bar{x}_2); n = 0, 1, 2, \ldots\}$ with $\bar{x}_2 = x_2/h$ is chosen for the description of the distribution of the field variables over the thickness of the plate. It is assumed that ϕ_n's form a complete set in the sense that a given function $f(x_2)$ in the interval $-h \le x_2 \le h$ can be represented in terms of ϕ_n's as

$$\lim_{N\to\infty} f_N(x_2) = f(x_2) \qquad \text{with} \qquad f_N(x_2) = \sum_{n=0}^{N} \alpha_n \phi_n(\bar{x}_2)$$

where α_n's are some constants and f_N defines the approximation obtained by retaining (N+1) terms in the series. Further, it is assumed that ϕ_n's are ordered such that

$$E_{N+1} \le E_N \quad ; \; N = 0, 1, \ldots$$

where E_N is the error associated with the approximation f_N defined by

$$E_N = \int_{-h}^{h} \left| f - f_N \right| dx_2$$

Finally, without loss of generality, it is assumed that ϕ_n is an even function of $\bar{x}_2$ for even n, and odd function of $\bar{x}_2$ for odd n. It goes without saying that there are no preconditions of orthogonality for the functions ϕ_n's.

To develop mth order approximate theory, the elements $\{\phi_0,\ \phi_1,\dots,\phi_m;\ \phi_{m+1},\ \phi_{m+2}\}$ of the set are retained, where the inclusion of the last two elements ϕ_{m+1}, ϕ_{m+2} is necessary for establishing constitutive equations for FV's. This explains why the terminology "modified Galerkin's method" is used for the method employed in the study.

2.1 Weighted Integrated Forms of Field Equations

To develop mth order theory, the operator

$$L_n = \frac{1}{2h}\int_{-h}^{h}(\bullet)\phi_n dx_2 \qquad (n = 0\dots m) \tag{2.5}$$

is applied to Eqs.(2.1–2.4), which gives

equations of motion :

$$\partial_\alpha \tau^n_{\alpha i} + (R^n_i - \bar{\tau}^n_{2i}) + b^n_i = \rho \ddot{u}^n_i \tag{2.6}$$

constitutive equations:

$$\tau^n_{ij} = C_{ij\alpha k}\partial_\alpha u^n_k + C_{ij2k}(S^n_k - \bar{u}^n_k) - \beta_{ij}\theta^n \tag{2.7}$$

energy equation:

$$-(\partial_\alpha q^n_\alpha + (Q^n - \bar{q}^n_2)) + r^n = c_v\dot{\theta}^n + T_0\beta_{\alpha j}\partial_\alpha u^n_j + T_0\beta_{2j}(\dot{S}^n_j - \dot{\bar{u}}^n_j) \tag{2.8}$$

modified Fourier's law:

$$\tau\dot{q}^n_i + q^n_i = -k_{i\alpha}\partial_\alpha\theta^n + k_{i2}(\Psi^n - \bar{\theta}^n) \tag{2.9}$$

where it is assumed that the range of Latin subscripts i, j, k, etc. is from 1 to 3, and Greek subscripts α, β, etc., take the values 1 and 3 only. The field variables appearing in the above equations are defined by

$$f^n = L_n(f) \quad \text{with} \quad f = (u_i, \tau_{ij}, q_i, \theta, b_i, r) \tag{2.10}$$

$$\bar{f}^n = \frac{1}{h}\sum_{j=0}^{m} c_{nj} f^j \quad \text{with} \quad \bar{f}^n = (\bar{u}^n_i, \bar{\tau}^n_{2i}, \bar{q}^n_2, \bar{\theta}^n) \tag{2.11}$$

where c_{nj} coefficients relate $\phi'_n = d\phi_n / d\bar{x}_2$ to ϕ_j by

$$\phi'_n = \sum_{j=0}^{m} c_{nj}\phi_j \tag{2.12}$$

and

$$F^n = \frac{\overset{*}{F}{}^n \phi_n(1)}{2h} \quad \text{with } F^n = (S_i^n, R_i^n, Q^n, \Psi^n)$$

$$\overset{*}{F}{}^n = \begin{cases} F^- = f^+ - f^- & \text{for even n} \\ F^+ = f^+ + f^- & \text{for odd n} \end{cases} \tag{2.13}$$

$$f^{\mp} = f\Big|_{x_2=\mp h}, \quad f = (u_i, \tau_{2i}, q_2, \theta), \quad F^{\mp} = (S_i^{\mp}, R_i^{\mp}, Q^{\mp}, \Psi^{\mp})$$

In view of Eqs.(2.10-2.13), it is observed that the equations of approximate theory, Eqs.(2.5-2.9), contain two types of variables:

$$f^n = (u_i^n, \tau_{ij}^n, q_i^n, \theta^n)$$

representing weighted averages of displacements etc., which are called generalized variables (GV) and

$$F^{\mp} = \begin{pmatrix} S_i^{\mp} = u_i^+ \mp u_i^- \\ R_i^{\mp} = \tau_{2i}^+ \mp \tau_{2i}^- \\ Q^{\mp} = q_2^+ \mp q_2^- \\ \Psi^{\mp} = \theta^+ \mp \theta^- \end{pmatrix}$$

are defined on the flat faces of the plate, which are called face variables (FV).

It is easy to see that the number of available equations in Eqs.(2.5-2.9) together with prescribed lateral boundary conditions of the plate is [13(m+1)+8]. On the other hand the number of unknown variables (u_i^n, τ_{ij}^n, q_i^n, θ^n, $S_i^{\mp}$, $R_i^{\mp}$, $Q^{\mp}$, $\Psi^{\mp}$) in Eqs.(2.5-2.9) is [13(m+1)+16]; thus, 8 equations will be needed for the completion of the approximate theory, which will come from constitutive equations of FV's.

2.2 Constitutive Equations for FV's

To establish these equations, u_i and θ are expanded in terms of ϕ_n as

$$(u_i, \theta) = \sum_{k=0}^{m+2} (a_k^i, b_k)\,\phi_k \qquad (2.14)$$

where the coefficients a_k^i and b_k are some functions of x_α (α = 1, 3) and time "t". To relate these coefficients to GV's and FV's of the approximate theory, the operator L_n (n = 0…m) and the operations $(S_i^{\mp} = u_i^{+} \mp u_i^{-}\,,\ \Psi^{\mp} = \theta^{+} \mp \theta^{-})$ are applied to Eq.(2.14) which gives

$$\begin{aligned} &\sum_{k=0,2,..}^{p+2} d_{nk}\,(a_k^i, b_k) = (u_i^n, \theta^n) \qquad (n = 0,2,\ldots,p) \\ &\sum_{k=0,2,..}^{p+2} \phi_k(1)\,(a_k^i, b_k) = \frac{1}{2}(S_i^{+}, \Psi^{+}) \end{aligned} \qquad (2.15)$$

and

$$\begin{aligned} &\sum_{k=1,3,..}^{p'+2} d_{nk}\,(a_k^i, b_k) = (u_i^n, \theta^n) \qquad (n = 1,3,\ldots,p') \\ &\sum_{k=1,3,..}^{p'+2} \phi_k(1)\,(a_k^i, b_k) = \frac{1}{2}(S_i^{-}, \Psi^{-}) \end{aligned} \qquad (2.16)$$

where $p = m,\ p' = m-1$ for even m, and $p' = m\,,\ p = m-1$ for odd m, and

$$d_{nk} = d_{kn} = L_n(\phi_k)$$

Solutions of Eqs.(2.15) and (2.16) determine a_k^i and b_k with the form

$$\begin{aligned} (a_k^i, b_k) &= \sum_{j=0,2,..}^{p} f_{kj}(u_i^j, \theta^j) + f_{k,p+2}(S_i^{+}, \Psi^{+}) \quad \text{for} \quad k = 0,2,\ldots,p+2 \\ (a_k^i, b_k) &= \sum_{j=1,3,..}^{p'} f_{kj}(u_i^j, \theta^j) + f_{k,p'+2}(S_i^{-}, \Psi^{-}) \quad \text{for} \quad k = 1,3,\ldots,p'+2 \end{aligned} \qquad (2.17)$$

where the coefficients f_{kj} (k,j = (0…(m+2)) may be evaluated when the distribution functions ϕ_n are selected.

Finally, Eq.(2.14) is inserted into the right hand sides of equations

$$\begin{aligned} \tau_{2i} &= C_{2i\alpha r}\partial_\alpha u_r + C_{2i2r}\partial_2 u_r - \beta_{2i}\theta \\ q_2 + \tau\,\dot{q}_2 &= -k_{2\alpha}\partial_\alpha\theta - k_{22}\partial_2\theta \qquad (\alpha = 1,3;\ \ i,r = 1\ldots3) \end{aligned}$$

and the resulting expressions are used in $R_i^{\mp} = \tau_{2i}^{+} \mp \tau_{2i}^{-}$; $Q^{\mp} = q_2^{+} \mp q_2^{-}$. This gives the constitutive equations of FV's as, in view of Eqs.(2.17) and the definitions ($S_i^{\mp} = u_i^{+} \mp u_i^{-}$, $\Psi^{\mp} = \theta^{+} \mp \theta^{-}$),

$$R_i^{+} = C_{2i\alpha r}\partial_\alpha S_r^{+} + \frac{2}{h}C_{2i2r}\Big(\sum_{j=1,3,..}^{p'}\gamma_j u_r^j + \gamma^{-}S_r^{-}\Big) - \beta_{2i}\Psi^{+}$$

$$R_i^{-} = C_{2i\alpha r}\partial_\alpha S_r^{-} + \frac{2}{h}C_{2i2r}\Big(\sum_{j=0,2,..}^{p}\gamma_j u_r^j + \gamma^{+}S_r^{+}\Big) - \beta_{2i}\Psi^{-}$$

$$Q^{+} + \tau\dot{Q}^{+} = -k_{2\alpha}\partial_\alpha\Psi^{+} - \frac{2}{h}k_{22}\Big(\sum_{j=1,3,..}^{p'}\gamma_j\theta^j + \gamma^{-}\Psi^{-}\Big)$$

$$Q^{-} + \tau\dot{Q}^{-} = -k_{2\alpha}\partial_\alpha\Psi^{-} - \frac{2}{h}k_{22}\Big(\sum_{j=0,2,..}^{p}\gamma_j\theta^j + \gamma^{+}\Psi^{+}\Big) \qquad (2.18)$$

where the coefficients γ_j and $\gamma^{\mp}$ may be computed from

$$\begin{aligned}
\gamma_j &= \sum_{k=0,2,..}^{p+2} f_{kj}\phi_k'(1) \quad \text{for even } j\\
\gamma_j &= \sum_{k=1,3,..}^{p'+2} f_{kj}\phi_k'(1) \quad \text{for odd } j\\
\gamma^{+} &= \sum_{k=0,2,..}^{p+2} f_{k,p+2}\phi_k'(1)\\
\gamma^{-} &= \sum_{k=1,3,..}^{p'+2} f_{k,p'+2}\phi_k'(1)
\end{aligned} \qquad (2.19)$$

2.3 Symbolic Description of the Approximate Plate Theory

To facilitate the discussions for the models which will be developed for layered composites in subsequent sections, the equations of the approximate plate theory will be written here symbolically as

$$(M_i^n,\ C_{ij}^n,\ E^n,\ T_i^n,\ Z_i^{\mp},\ P^{\mp}) = 0 \qquad (n = 0\ldots m) \qquad (2.20)$$

where the first four equations represent, respectively, equations of motion (Eq.(2.6)), constitutive equations for GV's (Eq.(2.7)), energy equation (Eq.(2.8)), modified Fourier's law (Eq.(2.9)) and the last two describe, respectively, the constitutive equations for the FV's $R_i^{\mp}$ and $Q^{\mp}$ in Eqs.(2.18). It is worth to note that the proposed theory reduces the dimension of the plate theory by "one" through the integration over its thickness; thus, both GV's and FV's would be functions of x_1 , x_3 and time "t" only.

The appearance of FV's in the approximate theory is crucial in construction of consistent dynamic models for layered composites accommodating their refractive properties at interfaces properly.

2.4 Case of Orthogonal ϕ_n's

As mentioned earlier, orthogonality is not a requirement for the constitutive relations in Eqs.(2.18). However, taking the distribution functions as orthogonal simplifies the computation of the constants appearing in these equations. For this special case, the expression $d_{nk} = L_n(\phi_k)$ takes the form

$$d_{nk} = \delta_{n\underline{k}} d_{\underline{k}} \tag{2.21}$$

with δ_{nk} being Kronecker's delta and

$$d_k = \frac{1}{2}\int_{-1}^{1} \phi_k^2 d\bar{x}_2 \tag{2.22}$$

In Eq.(2.21), the underlined index implies that there is no summation over it. In view of Eq.(2.21), the solutions of Eqs.(2.15) and (2.16) for a_k^i and b_k can be obtained as

$$\begin{aligned}
(a_k^i, b_k) &= \frac{1}{d_k}(u_i^k, \theta^k), \quad k = (0 \ldots m) \\
(a_{p+2}^i, b_{p+2}) &= \frac{\left(\dfrac{(S_i^+, \Psi^+)}{2} - \sum\limits_{k=0,2,..}^{p} \dfrac{\phi_k(1)}{d_k}(u_i^k, \theta^k) \right)}{\phi_{p+2}(1)} \\
(a_{p'+2}^i, b_{p'+2}) &= \frac{\left(\dfrac{(S_i^-, \Psi^-)}{2} - \sum\limits_{k=1,3,..}^{p'} \dfrac{\phi_k(1)}{d_k}(u_i^k, \theta^k) \right)}{\phi_{p'+2}(1)}
\end{aligned} \tag{2.23}$$

Comparison of these equations with Eqs.(2.17) gives, for f_{kj} coefficients,

$$f_{kj} = \begin{cases} \frac{1}{d_{\underline{k}}} \delta_{\underline{kj}} & \text{for } (k, j) = 1 \ldots m \\ 0 & \text{for } k = 1 \ldots m;\ j = p + 2,\ p' + 2 \\ -\frac{\phi_{\underline{j}}(1)}{d_{\underline{j}} \phi_k (1)} & \text{for } k = p + 2,\ p' + 2;\ j = 0 \ldots m \\ 1/2 & \text{for } (k, j) = p + 2,\ p' + 2 \quad \text{with } k = j \\ 0 & \text{for } (k, j) = p + 2,\ p' + 2 \quad \text{with } k \neq j \end{cases} \tag{2.24}$$

Insertion of Eqs.(2.24) into (2.19) determines the constants γ_j and $\gamma^{\mp}$ which appear in the constitutive equations of FV's. These constants together with c_{nj} coefficients are given in Tables 2.1 and 2.2 when ϕ_n's are Legendre polynomials (which form a complete orthogonal set), i.e., when

$$\phi_n (\bar{x}_2) = P_n (\bar{x}_2)$$

where $P_n (\bar{x}_2)$ is the Legendre polynomial of order "n" defined by

$$P_n (\bar{x}_2) = \frac{1}{2^n\, n!} \frac{d^n}{d\bar{x}_2^n} (\bar{x}_2^2 - 1)^n \tag{2.25}$$

It may be noted that selection of Legendre polynomials for ϕ_n's results in $d_n = 1/(2n+1)$, $\phi_n(1) = 1$ and $\phi'_n(1) = (0, 1, 3, 6, 10)$ for $n = (0 \ldots 4)$, respectively. The dash in Table 2.2 indicates that the corresponding constant does not appear in the theory.

Table 2.1 c_{nj} coefficients (ϕ_n's are Legendre polynomials)

n \ j	0	1	2	3	4
1	1	0	0	0	0
2	0	3	0	0	0
3	1	0	5	0	0
4	0	3	0	7	0

Table 2.2 γ_j and $\gamma^{\pm}$ coefficients (ϕ_n's are Legendre polynomials)

order (m)	γ_0	γ_1	γ_2	γ_3	γ_4	γ^+	γ^-
0	-3	–	–	–	–	3/2	1/2
1	-3	-15	–	–	–	3/2	3
2	-10	-15	-35	–	–	5	3
4	-21	-42	-90	-63	-99	21/2	15/2

2.5 Reduction of the Approximate Theory to Generally Orthotropic Plates

A generally orthotropic plate will be taken here, for illustrative purposes, as a fiber-reinforced plate with a ply angle θ as shown in Fig. 2.1, where x_i axes define the global coordinate system (the body coordinates); whereas (1,2,3)-axes denote the material frame (the principal material directions, i.e., the fiber direction and the directions normal to it). Constitutive equations in x_i frame would be, in matrix form

$$\boldsymbol{\sigma} = \mathbf{C}\,\boldsymbol{\varepsilon} - \boldsymbol{\beta}\,\theta$$

or

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} - \begin{bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ \beta_{13} \\ 0 \end{bmatrix} \theta \tag{2.26}$$

where $\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ are strain components. It may be noted that the first and last indices, "i" and "j", in C_{ij} in Eq.(2.26) are related to first two and last two indices, "pq" and "mn", in C_{pqmn} in Eq.(2.2) by the following rule

$$1\to 11,\ 2\to 22,\ 3\to 33,\ 4\to 23,\ 5\to 13,\ 6\to 12$$

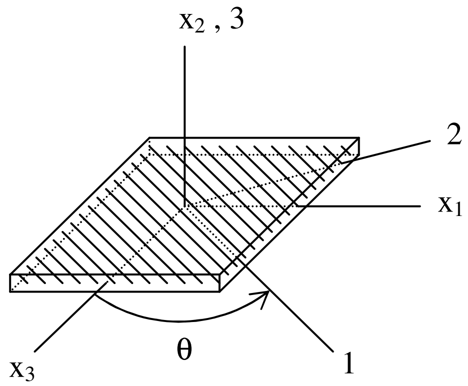


Figure 2.1 Fiber-reinforced layer

On the other hand, nonzero components of the heat conduction coefficient matrix $\mathbf{k} = (k_{ij})$ and thermal coefficient matrix $\boldsymbol{\beta} = (\beta_{ij})$ are , for the generally orthotropic plate in x_i frame, (k_{11}, k_{22}, k_{33}, k_{13}) and (β_{11}, β_{22}, β_{33}, β_{13}).

The elastic, heat conduction and thermal coefficients of generally orthotropic plate in x_i frame are related, through the ply angle θ, to the coefficients in material frame by the usual tensor transformation rules:

$$\begin{aligned} &C_{pqmn} = a_{pr} a_{qs} a_{mu} a_{nv} \hat{C}_{rsuv} \\ &k_{ij} = a_{im} a_{jn} \hat{k}_{mn} \;, \qquad \beta_{ij} = a_{im} a_{jn} \hat{\beta}_{mn} \end{aligned} \tag{2.27}$$

with

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \tag{2.28}$$

being direction cosine matrix and overhead denotes the values of coefficients in material frame. The transformed coefficients in Eq.(2.26) are given in Appendix A.

The equations of the approximate theory for generally orthotropic plates can be derived from those of triclinic anisotropic plates, Eqs.(2.6-2.9) and (2.18), by specializing them for the properties of orthotropic plates discussed above. The resulting equations in extended form are

equations of motion:

$$\partial_\alpha \tau^n_{\alpha i} + (R^n_i - \overline{\tau}^n_{2i}) + b_i = \rho\, \ddot{u}^n_i \tag{2.29}$$

constitutive equations:

$$\begin{aligned}
\tau^n_{11} &= C_{11}\partial_1 u^n_1 + C_{15}\partial_1 u^n_3 + C_{15}\partial_3 u^n_1 + C_{13}\partial_3 u^n_3 + C_{12}(S^n_2 - \overline{u}^n_2) - \beta_{11}\theta^n \\
\tau^n_{22} &= C_{12}\partial_1 u^n_1 + C_{25}\partial_1 u^n_3 + C_{25}\partial_3 u^n_1 + C_{23}\partial_3 u^n_3 + C_{22}(S^n_2 - \overline{u}^n_2) - \beta_{22}\theta^n \\
\tau^n_{33} &= C_{13}\partial_1 u^n_1 + C_{35}\partial_1 u^n_3 + C_{35}\partial_3 u^n_1 + C_{33}\partial_3 u^n_3 + C_{32}(S^n_2 - \overline{u}^n_2) - \beta_{33}\theta^n \\
\tau^n_{23} &= C_{46}\partial_1 u^n_2 + C_{44}\partial_3 u^n_2 + C_{46}(S^n_1 - \overline{u}^n_1) + C_{44}(S^n_3 - \overline{u}^n_3) \\
\tau^n_{13} &= C_{15}\partial_1 u^n_1 + C_{55}\partial_1 u^n_3 + C_{55}\partial_3 u^n_1 + C_{53}\partial_3 u^n_3 + C_{25}(S^n_2 - \overline{u}^n_2) - \beta_{13}\theta^n \\
\tau^n_{12} &= C_{66}\partial_1 u^n_2 + C_{46}\partial_3 u^n_2 + C_{66}(S^n_1 - \overline{u}^n_1) + C_{46}(S^n_3 - \overline{u}^n_3)
\end{aligned} \tag{2.30}$$

energy equation:

$$-(\partial_\alpha q^n_\alpha + (Q^n - \overline{q}^n_2)) + r^n = c_v \dot{\theta}^n + T_0(\beta_{\alpha\mu}\, \partial_\alpha u^n_\mu) + T_0\beta_{22}(\dot{S}^n_2 - \dot{\overline{u}}^n_2) \tag{2.31}$$

modified Fourier's law:

$$\begin{aligned}
\tau\, \dot{q}^n_\mu + q^n_\mu &= -k_{\mu\alpha}\partial_\alpha \theta^n \\
\tau\, \dot{q}^n_2 + q^n_2 &= k_{22}(\Psi^n - \overline{\theta}^n)
\end{aligned} \qquad (n = 0\ldots m) \tag{2.32}$$

constitutive relations for FV's:

$$R^+_1 = C_{66}\partial_1 S^+_2 + C_{46}\partial_3 S^+_2 + C_{66}\frac{2}{h}\Big(\sum_{j=1,3,..}^{p'} \gamma_j u^j_1 + \gamma^- S^-_1\Big) + C_{46}\frac{2}{h}\Big(\sum_{j=1,3,..}^{p'} \gamma_j u^j_3 + \gamma^- S^-_3\Big)$$

$$R^-_1 = C_{66}\partial_1 S^-_2 + C_{46}\partial_3 S^-_2 + C_{66}\frac{2}{h}\Big(\sum_{j=0,2,..}^{p} \gamma_j u^j_1 + \gamma^+ S^+_1\Big) + C_{46}\frac{2}{h}\Big(\sum_{j=0,2,..}^{p} \gamma_j u^j_3 + \gamma^+ S^+_3\Big)$$

$$R^+_2 = C_{21}\partial_1 S^+_1 + C_{25}\partial_1 S^+_3 + C_{25}\partial_3 S^+_1 + C_{23}\partial_3 S^+_3 + C_{22}\frac{2}{h}\Big(\sum_{j=1,3,..}^{p'} \gamma_j u^j_2 + \gamma^- S^-_2\Big) - \beta_{22}\Psi^+$$

$$R^-_2 = C_{21}\partial_1 S^-_1 + C_{25}\partial_1 S^-_3 + C_{25}\partial_3 S^-_1 + C_{23}\partial_3 S^-_3 + C_{22}\frac{2}{h}\Big(\sum_{j=0,2,..}^{p} \gamma_j u^j_2 + \gamma^+ S^+_2\Big) - \beta_{22}\Psi^-$$

$$R_3^+ = C_{46}\partial_1 S_2^+ + C_{44}\partial_3 S_2^+ + C_{46}\frac{2}{h}(\sum_{j=1,3,..}^{p'}\gamma_j u_1^j + \gamma^- S_1^-) + C_{44}\frac{2}{h}(\sum_{j=1,3,..}^{p'}\gamma_j u_3^j + \gamma^- S_3^-)$$

$$R_3^- = C_{46}\partial_1 S_2^- + C_{44}\partial_3 S_2^- + C_{46}\frac{2}{h}(\sum_{j=0,2,..}^{p}\gamma_j u_1^j + \gamma^+ S_1^+) + C_{44}\frac{2}{h}(\sum_{j=0,2,..}^{p}\gamma_j u_3^j + \gamma^+ S_3^+)$$

$$Q^+ + \tau\dot{Q}^+ = -k_{22}\frac{2}{h}(\sum_{j=1,3,..}^{p'}\gamma_j \theta^j + \gamma^- \Psi^-)$$

$$Q^- + \tau\dot{Q}^- = -k_{22}\frac{2}{h}(\sum_{j=0,2,..}^{p}\gamma_j \theta^j + \gamma^+ \Psi^+)$$

(2.33)

where $p = m$, $p' = m - 1$ for even m, and $p' = m$, $p = m - 1$ for odd m and the range of Latin subscripts i, j, k, etc. is from 1 to 3, and Greek subscripts α, μ, etc., take the values 1 and 3, only.

CHAPTER 3

ASSESSMENT OF THE APPROXIMATE PLATE THEORY

To appraise the approximate theory developed in the previous chapter, dispersion curves, as well as displacement mode shapes at some cut-off frequencies, of the approximate theory will be compared with those of exact theory. For this purpose, guided axial waves propagating in x_1-direction of the generally orthotropic plate shown in Fig. 2.1 are considered. With guided axial waves in x_1-direction, it is meant that the waves carrying the displacement disturbances in that direction with the property: the in-plane displacement components u_1 and u_3 being symmetric and transverse displacement u_2 being antisymmetric about midplane of the plate. It goes without saying that these waves are formed by wave components reflected at lateral boundaries of the plate with overall propagation direction in x_1-direction. For the waves under consideration, the nonzero elements of generalized displacements and FV's of the approximate theory would be u_α^n $(\alpha = 1,3\ ;\ n = 0,\ 2, \ldots)$; u_2^n $(n = 1,\ 3, \ldots)$; $S_\alpha^+ (\alpha = 1,3)$; S_2^- and we note that $R_i^\mp = 0$ due to free flat faces of the plate.

3.1 Dispersion Relations

3.1.1 Approximate dispersion relations

Dispersion curves predicted by the approximate theory for guided axial waves can be obtained by assuming the form

$$Ae^{i(\omega t - kx_1)} \tag{3.1}$$

for the nonzero field variables of the approximate theory and requiring that the equations possess a nontrivial solution. In Eq.(3.1) ω, k and A are, respectively, frequency, wave number (in x_1-direction) and amplitude. The resulting frequency equations for 2nd order and 4th order theory together with relevant equations, for isothermal case, are given below.

2nd order approximate theory :

For the second order theory one has m=2 (n=0,1,2) and p=m=2, p′=m−1=1. After removing the vanishing generalized displacements and applying the boundary conditions on the flat faces of the plate, the equations of the 2nd order theory reduce to

equations of motion:

$$
\begin{aligned}
&\partial_1 \tau_{11}^0 = \rho \ddot{u}_1^0 \\
&\partial_1 \tau_{11}^2 - \frac{3}{h}\tau_{21}^1 = \rho \ddot{u}_1^2 \\
&\partial_1 \tau_{12}^1 - \frac{1}{h}\tau_{22}^0 = \rho \ddot{u}_2^1 \\
&\partial_1 \tau_{13}^0 = \rho \ddot{u}_3^0 \\
&\partial_1 \tau_{13}^2 - \frac{3}{h}\tau_{23}^1 = \rho \ddot{u}_3^2
\end{aligned}
\tag{3.2}
$$

constitutive equations:

$$
\begin{aligned}
&\tau_{11}^0 = C_{11}\partial_1 u_1^0 + C_{15}\partial_1 u_3^0 + C_{12}(\frac{1}{2h}S_2^-) \\
&\tau_{11}^2 = C_{11}\partial_1 u_1^2 + C_{15}\partial_1 u_3^2 + C_{12}(\frac{1}{2h}S_2^- - \frac{3}{h}u_2^1) \\
&\tau_{12}^1 = C_{66}\partial_1 u_2^1 + C_{66}(\frac{1}{2h}S_1^+ - \frac{1}{h}u_1^0) + C_{46}(\frac{1}{2h}S_3^+ - \frac{1}{h}u_3^0) \\
&\tau_{22}^0 = C_{12}\partial_1 u_1^0 + C_{25}\partial_1 u_3^0 + C_{22}(\frac{1}{2h}S_2^-) \\
&\tau_{13}^0 = C_{15}\partial_1 u_1^0 + C_{55}\partial_1 u_3^0 + C_{25}(\frac{1}{2h}S_2^-) \\
&\tau_{13}^2 = C_{15}\partial_1 u_1^2 + C_{55}\partial_1 u_3^2 + C_{25}(\frac{1}{2h}S_2^- - \frac{3}{h}u_2^1) \\
&\tau_{23}^1 = C_{46}\partial_1 u_2^1 + C_{46}(\frac{1}{2h}S_1^+ - \frac{1}{h}u_1^0) + C_{44}(\frac{1}{2h}S_3^+ - \frac{1}{h}u_3^0)
\end{aligned}
\tag{3.3}
$$

constitutive equations of FV's:

$$
\begin{aligned}
R_1^- &= C_{66}\partial_1 S_2^- + C_{66}\frac{2}{h}(\gamma_0 u_1^0 + \gamma_2 u_1^2 + \gamma^+ S_1^+) + C_{46}\frac{2}{h}(\gamma_0 u_3^0 + \gamma_2 u_3^2 + \gamma^+ S_3^+) \\
R_2^+ &= C_{12}\partial_1 S_1^+ + C_{25}\partial_1 S_3^+ + C_{22}\frac{2}{h}(\gamma_1 u_2^1 + \gamma^- S_2^-) \\
R_3^- &= C_{46}\partial_1 S_2^- + C_{46}\frac{2}{h}(\gamma_0 u_1^0 + \gamma_2 u_1^2 + \gamma^+ S_1^+) + C_{44}\frac{2}{h}(\gamma_0 u_3^0 + \gamma_2 u_3^2 + \gamma^+ S_3^+)
\end{aligned}
\qquad (3.4)
$$

To obtain the dispersion relation, equations of motion, Eqs.(3.2), are written in terms of generalized displacements by using the constitutive equations in Eqs.(3.3). Substituting the trial solution of the form in Eq.(3.1) into the resulting equations and constitutive equations of FV's, Eqs.(3.4), a system of 8 homogeneous algebraic equations with the coefficients dependent on ω and k is obtained. Equating the determinant of the coefficient matrix to zero, which is the condition for having a nontrivial solution, the frequency equation (dispersion relation) relating ω and k is found. Each pair (ω, k) satisfying the dispersion relation gives a plane wave solution. The frequency equation is given below in determinant form:

$$
\begin{vmatrix}
\rho\omega^2 - C_{11}k^2 & 0 & 0 & \frac{C_{12}}{2h}ik & 0 & -C_{15}k^2 & 0 & 0 \\
\frac{-(C_{12}+C_{66})}{h}ik & \rho\omega^2 - C_{66}k^2 & 0 & \frac{-C_{22}}{2h^2} & \frac{C_{66}}{2h}ik & \frac{-(C_{25}+C_{46})}{h}ik & 0 & \frac{C_{46}}{2h}ik \\
\frac{3C_{66}}{h^2} & \frac{-3(C_{12}+C_{66})}{h}ik & \rho\omega^2 - C_{11}k^2 & \frac{C_{12}}{2h}ik & \frac{-3C_{66}}{2h^2} & \frac{3C_{46}}{h^2} & -C_{15}k^2 & \frac{-3C_{46}}{2h^2} \\
\frac{2C_{66}}{h}\gamma_0 & 0 & \frac{2C_{66}}{h}\gamma_2 & C_{66}ik & \frac{2C_{66}}{h}\gamma^+ & \frac{2C_{46}}{h}\gamma_0 & \frac{2C_{46}}{h}\gamma_2 & \frac{2C_{46}}{h}\gamma^+ \\
0 & \frac{2C_{22}}{h}\gamma_1 & 0 & \frac{2C_{22}}{h}\gamma^- & C_{12}ik & 0 & 0 & C_{25}ik \\
-C_{15}k^2 & 0 & 0 & \frac{C_{25}}{2h}ik & 0 & \rho\omega^2 - C_{55}k^2 & 0 & 0 \\
\frac{3C_{46}}{h^2} & \frac{-3(C_{25}+C_{46})}{h}ik & -C_{15}k^2 & \frac{C_{25}}{2h}ik & \frac{-3C_{46}}{2h^2} & \frac{3C_{44}}{h^2} & \rho\omega^2 - C_{55}k^2 & \frac{-3C_{44}}{2h^2} \\
\frac{2C_{46}}{h}\gamma_0 & 0 & \frac{2C_{46}}{h}\gamma_2 & C_{46}ik & \frac{2C_{46}}{h}\gamma^+ & \frac{2C_{44}}{h}\gamma_0 & \frac{2C_{44}}{h}\gamma_2 & \frac{2C_{44}}{h}\gamma^+
\end{vmatrix} = 0
\qquad (3.5)
$$

It should be noted here that, m^{th} order approximate theory accommodates 2(m+1) dispersion curves in the spectra.

4^{th} order approximate theory :

In a similar manner as in 2^{nd} order theory, the dispersion relations for 4^{th} order theory are obtained from the governing equations

equations of motion:

$$\partial_1 \tau_{11}^0 = \rho \ddot{u}_1^0$$

$$\partial_1 \tau_{11}^2 - \frac{3}{h}\tau_{21}^1 = \rho \ddot{u}_1^2$$

$$\partial_1 \tau_{11}^4 - \frac{3}{h}\tau_{21}^1 - \frac{7}{h}\tau_{21}^3 = \rho \ddot{u}_1^4$$

$$\partial_1 \tau_{12}^1 - \frac{1}{h}\tau_{22}^0 = \rho \ddot{u}_2^1 \tag{3.6}$$

$$\partial_1 \tau_{12}^3 - \frac{1}{h}\tau_{22}^0 - \frac{5}{h}\tau_{22}^2 = \rho \ddot{u}_2^3$$

$$\partial_1 \tau_{13}^0 = \rho \ddot{u}_3^0$$

$$\partial_1 \tau_{13}^2 - \frac{3}{h}\tau_{23}^1 = \rho \ddot{u}_3^2$$

$$\partial_1 \tau_{13}^4 - \frac{3}{h}\tau_{23}^1 - \frac{7}{h}\tau_{23}^3 = \rho \ddot{u}_3^4$$

constitutive equations:

$$\tau_{11}^0 = C_{11}\partial_1 u_1^0 + C_{15}\partial_1 u_3^0 + C_{12}(\frac{1}{2h}S_2^-)$$

$$\tau_{11}^2 = C_{11}\partial_1 u_1^2 + C_{15}\partial_1 u_3^2 + C_{12}(\frac{1}{2h}S_2^- - \frac{3}{h}u_2^1)$$

$$\tau_{11}^4 = C_{11}\partial_1 u_1^4 + C_{15}\partial_1 u_3^4 + C_{12}(\frac{1}{h}S_2^- - \frac{3}{h}u_2^1 - \frac{7}{h}u_2^3)$$

$$\tau_{12}^1 = C_{66}\partial_1 u_2^1 + C_{66}(\frac{1}{2h}S_1^+ - \frac{1}{h}u_1^0) + C_{46}(\frac{1}{2h}S_3^+ - \frac{1}{h}u_3^0)$$

$$\tau_{12}^3 = C_{66}\partial_1 u_2^3 + C_{66}(\frac{1}{2h}S_1^+ - \frac{1}{h}u_1^0 - \frac{5}{h}u_1^2) + C_{46}(\frac{1}{2h}S_3^+ - \frac{1}{h}u_3^0 - \frac{5}{h}u_3^2)$$

$$\tau_{22}^0 = C_{12}\partial_1 u_1^0 + C_{25}\partial_1 u_3^0 + C_{22}(\frac{1}{2h}S_2^-)$$

$$\tau_{22}^{2}=C_{12}\partial_1 u_1^2+C_{25}\partial_1 u_3^2+C_{22}(\frac{1}{2h}S_2^- -\frac{3}{h}u_2^1)$$

$$\tau_{13}^{0}=C_{15}\partial_1 u_1^0+C_{55}\partial_1 u_3^0+C_{25}(\frac{1}{2h}S_2^-)$$

$$\tau_{13}^{2}=C_{15}\partial_1 u_1^2+C_{55}\partial_1 u_3^2+C_{25}(\frac{1}{2h}S_2^- -\frac{3}{h}u_2^1)$$

$$\tau_{13}^{4}=C_{15}\partial_1 u_1^4+C_{55}\partial_1 u_3^4+C_{25}(\frac{1}{2h}S_2^- -\frac{3}{h}u_2^1-\frac{7}{h}u_2^3)$$

$$\tau_{23}^{1}=C_{46}\partial_1 u_2^1+C_{46}(\frac{1}{2h}S_1^+ -\frac{1}{h}u_1^0)+C_{44}(\frac{1}{2h}S_3^+ -\frac{1}{h}u_3^0)$$

$$\tau_{23}^{3}=C_{46}\partial_1 u_2^3+C_{46}(\frac{1}{2h}S_1^+ -\frac{1}{h}u_1^0-\frac{5}{h}u_1^2)+C_{44}(\frac{1}{2h}S_3^+ -\frac{1}{h}u_3^0-\frac{5}{h}u_3^2)$$

(3.7)

constitutive equations of FV's:

$$R_1^-=C_{66}\partial_1 S_2^-+C_{66}\frac{2}{h}(\gamma_0 u_1^0+\gamma_2 u_1^2+\gamma_4 u_1^4+\gamma^+S_1^+)+C_{46}\frac{2}{h}(\gamma_0 u_3^0+\gamma_2 u_3^2+\gamma_4 u_3^4+\gamma^+S_3^+)$$

$$R_2^+=C_{12}\partial_1 S_1^++C_{25}\partial_1 S_3^++C_{22}\frac{2}{h}(\gamma_1 u_2^1+\gamma_3 u_2^3+\gamma^-S_2^-)$$

$$R_3^-=C_{46}\partial_1 S_2^-+C_{46}\frac{2}{h}(\gamma_0 u_1^0+\gamma_2 u_1^2+\gamma_4 u_1^4+\gamma^+S_1^+)+C_{44}\frac{2}{h}(\gamma_0 u_3^0+\gamma_2 u_3^2+\gamma_4 u_3^4+\gamma^+S_3^+)$$

(3.8)

The frequency equation for this order is given in Appendix B.

3.1.2 Exact dispersion relations

Here, the dispersion relations will be obtained through the use of exact elasticity equations for the axial waves propagating in x_1-direction of the plate in Fig. 2.1. Lateral surfaces of the plate is free of forces (i.e., $\tau_{2i}\big|_{x_2=\mp h}=0$). The displacements u_1 and u_3 would be symmetric and u_2 would be antisymmetric about midplane of the plate, and the displacements are assumed to be independent of x_3, so that $u_i = u_i\ (x_1\ , x_2, t)$ and $\partial_3(\cdot)=0$. Then, the equations of the elasticity theory yield:

equations of motion:

$$\begin{aligned}\partial_1\tau_{11}+\partial_2\tau_{21}&=\rho\ddot{u}_1\\ \partial_1\tau_{12}+\partial_2\tau_{22}&=\rho\ddot{u}_2\\ \partial_1\tau_{13}+\partial_2\tau_{23}&=\rho\ddot{u}_3\end{aligned} \tag{3.9}$$

constitutive equations:

$$\begin{aligned}\tau_{11}&=C_{11}\partial_1u_1+C_{12}\partial_2u_2+C_{15}\partial_1u_3\\ \tau_{22}&=C_{12}\partial_1u_1+C_{22}\partial_2u_2+C_{25}\partial_1u_3\\ \tau_{33}&=C_{13}\partial_1u_1+C_{23}\partial_2u_2+C_{35}\partial_1u_3\\ \tau_{23}&=C_{44}\partial_2u_3+C_{46}(\partial_1u_2+\partial_2u_1)\\ \tau_{13}&=C_{15}\partial_1u_1+C_{25}\partial_2u_2+C_{55}\partial_1u_3\\ \tau_{12}&=C_{46}\partial_2u_3+C_{66}(\partial_1u_2+\partial_2u_1)\end{aligned} \tag{3.10}$$

For harmonic waves propagating in x_1-direction, one has for displacements: $u_i=\tilde{u}_i e^{i(kx_1-\omega t)}$, where $\tilde{u}_i=\tilde{u}_i(x_2)$ are displacement amplitudes. In accordance with the assumed displacement shapes, one can write for $\tilde{u}_i$,

$$\begin{aligned}\tilde{u}_1&=A_1\cosh\lambda x_2\\ \tilde{u}_2&=A_2\sinh\lambda x_2\\ \tilde{u}_3&=A_3\cosh\lambda x_2\end{aligned} \tag{3.11}$$

where λ is the wave number in x_2-direction and A_i are some constants. Substitution of this form of displacements to the equations of motion yields the eigenvalue problem:

$$\begin{bmatrix} C_{66}\lambda^2+(\rho\omega^2-C_{11}k^2) & ik\lambda(C_{12}+C_{66}) & C_{46}\lambda^2-C_{15}k^2 \\ ik\lambda(C_{12}+C_{66}) & C_{22}\lambda^2+(\rho\omega^2-C_{66}k^2) & ik\lambda(C_{12}+C_{66}) \\ C_{46}\lambda^2-C_{15}k^2 & ik\lambda(C_{25}+C_{46}) & C_{44}\lambda^2+(\rho\omega^2-C_{55}k^2) \end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix}=\mathbf{0} \tag{3.12}$$

or $\mathbf{D}\mathbf{A}=\mathbf{0}$. To have a nontrivial solution of this set of equations, determinant of the coefficients must vanish (i.e., det $\mathbf{D}$=0). This determines the eigenvalues λ_i and the corresponding eigenvectors $\mathbf{A}^i$ in terms of ω and k (i=1…3).

The solution for $\tilde{\mathbf{u}}$ can be obtained by using superposition principle as $\tilde{\mathbf{u}} = \sum_{r=1}^{3} B_r \boldsymbol{\phi}^r$, where B_r are arbitrary constants and $\boldsymbol{\phi}^r$ are shape functions of the form (in view of Eq.(3.11)):

$$\begin{bmatrix} \phi_1^r \\ \phi_2^r \\ \phi_3^r \end{bmatrix} = \begin{bmatrix} A_1^r \cosh \lambda_r x_2 \\ A_2^r \sinh \lambda_r x_2 \\ A_3^r \cosh \lambda_r x_2 \end{bmatrix} \qquad (3.13)$$

Thus, $\tilde{\mathbf{u}}$ can be written explicitly as

$$\begin{aligned} \tilde{u}_1 &= B_1 A_1^1 \cosh \lambda_1 x_2 + B_2 A_1^2 \cosh \lambda_2 x_2 + B_3 A_1^3 \cosh \lambda_3 x_2 \\ \tilde{u}_2 &= B_1 A_2^1 \sinh \lambda_1 x_2 + B_2 A_2^2 \sinh \lambda_2 x_2 + B_3 A_2^3 \sinh \lambda_3 x_2 \\ \tilde{u}_3 &= B_1 A_3^1 \cosh \lambda_1 x_2 + B_2 A_3^2 \cosh \lambda_2 x_2 + B_3 A_3^3 \cosh \lambda_3 x_2 \end{aligned} \qquad (3.14)$$

On the other hand, the substitution of $u_i = \tilde{u}_i(x_2)\, e^{i(kx_1 - \omega t)}$ into the constitutive equations in Eqs.(3.10) gives for τ_{2i}: $\tau_{2i} = \tilde{\tau}_{2i} e^{i(kx_1-\omega t)}$, where $\tilde{\tau}_{2i}$ are stress amplitudes defined by

$$\begin{aligned} \tilde{\tau}_{21} &= C_{46} \partial_2 \tilde{u}_3 + C_{66}(ik\tilde{u}_2 + \partial_2 \tilde{u}_1) \\ \tilde{\tau}_{22} &= C_{12} ik\tilde{u}_1 + C_{22} \partial_2 \tilde{u}_2 + C_{25} ik\tilde{u}_3 \\ \tilde{\tau}_{23} &= C_{44} \partial_2 \tilde{u}_3 + C_{46}(ik\tilde{u}_2 + \partial_2 \tilde{u}_1) \end{aligned} \qquad (3.15)$$

with $\tilde{u}_i$ being given in Eqs.(3.14).

The free lateral boundary conditions of the plate imply that $\tilde{\tau}_{2i}\big|_{x_2 = \mp h} = 0$. Since $\tilde{\tau}_{21}$ and $\tilde{\tau}_{23}$ are antisymmetric and $\tilde{\tau}_{22}$ is symmetric about the midplane of the plate, these boundary conditions are needed to be satisfied only at $x_2 = +h$, i.e., $\tilde{\tau}_{2i}\big|_{x_2=+h} = 0$, which yields, in view of Eqs.(3.15) and (3.14),

$$
\left[\begin{array}{c:c:c}
\left\{\begin{matrix} C_{66}A_1^1\lambda_1 + \\ C_{66}ikA_2^1 + \\ C_{46}A_3^1\lambda_1 \end{matrix}\right\}\sinh\lambda_1 h &
\left\{\begin{matrix} C_{66}A_1^2\lambda_2 + \\ C_{66}ikA_2^2 + \\ C_{46}A_3^2\lambda_2 \end{matrix}\right\}\sinh\lambda_2 h &
\left\{\begin{matrix} C_{66}A_1^3\lambda_3 + \\ C_{66}ikA_2^3 + \\ C_{46}A_3^3\lambda_1 \end{matrix}\right\}\sinh\lambda_3 h \\
\hdashline
\left\{\begin{matrix} C_{12}ikA_1^1 + \\ C_{22}A_2^1\lambda_1 + \\ C_{25}ikA_3^1 \end{matrix}\right\}\cosh\lambda_1 h &
\left\{\begin{matrix} C_{12}ikA_1^2 + \\ C_{22}A_2^2\lambda_2 + \\ C_{25}ikA_3^2 \end{matrix}\right\}\cosh\lambda_2 h &
\left\{\begin{matrix} C_{12}ikA_1^3 + \\ C_{22}A_2^3\lambda_3 + \\ C_{25}ikA_3^3 \end{matrix}\right\}\cosh\lambda_3 h \\
\hdashline
\left\{\begin{matrix} C_{46}A_1^1\lambda_1 + \\ C_{46}ikA_2^1 + \\ C_{44}A_3^1\lambda_1 \end{matrix}\right\}\sinh\lambda_1 h &
\left\{\begin{matrix} C_{46}A_1^2\lambda_2 + \\ C_{46}ikA_2^2 + \\ C_{44}A_3^2\lambda_2 \end{matrix}\right\}\sinh\lambda_2 h &
\left\{\begin{matrix} C_{46}A_1^3\lambda_3 + \\ C_{46}ikA_2^3 + \\ C_{44}A_3^3\lambda_3 \end{matrix}\right\}\sinh\lambda_3 h
\end{array}\right]
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \mathbf{0}
\tag{3.16}
$$

or $\mathbf{EB} = \mathbf{0}$. For the system of equations given above, to have a nontrivial solution, determinant of $\mathbf{E}$ must vanish, which gives the exact spectrum. The resulting dispersion equation is transcendental and can not be solved analytically; therefore, a numerical search and find algorithm is used in the determination of exact dispersion curves.

3.2 Dispersion Curves

Approximate and exact dispersion curves are compared in Figs. 3.1-3.8. The dispersion curves in the figures are obtained for graphite fabric-carbon matrix composite with the properties [15]:

$$
\begin{aligned}
&E_1 = 173.058,\ \ E_2 = 33.095,\ \ E_3 = 5.171 \\
&G_{12} = 9.377,\ \ G_{13} = 8.274,\ \ G_{23} = 3.241 \\
&\nu_{12} = 0.036,\ \nu_{13} = 0.25,\ \nu_{23} = 0.171 \qquad \text{(moduli are in GPa)}
\end{aligned}
\tag{3.17}
$$

The expressions relating $\hat{C}_{ij}$ to the elastic properties given in Eq.(3.17) are, for the orthotropic representation of the fiber-reinforced composite,

$$\hat{C}_{11} = E_1 \frac{1-\nu_{23}\nu_{32}}{\Delta}\ , \ \hat{C}_{12} = E_1 \frac{\nu_{21}+\nu_{31}\nu_{23}}{\Delta}\ , \ \hat{C}_{13} = E_1 \frac{\nu_{31}+\nu_{21}\nu_{32}}{\Delta}$$

$$\hat{C}_{22} = E_2 \frac{1-\nu_{13}\nu_{31}}{\Delta}\ , \ \hat{C}_{13} = E_2 \frac{\nu_{32}+\nu_{12}\nu_{31}}{\Delta}\ , \ \hat{C}_{33} = E_3 \frac{1-\nu_{12}\nu_{21}}{\Delta} \qquad (3.18)$$

$$\hat{C}_{44} = G_{23}\ , \quad \hat{C}_{55} = G_{13}\ , \quad \hat{C}_{66} = G_{12}$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}$$

subject to the following reciprocal relation: $\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$ [15].

The nondimensional frequency $\overline{\omega}$ and wave number $\overline{k}$ which appear in the figures are defined by

$$\overline{\omega} = \frac{2h}{\pi c_s}\omega \ \text{ with } \ c_s = \sqrt{\frac{G_{13}}{\rho}}\ , \qquad \overline{k} = \frac{2h}{\pi}k$$

Figs.(3.1,3.2) and (3.3,3.4) give, respectively, the comparison for dispersion curves when θ = 0° and 90°, that is, when the waves propagate parallel and perpendicular to fiber direction. For these directions, the wave motion in x_1x_2-plane and in out-of-plane (x_3) direction would be uncoupled; this is the reason why the spectra for these uncoupled wave motions are given separately. Figs. (3.5,3.6) and (3.7,3.8) contain, respectively, the spectra for inclined guided axial waves when θ = 30° and 60°, where it is to be noted that the wave motion in x_1x_2-plane and in out-of-plane (x_3) direction would be coupled. From the figures, it may be observed that exact and approximate dispersion curves compare very well, and that as the order of the theory increases, the comparison improves and the number of dispersion curves, that is, the frequency range accounted for by the approximate theory also increases.

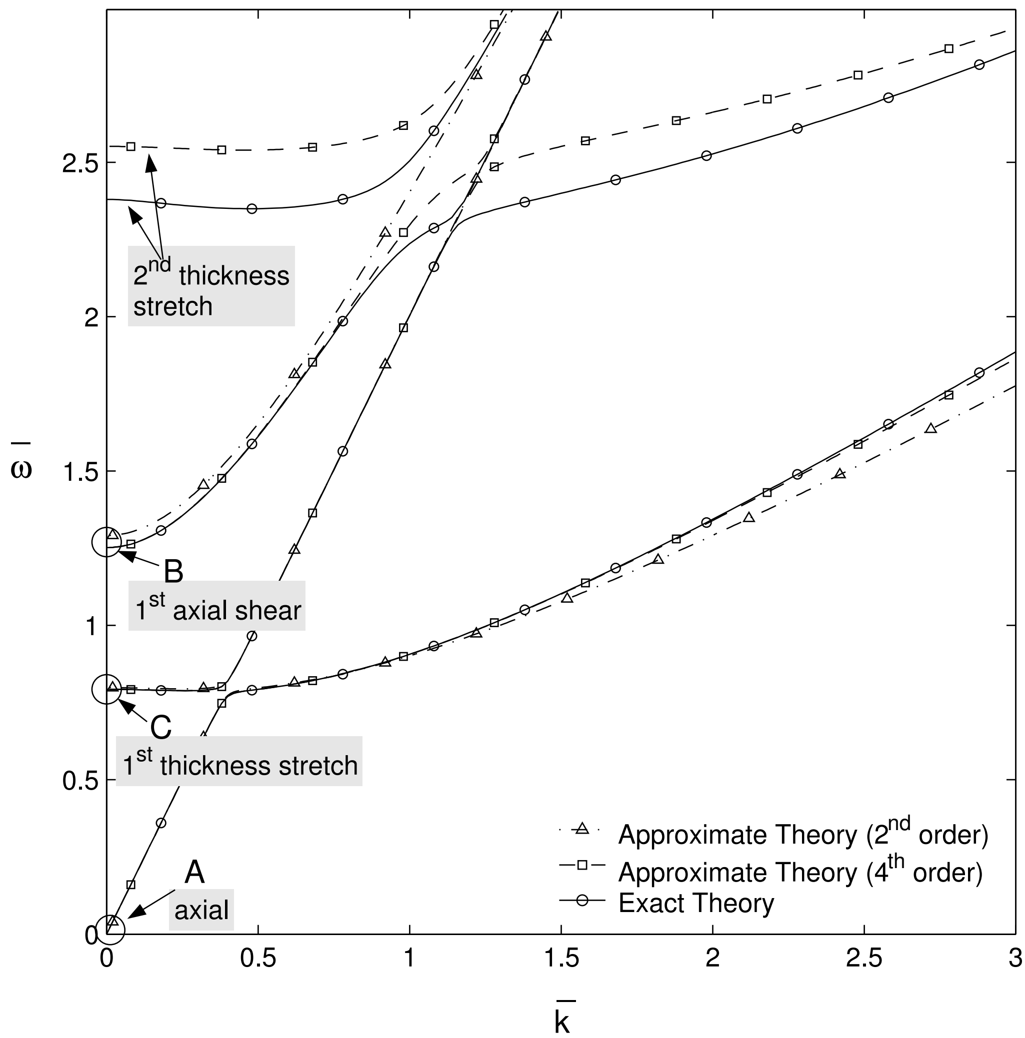


Figure 3.1 Comparison of dispersion curves for guided axial waves in x_1-direction when $\theta=0°$ (wave motion in x_1x_2-plane)

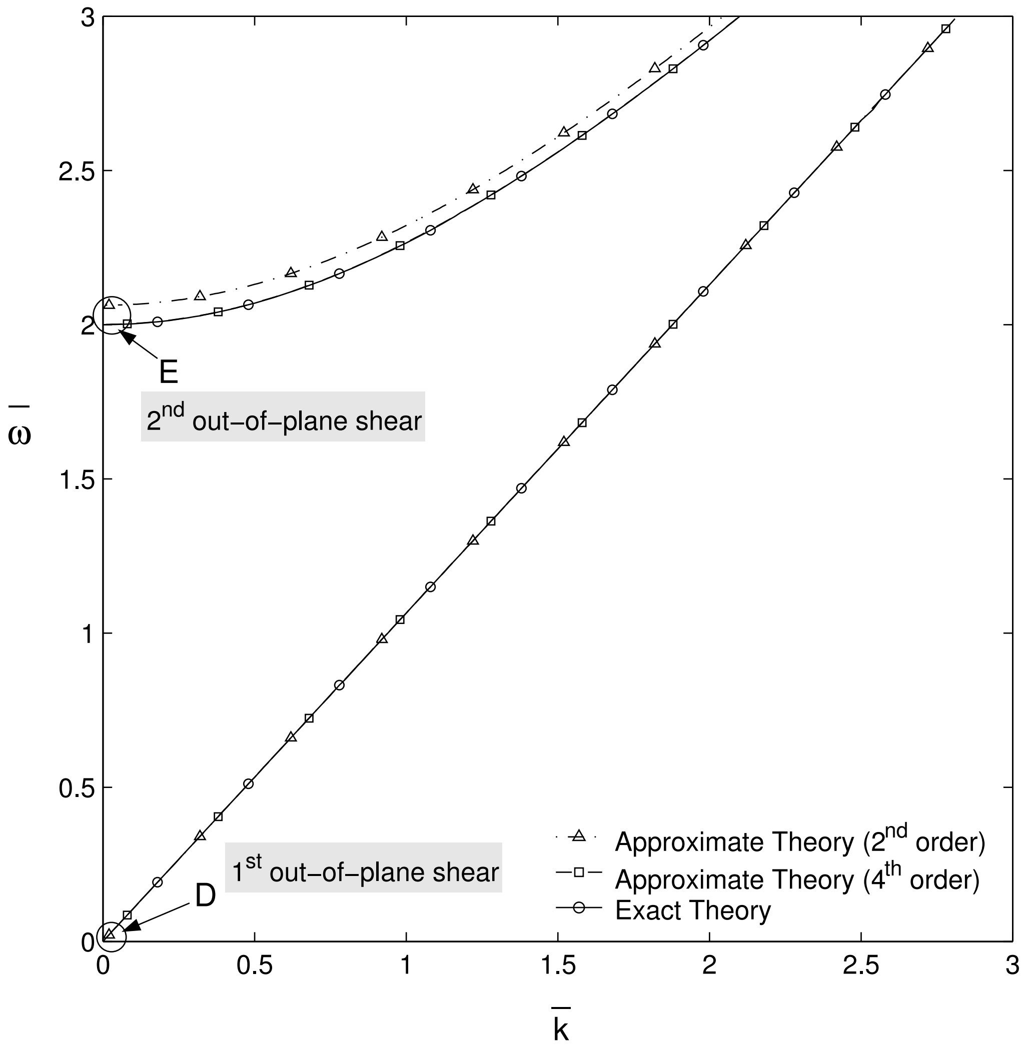


Figure 3.2 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=0° (wave motion in out-of-plane (x_3) direction)

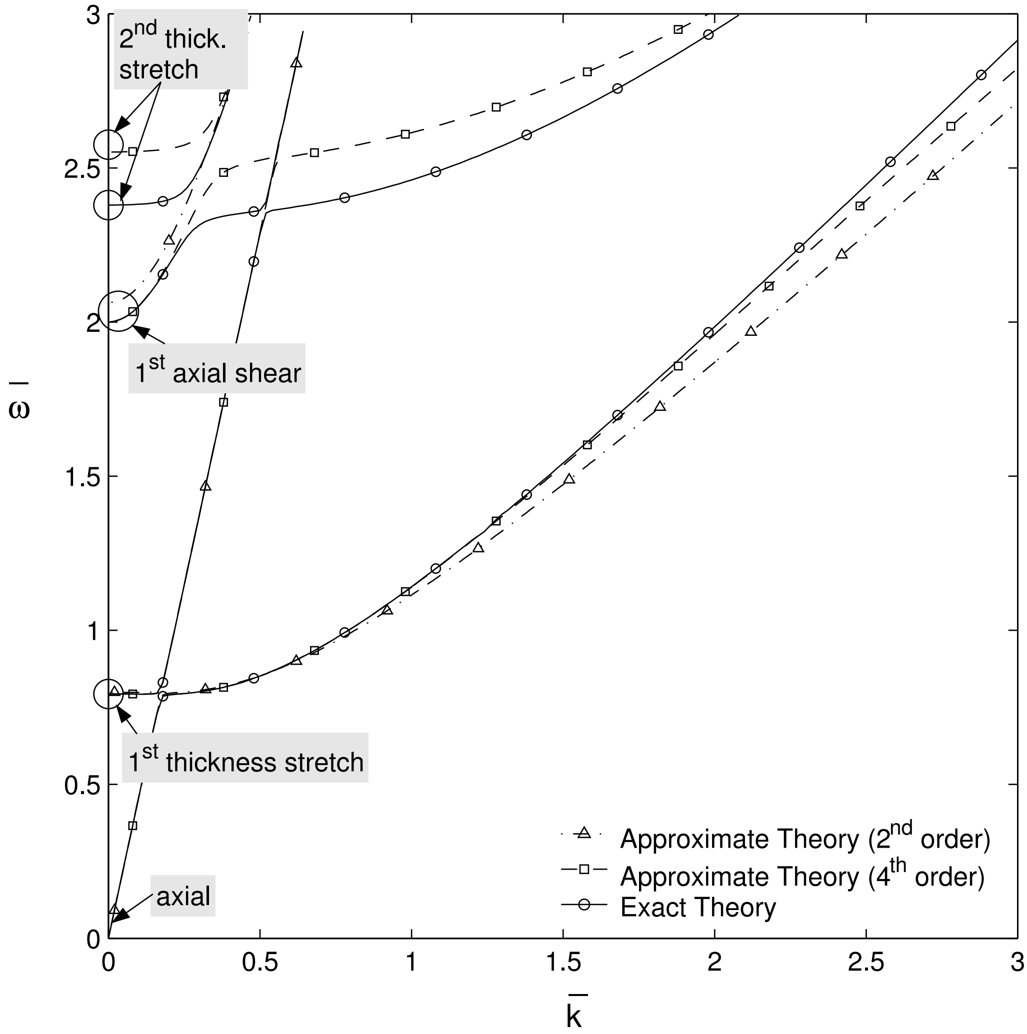


Figure 3.3 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=90° (wave motion in x_1x_2-plane)

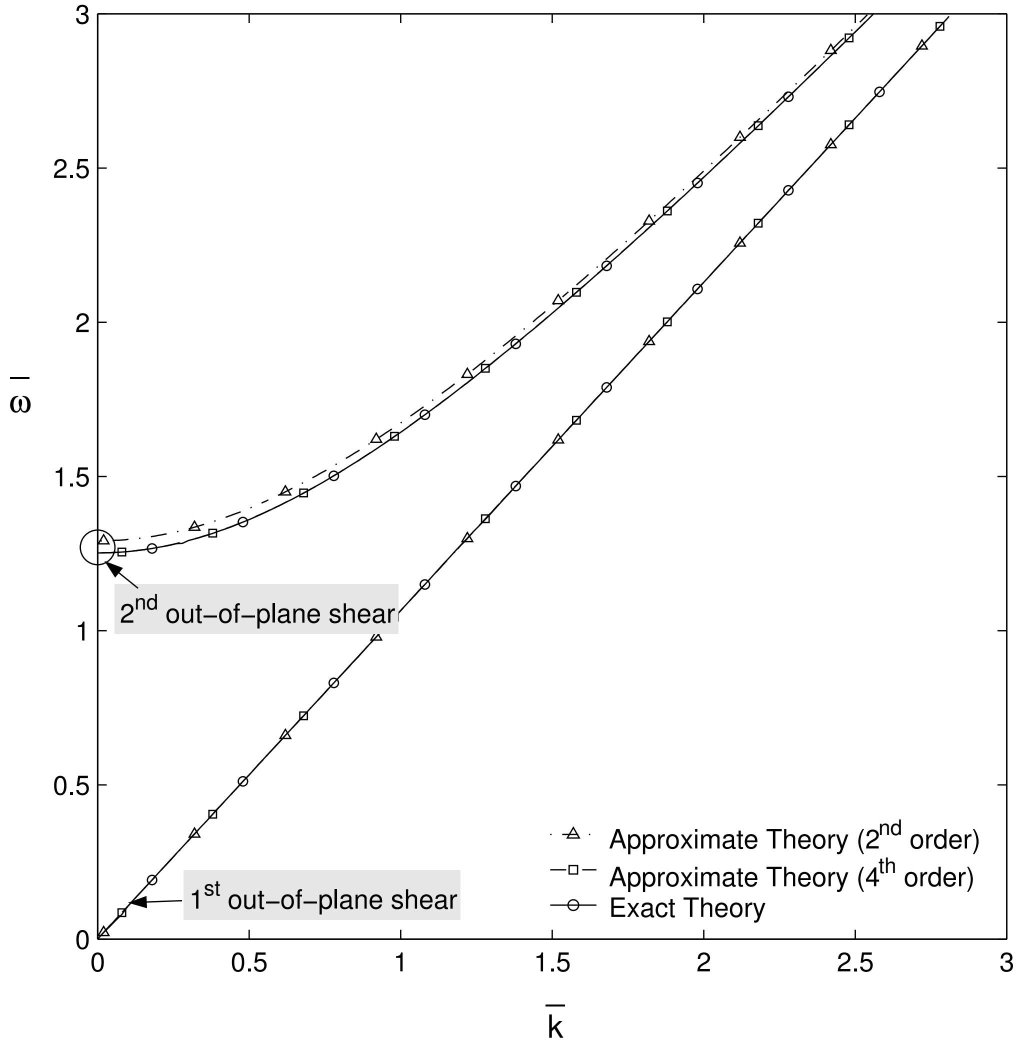


Figure 3.4 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=90° (wave motion in out-of-plane (x_3) direction)

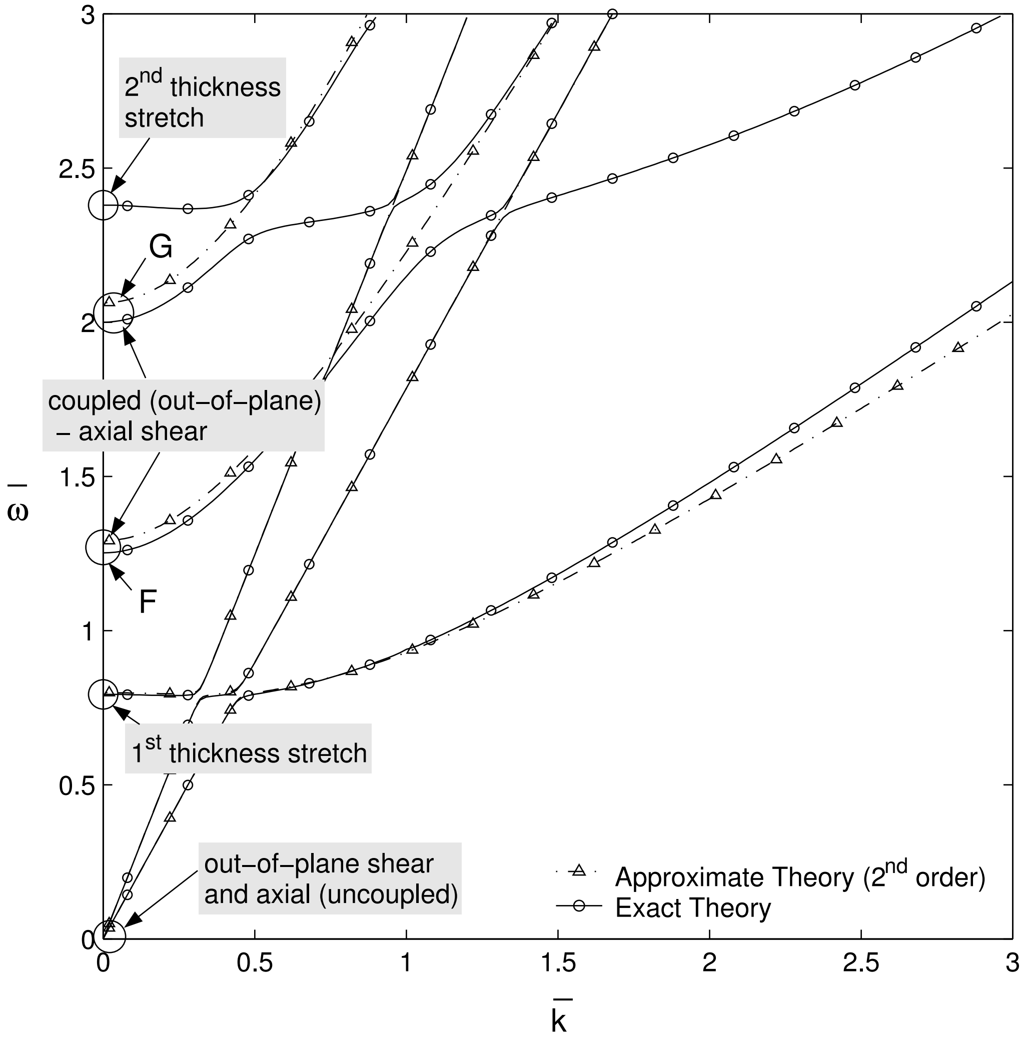


Figure 3.5 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=30° (the order of the approximate theory = 2)

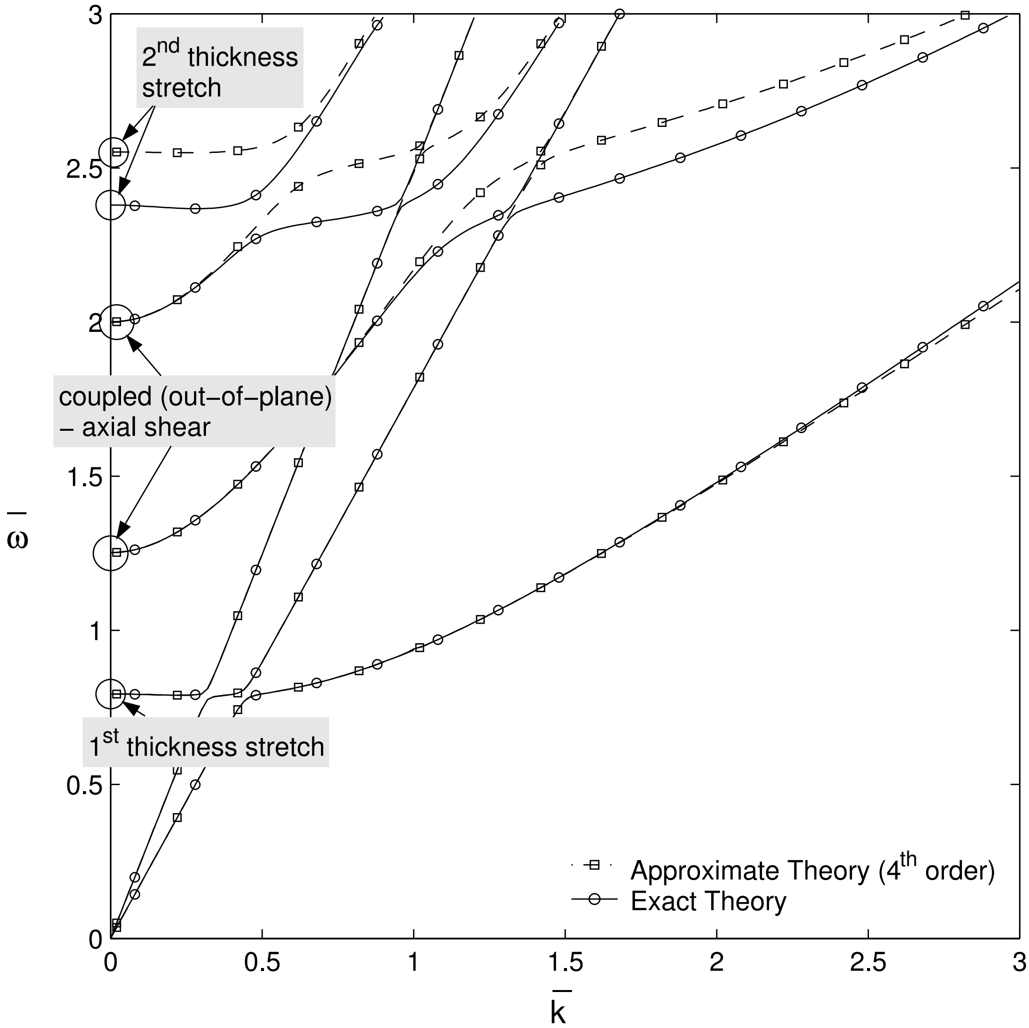


Figure 3.6 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=30° (the order of the approximate theory = 4)

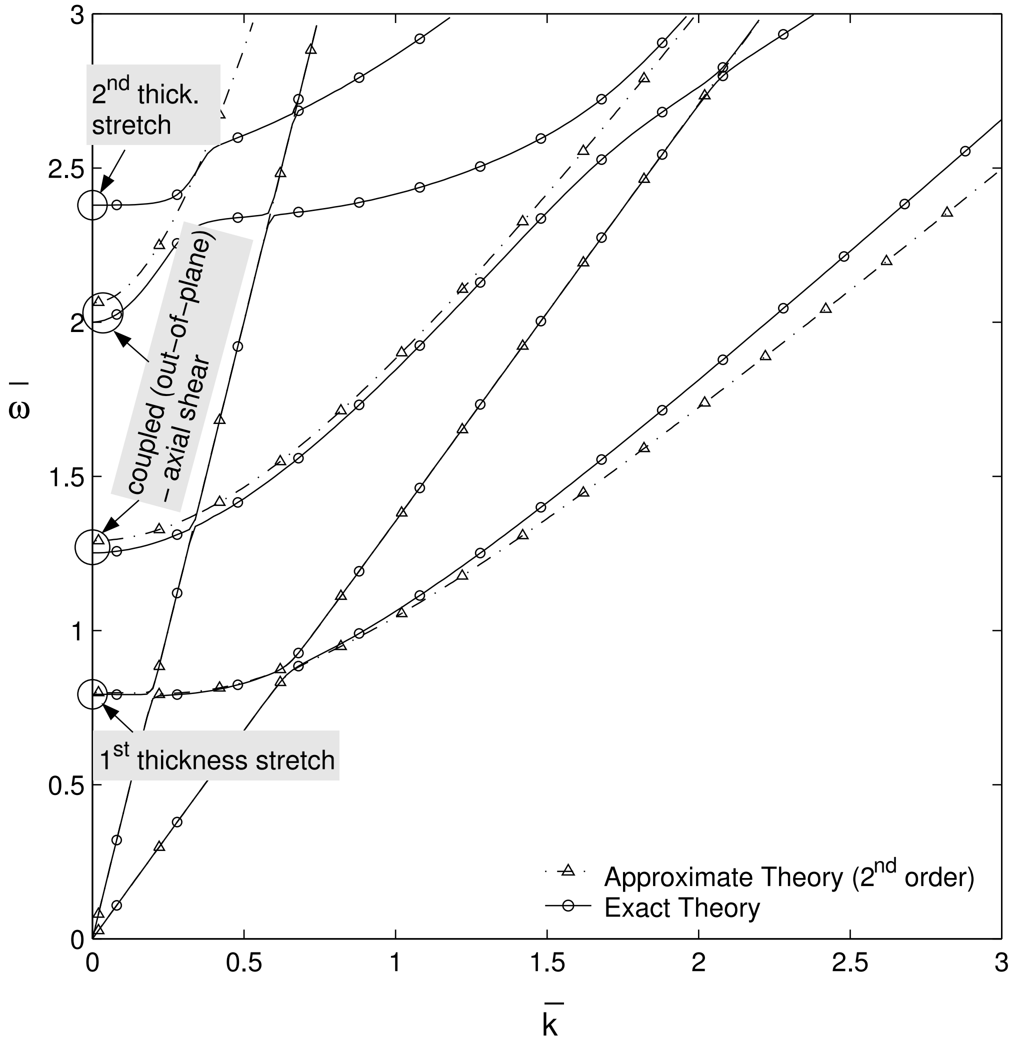


Figure 3.7 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=60° (the order of the approximate theory = 2)

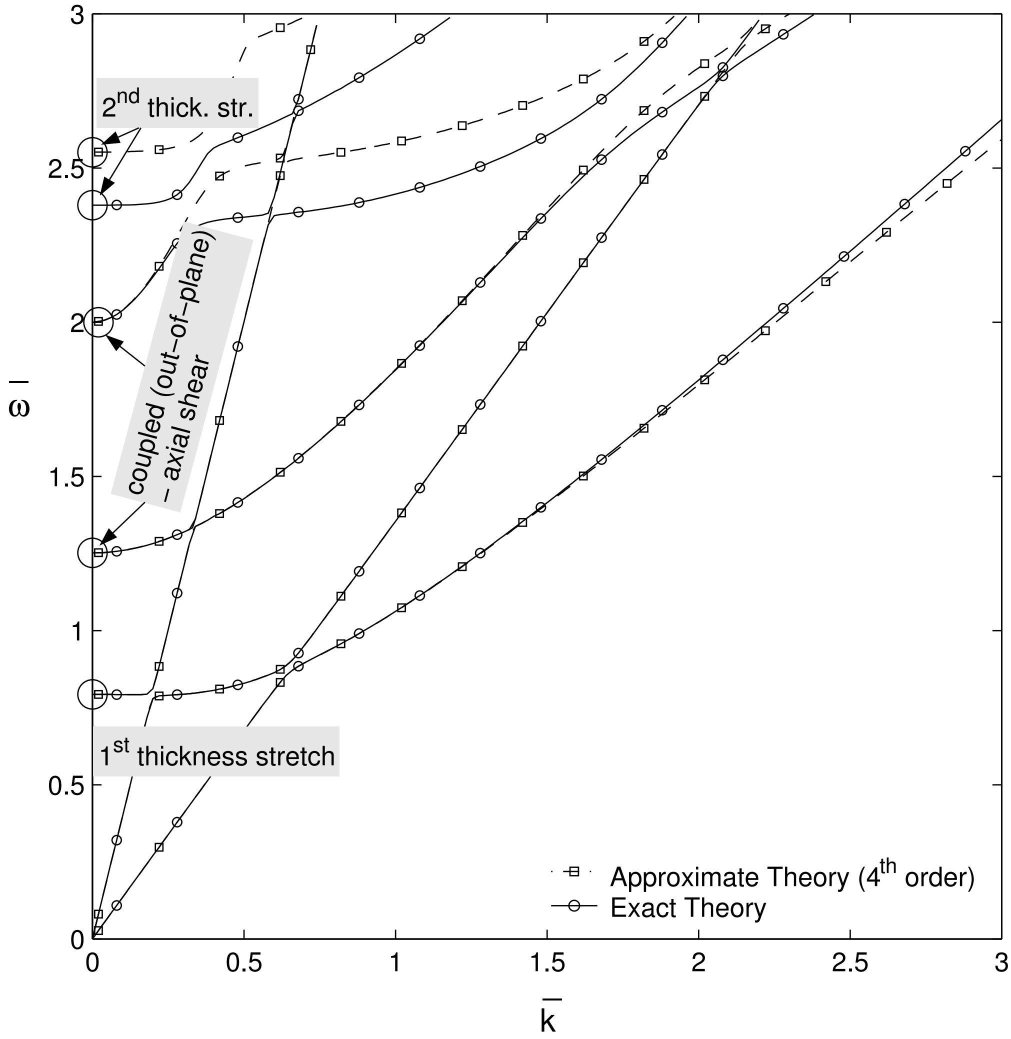


Figure 3.8 Comparison of dispersion curves for guided axial waves in x_1-direction when θ=60° (the order of the approximate theory = 4)

3.3 Mode Shapes

Mode shapes are very important for understanding the propagation of waves in plates. In what follows, the displacement mode shapes of approximate and exact theories at some cut-off frequencies (at k=0) in Figs. (3.1, 3.2, 3.5) will be obtained and compared.

3.3.1 Approximate mode shapes

Approximate mode shapes are obtained for uncoupled $(\theta = 0^\circ)$ and coupled $(\theta \neq 0^\circ, \frac{\pi}{2})$ cases separately. Procedure for obtaining the modes of 2nd order theory is explicitly given below.

<u>Uncoupled case (θ = 0°):</u>

In order to determine the mode shapes, the system of algebraic equations associated with Eqs.(3.2) and (3.4) is reconsidered, which can be obtained after the substitution of trial solution, Eq.(3.1), into them. Setting k = 0 and θ = 0° in these equations yields

$$\begin{bmatrix} \rho\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3C_{66}}{h^2} & \rho\omega^2 & \frac{-3C_{66}}{2h^2} & 0 & 0 & 0 & 0 & 0 \\ \frac{2C_{66}}{h}\gamma_0 & \frac{2C_{66}}{h}\gamma_2 & \frac{2C_{66}}{h}\gamma^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho\omega^2 & \frac{-C_{22}}{2h^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2C_{22}}{h}\gamma_1 & \frac{2C_{22}}{h}\gamma^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3C_{44}}{h^2} & \rho\omega^2 & \frac{-3C_{44}}{2h^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{2C_{44}}{h}\gamma_0 & \frac{2C_{44}}{h}\gamma_2 & \frac{2C_{44}}{h}\gamma^+ \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{bmatrix} = \mathbf{0} \tag{3.19}$$

where $A_1, A_2, \ldots, A_8$ are the amplitudes in Eq.(3.1) associated to the unknowns u_1^0, u_1^2, S_1^+, u_2^1, S_2^-, u_3^0, u_3^2 and S_3^+, respectively. As seen from Eq.(3.19) this system of equations can be separated into three uncoupled subsystems as

$$\text{for axial modes:}\quad \underbrace{\begin{bmatrix} \rho\omega^2 & 0 & 0 \\ \dfrac{3C_{66}}{h^2} & \rho\omega^2 & \dfrac{-3C_{66}}{2h^2} \\ \dfrac{2C_{66}}{h}\gamma_0 & \dfrac{2C_{66}}{h}\gamma_2 & \dfrac{2C_{66}}{h}\gamma^+ \end{bmatrix}}_{\mathbf{aa}} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \mathbf{0} \qquad (3.20)$$

$$\text{for thickness mode:}\quad \underbrace{\begin{bmatrix} \rho\omega^2 & \dfrac{-C_{22}}{2h^2} \\ \dfrac{2C_{22}}{h}\gamma_1 & \dfrac{2C_{22}}{h}\gamma^- \end{bmatrix}}_{\mathbf{bb}} \begin{bmatrix} A_4 \\ A_5 \end{bmatrix} = \mathbf{0} \qquad (3.21)$$

$$\text{for out-of-plane modes:}\quad \underbrace{\begin{bmatrix} \rho\omega^2 & 0 & 0 \\ \dfrac{3C_{44}}{h^2} & \rho\omega^2 & \dfrac{-3C_{44}}{2h^2} \\ \dfrac{2C_{44}}{h}\gamma_0 & \dfrac{2C_{44}}{h}\gamma_2 & \dfrac{2C_{44}}{h}\gamma^+ \end{bmatrix}}_{\mathbf{cc}} \begin{bmatrix} A_6 \\ A_7 \\ A_8 \end{bmatrix} = \mathbf{0} \qquad (3.22)$$

It is obvious that the frequencies making the determinant of the matrices **aa**, **bb** and **cc** zero determine the cut-off frequencies for the modes indicated in Eqs.(3.20-22). Solutions for the amplitudes of GV's and FV's can be obtained after substituting these cut-off frequencies into the associated uncoupled system. The displacement distributions can then be evaluated from Eq.(2.14):

$$u_i = \sum_{k=0}^{m+2} a_k^i \, \phi_k$$

where ϕ_k's are chosen as Legendre polynomials and a_k^i coefficients are related to GV's and FV's by Eq.(2.23). Cut-off frequencies, normalized nonzero variables for each cut-off frequency and the resulting displacement expressions are presented below:

Axial modes:

cut-off frequencies:

$$\omega_1 = 0 \text{ and } \omega_2 = \sqrt{\frac{aa_{23}\, aa_{32}}{\rho\, aa_{33}}} \tag{3.23}$$

variables:

for $\omega=\omega_1$ $\quad u_1^0 = 1,\ S_1^+ = -\dfrac{aa_{21}}{aa_{23}},\ u_1^2 = \dfrac{-aa_{31} - aa_{33} S_1^+}{aa_{32}}$

for $\omega=\omega_2$ $\quad u_1^0 = 0,\ u_1^2 = 1 \text{ and } S_1^+ = -\dfrac{aa_{32}}{aa_{33}}$

displacement distributions:

$$u_1(x_2) = \frac{u_1^0}{d_0} + \frac{u_1^2}{d_2}\phi_2 + \left(\frac{S_1^+}{2} - \frac{u_1^0}{d_0} - \frac{u_1^2}{d_2}\right)\phi_4$$
$$u_2 = u_3 = 0 \tag{3.24}$$

Thickness mode:

cut-off frequency:

$$\omega_1 = \sqrt{\frac{bb_{12}\, bb_{21}}{\rho\, bb_{22}}} \tag{3.25}$$

variables:

for $\omega=\omega_1$ $\quad u_2^1 = 1,\ S_2^- = -\dfrac{bb_{21}}{bb_{22}}$

displacement distributions:

$$u_2 = \frac{u_2^1}{d_1}\phi_1 + \left(\frac{S_2^-}{2} - \frac{u_2^1}{d_1}\right)\phi_3$$
$$u_1 = u_3 = 0 \tag{3.26}$$

Out-of-plane modes:

cut-off frequencies:

$$\omega_1 = 0 \text{ and } \omega_2 = \sqrt{\frac{cc_{23}\, cc_{32}}{\rho\, cc_{33}}} \tag{3.27}$$

variables:

for $\omega=\omega_1$ $\quad u_3^0 = 1,\ S_1^+ = -\dfrac{cc_{21}}{cc_{23}},\ u_1^2 = \dfrac{-cc_{31} - cc_{33} S_3^+}{cc_{32}}$

for $\omega=\omega_2$ $\quad u_3^0 = 0,\ u_3^2 = 1$ and $S_3^+ = -\dfrac{cc_{32}}{cc_{33}}$

displacement distributions:

$$u_3(x_2) = \frac{u_3^0}{d_0} + \frac{u_3^2}{d_2}\phi_2 + \left(\frac{S_3^+}{2} - \frac{u_3^0}{d_0} - \frac{u_3^2}{d_2}\right)\phi_4 \qquad (3.28)$$

$$u_2 = u_1 = 0$$

Coupled case:

For this case, the system of algebraic equations associated with Eqs.(3.2) and (3.4), after setting $k = 0$ and $(\theta \neq 0^\circ, \frac{\pi}{2})$, will be

$$\begin{bmatrix} \rho\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3C_{66}}{h^2} & \rho\omega^2 & \frac{-3C_{66}}{2h^2} & 0 & 0 & \frac{3C_{46}}{h^2} & 0 & -\frac{3C_{46}}{2h^2} \\ \frac{2C_{66}}{h}\gamma_0 & \frac{2C_{66}}{h}\gamma_2 & \frac{2C_{66}}{h}\gamma^+ & 0 & 0 & \frac{2C_{46}}{h}\gamma_0 & \frac{2C_{46}}{h}\gamma_2 & \frac{2C_{46}}{h}\gamma^+ \\ 0 & 0 & 0 & \rho\omega^2 & \frac{-C_{22}}{2h^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2C_{22}}{h}\gamma_1 & \frac{2C_{22}}{h}\gamma^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\omega^2 & 0 & 0 \\ \frac{3C_{46}}{h^2} & 0 & -\frac{3C_{46}}{2h^2} & 0 & 0 & \frac{3C_{44}}{h^2} & \rho\omega^2 & \frac{-3C_{44}}{2h^2} \\ \frac{2C_{46}}{h}\gamma_0 & \frac{2C_{46}}{h}\gamma_2 & \frac{2C_{46}}{h}\gamma^+ & 0 & 0 & \frac{2C_{44}}{h}\gamma_0 & \frac{2C_{44}}{h}\gamma_2 & \frac{2C_{44}}{h}\gamma^+ \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{bmatrix} = \mathbf{0} \qquad (3.29)$$

From this equation it is obvious that, axial modes and out-of plane modes are coupled and thickness modes are uncoupled from them. After separating Eq.(3.29) into subsystems, resulting equations would be for thickness mode:

$$\underbrace{\begin{bmatrix} \rho\omega^2 & \dfrac{-C_{22}}{2h^2} \\ \dfrac{2C_{22}}{h}\gamma_1 & \dfrac{2C_{22}}{h}\gamma^- \end{bmatrix}}_{\mathbf{dd}} \begin{bmatrix} A_4 \\ A_5 \end{bmatrix} = \mathbf{0} \qquad (3.30)$$

for axial – (out-of-plane) modes:

$$\underbrace{\begin{bmatrix} \rho\omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho\omega^2 & 0 & 0 & 0 & 0 \\ \dfrac{3C_{66}}{h^2} & \dfrac{3C_{46}}{h^2} & \rho\omega^2 & 0 & \dfrac{-3C_{66}}{2h^2} & -\dfrac{3C_{46}}{2h^2} \\ \dfrac{3C_{46}}{h^2} & \dfrac{3C_{44}}{h^2} & 0 & \rho\omega^2 & -\dfrac{3C_{46}}{2h^2} & \dfrac{-3C_{44}}{2h^2} \\ \dfrac{2C_{66}}{h}\gamma_0 & \dfrac{2C_{46}}{h}\gamma_0 & \dfrac{2C_{66}}{h}\gamma_2 & \dfrac{2C_{46}}{h}\gamma_2 & \dfrac{2C_{66}}{h}\gamma^+ & \dfrac{2C_{46}}{h}\gamma^+ \\ \dfrac{2C_{46}}{h}\gamma_0 & \dfrac{2C_{44}}{h}\gamma_0 & \dfrac{2C_{46}}{h}\gamma_2 & \dfrac{2C_{44}}{h}\gamma_2 & \dfrac{2C_{46}}{h}\gamma^+ & \dfrac{2C_{44}}{h}\gamma^+ \end{bmatrix}}_{\mathbf{ee}} \begin{bmatrix} A_1 \\ A_6 \\ A_2 \\ A_7 \\ A_3 \\ A_8 \end{bmatrix} = \mathbf{0} \qquad (3.31)$$

ff **gg**

where the submatrices **ff** and **gg** are marked in Eq.(3.31) for a later use.

In view of Eqs.(3.30,31), the cut-off frequencies, the displacement distributions and nonzero variables appearing in them would be, for each mode,

Coupled axial – (out-of-plane) modes:

cut-off frequencies:

$$\begin{aligned} &\omega_{1,2} = 0 \\ &\omega_3 \text{ and } \omega_4 \text{ are obtained from } \det(\mathbf{ff}) = 0 \end{aligned} \qquad (3.32)$$

variables:

for $\omega=\omega_1$ $\quad u_1^0 = 1,\ \ u_3^0 = 0$

$$\begin{bmatrix} u_1^2 \\ u_1^2 \\ S_1^+ \\ S_3^+ \end{bmatrix} = -\mathbf{ff}^{-1} \begin{bmatrix} ee_{31} \\ ee_{41} \\ ee_{51} \\ ee_{61} \end{bmatrix}$$

for $\omega=\omega_2$ $\quad u_1^0 = 0, \quad u_3^0 = 1$

$$\begin{bmatrix} u_1^2 \\ u_1^2 \\ S_1^+ \\ S_3^+ \end{bmatrix} = -\mathbf{ff}^{-1} \begin{bmatrix} ee_{32} \\ ee_{42} \\ ee_{52} \\ ee_{62} \end{bmatrix}$$

for $\omega=\omega_{3,4}$ $\quad u_1^0 = 0, \quad u_3^0 = 0, \quad u_1^2 = 1$

$$\begin{bmatrix} u_3^2 \\ S_1^+ \\ S_3^+ \end{bmatrix} = -\mathbf{gg}^{-1} \begin{bmatrix} ff_{11} \\ ff_{21} \\ ff_{31} \end{bmatrix}$$

displacement distributions:

$$u_1(x_2) = \frac{u_1^0}{d_0} + \frac{u_1^2}{d_2}\phi_2 + \left(\frac{S_1^+}{2} - \frac{u_1^0}{d_0} - \frac{u_1^2}{d_2} \right)\phi_4$$

$$u_2 = 0 \tag{3.33}$$

$$u_3(x_2) = \frac{u_3^0}{d_0} + \frac{u_3^2}{d_2}\phi_2 + \left(\frac{S_3^+}{2} - \frac{u_3^0}{d_0} - \frac{u_3^2}{d_2} \right)\phi_4$$

Thickness mode: Since this mode is uncoupled from the coupled axial – (out-of-plane) modes, it remains the same as that presented previously for $\theta = 0^\circ$.

3.3.2 Exact mode shapes

Exact mode shapes are again obtained for uncoupled ($\theta = 0°$) and coupled $(\theta \neq 0^\circ, \frac{\pi}{2})$ cases separately at the cut-off frequencies.

Uncoupled case ($\theta = 0°$):

For $k = 0$ and $\theta = 0°$, eigenvalue problem in Eq.(3.12) takes the form

$$\begin{bmatrix} C_{66}\lambda^2+\rho\omega^2 & 0 & 0 \\ 0 & C_{22}\lambda^2+\rho\omega^2 & 0 \\ 0 & 0 & C_{55}\lambda^2+\rho\omega^2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \mathbf{0} \tag{3.34}$$

Thus, for the eigenvalues λ_k's and their corresponding eigenvectors $\mathbf{A}^k$, one has

$$\lambda_1 = i\frac{\omega}{\sqrt{\frac{C_{66}}{\rho}}}\ , \quad \lambda_2 = i\frac{\omega}{\sqrt{\frac{C_{22}}{\rho}}}\ , \quad \lambda_3 = i\frac{\omega}{\sqrt{\frac{C_{55}}{\rho}}} \tag{3.35}$$

$$\mathbf{A}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{A}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{A}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.36}$$

where “i” is the imaginary number.

Substituting these into Eqs.(3.14) and using the identites

$$\cosh(i\,x) = \cos(x) \quad \text{and} \quad \sinh(i\,x) = i\sin(x)$$

one obtains the displacement amplitudes as

$$\tilde{u}_1 = B_1\cos(\tilde{\lambda}_1 x_2)\ , \quad \tilde{u}_2 = i\,B_2\sin(\tilde{\lambda}_2 x_2)\ , \quad \tilde{u}_3 = B_3\cos(\tilde{\lambda}_3 x_2) \tag{3.37}$$

where

$$\tilde{\lambda}_k = \frac{\lambda_k}{i}$$

To determine the cut-off frequencies ω_k and the corresponding $\mathbf{B}^k$ vectors, Eq.(3.16) is utilized, which reduces to, for $k = 0$ and $\theta = 0°$,

$$\begin{bmatrix} C_{66}\lambda_1\sinh(\lambda_1 h) & 0 & 0 \\ 0 & C_{22}\lambda_2\cosh(\lambda_2 h) & 0 \\ 0 & 0 & C_{44}\lambda_3\sinh(\lambda_3 h) \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \mathbf{0} \tag{3.38}$$

Which will have a nontrivial solution when the diagonal elements of the coefficient matrix vanish, that is

$$\sinh(\lambda_1 h) = 0 \quad \text{or} \quad \cosh(\lambda_2 h) = 0 \quad \text{or} \quad \sinh(\lambda_3 h) = 0$$

It may be noted that, the exact theory generates infinitely many cut-off frequencies and mode shapes. The resulting frequencies and corresponding displacement distributions are as follows:

Axial modes: ($\underline{\sinh(\lambda_1 h) = 0}$ $\Rightarrow \tilde{\lambda}_1 h = j\pi \qquad (j = 0,1,2,\ldots)$)

cut-off frequencies:

$$\omega_j = j\pi \frac{\sqrt{\frac{C_{66}}{\rho}}}{h} \qquad (j = 0,1,2,\ldots) \tag{3.39}$$

solution of Eq.(3.38):

$$B_1 \neq 0,\ B_2 = B_3 = 0$$

displacement distributions:

$$\begin{aligned} &\tilde{u}_1 = \cos(\frac{j\pi}{h} x_2) \qquad (j = 0,1,\ldots) \\ &\tilde{u}_2 = \tilde{u}_3 = 0 \end{aligned} \tag{3.40}$$

Thickness modes: ($\underline{\cosh(\lambda_2 h) = 0}$ $\Rightarrow \tilde{\lambda}_2 h = \frac{2j-1}{2}\pi \qquad (j = 1,2,\ldots)$)

cut-off frequencies:

$$\omega_j = \frac{2j-1}{2}\pi \frac{\sqrt{\frac{C_{22}}{\rho}}}{h} \qquad (j = 1,2,\ldots) \tag{3.41}$$

solution of Eq.(3.38):

$$B_2 \neq 0,\ B_1 = B_3 = 0$$

displacement distributions:

$$\begin{aligned} &\tilde{u}_2 = \sin(\frac{2j-1}{2}\pi \frac{x_2}{h}) \qquad (j = 1,2\ldots) \\ &\tilde{u}_1 = \tilde{u}_3 = 0 \end{aligned} \tag{3.42}$$

Out-of-plane modes: ($\underline{\sinh(\lambda_3 h) = 0}$ $\Rightarrow \tilde{\lambda}_3 h = j\pi \qquad (j = 0,1,2,\ldots)$)

cut-off frequencies:

$$\omega_j = j\pi \frac{\sqrt{\frac{C_{44}}{\rho}}}{h} \qquad (j = 0,1,2,\ldots) \tag{3.43}$$

solution of Eq.(3.38):

$$B_3 \neq 0,\ B_1 = B_2 = 0$$

displacement distributions:

$$\tilde{u}_3 = \cos(\frac{j\pi}{h} x_2) \qquad (j = 0,1,\ldots)$$
$$\tilde{u}_1 = \tilde{u}_2 = 0 \tag{3.44}$$

Coupled case:

For this case, eigenvalue problem in Eq.(3.12) takes the form

$$\begin{bmatrix} C_{66}\lambda^2 + \rho\omega^2 & 0 & C_{46}\lambda^2 \\ 0 & C_{22}\lambda^2 + \rho\omega^2 & 0 \\ C_{46}\lambda^2 & 0 & C_{44}\lambda^2 + \rho\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \underline{0} \tag{3.45}$$

Whose eigenvalues and their corresponding eigenvectors would be

$$\lambda_1^2 = \frac{-\rho\omega^2(C_{66} + C_{44}) + \sqrt{\Delta}}{2(C_{66}C_{44} - C_{46}^2)},\quad \lambda_2 = i\frac{\omega}{\sqrt{\frac{C_{22}}{\rho}}},\quad \lambda_3^2 = \frac{-\rho\omega^2(C_{66} + C_{44}) - \sqrt{\Delta}}{2(C_{66}C_{44} - C_{46}^2)} \tag{3.46}$$

$$\underline{A}^1 = \begin{bmatrix} 1 \\ 0 \\ a_1 \end{bmatrix},\qquad \underline{A}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\qquad \underline{A}^3 = \begin{bmatrix} a_3 \\ 0 \\ 1 \end{bmatrix} \tag{3.47}$$

where

$$\Delta = \rho^2\omega^4(C_{66} + C_{44})^2 - 4\rho^2\omega^4(C_{66}C_{44} - C_{46}^2)$$

$$a_1 = -\frac{C_{46}\lambda_1^2}{C_{44}\lambda_1^2 + \rho\omega^2},\quad a_3 = -\frac{C_{46}\lambda_3^2}{C_{66}\lambda_3^2 + \rho\omega^2}$$

With these eigenvalues and eigenvectors one obtains the displacement amplitudes from Eqs.(3.14) as

$$\tilde{u}_1 = B_1\cosh(\lambda_1 x_2) + B_3 a_3\cosh(\lambda_3 x_2)$$
$$\tilde{u}_2 = B_2\sinh(\lambda_1 x_2) \tag{3.48}$$
$$\tilde{u}_3 = B_1 a_1\cosh(\lambda_1 x_2) + B_3\cosh(\lambda_3 x_2)$$

Consequently, as in uncoupled case, the cut-off frequencies and mode shapes can be obtained from nontrivial solutions of Eq.(3.16). This reduces to, when k = 0 and in view of Eqs.(3.47),

$$\begin{bmatrix} (C_{46}a_1 + C_{66})\lambda_1 \sinh(\lambda_1 h) & 0 & (C_{44}a_3 + C_{46})\lambda_3 \sinh(\lambda_3 h) \\ 0 & C_{22}\lambda_2 \cosh(\lambda_2 h) & 0 \\ (C_{46} + C_{66}a_1)\lambda_1 \sinh(\lambda_1 h) & 0 & (C_{44} + C_{46}a_3)\lambda_3 \sinh(\lambda_3 h) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \underline{0} \tag{3.49}$$

Coupled axial – (out-of-plane) modes:

cut-off frequencies: are obtained from

$$\begin{vmatrix} (C_{46}a_1 + C_{66})\lambda_1 \sinh(\lambda_1 h) & (C_{44}a_3 + C_{46})\lambda_3 \sinh(\lambda_3 h) \\ (C_{46} + C_{66}a_1)\lambda_1 \sinh(\lambda_1 h) & (C_{44} + C_{46}a_3)\lambda_3 \sinh(\lambda_3 h) \end{vmatrix} = 0 \tag{3.50}$$

solution of Eq.(3.49):

$$B_2 = 0, \;\; B_1 = 1, \;\; B_3 = -\frac{(C_{46} + C_{66}a_1)\lambda_1 \sinh(\lambda_1 h)}{(C_{44} + C_{46}a_3)\lambda_3 \sinh(\lambda_3 h)} \quad (\text{for } \omega \neq 0) \tag{3.51}$$

displacement distributions: may be found from Eqs.(3.48) by substituting the frequencies and their corresponding solutions, for **B** vector.

Indeterminacy appearing in Eq.(3.51) for $\omega = 0$ may be avoided by canceling "ω^2" terms in a_1 and a_3. This gives two possible displacement distributions for $\omega = 0$

$$\tilde{u}_1 = 1, \;\; \tilde{u}_3 = 0$$

$$\tilde{u}_1 = 0, \;\; \tilde{u}_3 = 1$$

Thickness mode: This mode is uncoupled from axial – (out-of-plane) modes and it remains the same as that of uncoupled case.

In Fig. 3.9, the displacement distributions over the thickness of the plate at some cut-off frequencies (at $k = 0$), predicted from exact and 2nd order approximate theories, are compared, where $\bar{u}_i$ denotes normalized displacements. The figure shows that the comparisons are very good, which is achieved in spite of our using relatively lower order theory, that is, 2nd order theory.

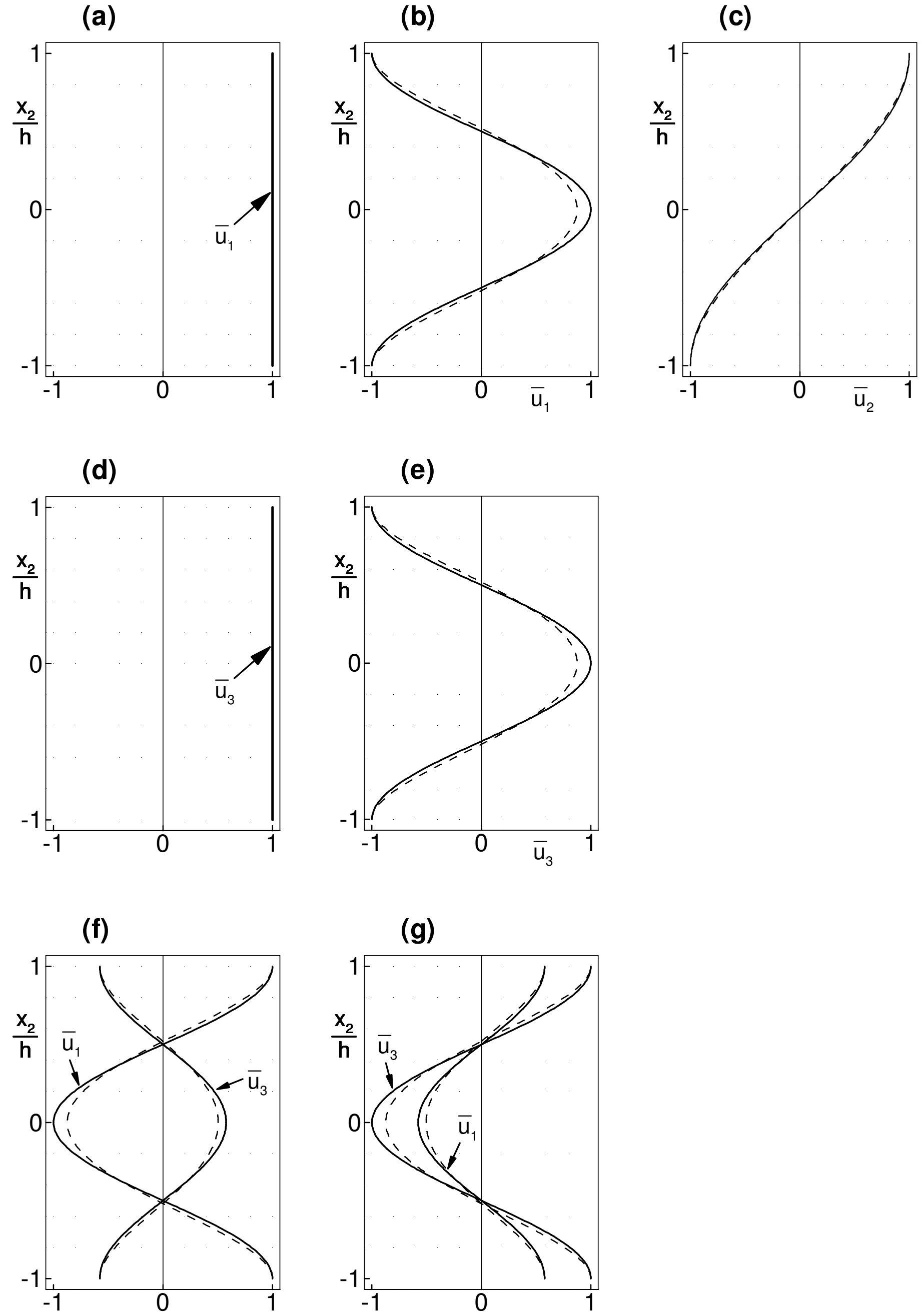


Figure 3.9 Comparison of mode shapes at cut-off frequencies: **(a)** axial mode (at point A in Fig. 3.1) **(b)** 1st axial shear mode (at point B in Fig. 3.1) **(c)** 1st thickness stretch mode (at point C in Fig. 3.1) **(d)** 1st out-of-plane shear mode (at point D in Fig. 3.2) **(e)** 2nd out-of-plane shear mode (at point E in Fig. 3.2) **(f)** 1st coupled (out-of-plane)-axial shear mode (at point F in Fig. 3.5) **(g)** 2nd coupled (out-of-plane)-axial shear mode (at point G in Fig. 3.5)

CHAPTER 4

APPROXIMATE DYNAMIC MODELS FOR A PERIODIC LAYERED COMPOSITE WITH TWO ALTERNATING LAMINAE

In this chapter, two dynamic models, discrete and continuum models, based on the approximate plate theory presented in Chapter 2, are proposed for a periodic layered composite with two alternating laminae. In Chapter 5, the continuum model will be extended to a more general layered composite with a unit cell containing arbitrary number of laminae.

The formulation is presented in general terms: the material of laminae is taken as triclinic elastic with no material symmetry, thermal effects are included, the order of the approximate theory is kept arbitrary, etc.

In the formulation of the models, the layered composite is referred to an x_i global coordinate system in which x_1x_3 plane is parallel to layering (see Fig. 4.1). The $x_2(x)$ axis in the figure (perpendicular to layering) is used to distinguish a lamina in the composite by specifying $x_2(x)$ coordinate of its midplane. The two different phases of the layered composite are indicated by circled numbers "1" and "2" in the figure. Without loss of generality, it is assumed that the equations of the approximate theory for a lamina are written at its midplane, that is, at $x = \overset{\alpha}{x}{}^{k}$, where $\overset{\alpha}{x}{}^{k}$ denotes the x coordinate of the midplane of a lamina pertaining to α phase ($\alpha = 1, 2$) of k^{th} pair of the composite (see Fig. 4.2). A typical field variable of the periodic composite defined discretely at midplanes of laminae will be designated by $\overset{\alpha}{f}(\overset{\alpha}{x}{}_2^{k})$, which belongs to α phase of the pairs k=1,2,... of the composite; here, x_1, x_3 and t dependencies of $\overset{\alpha}{f}$ are disregarded for notational convenience (see Fig. 4.3).

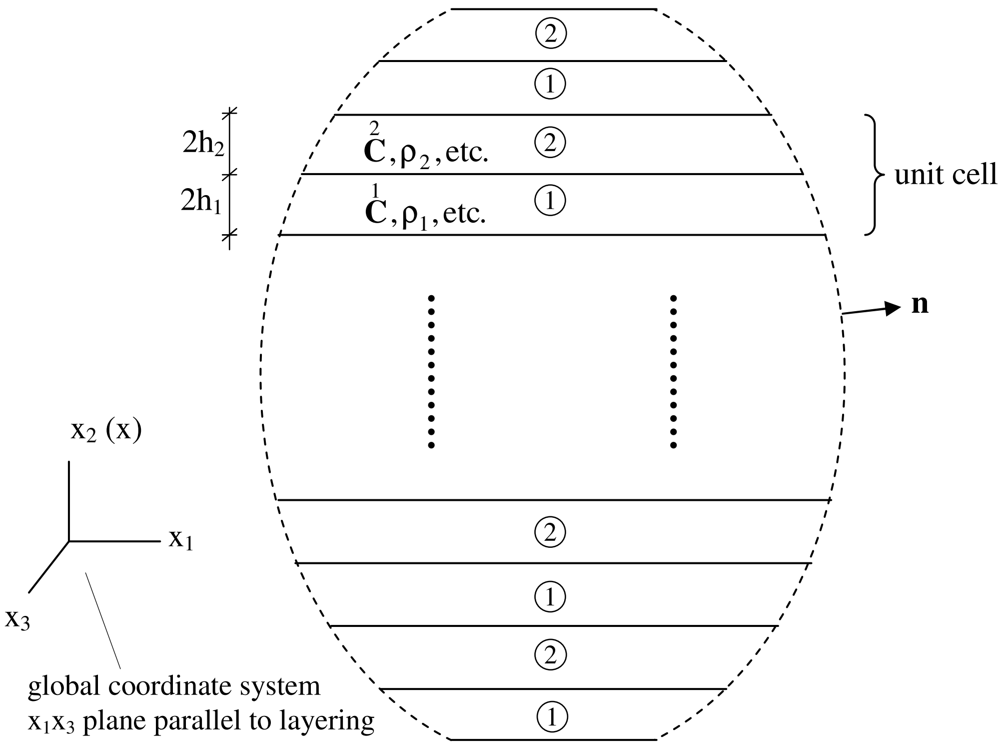


Figure 4.1 Layered composite body

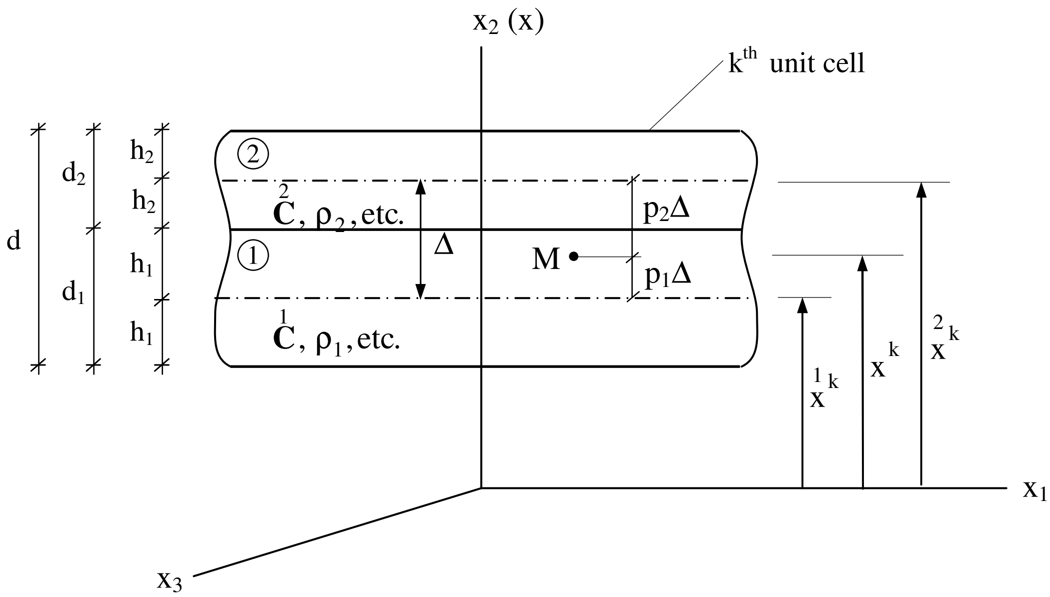


Figure 4.2 Typical unit cell

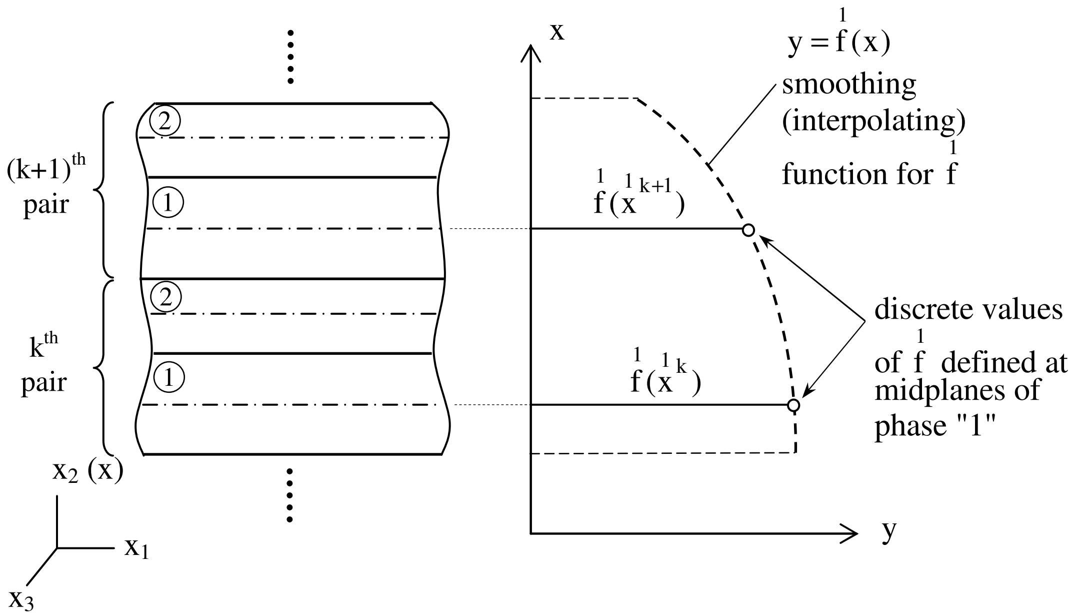


Figure 4.3 Description of smoothing operation

Equations of discrete and continuum models are presented in the following subsections.

4.1 Discrete Model (DM)

This model can be established by writing the equations of the approximate theory, Eq.(2.20), for each lamina of the composite and the continuity conditions at interfaces. The equations of DM for the two phase periodic composite take the form

$$(\overset{\alpha}{M}{}_i^n,\ \overset{\alpha}{C}{}_{ij}^n,\ \overset{\alpha}{E}{}^n,\ \overset{\alpha}{T}{}_i^n,\ \overset{\alpha}{Z}{}_i^{\mp},\ \overset{\alpha}{P}{}^{\mp})\Big|_{x=\overset{\alpha}{x}{}^k} = 0 \qquad (\alpha = 1,2\ ;\ n = 0 \ldots m) \quad (4.1)$$

which are to be written for the pairs k = 1, 2, … , together with the continuity conditions (CC) [see Fig. 4.3]:

for the interface following the layer 1:

$$\overset{1}{f}{}^{+}(\overset{1}{x}{}^{k}) = \overset{2}{f}{}^{-}(\overset{2}{x}{}^{k}) \qquad (4.2)$$

for the interface following the layer 2:

$$\overset{2}{f}{}^{+}(\overset{2}{x}{}^{k}) = \overset{1}{f}{}^{-}(\overset{1}{x}{}^{k+1}) \qquad (4.3)$$

with $f = (u_i,\ \tau_{2i},\ q_2,\ \theta)$ and k = 1, 2,… In Eq.(4.1), the superscript α denotes the phase, m stands for the order of the approximate theory and $f^{\mp}$ give the values of f at upper and lower faces of the layer. The continuity conditions in Eqs.(4.2) and (4.3) may be expressed in terms of FV's appearing in the approximate theory by using the relations, in view of Eqs.(2.13),

$$\overset{\alpha}{f}{}^{+} = \frac{\overset{\alpha}{F}{}^{+} + \overset{\alpha}{F}{}^{-}}{2} \quad , \quad \overset{\alpha}{f}{}^{-} = \frac{\overset{\alpha}{F}{}^{+} - \overset{\alpha}{F}{}^{-}}{2} \tag{4.4}$$

with $\overset{\alpha}{F}{}^{\mp} = (\overset{\alpha}{S}{}^{\mp},\ \overset{\alpha}{R}{}^{\mp},\ \overset{\alpha}{Q}{}^{\mp},\ \overset{\alpha}{\Psi}{}^{\mp})$ being FV's. The CC's in Eqs.(4.2) and (4.3) can be written explicitly in terms of the FV's $\overset{\alpha}{F}{}^{\mp}$ as

for the interface following the layer 1:

$$\overset{2}{S}{}_i^{+} - \overset{1}{S}{}_i^{+} = \overset{1}{S}{}_i^{-} + \overset{2}{S}{}_i^{-}$$

$$\overset{2}{R}{}_i^{+} - \overset{1}{R}{}_i^{+} = \overset{1}{R}{}_i^{-} + \overset{2}{R}{}_i^{-}$$

$$\overset{2}{Q}{}_i^{+} - \overset{1}{Q}{}_i^{+} = \overset{1}{Q}{}_i^{-} + \overset{2}{Q}{}_i^{-}$$

$$\overset{2}{\Psi}{}_i^{+} - \overset{1}{\Psi}{}_i^{+} = \overset{1}{\Psi}{}_i^{-} + \overset{2}{\Psi}{}_i^{-}$$

for the interface following the layer 2:

$$\overset{1}{S}{}_i^{+} - \overset{2}{S}{}_i^{+} = \overset{1}{S}{}_i^{-} + \overset{2}{S}{}_i^{-}$$

$$\overset{1}{R}{}_i^{+} - \overset{2}{R}{}_i^{+} = \overset{1}{R}{}_i^{-} + \overset{2}{R}{}_i^{-}$$

$$\overset{1}{Q}{}_i^{+} - \overset{2}{Q}{}_i^{+} = \overset{1}{Q}{}_i^{-} + \overset{2}{Q}{}_i^{-}$$

$$\overset{1}{\Psi}{}_i^{+} - \overset{2}{\Psi}{}_i^{+} = \overset{1}{\Psi}{}_i^{-} + \overset{2}{\Psi}{}_i^{-}$$

The equations of DM in Eqs.(4.1–4.3) are to be supplied by boundary conditions (bc). For a boundary portion parallel to layering , bc's can be expressed in terms of FV's. For example bc's at the top boundary of the composite in Fig. 4.1 involve specification of one member or combination from the pairs

$$(\overset{2}{u}{}_i^{+},\ \overset{2}{\tau}{}_{2i}^{+})\ ;\ (\overset{2}{\theta}{}^{+},\ \overset{2}{q}{}_2^{+}) \tag{4.5}$$

which, in view of Eqs.(4.4), can be expressed in terms of FV's. On the other hand, bc's on side boundary dictates that one member or combination is prescribed from the pairs

$$(u_i^s,\ n_j\tau_{ji}^s)\ ;\ (\theta^s,\ n_i q_i^s) \qquad (s=0\ldots m) \tag{4.6}$$

where **n** is outer unit normal of side boundary (see Fig. 4.1).

4.2 Continuum Model (CM)

The use of this model may be suggested when the number of laminae in the composite is very large. For establishing CM, a smoothing operation (SO) is employed, where a field variable $\overset{\alpha}{f}(\overset{\alpha}{x}{}^k)$ of DM defined discretely at midplanes of laminae is replaced by a continuous function of x, $\overset{\alpha}{f}(x)$, which interpolates discrete $\overset{\alpha}{f}$ values at midplanes of laminae, that is, at $x=\overset{\alpha}{x}{}^k$ (see Fig.4.3). Thus, the smoothed variable $\overset{\alpha}{f}(x)$ is now defined for all x; but, it has physical meaning only at midplanes of α phase, that is, at $x=\overset{\alpha}{x}{}^k$.

Now, in view of SO, the equations of CM are obtained from those of DM. It is observed that SO leaves Eq.(4.1) unchanged since the field variables appearing in it are defined at the midplane $x=\overset{\alpha}{x}{}^k$ of the same lamina belonging to α phase of k^{th} pair. For the sake of completeness, this equation will be restated here:

$$(\overset{\alpha}{M}{}_i^n,\ \overset{\alpha}{C}{}_{ij}^n,\ \overset{\alpha}{E}{}^n,\ \overset{\alpha}{T}{}_i^n,\ \overset{\alpha}{Z}{}_i^{\mp},\ \overset{\alpha}{P}{}^{\mp})=0 \qquad (\alpha=1,2\ ;\ n=0\ldots m) \tag{4.7}$$

which, in view of SO, holds for any x.

Now concentrate on deriving smoothed forms of CC's in Eqs.(4.2) and (4.3), where Eq.(4.2) representing the CC for the interface following layer 1 will be considered first. It is to be noted that the left and right hand sides of this equation are defined at the midplanes of two different (successive) laminae, namely, they are defined at $x = \overset{1}{x}{}^{k}$ and $\overset{2}{x}{}^{k}$ representing the midplanes of layers 1 and 2 of k^{th} pair (see Fig. 4.2). Therefore, in view of the idealization implied by SO, to obtain the smoothed form of CC in Eq.(4.2), it is necessary to write it at a common point M $(x = x^k)$, which will be taken in the present study as a point lying in the interval $(\overset{1}{x}{}^{k}, \overset{2}{x}{}^{k})$; thus, one can write for $\overset{1}{x}{}^{k}$ and $\overset{2}{x}{}^{k}$:

$$\overset{1}{x}{}^{k} = x^k - p_1\Delta \;; \quad \overset{2}{x}{}^{k} = x^k + p_2\Delta \tag{4.8}$$

where

$$\Delta = h_1 + h_2 \tag{4.9}$$

is half of the thickness of unit cell and p_α are nondimensional interpolation factors satisfying

$$p_1 + p_2 = 1 \quad \text{with} \quad 0 < p_\alpha < 1 \tag{4.10}$$

When Eqs.(4.8) are inserted into (4.2), both sides are expanded about $x = x^k$ by Taylor's formula and x^k is replaced by x, in view of SO, one obtains the smoothed form of CC in Eq.(4.2) as

$$E_1^- \overset{1}{f}{}^{+} = E_2^+ \overset{2}{f}{}^{-} \tag{4.11}$$

which holds for any "x", where E_1^- and E_2^+ are operators defined by

$$\begin{aligned} E_\alpha^- &= \exp(-\chi_\alpha) = 1 - \chi_\alpha + \frac{\chi_\alpha^2}{2!} - \cdots \\ E_\alpha^+ &= \exp(+\chi_\alpha) = 1 + \chi_\alpha + \frac{\chi_\alpha^2}{2!} + \cdots \qquad (\alpha = 1,2) \end{aligned} \tag{4.12}$$

with

$$\chi_\alpha = p_\alpha \Delta \frac{\partial}{\partial x} \tag{4.13}$$

Smoothed form of the second CC in Eq.(4.3) for the interface following layer 2 may be obtained from Eq.(4.11) by interchanging the indices "1" and "2"; it is

$$E_2^- \overset{2}{f}{}^{+} = E_1^+ \overset{1}{f}{}^{-} \tag{4.14}$$

where the operators E_2^- and E_1^+ are already defined in Eqs.(4.12) and (4.13). It may be noted that $\overset{\alpha}{f}{}^{\mp}$ in Eqs.(4.11) and (4.14) are related to the FV's $\overset{\alpha}{F}{}^{\mp}$ appearing in the approximate theory by Eqs.(4.4). Through the insertion of these equations into Eqs.(4.11) and (4.14), one can write the smoothed form of CC's in terms of the FV's $\overset{\alpha}{F}{}^{\mp}$ as, after some manipulations,

$$
\begin{aligned}
s_2 \overset{2}{S}{}_i^{+} + s_1 \overset{1}{S}{}_i^{+} &= c_2 \overset{2}{S}{}_i^{-} + c_1 \overset{1}{S}{}_i^{-} \\
c_2 \overset{2}{S}{}_i^{+} - c_1 \overset{1}{S}{}_i^{+} &= s_2 \overset{2}{S}{}_i^{-} - s_1 \overset{1}{S}{}_i^{-} \\
s_2 \overset{2}{R}{}_i^{+} + s_1 \overset{1}{R}{}_i^{+} &= c_2 \overset{2}{R}{}_i^{-} + c_1 \overset{1}{R}{}_i^{-} \\
c_2 \overset{2}{R}{}_i^{+} - c_1 \overset{1}{R}{}_i^{+} &= s_2 \overset{2}{R}{}_i^{-} - s_1 \overset{1}{R}{}_i^{-} \\
s_2 \overset{2}{Q}{}_i^{+} + s_1 \overset{1}{Q}{}_i^{+} &= c_2 \overset{2}{Q}{}_i^{-} + c_1 \overset{1}{Q}{}_i^{-} \\
c_2 \overset{2}{Q}{}_i^{+} - c_1 \overset{1}{Q}{}_i^{+} &= s_2 \overset{2}{Q}{}_i^{-} - s_1 \overset{1}{Q}{}_i^{-} \\
s_2 \overset{2}{\Psi}{}_i^{+} + s_1 \overset{1}{\Psi}{}_i^{+} &= c_2 \overset{2}{\Psi}{}_i^{-} + c_1 \overset{1}{\Psi}{}_i^{-} \\
c_2 \overset{2}{\Psi}{}_i^{+} - c_1 \overset{1}{\Psi}{}_i^{+} &= s_2 \overset{2}{\Psi}{}_i^{-} - s_1 \overset{1}{\Psi}{}_i^{-}
\end{aligned}
\tag{4.15}
$$

where s_α and c_α are the operators defined by

$$
\begin{aligned}
s_\alpha &= \chi_\alpha + \frac{\chi_\alpha^3}{3!} + \cdots = \sinh \chi_\alpha \\
c_\alpha &= 1 + \frac{\chi_\alpha^2}{2!} + \cdots = \cosh \chi_\alpha
\end{aligned}
\tag{4.16}
$$

It may be noted that the CC's given in Eqs.(4.11,4.14) or in Eqs.(4.15) have an invariant form with regard to the layer indices 1 and 2, i.e., they remain unchanged when the indices 1 and 2 are interchanged.

Regarding the constants p_1 and p_2 appearing in CM equations, the numerical results presented in Ref.[43] indicated that the best match between exact and approximate dispersion curves for waves propagating in isotropic elastic layered composites is obtained when $p_1 = p_2 = 0.5$. Hence, in this study, $p_1 = p_2 = 0.5$ is chosen in numerical analyses, which corresponds to taking point M in Fig. 4.2 at the midpoint of the interval $[\overset{1}{x}{}^{k}, \overset{2}{x}{}^{k}]$.

The order of CC's is determined by the number of terms kept in the expansions in Eqs.(4.12); it will be called pth order in the study when (p+1) terms are retained in the expansions. Thus, for zeroth order CC one has $\overset{\alpha}{E}{}^{\mp} = 1$, which will be exact only when the interpolating values of $\overset{\alpha}{f}{}^{\mp}$ have uniform distributions in x_2 (x) direction; which is the case, for example, for harmonic plane waves propagating parallel to layering in an infinite layered composite; otherwise, it would be approximate.

Boundary Conditions

Governing equations of CM are composed of Eqs.(4.7), (4.11) and (4.14). For a finite layered composite, these equations should be considered together with bc's to be prescribed on the boundary of the composite.

Side bc's can be obtained by writing Eqs.(4.6) for both phases of the composite; this yields, in view of idealization implied by CM: one member or combination from the pairs

$$(\overset{\alpha}{u}{}_i^{s},\ n_j \overset{\alpha}{\tau}{}_{ji}^{s}) \ ; \ (\overset{\alpha}{\theta}{}^{s},\ n_i \overset{\alpha}{q}{}_i^{s}) \qquad (s = 0 \ldots m \,; \alpha = 1,2) \qquad (4.17)$$

is specified at each point of side boundary.

On the other hand, the bc's of CM for a portion of boundary parallel to layering may be expressed in terms of FV's. For example, the top bc's for the composite body in Fig. 4.1 would be: one member or combination from the pairs

$$(\overset{\alpha}{u}{}_i^{+},\ \overset{\alpha}{\tau}{}_{2i}^{+}) \ ; \ (\overset{\alpha}{\theta}{}^{+},\ \overset{\alpha}{q}{}_2^{+}) \quad (\alpha = 1,2) \qquad (4.18)$$

be specified at the points of the midplane of the top layer. It is to be noted that the bc's in Eq.(4.18) may be expressed in terms of FV's of the approximate theory by using Eqs.(4.4).

Floquet Periodicity Condition

The proposed CM contains inherently the Floquet periodicity condition (FPC) induced by the periodic structure of the layered composite. This can be verified by writing CC's of CM in Eqs.(4.11) and (4.14) at point $x = x^k$ in Fig. 4.2:

$$
\begin{aligned}
E_1^- \overset{1}{f}{}^+(x^k) &= E_2^+ \overset{2}{f}{}^-(x^k) \\
E_2^- \overset{2}{f}{}^+(x^k) &= E_1^+ \overset{1}{f}{}^-(x^k)
\end{aligned}
\tag{4.19}
$$

or, explicitly as

$$
\begin{aligned}
e^{-p_1\Delta\partial_x} \overset{1}{f}{}^+(x^k) &= e^{p_2\Delta\partial_x} \overset{2}{f}{}^-(x^k) \\
e^{-p_2\Delta\partial_x} \overset{2}{f}{}^+(x^k) &= e^{p_1\Delta\partial_x} \overset{1}{f}{}^-(x^k)
\end{aligned}
\tag{4.20}
$$

where it is noted that $p_1 + p_2 = 1$.

Through the use of Taylor's expansion formula, one obtains, for the configuration in Fig. 4.2 (that is, for the case in which layer 2 follows layer 1),

$$
\begin{aligned}
\overset{1}{f}{}^\mp(x^k) &= e^{p_1\Delta\partial_x} \overset{1}{f}{}^\mp(\overset{1}{x}{}^k) \\
\overset{2}{f}{}^\mp(x^k) &= e^{-p_2\Delta\partial_x} \overset{2}{f}{}^\mp(\overset{2}{x}{}^k)
\end{aligned}
\tag{4.21}
$$

where, as indicated before, $\overset{\alpha}{x}{}^k$ denotes the midplane of layer α. When Eqs.(4.21) are inserted into (4.20), one gets

$$
\begin{aligned}
\overset{1}{f}{}^+(\overset{1}{x}{}^k) &= \overset{2}{f}{}^-(\overset{2}{x}{}^k) \\
\overset{2}{f}{}^+(\overset{2}{x}{}^k) &= e^{2\Delta\partial_x} \overset{1}{f}{}^-(\overset{1}{x}{}^k)
\end{aligned}
\tag{4.22}
$$

which represent, respectively, for the configuration in Fig. 4.2, the continuity condition between layers 1 and 2, and the FPC with the period $d = 2\Delta$. The usual form of FPC may be obtained from the second of Eqs.(4.22) in wave number space, as

$$
\overset{2}{f}{}^+(\overset{2}{x}{}^k) = e^{i\kappa d} \overset{1}{f}{}^-(\overset{1}{x}{}^k) \quad \text{with } d = 2\Delta \tag{4.23}
$$

where "i" is imaginary number and κ is Floquet wave number representing the wave number associated with harmonic function interpolating FV's at the points (along x axis) with increment "d". This point will be clarified further in numerical examples to be presented in Chapter 6.

Up to this point, the discussions are given for the case in which layer 2 follows layer 1. For the opposite situation, that is, when layer 1 follows layer 2, it is obvious that the physical interpretations of the equations in Eqs.(4.20) are to be reversed, that is, in this case the second of Eqs.(4.20) represents the continuity condition between layers 2 and 1, while the first gives FPC.

CHAPTER 5

CONTINUUM MODEL FOR A PERIODIC LAYERED COMPOSITE WITH A UNIT CELL CONTAINING ARBITRARY NUMBER OF LAMINAE

CM is developed for a two-phase layered composite in Chapter 4 and here this model is extended to a q-phase composite. For the discussions, we refer to Fig. 5.1 showing the unit cell of a periodic layered composite with arbitrary number ("q") of phases. The equations of CM for the layered composite under consideration are composed of two groups of equations. The first group comes from the equations of the approximate plate theory written for each lamina of the composite. Smoothed (continuum) form of these equations are given by Eq.(4.7), where the number of phases should be taken now as "q", that is, by

$$(\overset{\gamma}{M}{}_i^n,\ \overset{\gamma}{C}{}_{ij}^n,\ \overset{\gamma}{E}{}^n,\ \overset{\gamma}{T}{}_i^n,\ \overset{\gamma}{Z}{}_i^{\mp},\ \overset{\gamma}{P}{}^{\mp}) = 0 \qquad (\gamma = 1\ldots q\ ;\ n = 0\ldots m) \tag{5.1}$$

The second comes from continuity conditions (CC). To obtain the smoothed form of CC's, they are written in discrete form first, which are (see Fig. 5.1):

$$\begin{aligned} &\overset{\gamma}{f}{}^{+}(\overset{\gamma}{x}{}^{k}) = \overset{\gamma+1}{f}{}^{-}(\overset{\gamma+1}{x}{}^{k}) \quad , \quad \gamma = 1\ldots(q-1) \\ &\overset{q}{f}{}^{+}(\overset{q}{x}{}^{k}) = \overset{1}{f}{}^{-}(\overset{1}{x}{}^{k+1}) \end{aligned} \tag{5.2}$$

where $\overset{\gamma}{f}{}^{\mp}$ stands for a typical member of $(u_i^{\mp}, \tau_{2i}^{\mp}, \theta^{\mp}, q_2^{\mp})$ for γ-phase and $\overset{\gamma}{x}{}^{k}$ denotes x_2 coordinate of midplane of layer γ of the k^{th} unit cell. It is to be noted that the two sides of Eqs.(5.2) are defined at different points; to use them in CM, their left and right sides should be expressed at common points. These common points are marked in Fig. 5.1 as M_γ together with their x_2 coordinates as $\overset{\gamma}{x}{}_M^{k}$ $(\gamma = 1\ldots q)$. It may be noted that M_γ $(\gamma = 1\ldots(q-1))$ represent, respectively,

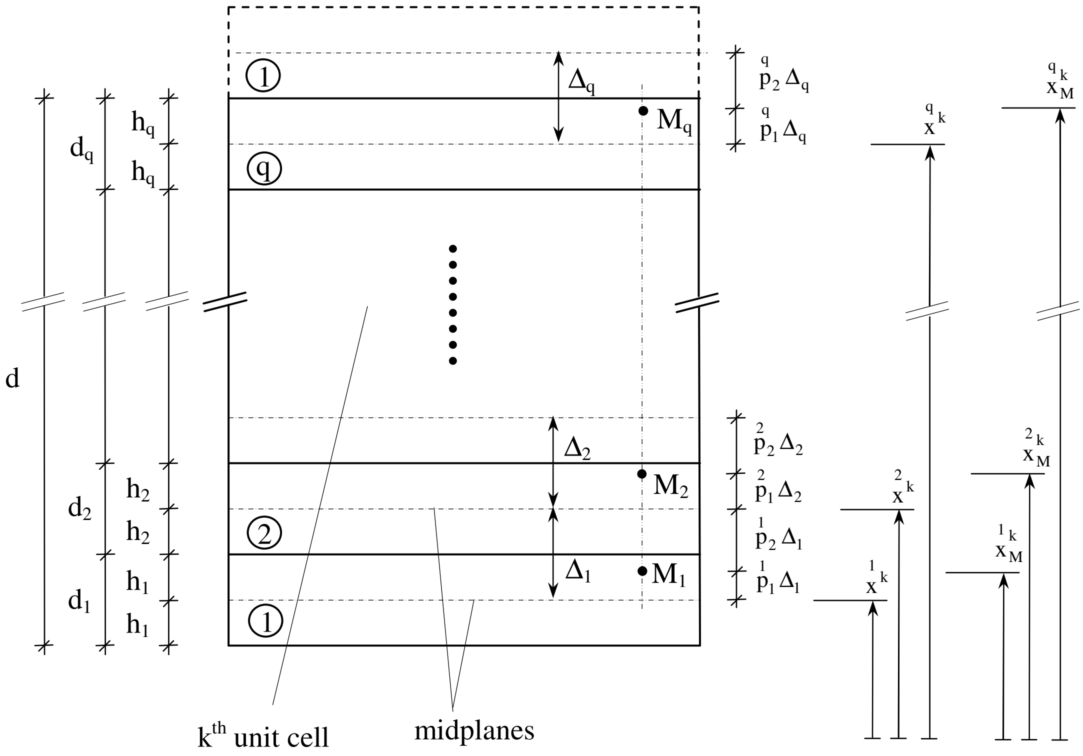


Figure 5.1 Typical unit cell of a periodic layered composite with arbitrary number of phases

the common points for CC's with $\gamma = 1 \ldots (q-1)$ in the first of Eqs.(5.2) and M_q is that for the CC in the second. From the figure, it is clear that for the x_2 coordinates of midplanes of laminae, one can write

$$\overset{\gamma}{x}{}^{k} = \overset{\gamma}{x}{}^{k}_{M} - \overset{\gamma}{p}_1 \Delta_\gamma \quad ; \quad \overset{\gamma+1}{x}{}^{k} = \overset{\gamma}{x}{}^{k}_{M} + \overset{\gamma}{p}_2 \Delta_\gamma \qquad \text{for } \gamma = 1 \ldots (q-1)$$
$$\overset{q}{x}{}^{k} = \overset{q}{x}{}^{k}_{M} - \overset{q}{p}_1 \Delta_q \quad ; \quad \overset{1}{x}{}^{k+1} = \overset{q}{x}{}^{k}_{M} + \overset{q}{p}_2 \Delta_q \tag{5.3}$$

where

$$\Delta_\gamma = h_\gamma + h_{\gamma+1} \quad \text{for } \gamma = 1 \ldots (q-1) \; ; \; \Delta_q = h_q + h_1 \tag{5.4}$$

and p_α^γ $(\alpha = 1,2 \; ; \; \gamma = 1 \ldots q)$ are interpolation factors satisfying

$$\overset{\gamma}{p}_1 + \overset{\gamma}{p}_2 = 1 \ , \quad (\gamma = 1 \ldots q) \tag{5.5}$$

(for geometric descriptions of Δ_γ and p_α^γ, see Fig. 5.1).

When Eqs.(5.3) are inserted in (5.2) and the variables appearing in it are expanded by Taylor's formula about $x = \overset{\gamma}{x}{}_M^k$ $(\gamma = 1, \ldots, q)$, one obtains the smoothed form of CC's as

$$\begin{aligned} \overset{\gamma}{E}{}_1^{-} \, \overset{\gamma}{f}{}^{+} &= \overset{\gamma}{E}{}_2^{+} \, \overset{\gamma+1}{f}{}^{-} \quad , \ \gamma = 1 \ldots (q-1) \\ \overset{q}{E}{}_1^{-} \, \overset{q}{f}{}^{+} &= \overset{q}{E}{}_2^{+} \, \overset{1}{f}{}^{-} \end{aligned} \tag{5.6}$$

where $\overset{\gamma}{E}{}_\alpha^{\mp}$ are operators defined by

$$\overset{\gamma}{E}{}_\alpha^{\mp} = \exp(\mp \overset{\gamma}{p}_\alpha \Delta_\gamma \partial_x) \qquad (\alpha = 1,2 \ ; \ \gamma = 1 \ldots q) \tag{5.7}$$

Eqs.(5.6) constitute the CC's to be used in CM and hold, in view of the idealization by SO, for any "x".

It is very easy to show that the first of Eqs.(5.6) represent, for the configuration of unit cell in Fig. 5.1, the CC's at interfaces following the layers "1" up to (q-1), while the second of Eqs.(5.6) yields the Floquet periodicity condition (FPC):

$$\overset{q}{f}{}^{+}(\overset{q}{x}{}^{k}) = e^{d\partial_x} \overset{1}{f}{}^{-}(\overset{1}{x}{}^{k}) \tag{5.8}$$

or, in wave number space,

$$\overset{q}{f}{}^{+}(\overset{q}{x}{}^{k}) = e^{i\kappa d} \overset{1}{f}{}^{-}(\overset{1}{x}{}^{k}) \tag{5.9}$$

where "d" is the period of periodic layered composite (see Fig. 5.1).

It is obvious that if the configuration of unit cell is taken as [2,3,…,q,1], then, Eq.$(5.6)_1$ with γ = 2…(q-1) and Eq.$(5.6)_2$ would represent the CC's at interfaces while Eq.$(5.6)_1$ with γ = 1 would give FPC.

In view of discussions given above, it is clear that the governing equations of a periodic q-phase layered composite are given by Eqs.(5.1) and (5.6). These equations should be considered together with the conditions to be prescribed on the boundary of the composite and at initial time.

CHAPTER 6

ASSESSMENT OF CONTINUUM MODEL (CM)

In Chapter 4, two different models were proposed for periodic layered composites: discrete model (DM) and continuum model (CM). The use of DM in the analysis requires writing the governing equations in each layer and combining them using CC's. But, this kind of procedure involves lengthy computations and appears to be of no practical use, especially when the numbers of the layers making up the composite is large. Therefore, for the assessment only the CM will be considered.

For the assessment of CM, the use of spectra of harmonic plane waves propagating in an infinite layered composite is chosen. This selection is done since a dynamic model is described and characterized completely by its spectrum; thus, the match of spectrum with the exact may be used as a criterion for the validity of the model. This criterion is checked at the end of this chapter by considering transient dynamic behavior of a composite slab induced by waves propagating perpendicular to layering.

6.1 Spectral Assessment

The exact spectra for harmonic waves propagating in periodic layered composites are studied extensively in literature, which is reviewed in Chapter 1. The establishment of these spectra involve writing elasticity equations for each lamina, considering continuity conditions at interfaces and imposing Floquet periodicity conditions.

On the other hand, the establishment of the approximate spectra predicted by CM is very simple and straightforward. Since CM is a homogeneous model and contains inherently the Floquet periodicity conditions, the determination of approximate dispersion relation (for harmonic waves with propagation direction parallel to x_1x_2-plane) involves the insertion of

$$A\,e^{i(k\,x_1+\kappa x_2-\omega t)} \tag{6.1}$$

formwise for the variables of CM and imposing that the equations possess a nontrivial solution. In Eq.(6.1), A: amplitude, ω: angular frequency, k: wave number in x_1-direction, κ: Floquet wave number (FWN) in x_2 direction. For a real FWN κ, one can write

$$\kappa = k\tan(\phi) \tag{6.2}$$

where ϕ is inclination angle of wave from x_1-axis and each ϕ gives a different value of FWN.

In what follows, the dispersion relations of both approximate and exact theory are constructed for waves propagating perpendicular, parallel and obliquely to layering and the dispersion curves obtained from these relations are compared. The comparison involves also the displacement mode shapes at some selected points on dispersion curves. In the analyses, it is assumed that the layered composite is made of generally orthotropic laminae and thermal effects are neglected.

6.1.1 Approximate dispersion relations

They will be presented for zeroth, first and second order theories.

Shear waves propagating perpendicular to layering

For such waves, propagation is in the direction normal to layering and wave motion is parallel to layering. Due to their highly dispersive characteristics, the assessment of CM for these waves is crucial. To derive the dispersion relations, a trial solution of the form (which is obtained by putting k=0 in Eq.(6.1))

$$Ae^{i(\kappa x_2-\omega t)} \tag{6.3}$$

is assumed for the unknown field variables, where the FWN κ, if real, corresponds to the wave number of harmonic wave interpolating the actual distribution of a variable at discrete points along $x_2(x)$ axis with the increment “d”, “d” being the period of the layered composite in $x_2(x)$ direction. This point will be clearer after presenting numerical results in a later section of this chapter.

The nonzero field variables of CM for the waves under consideration would be

$$\overset{\alpha}{u}{}_1^{n}\,,\ \overset{\alpha}{u}{}_3^{n} \qquad (\, n = 0\ldots m,\ \ m = 0,1,2)$$

and

$$\overset{\alpha}{S}{}_1^{\mp}\,,\ \overset{\alpha}{S}{}_3^{\mp} \quad ; \quad \overset{\alpha}{R}{}_1^{\mp}\,,\ \overset{\alpha}{R}{}_3^{\mp}$$

which are functions of x_2 and time "t" only.

In view of these conditions, the governing equations of CM would reduce to

equations of motion:

$$\overset{\alpha}{R}{}_1^{n} - \overset{\alpha}{\bar{\tau}}{}_{21}^{n} = \rho_\alpha \,\overset{\alpha}{\ddot{u}}{}_1^{n} \quad (\, n = 0\ldots m,\ \ m = 0,1,2)$$

$$\overset{\alpha}{R}{}_3^{n} - \overset{\alpha}{\bar{\tau}}{}_{23}^{n} = \rho_\alpha \,\overset{\alpha}{\ddot{u}}{}_3^{n} \tag{6.4}$$

where

$$\overset{\alpha}{F}{}_i^{n} = \frac{\overset{\alpha}{F}{}_1^{-}}{2h_\alpha} \quad \text{for even } n$$

$$\overset{\alpha}{F}{}_i^{n} = \frac{\overset{\alpha}{F}{}_1^{+}}{2h_\alpha} \quad \text{for odd } n \qquad \text{with } \overset{\alpha}{F}{}_i^{n} = (\overset{\alpha}{S}{}_i^{n}, \overset{\alpha}{R}{}_i^{n}) \quad (i = 1...3)$$

and

$$\overset{\alpha}{\bar{f}}{}^{n} = \frac{1}{h_\alpha}\sum_{j=0}^{m} c_{nj}\,\overset{\alpha}{f}{}^{j} \qquad (\overset{\alpha}{f} = \overset{\alpha}{u}_i\,,\ \overset{\alpha}{\tau}_{2i}\,)$$

constitutive equations:

$$\overset{\alpha}{\tau}{}_{21}^{0} = \frac{1}{2h_\alpha}\overset{\alpha}{C}_{66}\,\overset{\alpha}{S}{}_1^{-} + \frac{1}{2h_\alpha}\overset{\alpha}{C}_{46}\,\overset{\alpha}{S}{}_3^{-}$$

$$\overset{\alpha}{\tau}{}_{21}^{1} = \frac{1}{2h_\alpha}\overset{\alpha}{C}_{66}\,\overset{\alpha}{S}{}_1^{+} + \frac{1}{2h_\alpha}\overset{\alpha}{C}_{46}\,\overset{\alpha}{S}{}_3^{+} - \frac{1}{2h_\alpha}\overset{\alpha}{C}_{66}\,\overset{\alpha}{u}{}_1^{0} - \frac{1}{2h_\alpha}\overset{\alpha}{C}_{46}\,\overset{\alpha}{u}{}_3^{0}$$

$$\overset{\alpha}{\tau}{}_{23}^{0} = \frac{1}{2h_\alpha}\overset{\alpha}{C}_{46}\,\overset{\alpha}{S}{}_1^{-} + \frac{1}{2h_\alpha}\overset{\alpha}{C}_{44}\,\overset{\alpha}{S}{}_3^{-}$$

$$\overset{\alpha}{\tau}{}_{23}^{1} = \frac{1}{2h_\alpha}\overset{\alpha}{C}_{46}\,\overset{\alpha}{S}{}_1^{+} + \frac{1}{2h_\alpha}\overset{\alpha}{C}_{44}\,\overset{\alpha}{S}{}_3^{+} - \frac{1}{2h_\alpha}\overset{\alpha}{C}_{46}\,\overset{\alpha}{u}{}_1^{0} - \frac{1}{2h_\alpha}\overset{\alpha}{C}_{44}\,\overset{\alpha}{u}{}_3^{0} \tag{6.5}$$

constitutive equations for FV's:

0th order

$$\overset{\alpha}{R}{}_1^{+} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{46}\,(\gamma^- \overset{\alpha}{S}{}_3^{-})\ \right]$$

$$\overset{\alpha}{R}{}_3^{+} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{44}\,(\gamma^- \overset{\alpha}{S}{}_3^{-})\ \right]$$

$$\overset{\alpha}{R}{}_1^{-} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_0\, \overset{\alpha}{u}{}_1^{0} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{46}\,(\gamma_0\, \overset{\alpha}{u}{}_3^{0} + \gamma^+ \overset{\alpha}{S}{}_3^{+})\ \right]$$

$$\overset{\alpha}{R}{}_3^{-} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_0\, \overset{\alpha}{u}{}_1^{0} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{44}\,(\gamma_0\, \overset{\alpha}{u}{}_3^{0} + \gamma^+ \overset{\alpha}{S}{}_3^{+})\ \right] \tag{6.6}$$

1st order

$$\overset{\alpha}{R}{}_1^{+} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_1\, \overset{\alpha}{u}{}_1^{1} + \gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{46}\,(\gamma_1\, \overset{\alpha}{u}{}_3^{1} + \gamma^- \overset{\alpha}{S}{}_3^{-})\ \right]$$

$$\overset{\alpha}{R}{}_3^{+} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_1\, \overset{\alpha}{u}{}_1^{1} + \gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{44}\,(\gamma_1\, \overset{\alpha}{u}{}_3^{1} + \gamma^- \overset{\alpha}{S}{}_3^{-})\ \right]$$

$$\overset{\alpha}{R}{}_1^{-} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_0\, \overset{\alpha}{u}{}_1^{0} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{46}\,(\gamma_0\, \overset{\alpha}{u}{}_3^{0} + \gamma^+ \overset{\alpha}{S}{}_3^{+})\ \right]$$

$$\overset{\alpha}{R}{}_3^{-} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_0\, \overset{\alpha}{u}{}_1^{0} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{44}\,(\gamma_0\, \overset{\alpha}{u}{}_3^{0} + \gamma^+ \overset{\alpha}{S}{}_3^{+})\ \right] \tag{6.7}$$

2nd order

$$\overset{\alpha}{R}{}_1^{+} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_1\, \overset{\alpha}{u}{}_1^{1} + \gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{46}\,(\gamma_1\, \overset{\alpha}{u}{}_3^{1} + \gamma^- \overset{\alpha}{S}{}_3^{-})\ \right]$$

$$\overset{\alpha}{R}{}_3^{+} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_1\, \overset{\alpha}{u}{}_1^{1} + \gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{44}\,(\gamma_1\, \overset{\alpha}{u}{}_3^{1} + \gamma^- \overset{\alpha}{S}{}_3^{-})\ \right]$$

$$\overset{\alpha}{R}{}_1^{-} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_0\, \overset{\alpha}{u}{}_1^{0} + \gamma_2\, \overset{\alpha}{u}{}_1^{2} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{46}\,(\gamma_0\, \overset{\alpha}{u}{}_3^{0} + \gamma_2\, \overset{\alpha}{u}{}_3^{2} + \gamma^+ \overset{\alpha}{S}{}_3^{+})\ \right]$$

$$\overset{\alpha}{R}{}_3^{-} = \frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_0\, \overset{\alpha}{u}{}_1^{0} + \gamma_2\, \overset{\alpha}{u}{}_1^{2} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{44}\,(\gamma_0\, \overset{\alpha}{u}{}_3^{0} + \gamma_2\, \overset{\alpha}{u}{}_3^{2} + \gamma^+ \overset{\alpha}{S}{}_3^{+})\ \right] \tag{6.8}$$

continuity conditions: (with $p_1 = p_2 = 0.5$)

<u>1st order</u>

$$\frac{\Delta}{2}\partial_x \overset{2+}{S}_1 + \frac{\Delta}{2}\partial_x \overset{1+}{S}_1 - \overset{2-}{S}_1 - \overset{1-}{S}_1 = 0$$

$$\frac{\Delta}{2}\partial_x \overset{2+}{S}_3 + \frac{\Delta}{2}\partial_x \overset{1+}{S}_3 - \overset{2-}{S}_3 - \overset{1-}{S}_3 = 0$$

$$\overset{2+}{S}_1 - \overset{1+}{S}_1 - \frac{\Delta}{2}\partial_x \overset{2-}{S}_1 + \frac{\Delta}{2}\partial_x \overset{1-}{S}_1 = 0$$

$$\overset{2+}{S}_3 - \overset{1+}{S}_3 - \frac{\Delta}{2}\partial_x \overset{2-}{S}_3 + \frac{\Delta}{2}\partial_x \overset{1-}{S}_3 = 0$$

$$\frac{\Delta}{2}\partial_x \overset{2+}{R}_1 + \frac{\Delta}{2}\partial_x \overset{1+}{R}_1 - \overset{2-}{R}_1 - \overset{1-}{R}_1 = 0$$

$$\frac{\Delta}{2}\partial_x \overset{2+}{R}_3 + \frac{\Delta}{2}\partial_x \overset{1+}{R}_3 - \overset{2-}{R}_3 - \overset{1-}{R}_3 = 0$$

$$\overset{2+}{R}_1 - \overset{1+}{R}_1 - \frac{\Delta}{2}\partial_x \overset{2-}{R}_1 + \frac{\Delta}{2}\partial_x \overset{1-}{R}_1 = 0$$

$$\overset{2+}{R}_3 - \overset{1+}{R}_3 - \frac{\Delta}{2}\partial_x \overset{2-}{R}_3 + \frac{\Delta}{2}\partial_x \overset{1-}{R}_3 = 0 \qquad (6.9)$$

<u>2nd order</u>

$$\frac{\Delta}{2}\partial_x \overset{2+}{S}_1 + \frac{\Delta}{2}\partial_x \overset{1+}{S}_1 - \overset{2-}{S}_1 - \overset{1-}{S}_1 - \frac{\Delta^2}{8}\partial_{xx} \overset{2-}{S}_1 - \frac{\Delta^2}{8}\partial_{xx} \overset{1-}{S}_1 = 0$$

$$\frac{\Delta}{2}\partial_x \overset{2+}{S}_3 + \frac{\Delta}{2}\partial_x \overset{1+}{S}_3 - \overset{2-}{S}_3 - \overset{1-}{S}_3 - \frac{\Delta^2}{8}\partial_{xx} \overset{2-}{S}_3 - \frac{\Delta^2}{8}\partial_{xx} \overset{1-}{S}_3 = 0$$

$$\overset{2+}{S}_1 - \overset{1+}{S}_1 + \frac{\Delta^2}{8}\partial_{xx} \overset{2+}{S}_1 - \frac{\Delta^2}{8}\partial_{xx} \overset{1+}{S}_1 - \frac{\Delta}{2}\partial_x \overset{2-}{S}_1 + \frac{\Delta}{2}\partial_x \overset{1-}{S}_1 = 0$$

$$\overset{2+}{S}_3 - \overset{1+}{S}_3 + \frac{\Delta^2}{8}\partial_{xx} \overset{2+}{S}_3 - \frac{\Delta^2}{8}\partial_{xx} \overset{1+}{S}_3 - \frac{\Delta}{2}\partial_x \overset{2-}{S}_3 + \frac{\Delta}{2}\partial_x \overset{1-}{S}_3 = 0$$

$$\frac{\Delta}{2}\partial_x \overset{2+}{R}_1 + \frac{\Delta}{2}\partial_x \overset{1+}{R}_1 - \overset{2-}{R}_1 - \overset{1-}{R}_1 - \frac{\Delta^2}{8}\partial_{xx} \overset{2-}{R}_1 - \frac{\Delta^2}{8}\partial_{xx} \overset{1-}{R}_1 = 0$$

$$\frac{\Delta}{2}\partial_x \overset{2+}{R}_3 + \frac{\Delta}{2}\partial_x \overset{1+}{R}_3 - \overset{2-}{R}_3 - \overset{1-}{R}_3 - \frac{\Delta^2}{8}\partial_{xx} \overset{2-}{R}_3 - \frac{\Delta^2}{8}\partial_{xx} \overset{1-}{R}_3 = 0$$

$$\overset{2}{R}{}_1^{+} - \overset{1}{R}{}_1^{+} + \frac{\Delta^2}{8}\partial_{xx}\overset{2}{R}{}_1^{+} - \frac{\Delta^2}{8}\partial_{xx}\overset{1}{R}{}_1^{+} - \frac{\Delta}{2}\partial_x\overset{2}{R}{}_1^{-} + \frac{\Delta}{2}\partial_x\overset{1}{R}{}_1^{-} = 0$$

$$\overset{2}{R}{}_3^{+} - \overset{1}{R}{}_3^{+} + \frac{\Delta^2}{8}\partial_{xx}\overset{2}{R}{}_3^{+} - \frac{\Delta^2}{8}\partial_{xx}\overset{1}{R}{}_3^{+} - \frac{\Delta}{2}\partial_x\overset{2}{R}{}_3^{-} + \frac{\Delta}{2}\partial_x\overset{1}{R}{}_3^{-} = 0 \qquad (6.10)$$

Substituting trial solutions of the form in Eq.(6.3) into the equations of motion, constitutive equations for FV's and continuity conditions, a system of homogeneous algebraic equations (SHAE) with the coefficients dependent on ω and κ is obtained. Equating the determinant of the coefficient matrix of SHAE to zero, the characteristic equation (dispersion relation), relating ω and κ, is found. Each pair (ω,κ) satisfying the dispersion relation gives a plane wave solution and corresponds to a propagation mode. The dispersion equation for zeroth order theory and first order CC is given in Appendix C. To facilitate the discussions involving the displacement mode shapes, the SHAE will be written symbolically as

$$\mathbf{M}\,\mathbf{a} = \mathbf{0}$$

where the column vector **a** contains field variables of CM in frequency-wave number space; thus, the frequency equation would be

$$\det(\mathbf{M}) = 0$$

<u>Axial waves propagating parallel to layering</u>

For this type of waves, which are on the average longitudinal, the displacements $\overset{\alpha}{u}_1$ and $\overset{\alpha}{u}_3$ are symmetric and $\overset{\alpha}{u}_2$ are antisymmetric with respect to the midplanes of the layers. The nonzero components of generalized displacements and FV's are

$$\overset{\alpha}{u}{}_1^{n},\ \overset{\alpha}{u}{}_3^{n} \quad \text{for even n}$$

$$\overset{\alpha}{u}{}_2^{n} \qquad \text{for odd n} \qquad (\, n = 0\ldots m,\ \ m = 0,1,2)$$

and

$$\overset{\alpha}{S}{}_1^{+},\ \overset{\alpha}{S}{}_2^{-},\ \overset{\alpha}{S}{}_3^{+} \quad ; \quad \overset{\alpha}{R}{}_1^{-},\ \overset{\alpha}{R}{}_2^{+},\ \overset{\alpha}{R}{}_3^{-}$$

For these unknown variables assume a trial solution of the form (which is obtained by putting $\kappa = 0$ in Eq.(6.1))

$$Ae^{i(kx_1-\omega t)} \tag{6.11}$$

Thus, the variables are functions of x_1 and time "t" only, hence, $\partial_3(\cdot) = 0$. The governing equations of CM reduce to, for this case,

equations of motion:

$$\partial_1 \overset{\alpha}{\tau}{}^{n}_{11} + \overset{\alpha}{R}{}^{n}_{1} - \overset{\alpha}{\bar{\tau}}{}^{n}_{21} = \rho_\alpha \overset{\alpha}{\ddot{u}}{}^{n}_{1}$$

$$\partial_1 \overset{\alpha}{\tau}{}^{n}_{12} + \overset{\alpha}{R}{}^{n}_{2} - \overset{\alpha}{\bar{\tau}}{}^{n}_{22} = \rho_\alpha \overset{\alpha}{\ddot{u}}{}^{n}_{2}$$

$$\partial_1 \overset{\alpha}{\tau}{}^{n}_{13} + \overset{\alpha}{R}{}^{n}_{3} - \overset{\alpha}{\bar{\tau}}{}^{n}_{23} = \rho_\alpha \overset{\alpha}{\ddot{u}}{}^{n}_{3} \quad (\, n = 0 \ldots m,\ m = 1{,}2) \tag{6.12}$$

constitutive equations:

$$\overset{\alpha}{\tau}{}^{0}_{11} = \overset{\alpha}{C}_{11}\,\partial_1 \overset{\alpha}{u}{}^{0}_{1} + \overset{\alpha}{C}_{15}\,\partial_1 \overset{\alpha}{u}{}^{0}_{3} + \overset{\alpha}{C}_{12}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{-}_{2}\Big)$$

$$\overset{\alpha}{\tau}{}^{2}_{11} = \overset{\alpha}{C}_{11}\,\partial_1 \overset{\alpha}{u}{}^{2}_{1} + \overset{\alpha}{C}_{15}\,\partial_1 \overset{\alpha}{u}{}^{2}_{3} + \overset{\alpha}{C}_{12}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{-}_{2} - \frac{3}{h_\alpha}\overset{\alpha}{u}{}^{1}_{2}\Big)$$

$$\overset{\alpha}{\tau}{}^{1}_{12} = \overset{\alpha}{C}_{66}\,\partial_1 \overset{\alpha}{u}{}^{1}_{2} + \overset{\alpha}{C}_{66}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{+}_{1} - \frac{1}{h_\alpha}\overset{\alpha}{u}{}^{0}_{1}\Big) + \overset{\alpha}{C}_{46}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{+}_{3} - \frac{1}{h_\alpha}\overset{\alpha}{u}{}^{0}_{3}\Big)$$

$$\overset{\alpha}{\tau}{}^{0}_{22} = \overset{\alpha}{C}_{12}\,\partial_1 \overset{\alpha}{u}{}^{0}_{1} + \overset{\alpha}{C}_{25}\,\partial_1 \overset{\alpha}{u}{}^{0}_{3} + \overset{\alpha}{C}_{22}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{-}_{2}\Big)$$

$$\overset{\alpha}{\tau}{}^{0}_{13} = \overset{\alpha}{C}_{15}\,\partial_1 \overset{\alpha}{u}{}^{0}_{1} + \overset{\alpha}{C}_{55}\,\partial_1 \overset{\alpha}{u}{}^{0}_{3} + \overset{\alpha}{C}_{25}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{-}_{2}\Big)$$

$$\overset{\alpha}{\tau}{}^{2}_{13} = \overset{\alpha}{C}_{15}\,\partial_1 \overset{\alpha}{u}{}^{2}_{1} + \overset{\alpha}{C}_{55}\,\partial_1 \overset{\alpha}{u}{}^{2}_{3} + \overset{\alpha}{C}_{25}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{-}_{2} - \frac{3}{h_\alpha}\overset{\alpha}{u}{}^{1}_{2}\Big)$$

$$\overset{\alpha}{\tau}{}^{1}_{23} = \overset{\alpha}{C}_{46}\,\partial_1 \overset{\alpha}{u}{}^{1}_{2} + \overset{\alpha}{C}_{46}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{+}_{1} - \frac{1}{h_\alpha}\overset{\alpha}{u}{}^{0}_{1}\Big) + \overset{\alpha}{C}_{44}\Big(\frac{1}{2h_\alpha}\overset{\alpha}{S}{}^{+}_{3} - \frac{1}{h_\alpha}\overset{\alpha}{u}{}^{0}_{3}\Big) \tag{6.13}$$

constitutive equations for FV's:

<u>0^{th} order</u>

$$\overset{\alpha}{R}{}_1^{-}=\overset{\alpha}{C}_{66}\,\partial_1\overset{\alpha}{S}{}_2^{-}+\frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_0\,\overset{\alpha}{u}{}_1^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_1^{+})+\overset{\alpha}{C}_{46}\,(\gamma_0\,\overset{\alpha}{u}{}_3^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_3^{+})\ \right]$$

$$\overset{\alpha}{R}{}_2^{+}=\overset{\alpha}{C}_{12}\,\partial_1\overset{\alpha}{S}{}_1^{+}+\overset{\alpha}{C}_{25}\,\partial_1\overset{\alpha}{S}{}_3^{+}+\frac{2}{h_\alpha}\overset{\alpha}{C}_{22}\,(\gamma^{-}\,\overset{\alpha}{S}{}_2^{-})$$

$$\overset{\alpha}{R}{}_3^{-}=\overset{\alpha}{C}_{46}\,\partial_1\overset{\alpha}{S}{}_2^{-}+\frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_0\,\overset{\alpha}{u}{}_1^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_1^{+})+\overset{\alpha}{C}_{44}\,(\gamma_0\,\overset{\alpha}{u}{}_3^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_3^{+})\ \right] \tag{6.14}$$

<u>1^{st} order</u>

$$\overset{\alpha}{R}{}_1^{-}=\overset{\alpha}{C}_{66}\,\partial_1\overset{\alpha}{S}{}_2^{-}+\frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_0\,\overset{\alpha}{u}{}_1^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_1^{+})+\overset{\alpha}{C}_{46}\,(\gamma_0\,\overset{\alpha}{u}{}_3^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_3^{+})\ \right]$$

$$\overset{\alpha}{R}{}_2^{+}=\overset{\alpha}{C}_{12}\,\partial_1\overset{\alpha}{S}{}_1^{+}+\overset{\alpha}{C}_{25}\,\partial_1\overset{\alpha}{S}{}_3^{+}+\frac{2}{h_\alpha}\overset{\alpha}{C}_{22}\,(\gamma_1\,\overset{\alpha}{u}{}_2^{1}+\gamma^{-}\,\overset{\alpha}{S}{}_2^{-})$$

$$\overset{\alpha}{R}{}_3^{-}=\overset{\alpha}{C}_{46}\,\partial_1\overset{\alpha}{S}{}_2^{-}+\frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_0\,\overset{\alpha}{u}{}_1^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_1^{+})+\overset{\alpha}{C}_{44}\,(\gamma_0\,\overset{\alpha}{u}{}_3^{0}+\gamma^{+}\,\overset{\alpha}{S}{}_3^{+})\ \right] \tag{6.15}$$

<u>2^{nd} order</u>

$$\overset{\alpha}{R}{}_1^{-}=\overset{\alpha}{C}_{66}\,\partial_1\overset{\alpha}{S}{}_2^{-}+\frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{66}\,(\gamma_0\,\overset{\alpha}{u}{}_1^{0}+\gamma_2\,\overset{\alpha}{u}{}_1^{2}+\gamma^{+}\,\overset{\alpha}{S}{}_1^{+})+\overset{\alpha}{C}_{46}\,(\gamma_0\,\overset{\alpha}{u}{}_3^{0}+\gamma_2\,\overset{\alpha}{u}{}_3^{2}+\gamma^{+}\,\overset{\alpha}{S}{}_3^{+})\ \right]$$

$$\overset{\alpha}{R}{}_2^{+}=\overset{\alpha}{C}_{12}\,\partial_1\overset{\alpha}{S}{}_1^{+}+\overset{\alpha}{C}_{25}\,\partial_1\overset{\alpha}{S}{}_3^{+}+\frac{2}{h_\alpha}\overset{\alpha}{C}_{22}\,(\gamma_1\,\overset{\alpha}{u}{}_2^{1}+\gamma^{-}\,\overset{\alpha}{S}{}_2^{-})$$

$$\overset{\alpha}{R}{}_3^{-}=\overset{\alpha}{C}_{46}\,\partial_1\overset{\alpha}{S}{}_2^{-}+\frac{2}{h_\alpha}\left[\ \overset{\alpha}{C}_{46}\,(\gamma_0\,\overset{\alpha}{u}{}_1^{0}+\gamma_2\,\overset{\alpha}{u}{}_1^{2}+\gamma^{+}\,\overset{\alpha}{S}{}_1^{+})+\overset{\alpha}{C}_{44}\,(\gamma_0\,\overset{\alpha}{u}{}_3^{0}+\gamma_2\,\overset{\alpha}{u}{}_3^{2}+\gamma^{+}\,\overset{\alpha}{S}{}_3^{+})\ \right]$$

(6.16)

continuity conditions:

Continuity conditions assume a simpler form for the waves under consideration. For such waves the field variables become independent of x_2 and consequently the terms which involve derivatives of x_2 vanish. Thus, the operators in $E_\alpha^{\mp}$ in Eq.(4.12) reduce to

$$E_\alpha^{-}=E_\alpha^{+}=1$$

With this form of the operators, the continuity conditions, which are exact, become

$$\overset{2}{S}{}_1^{+} - \overset{1}{S}{}_1^{+} = 0$$

$$\overset{2}{S}{}_2^{-} + \overset{1}{S}{}_2^{-} = 0$$

$$\overset{2}{S}{}_3^{+} - \overset{1}{S}{}_3^{+} = 0$$

$$\overset{2}{R}{}_1^{-} + \overset{1}{R}{}_1^{-} = 0$$

$$\overset{2}{R}{}_2^{+} - \overset{1}{R}{}_2^{+} = 0$$

$$\overset{2}{R}{}_3^{-} + \overset{1}{R}{}_3^{-} = 0 \qquad (6.17)$$

Following the procedure outlined in the previous section, the dispersion equation for the axial waves propagating parallel to layering can be determined through the substitution of Eq.(6.11) into (6.12-17). This equation is given in Appendix D for 1st order theory.

Waves propagating obliquely to layering

For this case, the trial solutions of the field variables of CM would have the form of Eq.(6.1), that is, of the form

$$A e^{i(k x_1 + \kappa x_2 - \omega t)}$$

Here, only the second order theory and second order CC's are considered. All of the field variables

$$\overset{\alpha}{u}{}_1^{n},\ \overset{\alpha}{u}{}_2^{n},\ \overset{\alpha}{u}{}_3^{n} \qquad (n = 0 \ldots m,\ m = 2)$$

and

$$\overset{\alpha}{S}{}_i^{\mp},\ \overset{\alpha}{R}{}_i^{\mp}$$

would be nonzero and functions of x_1, x_2 and time "t" for oblique case; hence, $\partial_3(\cdot) = 0$.

The governing equations of the CM for such waves are as follows:

equations of motion:

$$\partial_1 \overset{\alpha}{\tau}{}_{11}^{n} + \overset{\alpha}{R}{}_1^{n} - \overset{\alpha}{\bar{\tau}}{}_{21}^{n} = \rho_\alpha \, \overset{\alpha}{\ddot{u}}{}_1^{n}$$

$$\partial_1 \overset{\alpha n}{\tau}_{12} + \overset{\alpha n}{R}_2 - \overset{\alpha n}{\bar{\tau}}_{22} = \rho_\alpha \overset{\alpha n}{\ddot{u}}_2$$

$$\partial_1 \overset{\alpha n}{\tau}_{13} + \overset{\alpha n}{R}_3 - \overset{\alpha n}{\bar{\tau}}_{23} = \rho_\alpha \overset{\alpha n}{\ddot{u}}_3 \quad (\, n = 0 \ldots m,\ m = 2) \tag{6.18}$$

constitutive equations:

$$\begin{aligned}
\overset{\alpha n}{\tau}_{11} &= \overset{\alpha}{C}_{11} \partial_1 \overset{\alpha n}{u}_1 + \overset{\alpha}{C}_{15} \partial_1 \overset{\alpha n}{u}_3 + \overset{\alpha}{C}_{12} (\overset{\alpha n}{S}_2 - \overset{\alpha n}{\bar{u}}_2) \\
\overset{\alpha n}{\tau}_{12} &= \overset{\alpha}{C}_{66} \partial_1 \overset{\alpha n}{u}_2 + \overset{\alpha}{C}_{66} (\overset{\alpha n}{S}_1 - \overset{\alpha n}{\bar{u}}_1) + \overset{\alpha}{C}_{46} (\overset{\alpha n}{S}_3 - \overset{\alpha n}{\bar{u}}_3) \\
\overset{\alpha n}{\tau}_{13} &= \overset{\alpha}{C}_{15} \partial_1 \overset{\alpha n}{u}_2 + \overset{\alpha}{C}_{55} \partial_1 \overset{\alpha n}{u}_3 + \overset{\alpha}{C}_{25} (\overset{\alpha n}{S}_2 - \overset{\alpha n}{\bar{u}}_2) \\
\overset{\alpha n}{\tau}_{22} &= \overset{\alpha}{C}_{12} \partial_1 \overset{\alpha n}{u}_1 + \overset{\alpha}{C}_{25} \partial_1 \overset{\alpha n}{u}_3 + \overset{\alpha}{C}_{22} (\overset{\alpha n}{S}_2 - \overset{\alpha n}{\bar{u}}_2) \\
\overset{\alpha n}{\tau}_{23} &= \overset{\alpha}{C}_{46} \partial_1 \overset{\alpha n}{u}_2 + \overset{\alpha}{C}_{46} (\overset{\alpha n}{S}_1 - \overset{\alpha n}{\bar{u}}_1) + \overset{\alpha}{C}_{44} (\overset{\alpha n}{S}_3 - \overset{\alpha n}{\bar{u}}_3)
\end{aligned} \tag{6.19}$$

constitutive equations for FV's:

<u>2^{nd} order</u>

$$\begin{aligned}
\overset{\alpha}{R}{}_1^{+} &= \overset{\alpha}{C}_{66} \partial_1 \overset{\alpha}{S}{}_2^{+} + \frac{2}{h_\alpha} \Big[\overset{\alpha}{C}_{66} (\gamma_1 \overset{\alpha}{u}{}_1^{1} + \gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{46} (\gamma_1 \overset{\alpha}{u}{}_3^{1} + \gamma^- \overset{\alpha}{S}{}_3^{-}) \Big] \\
\overset{\alpha}{R}{}_1^{-} &= \overset{\alpha}{C}_{66} \partial_1 \overset{\alpha}{S}{}_2^{-} + \frac{2}{h_\alpha} \Big[\overset{\alpha}{C}_{66} (\gamma_0 \overset{\alpha}{u}{}_1^{0} + \gamma_2 \overset{\alpha}{u}{}_1^{2} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{46} (\gamma_0 \overset{\alpha}{u}{}_3^{0} + \gamma_2 \overset{\alpha}{u}{}_3^{2} + \gamma^+ \overset{\alpha}{S}{}_3^{+}) \Big] \\
\overset{\alpha}{R}{}_2^{+} &= \overset{\alpha}{C}_{12} \partial_1 \overset{\alpha}{S}{}_1^{+} + \overset{\alpha}{C}_{25} \partial_1 \overset{\alpha}{S}{}_3^{+} + \frac{2}{h_\alpha} \overset{\alpha}{C}_{22} (\gamma_1 \overset{\alpha}{u}{}_2^{1} + \gamma^- \overset{\alpha}{S}{}_2^{-}) \\
\overset{\alpha}{R}{}_2^{-} &= \overset{\alpha}{C}_{12} \partial_1 \overset{\alpha}{S}{}_1^{-} + \overset{\alpha}{C}_{25} \partial_1 \overset{\alpha}{S}{}_3^{-} + \frac{2}{h_\alpha} \overset{\alpha}{C}_{22} (\gamma_0 \overset{\alpha}{u}{}_2^{0} + \gamma_2 \overset{\alpha}{u}{}_2^{2} + \gamma^+ \overset{\alpha}{S}{}_2^{+}) \\
\overset{\alpha}{R}{}_3^{+} &= \overset{\alpha}{C}_{46} \partial_1 \overset{\alpha}{S}{}_2^{+} + \frac{2}{h_\alpha} \Big[\overset{\alpha}{C}_{46} (\gamma_1 \overset{\alpha}{u}{}_1^{1} + \gamma^- \overset{\alpha}{S}{}_1^{-}) + \overset{\alpha}{C}_{44} (\gamma_1 \overset{\alpha}{u}{}_3^{1} + \gamma^- \overset{\alpha}{S}{}_3^{-}) \Big] \\
\overset{\alpha}{R}{}_3^{-} &= \overset{\alpha}{C}_{46} \partial_1 \overset{\alpha}{S}{}_2^{-} + \frac{2}{h_\alpha} \Big[\overset{\alpha}{C}_{46} (\gamma_0 \overset{\alpha}{u}{}_1^{0} + \gamma_2 \overset{\alpha}{u}{}_1^{2} + \gamma^+ \overset{\alpha}{S}{}_1^{+}) + \overset{\alpha}{C}_{44} (\gamma_0 \overset{\alpha}{u}{}_3^{0} + \gamma_2 \overset{\alpha}{u}{}_3^{2} + \gamma^+ \overset{\alpha}{S}{}_3^{+}) \Big]
\end{aligned} \tag{6.20}$$

continuity conditions:

They are the same as those in Eqs.(6.10).

The substitution of Eq.(6.1) into Eqs.(6.18-20) and (6.10) yields the frequency equation for oblique case, which is given in Appendix E.

6.1.2 Exact dispersion relations

Exact solutions for the propagation of harmonic elastic waves in layered anisotropic media are already established in the literature. Exact treatments involve writing the exact field equations in each layer of the composite and satisfying the continuity conditions at the interfaces. To facilitate this analysis, the transfer matrix method, which is first introduced by Thomson [54], is employed. In this method, a system of equations for the layered media is constructed from the field equations of each layer by satisfying appropriate interface conditions. For the composite medium with periodic structure, as in our case, displacements and stresses under harmonic waves can be represented by periodic functions . This representation is implemented through the use of Floquet (or Bloch) theory (see [58,59,68]).

Here, a periodic layered composite with two alternating generally orthotropic laminae is considered. It is under the influence of plane harmonic waves with overall propagating direction in x_1x_2-plane at an angle ϕ with x_1-axis (see Fig. 6.1).

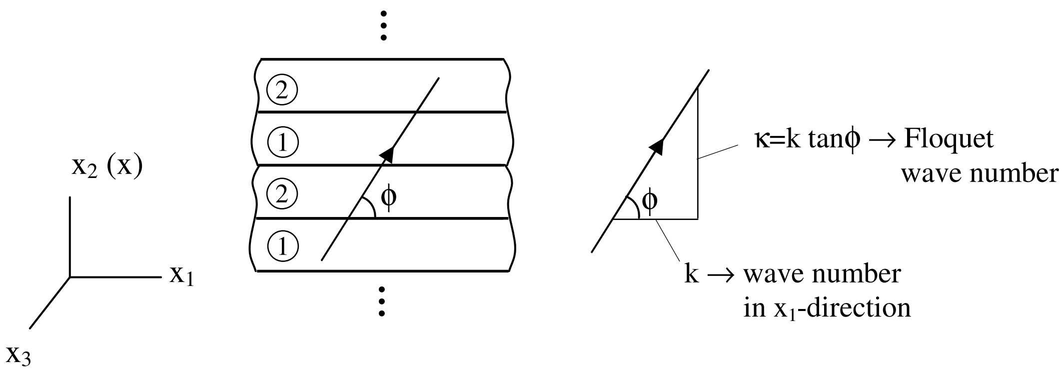


Figure 6.1 Wave propagation direction

Propagation of these waves involves reflections and refractions at the interfaces. For such waves, the solution for the displacements would be of the form

$$u_i = \hat{u}_i \, e^{i(kx_1 + \alpha x_2 - \omega t)} \tag{6.21}$$

where, $\hat{u}_i$ are amplitudes, ω is frequency and k and α are wave numbers in x_1 and x_2 directions, respectively. With this form of displacements, the wave propagation takes place in x_1x_2-plane; but, we note that, $u_3 \neq 0$ due to anisotropy, and we also note that displacements are independent of x_3, that is, $\partial_3(\cdot) = 0$.

Now, we consider a typical layer of the composite, which is taken as generally orthotropic (Fig. 6.2a). The field equations for this single layer are

stress equations of motion:

$$\partial_j \tau_{ji} = \rho \ddot{u}_i \quad \Rightarrow \quad \begin{aligned} \partial_1 \tau_{11} + \partial_2 \tau_{21} &= \rho \ddot{u}_1 \\ \partial_1 \tau_{12} + \partial_2 \tau_{22} &= \rho \ddot{u}_2 \\ \partial_1 \tau_{13} + \partial_2 \tau_{23} &= \rho \ddot{u}_3 \end{aligned} \tag{6.22}$$

constitutive equations:

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} \tag{6.23}$$

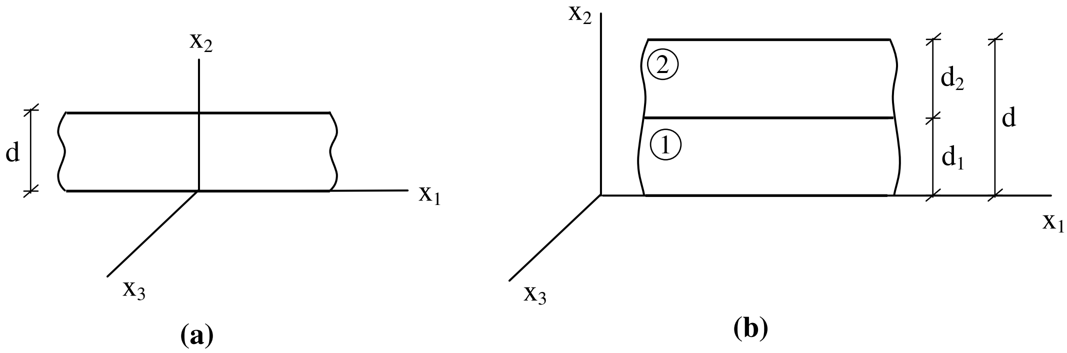


Figure 6.2 **(a)** Typical single layer **(b)** unit cell used in exact analysis

strain-displacement relations:

$$\begin{aligned}&\varepsilon_{11}=\partial_1 u_1 \quad ; \quad \varepsilon_{22}=\partial_2 u_2 \quad ; \quad \varepsilon_{33}=0\\&\varepsilon_{23}=\frac{1}{2}(\partial_2 u_3) \quad ; \quad \varepsilon_{13}=\frac{1}{2}(\partial_1 u_3) \quad ; \quad \varepsilon_{12}=\frac{1}{2}(\partial_1 u_2+\partial_2 u_1)\end{aligned} \tag{6.24}$$

When the stresses in Eq.(6.23) are substituted into Eqs.(6.22), the displacement equations of motion may be obtained as

$$\begin{aligned}&C_{11}\partial_{11}u_1+C_{12}\partial_{12}u_2+C_{15}\partial_{11}u_3+C_{46}\partial_{22}u_3+C_{66}(\partial_{12}u_2+\partial_{22}u_1)=\rho\ddot{u}_1\\&C_{46}\partial_{12}u_3+C_{66}(\partial_{11}u_2+\partial_{12}u_1)+C_{12}\partial_{12}u_1+C_{22}\partial_{22}u_2+C_{25}\partial_{12}u_3=\rho\ddot{u}_2\\&C_{15}\partial_{11}u_1+C_{25}\partial_{12}u_2+C_{55}\partial_{11}u_3+C_{44}\partial_{22}u_3+C_{46}(\partial_{12}u_2+\partial_{22}u_1)=\rho\ddot{u}_3\end{aligned} \tag{6.25}$$

Substitution of the trial solution in Eq.(6.21) into Eq.(6.25) gives

$$\underbrace{\begin{bmatrix}\rho\omega^2-C_{11}k^2-C_{66}\alpha^2 & -(C_{12}+C_{66})\alpha k & -C_{15}k^2-C_{46}\alpha^2\\ -(C_{12}+C_{66})\alpha k & \rho\omega^2-C_{66}k^2-C_{22}\alpha^2 & -(C_{25}+C_{46})\alpha k\\ -C_{15}k^2-C_{46}\alpha^2 & -(C_{25}+C_{46})\alpha k & \rho\omega^2-C_{55}k^2-C_{44}\alpha^2\end{bmatrix}}_{\mathbf{B}}\underbrace{\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\\ \hat{u}_3\end{bmatrix}}_{\hat{\mathbf{u}}}=\mathbf{0} \tag{6.26}$$

The characteristic equation would be $\det(\mathbf{B})=0$, which gives

$$c_0\alpha^6+c_1\alpha^4+c_2\alpha^2+c_3=0 \tag{6.27}$$

Here, the coefficients c_i ($i = 0\ldots3$) are function of ω and k, which are given by

$$\begin{aligned}c_0&=C_{22}C_{46}^2-C_{22}C_{44}C_{66}\\ c_1&=k^2C_{12}^2C_{44}-\rho\omega^2C_{46}^2+k^2C_{25}^2C_{66}+\rho\omega^2C_{44}C_{66}-2k^2C_{12}(C_{25}C_{46}+C_{46}^2\\&\quad-C_{44}C_{66})+C_{22}(C_{44}(\rho\omega^2-k^2C_{11})+2k^2C_{15}C_{46}+C_{66}(\rho\omega^2-k^2C_{55})\\ c_2&=-k^2\rho\omega^2C_{25}^2+k^4C_{11}C_{25}^2-\rho^2\omega^4C_{44}+k^2\rho\omega^2C_{11}C_{44}-2k^2\rho\omega^2C_{15}C_{46}\\&\quad-2k^2\rho\omega^2C_{25}C_{46}+2k^4C_{11}C_{25}C_{46}-k^2\rho\omega^2C_{46}^2+k^4C_{11}C_{46}^2+C_{12}^2(-k^2\rho\omega^2\\&\quad+k^4C_{55})+C_{22}(-\rho^2\omega^4+k^4C_{15}^2+k^2\rho\omega^2C_{55}+C_{11}(k^2\rho\omega^2-k^4C_{55}))\\&\quad-\rho^2\omega^4C_{66}-2k^4C_{15}C_{25}C_{66}+k^2\rho\omega^2C_{44}C_{66}-k^4C_{11}C_{44}C_{66}\\&\quad+k^2\rho\omega^2C_{55}C_{66}-2C_{12}(k^4C_{15}(C_{25}+C_{46})+k^2C_{66}(\rho\omega^2-k^2C_{55}))\end{aligned}$$

$$c_3 = (-\rho\omega^2 + k^2 C_{66})(k^4 C_{15}^2 + \rho\omega^2(-\rho\omega^2 + k^2 C_{55}) + C_{11}(k^2\rho\omega^2 - k^4 C_{55})) \tag{6.28}$$

The roots of Eq.(6.27) would be of the form

$$\alpha_1 = \sqrt{\lambda_1}\ ; \quad \alpha_3 = \sqrt{\lambda_2}\ ; \quad \alpha_5 = \sqrt{\lambda_3}$$

$$\alpha_2 = -\sqrt{\lambda_1}\ ; \ \alpha_4 = -\sqrt{\lambda_2}\ ; \ \alpha_6 = -\sqrt{\lambda_3}$$

where $\lambda = \alpha^2$.

Now, the normalized eigensolution of Eq.(6.26) can be obtained in the form

$$\hat{\mathbf{u}}^j = \begin{bmatrix} 1 \\ r_j \\ p_j \end{bmatrix} \qquad (j = 1 \ldots 6) \tag{6.29}$$

then, r_j and p_j are to be found from

$$\begin{bmatrix} r_j \\ p_j \end{bmatrix} = -\begin{bmatrix} B_{22}^j & B_{23}^j \\ B_{23}^j & B_{33}^j \end{bmatrix}^{-1} \begin{bmatrix} B_{21}^j \\ B_{31}^j \end{bmatrix} \tag{6.30}$$

where $B_{mn}^j = B_{mn}\Big|_{\alpha=\alpha_j}$. From Eq.(6.30), one may deduce

$$r_2 = -r_1 \quad ; \quad p_2 = p_1$$

$$r_4 = -r_3 \quad ; \quad p_4 = p_3$$

$$r_6 = -r_5 \quad ; \quad p_6 = p_5$$

The solution for **u** can be obtained by combining the solutions corresponding to each root α_j as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \sum_{j=1}^{6} A_j \begin{bmatrix} 1 \\ r_j \\ p_j \end{bmatrix} e^{i(kx_1 + \alpha_j x_2 - \omega t)}$$

$$= \left(\sum_{j=1}^{6} A_j \hat{\mathbf{u}}^j e^{i\alpha_j x_2} \right) e^{i(kx_1 - \omega t)} \tag{6.31}$$

where A_j $(j = 1 \ldots 6)$ are some constants.

To simplify the analysis, the common factor $e^{i(kx_1-\omega t)}$ is dropped in Eq.(6.31) and a new displacement variable $\tilde{\mathbf{u}}$ is introduced by

$$\tilde{\mathbf{u}} = \sum_{j=1}^{6} A_j \hat{\mathbf{u}}^j e^{i\alpha_j x_2} \tag{6.32}$$

The solutions for the stresses τ_{2i} can be obtained by inserting Eq.(6.31) into (6.23), which gives

$$\begin{bmatrix} \tau_{21} \\ \tau_{22} \\ \tau_{23} \end{bmatrix} = \left(\sum_{j=1}^{6} i A_j \begin{bmatrix} C_{46}\alpha_j p_j + C_{66} k r_j + C_{66}\alpha_j \\ C_{12} k + C_{22}\alpha_j r_j + C_{25} k p_j \\ C_{44}\alpha_j p_j + C_{46} k r_j + C_{46}\alpha_j \end{bmatrix} e^{i\alpha_j x_2} \right) e^{i(kx_1-\omega t)} \tag{6.33}$$

Here, again the common factor $e^{i(kx_1-\omega t)}$ is suppressed, which results in

$$\begin{bmatrix} \tilde{\tau}_{21}/i \\ \tilde{\tau}_{22}/i \\ \tilde{\tau}_{23}/i \end{bmatrix} = \sum_{j=1}^{6} A_j \mathbf{s}^j e^{i\alpha_j x_2} \tag{6.34}$$

where

$$\mathbf{s}^j = \begin{bmatrix} C_{46}\alpha_j p_j + C_{66} k r_j + C_{66}\alpha_j \\ C_{12} k + C_{22}\alpha_j r_j + C_{25} k p_j \\ C_{44}\alpha_j p_j + C_{46} k r_j + C_{46}\alpha_j \end{bmatrix} \tag{6.35}$$

When Eqs.(6.32) and (6.34) are combined, one obtains, in expanded form,

$$\underbrace{\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{\tau}_{21}/i \\ \tilde{\tau}_{22}/i \\ \tilde{\tau}_{23}/i \end{bmatrix}}_{\tilde{\mathbf{f}}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\ p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ s_1^1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 \\ s_2^1 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 \\ s_3^1 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & s_3^6 \end{bmatrix}}_{\mathbf{U}} \cdot \underbrace{\begin{bmatrix} e^{i\alpha_1 x_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\alpha_2 x_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_3 x_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_4 x_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_5 x_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_6 x_2} \end{bmatrix}}_{\mathbf{E}} \cdot \underbrace{\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix}}_{\mathbf{A}} \tag{6.36}$$

or in matrix form

$$\tilde{\mathbf{f}}(x_2) = \mathbf{U}\,\mathbf{E}(x_2)\,\mathbf{A}$$

Tansfer matrix for a single layer

Eq.(6.36) can be used to relate, through a transfer matrix, the displacements and stresses at the top face of the layer, at x_2=d, to those at its lower face, at x_2=0 (see Fig. 6.2a). To obtain this transfer matrix we proceed by writing Eq.(6.36) at x_2=0 and x_2=d:

at x_2=0:

$$\tilde{\mathbf{f}}(0) = \mathbf{U}\,\mathbf{E}(0)\,\mathbf{A} = \mathbf{U}\,\mathbf{A} \tag{6.37}$$

which yields

$$\mathbf{A} = \mathbf{U}^{-1}\,\tilde{\mathbf{f}}(0) \tag{6.38}$$

at x_2=d:

$$\tilde{\mathbf{f}}(d) = \mathbf{U}\ \mathbf{E}(d)\ \mathbf{A} \tag{6.39}$$

Insertion of Eq.(6.38) into (6.39) leads to

$$\tilde{\mathbf{f}}(d) = \underbrace{(\mathbf{U}\ \mathbf{E}(d)\ \mathbf{U}^{-1})}_{\mathbf{T}(d)}\ \tilde{\mathbf{f}}(0) \tag{6.40}$$

where **T**(d) defined by

$$\mathbf{T}(d) = \mathbf{U}\ \mathbf{E}(d)\ \mathbf{U}^{-1} \tag{6.41}$$

is the transfer matrix of the layer relating $\tilde{\mathbf{f}}(d)$ to $\tilde{\mathbf{f}}(0)$.

Transfer matrix for unit cell

By applying Eq.(6.40) to each layer in the unit cell (see Fig. 6.2b) and invoking the continuity of the displacements u_i and stress components τ_{2i} at layer interfaces, we can relate $\tilde{\mathbf{f}}$ at the top face of unit cell to that at its lower face. For that, one may proceed as follows:

$$\text{Layer 1} \rightarrow \overset{1}{\tilde{\mathbf{f}}}(d_1) = \overset{1}{\mathbf{T}}(d_1)\ \overset{1}{\tilde{\mathbf{f}}}(0) \tag{6.42}$$

$$\text{Layer 2} \rightarrow \overset{2}{\tilde{\mathbf{f}}}(d) = \overset{2}{\mathbf{T}}(d_2)\ \overset{2}{\tilde{\mathbf{f}}}(d_1) \tag{6.43}$$

and use the interface condition

$$\overset{2}{\tilde{\mathbf{f}}}(d_1) = \overset{1}{\tilde{\mathbf{f}}}(d_1) \tag{6.44}$$

which leads to

$$\overset{2}{\tilde{\mathbf{f}}}(d) = \overset{2}{\mathbf{T}}(d_2)\ \overset{1}{\mathbf{T}}(d_1)\ \overset{1}{\tilde{\mathbf{f}}}(0)$$

or

$$\overset{2}{\tilde{\mathbf{f}}}(d) = \mathbf{T}^*\ \overset{1}{\tilde{\mathbf{f}}}(0) \tag{6.45}$$

where

$$\mathbf{T}^* = \overset{2}{\mathbf{T}}(d_2)\ \overset{1}{\mathbf{T}}(d_1) \tag{6.46}$$

is the 6x6 transfer matrix for the unit cell.

For the waves propagating obliquely to the layering, the Floquet wave number (FWN) can be related to the wave number in x_1-direction by, if FWN is real,

$$\kappa = k \cdot \tan\phi \quad \text{(see Fig. 6.1)} \tag{6.47}$$

Floquet periodicity condition can be written for the periodic layered composite under consideration as

$$\overset{2}{\tilde{\mathbf{f}}}(d) = \overset{1}{\tilde{\mathbf{f}}}(0)\, e^{i\kappa d} \tag{6.48}$$

Combining Eqs.(6.45), (6.47) and (6.48), one obtains

$$(\mathbf{T}^* - \mathbf{I}\, e^{i(k\tan\phi)d})\overset{1}{\tilde{\mathbf{f}}}(0) = \mathbf{0} \tag{6.49}$$

whose characteristic equation would be

$$\det(\mathbf{T}^* - \mathbf{I}\, e^{i(k\tan\phi)d}) = 0 \tag{6.50}$$

Eq.(6.50) is the desired exact dispersion equation relating ω, k and the FWN $\kappa = k \cdot \tan\phi$.

Special case 1: propagation perpendicular to layering

In this case $\phi = \dfrac{\pi}{2}$ and k = 0.Therefore κ should be taken as it is and the dispersion relation between ω and κ is given by

$$\det(\mathbf{T}^* - \mathbf{I}\, e^{i\kappa d}) = 0 \tag{6.51}$$

Special case 2: propagation parallel to layering

For such waves, $\phi = 0$, thus κ = 0. The dispersion relation for this case becomes

$$\det(\mathbf{T}^* - \mathbf{I}) = 0 \tag{6.52}$$

6.1.3 Dispersion curves

In obtaining dispersion curves, the two alternating laminae are assumed to be fiber-reinforced layered composites with the ply angles $\theta = \mp 30^\circ$, but with the same thicknesses, mass densities and material properties (in material coordinates):

$$\begin{aligned} &E_1 = 173.058,\ E_2 = 33.095,\ E_3 = 5.171 \\ &G_{12} = 9.377,\ G_{13} = 8.274,\ G_{23} = 3.241 \\ &\nu_{12} = 0.036,\ \nu_{13} = 0.25,\ \nu_{23} = 0.171 \qquad \text{(moduli are in GPa)} \end{aligned} \tag{6.53}$$

which represent graphite fabric-carbon matrix composite [15] (for the definition of material coordinates and ply angle, see Fig. 2.1). E_i, G_{ij} and ν_{ij} in Eqs.(6.53) denote, respectively, Young's and shear moduli, and Poisson's ratios of the orthotropic lamina under consideration. These material properties are related to the elastic coefficients $\hat{C}_{ij}$ by Eq.(3.18). The transformation relations for the elastic coefficients C_{ij} in global coordinates are given in Appendix A.

The dispersion curves of CM are compared with the exact in Figs.6.3–6.7 for harmonic waves propagating in the two-phase layered composite described above. The nondimensional frequency $\overline{\omega}$, and wave numbers $\overline{k}$ and $\overline{\kappa}$ appearing in the figures are defined by

$$\overline{\omega} = \frac{2h_1}{\overset{1}{c}_s}\omega \quad \text{with } \overset{1}{c}_s = \sqrt{\frac{\overset{1}{G}_{13}}{\overset{1}{\rho}}} \quad ; \ \overline{k} = 2h_1 k \quad ; \ \overline{\kappa} = 2h_1 k \tag{6.54}$$

and nondimensional geometric properties of the unit cell are

$$\bar{h}_1 = \bar{h}_2 = 0.5 \ ; \bar{\Delta} = \bar{h}_1 + \bar{h}_2 = 1$$
$$\bar{d}_1 = \bar{d}_2 = 1 \ ; \ \bar{d} = \bar{d}_1 + \bar{d}_2 = 2 \qquad (6.55)$$

The symbol "app(mn)" in the figures stands for the results obtained by CM, where the first digit "m" denotes the order of the approximate theory while the second digit "n" is the order of CC.

Fig.6.3 pertains to the results for the shear waves propagating perpendicular to the layering. The assessment of CM for this case is very crucial and important since the spectra for these waves have periodic structure along FWN (κ) axis, and possess stopping and passing bands along frequency (ω) axis. The figure shows that these properties of spectra are well predicted by CM and that increase in the order of the theory and CC improves this prediction in ω and κ directions, respectively. In fact, the prediction approaches the exact as the orders of the theory and CC go to infinity. It may be observed that the period in FWN direction in the figure is π, which corresponds to the nondimensional value of the period $(\frac{2\pi}{d})$ for the layered composite under consideration. $\bar{\kappa} = \frac{\pi}{2}$ in the figure ($\kappa = \frac{\pi}{d}$ in dimensional form) represents the cut-off value of FWN in (ω-κ) space.

For the sake of completeness, the performance of CM for the harmonic waves propagating parallel and obliquely to layering is assessed in Figs.6.4–6.7. Fig.6.4 contains the spectra for the axial waves with propagation direction parallel to x_1-axis. With axial waves in x_1-direction, it is implied that the waves carrying the displacement disturbances in that direction with the property: the in plane displacements u_1 and u_3 being symmetric and transverse displacement u_2 being antisymmetric about midplanes of laminae. It is obvious that, in conjunction with the analysis of these waves by CM, the FWN κ in Eq.(6.1) should be taken as zero which reduces the order of CC to zero and implies the periodicity of displacement distributions in x_2 (x) direction with the period "d". Fig.6.4 shows that the approximate dispersion curves compare very well with the exact and the comparison improves as the order of the theory increases.

A similar comparison may be observed in Figs.6.5–6.7 for oblique waves with inclined propagation direction of $\phi = 15^\circ$, $\phi = 45^\circ$, $\phi = 75^\circ$ from x_1 axis. In these figures, all propagation modes are included. It may be seen that as the inclination angle increases the shape of the dispersion curves approaches, as anticipated, into periodic form.

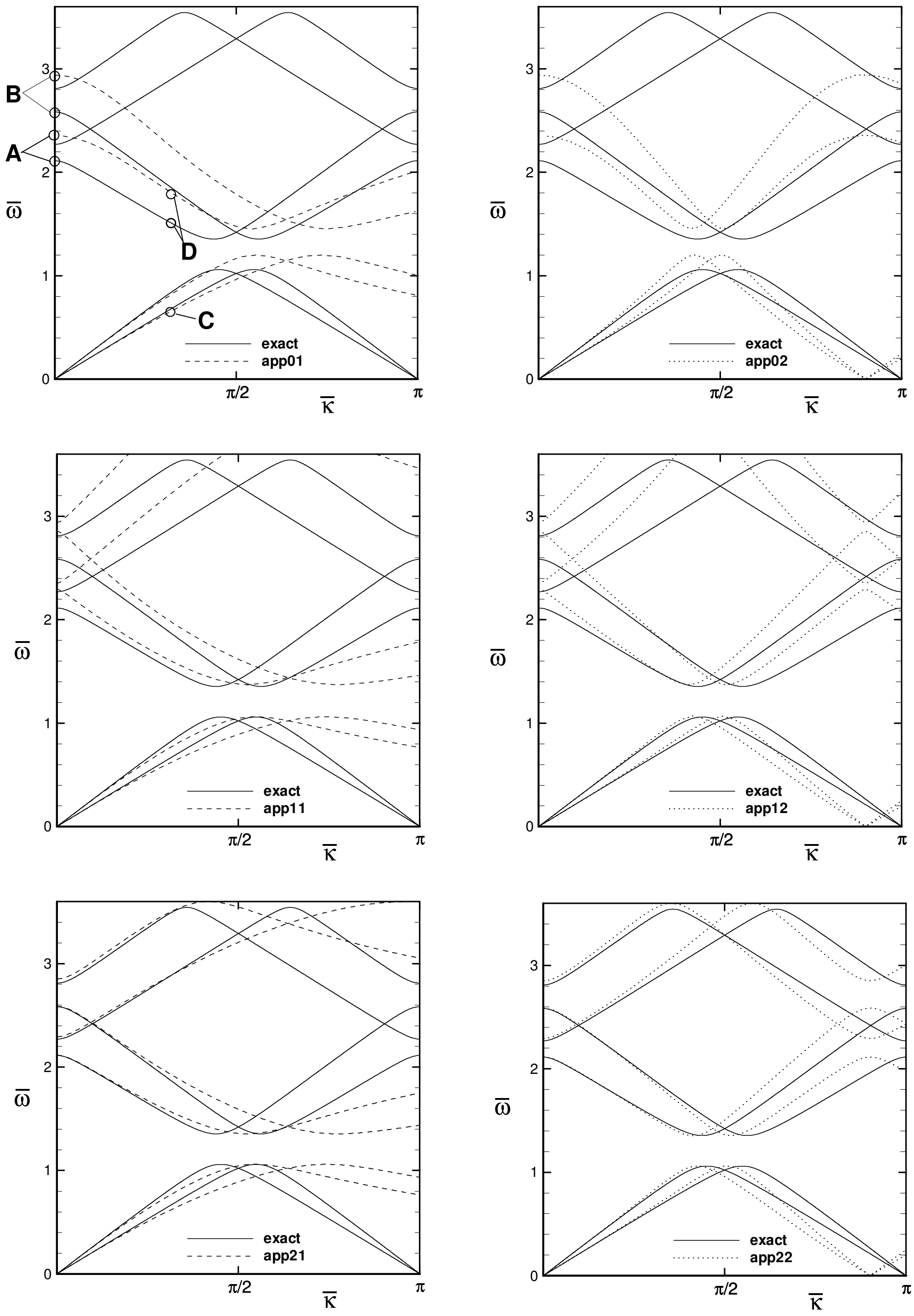


Figure 6.3 Comparison of spectra for harmonic shear waves propagating in a two-phase periodic layered composite in the direction perpendicular to layering

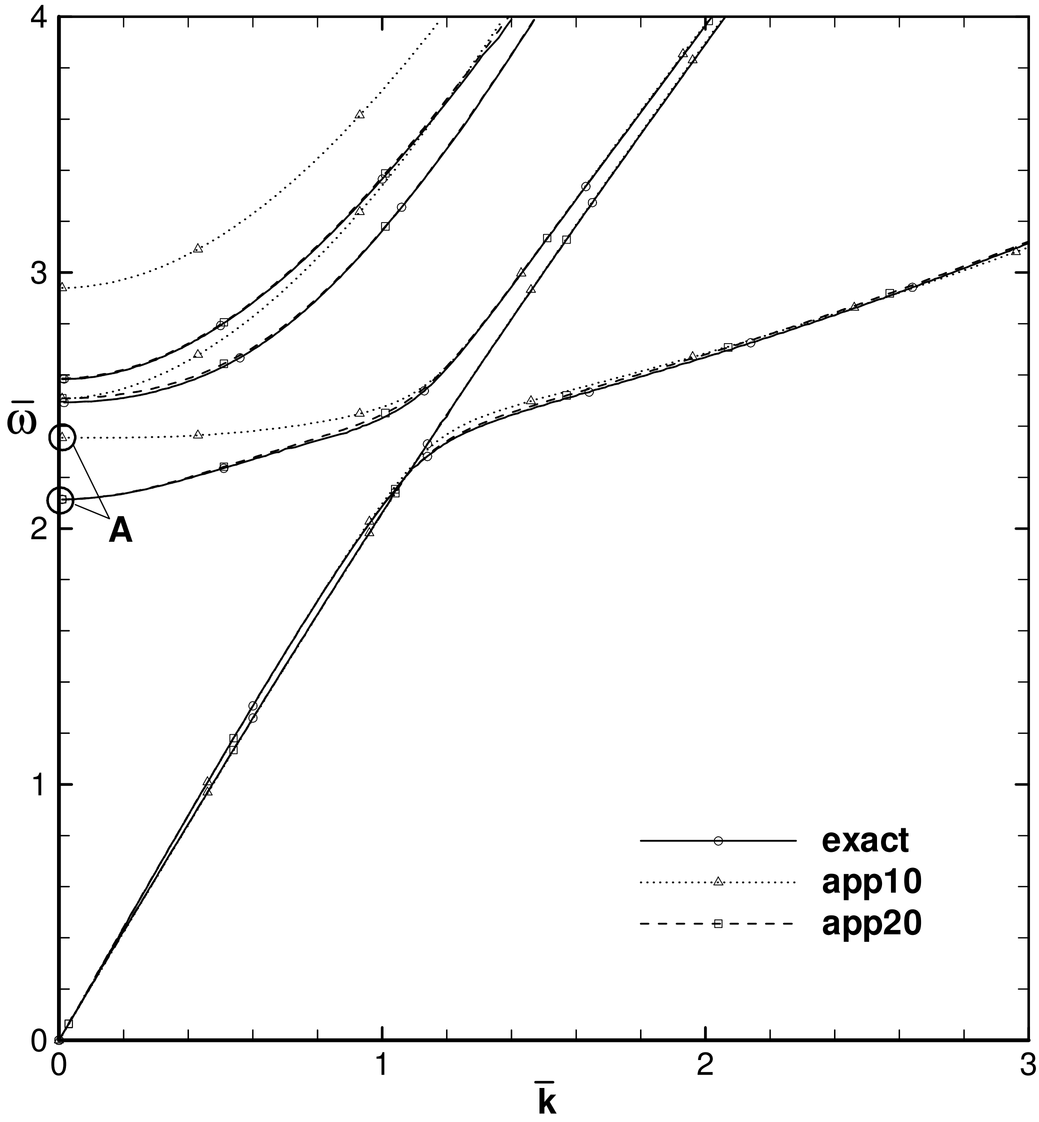


Figure 6.4 Comparison of spectra for axial harmonic waves propagating in a two-phase periodic layered composite in the direction parallel to layering

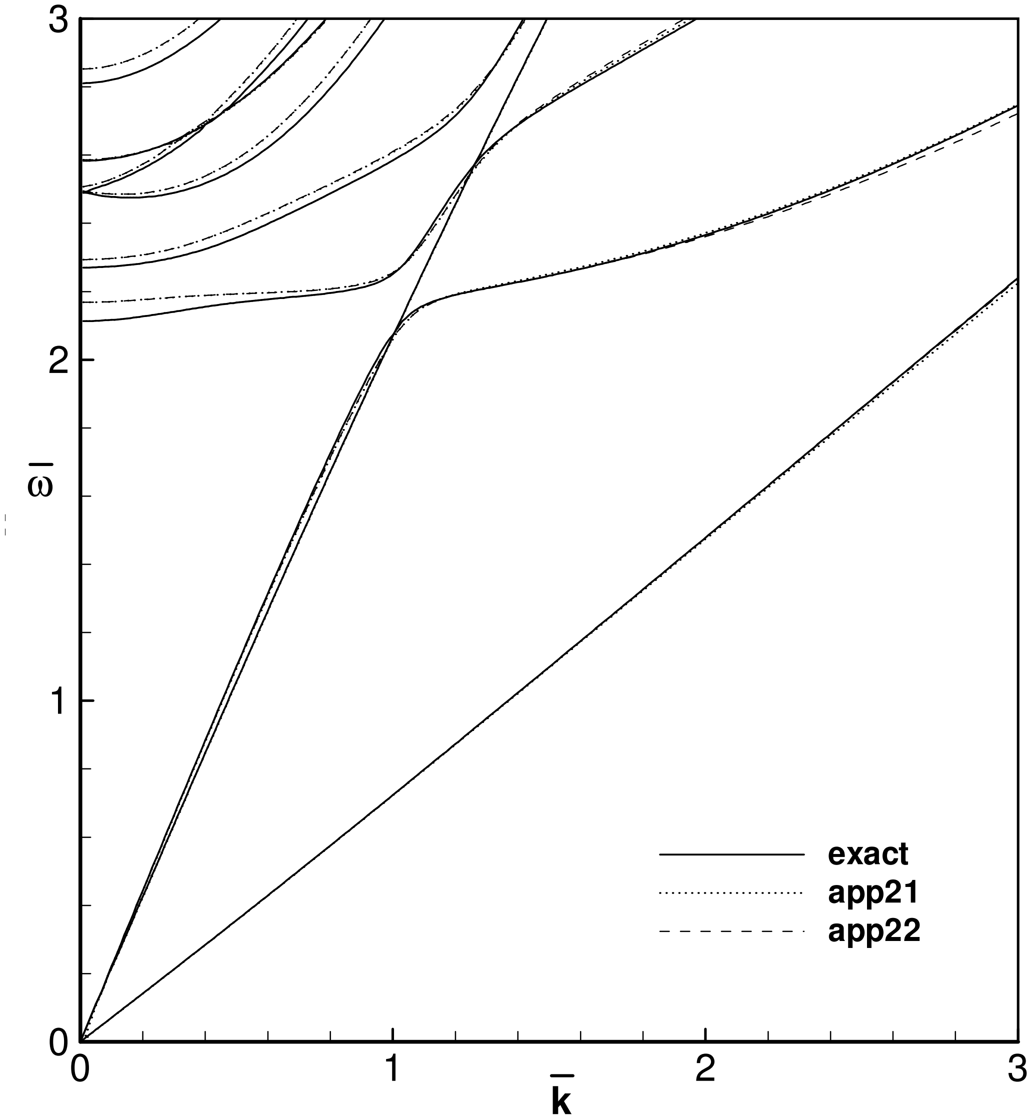


Figure 6.5 Comparison of spectra for inclined harmonic waves propagating in a two-phase periodic layered composite (inclination angle with x_1-axis: $\phi = 15^\circ$)

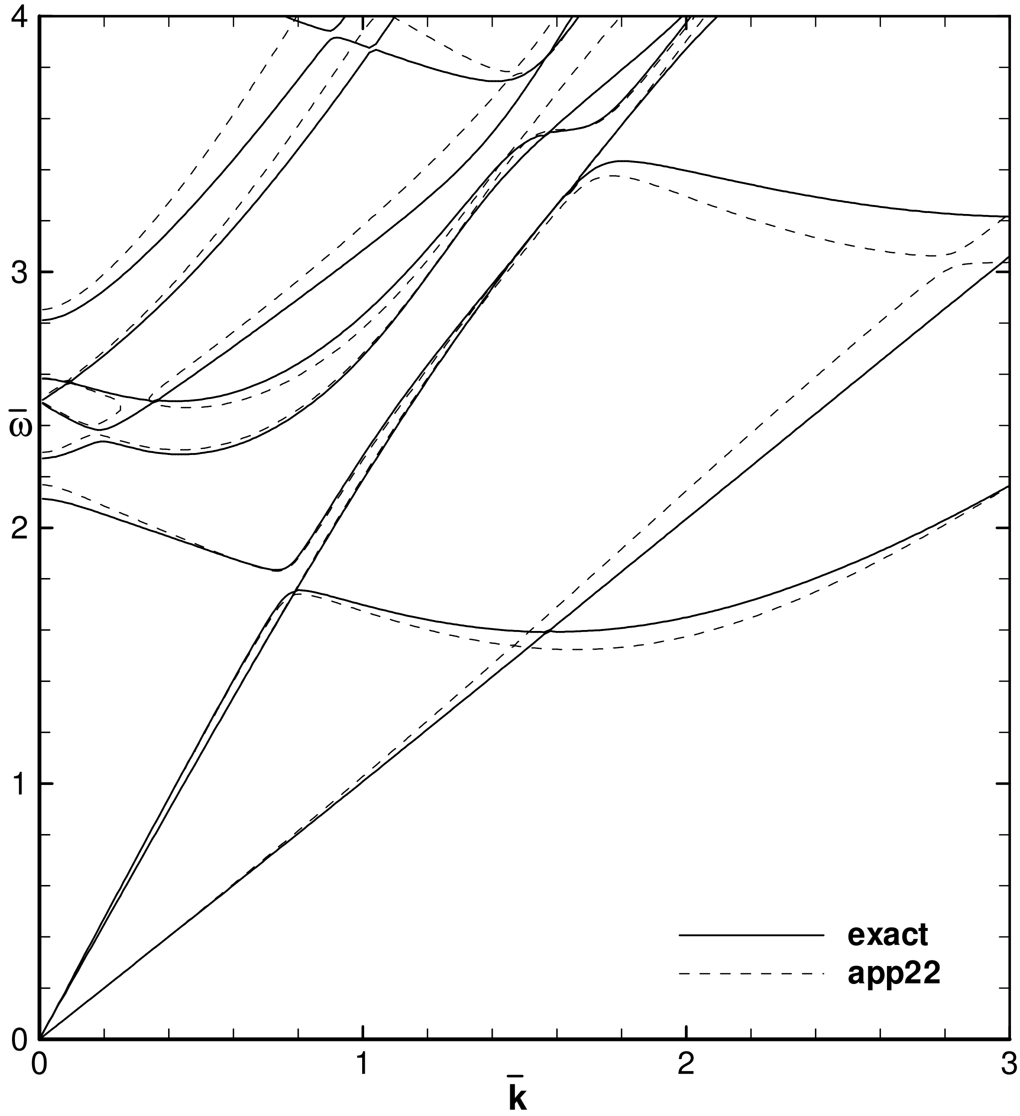


Figure 6.6 Comparison of spectra for inclined harmonic waves propagating in a two-phase periodic layered composite (inclination angle with x_1-axis: $\phi = 45^\circ$)

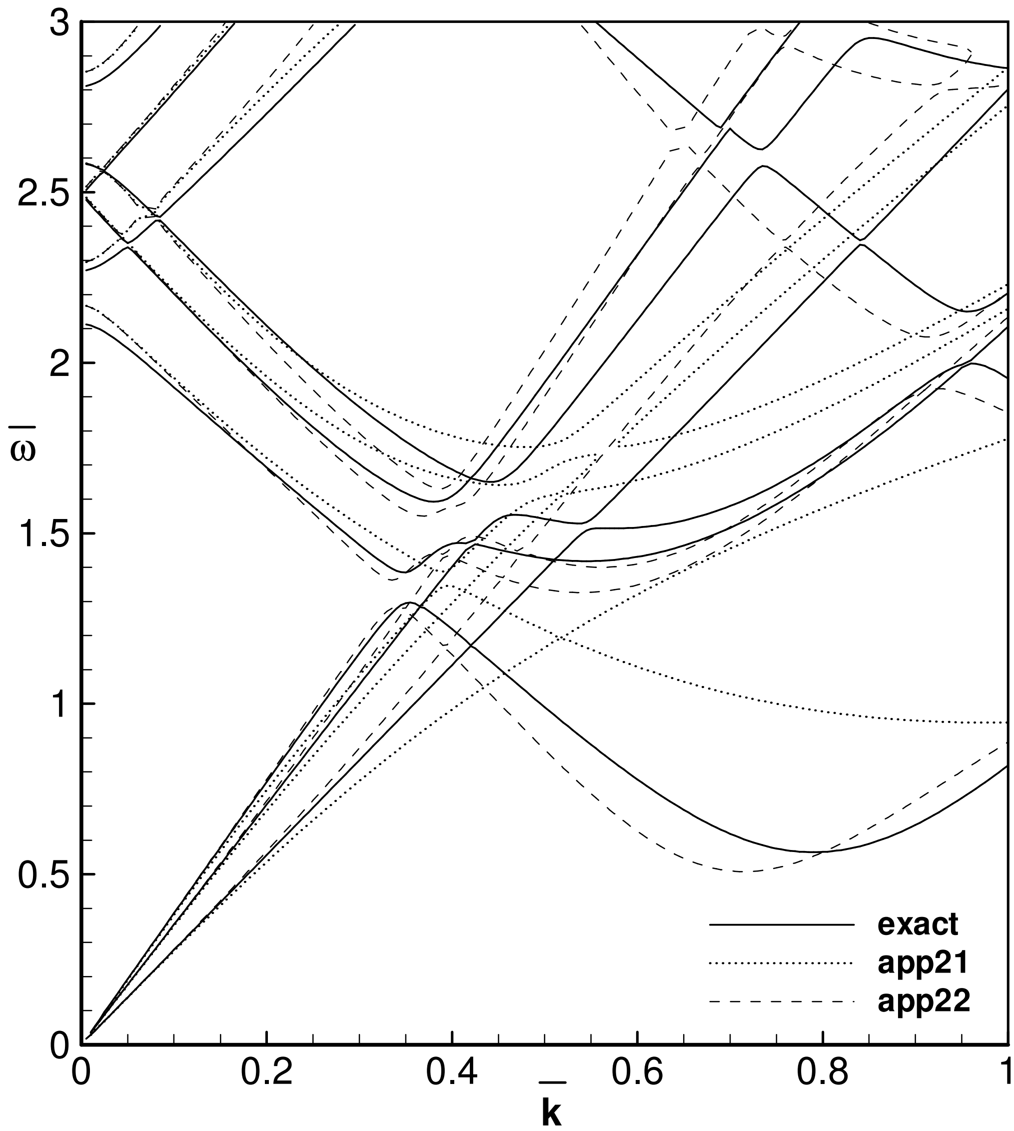


Figure 6.7 Comparison of spectra for inclined harmonic waves propagating in a two-phase periodic layered composite (inclination angle with x_1-axis: $\phi = 75^\circ$)

6.1.4 Mode shapes

Displacement mode shapes of approximate and exact theory are compared as a part of spectral assessment. These comparisons are made at points A, B, C and D of the shear wave spectrum in Fig. 6.3, and at point A of the axial wave spectrum in Fig. 6.4. The procedures obtaining the approximate and exact mode shapes are outlined below briefly.

6.1.4.1 Approximate mode shapes

At a given point P(ω,k,κ) of a dispersion curve of spectrum, the displacement mode shapes as predicted by CM are determined through the following steps:

1. Through the solution of SHAE

$$\mathbf{M}\,\mathbf{a} = \mathbf{0}$$

   determine the eigensolution **a** at point P; note that **a** contains the field variables of CM as indicated in Appendices C, D and E.

2. Evaluate the displacement mode shapes for the unit cell from

$$\overset{\alpha}{u}_i = \sum_{k=0}^{m+2} \overset{\alpha}{a}{}^{i}_{k}\, \phi_k \tag{6.56}$$

   where we note that the coefficients $\overset{\alpha}{a}{}^{i}_{k}$ are related to the field variables of CM by Eqs.(2.17).

When $\kappa \neq 0$, Eq.(6.56) should be multiplied by the factor $e^{i\kappa x}$, where "x" is the x_2 coordinate of the midplane of the layer for which the displacement distribution is being evaluated. It should be noted that $\overset{\alpha}{u}_i$ as determined from Eq.(6.56) is complex; in view of the properties of complex Fourier series the real part of Eq.(6.56) is to be considered for the evaluation of mode shapes.

As a sample, we give below the displacement expressions in Eq.(6.56) in extended form for the shear waves propagating perpendicular to layering when the order of theory is zero:

$$\overset{1}{u}_1(x_2)=\frac{\overset{1}{u}{}_1^0}{d_0}\phi_0+\frac{\overset{1}{S}{}_1^-}{2}\phi_1+\left(\frac{\overset{1}{S}{}_1^+}{2}-\frac{\overset{1}{u}{}_1^0}{d_0}\right)\phi_2$$

$$\overset{1}{u}_3(x_2)=\frac{\overset{1}{u}{}_3^0}{d_0}\phi_0+\frac{\overset{1}{S}{}_3^-}{2}\phi_1+\left(\frac{\overset{1}{S}{}_3^+}{2}-\frac{\overset{1}{u}{}_3^0}{d_0}\right)\phi_2$$

$$\overset{2}{u}_1(x_2)=\frac{\overset{2}{u}{}_1^0}{d_0}\phi_0+\frac{\overset{2}{S}{}_1^-}{2}\phi_1+\left(\frac{\overset{2}{S}{}_1^+}{2}-\frac{\overset{2}{u}{}_1^0}{d_0}\right)\phi_2$$

$$\overset{2}{u}_3(x_2)=\frac{\overset{2}{u}{}_3^0}{d_0}\phi_0+\frac{\overset{2}{S}{}_3^-}{2}\phi_1+\left(\frac{\overset{2}{S}{}_3^+}{2}-\frac{\overset{2}{u}{}_3^0}{d_0}\right)\phi_2 \quad , \quad \overset{2}{u}_2=0 \qquad (6.57)$$

6.1.4.2 Exact mode shapes

To obtain the exact mode shapes at a point P(ω,k,κ) of the spectrum, the eigensystem in Eq.(6.49), that is,

$$(\mathbf{T}^*-\mathbf{I}\,e^{i(k\tan\phi)d})\overset{1}{\tilde{\mathbf{f}}}(0)=\mathbf{0}$$

is to be solved for $\overset{1}{\tilde{\mathbf{f}}}(0)$ at point P, where $\overset{1}{\tilde{\mathbf{f}}}(0)$ contains the displacement and stresses at lower end of unit cell (see Fig. 6.2b).

Subsequently, the amplitudes are found from Eq.(6.38) for layer 1 as

$$\overset{1}{\mathbf{A}}=\overset{1}{\mathbf{U}}{}^{-1}\,\overset{1}{\tilde{\mathbf{f}}}(0) \qquad (6.58)$$

and the displacement and stress mode shapes for layer 1 from, in view of Eq.(6.36),

$$\overset{1}{\tilde{\mathbf{f}}}(x_2)=\overset{1}{\mathbf{U}}\,\overset{1}{\mathbf{E}}(x_2)\,\overset{1}{\mathbf{A}} \qquad (6.59)$$

Distributions for layer 2 may be found in an analogous manner as layer 1, but for that, displacement and stresses $\overset{2}{\tilde{\mathbf{f}}}(d_1)$ at the lower end of layer 2 are needed (see Fig. 6.2b). This can be obtained from, in view of Eq.(6.42) and the continuity of $\tilde{\mathbf{f}}$ at interface,

$$\overset{2}{\tilde{\mathbf{f}}}(d_1) = \overset{1}{\mathbf{T}}(d_1)\ \overset{1}{\tilde{\mathbf{f}}}(0) \tag{6.60}$$

Amplitudes of the variables for layer 2 may be evaluated from, in view of Eq.(6.38),

$$\overset{2}{\mathbf{A}} = \overset{2}{\mathbf{U}}^{-1}\ \overset{2}{\tilde{\mathbf{f}}}(d_1) \tag{6.61}$$

and the displacement and stress distributions for layer 2 are determined by

$$\overset{2}{\tilde{\mathbf{f}}}(x_2) = \overset{2}{\mathbf{U}}\ \overset{2}{\mathbf{E}}(x_2)\ \overset{2}{\mathbf{A}} \tag{6.62}$$

The distributions in the next unit cell can be obtained by multiplying the distributions of the preceding unit cell by $e^{i\kappa d}$ because of the Floquet periodicity condition.

6.1.4.3 Comparison of mode shapes and physical significance of Floquet wave number

Exact and approximate displacement mode shapes are compared in Figs.(6.8-6.11).

For the physical significance of FWN (κ), we refer to Fig. 6.8 showing the displacement mode shapes at point C of the spectrum in Fig. 6.3, where we note that $\bar{\kappa} = 1$. In the figure, we first observe that the horizontal displacements u_1 and u_3 are coupled at point C, and that CM results (obtained by zeroth order theory, and first and second order CC's) match very well with the exact. The solid line in the figure represents the actual displacement distributions produced by multiple reflections and/or refractions of perpendicular shear wave components at interfaces. On the other hand, the dotted line is harmonic envelope (interpolating) curve connecting the displacement values at **discrete** interface points with the increment "d". The FWN is the wave number associated with this envelope curve. The wave length for the envelope curve is

$$\bar{\lambda} = \frac{2\pi}{\bar{\kappa}} = 2\pi$$

which is consistent with the results presented in Fig. 6.8.

Mode shapes at point D of the spectrum in Fig. 6.3 are compared in Fig. 6.9. We note that points D and C have the same FWN, $\bar{\kappa}=1$; but, frequency of D is higher than that of C. The figure shows that the match of exact and approximate curves for point D (with higher frequency) is not as good as point C; however, it should be noted that this result is obtained by the lowest (zeroth) order theory and improvement should be expected when the order is increased.

Fig. 6.10 contains the displacement mode shapes at second and third cut-off frequencies, that is, at points A and B of spectrum in Fig. 6.3. Here, approximate and exact mode shapes compare quite well, in spite of our using lower order theory and CC. We note that at cut-off points A and B: $\bar{\kappa}=0$, which corresponds to infinite wave length. This, in view of physical interpretation of FWN, implies that displacement values at **discrete** points along x_2 (x) axis with the increment "d" should have uniform distribution, which is in agreement with the results in Fig. 6.10 (in view of periodicity of the actual displacement distributions with the period "d").

For axial waves propagating parallel to layering, the displacement mode shapes at point A of the spectrum in Fig. 6.4 are given in Fig. 6.11. In this case, the continuity conditions are exact and the curves in the figure converge rapidly to the exact as the order of theory increases. Here, the FWN is zero, which gives rise to uniform distribution for envelope curves interpolating the displacement values at discrete points with period “d”.

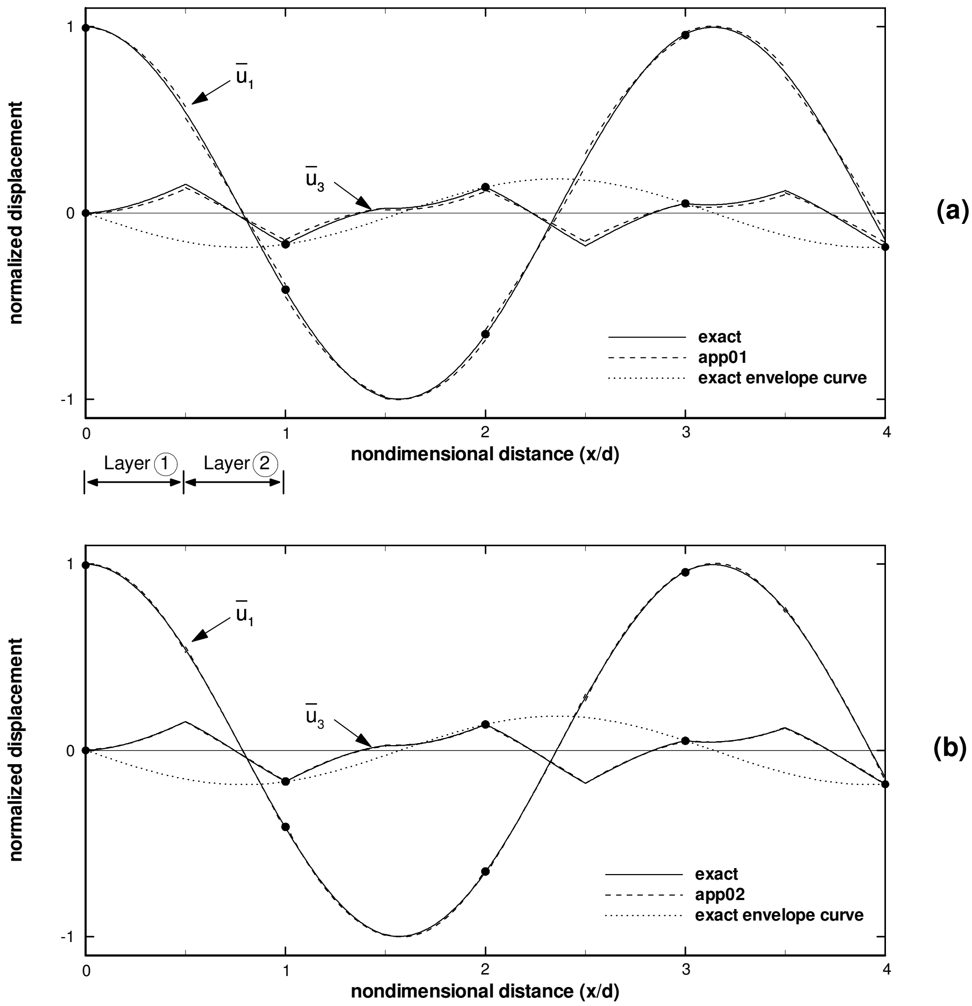


Figure 6.8 Comparison of shear mode shapes at location C of spectrum in Fig.6.3 for **(a)** 1st order **(b)** 2nd order continuity conditions ($\bar{\kappa}$ (nondimensional Floquet wave number) at point C is 1)

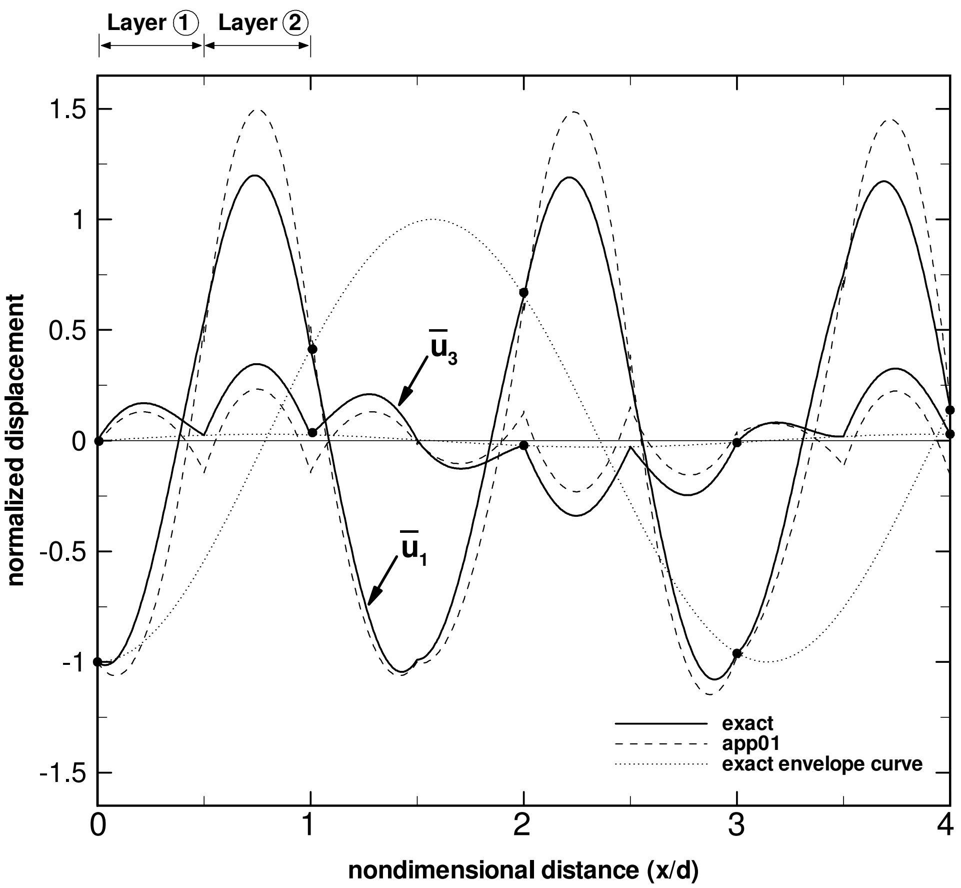


Figure 6.9 Comparison of shear mode shapes at location D of spectrum in Fig.6.3 ($\bar{\kappa}$ (nondimensional Floquet wave number) at point D is 1)

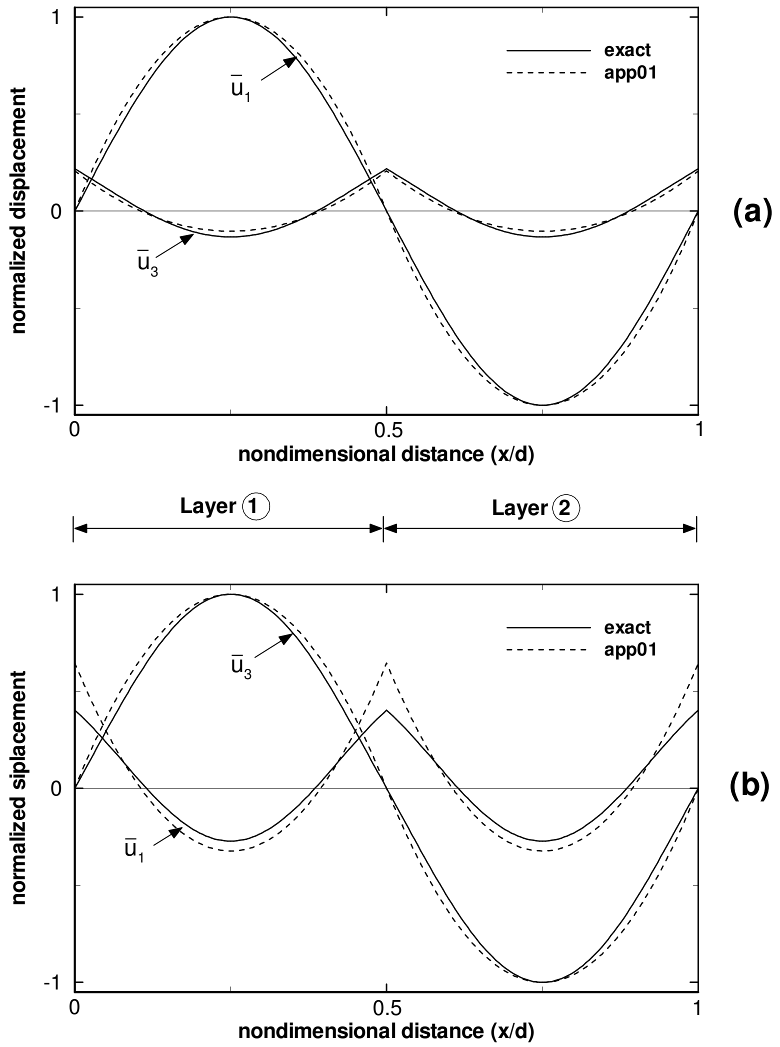


Figure 6.10 Comparison of shear mode shapes at **(a)** second **(b)** third cut-off frequencies (at locations A and B in Fig.6.3, respectively)

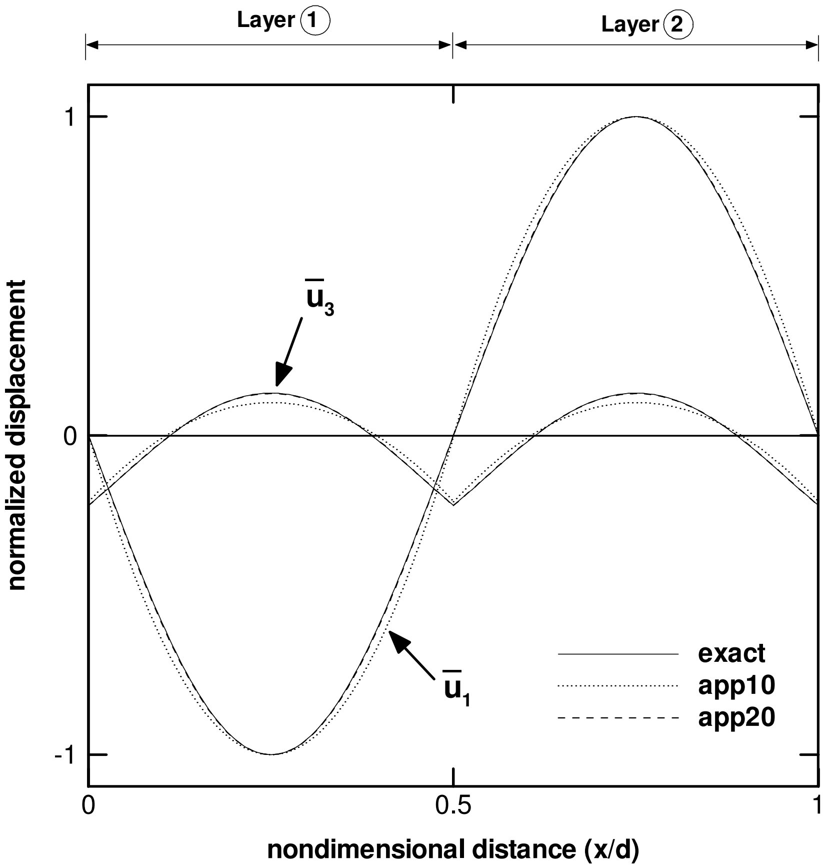


Figure 6.11 Comparison of coupled (out-of-plane)-axial shear mode shapes at location A of spectrum in Fig.6.4

6.2 Check of Spectral Criterion for a Transient Case

In view of the spectral criterion stated previously and the results presented in Section 6.1, it is expected that CM should be capable to predict the dynamic behavior of a periodic layered composite body when it is subjected to transient inputs. For checking this, a two-phase layered slab of a finite thickness shown in Fig. 6.12 is considered. The composite slab is subjected on the left end to a uniform shear stress “s” with a stepwise time variation of intensity s_0 as shown in the figure. This stress input generates transient shear waves propagating perpendicular to layering. The right face of the slab is free of tractions and on this face, the wave profiles for the particle velocity are obtained by using CM and exact theory. In the numerical analysis, the thermal effects are neglected. The equations of the exact theory are integrated exactly using the method of characteristics. On the other hand, since the equations of the approximate theory are not hyperbolic, they are integrated employing a different method, namely, the method of lines. The material of the layers constructing the slab are taken as orthotropic.

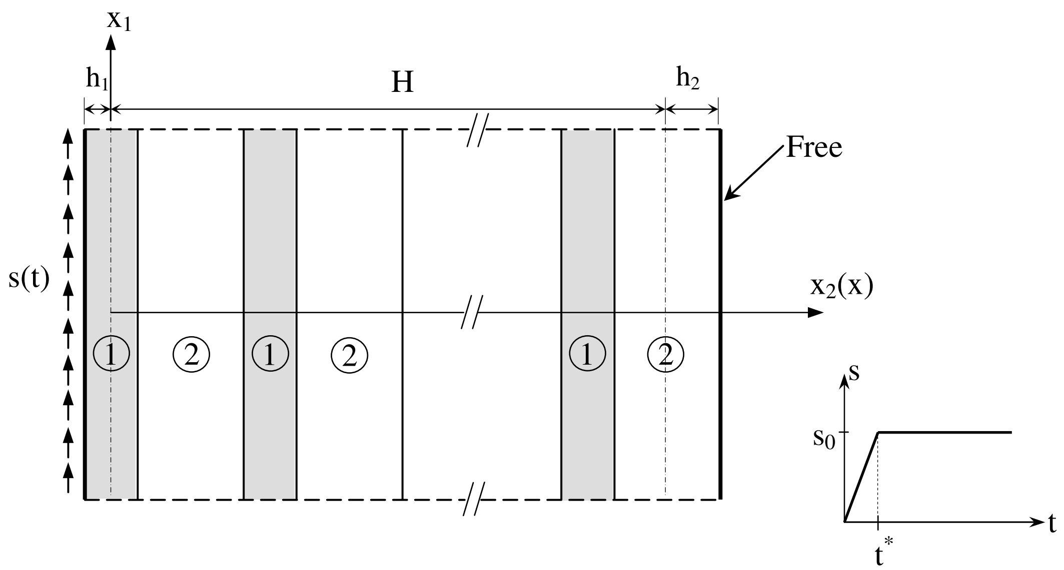


Figure 6.12 Two-phase layered slab

6.2.1 Approximate formulation and solution

The prediction of CM is evaluated by choosing the orders of the theory and continuity conditions (CC) as low as possible, namely, zeroth order for the theory and first order for CC's. The governing equations of CM for shear waves propagating perpendicular to layering are given in Eqs.(6.4–6.10), which reduce to, for the transient problem under consideration (in view of $\partial_3(\cdot)=0$)

equations of motion:

$$\overset{\alpha}{R}{}_1^{-} = 2h_\alpha \rho_\alpha \overset{\alpha}{\ddot{u}}{}_1^{0} \qquad (\alpha=1,2) \tag{6.63}$$

constitutive equations for face variables:

$$\overset{\alpha}{R}{}_1^{+} = 2\gamma^{-} \overset{\alpha}{C}_{66} \overset{\alpha}{S}{}_1^{-} / h_\alpha$$

$$\overset{\alpha}{R}{}_1^{-} = 2(\gamma_0 \overset{\alpha}{C}_{66} \overset{\alpha}{u}{}_1^{0} + \gamma^{+} \overset{\alpha}{C}_{66} \overset{\alpha}{S}{}_1^{+})/h_\alpha \tag{6.64}$$

continuity conditions:

$$\frac{\Delta}{2}\partial_x \overset{2}{S}{}_1^{+} + \frac{\Delta}{2}\partial_x \overset{1}{S}{}_1^{+} = \overset{2}{S}{}_1^{-} + \overset{1}{S}{}_1^{-}$$

$$\frac{\Delta}{2}\partial_x \overset{2}{S}{}_1^{-} - \frac{\Delta}{2}\partial_x \overset{1}{S}{}_1^{-} = \overset{2}{S}{}_1^{+} + \overset{1}{S}{}_1^{+}$$

$$\frac{\Delta}{2}\partial_x \overset{2}{R}{}_1^{+} + \frac{\Delta}{2}\partial_x \overset{1}{R}{}_1^{+} = \overset{2}{R}{}_1^{-} + \overset{1}{R}{}_1^{-}$$

$$\frac{\Delta}{2}\partial_x \overset{2}{R}{}_1^{-} - \frac{\Delta}{2}\partial_x \overset{1}{R}{}_1^{-} = \overset{2}{R}{}_1^{+} + \overset{1}{R}{}_1^{+} \tag{6.65}$$

where $\overset{1}{C}_{66}$ and $\overset{2}{C}_{66}$ represent, respectively, the shear moduli of layers 1 and 2 associated with x_1x_2-plane.

<u>Boundary conditions</u>

The left end of the slab is subjected to the shear stress s(t); accordingly, the exact boundary condition at the left end is

$$\overset{1}{\tau}_{21}\Big|_{x=-h_1} = -s(t) \tag{6.66}$$

Using the relations in Eq.(2.13), the boundary condition in Eq.(6.66) can be written in terms of stress face variables as

$$(\overset{1}{R}{}_1^{+} - \overset{1}{R}{}_1^{-})\Big|_{x=0} = -2s \tag{6.67}$$

The latter boundary condition is written at $x = 0$ instead of $x = -h_1$, because the face variables in CM have physical meanings only at the midplanes of layers. In view of idealization implied by smoothing operations, implying that the two types of interfaces following layer 1 and 2 exist at the same point of the composite, the other boundary condition at the left end may be written as

$$(\overset{2}{R}{}_1^{+} - \overset{2}{R}{}_1^{-})\Big|_{x=0} = -2s \tag{6.68}$$

With the use of the same procedure, the boundary conditions at the right end (traction free) of the slab can be found as

$$(\overset{1}{R}{}_1^{+} + \overset{1}{R}{}_1^{-})\Big|_{x=H} = 0 \ ; \qquad (\overset{2}{R}{}_1^{+} + \overset{2}{R}{}_1^{-})\Big|_{x=H} = 0 \tag{6.69}$$

where H is the length between the midplanes of the first and last layers of the slab (see Fig. 6.12).

<u>Initial conditions</u>

Since the slab is at rest initially, all of the field variables of the approximate theory should be zero at the initial time $t = 0$.

The formulation of the CM is now complete. Eqs.(6.63–65) constitute ten partial differential equations for the ten unknown variables $\overset{\alpha}{u}{}_1^{0}$, $\overset{\alpha}{S}{}_1^{\mp}$ and $\overset{\alpha}{R}{}_1^{\mp}$. The unknowns can be determined uniquely by solving them subject to boundary conditions, Eqs.(6.67–69), and zero initial conditions.

The CM equations are integrated by method of lines. With the object of explaining this method, the solution domain $0 \le x \le H$ and $t \ge 0$ is referred to an x-t rectangular coordinate system in which x and t are chosen to be as horizontal and vertical axes, respectively. The solution domain is subdivided by a rectangular network with the space mesh size Δx and time mesh size Δt. The vertical and horizontal grid lines are numbered in increasing order (starting with zero) in the positive directions of x and t axes. Thus, the coordinates of a point located at the intersection of i[th] vertical and j[th] horizontal grid lines become $(x_i, t_j) = (i\Delta x,\ j\Delta t)$. In the discussions that follow, the value of a function $f(x,t)$ at the nodal point (x_i, t_j) is designated by ${}_{ij}f$.

For establishing the unknowns in the solution domain, the analysis is started from the first horizontal grid line $t = t_0 = 0$, where the unknowns are known from the initial conditions. Using the values of unknowns along $t = t_0$ and a technique to be explained shortly, the unknowns along the next horizontal line $t = t_1$ are found. After the unknowns are determined along $t = t_1$ the same technique is used to find them along $t = t_2$ and so forth.

In order to explain the technique mentioned above, two consecutive horizontal grid lines, say $t = t_{j-1}$ and $t = t_j$, are considered. The dependent variables are known at the nodal points along $t = t_{j-1}$. To establish them along $t = t_j$ an iterative procedure is used. In the first cycle of iterations, Eq.(6.63) and $\dot{\overset{\alpha}{u}}{}_1^0 = \overset{\alpha}{v}{}_1^0$ are integrated with respect to time (i.e., along $x = x_i$ $(i = 0,1,\ldots)$)from $t = t_{j-1}$ to $t = t_j$ using rectangular and trapezoidal rules, respectively. This gives

$$\overset{1}{{}_{ij}\overset{\alpha}{v}{}_1^0} = {}_{i,j-1}\overset{\alpha}{v}{}_1^0 + \frac{\Delta t}{2h_\alpha \rho_\alpha}\, {}_{i,j-1}\overset{\alpha}{R}{}_1^-$$

$$\overset{1}{{}_{ij}\overset{\alpha}{u}{}_1^0} = {}_{i,j-1}\overset{\alpha}{u}{}_1^0 + \frac{\Delta t}{2}\left({}_{i,j-1}\overset{\alpha}{v}{}_1^0 + \overset{1}{{}_{ij}\overset{\alpha}{v}{}_1^0}\right) \qquad (i = 0,1,\ldots) \qquad (6.70)$$

where the number over the variables indicates the iteration number. Since $\overset{1}{\overset{\alpha}{u}_1}$ along $t = t_j$ is determined, Eqs.(6.64) and (6.65) constitute eight ordinary differential equations in space (along $t = t_j$) for the remaining unknowns $\mathbf{w} = (\overset{\alpha}{R}{}_1^{\mp}, \overset{\alpha}{S}{}_1^{\mp})$. They are solved using the method of complementary functions to be discussed shortly. Having known now $\overset{1}{\overset{\alpha}{R}{}_1^-}$ along $t = t_j$, Eq.(6.63) and $\dot{\overset{\alpha}{u}}_1 = \overset{\alpha}{v}_1$

are again integrated with respect to time from $t = t_{j-1}$ to $t = t_j$, but this time using trapezoidal rule for both of them:

$$
{}_{ij}\overset{2}{\overset{\alpha}{v}}{}^{0}_{1} = {}_{i,j-1}\overset{\alpha}{v}{}^{0}_{1} + \frac{\Delta t}{4h_{\alpha}\rho_{\alpha}}\left({}_{i,j-1}\overset{\alpha}{R}{}^{-}_{1} + {}_{i,j}\overset{1}{\overset{\alpha}{R}}{}^{-}_{1} \right)
$$

$$
{}_{ij}\overset{2}{\overset{\alpha}{u}}{}^{0}_{1} = {}_{i,j-1}\overset{\alpha}{u}{}^{0}_{1} + \frac{\Delta t}{2}\left({}_{i,j-1}\overset{\alpha}{v}{}^{0}_{1} + {}_{ij}\overset{2}{\overset{\alpha}{v}}{}^{0}_{1} \right) \qquad (i = 0,1,\ldots) \tag{6.71}
$$

The second cycle of iterations is completed using the improved $\overset{2}{\overset{\alpha}{u}}{}^{0}_{1}$. The iterations are continued until a prescribed accuracy is achieved.

We explain now the method of complementary functions used to solve the system of ordinary differential equations in space, Eqs.(6.64) and (6.65), for the unknowns $\mathbf{w} = (\overset{\alpha}{R}{}^{\mp}_{1}, \overset{\alpha}{S}{}^{\mp}_{1})$ along $t = t_j$. In this method, first the solution $\mathbf{w}$ is written in the form

$$
\mathbf{w} = \mathbf{w}^{p} + \sum_{k=1}^{4} \beta_k \, \mathbf{w}^{k} \tag{6.72}
$$

where β_k's are some constants. The particular solution $\mathbf{w}^p$ is governed by Eqs.(6.64) and (6.65), and the complementary solutions $\mathbf{w}^k$'s are governed by the same equations with $\overset{\alpha}{u}{}^{0}_{1} = 0$. Further, $\mathbf{w}^p$ and $\mathbf{w}^k$ satisfy the conditions at $x = 0$ indicated in Table 6.1. The solution in Eq.(6.72) satisfies then the governing equations, Eqs.(6.64) and (6.65), and the boundary conditions at $x = 0$ given by Eqs.(6.67) and (6.68) exactly. The initial value problems associated with $\mathbf{w}^p$ and $\mathbf{w}^k$ are solved by integrating the governing equations numerically subject to the conditions at $x = 0$ shown in Table 6.1. For integration, the trapezoidal rule formula with the space mesh size Δx is employed. Having established $\mathbf{w}^p$ and $\mathbf{w}^k$ along $t = t_j$ the four constants β_1, β_2, β_3 and β_4 appearing in Eq.(6.72) are determined so that the traction free boundary conditions in Eqs.(6.69) at $x = H$ are satisfied.

Table 6.1 The conditions at $x=0$ for $\mathbf{w}^p$ and $\mathbf{w}^k$

	$\mathbf{w}^p$	$\mathbf{w}^1$	$\mathbf{w}^2$	$\mathbf{w}^3$	$\mathbf{w}^4$
$\overset{1}{R}{}_1^+$	0	1	0	0	0
$\overset{1}{R}{}_1^-$	0	0	1	0	0
$\overset{2}{R}{}_1^+$	0	0	0	1	0
$\overset{2}{R}{}_1^-$	0	0	0	0	1

6.2.2 Exact formulation and solution

In the exact formulation, equations of elasticity theory are applied to each layer and the solutions are required to satisfy the continuity conditions at the interfaces of the layers and the boundary conditions at the bounding surfaces. Method of characteristics will be employed to obtain the solutions. Elasticity equations for a typical layer of the problem under consideration are

stress equation of motion:

$$\partial_2 \tau_{21} = \rho \ddot{u}_1 \tag{6.73}$$

constitutive equation:

$$\tau_{21} = C_{66}\, \partial_2 u_1 \tag{6.74}$$

Applying change of variables $v_1 = \dot{u}_1$ and $\partial_2 u_1 = \gamma_1$ (or $\partial_2 v_1 = \dot{\gamma}_1$) yields the governing first order partial differential equations in matrix form as

$$\mathbf{A}\,\mathbf{U},_t + \mathbf{B}\,\mathbf{U},_x = \mathbf{0} \tag{6.75}$$

or in expanded form as

$$\begin{bmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \gamma_1 \\ \tau_{12} \end{bmatrix},_t + \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ -C_{66} & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \gamma_1 \\ \tau_{12} \end{bmatrix},_x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.76}$$

where comma denotes partial differentiation with respect to the subscript following it. Since these equations are hyperbolic partial differential equations, they can be transformed into a system of ordinary differential equations each of which is valid along a different family of characteristic lines. These equations, called the canonical equations, are suitable for numerical analysis because the use of the canonical form makes it possible to obtain the solution by a step-by-step integration procedure. The convergence and the numerical stability of the method of characteristics are well established (e.g., see [69])

The characteristic lines, along which the canonical equations are valid, are governed by the characteristic equation

$$\det(\mathbf{B} - \lambda \mathbf{A}) = 0 \tag{6.77}$$

where $\lambda = dx/dt$. Eq.(6.77) yields the eigenvalues

$$\lambda_1 = c\ ,\ \lambda_2 = -c\ ,\ \lambda_3 = 0 \tag{6.78}$$

where $c = \sqrt{C_{66}/\rho}$ is the shear wave velocity. The characteristic manifold is thus composed of families of lines $dx/dt = \lambda_i$ $(i = 1\ldots3)$. $dx/dt = \lambda_1 = c$ and $dx/dt = \lambda_2 = -c$ describe two characteristic families of straight lines with slopes (c) and (−c), respectively, on the (x-t) plane; whereas, $dx/dt = \lambda_3 = 0$ defines straight lines parallel to the t-axis, see Fig. 6.13. The canonical equations are given by

$$\mathbf{l}_i^T \mathbf{A} \frac{d\mathbf{U}}{dt} = 0 \tag{6.79}$$

along the characteristic lines $dx/dt = \lambda_i$ $(i = 1\ldots3)$. In Eq.(6.79), d/dt describes the total time derivative along a characteristic line and $\mathbf{l}_i$ is the left-hand eigenvector corresponding to i^{th} eigenvalue λ_i of the eigenvalue problem

$$(\mathbf{B}^T - \lambda_i \mathbf{A}^T)\mathbf{l}_i = \mathbf{0} \tag{6.80}$$

The left-hand eigenvectors are found as

$$\mathbf{l}_1 = \begin{bmatrix} 1 \\ 0 \\ -1/c \end{bmatrix},\quad \mathbf{l}_2 = \begin{bmatrix} 1 \\ 0 \\ 1/c \end{bmatrix},\quad \mathbf{l}_3 = \begin{bmatrix} 0 \\ 1 \\ -1/C_{66} \end{bmatrix} \tag{6.81}$$

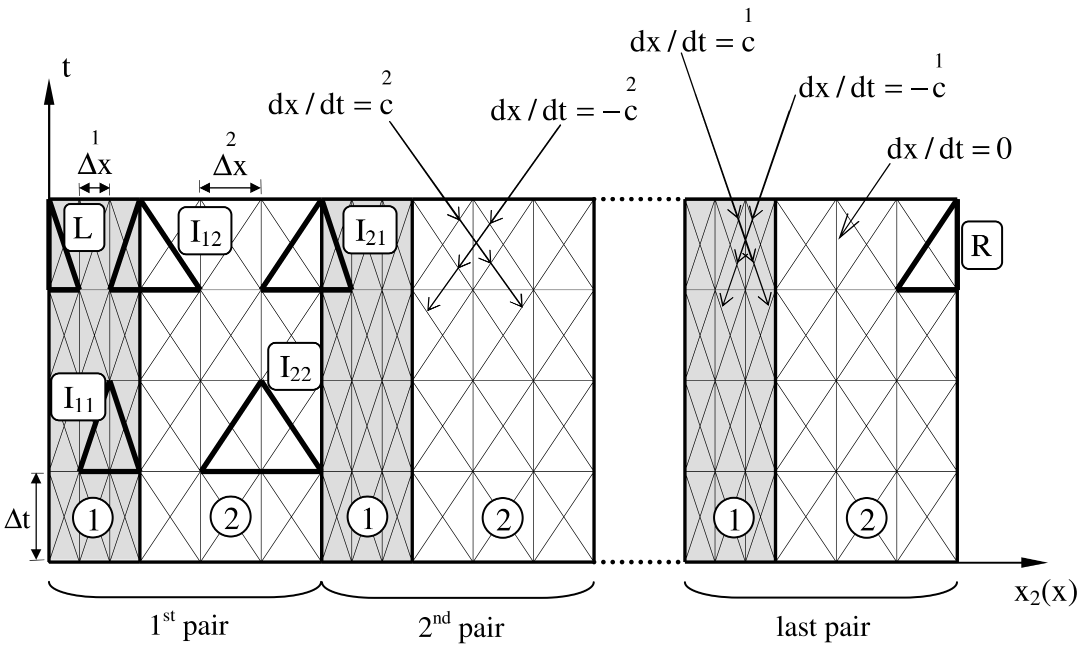


Figure 6.13 Network of characteristic lines on $(x_2 - t)$ plane

When these eigenvectors are substituted into Eq.(6.79), the canonical equations can be obtained explicitly as

$$
\begin{aligned}
&\rho c\, dv_1 - d\tau_{12} = 0 \quad && \text{along} \quad dx/dt = c \\
&\rho c\, dv_1 + d\tau_{12} = 0 \quad && \text{along} \quad dx/dt = -c \\
&C_{66}\, d\gamma_1 - d\tau_{12} = 0 \quad && \text{along} \quad dx/dt = 0
\end{aligned}
\tag{6.82}
$$

Canonical equations can be integrated easily by using a representative network shown in Fig. 6.13, which is composed of characteristic lines for layer 1 and 2. Different types of integration elements appear in the network, which are shown with thicker lines and marked with an explanatory label. Each layer of the composite slab is divided into three sublayers for illustrative purposes and during the integration along the characteristic lines of the integration elements, coefficients of the related layers must be considered. These elements together with the resulting integrated canonical equations are given in a tabular form in Table 6.2. The quantities pertaining to layers 1 and 2 in the integrated canonical equations are denoted by overhead numbers 1 and 2, see Table 6.2.

Table 6.2 Integration elements and the resulting integrated canonical equations

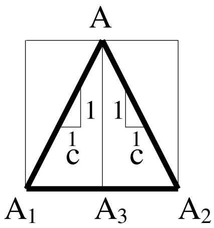
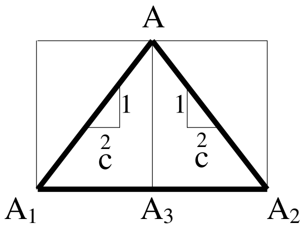
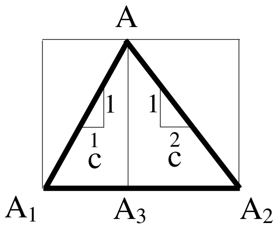
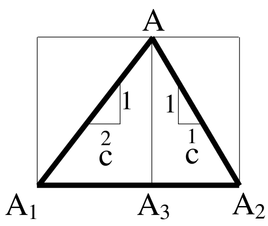
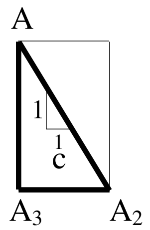
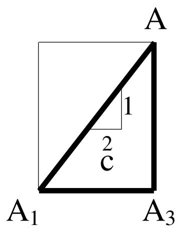
I_{11}		$\rho_1 \overset{1}{c}\,\overset{1}{v}_1(A) - \overset{1}{\tau}_{12}(A) = \rho_1 \overset{1}{c}\,\overset{1}{v}_1(A_1) - \overset{1}{\tau}_{12}(A_1)$ $\rho_1 \overset{1}{c}\,\overset{1}{v}_1(A) + \overset{1}{\tau}_{12}(A) = \rho_1 \overset{1}{c}\,\overset{1}{v}_1(A_2) + \overset{1}{\tau}_{12}(A_2)$ $\overset{1}{C}_{66}\,\overset{1}{\gamma}_1(A) - \overset{1}{\tau}_{12}(A) = \overset{1}{C}_{66}\,\overset{1}{\gamma}_1(A_3) - \overset{1}{\tau}_{12}(A_3)$
I_{22}		$\rho_2 \overset{2}{c}\,\overset{2}{v}_1(A) - \overset{2}{\tau}_{12}(A) = \rho_2 \overset{2}{c}\,\overset{2}{v}_1(A_1) - \overset{2}{\tau}_{12}(A_1)$ $\rho_2 \overset{2}{c}\,\overset{2}{v}_1(A) + \overset{2}{\tau}_{12}(A) = \rho_2 \overset{2}{c}\,\overset{2}{v}_1(A_2) + \overset{2}{\tau}_{12}(A_2)$ $\overset{2}{C}_{66}\,\overset{2}{\gamma}_1(A) - \overset{2}{\tau}_{12}(A) = \overset{2}{C}_{66}\,\overset{2}{\gamma}_1(A_3) - \overset{2}{\tau}_{12}(A_3)$
I_{12}		$\rho_1 \overset{1}{c}\,\overset{1}{v}_1(A) - \overset{1}{\tau}_{12}(A) = \rho_1 \overset{1}{c}\,\overset{1}{v}_1(A_1) - \overset{1}{\tau}_{12}(A_1)$ $\overset{1}{C}_{66}\,\overset{1}{\gamma}_1(A) - \overset{1}{\tau}_{12}(A) = \overset{1}{C}_{66}\,\overset{1}{\gamma}_1(A_3) - \overset{1}{\tau}_{12}(A_3)$ $\overset{1}{v}_1(A) = \overset{2}{v}_1(A)$ $\overset{1}{\tau}_{12}(A) = \overset{2}{\tau}_{12}(A)$ $\rho_2 \overset{2}{c}\,\overset{2}{v}_1(A) + \overset{2}{\tau}_{12}(A) = \rho_2 \overset{2}{c}\,\overset{2}{v}_1(A_2) + \overset{2}{\tau}_{12}(A_2)$ $\overset{2}{C}_{66}\,\overset{2}{\gamma}_1(A) - \overset{2}{\tau}_{12}(A) = \overset{2}{C}_{66}\,\overset{2}{\gamma}_1(A_3) - \overset{2}{\tau}_{12}(A_3)$
I_{21}		$\rho_2 \overset{2}{c}\,\overset{2}{v}_1(A) - \overset{2}{\tau}_{12}(A) = \rho_2 \overset{2}{c}\,\overset{2}{v}_1(A_1) - \overset{2}{\tau}_{12}(A_1)$ $\overset{2}{C}_{66}\,\overset{2}{\gamma}_1(A) - \overset{2}{\tau}_{12}(A) = \overset{2}{C}_{66}\,\overset{2}{\gamma}_1(A_3) - \overset{2}{\tau}_{12}(A_3)$ $\overset{1}{v}_1(A) = \overset{2}{v}_1(A)$ $\overset{1}{\tau}_{12}(A) = \overset{2}{\tau}_{12}(A)$ $\rho_1 \overset{1}{c}\,\overset{1}{v}_1(A) + \overset{1}{\tau}_{12}(A) = \rho_1 \overset{1}{c}\,\overset{1}{v}_1(A_2) + \overset{1}{\tau}_{12}(A_2)$ $\overset{1}{C}_{66}\,\overset{1}{\gamma}_1(A) - \overset{1}{\tau}_{12}(A) = \overset{1}{C}_{66}\,\overset{1}{\gamma}_1(A_3) - \overset{1}{\tau}_{12}(A_3)$

Table 6.2 (cont'd)

L		$\overset{1}{\tau}_{12}(A) = -s$ $\rho_1 \overset{1}{c}\, \overset{1}{v}_1(A) + \overset{1}{\tau}_{12}(A) = \rho_1 \overset{1}{c}\, \overset{1}{v}_1(A_2) + \overset{1}{\tau}_{12}(A_2)$ $\overset{1}{C}_{66}\, \overset{1}{\gamma}_1(A) - \overset{1}{\tau}_{12}(A) = \overset{1}{C}_{66}\, \overset{1}{\gamma}_1(A_3) - \overset{1}{\tau}_{12}(A_3)$
R		$\overset{2}{\tau}_{12}(A) = 0$ $\rho_2 \overset{2}{c}\, \overset{2}{v}_1(A) - \overset{2}{\tau}_{12}(A) = \rho_2 \overset{2}{c}\, \overset{2}{v}_1(A_1) - \overset{2}{\tau}_{12}(A_1)$ $\overset{2}{C}_{66}\, \overset{2}{\gamma}_1(A) - \overset{2}{\tau}_{12}(A) = \overset{2}{C}_{66}\, \overset{2}{\gamma}_1(A_3) - \overset{2}{\tau}_{12}(A_3)$

To obtain the solution in (x-t) plane, the analysis is started along the x-axis, where the unknown variables are all zero because of the quiescent initial conditions of the problem. Then, the governing equations in each element of the network lying just above the x-axis are integrated. This establishes the unknown variables at the points of the horizontal line $t = \Delta t$ of the (x-t) plane. Having determined the unknowns at $t = \Delta t$, the same procedure is used to establish them at the times $t = 2\Delta t$, $t = 3\Delta t$, etc.

6.2.3 Numerical results

The numerical results are obtained for the composite slab with the layer properties

$$\frac{\rho_2}{\rho_1} = 10 \;; \quad \frac{h_2}{h_1} = 2 \;; \quad \frac{\overset{2}{C}_{66}}{\overset{1}{C}_{66}} = 40 \qquad (6.83)$$

The reason for selecting the properties of layers 1 and 2 in Eq.(6.83) very different is to see the influence of refractive properties of interfaces on the transient response of the composite slab.

The results in Figs.6.14 and 6.15 are presented in nondimensional form, where nondimensional time is defined by $\bar{t} = \overset{2}{c}\, t / h_2$ with $\overset{2}{c} = \sqrt{\overset{2}{C}_{66} / \rho_2}$ being shear wave velocity in layer 2 and bar being used to denote a nondimensional quantity. The nondimensional rise time in the applied shear stress "s" (see Fig.6.12) is taken as $\bar{t}^* = 1$. The nondimensional velocity at the right end of the slab is normalized by $2s_0 / \rho c_0$, where ρ is the mass density per unit volume of the composite slab which is given by $\rho = n_1\rho_1 + n_2\rho_2$ with n_α being volume ratios given by $n_\alpha = h_\alpha / \Delta$. c_0 is wave velocity as predicted by effective modulus theory [29], which is defined by

$$c_0 = \sqrt{\frac{E_1 E_2}{\rho E}} \tag{6.84}$$

where $E_1 = \overset{1}{C}_{66} / n_1$, $E_2 = \overset{2}{C}_{66} / n_2$ and $E = E_1 + E_2$. Examination of the figures, which also include the prediction of effective modulus theory, reveals that

1. CM predicts the exact wave profiles closely not only around the head of the pulse, but also, for larger times after the disturbance reaches the station
2. the prediction of CM improves, as expected, as the number of pairs in the slab increases
3. CM is capable to predict the transient response of the slab with small number of pairs; but, its use becomes advantageous and should be suggested, in view of arguments given previously, when the number of the pairs is large.

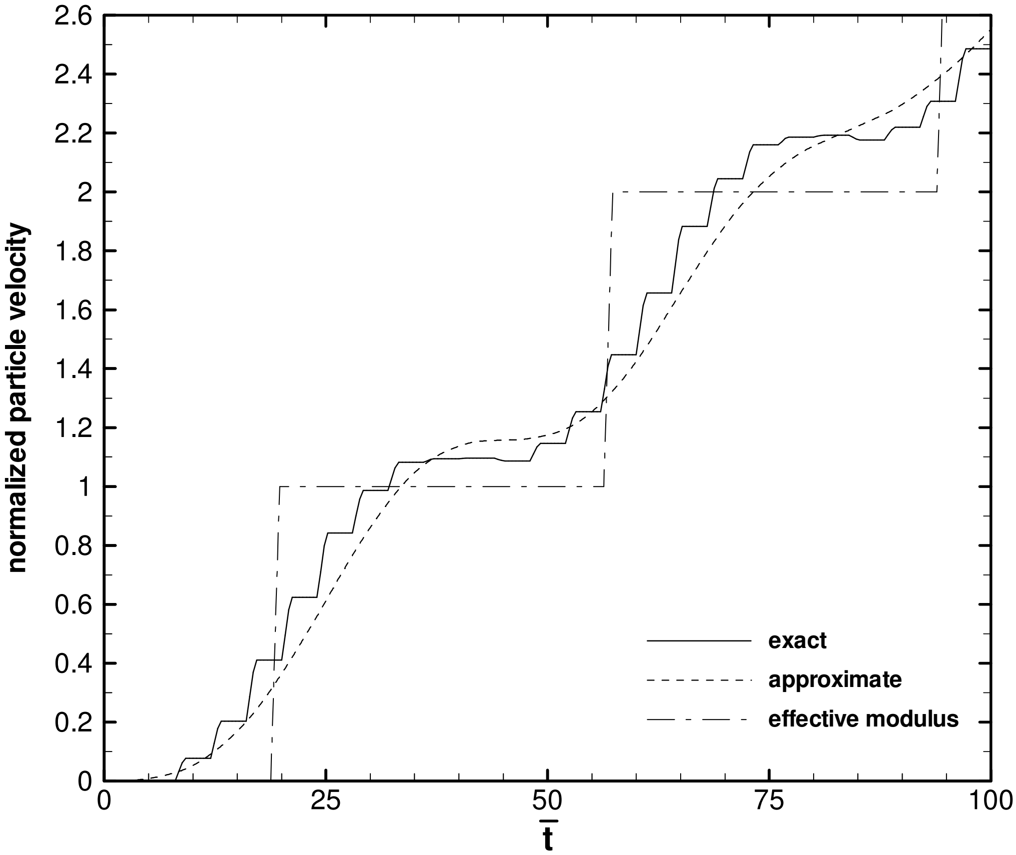


Figure 6.14 The wave profile for the particle velocity at the free end of the slab consisting of two pairs of alternating layers

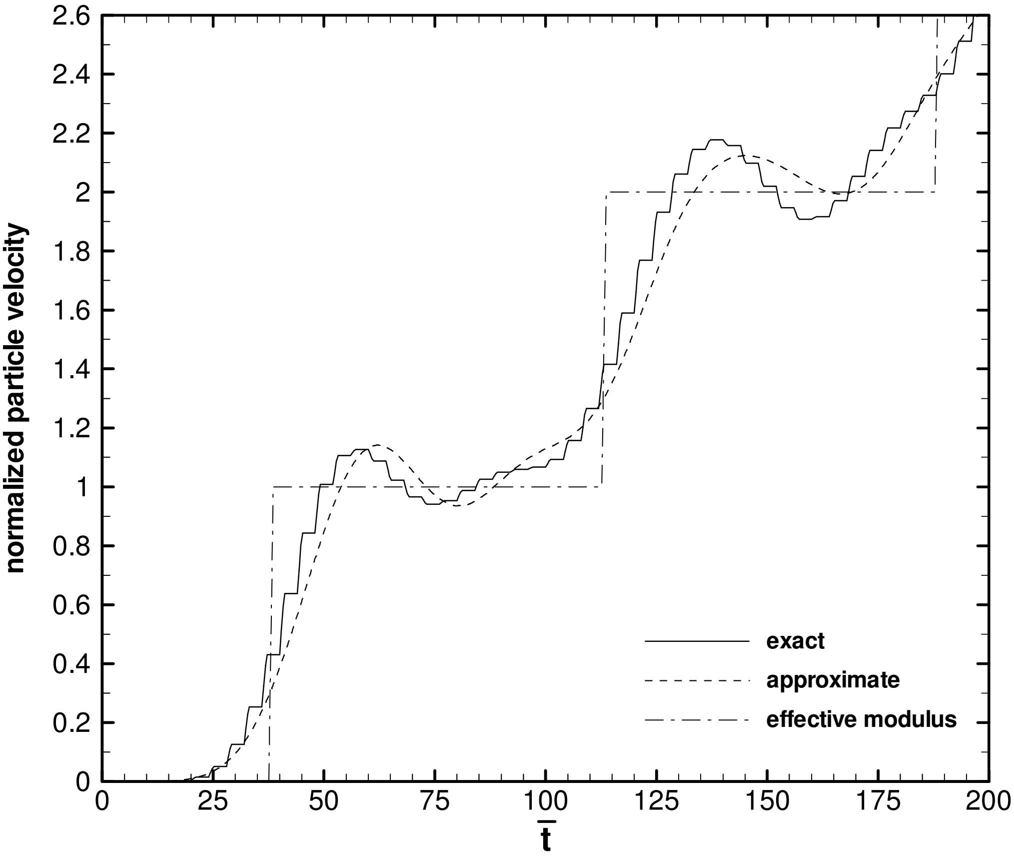


Figure 6.15 The wave profile for the particle velocity at the free end of the slab consisting of four pairs of alternating layers

CHAPTER 7

CONCLUSIONS

In this thesis, first, a higher order dynamic theory is developed for anisotropic thermoelastic plates; then, two approximate models based on this plate theory, namely discrete and continuum models, are proposed for periodic layered composites. The models are general in the sense that: they accommodate all kinds of deformation modes in the composite; dynamical and thermal effects are included in the formulation; the lamina material is assumed to be triclinic with no material symmetry. It is to be noted that the proposed models may also be used in the analysis of viscoelastic layered composites through the use of the correspondence principle [70].

In view of the assessments presented in Chapters 3 and 6, some of the important features of the proposed plate theory and composite models are stated below in itemized form.

1. The most important aspect of the plate theory is that it contains, in addition to GV's, also FV's as field variables. This feature of the theory permits satisfying the lateral boundary conditions of the plate correctly, thus, it improves the prediction of the theory for the dispersion characteristics of waves propagating in a plate. This is verified for a generally orthotropic plate considered in Chapter 3.
2. The order of the proposed approximate plate theory is arbitrary and by increasing it, the frequency range of the theory may be enlarged as observed in Chapter 3; in fact, as the order increases the prediction of the approximate theory approaches that of exact.
3. The form of the approximate plate theory is suitable for extending it to laminated composites; for that, one needs to add only, to the equations of

laminae, the interface conditions which can be written readily between the FV's appearing in the plate theory.

4. Of the two models proposed for layered composites, the continuum model (CM) is more important, which is developed in the study for **periodic** layered composites using smoothing operations. It is formwise a mixture theory with higher order microstructure and has the property: it contains asymptotically the Floquet periodicity conditions as the order of continuity conditions get larger (for proof, see Section 4.2). To the author's best knowledge, for periodic anisotropic layered composites there is no CM available in literature with this property.
5. The reflective and refractive properties of interfaces are well accounted for by CM, which is due to accommodating in the model the interface conditions correctly with the use of face variables appearing in the approximate plate theory. This property together with that stated in item 4 permits, as verified in Fig. 6.3, the CM to predict the periodic structure of spectra with passing and stopping bands for harmonic waves normal to layering. The prediction of CM improves along frequency and wave number axes with the orders of the theory and continuity conditions respectively, approaching the exact asymptotically in the limit.
6. The physical significance of real Floquet wave number is discussed through the use of some numerical results obtained in the study (see Figs. 6.8 and 6.9): it is the wave number of harmonic envelope curve interpolating the displacement values at discrete points along the axis perpendicular to layering with the increment "d" ("d" being the period of periodic layered composite in the direction perpendicular to layering).
7. CM is assessed by comparing its prediction with the exact for the spectra of harmonic waves propagating in various directions in a two-phase periodic layered composite. Choosing the comparison of spectra as a criterion for assessment may be justified on the basis that a dynamic model is described completely by its spectrum. Good matches in Figs.(6.3-6.11) give an indication for the reliability of CM in the analysis of periodic layered

composites. Figs. (6.14) and (6.15) verify, as example, the use of spectral criterion for assessment.

As a future development of this research, layered composites with delaminations may be considered. Handling these delaminations within the framework of the proposed CM requires only some modifications to be made in CC's.

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APPENDIX A

TRANSFORMATION RELATIONS

The following transformation relations hold in the global x_i coordinate system which is obtained by rotating the material coordinate system (1,2,3) in Fig.2.1 by an angle θ in (1-2) plane.

Elastic coefficients (C_{ij}):

$$C_{11} = \sin(\theta)^4 \hat{C}_{11} + \cos(\theta)^2 \left(2\sin(\theta)^2 \hat{C}_{12} + \cos(\theta)^2 \hat{C}_{22} + 4\sin(\theta)^2 \hat{C}_{66}\right)$$

$$C_{12} = \sin(\theta)^2 \hat{C}_{13} + \cos(\theta)^2 \hat{C}_{23}$$

$$C_{13} = \cos(\theta)^2 \sin(\theta)^2 \hat{C}_{11} + \left(3 + \cos(4\theta)\right)\hat{C}_{12} / 4 + \cos(\theta)^2 \sin(\theta)^2 \left(\hat{C}_{22} - 4\hat{C}_{66}\right)$$

$$C_{15} = -\sin(2\theta)\left(-2\sin(\theta)^2 \hat{C}_{11} - 2\cos(2\theta)\hat{C}_{12} + \hat{C}_{22} + \cos(2\theta)\hat{C}_{22} - 4\cos(2\theta)\hat{C}_{66}\right)/4$$

$$C_{22} = \hat{C}_{33}$$

$$C_{23} = \cos(\theta)^2 \hat{C}_{13} + \sin(\theta)^2 \hat{C}_{23}$$

$$C_{25} = \cos(\theta)\sin(\theta)\left(\hat{C}_{13} - \hat{C}_{23}\right)$$

$$C_{33} = \cos(\theta)^4 \hat{C}_{11} + \sin(\theta)^2 \left(2\cos(\theta)^2 \hat{C}_{12} + \sin(\theta)^2 \hat{C}_{22} + 4\cos(\theta)^2 \hat{C}_{66}\right)$$

$$C_{35} = \sin(2\theta)\left(2\cos(\theta)^2 \hat{C}_{11} - 2\cos(2\theta)\hat{C}_{12} - \hat{C}_{22} + \cos(2\theta)\hat{C}_{22} - 4\cos(2\theta)\hat{C}_{66}\right)/4$$

$$C_{44} = \sin(\theta)^2 \hat{C}_{44} + \cos(\theta)^2 \hat{C}_{55}$$

$$C_{46} = \cos(\theta)\sin(\theta)\left(\hat{C}_{55} - \hat{C}_{44}\right)$$

$$C_{55} = \left(2\sin(2\theta)^2 \hat{C}_{11} - 4\sin(2\theta)^2 \hat{C}_{12} + \hat{C}_{22} - \cos(4\theta)\hat{C}_{22} + 4\hat{C}_{66} + 4\cos(4\theta)\hat{C}_{66}\right)/8$$

$$C_{66} = \cos(\theta)^2 \hat{C}_{44} + \sin(\theta)^2 \hat{C}_{55}$$

other $C_{ij} = 0$

where C_{ij} and $\hat{C}_{ij}$ denote, respectively, the elastic coefficients in global and material frames.

Heat conduction and thermal coefficients $(k_{ij} \text{ and } \beta_{ij})$:

$$k_{11} = \hat{k}_{11} \sin^2(\theta) + \hat{k}_{22} \cos^2(\theta)$$

$$k_{22} = \hat{k}_{33}$$

$$k_{33} = \hat{k}_{11} \cos^2(\theta) + \hat{k}_{22} \sin^2(\theta)$$

$$k_{12} = (\hat{k}_{11} - \hat{k}_{22}) \sin(\theta) \cos(\theta) \quad , \text{ other } k_{ij} = 0$$

$$\beta_{11} = \hat{\beta}_{11} \sin^2(\theta) + \hat{\beta}_{22} \cos^2(\theta)$$

$$\beta_{22} = \hat{\beta}_{33}$$

$$\beta_{33} = \hat{\beta}_{11} \cos^2(\theta) + \hat{\beta}_{22} \sin^2(\theta)$$

$$\beta_{12} = (\hat{\beta}_{11} - \hat{\beta}_{22}) \sin(\theta) \cos(\theta) \quad , \text{ other } \beta_{ij} = 0$$

where (k_{ij}, β_{ij}) with and without overhead denote them, respectively, in material and global frames.

APPENDIX B

FREQUENCY EQUATION FOR 4th ORDER THEORY

$$
\begin{vmatrix}
\rho\omega^2 - C_{11}k^2 & 0 & 0 & \frac{C_{12}}{2h}ik & 0 & -C_{15}k^2 & 0 & 0 & 0 & 0 & 0 \\
\frac{-(C_{12}+C_{66})}{h}ik & \rho\omega^2 - C_{66}k^2 & 0 & \frac{-C_{22}}{2h^2} & \frac{C_{66}}{2h}ik & \frac{-(C_{25}+C_{46})}{h}ik & 0 & \frac{C_{46}}{2h}ik & 0 & 0 & 0 \\
\frac{3C_{66}}{h^2} & \frac{-3(C_{12}+C_{66})}{h}ik & \rho\omega^2 - C_{11}k^2 & \frac{C_{12}}{2h}ik & \frac{-3C_{66}}{2h^2} & \frac{3C_{46}}{h^2} & -C_{15}k^2 & \frac{-3C_{46}}{2h^2} & 0 & 0 & 0 \\
\frac{2C_{66}}{h}\gamma_0 & 0 & \frac{2C_{66}}{h}\gamma_2 & C_{66}ik & \frac{2C_{66}}{h}\gamma^+ & \frac{2C_{46}}{h}\gamma_0 & \frac{2C_{46}}{h}\gamma_2 & \frac{2C_{46}}{h}\gamma^+ & \frac{2C_{66}}{h}\gamma_4 & 0 & \frac{2C_{46}}{h}\gamma_4 \\
0 & \frac{2C_{22}}{h}\gamma_1 & 0 & \frac{2C_{22}}{h}\gamma^- & C_{12}ik & 0 & 0 & C_{25}ik & 0 & \frac{2C_{22}}{h}\gamma_3 & 0 \\
-C_{15}k^2 & 0 & 0 & \frac{C_{25}}{2h}ik & 0 & \rho\omega^2 - C_{55}k^2 & 0 & 0 & 0 & 0 & 0 \\
\frac{3C_{46}}{h^2} & \frac{-3(C_{25}+C_{46})}{h}ik & -C_{15}k^2 & \frac{C_{25}}{2h}ik & \frac{-3C_{46}}{2h^2} & \frac{3C_{44}}{h^2} & \rho\omega^2 - C_{55}k^2 & \frac{-3C_{44}}{2h^2} & 0 & 0 & 0 \\
\frac{2C_{46}}{h}\gamma_0 & 0 & \frac{2C_{46}}{h}\gamma_2 & C_{46}ik & \frac{2C_{46}}{h}\gamma^+ & \frac{2C_{44}}{h}\gamma_0 & \frac{2C_{44}}{h}\gamma_2 & \frac{2C_{44}}{h}\gamma^+ & \frac{2C_{46}}{h}\gamma_4 & 0 & \frac{2C_{44}}{h}\gamma_4 \\
\frac{10C_{66}}{h^2} & \frac{-3(C_{12}+C_{66})}{h}ik & \frac{35C_{66}}{h^2} & \frac{C_{12}}{2h}ik & \frac{-10C_{66}}{2h^2} & \frac{10C_{46}}{h^2} & \frac{35C_{46}}{h^2} & \frac{-10C_{46}}{2h^2} & \rho\omega^2 - C_{11}k^2 & \frac{-7(C_{12}+C_{66})}{h}ik & -C_{15}k^2 \\
\frac{-(C_{12}+C_{66})}{h}ik & \frac{15C_{22}}{h^2} & \frac{-5(C_{12}+C_{66})}{h}ik & \frac{-6C_{22}}{2h^2} & \frac{C_{66}}{2h}ik & \frac{-(C_{25}+C_{46})}{h}ik & \frac{-5(C_{25}+C_{46})}{h}ik & \frac{C_{46}}{2h}ik & 0 & \rho\omega^2 - C_{66}k^2 & 0 \\
\frac{10C_{46}}{h^2} & \frac{-3(C_{25}+C_{46})}{h}ik & \frac{35C_{46}}{h^2} & \frac{C_{25}}{2h}ik & \frac{-10C_{46}}{2h^2} & \frac{10C_{44}}{h^2} & \frac{35C_{44}}{h^2} & \frac{-10C_{44}}{2h^2} & -C_{15}k^2 & \frac{-7(C_{25}+C_{46})}{h}ik & \rho\omega^2 - C_{55}k^2
\end{vmatrix} = 0
$$

APPENDIX C

FREQUENCY EQUATION FOR SHEAR WAVES PROPAGATING PERPENDICULAR TO LAYERING AS DETERMINED BY CM (0th order theory and 1st order CC)

$$\text{SHAE}: \quad \mathbf{M}\,\mathbf{a} = \mathbf{0}$$

$$\text{Frequency equation}: \quad \det(\mathbf{M}) = 0$$

$$\mathbf{a} \Rightarrow \quad {}^{1}u_1^{0} \quad {}^{2}u_1^{0} \quad {}^{1}u_3^{0} \quad {}^{2}u_3^{0} \quad {}^{1}R_1^{+} \quad {}^{2}R_1^{+} \quad {}^{1}R_1^{-} \quad {}^{2}R_1^{-} \quad {}^{1}R_3^{+} \quad {}^{2}R_3^{+} \quad {}^{1}R_3^{-} \quad {}^{2}R_3^{-}$$

$$\mathbf{M} \Rightarrow \left[\begin{array}{cccc|cccccccc}
\rho_1\omega^2 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2h_1} & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2\omega^2 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2h_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_1\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2h_1} & 0 \\
0 & 0 & 0 & \rho_2\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2h_2} \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\frac{2}{h_1}\gamma_0\,{}^{1}C_{66} & 0 & \frac{2}{h_1}\gamma_0\,{}^{1}C_{46} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{h_2}\gamma_0\,{}^{2}C_{66} & 0 & \frac{2}{h_2}\gamma_0\,{}^{2}C_{46} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\frac{2}{h_1}\gamma_0\,{}^{1}C_{46} & 0 & \frac{2}{h_1}\gamma_0\,{}^{1}C_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & \frac{2}{h_2}\gamma_0\,{}^{2}C_{66} & 0 & \frac{2}{h_2}\gamma_0\,{}^{2}C_{46} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta}{2}ik & \frac{\Delta}{2}ik & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Delta}{2}ik & \frac{\Delta}{2}ik & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}ik & -\frac{\Delta}{2}ik & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}ik & -\frac{\Delta}{2}ik
\end{array}\right]$$

$$
\begin{array}{cccccccc}
\overset{1}{S}{}_1^{+} & \overset{2}{S}{}_1^{+} & \overset{1}{S}{}_1^{-} & \overset{2}{S}{}_1^{-} & \overset{1}{S}{}_3^{+} & \overset{2}{S}{}_3^{+} & \overset{1}{S}{}_3^{-} & \overset{2}{S}{}_3^{-}
\end{array}
$$

$$
\left.
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \frac{2}{h_1}\gamma^{-}\overset{1}{C}_{66} & 0 & 0 & 0 & \frac{2}{h_1}\gamma^{-}\overset{1}{C}_{46} & 0 \\
0 & 0 & 0 & \frac{2}{h_2}\gamma^{-}\overset{2}{C}_{66} & 0 & 0 & 0 & \frac{2}{h_2}\gamma^{-}\overset{2}{C}_{46} \\
0 & 0 & \frac{2}{h_1}\gamma^{-}\overset{1}{C}_{46} & 0 & 0 & 0 & \frac{2}{h_1}\gamma^{-}\overset{1}{C}_{44} & 0 \\
0 & 0 & 0 & \frac{2}{h_2}\gamma^{-}\overset{2}{C}_{46} & 0 & 0 & 0 & \frac{2}{h_2}\gamma^{-}\overset{2}{C}_{44} \\
\frac{2}{h_1}\gamma^{+}\overset{1}{C}_{66} & 0 & 0 & 0 & \frac{2}{h_1}\gamma^{+}\overset{1}{C}_{46} & 0 & 0 & 0 \\
0 & \frac{2}{h_2}\gamma^{+}\overset{2}{C}_{66} & 0 & 0 & 0 & \frac{2}{h_2}\gamma^{+}\overset{2}{C}_{46} & 0 & 0 \\
\frac{2}{h_1}\gamma^{+}\overset{1}{C}_{46} & 0 & 0 & 0 & \frac{2}{h_1}\gamma^{+}\overset{1}{C}_{44} & 0 & 0 & 0 \\
0 & \frac{2}{h_2}\gamma^{+}\overset{2}{C}_{46} & 0 & 0 & 0 & \frac{2}{h_2}\gamma^{+}\overset{2}{C}_{44} & 0 & 0 \\
\hline
\frac{\Delta}{2}ik & \frac{\Delta}{2}ik & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta}{2}ik & \frac{\Delta}{2}ik & -1 & -1 \\
-1 & 1 & \frac{\Delta}{2}ik & -\frac{\Delta}{2}ik & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}ik & -\frac{\Delta}{2}ik \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right]
$$

APPENDIX D

FREQUENCY EQUATION FOR AXIAL WAVES PROPAGATING PARALLEL TO LAYERING AS DETERMINED BY CM (1st order theory with exact CC)

$$\text{SHAE}: \quad \mathbf{M\,a}=\mathbf{0}$$

$$\text{Frequency equation}: \quad \det(\mathbf{M})=0$$

where, for the sake of convenience, **M** is factorized as

$$\mathbf{M}=\begin{bmatrix}\mathbf{M1} & \mathbf{M2}\\ \mathbf{M3} & \mathbf{M4}\end{bmatrix}$$

Columns of the matrix **M** correspond to the following field variables respectively:

$$\mathbf{a}\Rightarrow\left({}^{1}u_1^{0},{}^{2}u_1^{0},{}^{1}u_3^{0},{}^{2}u_3^{0},{}^{1}u_2^{1},{}^{2}u_2^{1},\,{}^{1}R_1^{-},{}^{2}R_1^{-},{}^{1}R_2^{+},{}^{2}R_2^{+},{}^{1}R_3^{-},{}^{2}R_3^{-},{}^{1}S_1^{+},{}^{2}S_1^{+},{}^{1}S_2^{-},{}^{2}S_2^{-},{}^{1}S_3^{+},{}^{2}S_3^{+}\right)$$

M1, **M2**, **M3** and **M4** matrices are defined by

$$\mathbf{M1}\Rightarrow\begin{bmatrix}
\rho_1\omega^2-\overset{1}{C}_{11}k^2 & 0 & -\overset{1}{C}_{15}k^2 & 0 & 0 & 0\\
0 & \rho_2\omega^2-\overset{2}{C}_{11}k^2 & 0 & -\overset{2}{C}_{15}k^2 & 0 & 0\\
-\overset{1}{C}_{15}k^2 & 0 & \rho_1\omega^2-\overset{1}{C}_{55}k^2 & 0 & 0 & 0\\
0 & -\overset{2}{C}_{15}k^2 & 0 & \rho_2\omega^2-\overset{2}{C}_{55}k^2 & 0 & 0\\
\dfrac{-\overset{1}{C}_{66}-\overset{1}{C}_{12}}{h_1}ik & 0 & \dfrac{-\overset{1}{C}_{46}-\overset{1}{C}_{25}}{h_1}ik & 0 & \rho_1\omega^2-\overset{1}{C}_{66}k^2 & 0\\
0 & \dfrac{-\overset{2}{C}_{66}-\overset{2}{C}_{12}}{h_2}ik & 0 & \dfrac{-\overset{2}{C}_{46}-\overset{2}{C}_{25}}{h_2}ik & 0 & \rho_2\omega^2-\overset{2}{C}_{66}k^2
\end{bmatrix}$$

$$\mathbf{M2}\Rightarrow\begin{bmatrix}
\frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{12}}{2h_1}ik & 0 & 0 & 0\\
0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{12}}{2h_2}ik & 0 & 0\\
0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & \frac{\overset{1}{C}_{25}}{2h_1}ik & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & \frac{\overset{2}{C}_{25}}{2h_2}ik & 0 & 0\\
0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & \frac{\overset{1}{C}_{66}}{2h_1}ik & 0 & \frac{\overset{1}{C}_{22}}{2h_1^2} & 0 & \frac{\overset{1}{C}_{46}}{2h_1}ik & 0\\
0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & \frac{\overset{2}{C}_{66}}{2h_2}ik & 0 & \frac{\overset{2}{C}_{22}}{2h_2^2} & 0 & \frac{\overset{2}{C}_{46}}{2h_2}ik
\end{bmatrix}$$

$$
\mathbf{M3} \Rightarrow \begin{bmatrix}
\overset{1}{C}_{66}\,\gamma_0 & 0 & \overset{1}{C}_{46}\,\gamma_0 & 0 & 0 & 0 \\
0 & \overset{2}{C}_{66}\,\gamma_0 & 0 & \overset{2}{C}_{46}\,\gamma_0 & 0 & 0 \\
0 & 0 & 0 & 0 & \overset{1}{C}_{22}\,\gamma_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{22}\,\gamma_1 \\
\overset{1}{C}_{46}\,\gamma_0 & 0 & \overset{1}{C}_{44}\,\gamma_0 & 0 & 0 & 0 \\
0 & \overset{2}{C}_{46}\,\gamma_0 & 0 & \overset{2}{C}_{44}\,\gamma_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{M4} \Rightarrow \begin{bmatrix}
\frac{-h_1}{2} & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{66}\,\gamma^{+} & 0 & \frac{h_1 \overset{1}{C}_{66}}{2}\,ik & 0 & \overset{1}{C}_{46}\,\gamma^{+} & 0 \\
0 & \frac{-h_2}{2} & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{66}\,\gamma^{+} & 0 & \frac{h_2 \overset{2}{C}_{66}}{2}\,ik & 0 & \overset{2}{C}_{46}\,\gamma^{+} \\
0 & 0 & \frac{-h_1}{2} & 0 & 0 & 0 & \frac{h_1 \overset{1}{C}_{12}}{2}\,ik & 0 & \overset{1}{C}_{22}\,\gamma^{-} & 0 & \frac{h_1 \overset{1}{C}_{25}}{2}\,ik & 0 \\
0 & 0 & 0 & \frac{-h_2}{2} & 0 & 0 & 0 & \frac{h_2 \overset{2}{C}_{12}}{2}\,ik & 0 & \overset{2}{C}_{22}\,\gamma^{-} & 0 & \frac{h_2 \overset{2}{C}_{25}}{2}\,ik \\
0 & 0 & 0 & 0 & \frac{-h_1}{2} & 0 & \overset{1}{C}_{46}\,\gamma^{+} & 0 & \frac{h_1 \overset{1}{C}_{46}}{2}\,ik & 0 & \overset{1}{C}_{44}\,\gamma^{+} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-h_2}{2} & 0 & \overset{2}{C}_{46}\,\gamma^{+} & 0 & \frac{h_2 \overset{2}{C}_{46}}{2}\,ik & 0 & \overset{2}{C}_{44}\,\gamma^{+} \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

APPENDIX E

FREQUENCY EQUATION FOR WAVES PROPAGATING OBLIQUELY TO LAYERING AS DETERMINED BY CM (2nd order theory and 2nd order CC)

$$\text{SHAE}: \quad \mathbf{M\,a} = \mathbf{0}$$

$$\text{Frequency equation}: \quad \det(\mathbf{M}) = 0$$

where, for the sake of convenience, **M** is factorized as

$$\mathbf{M} = \begin{array}{|c|c|c|} \hline \mathbf{M1} & \mathbf{M2} & \mathbf{M3} \\ \hline \mathbf{M4} & \mathbf{M5} & \mathbf{M6} \\ \hline \mathbf{M7} & \mathbf{M8} & \mathbf{M9} \\ \hline \end{array} \quad \text{and} \quad \mathbf{M9} = \begin{array}{|c|c|} \hline \mathbf{M9}_1 & \mathbf{M9}_2 \\ \hline \mathbf{M9}_3 & \mathbf{M9}_4 \\ \hline \end{array}$$

Columns of the matrix **M** correspond, respectively, to the following field variables

$$\mathbf{a} \Rightarrow \left({}^1u_1^0, {}^2u_1^0, {}^1u_3^0, {}^2u_3^0, {}^1u_2^1, {}^2u_2^1, {}^1R_1^-, {}^2R_1^-, {}^1R_2^+, {}^2R_2^+, {}^1R_3^-, {}^2R_3^-, {}^1S_1^+, {}^2S_1^+, {}^1S_2^-, {}^2S_2^-, {}^1S_3^+, {}^2S_3^+ \right)$$

The submatrices in the matrices **M** and **M9** are defined by

$$\mathbf{M1} \Rightarrow \begin{array}{|cccccc|} \hline \rho_1\omega^2 - \overset{1}{C}_{11}k^2 & 0 & 0 & 0 & -\overset{1}{C}_{15}k^2 & 0 \\ 0 & \rho_2\omega^2 - \overset{2}{C}_{11}k^2 & 0 & 0 & 0 & -\overset{2}{C}_{15}k^2 \\ 0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{66}k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{66}k^2 & 0 & 0 \\ -\overset{1}{C}_{15}k^2 & 0 & 0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{55}k^2 & 0 \\ 0 & -\overset{2}{C}_{15}k^2 & 0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{55}k^2 \\ \hline \end{array}$$

$$\mathbf{M2} \Rightarrow \begin{array}{|cccccccccccc|} \hline 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ \hline \end{array}$$

$$\mathbf{M3} \Rightarrow \boxed{\begin{matrix}
0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{12}}{2h_1}ik & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{12}}{2h_2}ik & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{66}}{2h_1}ik & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{46}}{2h_1}ik & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{66}}{2h_2}ik & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{46}}{2h_2}ik \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{25}}{2h_1}ik & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{25}}{2h_2}ik & 0 & 0 & 0 & 0
\end{matrix}}$$

$$
\mathbf{M4} \Rightarrow
\begin{array}{|cccccc|}
\hline
0 & 0 & \dfrac{\overset{1}{C}_{12}+\overset{1}{C}_{66}}{-h_1}ik & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{\overset{2}{C}_{12}+\overset{2}{C}_{66}}{-h_2}ik & 0 & 0 \\
\dfrac{\overset{1}{C}_{12}+\overset{1}{C}_{66}}{-h_1}ik & 0 & 0 & 0 & \dfrac{\overset{1}{C}_{25}+\overset{1}{C}_{46}}{-h_1}ik & 0 \\
0 & \dfrac{\overset{2}{C}_{12}+\overset{2}{C}_{66}}{-h_2}ik & 0 & 0 & 0 & \dfrac{\overset{2}{C}_{25}+\overset{2}{C}_{46}}{-h_2}ik \\
0 & 0 & \dfrac{\overset{1}{C}_{25}+\overset{1}{C}_{46}}{-h_1}ik & 0 & 0 & 0 \\
0 & 0 & & \dfrac{\overset{2}{C}_{25}+\overset{2}{C}_{46}}{-h_2}ik & 0 & 0 \\
\dfrac{3\overset{1}{C}_{66}}{h_1^2} & 0 & 0 & 0 & \dfrac{3\overset{1}{C}_{46}}{h_1^2} & 0 \\
0 & \dfrac{3\overset{2}{C}_{66}}{h_2^2} & 0 & 0 & 0 & \dfrac{3\overset{2}{C}_{46}}{h_2^2} \\
0 & 0 & \dfrac{3\overset{1}{C}_{22}}{h_1^2} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{3\overset{2}{C}_{22}}{h_2^2} & 0 & 0 \\
\dfrac{3\overset{1}{C}_{46}}{h_1^2} & 0 & 0 & 0 & \dfrac{3\overset{1}{C}_{44}}{h_1^2} & 0 \\
0 & \dfrac{3\overset{2}{C}_{46}}{h_2^2} & 0 & 0 & 0 & \dfrac{3\overset{2}{C}_{44}}{h_2^2} \\
\hline
\end{array}
$$

$$\mathbf{M5} \Rightarrow \begin{bmatrix}
\rho_1\omega^2 - \overset{1}{C}_{11}k^2 & 0 & 0 & 0 & -\overset{1}{C}_{15}k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2\omega^2 - \overset{2}{C}_{11}k^2 & 0 & 0 & 0 & -\overset{2}{C}_{15}k^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{66}k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{66}k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\overset{1}{C}_{15}k^2 & 0 & 0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{55}k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\overset{2}{C}_{15}k^2 & 0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{55}k^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3(\overset{1}{C}_{12}+\overset{1}{C}_{66})}{-h_1}ik & 0 & 0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{11}k^2 & 0 & 0 & 0 & -\overset{1}{C}_{15}k^2 & 0 \\
0 & 0 & 0 & \frac{3(\overset{2}{C}_{12}+\overset{2}{C}_{66})}{-h_2}ik & 0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{11}k^2 & 0 & 0 & 0 & -\overset{2}{C}_{15}k^2 \\
\frac{3(\overset{1}{C}_{12}+\overset{1}{C}_{66})}{-h_1}ik & 0 & 0 & 0 & \frac{3(\overset{1}{C}_{25}+\overset{1}{C}_{46})}{-h_1}ik & 0 & 0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{66}k^2 & 0 & 0 & 0 \\
0 & \frac{3(\overset{2}{C}_{12}+\overset{2}{C}_{66})}{-h_2}ik & 0 & 0 & 0 & \frac{3(\overset{2}{C}_{25}+\overset{2}{C}_{46})}{-h_2}ik & 0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{66}k^2 & 0 & 0 \\
0 & 0 & \frac{3(\overset{1}{C}_{25}+\overset{1}{C}_{46})}{-h_1}ik & 0 & 0 & 0 & -\overset{1}{C}_{15}k^2 & 0 & 0 & 0 & \rho_1\omega^2 - \overset{1}{C}_{55}k^2 & 0 \\
0 & 0 & 0 & \frac{3(\overset{2}{C}_{25}+\overset{2}{C}_{46})}{-h_2}ik & 0 & 0 & 0 & -\overset{2}{C}_{15}k^2 & 0 & 0 & 0 & \rho_2\omega^2 - \overset{2}{C}_{55}k^2
\end{bmatrix}$$

$$
\mathbf{M6} \Rightarrow
\begin{array}{|cccccccccccccccccccccccc|}
\hline
\frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{1}{C}_{66}}{2h_1^2} & 0 & \frac{\overset{1}{C}_{12}}{2h_1}ik & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{1}{C}_{46}}{2h_1^2} & 0 \\
0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{2}{C}_{66}}{2h_2^2} & 0 & \frac{\overset{2}{C}_{12}}{2h_2}ik & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{2}{C}_{46}}{2h_2^2} \\
0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{66}}{2h_1}ik & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{1}{C}_{22}}{2h_1^2} & 0 & \frac{\overset{1}{C}_{46}}{2h_1}ik & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{66}}{2h_2}ik & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{2}{C}_{22}}{2h_2^2} & 0 & \frac{\overset{2}{C}_{46}}{2h_2}ik & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{1}{C}_{46}}{2h_1^2} & 0 & \frac{\overset{1}{C}_{25}}{2h_1}ik & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{1}{C}_{44}}{2h_1^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{2}{C}_{46}}{2h_2^2} & 0 & \frac{\overset{2}{C}_{25}}{2h_2}ik & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{2}{C}_{44}}{2h_2^2} \\
0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3\overset{1}{C}_{66}}{2h_1^2} & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{12}}{2h_1}ik & 0 & \frac{-3\overset{1}{C}_{46}}{2h_1^2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3\overset{2}{C}_{66}}{2h_2^2} & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{12}}{2h_2}ik & 0 & \frac{-3\overset{2}{C}_{46}}{2h_2^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{66}}{2h_1}ik & 0 & \frac{-3\overset{1}{C}_{22}}{2h_1^2} & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{1}{C}_{46}}{2h_1^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{66}}{2h_2}ik & 0 & \frac{-3\overset{2}{C}_{22}}{2h_2^2} & 0 & 0 & 0 & 0 & 0 & \frac{-\overset{2}{C}_{46}}{2h_2^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_1} & 0 & \frac{-3\overset{1}{C}_{46}}{2h_1^2} & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{25}}{2h_1}ik & 0 & \frac{-3\overset{1}{C}_{44}}{2h_1^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{0.5}{h_2} & 0 & \frac{-3\overset{2}{C}_{66}}{2h_2^2} & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{25}}{2h_2}ik & 0 & \frac{-3\overset{2}{C}_{44}}{2h_2^2} & 0 & 0 \\
\hline
\end{array}
$$

$$
\mathbf{M7} \Rightarrow
\begin{array}{|cccccc|}
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\overset{1}{C}_{66}\gamma_0 & 0 & 0 & 0 & \overset{1}{C}_{46}\gamma_0 & 0 \\
0 & \overset{2}{C}_{66}\gamma_0 & 0 & 0 & 0 & \overset{2}{C}_{46}\gamma_0 \\
0 & 0 & \overset{1}{C}_{22}\gamma_0 & 0 & 0 & 0 \\
0 & 0 & 0 & \overset{2}{C}_{22}\gamma_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\overset{1}{C}_{46}\gamma_0 & 0 & 0 & 0 & \overset{1}{C}_{44}\gamma_0 & 0 \\
0 & \overset{2}{C}_{46}\gamma_0 & 0 & 0 & 0 & \overset{2}{C}_{44}\gamma_0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

$$
\mathbf{M8} \Rightarrow
\begin{array}{|cccccccccccc|}
\hline
\overset{1}{C}_{66}\gamma_1 & 0 & 0 & 0 & \overset{1}{C}_{46}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overset{2}{C}_{66}\gamma_1 & 0 & 0 & 0 & \overset{2}{C}_{46}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{66}\gamma_2 & 0 & 0 & 0 & \overset{1}{C}_{46}\gamma_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{66}\gamma_2 & 0 & 0 & 0 & \overset{2}{C}_{46}\gamma_2 \\
0 & 0 & \overset{1}{C}_{22}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \overset{2}{C}_{22}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{22}\gamma_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{22}\gamma_2 & 0 & 0 \\
\overset{1}{C}_{46}\gamma_1 & 0 & 0 & 0 & \overset{1}{C}_{44}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overset{2}{C}_{46}\gamma_1 & 0 & 0 & 0 & \overset{2}{C}_{44}\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{46}\gamma_2 & 0 & 0 & 0 & \overset{1}{C}_{44}\gamma_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{46}\gamma_2 & 0 & 0 & 0 & \overset{2}{C}_{44}\gamma_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

$$
\mathbf{M9}_1 \Rightarrow \begin{bmatrix}
-0.5h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.5h_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5h_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.5h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.5h_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.5h_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5h_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5h_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5h_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5h_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5h_2
\end{bmatrix}
$$

$$
\mathbf{M9}_3 \Rightarrow \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\Delta}{2}i\kappa & \frac{\Delta}{2}i\kappa & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta}{2}i\kappa & \frac{\Delta}{2}i\kappa & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Delta}{2}i\kappa & \frac{\Delta}{2}i\kappa & -1 & -1 \\
-1 & 1 & \frac{\Delta}{2}i\kappa & -\frac{\Delta}{2}i\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}i\kappa & -\frac{\Delta}{2}i\kappa & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}i\kappa & -\frac{\Delta}{2}i\kappa
\end{bmatrix}
$$

$$
\mathbf{M9}_4 \Rightarrow \begin{bmatrix}
\frac{\Delta}{2}i\kappa & \frac{\Delta}{2}i\kappa & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\Delta}{2}i\kappa & \frac{\Delta}{2}i\kappa & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Delta}{2}i\kappa & \frac{\Delta}{2}i\kappa & -1 & -1 \\
-1 & 1 & \frac{\Delta}{2}i\kappa & -\frac{\Delta}{2}i\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}i\kappa & -\frac{\Delta}{2}i\kappa & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \frac{\Delta}{2}i\kappa & -\frac{\Delta}{2}i\kappa \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\mathbf{M9}_2 \Rightarrow \boxed{\begin{array}{cccccccccccc}
0 & 0 & \overset{1}{C}_{66}\gamma^- & 0 & \frac{\overset{1}{C}_{66}}{2}h_1 ik & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{46}\gamma^- & 0 \\
0 & 0 & 0 & \overset{2}{C}_{66}\gamma^- & 0 & \frac{\overset{2}{C}_{66}}{2}h_2 ik & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{46}\gamma^- \\
\overset{1}{C}_{66}\gamma^+ & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{66}}{2}h_1 ik & 0 & \overset{1}{C}_{46}\gamma^+ & 0 & 0 & 0 \\
0 & \overset{2}{C}_{66}\gamma^+ & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{66}}{2}h_2 ik & 0 & \overset{2}{C}_{46}\gamma^+ & 0 & 0 \\
\frac{\overset{1}{C}_{12}}{2}h_1 ik & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{22}\gamma^- & 0 & \frac{\overset{1}{C}_{25}}{2}h_1 ik & 0 & 0 & 0 \\
0 & \frac{\overset{2}{C}_{12}}{2}h_2 ik & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{22}\gamma^- & 0 & \frac{\overset{2}{C}_{25}}{2}h_2 ik & 0 & 0 \\
0 & 0 & \frac{\overset{1}{C}_{12}}{2}h_1 ik & 0 & \overset{1}{C}_{22}\gamma^+ & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{25}}{2}h_1 ik & 0 \\
0 & 0 & 0 & \frac{\overset{2}{C}_{12}}{2}h_2 ik & 0 & \overset{2}{C}_{22}\gamma^+ & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{25}}{2}h_2 ik \\
0 & 0 & \overset{1}{C}_{46}\gamma^- & 0 & \frac{\overset{1}{C}_{46}}{2}h_1 ik & 0 & 0 & 0 & 0 & 0 & \overset{1}{C}_{44}\gamma^- & 0 \\
0 & 0 & 0 & \overset{2}{C}_{46}\gamma^- & 0 & \frac{\overset{2}{C}_{46}}{2}h_2 ik & 0 & 0 & 0 & 0 & 0 & \overset{2}{C}_{44}\gamma^- \\
\overset{1}{C}_{46}\gamma^+ & 0 & 0 & 0 & 0 & 0 & \frac{\overset{1}{C}_{46}}{2}h_1 ik & 0 & \overset{1}{C}_{44}\gamma^+ & 0 & 0 & 0 \\
0 & \overset{2}{C}_{46}\gamma^+ & 0 & 0 & 0 & 0 & 0 & \frac{\overset{2}{C}_{46}}{2}h_2 ik & 0 & \overset{2}{C}_{44}\gamma^+ & 0 & 0
\end{array}}$$

VITA

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