SEQUENTIAL AND PARALLEL HEURISTIC ALGORITHMS FOR THE RECTILINEAR STEINER TREE PROBLEM

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ABSTRACT

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The Steiner Tree problem is one of the most popular graph problems and has many application areas. The rectilinear version of this problem, introduced by Hanan, has taken special attention since it addresses a fundamental matter in "Physical Design" phase of the Very Large Scale Integrated (VLSI) Computer Aided Design (CAD) process. The Rectilinear Steiner Tree Problem asks for a minimum length tree that interconnects a given set of points by only horizontal and vertical line segments, enabling the use of extra points. There are various exact algorithms. However the problem is NP-complete hence approximation algorithms have to be used especially for large instances. In this thesis work, first a survey on heuristics for the Rectilinear Steiner Tree Problem is conducted and then two recently developed successful algorithms, BGA by Kahng *et. al.* and RST by Zhou have been studied and investigated deeply. Both algorithms have subproblems, most of which have

individual backgrounds in literature. After an analysis of BGA and RST, the thesis proposes a modification on RST, which leads to a faster and non-recursive version. The efficiency of the modified algorithm has been validated by computational tests using both random and VLSI benchmark instances. A partially parallelized version of the modified algorithm is also proposed for distributed computing environments. It is implemented using MPI (message passing interface) middleware and the results of comparative tests conducted on a cluster of workstations are presented. The proposed distributed algorithm has also proved to be promising especially for large problem instances.

Keywords: Analysis of Algorithms, Approximation Algorithms, Distributed Algorithms, Graph Theory, Rectilinear Steiner Tree

ÖZ

DOĞRULU STEİNER AĞAÇ PROBLEMİ İÇİN YAKLAŞIK SONUÇ VEREN SERİ VE PARALEL ALGORİTMALAR

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Steiner Ağaç Problemi birçok uygulama alanı bulunan en gözde çizge problemlerinden biridir. Bu problemin Hanan tarafından tanımlanan doğrulu biçimi, Çok Büyük Ölçekli Tümleşik (VLSI) Bilgisayar Destekli Tasarım (CAD) işleminin fiziksel tasarım evresinde temel bir soruna çözüm olması nedeniyle özel bir ilgi çekmiştir. Doğrulu Steiner Ağaç Problemi, verilen bir nokta kümesindeki noktaları, fazladan noktalar da kullanabilerek, yalnızca yatay ve dikey doğrularla birleştiren en kısa uzunluktaki ağacı bulmaya çalışır. Problemi kesin sonuç bularak çözen çeşitli algoritmalar bulunmaktadır. Ancak problem NP-tamdır ve dolayısıyla özellikle büyük nokta kümeleri için yaklaşık çözüm veren algoritmalar kullanılmalıdır. Bu tez çalışmasında öncelikle yaklaşık çözüm veren algoritmalar üzerinde inceleme yapılmış ve sonrasında yakın zamanda Kahng *et. al.* tarafından geliştirilen RST ve Zhou tarafından geliştirilen BGA algoritmaları çalışılmış ve de

derinlemesine incelenmiştir. Her iki algoritma da teknik yazında çoğunun kendi arka planları bulunan daha küçük problemlere bölünmüştür. BGA ve RST üzerinde yapılan analiz sonucu tez çalışmasında RST üzerine daha hızlı ve tekrarlamasız bir değişiklik önerilmiştir. Değiştirilmiş algoritmanın verimliliği hem rasgele hem de VLSI referans örnekleri için test edilmiş ve de gösterilmiştir. Bu algoritmanın dağıtık hesaplama ortamı için kısmen paralel biçimi önerilmiştir. Bu algoritma MPI (mesaj gönderim arayüzü) altyapısı kullanılarak gerçeklenmiş ve de birbirine küme şeklinde bağlı iş istasyonları üzerinde karşılaştırmalı testler yapılmıştır. Önerilen dağıtık algoritmanın özellikle büyük problem örnekleri için umut verici olduğu gösterilmiştir.

Anahtar Kelimeler: Algoritma Analizi, Çizge Algoritmaları, Dağıtık Algoritmalar, Doğrulu Steiner Ağaç Problemi, Yaklaşık Algoritmalar

To my Family

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ABBREVIATIONS

B1S : Batched 1-Steiner

BGA : Batched Greedy Algorithm

DRC : Design Rule Checking

EDA : Electronic Design Automation

FST : Full Steiner Tree

GTCA : Greedy Triple Contraction Algorithm

HGP : Hierarchical Greedy Preprocessing

HTU : Hypothetical Taxonomic Units

I1S : Iterated 1-Steiner

IRV : Iterated Rajagopalan and Vazirani

LB : Left-Bottom

LCA : Least Common Ancestor

LHP : Left-Hand Plane

MPI : Message Passing Interface

MST : Minimum Spanning Tree

NOW : Network of Workstations

RHP : Right-Hand Plane

RMST : Rectilinear Minimum Spanning Tree

RSG : Rectilinear Spanning Graph

RSMT : Rectilinear Steiner Minimum Tree

RST : Rectilinear Steiner Tree Algorithm

SMT : Steiner Minimum Tree

STGP : Steiner Tree Problem in Graphs

TR : Top-Right

VLSI : Very Large Scaled Integrated

CHAPTER 1

INTRODUCTION

The Steiner Tree problem is one of the oldest optimization problems in graph theory literature. It has many application areas one of which is the VLSI physical design process. The rectilinear version of the Steiner Tree problem is used in VLSI physical design because all nets are defined horizontally or vertically in this field and the Rectilinear Steiner Tree Problem finds an interconnection of a net by using only vertical and horizontal wires. It is different from the well studied rectilinear minimum spanning tree (RMST) problem, which looks for the minimal total length tree for a given set of vertices. The rectilinear Steiner minimum tree (RSMT) achieves the possible minimum tree by using some extra points that are not in the given set of vertices. These extra points are called Steiner points. Figure 1-1 illustrates the rectilinear minimum spanning tree (RMST) and the rectilinear Steiner minimum tree (RSMT) for a given set of four nodes as an example.

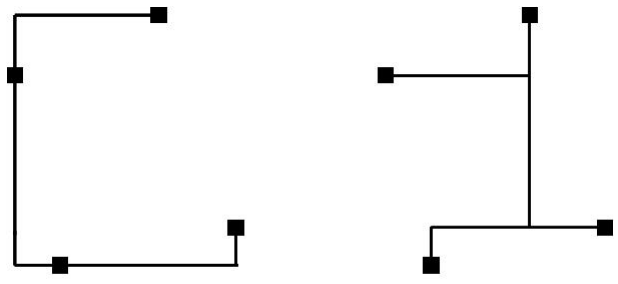


Figure 1-1 RMST vs. RSMT

Following the developments in the VLSI field, the designs became very complex and the number of transistors used in the designs has doubled every two years by the famous Moore's law. This brought up the result that up to millions of terminals need to be connected in a minimum rectilinear path. It is worth noting that the rectilinear Steiner minimum tree is calculated lots of times in a typical VLSI design cycle. Since lots of electronic designs exist involving this number of terminals and this number continuing to increase everyday, an efficient algorithm is needed to be found. However it has been shown that the rectilinear Steiner minimum tree is NP-complete. This result shows that the possibility of finding an efficient exact algorithm to solve this problem is not known, at least with our current state of knowledge. This leads the way to search for heuristic algorithms that solves the problem approximately. Most of the heuristics proposed up to now uses rectilinear minimum spanning tree as an initial step because it has been demonstrated that the length of the RMST is at most 1,5 times longer than the RSMT and it can be computed in polynomial time.

Although the rectilinear Steiner minimum tree has taken special interest, the algorithms proposed until a few years ago did not fulfill the rising requirements. Therefore several researchers have aimed at developing scalable algorithms but there still seems to be a lack of parallel algorithms in this field.

In this thesis work, first a detailed survey on rectilinear Steiner minimum tree problem has been presented. The major heuristics as well as exact algorithms that are developed until now have been examined. The algorithms have been compared mainly in terms of their performance and their time complexity. The performance of the heuristics is given mostly in terms of their improvements to the MST.

Following the literature survey, two recently proposed heuristic algorithms have been identified as efficient and satisfactory for relatively large input sizes. These are Kahng, Mandoiu and Zelikovsky's BGA algorithm [1] and Zhou's RST algorithm [2]. The other reason for selecting these two algorithms is the potential for their parallel implementation in a distributed environment.

The main contributions of this study can be listed as follows:

- The thesis conducts a comprehensive literature survey on RSMT.
- It proposes a hybrid approach for solving the RSMT by modifying the Zhou's RST algorithm with some parts borrowed from Kahng *et.al.*'s BGA algorithm.
- It proposes a partially distributed version of the modified algorithm.
- Following the implementation of the three algorithms, i.e., RST, BGA and modified RST, comparisons on their performances and convergence times are presented. The proposed modification approach, which is based on both analysis and profiling results of the two known algorithms, has proved to be effective.
- The proposed distributed algorithm is also implemented using the MPI (message passing interface) middleware and the results of comparative tests conducted on a cluster of workstations have been presented. The proposed distributed algorithm has also proved to be promising and useful especially for large problem instances.

This thesis work has been organized as follows:

In Chapter 2, the Steiner Tree Problem is defined and the historical background of the problem is briefly given. Then different variations of the problem are presented and the chapter ends by stating some application areas of the Steiner trees.

In Chapter 3, VLSI physical design cycle, which is one of the most important application areas of Steiner trees, is reviewed. Since rectilinear Steiner trees are heavily used in the physical design of VLSI systems, studying the Steiner trees and consequently improving the VLSI physical design cycle is among the main motivations and major interests of this thesis work.

In Chapter 4, the fundamental properties of rectilinear Steiner trees are defined including the basic definitions used in the literature. Then the exact algorithms are reviewed first, summarizing the main ideas that have created them. Afterwards

starting from the simplest algorithm, important heuristics are explained briefly. A comparison of the reviewed heuristics is given and the purpose of selecting the BGA and RST algorithms in this thesis is stated.

Chapter 5 first describes and explains in detail the components that form the BGA algorithm. Secondly, it repeats the same type of decomposition for the RST algorithm and it proposes the Modified RST algorithm finally.

In Chapter 6, parallel computing basics and a distributed environment are introduced first. Then the modified RST algorithm components, which can be parallelized, are discussed and the adopted method of parallelization is presented.

In Chapter 7, the data structures used in the implementations are explained. Then the implementation results of sequential BGA, RST and the Modified RST are presented. Chapter 7 also gives the implementation results of the distributed version of the Modified RST.

Chapter 8 concludes the thesis work.

CHAPTER 2

THE STEINER TREE PROBLEM

2.1. Problem Description

The Steiner problem asks for a shortest tree network spanning a given set of points but it is different from the classical minimum spanning tree problem in which all connections are required to be between the given set of points. The novel property of Steiner tree problem is that new points other than the original points can be introduced making the spanning tree as short as possible. These new points are called Steiner points and the inclusion of these extra points makes the construction of Steiner tree an NP-hard problem.

2.2. Historical Background of Steiner Tree Problem

The historical background of the Steiner Tree Problem goes to 1600's. Fermat proposed the problem of finding a point in the plane, the sum of whose distances from three points is minimal. Torricelli proposed a geometric solution to this problem before 1640. The general Fermat problem, which seeks a point in plane the sum of whose distances from n given points is minimal, has attracted the attention of many well-known mathematicians including Jacob Steiner [3].

In 1934 Jarnik and Kössler [4] raised the following question in a Czech journal: Find a shortest network which interconnects *n* points in the plane?. But they did not give any reference to Fermat.

Courant and Robbins first made the connection that the Fermat problem is the shortest interconnection network with *n*=3 [5]. They have called the former Fermat problem as 'Steiner problem' and called the Jarnik and Kössler problem as 'street network problem'. The popularity of their book made the problem called as the Steiner problem afterwards.

Melzak established many basic properties of a shortest interconnecting network and gave a finite solution to the Steiner problem [6]. Gilbert and Pollak introduced the name *Steiner minimal trees* (SMT) for shortest interconnecting networks and *Steiner points* for vertices of an SMT that are not among the *n* original points [7].

2.3. Variations of Steiner Tree Problem

There are many variations of the Steiner Tree problem for different metrics. These different variations emerged from different types of applications. Some special geometrical properties can be taken into account for special metrics.

Let $u = (u_x, u_y)$ and $v = (v_x, v_y)$ be a pair of points. The distance in the L_p metric, where $1 \le p \le \infty$, between u and v is $\|uv\|_p = \left(|u_x - v_x|^p + |u_y - v_y|^p\right)^{1/p}$. As special cases L_1-distance (called as rectilinear or Manhattan distance) equals to $\|uv\|_1 = \left(|u_x - v_x| + |u_y - v_y|\right)$ and L_2-distance (called as the Euclidean distance) equals to $\|uv\|_2 = \sqrt{|u_x - v_x|^2 + |u_y - v_y|^2}$.

Three major versions have emerged for the Steiner tree problem. These are the *Euclidean Steiner problem*, the *Rectilinear Steiner problem* and the *Steiner problem in networks*. It can be shown that Steiner points in the Euclidean and the Rectilinear cases belong to a finite set of points [3]. The Euclidean and the Rectilinear cases can be thought as special cases of the network problem. The Euclidean Steiner problem was the first identified version of the problem as was introduced in the previous Section 2.2. The rectilinear Steiner problem is identified by Hanan in 1966

[8] and the Steiner problem in networks is defined by Hakimi in 1971 [9] as a combinatorial version of the Euclidean Steiner problem. Other versions such as octilinear Steiner minimum tree are defined afterwards.

Other variations also exist. One of them is the *k-SMT* where the Steiner minimum tree spans exactly *k* points. Also *Steiner Arborescence* problem is defined as a directed tree rooted at the origin, spanning all the given points. Another version of the problem is the *group Steiner tree problem,* which is a generalization of the Steiner tree problem where several subsets of vertices are given in a weighted graph, where the goal is to find a minimum-weight connected sub-graph containing at least one vertex from each group.

Although the above variations are among the most popular ones, other versions are still possible and extensive surveys on the subject exists (for example [3]).

2.4. Application Areas of Steiner Trees

The Steiner tree problem is one of the most popular graph problems. Its popularity depends on the fact that it has many application areas some of which are given below.

One application of the Steiner tree stems from the minimal network theory. Minimal networks are applied in many other fields such as cluster theory, calculation of the characteristic dimension of a point set and minimization of the length of conductors for electronic equipment manufacture. The most popular problem in minimal network theory is finding the absolute minimal networks spanning a given set of points. It is shown that any absolute minimal network spanning a fixed set of points of the plane including some additional points is always a tree and this tree is known as *Minimal Steiner Tree* [10].

Another application area arising from the network theory is the Quality of Service Multicast Tree Problem, which appears in the context of multimedia multicast and

network design [11] and which is a generalization of the Steiner tree problem. The aim of Multicast Routing is to efficiently interconnect a set of destinations in a network for group communications like teleconferences. The resulting sub network, known as a multicast tree, avoids unnecessary duplication of data while optimizing a cost parameter such as bandwidth.

Other applications can also be found in different fields of science such as biology. A phylogenetic tree is a tree showing the evolutionary interrelationship among various species or other entities that are believed to have a common ancestor. In a phylogenetic tree, each node with descendants represents the most recent common ancestor of the descendants and edge lengths correspond to time estimates. Each node in a phylogenetic tree is called a taxonomic unit. Internal nodes are generally referred to as Hypothetical Taxonomic Units (HTUs) as they cannot be directly observed [12]. These internal nodes correspond to Steiner points.

There may be several other applications of Steiner trees but the most important one is its use in VLSI design. The next chapter presents the concepts of VLSI design very briefly, to illustrate the motivations behind this thesis work.

CHAPTER 3

VLSI PHYSICAL DESIGN AUTOMATION

3.1. VLSI Design Problem

Very Large Scale Integration (VLSI) refers to those integrated circuits that contain more than 10^5 transistors [13]. Since the VLSI chips today can contain more than a hundred million transistors, a research field called electronic design automation (EDA) has emerged. EDA is concerned with the tools that are used in the design and production of VLSI systems.

In creating a VLSI system, six major steps have to be followed [14]. In *specification* phase, a functional specification of the system under development is produced. In *logic design* phase, the functional specification is transformed into a logical representation. In *circuit design* phase, the logic representation is converted to circuit elements like gates or standard cells. The *physical design* phase translates the circuit design into a physical package representation also known as the layout. In *fabrication* phase, the physical package representation is used to generate an actual integrated circuit. In *testing* phase, the manufactured integrated circuit is examined to determine whether there are manufacturing errors that prevent the integrated circuit to work correctly in accordance with the functional specification.

VLSI Physical Design phase is within the scope of the current thesis work.

3.2. VLSI Physical Design Problem

The input to the physical design cycle is a circuit diagram and the output is the layout of that circuit. It is accomplished by converting each logic component into a geometric representation. Geometrical representation identifies the dimension and location of the transistors and wires on a silicone surface [15]. Layouts of the designs that need high performance may be partially manual designed but layouts of most designs are automated. To efficiently solve the problem with automated methods, physical design is accomplished in several stages main properties of which are introduced in the following sections.

3.2.1. Circuit Partitioning

A chip can contain several millions of transistors, so it may not be possible to layout the entire chip at one single step. Therefore the entire chip is normally partitioned into sub-blocks. The partitioning process considers many factors such as the size of blocks, the number of blocks and the number of interconnections between the blocks. The output of partitioning is a set of blocks and the interconnections required between the blocks [16].

3.2.2. Floorplanning and Placement

Floorplanning is concerned with selecting good layout alternatives for each block coming from the previous step. The area can be estimated approximately for each block after partitioning. This step is very critical because it constructs the basis for a good layout. During the placement step the blocks are exactly positioned on the chip. The goal of partitioning is to find a minimum area arrangement for the blocks while meeting the performance constraints. The placement is done in two phases. In the first phase an initial placement is created. In the second phase, the initial placement is evaluated and iterative improvements are made until layout has minimum area or best performance conforming the design specifications [16]. The quality of placement will be determined after the routing is performed.

3.2.3. Routing

The objective of the routing phase is to complete the interconnections between blocks according to the specified net list. First, the space that is not occupied by the blocks is partitioned into rectangular regions. A router completes all interconnections with shortest possible wire length using only rectangular regions. The vertices of this grid graph represent potential pins and vias and the edges represent the capacity of a channel, which can be defined as the routing space between two channels [15]. The routing is realized in two phases called *Global Routing* and *Detailed Routing*. In global routing phase, connections are made between blocks but the details of each wire and pin are not taken into account. Alternatively, the global routing is said to specify the different regions in the routing space, which a wire should be routed. Then the point-to-point connections between pins on each block is completed [16] in detailed routing.

3.2.4. Layout Compaction

Compaction phase can be identified by the task in which the layout is compressed in all directions such that the total area is reduced. By making the chip even smaller wire lengths are reduced, which in turn reduces the capacitances emerging from long wires and so the signal delay. By reducing the area also the number of chips produced from a wafer is increased, which may mean a significant cost decrease.

3.2.5. Extraction and Verification

This is a phase which checks the correctness of the layout. The Design Rule Checking (DRC) is a process which verifies that all geometric patterns meet the design rules imposed by the fabrication process [16]. After removing the design rule violations, the functionality of the layout is verified by circuit extraction which is a reverse engineering process. It generates a circuit from the layout to compare with the original net list. In performance verification phase the geometric information to compute resistance, capacitance, delay, etc is checked.

It is worth noting that physical design is an iterative process. That is to say, many phases such as global routing and detailed routing are repeated several times to obtain a better output. The quality of the design in some phase heavily depends on the quality of the solution obtained in earlier phases. For example if a poor quality solution is offered in the placement phase, even a high quality routing may not produce a satisfactory and good result. In general whole design phases may be repeated several times to accomplish the objectives of the design.

3.3. Using Rectilinear Steiner Trees in VLSI Physical Design Problem

After defining the major concepts of VLSI physical design in the previous section, this section will answer the question of 'where does the Steiner tree problem fit in the physical design process?'.

It has to be noted again that the geometry of VLSI, which usually allows only vertical and horizontal wiring directions, has motivated the studies of the rectilinear version of the problem [17]. The Steiner trees can be used in two phases of the VLSI physical design process, namely the placement and global routing phases.

The quality of a placement solution is evaluated by estimating the total wire length [13]. The total wire length is estimated by first estimating a length for each net and then by summing them up. In the next step total wiring area can be derived from this length by assuming a certain wire width and a wire separation distance. For timing critical nets, the minimization of wire concept defined here is not enough, but for most of the nets in a typical design is not that critical. Therefore, SMT can be used as an accurate estimation of wire length for the placement phase and this implies that the Steiner tree will be invoked millions of time [2].

The object of routing problem for a general purpose chip is to minimize the total wire length [16]. This is because VLSI design rules dictate a minimum separation

between wires and therefore the area occupied by the routing on a chip is roughly proportional to the total wire length of the routing [17]. Added wire length generally increases signal delay and power consumption due to increased resistance and capacitance. For global routing two types of nets exists; two terminal nets and nets with more than two terminals. Multi-terminal nets can be formulated as Steiner tree problems [3]. The size of the nets becomes larger with the improvements in the VLSI technology, so does the size of the Steiner trees involved.

In Chapter 2, Steiner tree problem has been defined and in this chapter one important use of the rectilinear Steiner tree problem is stated. The next section will investigate the problem in more detail by giving its properties and the algorithms that solve it.

CHAPTER 4

THE RECTILINEAR STEINER TREE PROBLEM

4.1. Problem Description

Hanan is the first author who considered the rectilinear version of the Steiner tree problem [8]. This version constitutes all definitions that have been made in the Euclidean version of the problem but all distances are measured in rectilinear metric. The rectilinear Steiner minimum tree (RSMT) is a tree that interconnects a set of terminals consisting of horizontal and vertical line segments only while having minimum total length [18]. This is equivalent to saying 'construct a Steiner minimum tree for the given set of terminals under the L_1 metric'. Given two points $u=(u_x,u_y)$ and $v=(v_x,v_y)$, the L_1 distance between them is $\|uv\|=|u_x-v_x|+|u_y-v_y|$. In other words, it is equal to the sum of distances in each of the two dimensions.

The problem is proved to be NP-complete by Garey and Johnson [19]. They have transformed the problem of *node cover in planar graphs*, which was proved to be in NP-complete class earlier, to rectilinear Steiner minimum tree (RSMT) polynomially [20]. Therefore no polynomial-time algorithm is found up to now for this problem and more effort is given to heuristic solutions after this proof.

Some definitions and properties about Rectilinear Steiner Minimum Tree will be given in the next section. Then some important exact algorithms and heuristics proposed up to now will be investigated.

4.2. Definitions and Basic Properties

A *rectilinear segment* is a horizontal or vertical line segment connecting its two endpoints in the plane. The intersection points of these segments are called *nodes*. The *degree of a node* is the number of segments incident to it. The nodes are either *terminals* (from the given set of points) or *non-terminals*. There are three types of non-terminals. *Corner points* have degree two and thus have exactly two incident perpendicular segments. *T-points* have degree three and *cross points* have degree four. *T-points* and *cross points* are also called Steiner points. Corner points are not defined as Steiner points because they do not have a distinct position on the tree. Namely, if the point that is in the opposite diagonal of the corner point is included on the tree; the length of the tree does not change. It can be noted that each endpoint of a segment is a terminal, a Steiner point or a corner point.

A *line of segments* is defined as a sequence of one or more adjacent, collinear segments with no terminal nodes in its relative interior. A *complete line* is defined as a line of maximal length. A *corner point* is incident to exactly one horizontal complete line and one vertical complete line where these complete lines are referred as the legs of the corner. If the other endpoints of the legs are terminals, the corner is referred as a complete corner.

A rectilinear Steiner tree in which every terminal is a leaf is called a *Full Steiner Tree (FST)*. It is found that every SMT is a union of FSTs. A fulsome SMT is defined as an SMT in which the number of FSTs is maximized. The number of FSTs is equals to $1+\sum_{z\in Z}\left(\deg(z)-1\right)$ where Z is the set of terminals and deg(z) is the degree of a terminal $z \in Z$. Therefore it can be said that in a fulsome SMT, sum of the degree of all terminals in the set is maximized.

In general there exists nearly an infinite number of SMTs for a given terminal set because by performing *sliding* and *flipping* operations given in Figure 4-1 an SMT can be transformed into another SMT.

Figure 4-1 Sliding and Flipping Transformations

In order to generate an efficient algorithm a concept called canonical FSTs are identified used. An FST *F* in a fulsome SMT is said to be *canonical* one if no vertical segment in it can be moved to the right using sliding and/or flipping operations. If every FST in a fulsome SMT is canonical, then the SMT is said to be canonical. In Figure 4-2 demonstrates all these definitions. The topmost figure is an SMT, which is neither fulsome nor canonical. The middle SMT is fulsome but not canonical, while the bottom SMT is both fulsome and canonical. These properties lead the way to the exact algorithms.

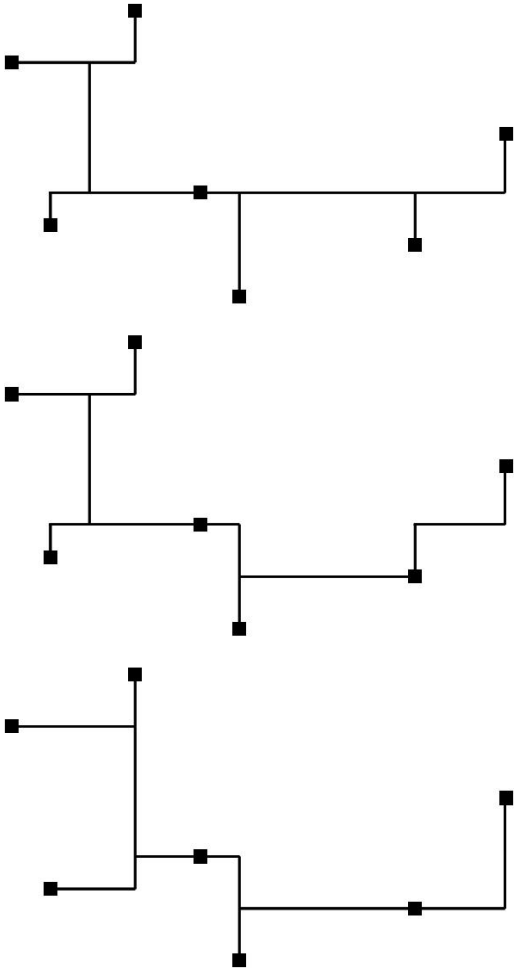


Figure 4-2 Fulsome and Canonical SMTs

Hanan, who has defined the rectilinear Steiner minimum tree problem for the first time, also gave a fundamental structural result. He has defined a grid in [8], also called as the *Hanan grid*, by drawing horizontal and vertical lines through all terminals of the given set *Z*. Let *H(Z)* be this grid as can be seen in Figure 4-3 and let $I_{H(Z)}$ be the set of intersection points in *H(Z)*. $|I_{H(Z)}| = O(n^2)$ where *n* is equals to the number of terminals. It is shown in [8] that there exists an SMT for *Z* such that every Steiner point belongs to $I_{H(Z)}$.

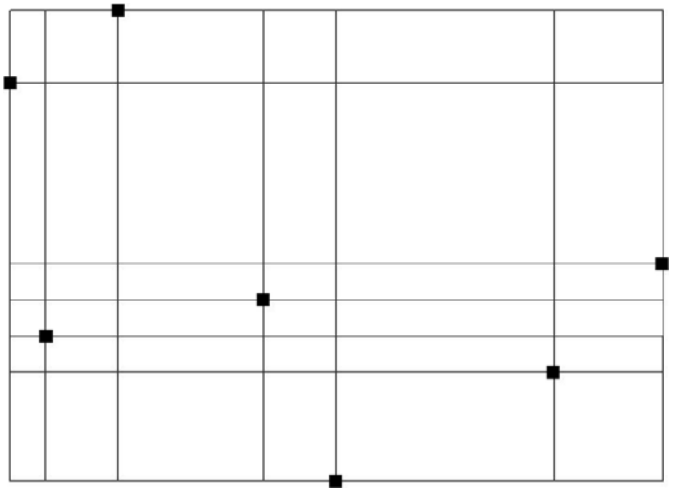


Figure 4-3 Hanan Grid for a Set of Terminals

This result of Hanan can be interpreted as follows: only the intersection points in the Hanan grid can be Steiner point candidates. This is a very important finding because only a polynomial number of points need to be considered as possible Steiner points. Another bound on the Steiner points is given by Gilbert and Pollak [7] by proving that that any Steiner tree may contain at most *n-2* Steiner points.

Another important property is given by Hwang [21] for the bound on the length of RSMT. For a given set of terminals *Z*, the length of SMT and MST are represented as |*SMT(Z)*| and |*MST(Z)*|, respectively. It is trivial that $|SMT(Z)| \leq |MST(Z)|$ but the question of 'how much can an SMT be shorter than MST?' arises naturally. The smallest possible ratio between SMT and MST lengths for any set of terminals is called the *Steiner ratio* and it changes with the specified metric. The *Steiner ratio* in the rectilinear plane, is given as:

$$\frac{|SMT(Z)|}{|MST(Z)|} = \frac{2}{3} \tag{4.1}$$

The property given in Equation (4.1) is proved in [21] and alternative proof exists in [18]. In Figure 4-4 the graph on the left is an RMST and the graph on the right is an RSMT. The lengths of RMST and RSMT are 6 and 4, respectively thus making the Steiner ratio 2/3.

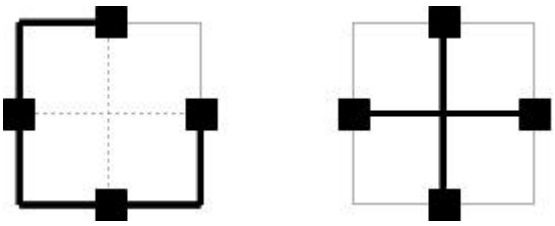


Figure 4-4 Steiner Ratio Example

The Steiner ratio, stating alternatively that the length of an MST is at most 1,5 times of the length of an SMT for a given set of terminals, is used by many algorithms including heuristics because MST can be calculated in polynomial time. An MST can also be viewed as an approximation to SMT, which has a worst case performance of 1,5 times. The heuristics usually relies on this fact and after starting from RMST, they try to improve this ratio.

4.3. Exact Algorithms

Since the rectilinear Steiner tree problem is NP-hard, little hope was left for any polynomial time exact algorithm to solve the problem. However, recent research results have still appeared on exact algorithms. In the present section, the history of exact algorithms for the rectilinear Steiner minimum tree will be investigated first and then two more recent algorithms will be introduced in more detail.

The first optimal algorithm in the literature is given by Yang and Wing [22]. It is a branch and bound type algorithm and the largest problem that was reported to be solved consisted of only nine vertices taking 255 seconds. In 1995, Hetzel [23] proposed an algorithm that could solve a 50 terminal problem in one hour and Salowe and Warme [24] simultaneously described an algorithm that could solve an 35-terminal problem in one day. In 1997, Fößmeier and Kaufmann [25] have nearly doubled this performance.

All these algorithms were based on the same method that was suggested by Winter [26] for the Euclidean Steiner tree problem in the plane. This method is adopted to the rectilinear case. It uses the fact that there exists an SMT, which is a union of FSTs having Hwang topology, which is described in Section 4.3.2.1. Thus first, they generate all Full Steiner Trees that could have been found in the terminal set. Then they concatenate these trees in order to form a rectilinear Steiner minimum tree. The major bottleneck of the algorithm was the concatenation phase until Warme [27] introduced a major breakthrough in the process.

In the next section the properties that an algorithm has to satisfy in order to be optimal will be discussed. Then the fastest algorithm that has been designed until now will be introduced. And Hanan Grid based algorithms will be explained afterwards.

4.3.1. Necessary Optimality Conditions

An edge $e= (u, v)$ in a Steiner minimum tree is a direct connection between a pair of nodes u and v, which are either terminal or Steiner points. In fulsome and canonical SMT an edge is either a single segment or a pair of perpendicular segments adjacent at a corner point. The length of it is calculated in rectilinear metric and called $||e||$. It is also worth noting that any sub-tree of an SMT must be an SMT for the nodes that it spans and the same condition occurs for FST. Now some upper bounds about the length of edges in MST will be discussed.

Consider an SMT and two terminals called i and j. The unique path between i and j in the SMT is denoted by $P(i, j)$. If any edge called e on this path is removed from SMT, the tree will be broken into two connected components. Now an edge in the MST, say f, will reconnect these broken components. Clearly these edges have to satisfy $\|e\| < \|f\|$ since otherwise SMT will not be optimal. Therefore the bottleneck Steiner distance, $b_{i,j}$, between a pair of terminals i and j is equal to the length of the longest edge in the MST between i and j [3]. After identifying the bottleneck

Steiner distance, it can be proved that for any edge $e \in P(i, j)$ in the SMT the rectilinear length of $||e||$ must be smaller from $b_{i,j}$.

After giving an upper bound on the length of the edges, we will now discuss the conditions on how close are the other terminals to an edge. Again assume that (u, v) is an edge of SMT, then the *lune* for the edge (u,v) is described as $L(u,v) = \{ p \in \mathbb{R}^2 : \|pu\| < \|uv\| \wedge \|pv\| < \|uv\| \}$. The lune can also be described as the intersection between the interior of two L_1 circles with radius $||uv||$ and centered at u and v [18]. The lune for an edge can be seen in Figure 4-5.(a).

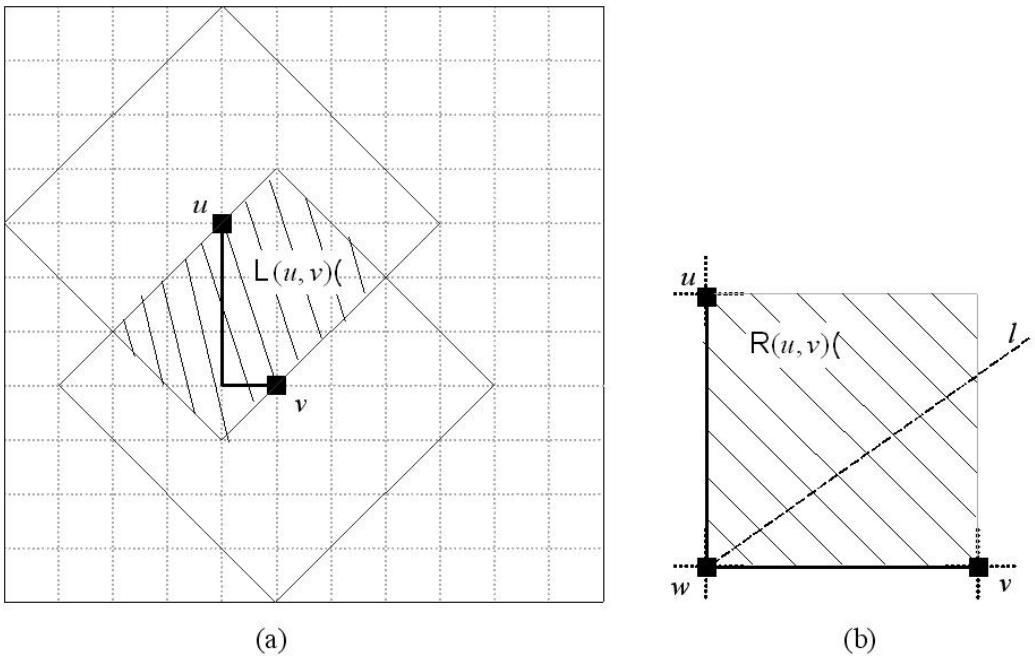


Figure 4-5 Empty Lune and Empty Corner Rectangle

By defining the lune, it is noted that if (u,v) is an edge in SMT then $L(u,v)$ contains no other point (terminal, Steiner point, or interior segment point) from SMT. Assume that such a point $p = L(u,v)$ exists in the SMT. Since the point is in the graph it has to be connected to u or v. If the edge (u,v) is removed from SMT, the

edge will be split into two connected components and one component will contain the point *p*. Supposing that the component that have point *u* contains point *p* also, by adding edge (*p*,*v*) the length of the SMT will be reduced which results in a contradiction. Therefore the lune has to be empty.

If two nodes *u, v* are not directly connected with an edge, but with another node, say *w*, such that the segments *uw* and *wv* are perpendicular as in Figure 4-5.(b), $\mathsf{R}(u,v)$ can be defined as the interior of the rectangle with sides *uw* and *wv* and then no other point can exist in $\mathsf{R}(u,v)$. This can be proved by assuming again that there exists such a point $p = \mathsf{R}(u,v)$. Let *l* be a line that passes through *w* and assume that *p* lies above the line *l*. Then when the edge (*u*,*w*) is removed from SMT, two connected components are formed. If *p* was connected to *u* before the deletion operation the SMT will be shortened by adding an edge from *p* to the segment *wv*, which is a contradiction.

4.3.2. Geosteiner

The GeoSteiner algorithm [28] depends on the method that was suggested by Winter [26] for the Euclidean Steiner tree problem in the plane as it was mentioned below. It uses the fact that there exists an SMT, which is a union of FSTs having Hwang topology. Thus first, they generate all Hwang topology Full Steiner Trees that fulfills the necessary optimality conditions. Then a subset of these FST's is selected in the concatenation phase.

4.3.2.1. FST Generation

It can be noted that FST concatenation phase was computationally harder than the FST generation phase. However, after Warme have introduced an effective way of concatenating the FSTs [27], the FST generation phase then posed an overhead to the whole algorithm. Therefore, a new method is also proposed by Zachariasen for FST generation phase [29].

Hwang [21] has proven that every FST in a canonical and fulsome SMT has a very particular shape denoted as the Hwang topology. The FSTs have one of the two generic shapes, and two degenerate cases, which can be seen in Figure 4-6 and can be stated as follows: An FST spanning k terminals consists of a corner (also denoted as the backbone) given by root z_0 and a tip z_{k-1}. The root is incident to the long leg, and the tip is incident to the short leg. Here long leg means, it has more incident segments than the short leg. There are two main types (i) and (ii) and two degenerate cases of type (i) [18]:

- *Type (i) has k-2 alternating segments incident to the short leg. The first degenerate case (i') has a zero-length short leg, that is, the corner is degenerated into a line. The second degenerate case (ii') is a cross spanning exactly four terminals.*
- *Type (ii) has k-3 alternating segments incident to the long leg and one segment incident to the short leg.*

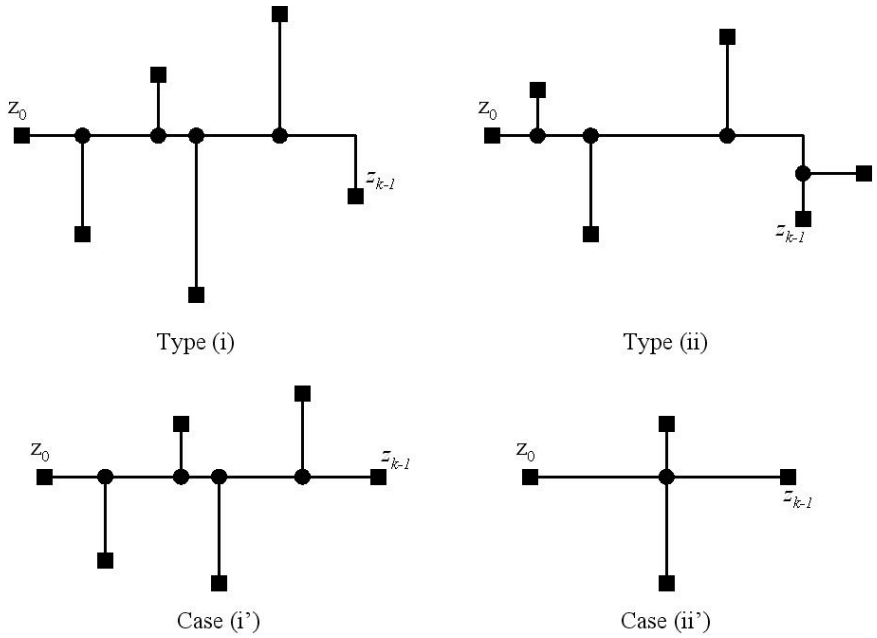


Figure 4-6 Hwang topology FSTs

The FSTs that are generated from the algorithm are all Hwang-topology FSTs. The algorithm works by growing FSTs. For a given terminal z_0 and a specific direction, an FST is grown out with long leg being in the specified direction. Let the growing direction be East and an example growing algorithm can be stated as follows:. All terminals to the right of the vertical line through z_0 are sorted by their x-coordinate. Also Z_a and Z_b denote the list of sorted terminals that are above the horizontal line through z_0 and the list of sorted terminals that are below this line respectively. By using the necessary optimality conditions and by selecting one terminal from Z_a and then from Z_b the algorithm saves the FSTs and recursively continues. It must be noted that the algorithm also backtracks if the optimality conditions can not be satisfied.

An independent preprocessing phase for FST generation, which runs in $O(n^2)$ time, is given in [29]. The main purpose of this algorithm is reducing the set of terminals that can be attached to a backbone. Bottleneck Steiner distances, empty lunes and empty corner rectangles are used to eliminate long-leg and short-leg terminal candidates which will reduce the overall complexity of the algorithm.

4.3.2.2. FST Concatenation

Warme gave an algorithm for finding the MST for the hypergraph problem [27]. He has motivated this algorithm to the FST concatenation phase. A hypergraph is generated with the set of terminals as its vertices and the generated FST in the previous step as its hyperedges. It has been shown in [27] that this problem is NP-hard when the hypergraph contains edges of cardinality four or more. Some methods to solve this problem have been tried in [28] such as backtrack search, dynamic programming or integer programming. Warme already gave an integer programming (IP) formulation, which is used in his branch-and-cut method. His IP formulation depends on three facts. First one is that the total length of the selected hyperedges has to be small. Then the hyperedges have to span all edges and the final one is that the resulting graph will have no cycles. The GeoSteiner code can be found in [30].

Later Emanet [31] has also proposed a new method for this concatenation phase by applying some modifications on Warme's ideas.

4.3.3. Hanan Grid Based Exact Algorithms

The first algorithms for the Rectilinear Steiner Tree Problem were based on the result that there exists an SMT in the Hanan grid that is composed for the given set of terminals. Hanan shown that RSTP problem reduces to Steiner Tree Problem in Graphs (STGP), which is stated as follows: Given an edge-weighted graph G=(V,E) and a set of terminals $Z \subseteq V$, find a tree in G that interconnects Z and has minimum total length [18].

The best algorithm proposed for this problem up to now uses an IP formulation [32]. The algorithm first generates a directed graph having the same vertices as G; where for every edge of G there are two directed edges and both of these edges costs as the edge in G. Then by selecting an arbitrary terminal as root, the problem changes into finding a rooted directed tree of minimum total length that contains all terminals, which is also called a *Steiner arborescence* [18].

The algorithm mentioned above uses the complete Hanan grid so it has problem when the number of terminals are more than 40. But the Hanan grid can be simplified first to improve its performance. Techniques of such reductions are presented in [33].

4.4. Approximation Algorithms

Since the Rectilinear Steiner Tree Problem is NP-hard, the major research effort is given to heuristic approximation algorithms. Having shown also that the rectilinear minimum spanning tree is at most 1,5 times longer than the rectilinear Steiner minimum tree, most of the heuristics starts with MST and tries to improve it. In the following, MST Embedding will be shown first and then the heuristics that improve the Steiner ratio from 3/2. Next, B1S and IRV heuristics, which add Steiner points

to the MST iteratively, will be introduced. Afterwards Borah's algorithm, which updates edges of MST iteratively and BGA algorithm, which merges tiny optimal Steiner trees to MST will be presented. And finally the RST algorithm, which improves Borah's algorithm with the help of the spanning graphs will be given.

4.4.1. MST Embeddings

It has been shown that the rectilinear minimum spanning tree can be found in O(*nlogn*) time by Hwang [34] and by Yao [35]. This leads to the question of how can an RMST be converted to a rectilinear Steiner minimum tree (RSMT). Each edge of an RMST can be represented by different rectilinear shortest paths in the plane between the corresponding terminals, unless the terminals are connected by a horizontal or vertical line, where only one possible way exists. This is because of the fact that the rectilinear distances are equal in both paths which can be seen with dots and dashes in Figure 4-7.(a). Each edge is assumed to be represented by a path with at most one corner point. In other words, each edge is realized on the graph either by straight wires or by one of two L-shaped wires. If there are n terminals given, an RMST has up to 2^{n-1} different embeddings in the plane. In a typical embedding there are pairs of wires from different edges that overlap. These overlaps can be removed by using Steiner points, like in the Figure 4-7.(b). Depending on how embeddings are selected, different heuristics are proposed.

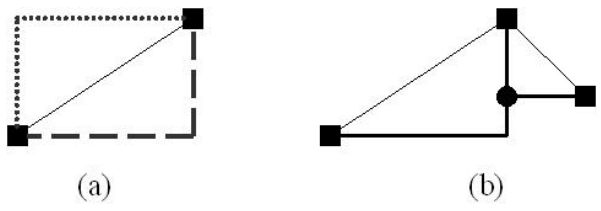


Figure 4-7 Different Embeddings and Insertion of a Steiner Point

Ho, Vijayan and Wong gave an algorithm for good MST embedding [36]. The key to their algorithm is not to start with an arbitrary RMST but to start with a separable

RMST. This is an MST for which the bounding boxes only overlap if the corresponding edges share a terminal. This definition can be made clear by the Figure 4-8. In the uppermost figure, overlapping bounding boxes can be seen. The middle figure shows an MST where no line segments overlap, which is the case for the separable MST. And also at the bottommost figure an optimal embedding can be seen. They have also proven that such an MST can be constructed in O(*nlogn*) time.

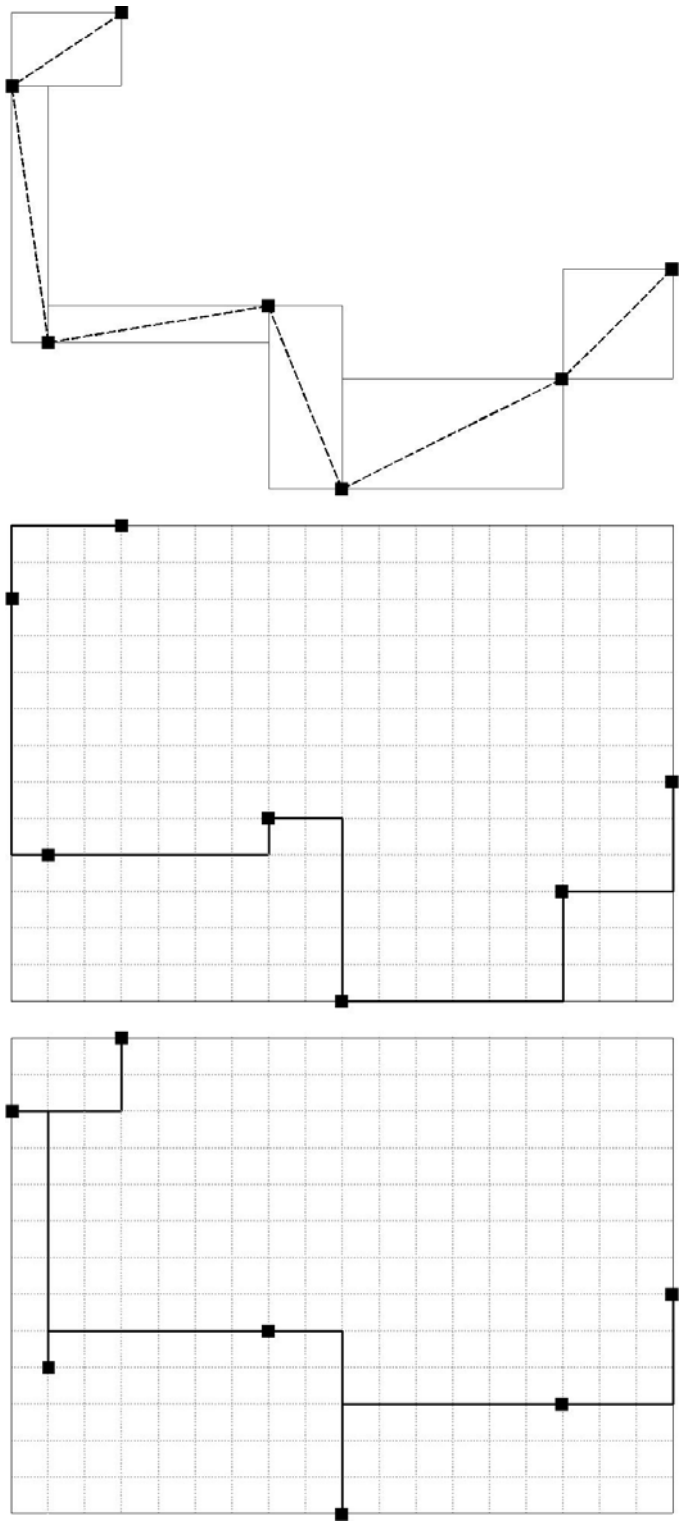


Figure 4-8 RMST, Separable RMST and Optimal Embedding

They have shown that starting with a separable MST; an optimal L-shaped embedding can be constructed in O(n) time. They have satisfied this with a O(n) time dynamic programming algorithm. They have begun by rooting the separable RMST at some leaf terminal and solve the sub-trees bottom-up. The key observation of the algorithm is that with a separable RMST, the optimal solution for a sub-tree depends only on the choice of which of the two embeddings of the L-shaped wire connecting the root of the sub-tree to its parent was chosen. It is worth noting that an optimal embedding for a given RMST is not necessarily an RSMT.

It can be shown that L-shaped wires are insufficient to find the optimal embeddings [3]. So Z-shaped wires, which have two corner points are also offered in [36]. Z-shaped optimal embeddings can achieve good improvements in $O(n^2)$ time generally but the worst case running time is $O(n^7)$.

4.4.2. Zelikovsky Based Heuristics

As it was noted before the rectilinear minimum Steiner tree is at most 3/2 times the rectilinear Steiner minimum tree. Zelikovsky worked generally on improving this ratio. He has improved this ratio to 11/6 and 11/8 [37, 38]. The algorithm proposed in [38] runs in $O(n^3)$ time and guarantees a performance of ratio 11/8. Afterwards Berman and Ramaiyer improved this time complexity to $O(n^{5/2})$ [39] and finally Fößmeier has given a new time bound of $O(n^{3/2})$ [40].

The main idea of the algorithm is to start with an initial rectilinear minimum spanning tree and then iteratively computing optimal Steiner trees for small subset of terminals and inserting these small Steiner trees into the current tree. In the algorithm 3-restricted full Steiner trees will be used, since the computation of bigger trees will be more complex than this one. A 3-restricted Steiner tree is composed of FST's each having at most 3 terminals. In the algorithm these small trees are called *stars* and it is proven that O(n) stars will be enough to achieve 11/8-approximation ratio. Also a method of finding O(n) stars in $O(n\log^2 n)$ time has been introduced in the paper.

Arora has also proposed an approximation algorithm but this has mainly a theoretical importance [41]. His algorithm, for any fixed value $\varepsilon > 0$, produces a rectilinear tree whose length is within a factor of $1+\varepsilon$ from optimum. His paper was actually a major breakthrough with its main focus on approximating the Euclidean traveling salesman problem [18]. The algorithm consists of three phases, which are perturbation, shifted quadtree construction and dynamic programming. As it was noted before this algorithm has mainly a theoretical importance but it offers a way to solve the problem in polynomial time with a performance nearly like the optimum.

4.4.3. B1S and IRV Heuristics

Kahng and Robins [42] presented a heuristics called Iterated 1-Steiner (I1S) that has been used the Rectilinear Steiner Tree Problem for long years [3]. This heuristic is based on the 1-Steiner Tree problem, which looks for the optimal Steiner tree if at most one Steiner point is permitted. This can alternatively be stated as follows: what is the location of a single point *p* such that RMST of N $\cap$ {*p*} is minimized? By taking this as a starting point, the I1S heuristic finds a set of Steiner points S such that MST(Z$\cap$S) has minimum length.

It has to be noted that the number of Steiner points can be at most *n-2* [7] and also the Steiner point candidates are only needed to be searched in the Hanan grid. Using these facts the I1S method repeatedly finds 1-Steiner points and includes them in the Steiner point set. This procedure continues until no points can be found that will improve MST (P $\cap$ S) where P represents the initial terminal set and S represents the Steiner point set. This procedure can add more than *n-2* points so that at each step any Steiner point having degree two or less will be deleted. A sample run of the I1S can be seen in Figure 4-9.

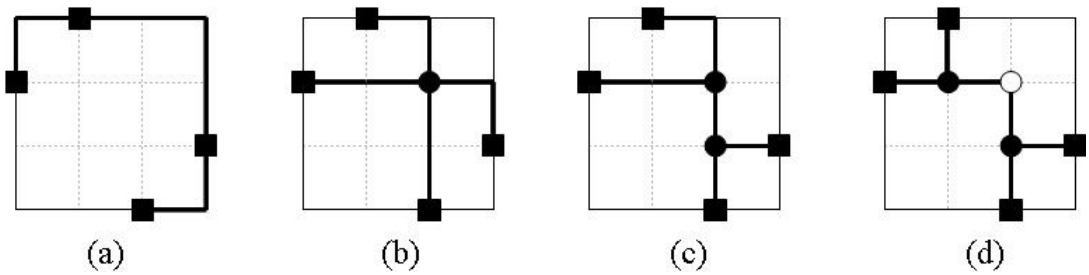


Figure 4-9 Sample Run of I1S Algorithm

In the figure, first a rectilinear minimum spanning tree on the set of terminals are given. Afterwards in steps (b) and (c), Steiner points are inserted sequentially and the added point that has a degree of two is removed from the Steiner point set in the last step.

The single 1-Steiner point can be found in $O(n^2)$ time using complicated techniques [17] but the trivial implementation takes $O(n^3 logn)$ time. Since this step takes so much time, a batched method is also generated which efficiently adds an entire set of *independent* Steiner points in one round [43]. Here the independence means no candidate Steiner point is allowed to reduce the potential MST cost savings of any other candidate. Therefore the total running time of the Batched 1-Steiner (B1S) heuristic becomes $O(n^4 logn)$.

In I1S and B1S, when a new Steiner point is found, MST of the new set is computed from scratch. This costs a lot. Therefore instead of computing the MST from scratch at each iteration, local updating of the MST may also be preferred. Using this observation a dynamic MST maintenance scheme is generated in [44]. According to this update, when a new Steiner point is added to the set, that point is connected to the 8 nearest octal points in the graph and the most expensive edges in the cycles formed are deleted. This reduces the time complexity to $O(n^3)$.

It is reported that all variants of the I1S algorithm solves the Rectilinear Steiner Tree Problem such that the solution is 0,5% away from the optimum on average,

and hence it can be called as the champion heuristic with respect to solution quality performance [18].

More recently Mandiou *et al.*[45] have proposed a new heuristic, IRV, which is similar to I1S concept. Similar to the previous ones, it also adds one or more points to MST until the MST does not improve, but it identifies the Steiner point candidates with a much more sophisticated algorithm. Their algorithm depends on the ideas given in [46] and it performs a little better than B1S but slightly worse when the empty rectangle test, which reduces the number of Steiner candidate points in the Hanan grid, is used for B1S.

4.4.4. Borah's Algorithm

The edge based heuristic that is proposed by Borah *et al.*[47] first calculates an initial rectilinear minimum spanning tree. The algorithm improves the cost of this MST by connecting a node to the rectangular layout of a neighboring edge and removing the longest edge in the loop that is formed by this process. By a straight forward implementation the complexity of Borah's algorithm is $O(n^2)$. The performance of the algorithm is shown to be close to the performances of other more complex heuristics.

This algorithm will be explained in much detail in Section 5.2.2, therefore only its general idea will be emphasized in the present section. In Figure 5-27 the basic operation of the algorithm is shown. This algorithm shortens the length of the MST by appending a terminal point to an edge in the MST. As a result a cycle is formed and by deleting the most expensive edge in this cycle an improvement can be achieved. The gain of this operation is positive if the cost between the terminal that was appended and the Steiner point that is grown out after this operation is shorter than the most expensive edge found in the cycle. It has to be noted that all costs are computed in rectilinear metric.

In the straight forward implementation all terminals will be appended to all edges of the MST. Thus the time complexity is $O(n^2)$ which is better than the previous proposed heuristics. In the previous algorithms mostly new nodes were connected to the graph, but in this heuristics edges are updated. Moreover it is mentioned in its associated paper that the heuristic achieves this improvement with no degradation in performance. It is noted that its performance, which is measured as the reduction of the length of the MST, is in the range of Iterated 1-Steiner heuristic.

Besides the above straightforward implementation, another method is also offered in [47]. First of all, for the initial minimum spanning tree phase they propose using Hwang's O(*nlgn*) time algorithm given in [34]. Then their new method relies on the fact that for an edge under consideration, not all terminals have to be connected to that edge. Considering only the terminals that are visible to the edge should be sufficient. For a point-edge pair to be able to result in a positive gain when merging into an MST, the point has to be visible to the edge meaning that there must be no edge that exists in the path from the point to the edge. Although this new method can result in a complexity of O(*nlgn*), it seems hard to implement it satisfactorily.

4.4.5. BGA

The BGA algorithm proposed by Kahng *et. al.* [1] starts with an initial minimum spanning tree and iteratively improves it. It is claimed to run in $O(nlg^2n)$ time producing high quality solutions. The BGA algorithm uses the implementations proposed in [40] for the GTCA algorithm of Zelikovsky [37] with the batched method introduced in [43].

In BGA algorithm, a sparse graph for initial minimum spanning tree computation is constructed by using the Guibas-Stolfi algorithm [48]. This algorithm relies on the fact that for the rectilinear metric a point can not be connected to two points in the same octal region. In Guibas-Stolfi's algorithm, first, all octal regions are mapped into the first quadrant. Then with a recursive method, all north-east nearest neighbors are found for all regions. Therefore all nearest points in all octal regions

are found and then Kruskal's MST algorithm is used in order to obtain the final rectilinear MST.

The GTCA algorithm finds an approximate minimum cost 3-restricted Steiner tree by greedily choosing 3-restricted full components, which are called *triples*. When a triple is merged into MST, two cycles are formed and then they are removed by deleting the most expensive edges in each cycle. The gain of a triple is the difference between two most expensive edges in each cycle formed and the cost of the triple. A triple is called empty if the minimum rectangle bounding the triple does not contain any other terminals and it is called a tree triple if the gain when the triple is merged into MST is positive. In BGA algorithm all empty tree triples are found and they are sorted according to the gains of the triples in non-increasing order. Then the greedy rule used in GTCA is changed to batched method. This means that starting from the triple with the biggest gain; all triples will be merged into the MST if both most expensive edges for the triple are not deleted yet. Finally all chosen triples will be applied to MST, the output being an approximate minimum Steiner tree.

In BGA, a new methodology is also given for the triple generation phase. The triples are divided into four types according to terminals' geometrical position with respect to each other. The terminals are partitioned recursively and four cases for each type of triples are generated. With these methods all empty tree triples are generated. The maximum number of such triples is O(n).

A maximum cost edge on the tree path is computed with a new method in BGA. Two arrays are computed for this purpose, which have a maximum length of 2n. These arrays are computed with a preprocessing algorithm, which runs in the same time with Kruskal's algorithm. This preprocessing algorithm takes O($nlgn$) time which is computed once. Then each maximum cost edge between two points is calculated in O(lgn) time.

4.4.6. RST

The RST algorithm proposed by Zhou [2] is also a heuristic which starts with an initial minimum spanning tree and iteratively improves it. This algorithm uses the basis of Borah's algorithm, which was introduced in Section 4.4.4. It is an enhanced algorithm that uses the spanning graph introduced by Zhou *et al.* [49] among other improvements. The algorithm runs in O(*nlgn*) time and takes O(*n*) storage without losing the performance of Borah's algorithm [2].

In Borah's algorithm, point-edge pairs are formed and used. In the original algorithm, all points in the given set are taken into account for all edges of the MST. Borah has already pointed out that this is not necessary. A "*spanning graph*" is a sparse graph in which a minimum spanning tree exists. Zhou noticed that a spanning graph can be used to generate point-edge pairs although there is no direct relationship with the visibility of the point and spanning graph. In Zhou's RST algorithm, for an edge in the MST, the neighbors in the spanning graph of the points forming that edge are taken as point components of the corresponding point-edge pair. Thus by forming point-edge pairs in this manner, O(*n*) point-edge pairs are formed. This is because of the fact that a point can be connected to at most 8 octal neighbors in the spanning graph. After a point-edge pair is updated in the MST, the most expensive edge in the formed cycle is then found.

RST algorithm uses the spanning graph concept also in the initial MST computation phase. The spanning graph depends on the octal partitioning concept. A point can not be connected to two points in the same octal region for the rectilinear metric. A sweep-line algorithm was designed by Zhou *et al.* [49] to generate the rectilinear spanning graph.

After the spanning graph is generated, Kruskal's algorithm is used to find the initial rectilinear minimum spanning tree. Kruskal's algorithm is used also in the longest edge computation for all point-edge pairs. In Kruskal's algorithm, the edges are initially sorted with respect to their costs and this information is used for a least

common ancestor algorithm (LCA). Then LCA queries will be generated according to this algorithm and most expensive edges will be calculated afterwards.

In RST, all point-edge pairs, the most expensive edges and the gain of each pair are calculated first. Then the pairs are sorted in non-increasing order of gains and a connection is made if both the connection edge and the deletion edge have not been deleted yet.

The algorithm has a time complexity of O(*nlgn*) and it has a good performance measured in terms of the improvements that it makes over the MST. Test results exits in [2] reflecting the efficiency of the algorithm.

4.4.7. Comparison of the Approximation Algorithms

The present section compares the approximation algorithms in terms of their performance, time complexity and ease of implementation. The performance of the heuristics is given mostly in terms of their improvements to the MST. Namely, the difference between the cost of the rectilinear Steiner minimum tree and the minimum rectilinear spanning tree using rectilinear distances is compared. The time complexity is calculated in terms of big-O notation. It is also very important because very large instances up to millions of nodes are within the scope and interest of the present thesis work.

The MST embeddings and the Zelikovsky's heuristics give good performance, but their time complexity is high in order to compute large instances. Also they are not very easy to implement.

B1S has been the champion heuristic for a long time achieving an %11 improvement on the MST on average and also being 0,5% away from the optimum. It is given in [44] that up to 10,000 nets can be computed efficiently with the O(n^3) implementation of the algorithm, which is fairly straightforward. The IRV heuristic which gives better performance than B1S is much complicated to implement.

Borah's algorithm has a performance that can be scaled with B1S, but can operate faster than B1S with its time complexity of $O(n^2)$. It is again not scalable for very large set of points.

The BGA algorithm is shown to match the solution quality of Borah's edge based heuristic and sometimes being better. With the $O(nlog^2n)$ implementation, introduced in [1], a 34,000 terminal net can be solved within 25 seconds instead of 86 minutes of Borah's time. The methods for the implementation of the algorithm are shown to be realized efficiently.

The RST algorithm is based on Borah's algorithm. It was shown that that the $O(n^2)$ point-edge pairs in Borah's algorithm decreased to $O(n)$ in RST. By the help of this improvement, a significant decrease in asymptotic time bound and hence a $O(nlogn)$ time algorithm has been achieved. However, the maximum update on the MST is at most the value that Borah's algorithm can get. It can also be said that the algorithm does not need very complex data structures.

When all algorithms are compared, the two most recent ones, namely, the BGA and the RST, seem to achieve very good performance. They can be used for very large point sets, which is the main interest of this thesis work. The improvement to MST of BGA and RST has been shown to be at most 1% worse than the optimal improvement in the literature. It is observed that the BGA has better solution performance and the RST has better running time. It is also envisioned that these algorithms can be further modified or parallelized to increase efficiency. In Chapter 5 these two heuristics will be studied and described in more detail. A modification on the RST using some parts of the BGA will be proposed. Afterwards in Chapter 6, the modified RST will be parallelized.

CHAPTER 5

BGA and RST

In Chapter 4, important Rectilinear Steiner Tree Heuristics have been examined according to their fundamental properties. After comparing these algorithms, BGA and RST algorithms have been chosen among the others as the best and also most promising to be used in larger instances. In the following sections, the two algorithms are explained in detail and are divided into smaller parts which in turn are investigated individually. First BGA and then RST are explained. At the end of the chapter, a modification on RST algorithm is proposed.

5.1. Detailed Description of BGA

The BGA algorithm [1] proposed by Kahng in 2003 is a heuristic for Rectilinear Steiner Tree Problem that starts from Rectilinear Minimum Spanning Tree (RMST) and then updates it. The algorithm first finds the RMST and merges optimal Full Steiner Triples into the RMST and then removes the most expensive edges in the formed cycles to find the Rectilinear Steiner Tree (RSMT).

The following figure presents the BGA algorithm in the form of a pseudo code first and the detailed explanation of the steps in the pseudo code follows next.

BGA Pseudocode

Input: Set of terminals called A

Output: Steiner tree S spanning terminals of A

1. Compute_MST(A) // Compute minimum spanning tree of A

SG← Generate_Sparse_Graph(A);

MST← Kruskal(SG);

2. Compute Hierarchical_Greedy_Preprocessing(MST)

/* Generate two arrays called parent and edge that will be used in the bottleneck edge computations This step is embedded in Kruskal algorithm step*/

3. Triples← Generate_Triples(A);

4. For each $\tau \in$ Triples find MST U τ and compute

$R(\tau)$← Find_Most_Expensive_Edges(τ)

$A(\tau)$← Zero cost edges between 3 terminals of τ

$Gain(\tau)$← $cost(R(\tau)) - cost(\tau)$

Discard triples with $Gain(\tau) < 0$

5. Sort triples according to $Gain(\tau)$ in decreasing order

6. Unmark all edges of MST

7. For each $\tau \in$ Triples

If both edges of $R(\tau)$ are unmarked

Mark edges of $R(\tau)$

MST← MST - $R(\tau)$ + $A(\tau)$

8. Return MST

Generate_Sparse_Graph(A)

Input: Set of terminals called A

Output: Sparse graph spanning terminals A that will be used in MST computation

/* This function will find the nearest points for all terminals in A in both of their 8 octal regions according to Guibas-Stolfi Algorithm */

1. For i=1 to 4 // first 4 octants

transformed← Transform the terminals ∈ octant(i) to first quadrant

Sort points according to x coordinates in ascending order

edges← Find_NE_SW_Nearest_Neighbors(transformed)

2. Return edges

Find_NE_SW_Nearest_Neighbors(transformed)

While transformed has more than 2 elements do

Find_NE_Nearest_Neighbors(1st half of transformed)

Find_NE_Nearest_Neighbors(2nd half of transformed)

Merge(1st half of transformed,2nd half of transformed)

// y-sorted list will also be calculated in parallel

Edges← Nearest_NE[transformed], Nearest_SW[transformed]

Merge(1st half of transformed,2nd half of transformed)

left← Biggest y coordinate in the 1st half

right← Biggest y coordinate in the 2nd half

min← Biggest y coordinate in the 2nd half

while(y[left]> y[right]) do

left← advance next point in the 1st half in decreasing y

while there are unprocessed elements in the 1st half do

while(y[right]> y[left]) do

if (||left,right||< ||left,min||) /* distances are calculated in L1 distances */

min← right

right← advance next point in the 2nd half

Nearest_NE[left] ← min

Repeat same procedure for South-West region also

Kruskal (SG)

Input: Set of terminals A and sparse graph SG

Output: Minimum spanning tree spanning of A

1. MST← 0

2. For each vertex v ∈ A

do MAKE-SET(v) // Generate disjoint sets for all points

3. Sort the edges ∈ SG in non-decreasing order of gain in L1 metric

4. For each edge (u,v)∈ SG in descending order of gain

if FIND-SET(u) ≠ FIND-SET(v)

MST← MST U (u,v)

UNION(u,v) // Merge the disjoint sets into one set

5. Return MST

Generate_Triples(A)

Input: Set of terminals A

Output: All empty tree triples

1. For i=1 to 4

converted← Convert the terminals to North-East triple format

Calculate (x+y) for converted and sort them //d1-sorted

Calculate_NE_Triples(converted[0], converted[n])

```
Calculate_NE_Triples(left,right)
    mid=(left+right)/2
    if(mid-left>=2)
     Calculate_NE_Triples(left,mid)
    if(right-mid>=2)
     Calculate_NE_Triples(mid,right)
    Combine_NE(left,mid,right)
    // y-sorted, x-sorted, d2-sorted lists will also be calculated in
    parallel
```

```
Combine_NE(left,mid,right)
    Compute_Case1_Triples(left,mid,right)
    Compute_Case2_Triples(left,mid,right)
    Compute_Case3_Triples(left,mid,right)
    Compute_Case4_Triples(left,mid,right)
```

```
Compute_Case1_Triples(left,mid,right)
    For all points i=(x-sorted[mid] to x-sorted[right]) ∈ TR region
        Find (leftmost_low_right[i]) ∈ TR region
    For all points ∈ TR region
      if(leftmost_low_right[i] is defined)
        Mark as stripe end
    For all stripe ends
      Compute highest points ∈ LB region
    For all points i= (y-sorted[right] to y-sorted[mid]) ∈ TR region
      Find the highest point in stripe ends and save the triples
```

```
Compute_Case2_Triples(left,mid,right)
    For all points i=(y-sorted[right] to y-sorted[mid]) ∈ TR region
    Find (highest_low_left[i]) ∈ TR region
    Process TR region in y-ascending and LB region in d2-ascending
        order and save the triples
```

```
Compute_Case3_Triples(left,mid,right)
    For all points i=(y-sorted[left] to y-sorted[mid]) ∈ LB region
      Find (highest_low_right[i]) ∈ LB region
    For all points ∈ LB region
      if(highest_low_right[i] is defined)
        Mark as stripe end
    For all stripe ends
      Compute leftmost points ∈ TR region
    For all points i= (x-sorted[left] to x-sorted[mid]) ∈ LB region
      Find the leftmost point in stripe ends and save the triples
```

```
Compute_Case4_Triples(left,mid,right)
    For all points i=(x-sorted[left] to x-sorted[mid]) ∈ LB region
    Find (leftmost_high_right[i]) ∈ LB region
    Process LB region in x-descending and TR region in d2-ascending
        order and save the triples
```

Hierarchical_Greedy_Preprocessing(MST)

Input: MST

Output: Two arrays called parent and edge that will be used in bottleneck edge computations

```
//Edges of MST were already sorted for Kruskal's algorithm
// This preprocessing function runs only one time and in nlogn time it
//collects information about your MST. Then each bottleneck
//computation takes logn time only.
1. next← n= # of terminals
   For i=1 to 2n-1 do
        parent[i]=NIL
        edge[i]=NIL
2. For each edge e_i=(u,v), i=1 to n-1 do
        While u≠v and parent[u]≠NIL and parent[v]≠NIL do
             u← parent[u]
             v← parent[v]
        If parent[u]=parent[v]=NIL, then
             next← next+1
             parent[u]← parent[v]← next
             edge[u]← edge[v]← i
        If parent[u]=NIL and parent[v]≠NIL, then
             parent[u]← parent[v]
             edge[u]← i
        If parent[u]≠NIL and parent[v]=NIL, then
             parent[v]← parent[u]
             edge[v]← i
3. Output parent[i] and edge[i]
```

Find_Most_Expensive_Edges(τ)

Input: MST, parent and edge arrays, **τ**

Output: Maximum cost edges on the tree path when the triples in

contracted

1. For edges (u,v) $\in \tau$

index← -∞

While u≠v do

index← max(index, edge[u], edge[v])

u← parent[u]

v← parent[v]

Return e_{index}

Figure 5-1 BGA Pseudocode

5.1.1. Minimum Spanning Tree Construction

The Minimum Spanning Tree for any metric can be expressed as a set of edges that connects all the given points and has minimum length. As large instances are considered in our work, efficient solutions to the Minimum Spanning Tree (MST) Problem are also required. The Minimum Spanning Tree is a well studied problem in the literature and it has a complexity of O(*mlgn*) in a graph G(V,E) where n is the number of vertices and m is the number of edges.

Efficient algorithms have been constructed for the MST Problem when Euclidean distance (L_2 metric) is used. The O(*nlgn*) time complexity had been achieved by using Voronoi diagrams [50]. Voronoi diagrams can be defined for a set of points S as the union of Voronoi cells for all points. A Voronoi cell for a point *c* is the region formed of points that are closer to *c* than any other points of S. Although there are some applications [51], Voronoi diagrams are not well defined for rectilinear metric (L_1 metric), in which the distance between two points *p* and *q* on the plane is computed by $|p_x - q_x| + |p_y - q_y|$. The reason for this will be clarified in the following sections soon.

Because of the problem that exists in the computation of the spanning tree directly in the L_1 metric, other methods have been proposed. The most efficient one that has been built is the sparse spanning graph concept. The main idea behind this method is to find efficiently first a sparse graph that contains the MST and then to work on this graph afterwards.

5.1.1.1. Sparse Spanning Graphs

In order to find the Minimum Spanning Tree for L_1 metric, first a spanning graph for the set of points is constructed. Input to the proposed algorithms to find the spanning graphs is the set of points, call V. The spanning graph G= (V,E) is called a spanning graph if it contains a minimum spanning tree of the complete graph. The cost of the edges E are all computed in L_1 metric. It is trivial that complete graph which have $n*(n-1)$ edges where n is equal to the number of elements of V, has the MST inside it. Actually what is needed are sparse graphs that include MST inside. The least dense graph obtainable that includes MST is the MST itself. So there must be a trade off between a complete graph, which is easy to implement but so crowded and MST, which is sparse but hard to implement.

The main idea in the sparse graph construction is to eliminate edges that are guaranteed not to exist in the MST and to include edges that are guaranteed to exist in the MST. The first property is the cut property. This property says that an edge of smallest weight crossing any partition of the vertex set V into two parts belong to MST. The second property is the cycle property, which states that an edge with largest weight in any formed cycle in the graph can be safely deleted.

In [52] it is shown that for any partition of V, say V_1 and V_2 if |qp| is the shortest segment between V_1 and V_2 then q is in the Voronoi diagram of V. By considering strong cut property a Voronoi diagram can be used as a spanning graph in L_2 metric. But in the L_1 metric consider the case where half of the points of V take place in the $x+y=c$ segment and half of them in the $x-y=c$ segment. Then all edges between these

two subsets have the same length and all must be included in the spanning graph according to the strong cut property.

According to the above results, new methodologies have been constructed for spanning graph construction in L_1 metric. In BGA, Guibas-Stolfi algorithm is used for the sparse graph computation [53]. In RST, a Rectilinear Spanning Graph is constructed for the same purpose and it will be explained in detail later. Both of these algorithms uses octal partitioning concept. In the next section octal partitioning and its use in sparse spanning graph computation is explained and afterwards Guibas-Stolfi's algorithm will be defined in the following section.

5.1.1.2. Use of Octal Partitions in Spanning Graph Computation

In order to explain the octal partitioning phenomena, first strict uniqueness property needs to be defined.

Strict Uniqueness Property: Given a point p, a region R has the strict uniqueness property with respect to p, if for every pair of points u, w $\in$ R either $dist(w,u) < dist(w,p)$ or $dist(u,w) < dist(u,p)$. A partition of space into a finite set of disjoint regions is said to have the strict uniqueness property if each of its regions have the strict uniqueness property [54].

There are different partition schemes that the strict uniqueness property can be satisfied. But now octal partitioning will be defined and how it can be used for spanning graph computations will be shown. Define the octal partition for point p as two rectilinear lines and two 45 degree lines as can be seen in Figure 5-2(a). The regions in the figure involve the line to the left of the region in, as illustrated in Figure 5-2(b).

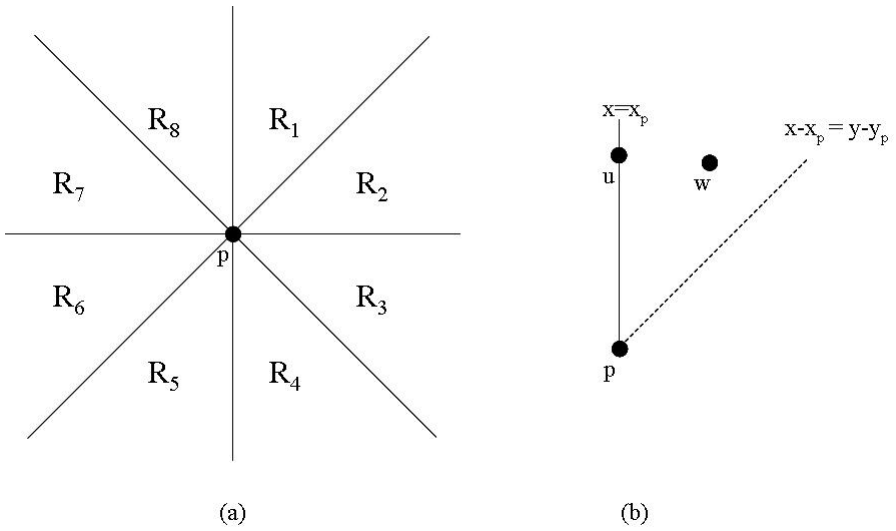


Figure 5-2 Octal Partitions for a point p

Now it must be proved that the octal partitioning have the strict uniqueness property. For this, all regions from R_1 to R_8 must have strict uniqueness property. Since they are similar, only R_1 region will be shown to have the uniqueness property. All others can then be shown to have it.

For the point p, the points (x, y) $\in$ R_1 region have the following inequalities:

$$\begin{aligned} x &\geq x_p \\ x - y &< x_p - y_p \end{aligned} \tag{5.1}$$

Now let us assume that we have two points u and w in R_1 region. And also assume that $x_u \leq x_w$. If $y_u \leq y_w$, then we have automatically $\|pw\| = \|pu\| + \|uw\| > \|uw\|$ where $\|ab\|$ represents the rectilinear distance between point a and b. And for $y_u > y_w$ we have

$$
\begin{aligned}
\|uw\| &= |x_u - x_w| + |y_u - y_w| && ,(x_u \le x_w, y_u > y_w) \\
&= x_w - x_u + y_u - y_w \\
&= (x_w - y_w) + y_u - x_u && ,(x_w - y_w < x_p - y_p, x_u \ge x_p) \\
&< (x_p - y_p) + y_u - x_p \\
&= y_u - y_p && ,(x_u - x_p \ge 0) \\
&\le y_u - y_p + x_u - x_p && ,(x_u \ge x_p, y_u > y_w) \\
&= |x_u - x_p| + |y_u - y_p| \\
&= \|pu\|
\end{aligned}
\tag{5.2}
$$

Same procedure is applied for other regions in the octal partition. Thus given two points u, w in the same octal region of point p, the strict uniqueness property states that $\|uw\| < \max(\|pu\|, \|pw\|)$.

Let us return back to the cycle property, which states that an edge with largest weight in any formed cycle in the graph can be safely deleted from the graph. In Figure 5-2.(b), a graph that includes edges (u,w), (p,u) and (p,w) includes a cycle and by using the cycle property and the previous results $\max(\|pu\|, \|pw\|)$ should be deleted from the graph. Thus for a given point p, only the closest points in L_1 distance in its octal partitions should exist in the graph.

5.1.1.3. Guibas-Stolfi's Algorithm

Guibas-Stolfi have used the octal partitioning method as the basis of their algorithm [48]. They calculate for all points the nearest neighbors in each of their eight octal regions. In order to calculate these nearest points, they initially map each octal region to first quadrant or north-east (NE) quadrant (Figure 5-3). They calculate the sparse graph making all the necessary calculations in this quadrant. The details of this algorithm are explained below.

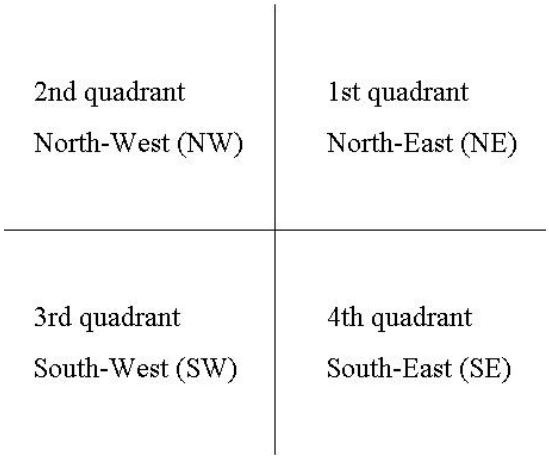


Figure 5-3 Quadrants of point p

The north-east nearest neighbor problem is a type of "all-nearest neighbors" problem for the sparse graph computation. This problem can be stated as for each point in a given set, determine which of the other points are closest to it. In our particular case for each point of the given set its nearest NE neighbor will be calculated.

The algorithm relies on the fact that when there are four points p ,q, r and s, where p and q are smaller than r and s, and $\|pr\| \leq \|ps\|$ then $\|qr\| \leq \|qs\|$. A detailed proof exists in [48].

Based on this fact a divide and conquer strategy for the problem can be devised. First step of the algorithm is sorting the points according to x. In this recursive process also a y-sorted list for future reference can be created. After sorting, an x value that can divide the points into two halves as left and right can be found. The algorithm repeats itself until two points remain in each half. The computed NE nearest neighbors for the right half are correct for the entire problem also, but the values on left half must be revised. This can be done according to the following one-pass procedure in O($nlgn$) time.

There will be three pointers used for this computation. Pointer *left* will advance down the y-sorted list for the left half starting from the biggest y-element. As it is mentioned above y-sorted list will be formed in the algorithm during recursion process. The purpose of this algorithm is to compute the NE nearest neighbor for the pointer *left*. Pointer *right* and *min* will advance down the y-sorted list for the right half starting from the biggest y-element. Pointer *right* must always be higher than *left*, so it will be revised to be bigger than *left* in the beginning of the algorithm. Pointer *min* is used to keep track of the nearest right half element for *left* seen until now.

Now for any *left* value that the NE nearest neighbor is being sought the procedure is started by finding the *right* element in the right half. Also set this element as *min* because it is the nearest point to *left* seen until now. Then advance *right* in the right hand plane as follows:

- If the point that *right* shows is higher than *left* in y, but the rectilinear distance between *right* and *left* is larger than the rectilinear distance between *min* and *left*, then advance *right* again
- If the point that *right* shows is higher than *left* in y, and the rectilinear distance between *right* and *left* is smaller than the rectilinear distance between *min* and *left*, then set *min* equal to *right* and advance *right* again
- If the point that *right* shows drops below the *left* in y, then set min as the nearest NE neighbor of left, and advance left.

Actually by applying this recursive procedure all nearest NE neighbors can be computed for a set of points. By the same procedure also the nearest SW neighbors can be calculated in parallel in order to reduce the computation time.

But the algorithm is not finished yet. We have to find the nearest NE neighbors for all points but what we need actually is to find the nearest octal neighbors of all points. This is done by making transformations of all octants into first quadrant. When making transformations some information about the points has to be protected.

- If q is in the k[th] octant of p, then Tq must be in the first quadrant of Tp.
- If q_1 and q_2 are two points in the k[th] octant of p, and $\|pq_1\| \leq \|pq_2\|$, then $\|T_pT_{q_1}\| \leq \|T_pT_{q_2}\|$.

where T_i represents the transformed point of i into first quadrant. Now the transfer functions for the octal regions will be calculated.

For octal region 1 which can be seen in Figure 5-4 the transformed function T can be $T:(x,y) \rightarrow (2x, y-x)$. This function will map octal region 1 to first quadrant which can be seen in Figure 5-5. Also the properties that the transformation has to satisfy are proved below.

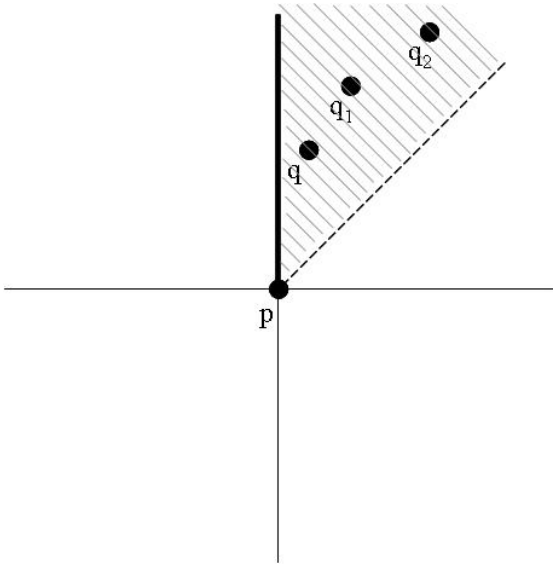


Figure 5-4 Octal Region 1

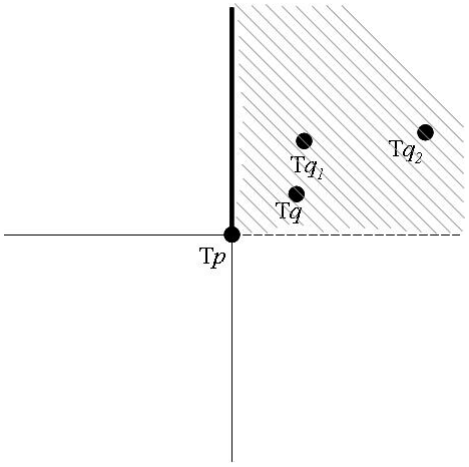


Figure 5-5 First Quadrant

For this octal region following properties occur:

$$\begin{aligned} &1) \quad x_q \geq x_p \\ &2) \quad x_q - x_p < y_q - y_p \\ &3) \quad \left|x_p - x_{q_1}\right| + \left|y_p - y_{q_1}\right| \leq \left|x_p - x_{q_2}\right| + \left|y_p - y_{q_2}\right| \\ &\qquad x_{q_1} + y_{q_1} \leq x_{q_2} + y_{q_2} \end{aligned} \tag{5.3}$$

For first quadrant following properties occur:

$$\begin{aligned} &1) \quad x_{Tq} > x_{Tp} \\ &2) \quad y_{Tq} > y_{Tp} \\ &3) \quad \left|x_{Tp} - x_{Tq_1}\right| + \left|y_{Tp} - y_{Tq_1}\right| \leq \left|x_{Tp} - x_{Tq_2}\right| + \left|y_{Tp} - y_{Tq_2}\right| \\ &\qquad x_{Tq_1} + y_{Tq_1} \leq x_{Tq_2} + x_{Tq_2} \end{aligned} \tag{5.4}$$

Now let's apply $T:(x,y)\rightarrow(2x,y-x)$ to Equation (5.4) in order to achieve Equation (5.3):

$$\begin{aligned} &1) \quad 2x_q \geq 2x_p \qquad \surd \\ &2) \quad y_q - x_q > y_p - x_p \qquad \surd \\ &3) \quad 2x_{q_1} + y_{q_1} - x_{q_1} \leq 2x_{q_2} + y_{q_2} - x_{q_2} \\ &\qquad x_{q_1} + y_{q_1} \leq x_{q_2} + y_{q_2} \qquad \surd \end{aligned} \tag{5.5}$$

For octal region 2 which can be seen in Figure 5-6 the transformed function T can be $T:(x,y)\rightarrow(x-y,2y)$. This function will map octal region 2 to first quadrant

which can be seen in Figure 5-5. Also the properties that the transformation has to satisfy are proved below.

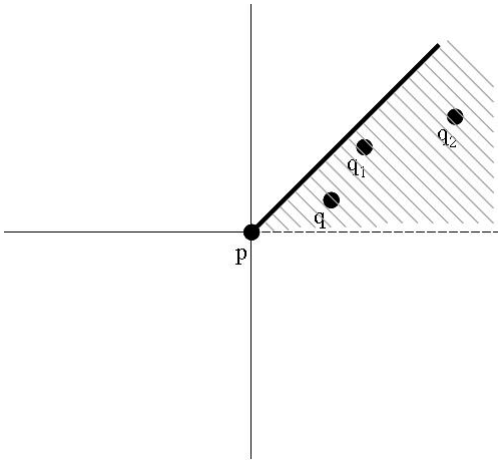


Figure 5-6 Octal Region 2

For this octal region following properties occur:

$$\begin{aligned} &1) \quad y_q > y_p \\ &2) \quad y_q - y_p \le x_q - x_p \\ &3) \quad \left|x_p - x_{q_1}\right| + \left|y_p - y_{q_1}\right| \le \left|x_p - x_{q_2}\right| + \left|y_p - y_{q_2}\right| \\ &\qquad x_{q_1} + y_{q_1} \le x_{q_2} + y_{q_2} \end{aligned} \tag{5.6}$$

For first quadrant, Equation (5.4) will not change. Now let's apply $T:(x,y)\to(x-y,2y)$ to Equation (5.4) in order to achieve Equation (5.6):

$$\begin{aligned} &1) \quad x_q - y_q \ge x_p - y_p \\ &\qquad x_q - x_p \ge y_q - y_p \qquad \surd \\ &2) \quad 2y_q > 2y_p \qquad \surd \\ &3) \quad x_{q_1} - y_{q_1} + 2y_{q_1} \le x_{q_2} - y_{q_2} + 2y_{q_2} \\ &\qquad x_{q_1} + y_{q_1} \le x_{q_2} + y_{q_2} \qquad \surd \end{aligned} \tag{5.7}$$

For octal region 3 which can be seen in Figure 5-7 the transformed function T can be $T:(x,y)\to(2y,-x-y)$. This function will map octal region 3 to first quadrant

which can be seen in Figure 5-5. Also the properties that the transformation has to satisfy are proved below.

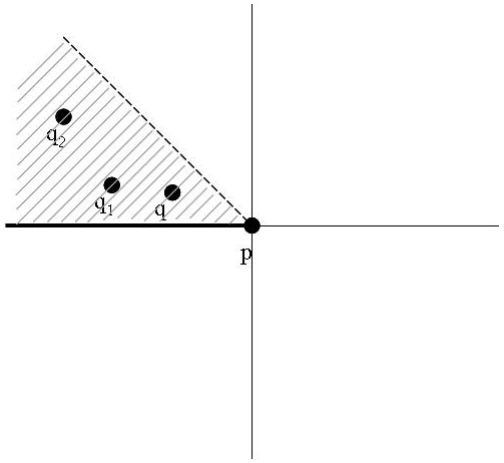


Figure 5-7 Octal Region 3

For this octal region following properties occur:

$$\begin{aligned} &1) \quad y_q > y_p \\ &2) \quad x_q + y_q \le x_p + y_p \\ &3) \quad \left|x_p - x_{q_1}\right| + \left|y_p - y_{q_1}\right| \le \left|x_p - x_{q_2}\right| + \left|y_p - y_{q_2}\right| \\ &\qquad y_{q_1} - x_{q_1} \le y_{q_2} - x_{q_2} \end{aligned} \tag{5.8}$$

For first quadrant, Equation (5.4) will not change. Now let's apply $T:(x,y)\to(2y,-x-y)$ to Equation (5.4) in order to achieve Equation (5.8):

$$\begin{aligned} &1) \quad 2y_q \ge 2y_p \qquad \surd \\ &2) \quad -x_q - y_q > -x_p - y_p \\ &\qquad x_q + y_q < x_p + y_p \qquad \surd \\ &3) \quad 2y_{q_1} - x_{q_1} - y_{q_1} \le 2y_{q_2} - x_{q_2} - y_{q_2} \\ &\qquad y_{q_1} - x_{q_1} \le y_{q_2} - x_{q_2} \qquad \surd \end{aligned} \tag{5.9}$$

For octal region 4 which can be seen in Figure 5-8 the transformed function T can be $T:(x,y)\to(x+y,-2x)$. This function will map octal region 4 to first quadrant

which can be seen in Figure 5-5. Also the properties that the transformation has to satisfy are proved below.

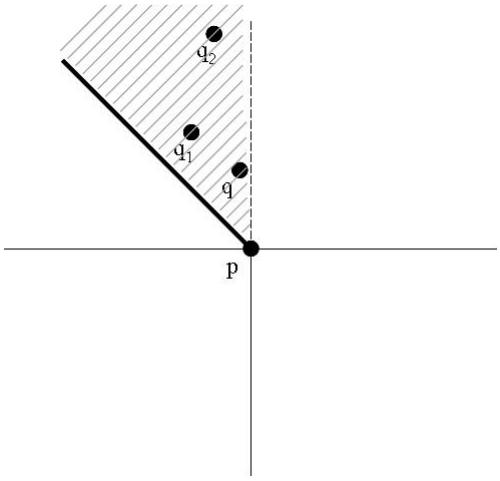


Figure 5-8 Octal Region 4

For this octal region following properties occur:

$$\begin{aligned} &1) \quad x_q < y_p \\ &2) \quad x_q + y_q \geq x_p + y_p \\ &3) \quad \left|x_p - x_{q_1}\right| + \left|y_p - y_{q_1}\right| \leq \left|x_p - x_{q_2}\right| + \left|y_p - y_{q_2}\right| \\ &\qquad y_{q_1} - x_{q_1} \leq y_{q_2} - x_{q_2} \end{aligned} \tag{5.10}$$

For first quadrant, Equation (5.4) will not change. Now let's apply $T:(x,y)\rightarrow(x+y,-2x)$ to Equation (5.4) in order to achieve Equation (5.10):

$$\begin{aligned} &1) \quad x_q + y_q \geq x_p + y_p \qquad \surd \\ &2) \quad -2x_q > -2x_p \\ &\qquad x_q < x_p \qquad \surd \\ &3) \quad x_{q_1} + y_{q_1} - 2x_{q_1} \leq x_{q_2} + y_{q_2} - 2x_{q_2} \\ &\qquad y_{q_1} - x_{q_1} \leq y_{q_2} - x_{q_2} \qquad \surd \end{aligned} \tag{5.11}$$

As a result, following the application of the specified algorithm above, all points in the given set are connected to all nearest neighbors in their octal partition as in the following figure:

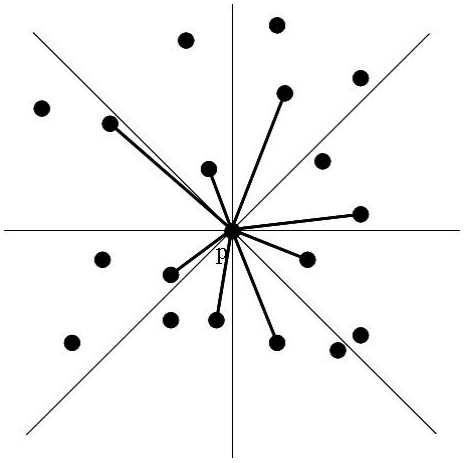


Figure 5-9 Nearest Octal Neighbors of a Point

Now the minimum spanning tree can be found using this sparse graph.

5.1.1.4. Kruskal's Algorithm

For the Minimum Spanning Tree computation two very popular algorithms exist. One of them is the Prim's algorithm [55] in which a single tree is formed and maintained at each iteration by always adding a safe edge to the tree. A safe edge is the one that connects the tree to a terminal that is not included in the tree yet.

The other one is the Kruskal's algorithm [56] in which a forest is formed first and at each iteration always a safe edge is added to the forest. A safe edge is the one that is the least-weight edge in the graph that connects two distinct components. Since at each step the algorithm the least possible weight edge is added to the forest, Kruskal's algorithm is a greedy one. In order to find the least weight edges at each iteration, edges in the graph have to be sorted in the beginning. This property of the Kruskal's algorithm was the reason for choosing it for the BGA code explained earlier.

As can be seen from the pseudocode of Kruskal's algorithm, a disjoint set data structure is used to maintain the disjoint set of elements [57]. Each set contains the connected terminals in the current forest in each iteration.

In the beginning of the algorithm *n* disjoint sets are created by the MAKE-SET(*v*) command where *n* is the number of terminals. Then the edges of the spanning graph calculated above are sorted in non-decreasing order of weight. Here it is worth noting again that that weight is calculated in L_1 metric. Then in this sorted order of edges, for each edge (*u,v*) it is checked whether *u* and *v* are in the same set or not. Here FIND-SET(*u*) function that returns the set that contains *u* is utilized. If these two terminals belong to the same disjoint set, inclusion of this edge will end up with a cycle in the growing tree and this is violate the cycle property of the MST. Thus an edge is included if the terminals of it belong to different disjoint sets. Then UNION (*u,v*) function merges the two sets that contain *u* and *v* into one set.

The complexity of Kruskal's algorithm is O(*mlgm*) where m is the number of edges in the graph [57].

5.1.2. Batched Greedy Triple Contraction Algorithm

In this phase of the BGA algorithm, the triples that were assumed to be found in the previous step will be merged into the MST. Thus some cycles will be formed and by breaking these cycles, in other words by removing the most expensive edge in the cycle, the MST will be improved. Batched Triple Contraction Algorithm is not a brand new algorithm indeed. It is based on the *Greedy Triple Contraction Algorithm* (GTCA) of Zelikovsky [37]. BGA uses the batch concept that was introduced by Kahng and Robins in *Iterated 1-Steiner heuristic*[43] which was also explained in Section 4.4.3. In this way, the greedy rule is relaxed and a single pass algorithm can be defined. All of these concepts are explained in detail below.

BGA uses some basic properties of Steiner trees. As a summary, in a Full Steiner Tree all terminals must be leaves of the tree. The parts of the Steiner tree which can

be split into edge-disjoint components are called Full Steiner Components [7]. If every full component of the Steiner tree has at most *k* terminals it is called *k*-restricted Steiner tree.

In the GTCA algorithm, an approximate minimum cost 3-restricted Steiner tree if found by greedily choosing 3-restricted full components which reduce the cost of the MST. GTCA algorithm calculates the 3-restricted full components because they can be found in linear time. When *k* is bigger than 3 in a *k*-restricted Steiner tree the quality of the resulting Steiner tree will be higher but these trees can not be calculated in linear time. Some definitions proposed in the algorithm are given below:

- Triple τ: An optimal Steiner tree consisting of three terminals.
- Center (τ): The single Steiner point of triple τ.
- Cost (τ): The cost of triple τ which is calculated in rectilinear metric.

For a given set of points A, when a triple τ is inserted into MST(A) two cycles are formed as in Figure 5-10. Please note that the edges are drawn in Euclidean form but this is only for clearness, all edges are actually calculated according to the rectilinear metric.

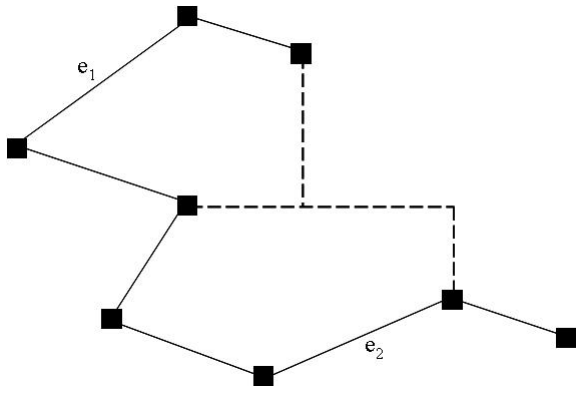


Figure 5-10 MST (A) U τ

In order to obtain the MST of the merged graph, the formed cycles should be broken. For this purpose, most expensive edges in the formed cycles have to be removed. Assume that the most expensive edges are e_1 and e_2 in the cycles formed in Figure 5-10 and let $R(\tau) = \{e_1, e_2\}$. When the triple τ is inserted into the graph, $R(\tau)$ must be removed, so the gain of the triple τ is $gain(\tau) = cost(R(\tau)) - cost(\tau)$.

GTCA repeatedly adds a triple τ with the largest gain to the MST and executes a process called contraction. Contraction means collapsing the three terminals of τ into a single new terminal. This is implemented by adding two zero cost edges between the triple terminals. By the contraction method it is guaranteed that the added triple will remain in the MST, because its edge costs will be zero and no edge can be smaller than these. In GTCA, all chosen triples according to the greedy rule are added to the MST and this union is given as output which is an approximate minimum Steiner tree.

It is shown in [58] that when empty tree triples are used in GTCA, the constructed rectilinear Steiner tree is at most 1,3125 times longer than the optimal one. The concept of triples will be explained further in the following section.

BGA has adopted the batch concept to the GTCA as it was mentioned above. At each iteration of GTCA, a new triple is found which is best among the others in terms of gain but resulting in a time consuming algorithm. In BGA the greedy rule used in selection of triples is relaxed. Instead of searching for a new triple at each step after contracting a triple, the best triple with unchanged gain will be the next triple to be merged into the new MST formed after contraction. In this batched method, the triples should be sorted with respect to non-increasing gain. It is obvious that there is no need to consider negative gain triples. From the gain equation, it is seen that a triple's gain can only be changed if the edges in $R(\tau)$ are removed during contraction of another triple. By this batch method only a single pass of triples has to be made which reduces the execution time significantly. But in

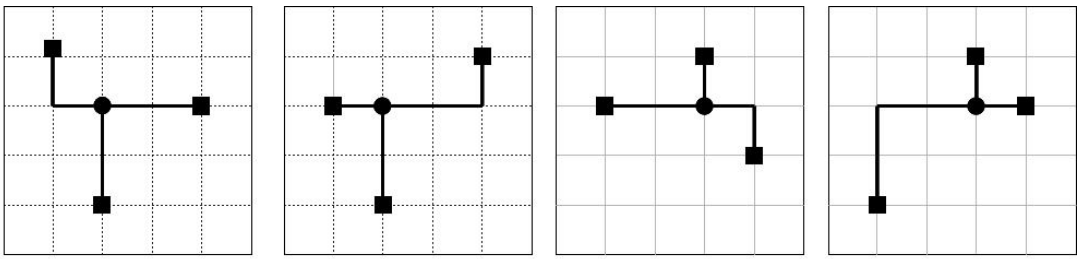
the batched method triples that have positive gain may still remain. More than a single pass of algorithm can be run in order to include them also.

5.1.3. Generation of Triples

The empty tree triples are used to shorten the MST. All empty tree triples will be merged into the MST and the cycles formed during this process will be broken which may reduce the size of MST. In this part of the algorithm, all empty tree triples will be formed for the given set of terminals. For this purpose, the triples will be classified into 4 distinct types and these distinct types will be solved individually in a recursive manner. Each type of these triples are divided to four distinct cases again and they are also solved individually. First, the empty triple concept, which was introduced in [40] will be explained and then the algorithm that calculates all empty tree triples in O(*nlgn*) time will be presented.

A triple is a sub-graph that consists of three terminals. If the minimum rectangle bounding the triple does not contain any other terminals it is called an *empty* triple. If the gain obtained by merging the triple into the MST is positive it is called a *tree* triple. A triple is an empty tree triple if it is an empty triple and a tree triple. It is shown in [40] that the number of all empty tree triples are at most 36*n*, where *n* is the number of terminals in the given set of points. In [40], all these empty tree triples are calculated in O(n^2lgn) by also maintaining dynamic minimum spanning trees. However this approach has been shown to be very impractical for large sets of points [59].

The triples are said to be classified to four distinct types earlier. The classification is made according to the position of the diagonal with respect to the center where the diagonal is the terminal of the triple which does not share the same *x* or *y* coordinates with the center. These types are called north-west, north-east, south-west and south-east and they are illustrated in Figure 5-11.



(a) North-West Triple (b) North-East Triple (c) South-East Triple (d) South-West Triple

Figure 5-11 Types of triples

In order to find all these types of triples, the terminals are partitioned into almost equal halves in a recursive manner. The lines which partition the terminals into equal halves are changing according to the types of triples. For the north-west and south-east types, the divisor line is parallel to line $y = -x$ and two halves are formed called LB (Left-Bottom) and TR (Top-Right). For the north-east and south-west type triples, the divisor line is parallel to $y = x$ and two halves are formed called TL (Top-Left) and BR (Bottom-Right). All these types and divisions are illustrated in Figure 5-12:

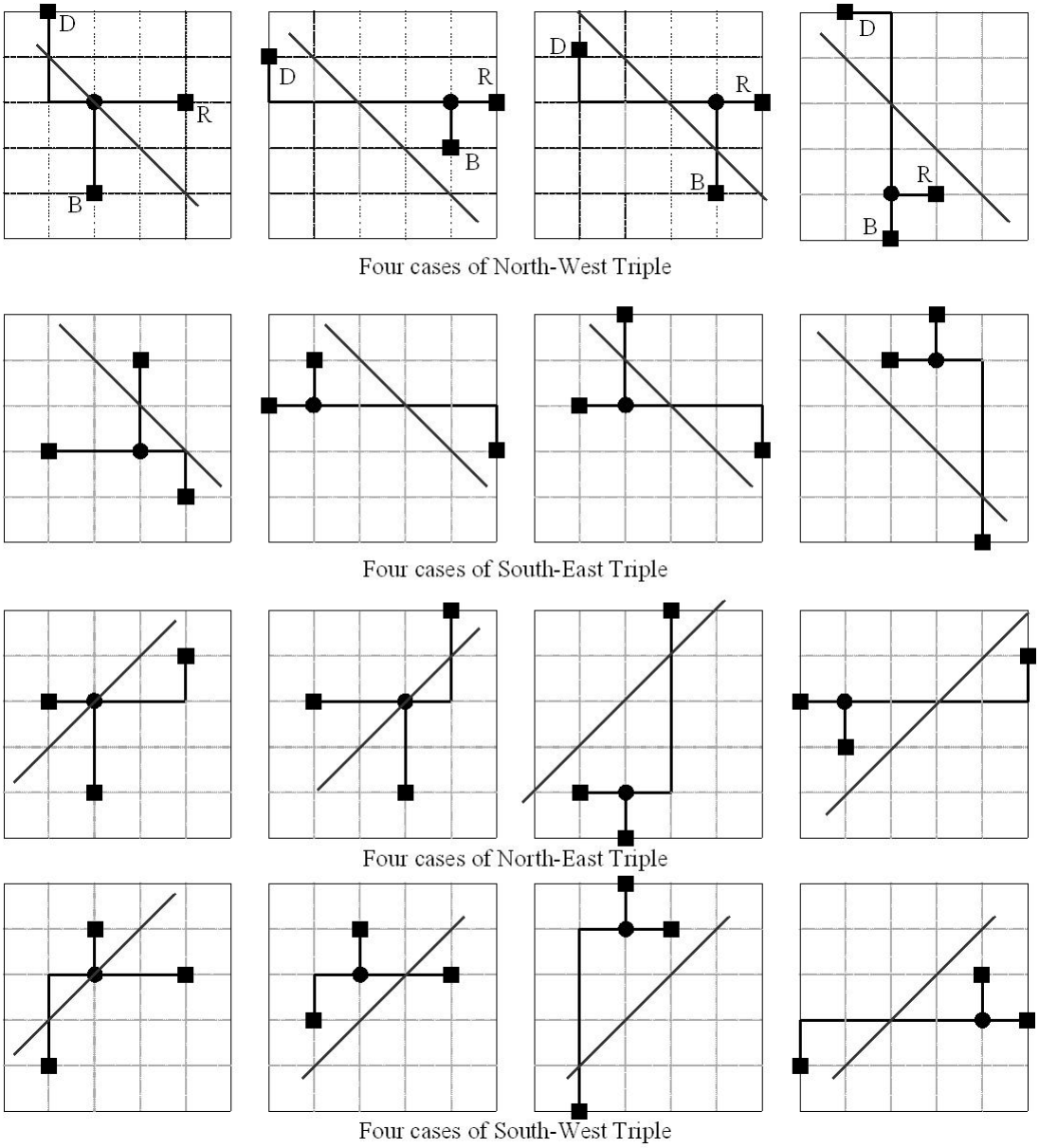


Figure 5-12 Types of Triples and Divisions

Indeed a similar type of conversion like in the one of Guibas-Stolfi algorithm can be used for this generation of triples case. An algorithm can be constructed for the north-west triple type, and the other types can be converted to this type in order to use the same algorithm. Therefore when the conversion takes place for each of other three types of triples (north-east, south-west and south-east), calculating the north-west type of triples will be enough. The conversions and the idea behind these

conversions can be shown with an example as in Figure 5-13 for all other three types of triples.

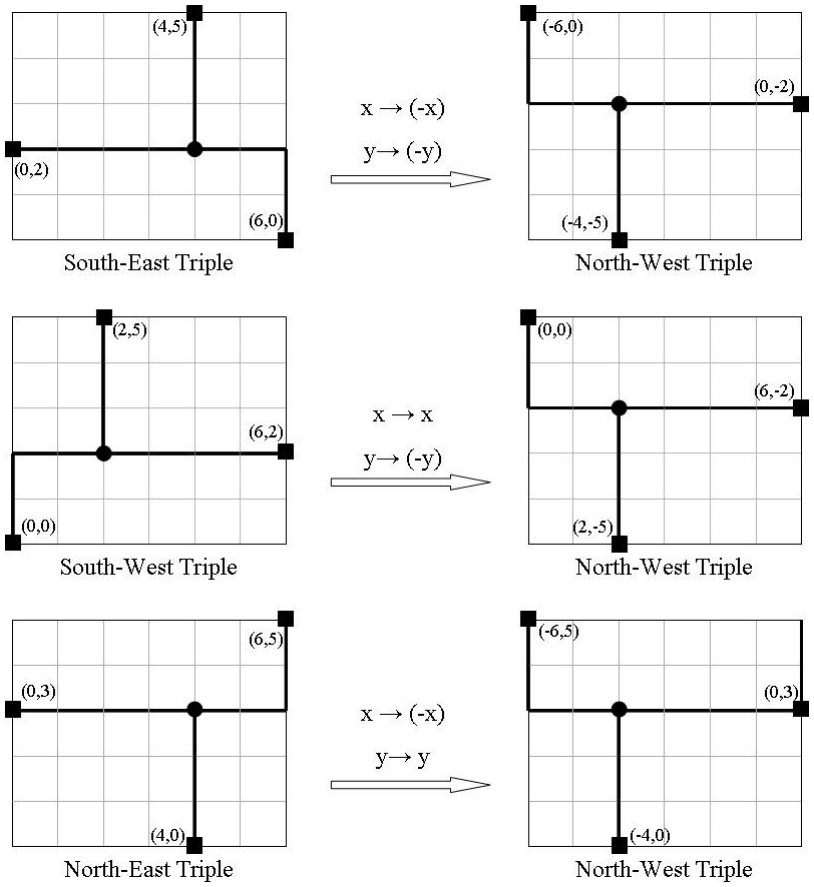


Figure 5-13 Mapping of terminals to North-West triple type

Now the algorithm of finding a north-west triple is to be examined. The terminals are already divided into almost equal halves according to $y = -x$ as can be seen in North-West type triple part of Figure 5-12. The north-west triples are divided into four distinct cases according to the position of the terminals in the halves. The terminals are called Bottom, Right and Diagonal in the north-west triples and each of these cases can be seen in the first line of Figure 5-12.

For the North-West triples the terminals must be sorted with respect to non-decreasing ($x+y$). Then a recursive procedure is applied until two terminals remain and the two partitions are combined afterwards. In Figure 5-14 an example of how recursion occurs for a random set of points. First the points are divided into two parts using the thick line, and then divided again into two parts with the thinner lines until two points remain in each part. All four case triples are searched and recorded in these partitions, and then the algorithm passes to the upper level partition which is represented with the thicker line. It searches and records the triples in each part and continues in the same manner till it ends up meaning that whole set of points are reached.

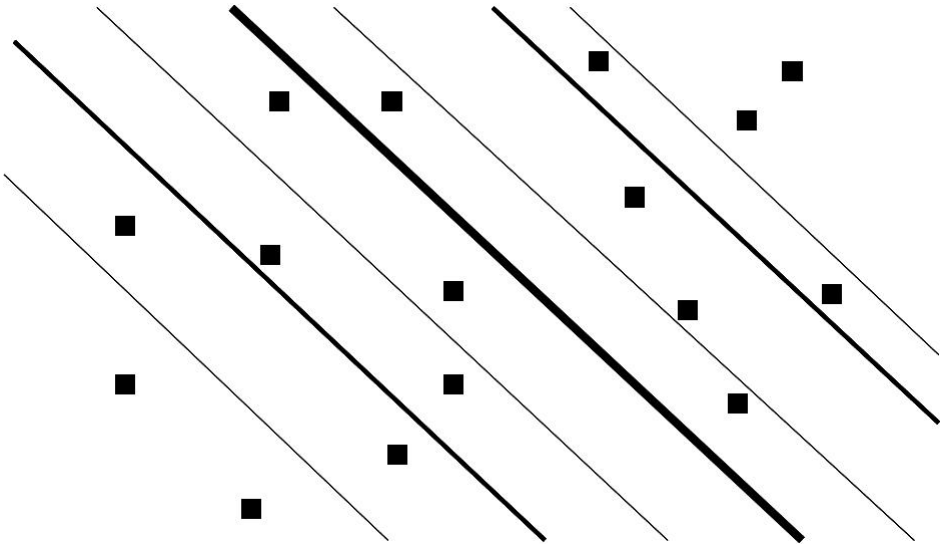


Figure 5-14 Divide and Conquer Algorithm Example

For the Case 1 triples D, R $\in$ TR and B $\in$ LB. In each step, for a Diagonal terminal, the algorithm finds the unique terminal R in the top-right region that can be in an empty north-west triple. In other words for all terminals in the TR, the points that are lower and righter than the terminal are found and the leftmost one is selected. As a result Diagonal-Right point pairs are formed and then, for each Diagonal-Right point pair, a Bottom point in the LB region is searched to complete the triple.

A sweep-line algorithm is constructed for the leftmost-low-right point calculation for all points in the TR region. So in only one pass D-R pairs can be constructed. For this purpose all the terminals in the TR region should be sorted according to increasing *x*. Starting from the first point in the sorted sequence the points are processed as follows:

- If the next terminal has *y* larger than the currently processed terminal then a dashed pointer is set from next to current terminal and the processing is advanced to the next terminal.
- If the next terminal has *y* smaller than the currently processed terminal then the processing is advanced back along the dashed pointer (if exists)

When all the points in TR are processed, each solid arc connects a terminal D to the leftmost terminal in TR that is low and right than the D. An example can be found in Figure 5-15. By this procedure all (D) terminals and associated leftmost-low-right (R) points can be found efficiently.

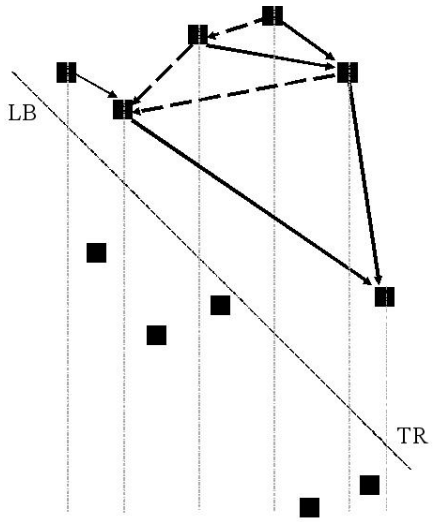


Figure 5-15 Algorithm for Case 1 NW Triple Calculation

Afterwards for all D-R point pairs in TR region, we need to find the node B in LB region. The node B is the maximum y-coordinate node that exists in the LB region

and in the vertical strip defined by D and R. This can also be done in a sweeping manner. First the stripe ends are marked and in ascending *x* order in LB the upper point in these stripe ends are computed. Then starting from the highest point in TR region, where we have the points in y-sorted order, all Bottom points are computed in order to complete the triples. In each recursion step, the calculated triples are saved and the recursion goes on. Therefore, the triples are not forming only in the last step.

For the Case 2 triples B, R $\in$ TR and D $\in$ LB. In each step, for a Right terminal, the recursive algorithm computes the unique terminal Bottom in the top-right region that can be in an empty north-west triple. This time, for all terminals in TR, the highest-low-left point is calculated that is in TR also. This can be made by a similar procedure that is explained above in the Case 1 triples. This time the points in the TR region must be sorted in non-increasing order for the sweeping process. Then in the formed Right-Bottom point pairs Diagonal points in the LB region are searched to complete the triple.

The Diagonal point in the LB region must be the closest point to R and also to B in order to complete the triple into an empty tree triple. Finding the D point for each pair of R-B is a little different from the procedure defined for Case-1 triples. This time a simultaneous traversing of terminals should be done in both TR and LB regions. The points in the TR must be sorted in *y*-ascending order and LB must be sorted in (*x*-*y*) ascending order during the traversal. The points in the TR will be advanced until a terminal R is reached that has an arc to a B whereas the points of LB will be advanced until a terminal higher than R is reached. Then that D is assigned to the B-R pair as a Diagonal point of the empty NW Case 2 triple.

For Case 3 triples R $\in$ TR and D, B $\in$ LB. The approach is similar to the one in Case-1. First, for all points in LB, a point that will serve as a Bottom point is found. This is equivalent to finding a point that is the highest low-right point of the currently processed point. This time the points in the LB must be *y*-sorted. For all

pairs D-B, R points are required to be calculated in the LB region. After the similar sweeping algorithm is used for the D-B pair computation, the stripe ends will be marked. Then for TR region, which is sorted according to *x* the leftmost points are found and then for each D-B pair the R point that will serve in the empty Case 3 triple is found.

For Case 4 triples D $\in$ TR and R, B $\in$ LB. In this case, the approach is similar to the Case 2 part. First, for all points in the LB region, associated R points will be searched. It is the rightmost-high-left point for all pairs. Similar sweeping procedure can be followed as in the preceding cases but the points in the LB should be x-sorted. After R-B pairs are constructed, a D in the TR region is required to be calculated. D will be the closest point to R and B and also D must be to the left of B. In order to complete the triple, LB region will be advanced in *x*-ascending order whereas TR region will be advanced in (*x*-*y*) ascending order.

5.1.4. Hierarchical Greedy Preprocessing Algorithm

It is shown in 5.1.2 that when a triple is merged into the MST two cycles are formed. These cycles in the merged graph have to be broken by removing the most expensive edges. In BGA, Hierarchical Greedy Preprocessing (HGP) algorithm is used for this purpose. In this algorithm two arrays are formed in O(*nlgn*) time first by analyzing the graph. Then each most expensive edge computation is done in O(*lgn*) time. The details are explained below.

The preprocessing phase of the algorithm is very similar with the ideas of Boruvka who has developed a method for constructing an efficient electricity network for Bohemia, in Czech Republic, in 1926 [60]. This algorithm was also a minimum spanning tree construction algorithm which was also applied by Sollin [61] later. The HGP algorithm works on the edges of MST which are sorted in ascending order of cost. Since the MST is formed by using Kruskal's algorithm the edges are sorted already. It has to be emphasized that this step is embedded into Kruskal computation. As the output of the preprocessing algorithm two arrays are formed

called *edge* and *parent* each of size at most *2n-1* where *n* equals the number of terminals.

HGP algorithm first draws a directed edge from every node to the least cost edge from that node and save the index of that edge to that node's place in array edge. Then a graph is formed which consists of some bi-directed, uni-directed and undirected edges. In this graph connected components are formed which consists of a bi-directed edge. An edge in MST is bi-directed when it is the least cost edge for each of its terminals. This kind of connected components will be collapsed into new nodes and these new nodes will be the *parent* of the elements of the components. The same procedure will be repeated until the components will be collapsed into a single component. Since each connected component have a bi-directed edge, at most *n*/2 component nodes are created. Hence the total running time of the HGP algorithm is O(*nlgn*). A sample run of HGP algorithm is given for a random set of points seen in Figure 5-16 in order to make the process more clear.

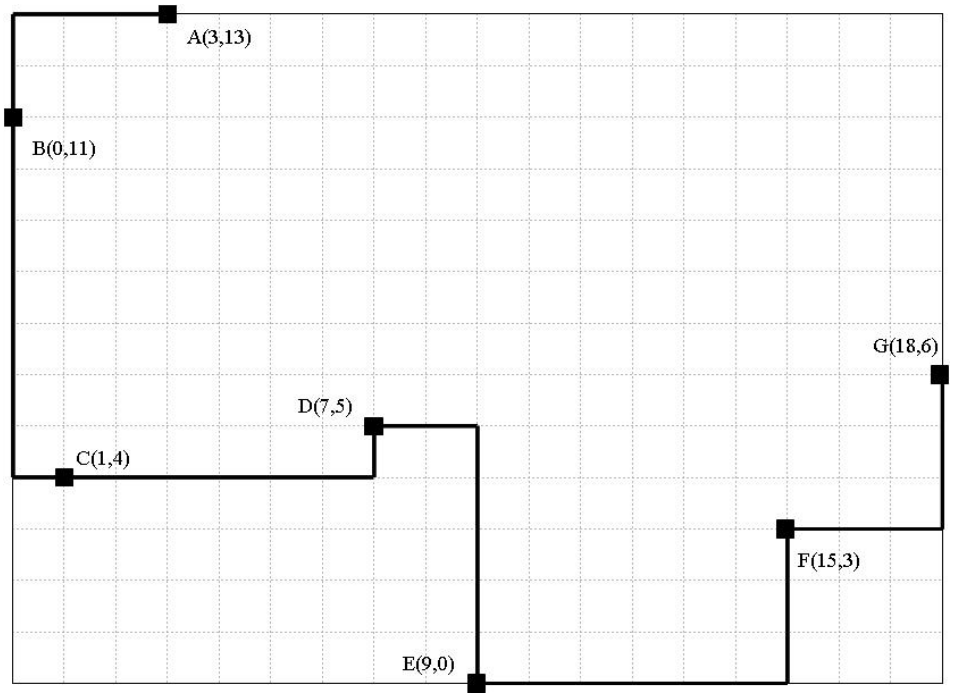


Figure 5-16 Random Set of Points and MST

From each node a directed edge will be drawn to the least cost edge from that node and this edge will be recorded to array *edge*. Then the connected components will be collapsed into a single component. In our example terminals *A, B* will be collapsed into *H*, terminals *C, D, E* will be collapsed into *J* and *F, G* will be collapsed into *I* as can be seen in Figure 5-17. The *parent* array will be $parent = [H, H, J, J, J, I, I, NIL, NIL, NIL, NIL, NIL, NIL]$ and edge array will be $edge = [AB, AB, CD, CD, DE, FG, FG, NIL, NIL, NIL, NIL, NIL, NIL]$ after the first iteration.

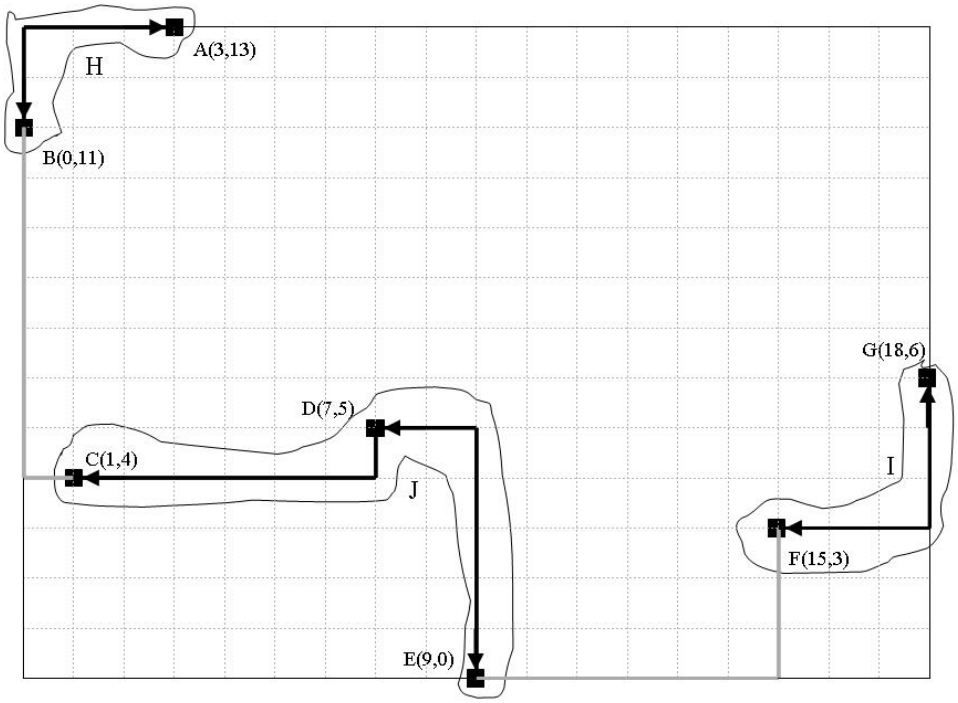


Figure 5-17 First Iteration of HGP Preprocessing Algorithm

The preprocessing algorithm is finished in the second iteration because a single node remains after collapsing. The output of the algorithm can be seen after the second iteration in Figure 5-18. The numbers on the edges represents the rectilinear distance from those edges. Also after the second iteration the *parent* array becomes

$parent = [H, H, J, J, J, I, I, K, K, NIL, NIL, NIL]$ and the edge array becomes $edge = [AB, AB, CD, CD, DE, FG, FG, BC, EF, BC, NIL, NIL, NIL]$.

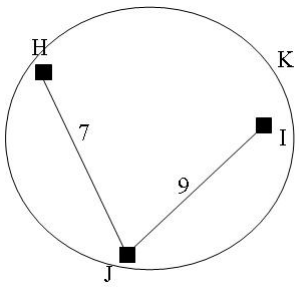


Figure 5-18 Second Iteration of HGP Preprocessing Algorithm

The most expensive edge on a given path will be found by using the results of the preprocessing algorithm. It can be followed by the HGP algorithm if two vertices *u* and *v* are in the same component then the maximum cost edge will be max{edge(*u*), edge(*v*)}. If *u* and *v* are in different components then the maximum cost edge will be the maximum of edges *u, v* and the maximum cost edge between the path of components. An example will be given again for the previous sample algorithm for clearness. The most expensive edge between node *A* and *G* will be calculated according to the following algorithm.

Now if *edge* array is examined edge(A)= |AB| and edge(G)= |FG|, and max{edge(*u*), edge(*v*)} = |FG|. Then the parents will be calculated which are parent(A)=H and parent(G)=I. Now again by looking at the *edge* array edge(H)= |BC| and edge(I)= |EF|. This time max {|FG|, |BC|, |EF|} will be calculated which is |EF|. At the next iteration parent(H)= K and the parent(I)= K, so the algorithm terminates and the most expensive edge will be |EF|.

5.2. Detailed Description of RST

Similar to BGA, the RST algorithm [2] proposed by Zhou in 2004 is also a heuristic for Rectilinear Steiner Tree Problem that starts from Rectilinear Minimum Spanning Tree (RMST) and tries then updates it. For each edge in RMST, a terminal from the set is connected to that edge and then the most expensive edges in the formed cycles are removed to find the Rectilinear Steiner Tree (RSMT).

The following figure presents the RST algorithm in the form of a pseudo code first and the detailed explanation of the steps in the pseudo code follows next.

RST Pseudocode
Input: Set of terminals called A **Output:** Steiner tree S spanning terminals of A

1. Compute_MST_RSG(A) // Compute minimum spanning tree of A
RSG← Generate_Sparse_Graph_RSG(A) /* Generate the sparse graph to compute MST and point edge pair candidates */
MST← Kruskal(RSG)
2. Binary tree ← Compute_Binary_Tree(RSG)
LCA queries
/* Compute both Binary tree and LCA queries with the same algorithm. This step will be embedded into Kruskal computation*/
3. point-edge pairs← LCA(Binary tree, LCA queries)
4. for all point-edge pairs in form (w,(u,v),(k,l))
st← Compute single steiner point in each point-edge pair
gain[point-edge pair]← ||st,w|| + ||st,u|| + ||st,v|| - ||u,v|| - ||k,l||
// All distances will be calculated in L1 metric
5. Sort point-edge pairs according to descending gains
6. for each point-edge pair (w,(u,v),(k,l)) in sorted order
if((u,v)and (k,l) has not been deleted from MST)
In MST Connect w to (u,v) with 3 edges

```
        //From the single steiner point to all 3 nodes
        Delete (u,v) and (k,l) from MST
7. Return MST
```

Generate_Sparse_Graph_RSG(A)

Input: Set of terminals called A

Output: Sparse graph spanning terminals A that will be used in MST and point edge pair candidate computation

```
1. for i=0; i<2;i++
     if(i=0)
        sort points according to (x+y)
        Find_R1_R2_Neighbors(A)
     else
        sort points according to (x-y)
        Find_R3_R4_Neighbors(A)
```

Find_R1_R2_Neighbors(A)

Input: Set of terminals called A sorted according to (x+y)

Output: Pair of terminals which are R1-R5 Nearest Neighbors of each other or R2-R6 Nearest Neighbors of each other

```
1. for i=1; i<=2;i++
     ASi= 0
     for each point p in non-decreasing order
        points← Find_Points_AS_Ri(p)
        /* Find the points in ASi such that the points are in the R(i+4)
        region of point p */
        if(points≠0)
          ||min,p||← ∞
          for all pt ∈ points
              if (||pt,p||<||min,p||)
                  min=pt
```

```
                Add_to_RSG(p,min)
                if(i=1)
                    Remove_from_AS_i(points) //Keep the x-increasing
            order
                else
                    Remove_from_AS_i(points) //Keep the y-increasing
            order
          if(i=1)
            Add_to_AS_R_i(p) // Keep the x-increasing order
          else
            Add_to_AS_R_i(p) // Keep the y-increasing order
          //Insert the current point under investigation to the active set
```

Find_R3_R4_Neighbors(A)

Input: Set of terminals called A sorted according to (x-y)

Output: Pair of terminals which are R3-R7 Nearest Neighbors of each other or R4-R8 Nearest Neighbors of each other

```
1.  for i=3; i<=4;i++
        AS_i= 0
        for each point p in non-decreasing order
            points← Find_Points_AS_R_i(p)
            /* Find the points in AS_i such that the points are in the R_(i+4)
            region of point p */
            if(points≠0)
              ||min,p||← ∞
              for all pt ∈ points
                  if (||pt,p||<||min,p||)
                      min=pt
              Add_to_RSG(p,min)
              if(i=3)
                  Remove_from_AS_i(points) //Keep the y-decreasing
```

order

else

Remove_from_AS_i(points) //Keep the x-increasing order

if(i=3)

Add_to_AS_R_i(p) // Keep the y-decreasing order

else

Add_to_AS_R_i(p) // Keep the x-increasing order

//Insert the current point under investigation to the active set

Find_Points_AS_R1(p)

Input: Active Set AS and point p under investigation

Output: Subset of active set that is in R5 region of p

1. Find the largest x in AS such that $x<=x_p$
2. While $((x-y)>(x_p-y_p))$

Add the current point to "points"

Move to the next point in AS in decreasing x-order

Find_Points_AS_R2(p)

Input: Active Set AS and point p under investigation

Output: Subset of active set that is in R6 region of p

1. Find the largest y in AS such that $y<=y_p$
2. While $((y-x)>=(y_p-x_p))$

Add the current point to "points"

Move to the next point in AS in decreasing y-order

Find_Points_AS_R3(p)

Input: Active Set AS and point p under investigation

Output: Subset of active set that is in R7 region of p

1. Find the smallest y in AS such that $y>y_p$
2. While $((x+y)<(x_p+y_p))$

Add the current point to “points” Move to the next point in AS in increasing y-order
Find_Points_AS_R4(p)
Input: Active Set AS and point p under investigation **Output:** Subset of active set that is in R8 region of p
1. Find the largest x in AS such that $x<x_p$ **2.** While $((x+y)>=(x_p+y_p))$ Add the current point to “points” Move to the next point in AS in decreasing x-order

Kruskal(RSG)
Input: Set of terminals A and sparse graph RSG **Output:** Minimum spanning tree spanning of A
//Same function as BGA

Compute_Binary_Tree (RSG)
Input: Set of terminals A and sparse graph RSG **Output:** LCA queries and merging binary tree

/*This step will be embedded into Kruskal computation. In Kruskal computation we add an edge when two disjoint sets are not equal. At the same time we do the following*/
1. For each neighbor w of (u,v) in RSG
if(s1== FIND-SET(w))
add(w,u,(u,v)) to lca queries
else
add(w,v(u,v)) to lca queries
2. Add_to_Binary_Tree((u,v), s1.edge)
3. Add_to_Binary_Tree((u,v), s2.edge)
/* Add the new edge to the merging binary tree. The new edge will be connected to both branches of the tree as parent. The branches that will be connected to this parent as children will be represented as s1.edge and

```
s2.edge */
//s= UNION(s1,s2) will be made in Kruskal
s.edge=(u,v) // New root of that branch will be (u,v)
```

LCA(Binary tree, LCA queries)

Input: Merging binary tree and LCA queries

Output: Point edge pairs

```
1. TOLCA(u)
        Make-Set(u)
        ancestor[ Find-Set(u) ] = u
        for each child v of u
            TOLCA(v)
            Union( u,v )
            ancestor[ Find-Set(u) ] = u
        Mark u
        for each v such that TOLCA( u,v ) is required
            if v is marked
                LCA of u and v is ancestor[ Find-Set(v) ]
```

Figure 5-19 RST Pseudocode

5.2.1. Minimum Spanning Tree Construction

In the RST algorithm first an MST has to be calculated. This part of the algorithm is the same as Section 5.1.1 but the sparse graph construction is different from the Guibas-Stolfi Algorithm. Instead it is based on the Rectilinear Spanning Graph algorithm [49] which is explained below.

5.2.1.1. Rectilinear Spanning Graph (RSG) Algorithm

RSG algorithm constructs a sparse graph on which the minimum spanning tree can be constructed on. It possesses all the properties of the sparse graph in Section

5.1.1.1. and it also possesses the use of the octal partitioning theme in Section 5.1.1.2.

The basics of the algorithm are very similar to the Guibas-Stolfi's algorithm. In Guibas-Stolfi's algorithm nearest octal neighbors are calculated for all points in the given set. Similarly in RSG algorithm, nearest points in octal partitions are computed. However the graph can be made sparser without violating the rule that the spanning graph should contain the MST. This is an improvement because Kruskal's algorithm depends on the number of edges in the graph. Since the octal partitioning have the strict uniqueness property for a point, only the closest points in L_1 distance in its octal partitions need to exist in the graph. Also for octal partitioning if a point *p* is in (R_1, R_2, R_3, R_4) region of point *q*, the point *q* is in (R_5, R_6, R_7, R_8) region of point *p* respectively. Now take R_1-R_5 pair as an example and assume point *s* is the nearest neighbor in R_1 region of points *p*, *q*, *t* and *v* as in Figure 5-20.

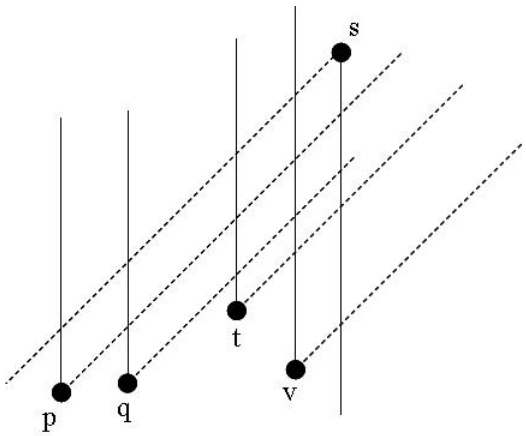


Figure 5-20 Example of Nearest Neighbors R_1-R_5 pair

It is clear that points *p*, *q*, *t* and *v* can not be the nearest neighbor in R_5 region of point *s*. So in the sparse graph, only the edge (*s*, *t*) needs to be included. Edges (*s*, *p*), (*s*, *q*), (*s*, *v*) can not take place in the MST, because *t* is the nearest point of *s* in its

R_5 region. It was proven before, that a point other than the nearest point in any octal partition will be eliminated in the MST according to the cycle property. In the Guibas-Stolfi's algorithm all edges like the ones mentioned above will be in the sparse graph which makes the sparse graph unnecessarily crowded. Therefore what needs to be done is finding the nearest neighbors of all points in all octal regions and taking the intersection of them. Intersection means including, if two points are the nearest neighbors of each other in the opposing regions.

RST algorithm is mainly based on the above concepts. The algorithm is designed as a sweep-line algorithm instead of a divide and conquer algorithm. It starts from a point, does calculations for that point and continues to the next one in order without any backtracking. It is explained in detail below.

The algorithm makes use of another property of octal partitioning. When a point and its octal partitions are investigated, the equi-distant points from that point forms a line segment in each region which can be seen in Figure 5-21. It can be also seen from the figure that for regions R_1, R_2, R_5 and R_6 these line segments can be represented by an equation in the form $x+y$=constant. For regions R_3, R_4, R_7 and R_8 the segments can be represented by an equation in the form $x-y$=constant.

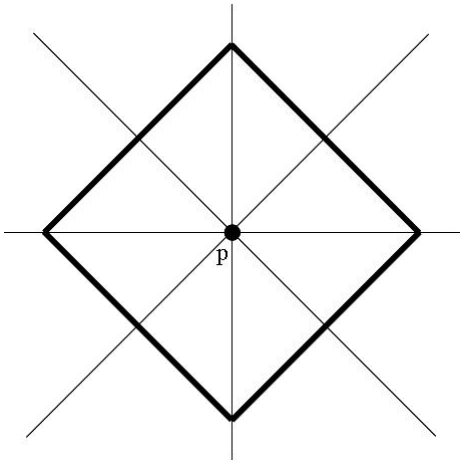


Figure 5-21 Equi-Distant Points for Octal Partitioning

Now take R_1, R2 regions. This property actually means that a point with (x+y) smaller than another one is closer to the origin when thinking distances in rectilinear metric. Thus the point with (x+y) smaller can not be in the R_1, R_2 region of the point with (x+y) that is larger. Similar situations occur for the points in the other regions.

By using this property the edges are sorted according to their (x+y) and (x-y) in the beginning of the algorithm. When the points are swept according to increasing (x+y) for regions R_1, R_2 what happens actually is the scanning of the region step by step as in Figure 5-22 without leaving any point behind. That means a point that is swept already can not be in the R_1, R_2 region for a point that will be processed later.

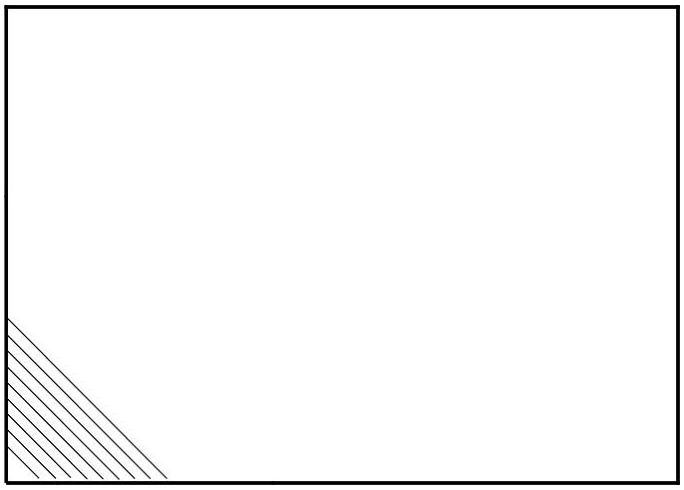


Figure 5-22 Scanning the Region Step by Step

Similar scanning hierarchy occurs in the R_3, R_4 regions when the points are swept according to increasing (x-y). This time the scanning starts from the upper left corner and resumes step by step without leaving any point behind.

In the algorithm what need is to be find the octal neighbors of each point, but the problem is approached in the reverse manner. Given a point, find all candidate points that the specified point can possibly be the nearest neighbor for a specified

octant. Now let's consider R_1 region in particular. As it was mentioned above, a sweep line algorithm for this octant is constructed according to (x+y). During this sweep an active set is constructed. The points in the active set are the ones whose nearest neighbors are still not discovered yet. When a point is processed, all the points in the active set are found which have that point in their R_1 regions. In this way, the whole point set is not searched for a point but only the points whose R_1 regions are not discovered yet are taken in to account. Now suppose that a point s is found in the active set when processing point p as p is in the R_1 region of s. Since the process is made in the form of non-decreasing (x+y) it is guaranteed that p is the nearest neighbor of s in its R_1 region. If there was another point that is closer to point s it will be discovered before the point p is processed. Then the edge sp will be included in the sparse graph and s will be removed from the active set. When the processing of the point p is finished then it is added to the active set. Each point will be added and deleted once from the active set.

The basic operation of the algorithm for a point p is to find in the sweeping sequence a subset of active points such that p is in their R_1 regions. Then the point p will be referred as the nearest neighbor in R_1 of the point q in this subset which is closer to p in rectilinear metric. In this way we actually make the intersection of R_1-R_5 regions which was discussed in the beginning of the section. Because when the subset of active points is found that have p in their R_1 region, actually p is found to be the nearest R_1 neighbor of those points. Then when the closest point in this subset is connected to the graph, actually the R_5 region of the point p is searched and that closest point is the one that is the nearest R_5 neighbor of point p. Also the subset of these active points must be removed from the active set. Point p is the nearest R_1 neighbor of the points in this subset, but for the points other than the closest one to point p, there is a closer point. So these points must be eliminated.

Since a point is deleted from the active set when a point that is in its R_1 region is found, no point in the active set can be in the R_1 region of each other. Then the following property holds for the points in the active set:

For any two points p and q that are in the active set it is guaranteed that $x_p \neq x_q$ and if $x_p < x_q$, then $x_p - y_p \leq x_q - y_q$. This can be seen in the Figure 5-23:

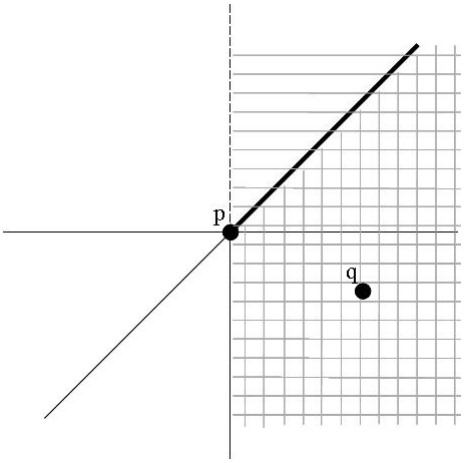


Figure 5-23 Two Points that are not in the R_1 Region of each other

Based on this property, the active set can be ordered in increasing order of x, which also implies a non-decreasing order in x-y. The active set is ordered according to this property and what needs to be done next is to find how it can be searched efficiently. When a point p is swept, the points which have p in their R_1 region needs to be found, which means the points in the R_5 region of p at the same time. These points are characterized according to the following inequalities:

$$\begin{aligned} x &\leq x_p \\ x - y &> x_p - y_p \end{aligned} \tag{5.12}$$

Now to find the subset of active points which have p in their R_1 regions, first the largest element that have $x \leq x_p$ in the active set is found. Then the other points in the decreasing x order in the active set are proceeded until x-y becomes smaller than $x_p - y_p$.

When the sweeping process ends for the R_1 region, also the R_5 octant's nearest neighbors are found. Similar process occurs for also the other regions. For R_2 region

the sweeping is kept in non-decreasing (x+y) also, and for R_3 and R_4 regions it is kept in non-decreasing (x-y).

The changes will be made on the ordering of the active set. For R_2 region any element that is in the active set will not be in the R_2 region of the other, so the following properties occur according to Figure 5-24:

$$x_q - x_p \neq y_q - y_p$$
$$if\ y_q \geq y_p, \quad y_p - x_p < y_q - x_q \tag{5.13}$$

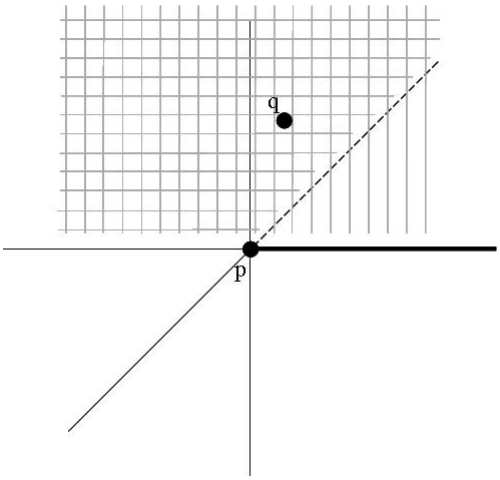


Figure 5-24 Two Points that are not in the R_2 Region of each other

According to the Equation(5.13) the active set for R_2 region can be kept in non-decreasing order of y. This also implies increasing order of y-x. Given a point p, the points which have p in their R_2 region must obey the following inequalities:

$$y < y_p$$
$$y - x \geq y_p - x_p \tag{5.14}$$

In order to find the subset of active points which have p in their R_2 region, first largest y such that $y < y_p$ can be found and then by proceeding in decreasing order of y in the active set until y-x becomes smaller than $y_p - x_p$.

For R_3 region any element that is in the active set will not be in the R_3 region of the other, so the following properties occur according to Figure 5-25:

$$y_p \neq y_q$$
$$if\ y_q < y_p, \quad x_q + y_q \leq x_p + y_p \tag{5.15}$$

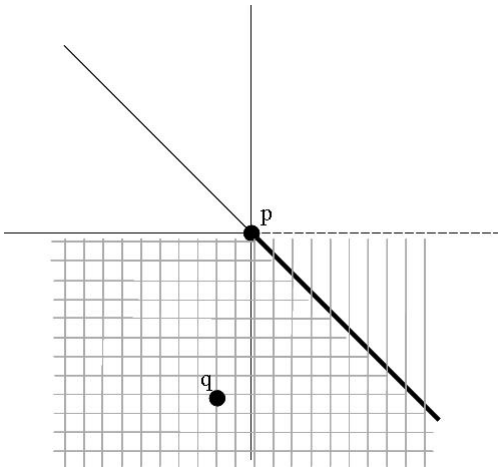


Figure 5-25 Two Points that are not in the R_3 Region of each other

According to the Equation(5.15) the active set for R_3 region can be kept in decreasing order of y. This also implies non-increasing order of $x+y$. Given a point p, the points which have p in their R_3 region must obey the following inequalities:

$$y \geq y_p$$
$$x + y < x_p + y_p \tag{5.16}$$

In order to find the subset of active points which have p in their R_3 region, first smallest y such that $y \geq y_p$ can be found and then by proceeding in increasing order of y in the active set until $x+y$ becomes larger than $x_p + y_p$.

For R_4 region any element that is in the active set will not be in the R_4 region of the other, so the following properties occur according to Figure 5-26:

$$x_p + y_p \neq x_q + y_q$$
$$if\ x_p \leq x_q, \quad x_p + y_p < x_q + y_q \tag{5.17}$$

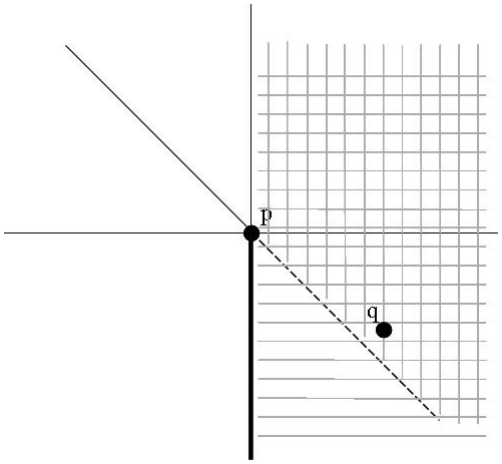


Figure 5-26 Two Points that are not in the R_4 Region of each other

According to the Equation(5.17) the active set for R_4 region can be kept in non-decreasing order of x. This also implies increasing order of $x+y$. Given a point p, the points which have p in their R_4 region must obey the following inequalities:

$$\begin{aligned} x &< x_p \\ x + y &\geq x_p + y_p \end{aligned} \tag{5.18}$$

In order to find the subset of active points which have p in their R_4 region, first largest x such that $x < x_p$ can be found and then by proceeding in decreasing order of x in the active set until $x+y$ becomes smaller than $x_p + y_p$.

By the procedure mentioned above the spanning graph is generated. And now the MST can be found using this sparse graph.

5.2.1.2. Calculating MST from RST

For calculating the MST from the spanning graph, the same procedure explained in 5.1.1.4. namely Kruskal's MST algorithm is used, In Kruskal's algorithm the edges of the graph are sorted according to their gains. This sorting is used in the *merging binary tree* step of the given pseudo code for the algorithm.

5.2.2. RST Edge Based Heuristics

The edge based heuristic used in RST takes its roots from the Edge-Based Heuristic of Borah *et al.* [47]. The idea of the algorithm is starting with an initial minimum spanning tree and then iteratively considering connecting a point to a nearby edge and deleting the longest edge on the formed cycle. Next the idea of Borah and how it is modified in the RST will be examined in more detail.

Borah's algorithm uses an edge based update on the MST. Its main step can be visualized easily with the help of Figure 5-27. It should be noted that the edges shown in the figure are in Euclidean metric, but the actual distances are measured by using the Rectilinear metric. This is preferred for clarity only.

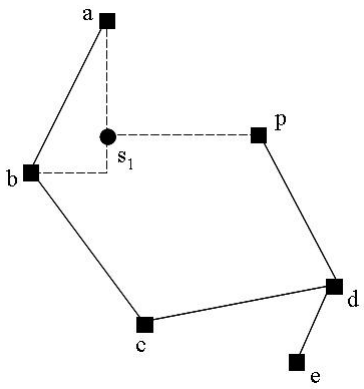


Figure 5-27 Edge-Based Update

When point *p* is connected to point s_1 in the upper figure, then a loop is formed in the tree. As in GTCA algorithm this loop must be broken by removing the most expensive edge in it. Now assume that for this example edge *(b,c)* is the most expensive edge on the path from point *b* to point *p*. Thus this edge has to be removed. Now by adding point s_1 to the tree, then following modifications occur:

- Remove edge *(a,b)*
- Remove edge *(b,c)*
- Add edge *(s1,a)*

- Add edge *(s1,b)*
- Add edge *(s1,p)*

Hence it is observed that the procedure has added a new node to the tree and two edges are replaced with three new edges in the tree. Following these modifications the graph becomes a spanning tree for a new set of points including this extra node. It can be observed that the total rectilinear costs of edge *(s1,a)* and edge *(s1,b)* is equal to the cost of edge *(a,b)*. Thus the total cost of these modifications is equal to:

$$gain\,(adding\ \ point\ \ p\ to\ edge\,(a,b)) = \|b,c\| - \|s_1,\, p\| \qquad (5.19)$$

It can be seen in Equation (5.19) that when the rectilinear distance between point *p* and point s_1 is smaller than the rectilinear distance between point *b* and *c*, the gain of adding point *p* to edge *(a,b)* reduces the cost of the total graph.

Borah's algorithm starts with an initial minimum spanning tree and considers all point-edge pairs in the graph whose size is equal to n^2. In order to find the most expensive edge in the cycle formed by each connected point-edge pair, a recursive routine similar to depth first search is computed. This procedure starts with the given edge as root and passes the maximum edge seen so far as a parameter to the recursive calls. Each depth first search takes O(*n*) time and by applying this edge-update procedure to all edges, this most expensive edge calculation procedure takes O(n^2) time. Then to keep the complexity at O(n^2) only one point with largest gain is held for each edge and O(*n*) point-edge pairs are sorted according to their gains. Afterwards each point-edge pair is applied to the initial MST in a batched manner starting from the one with the largest gain. Each point-edge pair is merged into the tree if their gains did not change, which means that the edges that will be removed for that particular point-edge pair is not deleted before.

Besides this implementation of the algorithm, Borah *et al.* have offered another method in order to achieve the time complexity of O(*nlgn*). This was based on the fact that, for a given edge, not all the points in the tree need to be considered as point-edge pairs. The idea can be seen in Figure 5-27. For the edge *(a,b)*, point *e*

will never end up with a positive gain, because in a way point *e* is blocked by edge *(c,d)*. This observation brings up a new concept of visible and blocked points. It can be noted from Figure 5-27 that for edge *(a,b)* the points *c*, *d* and *p* are visible to the edge, but the point *e* is blocked. Then the point-edge pairs are formed from the edges and the points which are visible to that edge. These blocked points will be very significant for a large point set and the idea decreases the point-edge pairs to O(*n*). Also for the initial MST computation they offered to use Hwang's algorithm [34], which is O(*nlgn*). But this visibility algorithm needs very complicated data structures and very complicated separate algorithms. Therefore, it remained only as a theoretical proposal and has never been implemented.

RST algorithm uses the idea of Borah's algorithm and the fact that there is no need to consider all points while forming the point-edge pairs. In RST, the geometrical information which is embedded in the spanning graph is used for visibility relationship. A point has its eight nearest points connected around it. The spanning graph is used for visibility relationship because if a point is visible to an edge then that point is *usually* connected in the spanning graph to one end point of the edge. This phenomena is illustrated in Figure 5-28 [2] for a point *p* and edge *(a,b)*. Here the dotted lines represents the octal regions for points *p, a* and *b*.

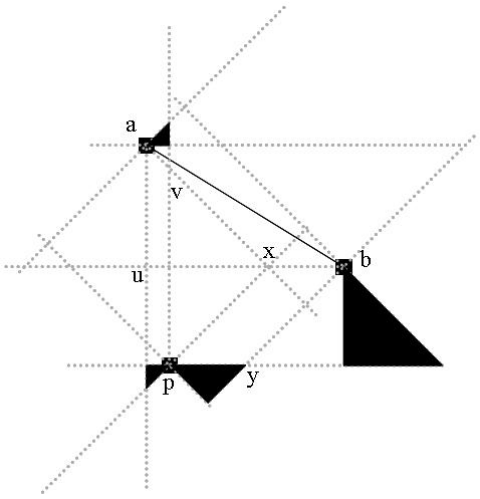


Figure 5-28 Visibility Concept in the Spanning Graph

Since *(a,b)* is in the graph, point *b* is the nearest point in R_3 region of point *a* and also point *a* is the nearest point in R_7 region of point *b*. So there can be no terminal in region *(a,x,b,z).* Afterwards if point *p* is connected to point *b* in the spanning graph, then point *b* is the nearest point in the R_2 region of point *p* and also point *p* is the nearest point in the R_6 region of point *b*. Since *(p,b)* edge is in the spanning graph there can be no terminal in region *(p,y,b,x)*. If point *p* is connected to point *a* in the spanning graph, then point *a* is the nearest point in the R_8 region of point *p* and also point *p* is the nearest point in the R_1 region of point *a*. Since *(p,a)* edge is in the spanning graph there can be no terminal in region *(p,y,b,x)*. Thus according to these properties point *p* is assumed to be visible to edge *(a,b)*. On the other hand, when there is no point between point *p* and edge *(a,b)*, point *p* may not be connected to edge *(a,b)* in the spanning graph when there is at least one point in each of the shaded regions. This is because the points in the shaded regions are nearer than the points mentioned. It is observed that such cases can cause loss of information.

In the RST algorithm spanning graph is used for point-edge pair generation. However it should be noted that there is no one-to-one correspondence between the visibility and connection to the end points. For each edge in the Rectilinear Minimum Spanning Tree (RMST), the end points of that edge are considered. Then all neighbors of both endpoints in the Rectilinear Spanning Graph (RSG) are considered as point components of point-edge pairs. Since the number of possible point edge pairs is O(n) then the complexity of RSG is also O(*n*).

The same edge update procedure of Borah also applies to RST. After the initial minimum spanning tree is formed by the Spanning Graph Algorithm, the point-edge pairs are calculated. All point-edge pairs will be merged into MST and therefore most expensive edges in the formed cycles should be found. Gains of these merging processes are calculated and sorted afterwards. As a last step if the gain of the point-edge pair in order has not been changed yet, three new edges will be connected to the tree and two most expensive edges will be removed from the tree. The gain of a

point-edge pair will be changed if the most expensive edge of that triple has not been deleted yet.

In RST, when a point is merged with an edge, a cycle is formed as was explained earlier. The most expensive edge in the formed cycle should be deleted and the procedure for most expensive edge calculation will be explained in the next section.

5.2.3. LCA Query Algorithm

As was explained earlier in Section 5.2.1.2. Kruskal's algorithm is used on the Rectilinear Spanning Graph. In the first phase of the Kruskal's algorithm the edges are sorted according to their rectilinear costs. Each edge is considered one by one and some of them are added to the tree and some of them are not. A new structure can easily be formed while computing Kruskal. The new structure is a binary tree where the leaves represent the points and the edges represent the internal nodes. When an edge is selected to be included in the MST, then a node is created in its Merging Binary Tree which has two children both of which represents one of the two components connected by this edge. In order to make the process more clear the merging binary tree of Figure 5-16 is given in Figure 5-29.

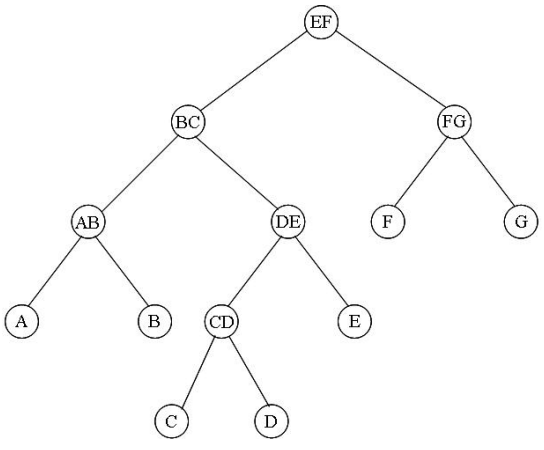


Figure 5-29 Merging Binary Tree for the Sample Set

It can be shown that the most expensive edge between two points is the least common ancestor of these points in the corresponding merging binary tree. As an example the most expensive edge between point *A* and point *G* is the edge *(E,F)* and the most expensive edge between point *A* and point *E* is the edge *(B,C)*. In RST in order to find this longest edge between two points, Tarjan's off-line least common ancestor algorithm [62] is used.

In RST algorithm the maximum cost edge has to be found between a point and an edge, which is a little different from Least Common Ancestor (LCA) procedure to apply directly. What we need to do is to find the end point of the edge that was in the same component in Kruskal computation before the edge is added to the MST. It should be noted that this algorithm is embedded in the Kruskal computation.

Thus when an edge is to be added to the MST, it is added to the merging binary tree. Then for each neighbor of the edge in the Rectilinear Spanning Graph, this edge's end point which is in the same component with the neighbor node and the neighbor node itself will be added to LCA queries. Therefore Kruskal's algorithm is modified according to RST and the merging binary tree and the LCA queries are ready.

5.2.4. Tarjan's Offline Least Common Ancestor Algorithm

Tarjan's Offline Least Common Ancestor algorithm is used in RST to calculate the least common ancestors for the computed LCA queries. The problem is called offline because the entire request sequence can be seen before providing the first answer. This algorithm is an important graph theory algorithm that has also applications in many other fields such as computational biology [63] .

The algorithm works by performing a post-order tree traversal. When the algorithm returns from processing a node, the pair list is examined to find out if there are any ancestor calculations to be performed in the pair list or not. Namely, if *u* is the current node under investigation and *(u,v)* is in the pair list, and also a recursive call

to *v* is already finished, then enough information is determined to calculate the least common ancestors between *u* and *v*.

The progress of the algorithm can be seen in Figure 5-30. In the figure it is assumed that a recursive call to *D* is about to finish. The nodes that have been visited are shaded and the recursive calls for all nodes except the nodes on the path to node *D* are already finished. When the recursive call to a node is finished that node is marked. Also the closest node on the current access path to a visited node *v* is called the *anchor* for node *v*. In Figure 5-30, *A* is the anchor of *p*, *B* is the anchor of *q* and *r* is unanchored. In the algorithm all nodes that have the same anchor are assumed to be on the same equivalence class and all nodes that are unvisited are assumed to be on their own equivalence class. In the figure assume that (*D*,*v*) is in the pair list. Then there are three possible cases:

- If *v* is unmarked, there is no information to compute the least common ancestor of *D* and *v*.
- If *v* is marked but not in *D*'s sub-tree, then LCA of (*D*,*v*) is the anchor of *v*.
- If *v* is in *D*'s sub-tree, then LCA(*D*,*v*) equals to *D*.

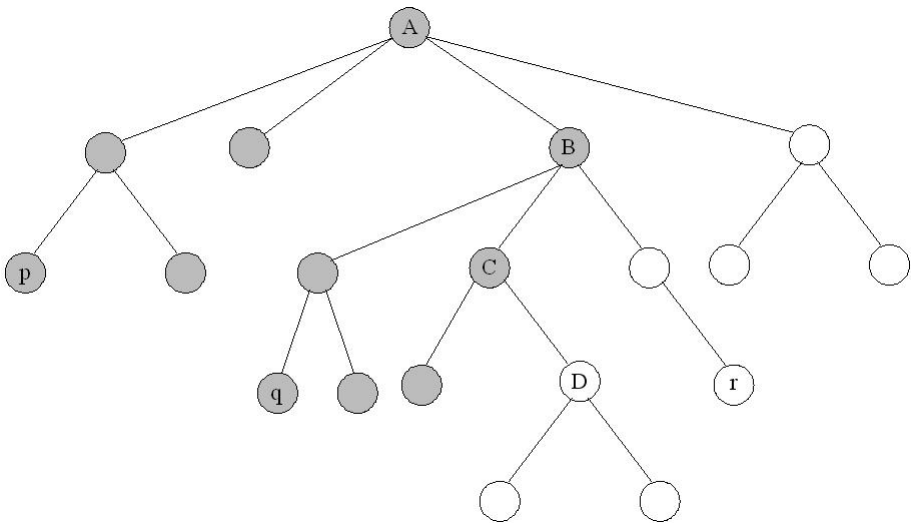


Figure 5-30 LCA Algorithm Progress

In order to determine the least common ancestors with the above algorithm, the anchor must be identified at each time. After each recursive call returns the sub-tree is combined into that point's equivalence class and the anchor is updated. For example after the recursive call to *D* returns as in Figure 5-30 all nodes that are in *D* have changed their anchors to *C* from *D*.

5.3. Modified RST Algorithm

Investigating the BGA and RST algorithms in detail, one can conclude that some parts of the two algorithms are very similar. For example both of them starts with an initial sparse graph on which they try to find the rectilinear minimum spanning tree. Then they add different structures to the minimum spanning tree and they both compute the longest edge formed after this addition. A closer investigation on both algorithms will be made and best parts of the algorithms in terms of performance will be merged in the rest of this work.

First, the construction of the sparse graph will be investigated. The number of selected edges is more in the BGA algorithm. When the initial graph gets a more sparse nature, the Kruskal algorithm will deal with less number of edges, which may cause it to run faster. In BGA, the edges are constructed through a recursive algorithm, whereas the RST algorithm uses an iterative approach in a sweeping manner. Recursive algorithms are not preferred when speed is desired because at each recursion step the values are stored and then reloaded again. Also stack overflows can occur at some point. This becomes a disadvantage when relatively large instances are solved. Taking these facts into account, the RSG algorithm should be taken as the initial spanning graph in the Modified RST algorithm.

The longest edge calculations are similar in both algorithms and they have been implemented using different approaches. In BGA, first, parent and edge arrays are constructed using Hierarchical Greedy Preprocessing (HGP) Algorithm. Then by using these arrays longest edge can be determined between any given two terminals. In RST merging binary tree are constructed initially and then Tarjan's Offline Least

Common Ancestor algorithm is applied. When these are examined, RST algorithm is conjectured to be more costly. Tarjan's Offline Least Common ancestor algorithm is also a recursive algorithm unlike the HGP algorithm. Therefore in the Modified RST algorithm, we will adopt the BGA approach for longest edge computations.

In BGA algorithm, triples are formed and then they are added to the tree. In RST algorithm the neighbors of the edge in RSG are added to the tree. Since the generation of triples requires a recursive approach again, it is not preferred in the Modified RST algorithm. Since RSG is selected as an initial sparse graph, there is no problem of selecting Borah's algorithm for the Modified RST algorithm.

These assumptions and proposals made for the Modified RST algorithm need to be verified. This is accomplished by profiling both of the algorithms where the results are presented in Chapter 7.

CHAPTER 6

DISTRIBUTED VERSION OF MODIFIED RST

The Rectilinear Steiner Minimum Tree may be required to be computed for a large number of times for a given physical design problem. In modern VLSI designs, also there exist many large pre-routes. Therefore a very large amount of time may be consumed for the Rectilinear Steiner Minimum Tree computation in a typical VLSI design cycle. One way to reduce this time may be to find better heuristics that calculates RSMT more efficiently. In another approach one may prefer to solve it using many computers simultaneously rather than solving it in one computer only.

In this chapter, a distributed computing environment will be identified first that can be used to compute the Rectilinear Steiner Minimum Tree in parallel. Then the modified RST presented in detail in Chapter 5 will be parallelized such that it can we can run it using many computers in parallel.

6.1. Computing Environment

Parallel programming consists of writing and running computer programs such that a program runs simultaneously on several processors. These processors work all together to achieve the computation task in a shorter time compared to one processor case. The potential speedup in the execution time takes the attention of many researchers for parallel computing [64].

There are mainly two types of parallel programming environments. One of them is the shared memory environment and the other one is the distributed memory

environment. A shared memory environment is the one in which there are many processors, each of which has access to the same common memory. This is an advantage since all processors have access to all the data. However, the biggest disadvantage is that it doesn't scale well to a large number of processors. Such shared memory systems are also very expensive compared to a distributed memory environment, which consists of many processors, each having its own local memory space.

Network of Workstation (NOW) is a system, which is a kind of distributed memory environment and is a good alternative to parallel computers since they can be set up at relatively low cost [65]. This is because of having cheaper, faster and commercially available of-the-shelf processors every day. In such an environment a high degree of parallelism can be achieved by dividing the computation into manageable subtasks and distributing these subtasks to the processor within the cluster [66]. A network of workstation setup can contain desktops and workstations, each of which is independent and autonomous systems themselves. The memory can vary on each system, but the minimum memory on any of the components should comply with the application. Theoretically, any number of nodes are acceptable for setting up a NOW application, but the scalability may not be linear as the number of nodes increases. The increase in performance should also be comparable with the increase in cost. In a NOW setup, nodes are connected through industry standard components and connections such as Ethernet [67]. An example system can be seen in Figure 6-1.

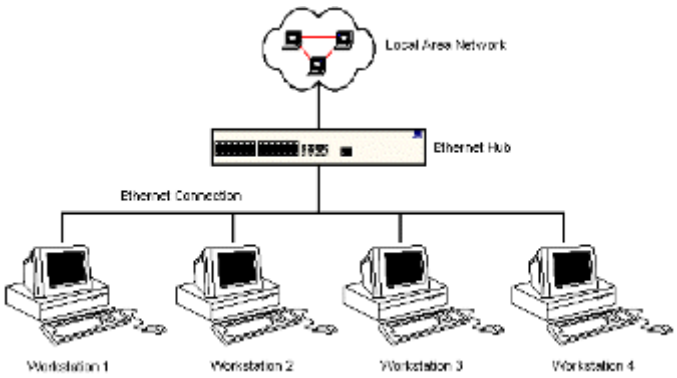


Figure 6-1 A Sample NOW Structure

In a NOW structure, besides their autonomous tasks, every processor may cooperate with each other for solving the same problem. Each processor may have its own local copy of the problem instance in its local memory. When the program proceeds, this data needs to be shared between the processors, which may be achieved by some kind of a message passing system. PVM and MPI are the most popular programs to implement message passing and synchronization among processes [68]. In this thesis work, MPI library will be used as a message passing library [69].

When an algorithm for distributed computing is designed, it is worth noting that data exchange between the processors will have a relatively high cost. Although the Ethernet is fast, a significant communication overhead may arise in the above described system of workstations.

6.2. Distributed Algorithm Proposed for Modified RST

A distributed algorithm for the Modified RST will be proposed in this section by parallelizing the major components of the Modified RST. Our proposal starts with the parallelization of the Rectilinear Spanning Graph algorithm whose sequential version is already explained in Section 5.2.1.1. and continues with the rest.

RSG algorithm is the backbone for the RST algorithm. Therefore, in order to construct an efficient distributed algorithm, RSG has to be parallelized efficiently.

When the algorithm is investigated some parts of it seems naturally parallelizable. For example octal regions can separately be solved because there is no relation between the computations of the separate octal regions. Therefore each pair R_1-R_5, R_2-R_6, R_3-R_7 and R_4-R_8 can be solved in different processors. However, this will cause a problem if there are more than four processors. A more general procedure, without any restriction on the number of processors will be more useful. The following paragraphs describe a possible parallelization of RSG such that the algorithm takes advantage of an increase in the number of processors.

In the first phase of the RSG algorithm for R_1 and R_2 regions the points have to be sorted according to $(x+y)$ and for R_3 and R_4 regions the points have to be sorted according to $(x-y)$. Then a master processor reads the data set and sends it to all processors. It examines the data set, finds the maximum and minimum points both in x and y-directions and also broadcasts these data to all processors, which will identify the regions. Using this data, the whole region may be partitioned into equal pieces depending on the number of processors. Each processor will process the points that lie inside its defined region. Region partitioning idea is illustrated in Figure 6-2.(a) for four processor case for R_1 and R_2 regions . A similar partitioning approach is applicable for R_3 and R_4 regions as illustrated in Figure 6-2.(b) which is drawn for number of processors is equal to 8. By assuming that the points are scattered across the region uniformly, the regions can be selected such that their area are equal.

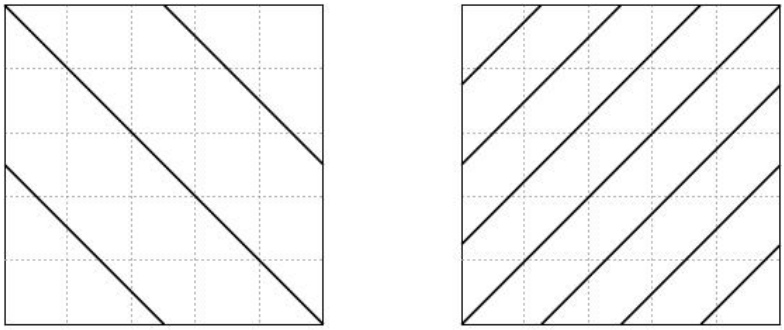


Figure 6-2 Division of the Points to Predefined Regions

Each worker processor sorts its data according to non-decreasing gain. R_1 region will be investigated first; the others will be handled in the same manner. The regions were divided to predefined halves depending on the number of processors above. Now assume that there are eight processor assigned to these region pair. Then the region is divided as can be seen in Figure 6-3. Starting from the regions that have an arrow pointing upwards, the R_1 nearest neighbors of the points will be calculated. The process is the same as RSG algorithm, namely an active set will be maintained whose elements consists of the points which have R_1 region still not have been discovered. Then for all points in that region it will be examined that if that point is in the R_5 region of the other. When all points in that region are processed, R_1 nearest neighbors of the points in that region are discovered except for the points in the active set. According to the octal partitioning properties no points are left behind other than the points in the active set whose R_1 region is not discovered because the points are sorted according to their *(x+y)*.

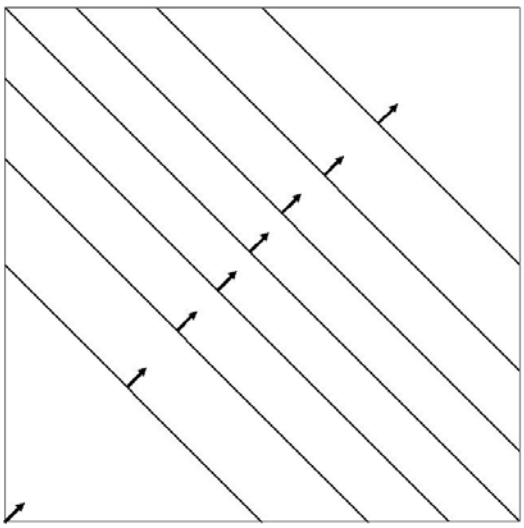


Figure 6-3 R1 Regions Divided for Separate Computation

Until this part of the algorithm it can be seen there is no problem with this process and nearly no communication is needed. The problem arises from the merging of the regions, both of which have been calculated in different processors. Merging will be realized in the boundaries which will be explained.

If the boundary is analyzed it has been determined that R_1 nearest neighbors of the points are discovered except for the points in the active set. If it was the sequential algorithm this process will continue with the next point in increasing $(x+y)$. In the distributed algorithm the next point in the $(x+y)$ direction is unknown since it belongs to the next neighbor processor. Thus if the processors have their next neighbor's sorted points the process can be continued. So after each processor sorts the terminals that belong to them they send this data to the preceding processor. These preceding processors will keep on running the algorithm in the crossing boundaries. This causes the last processor to be left idle during the boundary edge computation. This is the reason that a larger region is defined for the last processor.

The computed RSG will not be the same as the RSG computed sequentially. Since the algorithm is divided across the boundary, more edges can be formed when it is used. But this does not cause any problems because RSG is a method to make the

initial graph sparser. The extra edges will be eliminated in the Kruskal algorithm without causing any harm.

By the above methodology, a spanning graph can be realized in a distributed environment. Now the second step of the Modified RST algorithm is to run a special type of Kruskal computation which will find the parent, edge arrays and the LCA queries as well as the rectilinear minimum spanning tree. This step needs that the edges are sorted according to their gains. So in the previous step when the RSG algorithm is running, the processors will sort its points according to the rectilinear distance and will send this list to all processors. Then each processor will run this modified Kruskal algorithm after it has combined the list coming from the processors. This Kruskal algorithm will run on the all processors separately in this work because as disjoint set operations are used in the algorithm, the communication overhead will become the bottleneck of the computation. Thus the algorithm can be taught as a partially parallelized algorithm. It only recommends an idea and will try to test if it is working.

Now every processor have parent, edge arrays, the rectilinear minimum spanning tree and LCA queries which are the outputs of the previous algorithm. The LCA queries will be divided across the processors according to the number of the processors. Each LCA query is a point-edge pair indeed, so the processor will run the Borah's algorithm on them. The processor will merge the point to the edge and will calculate the most expensive edge in the formed cycle. Since the processor has the parent and edge arrays, it can calculate the most expensive edge easily and it can calculate the gain of connecting that point-edge pair as explained in the sequential algorithm. Then each processor sorts the point-edge pairs in terms of their gains and sends this sorted list to the master processor. It has to be noted that the most expensive edges that have been calculated are also been sent to the master processor.

The master processor combines the gains according to the sorted order and then applies these point-edge pairs to the minimum spanning tree as in sequential algorithm.

CHAPTER 7

COMPUTATIONAL WORK

Both algorithms defined in the previous chapters are implemented using Microsoft Visual C++ 6.0 on a Microsoft Windows machine. The stack size is not changed and remained as 1MB as default. BGA algorithm that can be run readily in a Linux machine was given in [53]. First this code is modified to run on Windows machines. Then the RST algorithm is implemented following the pseudocode presented in Chapter 5. We ran both programs and analyzed them in detail using a profiler. The time consuming parts of the algorithms are identified and the assumptions made on the algorithms are verified. Modified RST algorithm is then implemented as defined in Chapter 5. Both programs have been run and results have been taken on a 3,2 GHz Pentium IV processor that has 2GB memory.

The parallel code proposed for the modified algorithm in the previous chapter is implemented using the message passing interface (MPI). MPICH, an MPI implementation of Argonne National Laboratory, is used as the MPI library. The code is tested on a Windows cluster of eight machines all having a 2,4 GHz Pentium IV processor and 512 MB of memory. Test results have been obtained and will be presented below.

7.1. Implementation of RST

RST algorithm is implemented according to the pseudocode given in the previous chapter. However there is an addition to that code. After a single pass of the RST algorithm, there may still be a possibility of improvement. In the paper by Zhou [2]

this fact is taken into account and he has suggested to compute the sparse graph once and then run the rest of the program five times. However it is not clearly stated how the Steiner points that are found after the first pass will be handled in the following additional passes. For this part of the algorithm the following procedure is followed:

The first pass of the algorithm is carried out exactly according to the pseudocode. Then bypassing the sparse graph generation, the algorithm is applied five times if there are any improvements in the previous pass. In each pass Modified Kruskal Algorithm is slightly modified and adding the edge to the MST phase is skipped since the MST is generated already. Also it was noted that in the least common ancestor queries construction phase, the algorithm seeks for the RSG neighbors of the points that construct the edges. But after the first pass of the algorithm some edges have a Steiner point. Since the Steiner points do not have neighbors in the RSG, only the neighbors of the terminals in the edges will be considered for LCA queries.

Another modification is at the end of each pass of the algorithm, the Steiner points which have degree of two will be deleted and edges will be adjusted accordingly. This procedure was given in Section 4.4.3. The Steiner points with degree two or less are deleted because if they are not deleted, the algorithm will treat them like normal terminals but they will not necessarily be included in the graph. Of course after deleting these nodes the graph must not remain unconnected. For this reason if the deleted Steiner point has degree two, both of the edges connecting to that point will be deleted and the other points forming the deleted edges will be connected to each other. Here attention has to be taken for special cases. For example in some cases two Steiner points which have degree two can be connected to each other. Also for Steiner points with degree one, that edge containing it can be taken out of the graph.

After mentioning the differences when implementing the algorithm, now the basic data structures used throughout the algorithm will be explained. Using the right data

structures is very important for this work because the performance of the algorithms is measured in terms of size of the problem and the time to solve it. With a well-designed data structure, a variety of critical operations can be performed using fewer resources.

7.1.1. Balanced Binary Search Tree

The backbone of the algorithm is rectilinear spanning graph (RSG) construction part and the key concept there is to keep an efficient active set. The active set is defined as an ordered structure and insertion, deletion and search operations are performed on it. So it will be a good choice to select binary search trees to implement it. In a binary search tree all nodes to the left of the current node have key value smaller than that of the current node and all nodes to the right of the current node have key value larger than that of the current node. Most of the operations' complexity in a binary search tree depends on the height of the tree. For a binary tree of n nodes these operations run in O($n$lg$n$) average time but O(n) worst case time.

A self-balancing binary search tree or height-balanced binary search tree seams suitable for this thesis work. Balanced binary search tree is a binary search tree that attempts to keep its height, or the number of levels of the nodes beneath the root, as small as possible at all times, automatically. Namely with some effort to balance the tree, all operations are guaranteed to run in O(lg$n$) time in the worst case. The difference between the binary search tree (Figure 7-1.(a)) and balanced binary search tree (Figure 7-1.(b)) can be depicted in Figure 7-1.

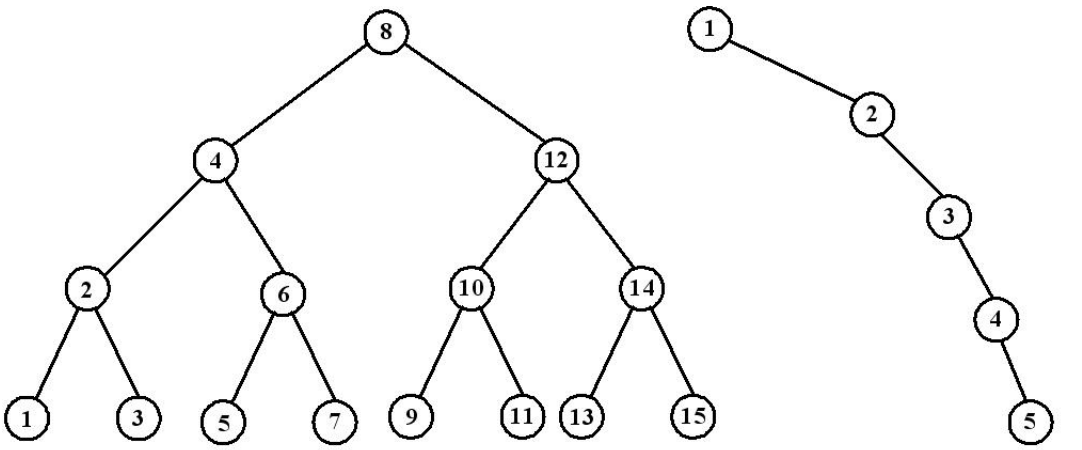


Figure 7-1 Binary Search Tree and Balanced Binary Search Tree

Some previously developed balanced binary search trees in the literature are AVL Trees, Red-Black Trees and AA-Trees [63]. AA-Trees [70] have been selected for their ease of implementation while providing a similar performance compared to others. AA-Trees store a value called level where level represents the number of left links of a node. In an AA-Tree the left child of a node must be one level lower than its parent, and the right child of a node may be at most one level lower than the parent. A binary search tree is shown in Figure 7-2.(a). In Figure 7-2.(b) besides the key values, the level values are also presented in each node and this tree is equivalent to the tree in Figure 7-2.(c).

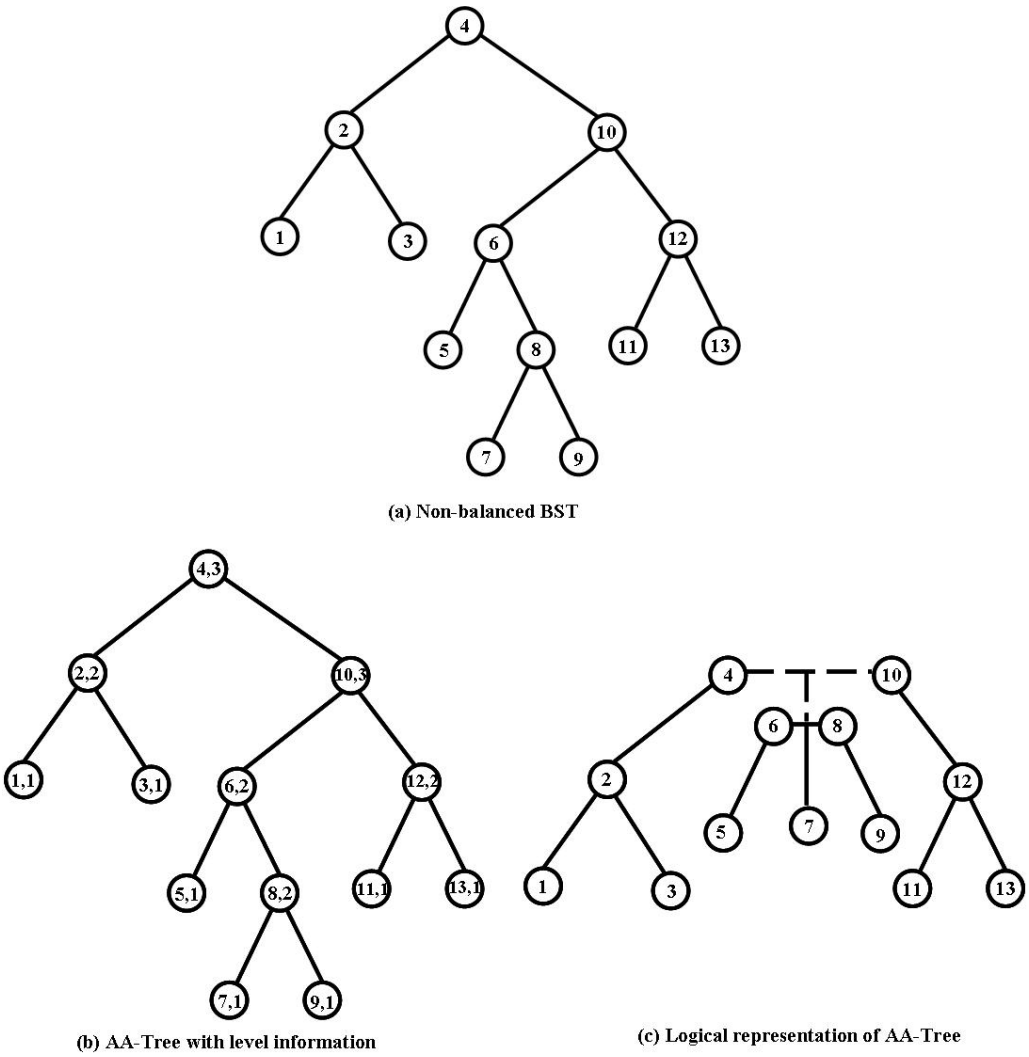


Figure 7-2 Binary Search Tree and its AA-Tree

In order to keep an AA-tree balanced in both insertion or deletion, two additional operations named as skew and split should be performed [71]. A skew removes left horizontal links by rotating right at the parent. No changes are needed to the levels after a skew because the operation simply turns a left horizontal link into a right horizontal link. A split removes consecutive horizontal links by rotating left and increasing the level of the parent. A split needs to change the level of a single node because if a skew is made first, a split will negate the changes made by doing the inverse of a skew. Therefore, a proper split will force the new parent to a higher

level. A skew operation can be seen in Figure 7-3.(a) and a split operation can be seen in Figure 7-3.(b).

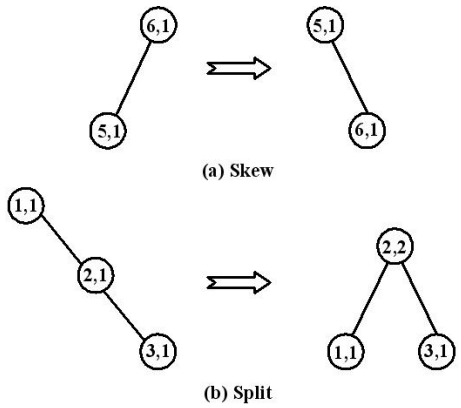


Figure 7-3 Skew and Split Operations

This basic definition of AA-Tree is not sufficient to be used in our algorithm and need some more additions. Besides addition and deletion operations, successor and predecessor information are also needed. Therefore parent information will be kept and updated in all operations. In standard AA-trees, when multiple values need to be inserted, they are normally discarded. To overcome this issue for RST a pointer that can hold duplicated values is inserted in each node and they are maintained at all operations.

Another data structure that has importance in RST algorithm is disjoint set classes. They are both used in Modified Kruskal's algorithm and Tarjan's offline least common ancestor algorithm.

7.1.2. Disjoint-Set Class

Some applications involve grouping elements into a collection of disjoint sets [57]. Two basic operations are finding which set a given element belongs to and uniting two sets. So the algorithm solving disjoint-set data structure is generally referred as

Union-Find algorithm. The simplest approaches that can be followed for this algorithm either needs O(n) time for Find operation or O(n) time for Union operation. Special effort is made for this algorithm to reduce this complexity in both Find and Union operation at the same time.

The first way, called “union by rank”, attaches the smaller tree to the root of the larger tree. To evaluate which tree is larger, a simple heuristic called rank is used. One-element trees have a rank of zero and when two trees of the same rank are merged, the result has one greater rank. The second improvement, called “path compression”, is a way of flattening the structure of the tree whenever Find operation runs on it. The idea is that each node visited on the way to a root node may as well be attached directly to the root node; they all share the same representative. One traversal up to the root node is made to find out what it is and then another traversal is made, making this root node the immediate parent of all nodes along the path. By applying these two operations together a complexity that is equal to the inverse of Ackermann’s function is achieved [72]. Ackerman’s function grows quickly, so its inverse is a very slowly growing function.

In the present work Union-Find algorithm with Union-by-Rank and Path Compression is applied. Besides these since most of the data size is determined while as the algorithm runs, vector data structure is used.

7.2. Implementation Results of BGA, RST and Modified RST

After the implementation of BGA, RST and modified RST algorithms, computational tests have been performed to evaluate their performance. Test results have been obtained both for randomly generated test cases and VLSI industry test cases [53]. The randomly generated test cases are generated using the program which can be found also in [53]. Random points are generated on a 1,000,000x1,000,000 grid and ten test cases are formed for each different data size. The performance of the algorithms depends on their improvements, which is

measured as their improvement to the length of minimum spanning tree, and with the time it consumes.

Average results for each test case are presented in Table 7-1. They are also shown in Figure 7-4 and Figure 7-5. The detailed results of all data sets are given in the appendix. The test results for VLSI industry test cases are presented in Table 7-2.

Table 7-1 Test Results for Random Test Cases

	BGA		RST		Modified RST	
Input Size	Improvement (%)	time (s)	Improvement (%)	time (s)	Improvement (%)	Time (s)
10000	11.052	3.424	10.427	0.358	10.422	0.267
50000	11.068	37.374	10.433	2.426	10.430	2.083
100000	11.084	100.607	10.451	5.582	10.452	5.155
500000	-	-	-	-	10.440	32.361
1000000	-	-	-	-	10.450	73.083

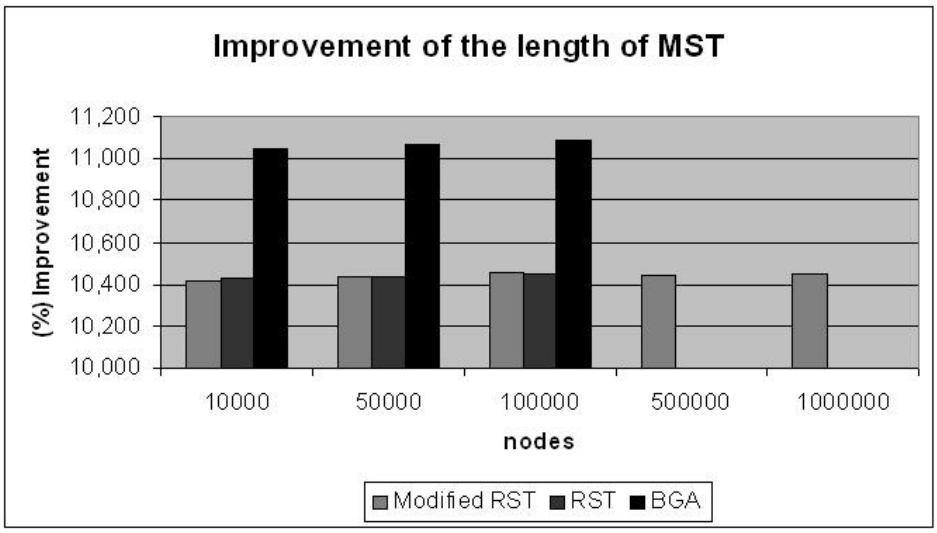


Figure 7-4 Improvement of MST for Random Instances

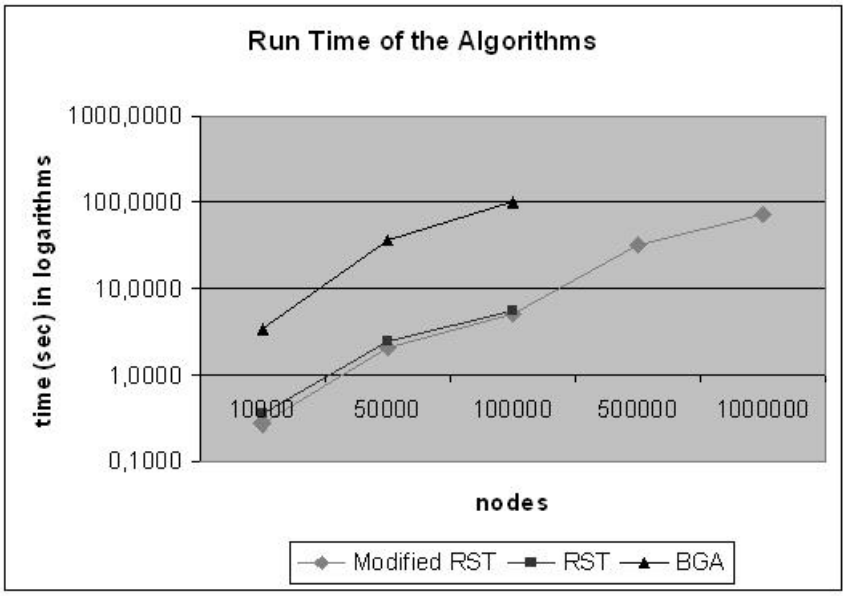


Figure 7-5 Run-time of Algorithms for Random Instances

Table 7-2 Test Results for VLSI Industry Test Cases

	BGA		RST		Modified RST	
Input Size	Improvement (%)	time (s)	Improvement (%)	Time (s)	Improvement (%)	Time (s)
337	6.434	0.034	6.301	0.008	6.330	0.007
830	3.202	0.067	2.999	0.014	2.983	0.012
1944	7.850	0.312	7.332	0.004	7.321	0.003
2437	7.965	0.511	7.611	0.066	7.597	0.050
2676	8.928	0.585	8.458	0.069	8.445	0.057
12052	8.450	3.438	8.149	0.373	8.118	0.295
22373	9.848	7.839	9.446	0.803	9.441	0.618
34728	9.046	14.241	8.662	1.211	8.661	0.983

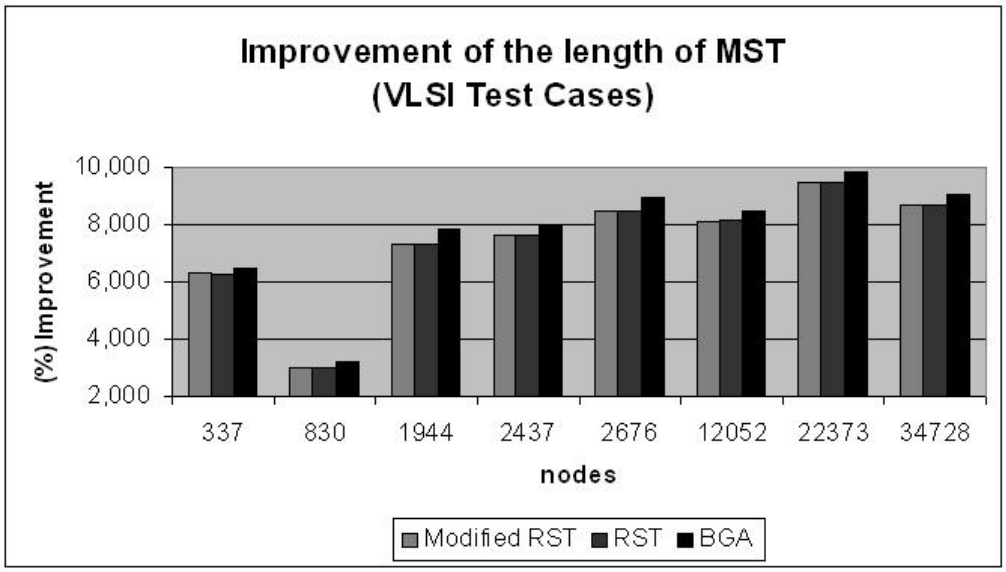


Figure 7-6 Improvement of MST for VLSI Test Cases

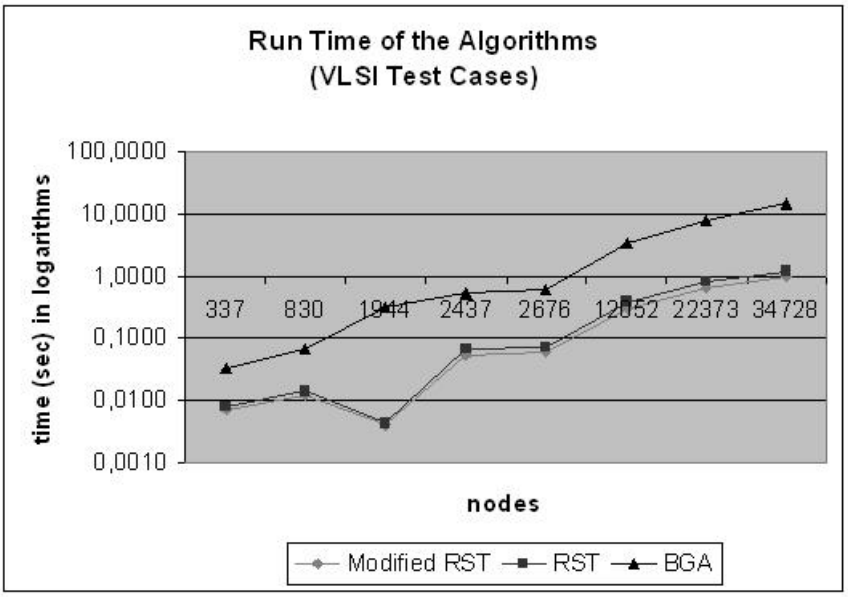


Figure 7-7 Run-time of Algorithms for VLSI Test Cases

The above tables show that the improvement of BGA is better than both RST and Modified RST. The improvement of the RST algorithm implemented in this thesis work is nearly the same as the RST algorithm given in the literature. It can be noted again that the improvement of both algorithms is shown to be at most 1% worse than the optimal solution. The observed slight difference is expected because there are unclear points in [2] which are implemented according to our interpretation in this thesis work. RST and Modified RST have almost the same improvements. The difference comes from the fact that the LCA queries are arranged in different order. Although the point edge pairs formed from the LCA queries are sorted afterwards, the point edge pairs providing the same gain can be sequenced in different order. This different ordering cause different Steiner points to be added to the tree, which may result in different improvements achieved in RST and Modified RST algorithms. It can be concluded that whose improvement is better changes randomly, but it can be seen that the differences between the improvements of RST and Modified RST is negligible and also no one can be said to be better than the other.

When it comes to execution time point of view, the story becomes different. Although the BGA algorithm results in better improvement, the time consumed by it is very high. Also when the number of nodes increases, the rate of the increase in the execution time rises even more. For a node set size of 100,000, the execution time nearly reaches to two minutes. The RST and Modified RST algorithms run much faster than the BGA. This can be observed even in small instances. When the number of nodes increases the difference becomes more noticeable. When building the Modified RST algorithm it was claimed that the Hierarchical Greedy Preprocessing algorithm that was defined in BGA will run faster when it is replaced with Merging Binary Tree and Tarjan's Offline Least Common Ancestor algorithm. This claim is proved by our implementations. It is worth noting that the Modified RST algorithm runs faster than the basic RST algorithm without degrading the performance. The modified RST algorithm have run 25% faster than the RST algorithm on average for 10,000 node case and run 10% faster than RST in 100,000 node case on average. Even 10% gain in execution time is important since this algorithm runs too many times in a design cycle.

When it comes to the problem size that the algorithm solves the BGA and RST algorithm fails above the 100,000 node cases in our implementation. The reason of this is both algorithms have heavy recursive parts. The stack reserved for this recursion gives an overflow when the problem size reaches these values. When building the Modified RST algorithm, it was also claimed that replacing the recursive part of the RST algorithm will result in a better performance when the data size gets bigger. This claim is also confirmed with our tests. It can be observed that larger data sets can now be solved using the Modified RST algorithm with no extra cost. This is an important outcome of this thesis work because solving such big sets has been the aim of this study from the very beginning.

7.3. Implementation Results of Distributed Modified RST Algorithm

After testing the performance of the Modified RST algorithm and concluding that it is better than the RST algorithm, its distributed version is implemented. As was noted in the previous chapter, this algorithm is a partially distributed algorithm. The mechanisms for passing of messages between processors are provided by the MPI library. The test cases used in sequential version of the algorithm are used also for the distributed algorithm. Since the sequential algorithm runs sufficiently fast and the distributed algorithm is proposed aiming large data sets, the test set sizes for distributed algorithm will start from 100,000. In this part the distributed Modified RST algorithm will make one pass only unlike its sequential version, which was run 5 times if there are possible improvements. Of course one pass version of Modified RST algorithm will then be compared with the distributed version. The distributed code is tested for 2, 4, 6 and 8 processors. Tests have been performed again on ten test cases for each data size and the average results are given in Table 7-3. The details of all other tests are given in the appendix.

Table 7-3 Distributed RST Algorithm Results

	# of proc=1		# of proc=2		# of proc=4		# of proc=6		# of proc=8	
Input Size	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	Time (s)	Impr. (%)	time (s)	Impr. (%)	Time (s)
100000	9.392	2.749	9.394	3.593	9.393	2.801	9.394	2.563	9.394	2.487
500000	9.390	17.932	9.390	20.567	9.391	15.078	9.391	14.593	9.391	14.230
1000000	9.399	302.036	9.400	43.053	9.400	32.030	9.399	30.374	9.400	29.792

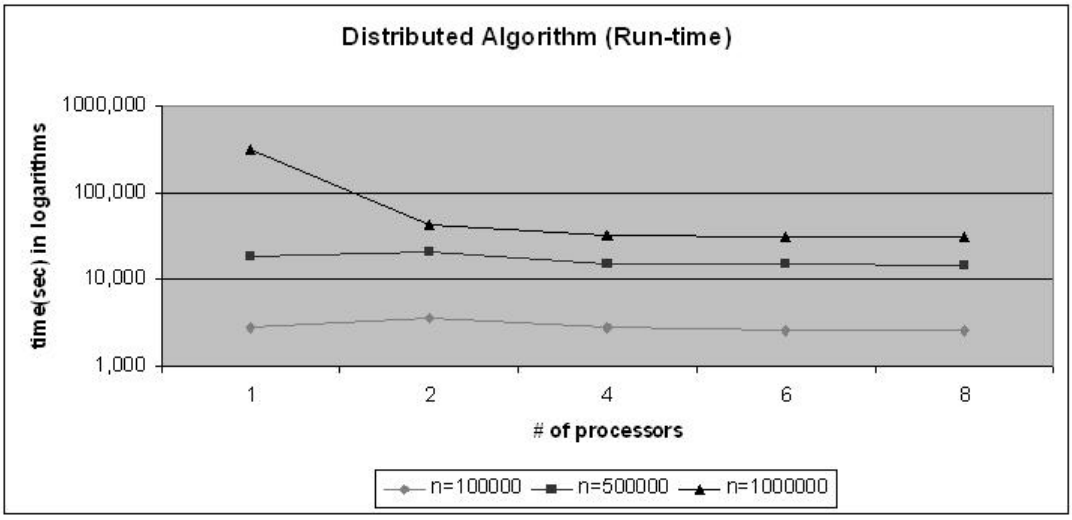


Figure 7-8 Run-time of the Distributed Algorithm

It can be observed from the above table that the improvement is pretty much the same for all tests. The difference again may come from the possible different orderings of the LCA queries and possibly from RSG part in the distributed version which can have some extra edges.

The execution time needs to be examined in detail. For relatively small instances, the algorithm for one processor runs faster than the two processor case. This is because of the fact that the communication time is higher than the computation time. As the number of processors increases the computation cost decreases but the communication cost does not. As a result the overall time cost of the algorithm decreases, but not in an efficient manner. In order to investigate this fact, a 1,000,000 terminal test case is taken and analyzed in detail. RSG is a backbone of this algorithm, so its performance is examined and given in Table 7-4. First, the result of RSG for the sequential algorithm and then the result of its distributed version are given. The first recorded time is the maximum of each processors completion time of RSG calculations in their partitions, and the second recorded time is the time when each processor gets the overall sparse graph.

Table 7-4 Performance of RSG algorithm in distributed algorithm

	# of proc=1	# of proc=2	# of proc=4	# of proc=6	# of proc=8
RSG Time (sec)	13.105	8.466	5.693	4.222	3.506
After Broadcast (sec)	-	22.220	13.556	12.671	12.471

As the number of processors increases, the RSG computation cost decreases almost linearly. But after the broadcast the overall cost is nearly the same. It can be concluded that as the number of processors increases the communication cost increases also. This is because such a broadcast is implemented as unicast operations in MPI. By using more appropriate methods or by using faster communication links, this performance can be improved. However since the RSG becomes faster it can still be concluded that the proposed algorithm is promising.

Another observation made from the above test results is that for the 1,000,000 terminal case the performance of one processor becomes bad. This is because for that size the 512 MB RAM becomes the bottleneck and swapping operations on memory to the hard disk start at this size. On the other hand for more than one processor this size of data does not cause any problem. This performance of clusters is the result of the increase in resources.

It can finally be concluded that although the distributed algorithm can not provide linear speedup, it is still useful when the data size is relatively big.

CHAPTER 8

CONCLUSION

The Steiner Tree Problem is one of the fundamental problems in “Graph Theory”. It is widely used in Physical Design phase of VLSI in which only horizontal and vertical lines can be used due to technological constraints. The Steiner Tree version used for VLSI Physical Design is the Rectilinear Steiner Tree which only consists of rectilinear lines.

The Rectilinear Steiner Minimum Tree (RSMT) is defined as a tree that interconnects a set of terminals by only using horizontal and vertical line segments aiming to achieve minimum total length. It is different than the famous spanning tree problem because extra nodes, called Steiner points, can be used in the solution. The problem is shown to be NP-complete which reduces the hope of finding efficient exact algorithms. In recent years successful exact algorithms have been designed for sizes up to a thousand nodes. However, in this thesis work the aim is solving the problem for sizes such as millions of nodes which eliminates the possibility of using exact algorithms. This is the reason of using heuristics in this study. The Rectilinear Steiner Minimum Tree Problem is solved many times throughout a design so such a scalable algorithm is needed also for big data sets.

In order to solve the problem an extensive literature survey has been made especially for approximation algorithms. Exact algorithms have also been studied to investigate the proposed methods. It has been observed that most of the heuristics start with minimum spanning tree and are updated with different algorithms. The algorithms have been compared in terms of their performance keeping the focus on

the solution of relatively big sized problems. The performance has been measured in terms of the length of the formed tree, the time that the algorithm takes to run and the size of the problem it can solve. Another issue that has been taken into account during the comparison between the algorithms is their adaptability for parallel implementations.

Two recently developed algorithms have been selected according to their performance namely Kahng's BGA algorithm and Zhou's RST algorithm. Both of which have been shown to be suitable for problem sizes such as 100,000 nodes. Both algorithms start with constructing a sparse graph on which initial minimum spanning tree will be calculated and reduce the size of the minimum spanning tree. To achieve this, the BGA algorithm uses Zelikovsky's GTCA algorithm whereas the RST algorithm uses Borah's algorithm. Similar blocks of the algorithms have been compared with each other first and their performances have been analyzed by using a profiler. In this way the faster parts of the two algorithms have been identified. The recursive parts have been removed because when a recursive routine is implemented on a computer the registers are stored in a stack which makes the algorithm slower. Besides as the input size becomes larger the stack can be overflowed. As a result a modified algorithm has been proposed and implemented in the thesis work.

In order to solve larger problems, a distributed version of the modified algorithm has been studied. Some parts of the algorithm appeared costly to be implemented on a distributed environment. So a partially distributed algorithm has been proposed and its performance has been investigated accordingly.

The BGA algorithm for Linux that has been obtained at the end of the literature survey has been modified to run on Windows. The RST algorithm has been implemented in C++ language. Using the profiler results it has been concluded that the modified algorithm can run faster than both of the algorithms. This fact has been verified by tests conducted using random input sets and also with VLSI test cases. The performance has been calculated in terms of time, improvement of MST and

the size of the node set that the algorithm solves. The BGA algorithm has been found to be the best performing algorithm in terms of improvement, where the improvements of RST and Modified RST can be rated as nearly equal. Their performances have been found 1% worse than the BGA algorithm on average. When the algorithms have been compared in terms of their timing performance, BGA algorithm becomes the worst. The difference between timing performance increases when the input set becomes larger. When the RST and Modified RST algorithms have been compared it is concluded that the Modified RST algorithm is faster in all data sizes. This speed up changes but it ranges from 10% to 25%. On the solvable data set size the BGA and RST seem to be worse than the Modified RST.

The first contribution of this work is proposing a modified algorithm that is formed by analyzing and profiling the BGA and RST algorithms and then determining faster blocks and removing recursive parts. Tests have been carried out on this algorithm and shown that it can solve bigger data sets while providing a speed up without degrading the performance.

The second contribution of this work is proposing a partially modified algorithm. The distributed algorithm has been implemented by using MPI. The algorithm is designed for large data sets so no tests have been made for small data sets. Tests have been conducted on an eight computer cluster with different number of processors. Results show that with increasing number of computers, the run time of the algorithm decreases. But the rate of the decrease has not been as expected. The main reason of it has been the communication overhead. It has been shown that the RSG can be computed efficiently on a distributed environment but when the edges have been gathered and broadcast through the communication medium the performance of RSG becomes nearly the same as sequential RSG algorithm. Since MPI converts broadcast operations to unicast operations, the communication increases compared with processor computation load. Thus by using a faster link and more importantly by using a more suitable message passing library than MPI the performance can be improved.

On the other hand as the input size becomes relatively large the memory overflow becomes a problem. At some point, memory gets full and swapping operations with the hard disk begins slowing down the computation. Here using a cluster have provided a gain in time by increasing the resources.

The partially modified algorithm can be improved as a future work. The idea proposed for the distributed algorithm is useful but some further modifications can be made on it. This partially parallelized algorithm can be modified to perform better by using a more suitable message passing library than MPI which handles collective communications better. Another improvement can be using a faster communication medium. With these two the communication overhead can be decreased. Also the sequential part of the algorithm that is not parallelized in this work can be made to run in parallel. This can be provided by implementing a successful Union-Find algorithm for the disjoint set data structure for distributed environment.

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APPENDIX

Table Appendix-I. BGA, RST and Modified RST Test Results

		BGA		RST		Modified RST	
Input Size	Input Name	Impr. (%)	Time (s)	Impr. (%)	time (s)	Impr. (%)	time (s)
10000	10000_1.xy	11.007	3.429	10.373	0.344	10.362	0.256
10000	10000_2.xy	11.008	2.908	10.353	0.362	10.375	0.274
10000	10000_3.xy	11.029	2.918	10.450	0.355	10.432	0.261
10000	10000_4.xy	11.040	3.400	10.408	0.360	10.408	0.272
10000	10000_5.xy	10.976	3.623	10.315	0.356	10.288	0.284
10000	10000_6.xy	11.124	4.011	10.502	0.360	10.500	0.261
10000	10000_7.xy	10.930	4.564	10.364	0.354	10.384	0.258
10000	10000_8.xy	11.121	3.047	10.431	0.359	10.430	0.266
10000	10000_9.xy	11.032	3.523	10.411	0.356	10.391	0.268
10000	10000_10.xy	11.253	2.815	10.666	0.376	10.654	0.269
50000	50000_1.xy	11.105	40.209	10.462	2.375	10.460	2.076
50000	50000_2.xy	11.120	33.994	10.472	2.398	10.467	1.970
50000	50000_3.xy	11.048	33.026	10.435	2.399	10.428	2.017
50000	50000_4.xy	11.133	45.858	10.456	2.382	10.459	2.054
50000	50000_5.xy	11.014	34.896	10.387	2.410	10.389	1.976
50000	50000_6.xy	11.096	34.946	10.497	2.422	10.495	2.013
50000	50000_7.xy	11.031	40.844	10.415	2.459	10.408	2.198
50000	50000_8.xy	10.977	35.868	10.326	2.593	10.324	2.173
50000	50000_9.xy	11.042	38.988	10.398	2.363	10.388	2.270
50000	50000_10.xy	11.117	35.106	10.478	2.459	10.481	2.086

Table Appendix-I cont.

		BGA		RST		Modified RST	
Input Size	Input Name	Impr. (%)	Time (s)	Impr. (%)	time (s)	Impr. (%)	time (s)
100000	100000_1.xy	11.099	108.243	10.457	5.369	10.454	5.223
100000	100000_2.xy	11.096	106.012	10.480	5.592	10.480	4.905
100000	100000_3.xy	11.081	93.349	10.427	5.393	10.435	5.044
100000	100000_4.xy	11.049	103.272	10.437	5.350	10.440	5.011
100000	100000_5.xy	11.098	96.114	10.444	6.336	10.438	5.742
100000	100000_6.xy	11.116	90.478	10.478	6.252	10.480	5.861
100000	100000_7.xy	11.046	120.761	10.407	5.380	10.413	4.940
100000	100000_8.xy	11.107	88.082	10.471	5.347	10.467	4.985
100000	100000_9.xy	11.101	93.505	10.470	5.470	10.470	4.924
100000	100000_10.xy	11.050	106.260	10.435	5.339	10.443	4.917
500000	500000_1.xy	-	-	-	-	10.437	32.150
500000	500000_2.xy	-	-	-	-	10.447	32.151
500000	500000_3.xy	-	-	-	-	10.453	32.139
500000	500000_4.xy	-	-	-	-	10.434	32.206
500000	500000_5.xy	-	-	-	-	10.455	32.306
500000	500000_6.xy	-	-	-	-	10.427	32.779
500000	500000_7.xy	-	-	-	-	10.434	32.187
500000	500000_8.xy	-	-	-	-	10.432	33.310
500000	500000_9.xy	-	-	-	-	10.449	32.188
500000	500000_10.xy	-	-	-	-	10.434	32.203
1000000	1000000_1.xy	-	-	-	-	10.442	72.005
1000000	1000000_2.xy	-	-	-	-	10.456	78.704
1000000	1000000_3.xy	-	-	-	-	10.457	72.478
1000000	1000000_4.xy	-	-	-	-	10.446	72.582
1000000	1000000_5.xy	-	-	-	-	10.446	72.712

Table Appendix-I cont.

		BGA		RST		Modified RST	
Input Size	Input Name	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	Time (s)
1000000	1000000_6.xy	-	-	-	-	10.470	72.492
1000000	1000000_7.xy	-	-	-	-	10.459	72.465
1000000	1000000_8.xy	-	-	-	-	10.442	72.485
1000000	1000000_9.xy	-	-	-	-	10.439	72.530
1000000	1000000_10.xy	-	-	-	-	10.442	72.378

Table Appendix-II. Distributed Algorithm Test Results

	# of proc=1		# of proc=2		# of proc=4		# of proc=6		# of proc=8	
Input Name	Impr. (%)	Time (s)	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	Time (s)
100000_1.xy	9.374	2.762	9.379	3.562	9.376	2.716	9.384	2.559	9.377	2.453
100000_2.xy	9.401	2.748	9.397	3.645	9.399	2.730	9.395	2.613	9.405	2.463
100000_3.xy	9.380	2.747	9.381	3.621	9.380	2.739	9.391	2.544	9.379	2.508
100000_4.xy	9.397	2.756	9.392	3.566	9.398	2.772	9.398	2.610	9.398	2.484
100000_5.xy	9.375	2.753	9.381	3.655	9.379	2.792	9.377	2.510	9.383	2.497
100000_6.xy	9.429	2.748	9.427	3.570	9.428	2.825	9.428	2.510	9.428	2.459
100000_7.xy	9.384	2.754	9.392	3.614	9.389	2.738	9.389	2.571	9.385	2.509
100000_8.xy	9.406	2.745	9.416	3.588	9.413	2.919	9.411	2.555	9.409	2.475
100000_9.xy	9.398	2.737	9.398	3.565	9.402	2.928	9.394	2.551	9.398	2.463
100000_10.xy	9.373	2.739	9.374	3.546	9.372	2.851	9.378	2.604	9.376	2.559
500000_1.xy	9.397	17.985	9.396	20.388	9.398	15.170	9.396	14.520	9.397	14.273
500000_2.xy	9.399	17.750	9.398	20.504	9.398	15.288	9.400	14.574	9.399	14.185
500000_3.xy	9.397	17.706	9.397	20.388	9.397	15.024	9.395	14.556	9.397	14.274
500000_4.xy	9.384	18.099	9.383	20.419	9.386	15.046	9.384	14.527	9.385	14.166
500000_5.xy	9.402	18.019	9.406	21.601	9.403	15.056	9.405	14.612	9.406	14.379
500000_6.xy	9.369	17.954	9.368	20.432	9.369	15.019	9.372	14.532	9.371	14.211
500000_7.xy	9.384	17.961	9.383	20.409	9.386	15.022	9.384	14.709	9.385	14.154

Table Appendix-II cont.

	# of proc=1		# of proc=2		# of proc=4		# of proc=6		# of proc=8	
Input Name	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	time (s)	Impr. (%)	Time (s)
500000_8.xy	9.386	17.999	9.382	20.584	9.384	15.218	9.388	14.450	9.385	14.236
500000_9.xy	9.402	17.928	9.405	20.484	9.404	14.965	9.404	14.927	9.405	14.276
500000_10.xy	9.384	17.916	9.383	20.456	9.386	14.977	9.384	14.521	9.385	14.145
1000000_1.xy	9.394	447.358	9.395	43.121	9.395	31.734	9.393	30.365	9.394	29.940
1000000_2.xy	9.400	434.304	9.400	42.931	9.398	32.087	9.401	30.456	9.398	29.676
1000000_3.xy	9.409	261.326	9.409	43.042	9.409	31.891	9.410	30.396	9.410	29.662
1000000_4.xy	9.395	275.076	9.395	42.883	9.394	32.096	9.391	30.207	9.394	29.709
1000000_5.xy	9.392	362.585	9.393	43.281	9.394	32.101	9.394	30.349	9.394	29.817
1000000_6.xy	9.416	227.625	9.416	42.992	9.415	32.097	9.416	30.227	9.417	29.913
1000000_7.xy	9.409	218.945	9.411	43.277	9.411	32.174	9.409	30.432	9.410	29.698
1000000_8.xy	9.394	211.065	9.395	43.042	9.395	32.130	9.393	30.367	9.394	29.870
1000000_9.xy	9.392	227.082	9.396	43.018	9.395	31.899	9.394	30.477	9.394	29.952
1000000_10.xy	9.394	354.998	9.393	42.944	9.395	32.090	9.393	30.463	9.394	29.681