EVOLUTIONARY ALGORITHMS IN DESIGN

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ABSTRACT

EVOLUTIONARY ALGORITHMS IN DESIGN

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Evolutionary Structural Optimization (ESO) is a relatively new design tool used to improve and optimise the design of structures. In this method, a few elements of an initial design domain of finite elements are iteratively removed. Such a process is carried out repeatedly until an optimum design is achieved, or until a desired given area or volume is reached.

In structural design, there is the demand for the development of design tools and methods that includes optimization. This need is the reason behind the development of methods like Evolutionary Structural Optimization (ESO). It is also this demand that this thesis seeks to satisfy. This thesis develops and examines the program named EVO, with the concept of structural optimization in the ESO process. Taking into account the stiffness and stress constraints, EVO allows a realistic and accurate approach to optimising a model in any given environment.

Finally, in verifying the ESO algorithm's and EVO program's usefulness to the practical aspect of design, the work presented herein applies the ESO method to case studies. They concern the optimization of 2-D frames, and the optimization of 3-D spatial frames and beams with the prepared program EVO. Comparisons of these optimised models are then made to those that exist in literature.

Keywords: ESO, evolutionary structural optimization, topology

ÖZ

TASARIMDA EVRİMSEL ALGORİTMALAR

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Evrimsel Yapısal Optimizasyon (EYO) yapı tasarımlarını iyileştirmede ve en uygun şekle getirmede kullanılan yeni sayılabilecek bir yöntemdir. Bu metodla, her tekrarlamada birkaç eleman tasarım evreninden çıkarılır. Bu iterasyonlar, sonunda optimum bir tasarım kalana dek ya da önceden belirlenmiş alan veya hacim elde edilene dek tekrar edilir.

Yapısal tasarımda, tasarım iyileştirmesi için kullanılacak metod ve programlara olan ihtiyaç, EYO yönteminin doğmasındaki sebeptir. Bu tezde yapılan çalışma da bu ihtiyacı karşılayabilme amaçlıdır. Bu çalışma, 3 boyutlu yapıların tasarımında evrimsel algoritmalar kullanarak yapısal optimizasyonu gerçekleştirecek EVO isimli bir program geliştirilmesini kapsar. EVO, verilen tasarım evreninde, gerilim veya rijitlik kısıtlamalarını kullanarak yapıyı gerçekçi ve doğru olarak en uygun tasarıma getiren bir yaklaşım sağlar.

Bu yöntemin ve geliştirilen programın yararlılığını ve pratik hayatta kullanılabilirliğini gösterebilmek için örnekler çözülmüştür. 2 boyutlu çerçeve optimizasyonları, ve 3 boyutlu çerçeve ve kiriş optimizasyonları çözülen örneklerdir. Elde edilen sonuçların diğer yöntemlerle olan karşılaştırmaları da literatürde yer alanlarla yapılmıştır.

Anahtar Kelimeler: Evrimsel yapısal optimizasyon, topoloji

To my family and my love Ayça …

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LIST OF SYMBOLS

C	mean compliance
C_{all}	mean compliance prescribed limit
crit	criterion
d	distance
u	global nodal displacement
e	subscript of element
E	Young's modulus
$f(x)$	vector of objective functions
i	element number
j	criterion number
K	stiffness
K	global stiffness matrix
F	nodal load vector
t	thickness
u_i	element displacement vector
V	volume
X_c, Y_c, Z_c,	centroid coordinates

Greek Symbols

α_i	strain energy sensitivity number
α_e^{crit}	element sensitivity number of *crit* criterion
$\alpha_{\max}^{crit}$	maximum value of element sensitivity number of *crit* criterion
$\alpha_{average}^{crit}$	average value of element sensitivity number of *crit* criterion
ε_j	criterion constraint
υ	poisson's ratio
σ_e^{vm}	von Mises stress of element
$\sigma_{\max}^{vm}$	maximum value of von Mises stress
ρ	density

Abbreviations

2-D	two dimensional
3-D	three dimensional
ESO	evolutionary structural optimization
FE	finite element
FEA	finite element analysis
GA	genetic algorithm
RR	ESO rejection ratio
SS	ESO steady state

CHAPTER 1

INTRODUCTION

1.1 Objective

The objective of this thesis is to use the Evolutionary Structural Optimization algorithm for the topology optimization of elastic solid bodies. Topology optimization is used to find the optimal mass distribution such that an objective takes a minimum or maximum while satisfying constraints for the given predefined domain with boundary conditions and load. Evolutionary Structural optimization is a relatively new technique used for the topology optimization. In this method the body is discretized by finite element modeling. Therefore, in this study two important topics in engineering, namely finite element method and evolutionary structural optimization have been studied.

Development of computers made it possible for engineers to analyze a structure in much more detail. FEM is a method where the structure is analysed by dividing it into very small elements and solving the equations of mechanics for each of those elements separately. This would be of course impossible to do by hand, so that is the reason why this method had to wait to be known until the development of computers. FEM is widely used especially in aerospace and automobile industries to analyse complex structures.

This thesis develops and examines the program named EVO, with the concept of structural optimization in the ESO process. Taking into account the stiffness and stress constraints, EVO allows a realistic and accurate approach to optimising a model in any given environment.

1.2 Methodology

The study includes a finite element analysis code and an optimization algorithm which is written in Fortran. Fortran is an easy to use robust and reliable programming language. The study also includes a preliminary user interface prepared in Matlab. The aim is to prepare the data file to be run by Fortran. The data file includes the node coordinates and element connectivities. The visualization is again done by a preliminary program written in Matlab making use of the program's already defined visualization subroutines.

1.3 Outline

The finite element formulation of the element used in this study is explained in chapter 2. The element stiffness matrix formulation is described in this chapter.

Chapter 3 identifies the evolutionary structural optimization algorithms used in this study. The algorithms used in reaching the fully stressed state and the algorithm to minimize compliance are given. To generalize the evolutionary optimization, the sensitivity based algorithm is also described

Chapter 4 contains various case studies in which the results obtained within this study are compared to those in literature. To show the programs efficiency, some 2D solid, 3D solid and beam examples are solved.

Chapter 5 reviews and concludes the results of the chapters and presents recommendations for future work.

In the Appendix , a typical data file, and a typical output file of the program is presented. The element stiffness matrix calculated in this study is also given.

1.4 Literature Review

The desire and drive towards the improvement in the performance of any given object, set in a structural environment of loads and constraints is known as structural optimization. This improvement may be in anything related to the structure ranging from the need to reduce the structural weight without compromising structural integrity (Schmidt, 1981) to the need to reduce the combined manufacturing cost of the structure and its operational cost throughout its expected lifetime (Sheu and Prager, 1968). Over the past four centuries, as the areas of engineering, mathematics, science and technology have become better established, the implementation of structural optimization has become more profound. Much of the structural optimization work carried out at the beginning of this period largely consisted of trials and unintentional experiments. This included the works of Leonardo da Vinci, Galileo and Euler (Wasiutynski and Brandt, 1963). At the end of the 19^{th} century and the turn of the 20^{th} century, came the capability of engineers to combine optimization principles and analytical prowess. This saw the likes of Maxwell's proved theorems (1872) leading the way for Michell (1904) to determine theories about the form of frames of minimum weight, known as Optimal Layout Theory. The next sixty years continued to contribute to the ever-growing database of structural optimization knowledge – specifically in the area of truss structures. It seemed to take on three directions: the minimization of the truss's weight, the minimization of the strain energy design for a given material volume and the optimization of statically indeterminate structures of uniform strength. Significant contributors to these ideas were: Rabinovich (1933), Wasiutynski (1939) and Prager (1956). Many of these techniques were addressed by classical optimization i.e. calculus based optimization (Haftka *et al.*, 1985). This work mainly concerned simple discrete or continuous structures that were optimized using classical techniques of ordinary differential calculus. Such work laid the foundation for certifying the validity of other more recent optimization methods. In the past fifty years, progress has seen the transition from this initial method to include a class of optimization where variables in the optimization equation are of a discrete nature. Mathematical programming has played a key role in this as seen by the contribution of methods such as linear and non-linear mathematical programming (Haftka *et al.*, 1985). Common to linear programming is the Simplex method (Van Der Veen, 1967). Constrained and non-constrained techniques have also been utilized in conjunction with mathematical programming. Such techniques have been presented in the form of the Lagrange Multiplier method and the Penalty Function method (Vanderplaats, 1984; Haftka *et al.*, 1985). Many structural optimization methods have emerged in recent decades with the development of computer technology. A large proportion of these methods uses discrete finite elements. They can be broadly arranged into three main areas of optimization: topology optimization, shape optimization and size optimization. A description of these areas and a sample of some of the methods that encapsulate these areas are as follows.

1.4.1 Topology Optimization

Topology optimization describes the process that defines the topology relationship in a structure. The resulting optimized structure can be vastly different from the initial starting design and so is independent of it. The implication of this is that there is no restriction on the final form of the structure relative to the initial form.

1.4.1.1 Optimality Criteria

The Optimality Criteria method is one example implementing topology optimization (Prager and Rozvany, 1977; Rozvany *et al.*, 1995). It is an alternative method to mathematical programming whereby it attempts to satisfy a set of criteria such as a fully stressed design or a set of Kuhn-Tucker conditions. The alteration or removal of elements in a finite element mesh achieves this. Such a method is capable of treating a large number of design variables with ease, but requires significant intuitive input from the user (Rozvany *et al.*, 1995).

1.4.1.2 Homogenisation Method

Topology optimization has greatly been impacted by the Homogenisation method (Bendsøe, 1995) in the past decade. This method simultaneously envelops the optimization of a structure's topology, shape and size. It does so by assigning finite elements (with numerous local variables) to the whole domain of the structure. For each element, parameters of size and orientation of internal rectangular holes are varied, having the effect of a varying porous material over the whole structure. The basis with which this optimal material distribution is found, is by the use of mathematical programming techniques with sequential quadratic programming. Many publications and contributions have been made to progress this method – see Bendsøe and Kikuchi (1988), Allaire and Kohn (1993) and Maute and Ramm (1995).

1.4.1.3 Evolutionary Structural Optimization

The Evolutionary Structural Optimization method (ESO) (Xie and Steven, 1997) is also an effective tool that is capable of handling topology optimization. It is a heuristic process that uses discrete finite elements as its foundation. It uses the Finite Element method as its analysis engine. Its approach to optimising a structure is to remove elements iteratively, which has been set up in a particular environment of loads and constraints. It is based on the simple concept that by slowly removing inefficient material from a structure, the topology of the structure evolves towards an optimum. Here "inefficiency" is a very general term, meaning the sensitivity of the alteration of an element in a FEA mesh to the optimality criterion. This sensitivity can be a composite of several performance measures and the optimality criterion can be a composite of several individual physical criteria. Much work has been done on ESO where many detailed studies have established systematic rules that make the method work for a full range of structural situations (Xie and Steven, 1997).

1.4.1.4 Genetic Algorithms

Topology optimization of structures can also be achieved using Genetic Algorithms (Goldberg, 1989). This involves the optimization of a population of chromosomes, where each chromosome represents a possible optimal solution. This is done by defining each chromosome with a character string of binary digits i.e. 0's and 1's. An artificial genetransformation mechanism is applied where these chromosomes are ranked, with the more favourable ones being selected and reproduced. Some of the poorly ranked members are selected and mutated with the more favourable ones. This occurs until the GA principle, over its successive generations, produces an optimum topology (Woon *et al.*, 2002).

1.4.2 Shape Optimization

Shape Optimization is a restricted form of topology optimization. It determines the optimal boundaries of a structure for the given fixed topology. In this form of optimization, the object is to find the best shape that will have the best objective outcome as defined by designer.

1.4.2.1 Evolutionary Structural Optimization

Shape optimization may be added to the ESO algorithm by adding a constraint to the method, which allows elements that exist only at the surface to be removed. This is known as the Nibbling constraint. In many design assignments, internal cavities are not allowed to be created, as only material is allowed to be nibbled away from the boundaries. Querin (1997) gives such an example, where shape optimization is applied to an object hanging under its own weight. There are also several benchmark types and illustrative examples in Xie and Steven (1997).

1.4.2.2 Mathematical Programming Approach

Mathematical programming is the most typical approach to shape optimization used in the 1970s and 1980s. In mathematical programming, the problem is defined mathematically by an objective function that is described in terms of a series of design variables. Differentials of the objective function are obtained directly or by computation with a finite difference form of the differential. Second differentials are obtained for the Hessian matrix. The design variables that fit the design criteria are then found using the Conjugate gradient, steepest decent or quadratic programming search engines (Kristensen *et al.*, 1976). Several categories of mathematical programming exist such as linear and non-linear, integer linear, sequential and stochastic programming (Haftka *et al.*, 1985).

1.4.2.3 Computer Aided Optimization

The Computer Aided Optimization method or alternatively, Simulated Biological Growth deals with the optimization of structures specifically in the context of shape (Mattheck and Moldenhauer, 1990). It uses a discretised model of finite elements, and volumetrically 'swells' these structural elements using a swelling operation. This is done iteratively by thermally loading the structure proportional to the stresses created in the domain by normal loading. In addition to shape optimising a structure, it removes notch stresses and promotes a stress-state at the surface of the structure.

1.4.3 Size Optimization

Size optimization defines the approach to change the sizes and dimensions of a structure to achieve the optimum design. This is obtained by finding the best possible combination of these sizes and dimensions.

As has been briefly reviewed, there are many structural optimization methods available. Each has their own advantages and disadvantages, and each is appropriate for specific optimization problems. In spite of this extensive array of methods in use, the studies conducted in this thesis shall all be based on Evolutionary Structural Optimization. This focus does not intend to undermine any of the other methods. Rather, it seeks to promote the capability and robustness of ESO amongst these other methods.

1.4.4 Evolutionary Structural Optimization

Amongst the many structural optimization methods that have been developed, one remains continually attractive due to its simplicity and continues to grow in its development in recent years. It is the Evolutionary Structural Optimization (ESO) method. Since its inception back in 1992 by Xie and Steven (1997), ESO has grown to be a robust, yet simple design tool growing in its capabilities to serve the designer in a complex range of environments and objectives. In its original form, it had as its removal technique a condition to remove elements based on the von Mises stress level of each element. A general description of the original stress based ESO process is briefly outlined in chapter three.

The optimality criterion used by ESO has included stiffness (Chu et al., 1996), stress minimization (Li et al., 1999a), strain (Xie and Steven, 1997), buckling (Manickarajah et al., 1998), torsional stiffness (Li et al., 1999b), heat transfer and conduction (Li et al., 1997), incompressible fluid flow problems (Li, 2000), electrostatic (Li, 2000) and magnetostatic (Li, 2000) problems.

Many other features have been integrated into the ESO process. Multiple load cases and multiple support environments were first reported by Xie and Steven (1994) and Steven *et al*. (1995). This allowed for the optimization of structures that were subject to different load cases at different times, and structures that were held or supported in different ways and at different times.

Similar to shape optimization, another innovation has been ESO Morphing (Querin, 1997). This is where, rather than completely remove elements as in classical ESO, the elements are removed gradually. For the case of beams this graduation could be applied to a variation in cross-sectional area: for plates – to a set of varying thickness', modulus of elasticity or density; and for bricks – to a range of modulus of elasticity or density.

To overcome any doubts about the question of material being inappropriately removed in ESO, a Bi-directional Evolutionary Structural Optimization (BESO) has been formulated (Querin, 1997; Young *et al.*, 1999). This method allows the addition of material as well as the removal of material to take place simultaneously. Those regions of high stress for example, are attended to by the addition of material to those areas in need. Thus the evolutionary process can start from the smallest possible structural kernel and grow towards an optimum. Such a final optimum design is the same as that obtained by removal evolution.

One of the latest innovations to ESO has been the introduction of Configurational Optimization - alternatively known as Group ESO (Lencus *et al.*, 1999a). This is where rather than considering each element as a design variable, groups or configurations of elements are put under scrutiny for removal, Morphing (Lencus *et al.*, 1999b) or Nibbling. This allows for layout optimization and can be used at an early stage of the design process where the configuration of the structural entities, holes, stiffeners and skin thickness values are not fixed.

Many other optimization characteristics have been created to exist inside the ESO regime to extend its capabilities. Some of these are ESO applied to composite panels (Falzon *et al.*, 1996), topology optimization with material and geometric non-linearities (Querin *et al.*, 1996), Intelligent Cavity Creation (ICC) (Kim *et al.*, 1998), Post processing of 2-D topologies (Kim *et al.*, 2000) and shape design for elastic contact problems (Li *et al.*, 1998a). The repertoire of ESO has been extensive in its practical applications as well. A sample of these applications include the optimization of wheels (Guan *et al.*, 1997), spanners (Steven *et al.*, 1997), bikes (Steven *et al.*, 1997), milk crates (Barton *et al.*, 1998), generic aircraft spoilers (Lencus *et al*, 1999b) and aircraft airframes (Lencus *et al.*, 2000).

As can be seen, ESO has been developed to be used in many different contexts and for many purposes. This section has sought to identify some of these developments.

CHAPTER 2

FINITE ELEMENT FORMULATION

Finite Element Analysis is a computer-based numerical technique for calculating the strength and behavior of structures. It can be used to calculate deflection, stress, vibration, buckling behavior and many other phenomena. Computers are required for solving these large numbers of calculations, needed to analyse a large structure. The power and low cost of modern computers has made finite element analysis available to many disciplines and companies.

In finite element method, a structure is divided into small and simple elements. The behavior of each element can be described with a simple set of equations. Just as the set of elements would be joined together to build the whole structure, the equations describing the behaviors of the individual elements are joined into an extremely large set of equations that describe the behavior of the whole structure. The computer can solve this large set of simultaneous equations. From the solution, the computer extracts the behavior of the individual elements. From this, it can get the stress and deflection of all the parts of the structure. The stresses will be compared to allowed values of stress for the materials to be used, to see if the structure is strong enough.

Finite Element Analysis made it possible to analyze complex structures in a computer before actually building the structure. Before the development of this method, analyzing the structures needed to be based on hand calculations only. For complex structures, the simplifying assumptions were required to make any calculation possible and this lead to conservative and heavy designs. Also a lot of expensive tests had to be performed.

With Finite Element Analysis, more efficient designs are possible with fewer experiments.

In FEA, the static behaviour of a structure is represented by the following equilibrium equation;

$$[K].\{u\} = \{F\}$$

where $[K]$ is the global stiffness matrix, $\{u\}$ is the nodal displacement vector and $\{F\}$ is the nodal load vector.

Once the global stiffness matrix is formed, this equilibrium equation is solved to get the displacements. Then the stresses and the strain energy are easily formed. This chapter will describe the element stiffness matrix calculation of the hexahedron element used in this study and stress and strain energy calculations will be explained in the next chapter.

The element used in this study is an eight noded hexahedron element. A three-dimensional (3D) solid element can be considered to be the most general of all solid finite elements because all the field variables are dependent of *x*, *y* and *z*. The force vectors can be in any arbitrary direction in space. A 3D solid can also have any arbitrary shape, material properties and boundary conditions in space. As such, there are altogether six possible stress components, three normal and three shear, that need to be taken into consideration. Typically, a 3D solid element can be a tetrahedron or hexahedron in shape with either flat or curved surfaces. Each node of the element will have three translational degrees of freedom. The element can thus deform in all three directions in space.

Since the 3D element is said to be the most general solid element, the truss, beam, plate, 2D solid and shell elements can all be considered to be special cases of the 3D element. Theoretically, the 3D element can actually be used to model all kinds of structural components, including trusses, beams, plates, shells and so on.

However, it is very tedious in geometry creation and meshing. Furthermore, it is also most demanding on computer resources. But for the sake of showing the programs applicability for solids, the hexahedron element is used in meshing the structures.

Consider a 3D domain, which is divided in a proper manner into a number of *hexahedron elements* with eight nodes and six surfaces, as shown in Figure 2.1. Each hexahedron element has nodes numbered 1, 2, 3, 4 and 5, 6, 7, 8 in a counter-clockwise manner, as shown in Figure 2.2.

As there are three DOF's at one node, there is a total of 24 DOF's in a hexahedron element. It is useful to define a *natural coordinate system* (ξ, η, ζ) with its origin at the centre of the transformed cube, as this makes it easier to construct the shape functions and to evaluate the matrix integration.

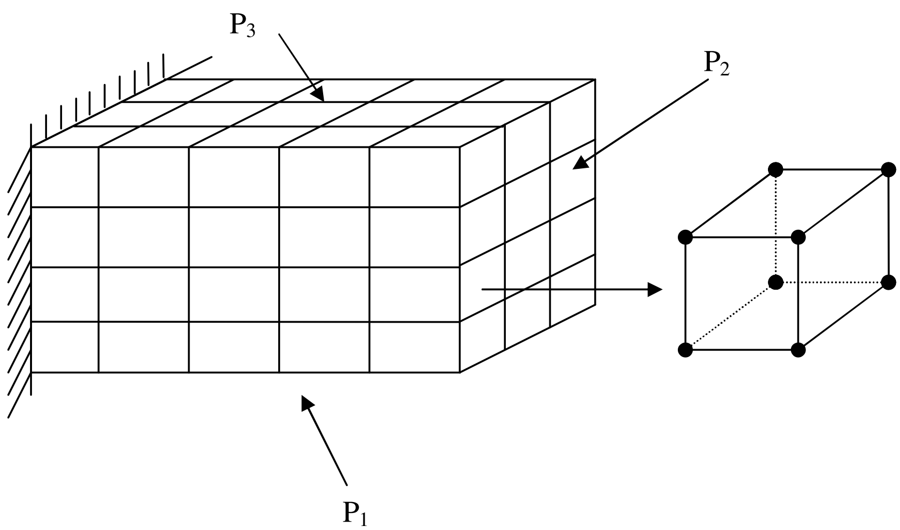

Figure 2.1. Solid block divided into eight nodal hexahedron elements

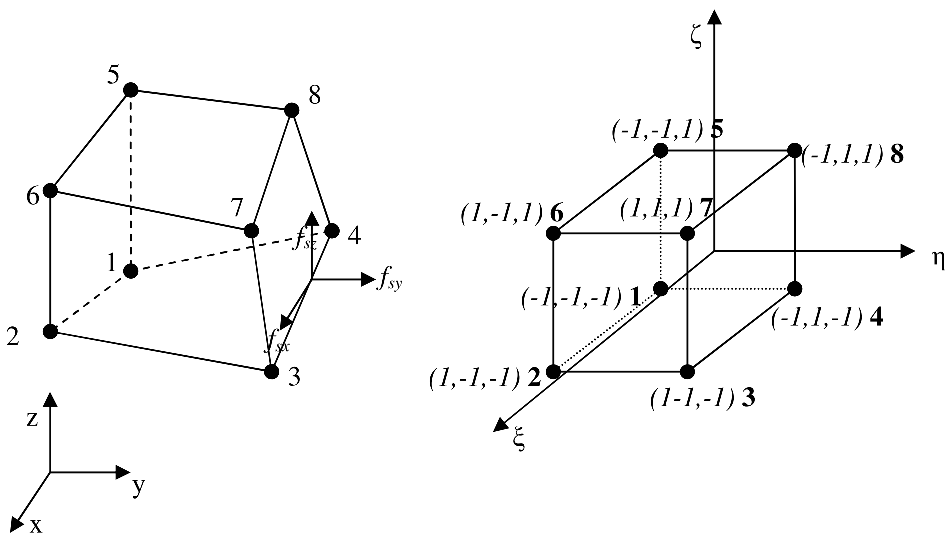

Figure 2.2. An eight nodal hexahedron element and its coordinate system

$$x = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) x_i$$

$$y = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) y_i$$

$$z = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) z_i \qquad (2.1)$$

$$N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$$

$$N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$$

$$N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$

$$N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$$

$$N_5 = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$$

$$N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \tag{2.2}$$

$$N_7 = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$$

$$N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$$

$$N_i = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

where (ξ_i, η_i, ζ_i) denotes the natural coordinates of node i . From Eq. (2.2), it can be seen that the shape functions vary linearly in the ξ, η and ζ directions. Therefore, these shape functions are sometimes called tri-linear functions. The tri-linear elements possess the delta function property. In addition, since all these shape functions can be formed using the common set of eight basis functions of

$$1, \xi, \eta, \zeta, \xi\eta, \xi\zeta, \eta\zeta, \xi\eta\zeta \tag{2.3}$$

which contain both constant and linear basis functions. Therefore, these shape functions can expect to possess both partitions of the unity property as well as the linear reproduction property.

In a hexahedron element, the displacement vector U is a function of the coordinates *x*, *y* and *z*, and as before, it is interpolated using the shape functions

$$U = Nd_e \tag{2.4}$$

where the nodal displacement vector, d_e is given by

$$d_e = \begin{pmatrix} d_{e1} \\ d_{e2} \\ d_{e3} \\ d_{e4} \\ d_{e5} \\ d_{e6} \\ d_{e7} \\ d_{e8} \end{pmatrix} \text{Displacement components at nodes 1 to 8} \tag{2.5}$$

in which

$$d_{ei} = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} (i = 1, 2, .., 8) \tag{2.6}$$

is the displacement at node i. The matrix of shape functions is given by

$$N_i = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 \end{bmatrix} \tag{2.7}$$

in which each sub-matrix, N_i, is given as

$$N_i = \begin{pmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{pmatrix} (i = 1, 2, ..., 8) \tag{2.8}$$

In this case, the strain matrix can be expressed as

$$B_i = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{bmatrix} \tag{2.9}$$

whereby

$$B_i = LN_i = \begin{bmatrix} \frac{\delta N_i}{\delta x} & 0 & 0 \\ 0 & \frac{\delta N_i}{\delta y} & 0 \\ 0 & 0 & \frac{\delta N_i}{\delta z} \\ 0 & \frac{\delta N_i}{\delta z} & \frac{\delta N_i}{\delta y} \\ \frac{\delta N_i}{\delta z} & 0 & \frac{\delta N_i}{\delta x} \\ \frac{\delta N_i}{\delta y} & \frac{\delta N_i}{\delta y} & 0 \end{bmatrix} \tag{2.10}$$

As the shape functions are defined in terms of the natural coordinates, ξ, η and ζ to obtain the derivatives with respect to x, y and z in the strain matrix, the chain rule of partial differentiation needs to be used:

$$\begin{aligned} \frac{\delta N_i}{\delta \xi} &= \frac{\delta N_i}{\delta x}\frac{\delta x}{\delta \xi} + \frac{\delta N_i}{\delta y}\frac{\delta y}{\delta \xi} + \frac{\delta N_i}{\delta z}\frac{\delta z}{\delta \xi} \\ \frac{\delta N_i}{\delta \eta} &= \frac{\delta N_i}{\delta x}\frac{\delta x}{\delta \eta} + \frac{\delta N_i}{\delta y}\frac{\delta y}{\delta \eta} + \frac{\delta N_i}{\delta z}\frac{\delta z}{\delta \eta} \\ \frac{\delta N_i}{\delta \zeta} &= \frac{\delta N_i}{\delta x}\frac{\delta x}{\delta \zeta} + \frac{\delta N_i}{\delta y}\frac{\delta y}{\delta \zeta} + \frac{\delta N_i}{\delta z}\frac{\delta z}{\delta \zeta} \end{aligned} \tag{2.11}$$

which can be expressed in the matrix form

$$\begin{Bmatrix} \dfrac{\delta N_i}{\delta \xi} \\ \dfrac{\delta N_i}{\delta \eta} \\ \dfrac{\delta N_i}{\delta \zeta} \end{Bmatrix} = J \begin{Bmatrix} \dfrac{\delta N_i}{\delta x} \\ \dfrac{\delta N_i}{\delta y} \\ \dfrac{\delta N_i}{\delta z} \end{Bmatrix} \tag{2.12}$$

where J is the Jacobian matrix defined by

$$J = \begin{bmatrix} \dfrac{\delta x}{\delta \xi} & \dfrac{\delta y}{\delta \xi} & \dfrac{\delta z}{\delta \xi} \\ \dfrac{\delta x}{\delta \eta} & \dfrac{\delta y}{\delta \eta} & \dfrac{\delta z}{\delta \eta} \\ \dfrac{\delta x}{\delta \zeta} & \dfrac{\delta y}{\delta \zeta} & \dfrac{\delta z}{\delta \zeta} \end{bmatrix} \tag{2.13}$$

The coordinates, x, y and z are interpolated by the shape functions from the nodal coordinates. Substituting the interpolation of the coordinates, Eq. (2.1), into Eq. (2.13), gives

$$J = \begin{bmatrix} \dfrac{\delta N_1}{\delta \xi} & \dfrac{\delta N_2}{\delta \xi} & \dfrac{\delta N_3}{\delta \xi} & \dfrac{\delta N_4}{\delta \xi} & \dfrac{\delta N_5}{\delta \xi} & \dfrac{\delta N_6}{\delta \xi} & \dfrac{\delta N_7}{\delta \xi} & \dfrac{\delta N_8}{\delta \xi} \\ \dfrac{\delta N_1}{\delta \eta} & \dfrac{\delta N_2}{\delta \eta} & \dfrac{\delta N_3}{\delta \eta} & \dfrac{\delta N_4}{\delta \eta} & \dfrac{\delta N_5}{\delta \eta} & \dfrac{\delta N_6}{\delta \eta} & \dfrac{\delta N_7}{\delta \eta} & \dfrac{\delta N_8}{\delta \eta} \\ \dfrac{\delta N_1}{\delta \zeta} & \dfrac{\delta N_2}{\delta \zeta} & \dfrac{\delta N_3}{\delta \zeta} & \dfrac{\delta N_4}{\delta \zeta} & \dfrac{\delta N_5}{\delta \zeta} & \dfrac{\delta N_6}{\delta \zeta} & \dfrac{\delta N_7}{\delta \zeta} & \dfrac{\delta N_8}{\delta \zeta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \\ x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \end{bmatrix} \tag{2.14}$$

or

$$j = \begin{bmatrix} \sum_{i=1}^{8} x_i \frac{\delta N_i}{\delta \xi} & \sum_{i=1}^{8} y_i \frac{\delta N_i}{\delta \xi} & \sum_{i=1}^{8} z_i \frac{\delta N_i}{\delta \xi} \\ \sum_{i=1}^{8} x_i \frac{\delta N_i}{\delta \eta} & \sum_{i=1}^{8} y_i \frac{\delta N_i}{\delta \eta} & \sum_{i=1}^{8} z_i \frac{\delta N_i}{\delta \eta} \\ \sum_{i=1}^{8} x_i \frac{\delta N_i}{\delta \zeta} & \sum_{i=1}^{8} y_i \frac{\delta N_i}{\delta \zeta} & \sum_{i=1}^{8} z_i \frac{\delta N_i}{\delta \zeta} \end{bmatrix} \tag{2.15}$$

Equation (2.12) can be re-written as

$$\begin{Bmatrix} \frac{\delta N_i}{\delta x} \\ \frac{\delta N_i}{\delta y} \\ \frac{\delta N_i}{\delta z} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\delta N_i}{\delta \xi} \\ \frac{\delta N_i}{\delta \eta} \\ \frac{\delta N_i}{\delta \zeta} \end{Bmatrix} \tag{2.16}$$

which is then used to compute the strain matrix, B, in Eqs. (2.9) and (2.10), by replacing all the derivatives of the shape functions with respect to x, y and z to those with respect to ξ, η and ζ.

Once the strain matrix, B, has been computed, the stiffness matrix, k_e, for 3D solid elements can be obtained by substituting B into Eq. $k_e = \int_{V_e} B^T cBdV$

$$k_e = \int_{V_e} B^T cBdV = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} B^T cB \det[J] d\xi d\eta d\zeta \tag{2.17}$$

Where c is the matrix of material constants and normally obtained through experiments. For a fully anisotropic material c matrix is given as;

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & \text{sym.} & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix} \tag{2.18}$$

Noting that $c_{ij} = c_{ji}$, there are altogether 21 independent material constants. For an isotropic material, c can be reduced to

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{11} & c_{12} & 0 & 0 & 0 \\ & & c_{11} & 0 & 0 & 0 \\ & & & l & 0 & 0 \\ & \text{sym.} & & & l & 0 \\ & & & & & l \end{bmatrix} \quad \text{where } l = (c_{11} - c_{12})/2 \tag{2.19}$$

$$c_{11} = \frac{E(1-v)}{(1-2v)(1+v)}; \quad c_{12} = \frac{Ev}{(1-2v)(1+v)}; \quad \frac{c_{11} - c_{12}}{2} = G \tag{2.20}$$

There are only two independent constants among these three constants. The relationship between them is

$$G = \frac{E}{2(1+v)} \tag{2.21}$$

The element stiffness matrix of the 1x1x1 element used in this study is obtained by following these steps and given in the appendix.

CHAPTER 3

DESIGN OF STRUCTURES WITH ESO

3.1. Overall Stress Constraint

The stress based version of ESO method uses the von Mises stress to guide removal. The aim is not only to remove low stressed material but to reach a fully stressed state.

It is interesting to notice that structural components of the skeleton of living beings in nature have optimum shapes. These shapes are such that they neither have stress concentrations nor weak places. The load is fairly distributed and there is a uniform stress distribution under the external loading. The bones of living beings have such a shape that the material is broken down in order not to carry excess load around. The idea of obtaining a structural form that has a uniform stress distribution is simulated into a numerical technique by making use of a finite element method. This technique is called evolutionary structural optimization due to that fact that optimum form under the external loading gradually evolves and takes its final shape during the design cycles. The design domain in this method is first discretized using finite element meshes and stresses are computed in each element. Those elements with lower stress density are then removed from the domain. The design domain with a new shape is once more analyzed and stresses are calculated in the finite elements and those elements with lower stresses are also removed from the design domain. This process is continued until all the remaining elements in the design domain have almost uniform stress distribution. The structural form obtained at this stage is accepted as the optimum shape.

This chapter describes the ESO procedures for the simultaneous shape and layout optimization of structures with stress constraints.

3.1.1. Determination of Elements to be Removed

The determination of elements to be removed was originally made by comparing the von Mises stress of each element σ_e^{vm} to the maximum von Mises stress that exists in the whole structure $\sigma_{\max}^{vm}$. At the end of each finite element analysis, all the elements that satisfy the following condition were deleted from the model:

$$\sigma_e^{vm} < RR.\sigma_{\max}^{vm} \quad (3.1)$$

Here, RR is the current Rejection Ratio. It is used to dampen or delay the element removal process and is confined to the condition $(0.0 \le RR \le 1.0)$. The same cycle of removing elements using the inequality of Equation (3.1) is repeated until no more elements are able to be removed (with the given *RR*). When this situation occurs, a Steady State has been reached. The *RR* is then updated with a counter function using a Steady State (*SS*) number:

$$RR = a_0 + a_1 SS + a_2 SS^2 + ... \quad (3.2)$$

The *SS* number is an integer counter that varies by increments of one, and is confined to the condition $(0 \le SS < \infty)$. The variables a_0, a_1, a_2 etc are coefficients that determine the nature of the variation in the *RR* number. Usually, they are set as $a_0 = a_2 = 0.0$ and $a_1 = 0.001$. Thus the increase of the *RR* is linear. Having updated the *RR* number, another comparison is made amongst the elements using Equation (3.1) to determine element removal. The *RR* is increased until elements are removed. When elements satisfy this inequality and are removed, another FEA is carried out on the now modified structure. This process is repeated until a desired volume fraction is obtained – for example, 50 % of the initial design domain. This process is illustrated in the flow chart given by Figure 3.1.

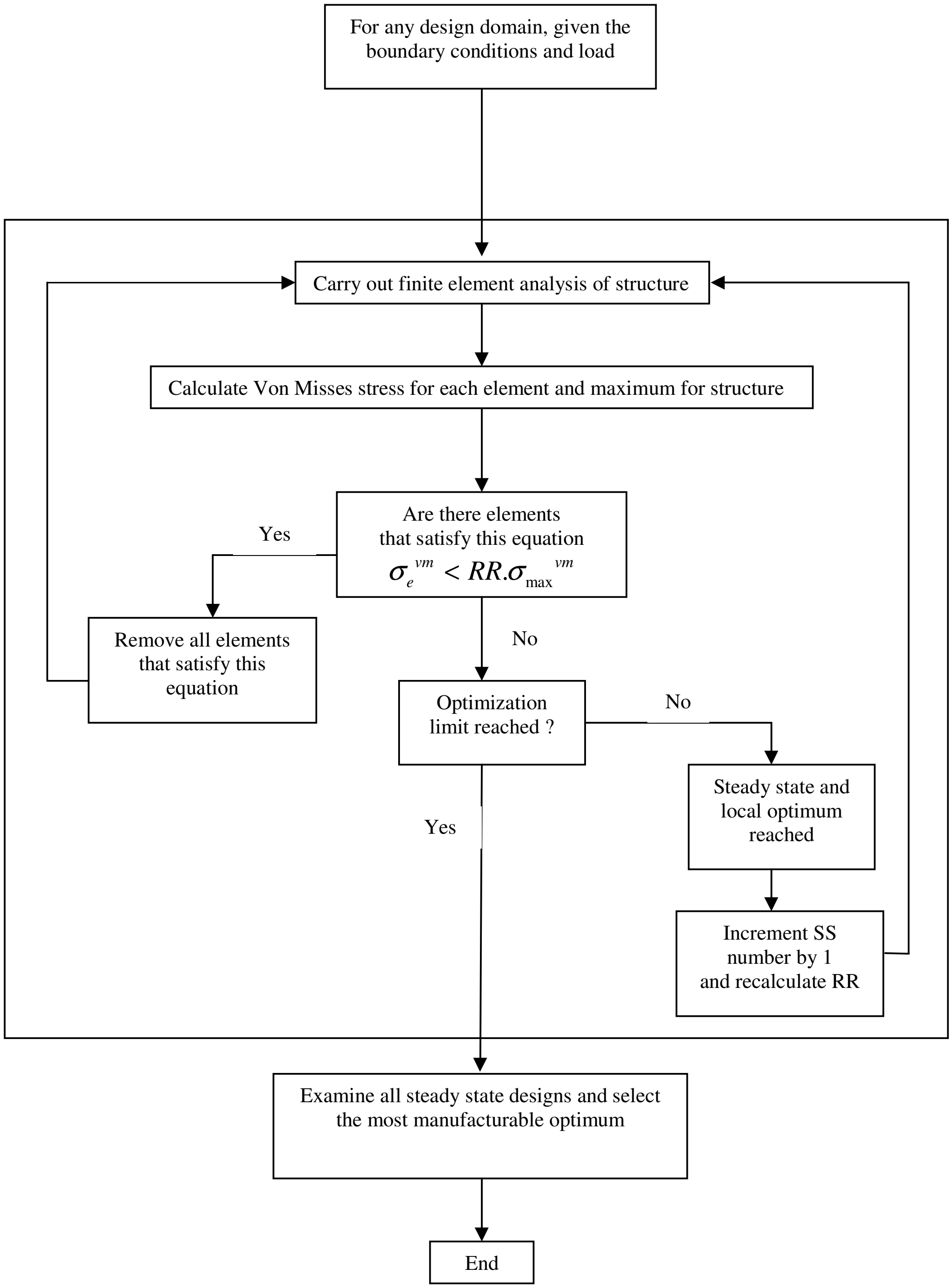

Figure 3.1. Flow chart depicting the logical steps of the stress based ESO

3.2. Overall Stiffness Constraint

Stiffness is one of the key factors that need to be taken into account in the design of structures such as bridges and buildings. It is often required that a structure be stiff enough so that the maximum deflection is within the prescribed limit. This chapter describes the ESO procedures for the simultaneous shape and layout optimization of structures with stiffness constraints.

3.2.2. Determination of Sensitivity Numbers for Element Removal

In FEA, the static behaviour of a structure is represented by the following equilibrium equation;

$$[K].\{u\} = \{F\} \tag{3.3}$$

where $[K]$ is the global stiffness matrix, $\{u\}$ is the nodal displacement vector and $\{F\}$ is the nodal load vector.

The strain energy of the structure is defined as;

$$C = \frac{1}{2}\{F\}^T.\{u\} \tag{3.4}$$

and is commonly used as the inverse measure of the overall stiffness of the structure. C is also known as the compliance. It is obvious that maximizing the overall stiffness is the same minimizing the strain energy.

Consider the removal of the i^{th} element from a structure comprising n finite elements. The stiffness matrix will change by

$$\Delta[K] = [K^*] - [K] = -[K^i] \tag{3.5}$$

where $[K^*]$ is the stiffness matrix after the removal of the i^{th} element and $[K^i]$ is the stiffness matrix of the i^{th} element.

It is assumed that the removal of the element has no effect on the load vector. By ignoring a higher order term, we obtain the change of the displacement vector from equation (3.5) as;

$$\Delta\{u\} = -[K]^{-1}.[\Delta K].\{u\} \tag{3.6}$$

From equations (3.4) and (3.6) we have

$$\Delta C = \frac{1}{2}[F]^T.\{\Delta u\} = -\frac{1}{2}[F]^T.[K]^{-1}.[\Delta K].\{u\} = \frac{1}{2}\{u^i\}^T.[K^i].\{u^i\} \tag{3.7}$$

The above equality indicates the change in the strain energy as a result of removing the i^{th} element. In fact it is the element strain energy and it can be easily calculated at the element level using the element stiffness matrix and the displacement vector of the element.

$$SE_i = \frac{1}{2}\{u^i\}^T.[K^i].\{u^i\} \tag{3.8}$$

It is worth pointing out that both C and SE_i always have positive values.

The optimization objective is to find the lightest structure while satisfying the stiffness constraints, typically given in the form

$$C \le C^* \tag{3.9}$$

where C^* is a prescribed limit for C. In general, when an element is removed, the overall stiffness of a structure reduces and correspondingly the strain energy C increases. To achieve the optimization objective through element removal, it is obviously most effective to remove the element which has the lowest value of SE_i so that the increase in C is minimum.

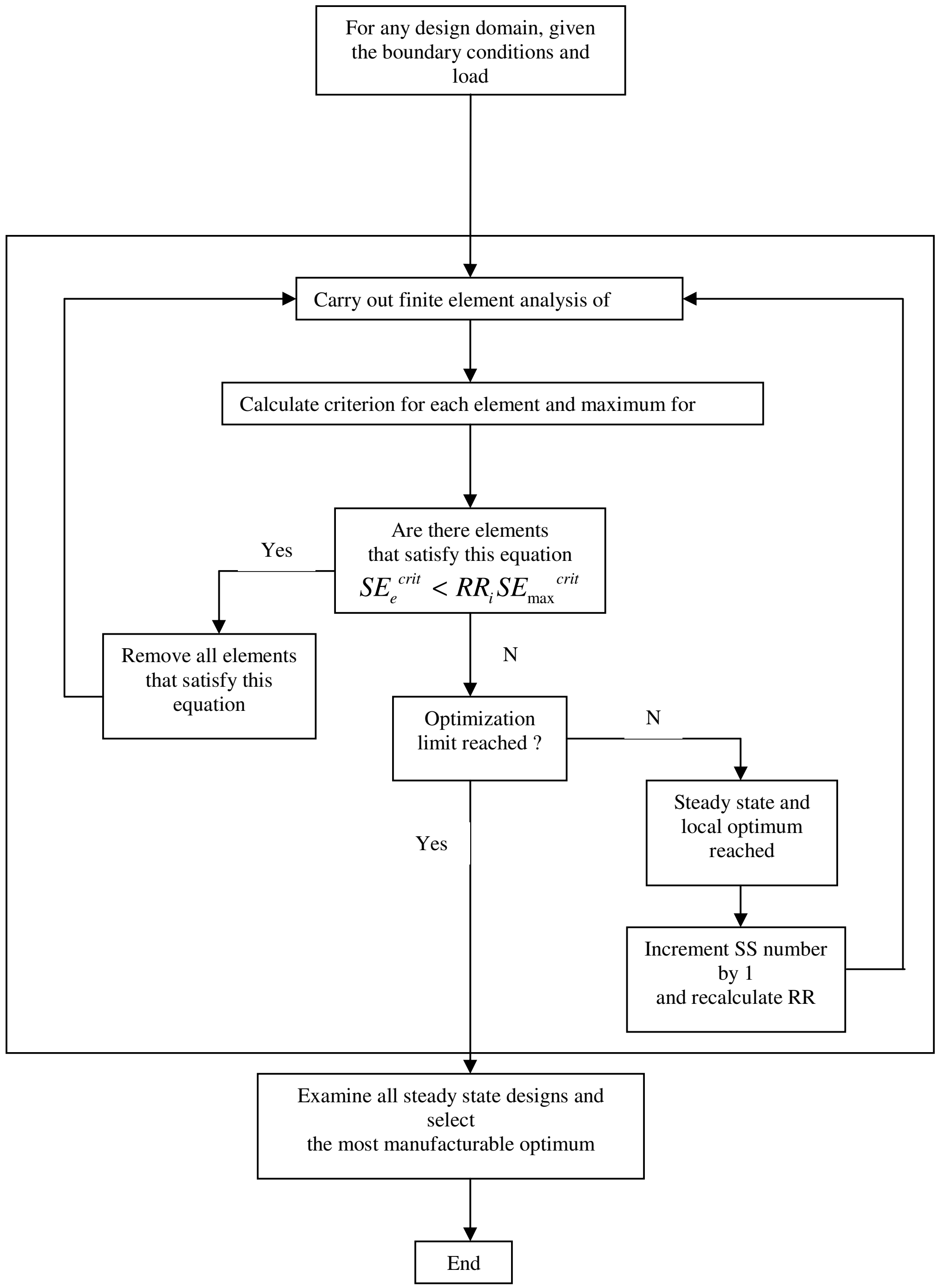

Figure 3.2. Flow chart depicting the logical steps of the strain energy based ESO

3.3. Overall Sensitivity Constraint

Today, ESO is still heavily based on the FEA computational engine, the basis with which to remove elements for stiffness or stress has progressed from that of the objective to examine individual performance characteristics of other engineering problems with optimum material distrubution throughout the structure. It now includes the option of sensitivity numbers (based on many individual criteria) for the element removal process. Some examples of other performance characteritics investigated to date are;

- Frequency
- Buckling
- Thermal stress
- Electrostatic
- Magnetostatic
- Shape optimization for fluid regions with optimality criterion of minimizing pressure drop for same mass flow rate.

Thus, Equation (3.1) has been converted to include the sensitivity numbers of one of many different optimality criteria. It may be presented as:

$$\alpha_e^{crit} < RR_i \alpha_{\max}^{crit} \tag{3.10}$$

The term α_e^{crit} is the sensitivity number of the eth element for the *crit* criterion in question, and the term $\alpha_{\max}^{crit}$ is the maximum sensitivity value that exists for that criterion. The sensitivity number calculated for each element represents the influence of that element on the overall magnitude of the structure's criterion.

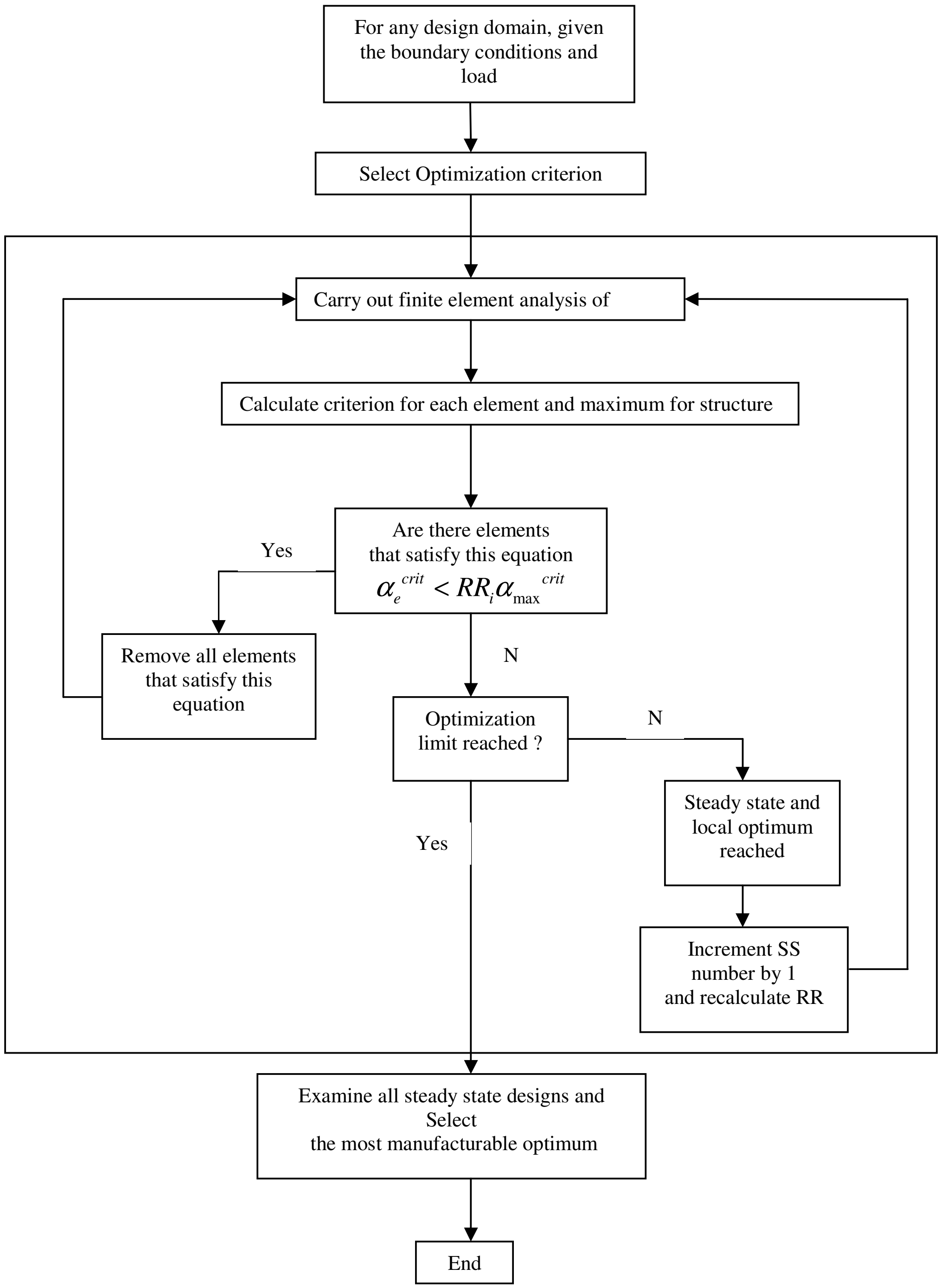

Figure 3.3. Flow chart depicting the logical steps of the sensitivity based ESO

CHAPTER 4

CASE STUDIES

4.1 Evolutionary Structural Optimization of 2 D Solids

One area of application where ESO finds play is in the design of bridges and buildings. Two different blocks of 2-D solids are optimized. In the first example, both of the stress based and strain energy based rejection criterion are applied to see their effects. The design domain is simply supported at the corner nodes of the bottom face and a load to the middle of the top face is applied. The topology for both rejection criterion evolved to the same 2-D frame.

In the second example, a larger block is loaded again at the middle of the top face. This time all the bottom face nodes are simply supported. Two cases are solved as follows. In the first case the boundary condition applied elements are not allowed to be removed. In the second case the boundary condition applied elements are allowed to be removed. These two cases are solved to see the effect of the design domain magnitude, on the final topology.

Finally the results are compared to those in literature.

4.1.1. Modelling Procedure

In the first example, the designable domain was discretized with 200 brick elements. Each element is of dimensions 1x1x1 while the material properties are given as E=200000 Mpa and v=0,3. The solid is a block of dimensions 20x10x1 in x, y and z axes respectively. The nodes on the four bottom corners are not allowed to move in any direction while the force is applied to the middle of the top face. The figure 4.1 displays the design domain. (1,2,3 meaning x,y,z displacements are zero)

In the second example, the designable domain was discretized with 300 brick elements. Each element is of dimensions 1x1x1 while the material properties are given as E=200000 Mpa and v=0,3. The solid is a block of dimensions 30x10x1 in x, y and z axes resectively. The figure 4.5 displays the design domain.

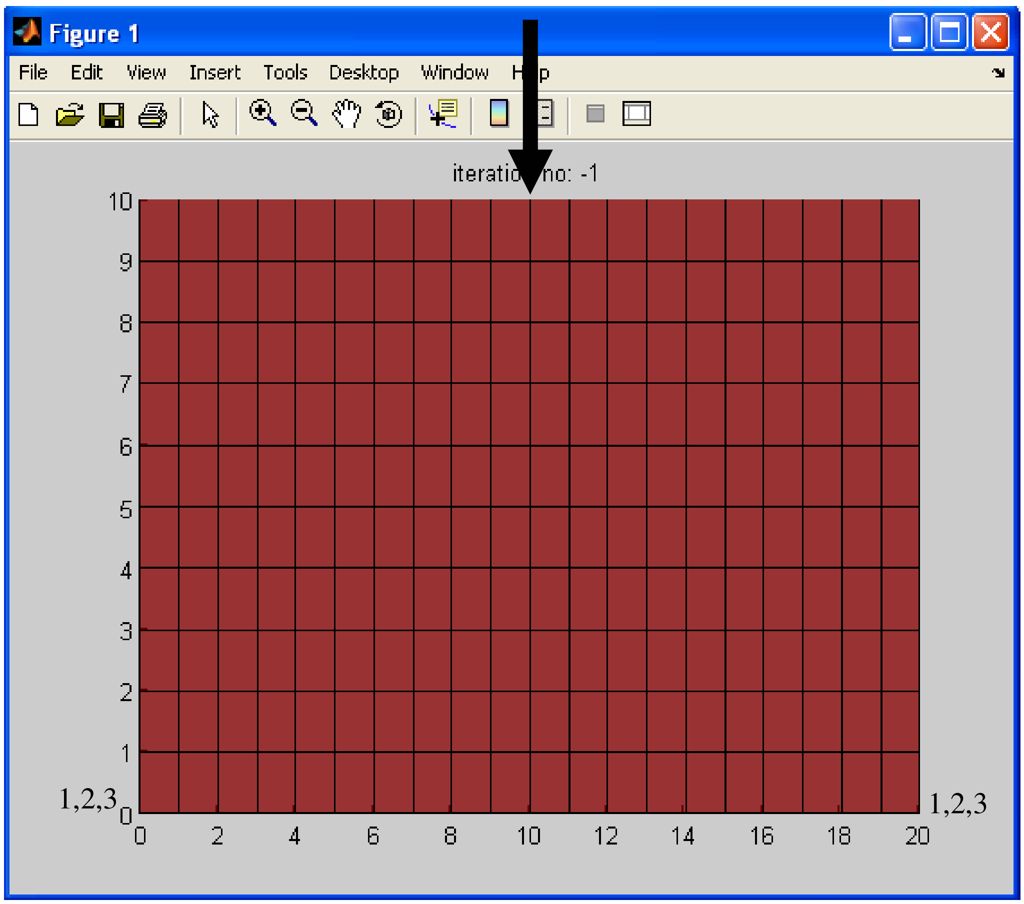

Figure 4.1. Design Domain of the 2-D Solid

4.1.2 Results

In both cases, although the mesh is very course , the results obtained are interpretable.

For the first example solved, the final topologies are displayed in the following figures. M is the Mass of the ersulting topology and M_0 as the initial Mass of the design domain.

Figure 4.2 shows the optimization results after 23 iterations when the M/M_0 value is equal to 0.25. ,the optimization result after 39 iterations when the M/M_0 value is equal to 0,50. and the optimization result after 78 iterations when the M/M_0 value is equal to 0,75.

Figure 4.3 shows the optimization results after 16 iterations when the M/M_0 value is equal to 0.25. ,the optimization result after 29 iterations when the M/M_0 value is equal to 0,50. and the optimization result after 131 iterations when the M/M_0 value is equal to 0,75.

As can be seen from the figures, the both criterion gave the same final topology however there are small differences in the evolution progresses. Evolutions of stress and strain energy based optimizations are given in figure 4.4.

One important point to note is that the final topology is strongly affected by the boundary conditions. The frame elements end on the bottom face are where the boundary conditons are applied. This is not the only possible method to define the boundary conditions in EVO. In EVO the optimum material distribution is done regarding the choice of the boundaries from a possible set of boundaries. The following example is given to illustrate this topic.

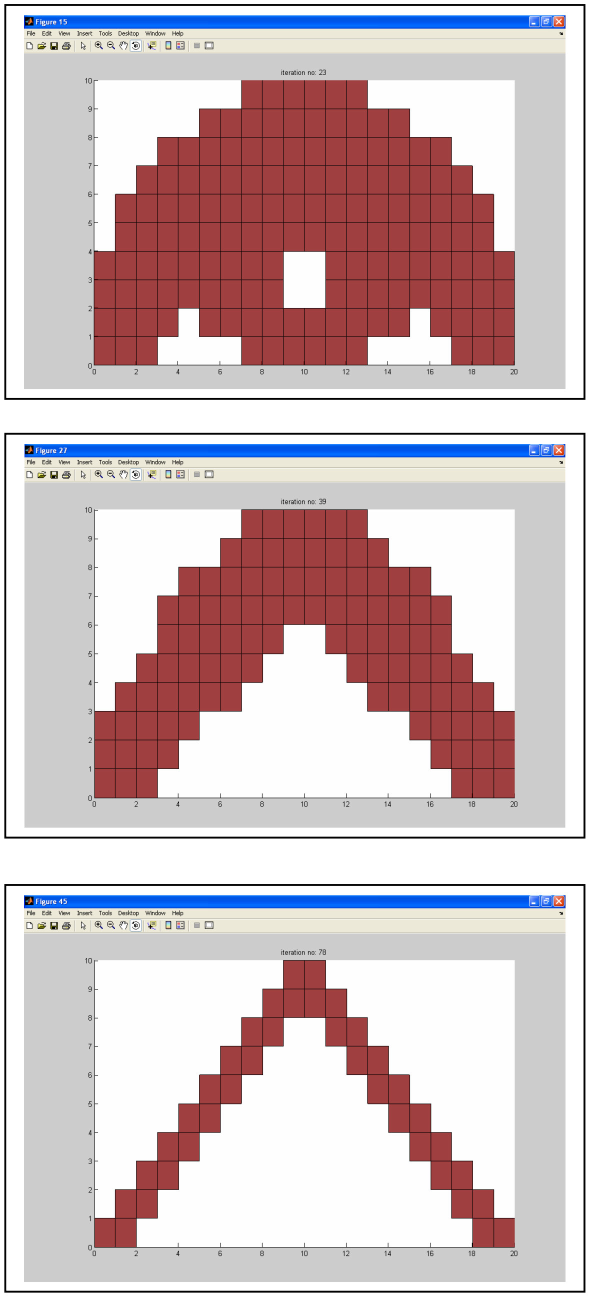

Figure 4.2. Obtained stress based topologies for 0.25, 0.50 and 0.80 M/M_0 values respectively

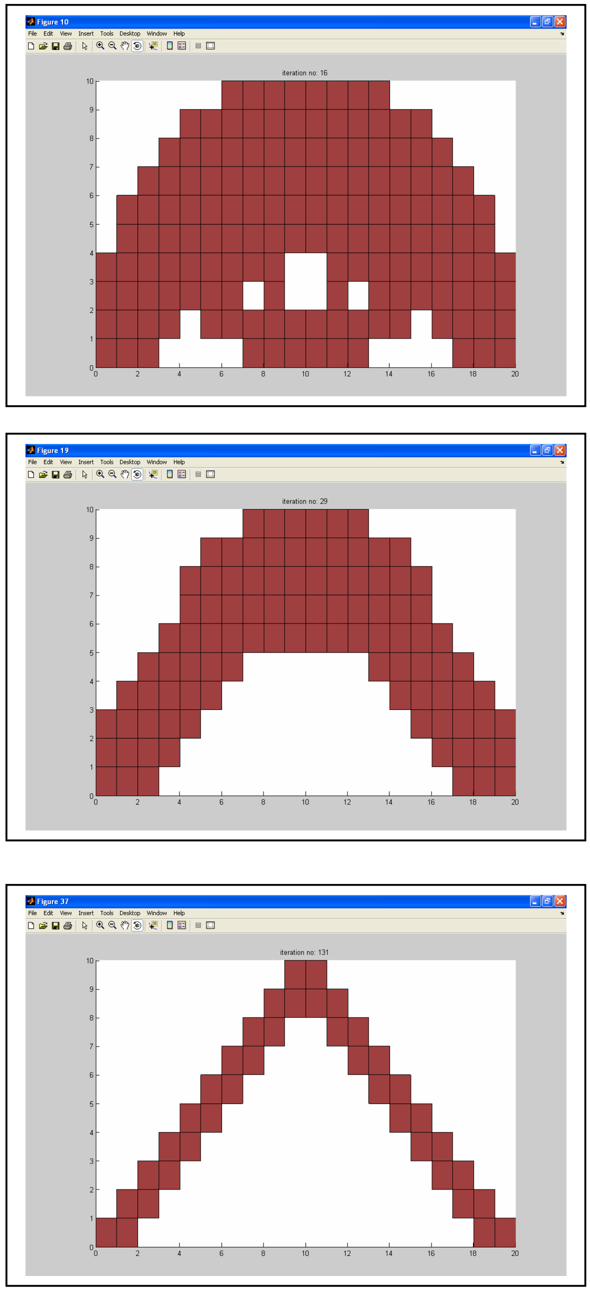

Figure 4.3 Obtained strain energy based topologies for 0.25, 0.50 and 0.80 M/M_0 values

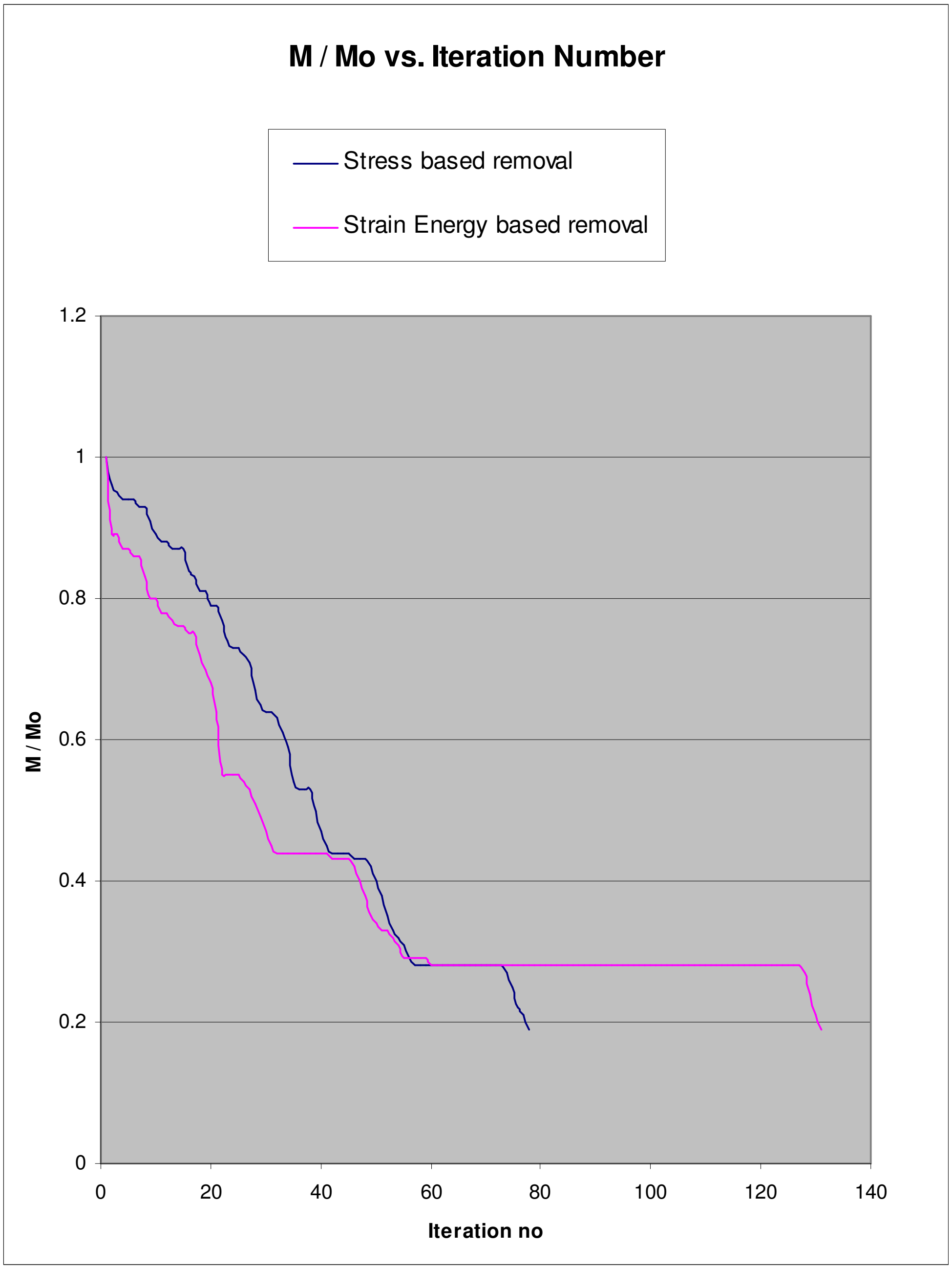

Figure 4.4. Evolutions of stress and strain energy based optimizations.

The second example design domain, the boundary conditions and forces are given in figure 4.5. The following figure 4.6 displays the topologies for 0.25, 0.50 and 0.75 M/M_0 values. As can easily be recognized, however the bottom face members still exist in the final design, they do not contribute to the overall stiffness.

In the second case, the boundary condition applied elements are allowed to be removed and the final topology is different than the one obtained before. The width of the design reduced to 20 and the elements not contributing to overall stiffness are removed. This example shows that, by using ESO, starting with an unnecessarily big design domain will not cause any problem for the final topology if the boundary conditions and forces are defined properly. However, creating such a big design domain will increase the solution time. With EVO, the best placement of boundary conditions is selected from the set of possible places defined by the user.

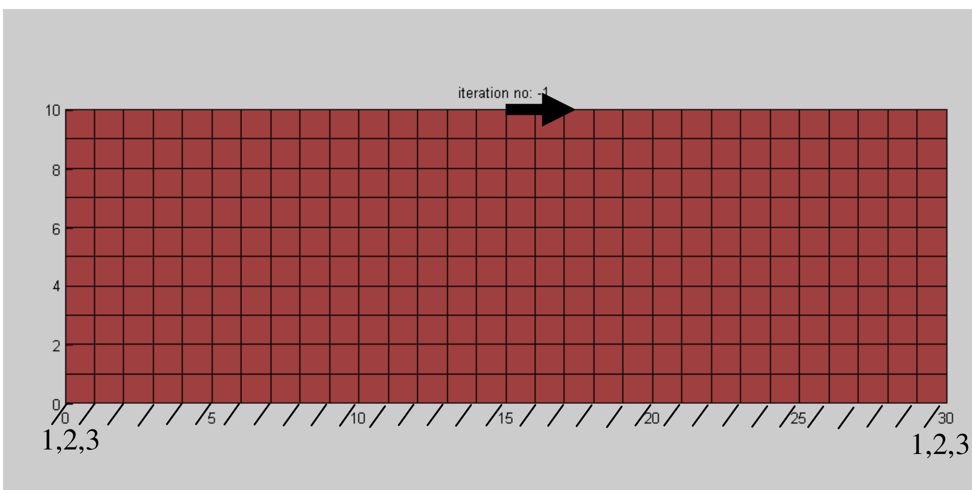

Figure 4.5 Design domain of the second example

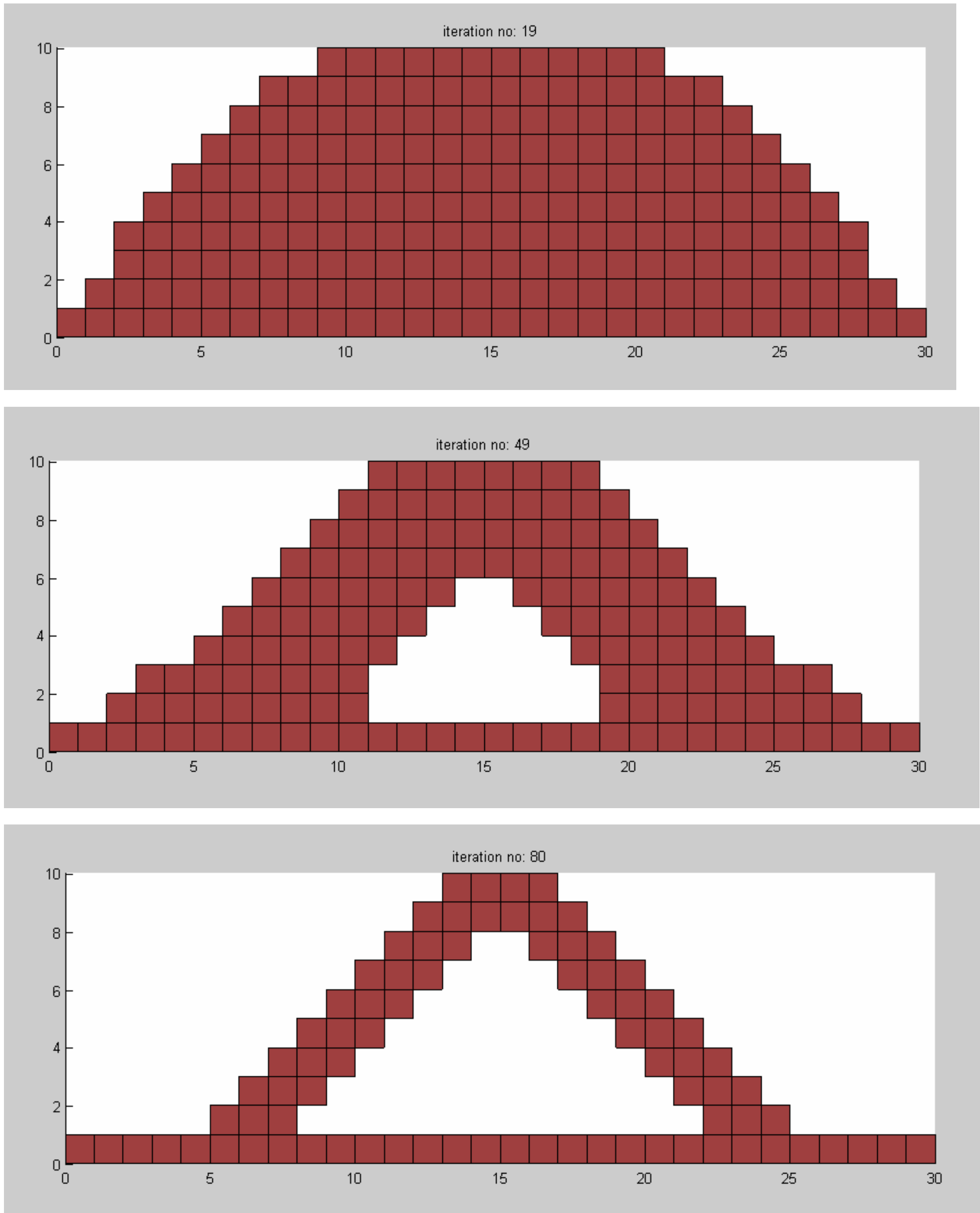

Figure 4.6 Obtained stress based topologies for 0.25, 0.50 and 0.80 M/M_0 values respectively

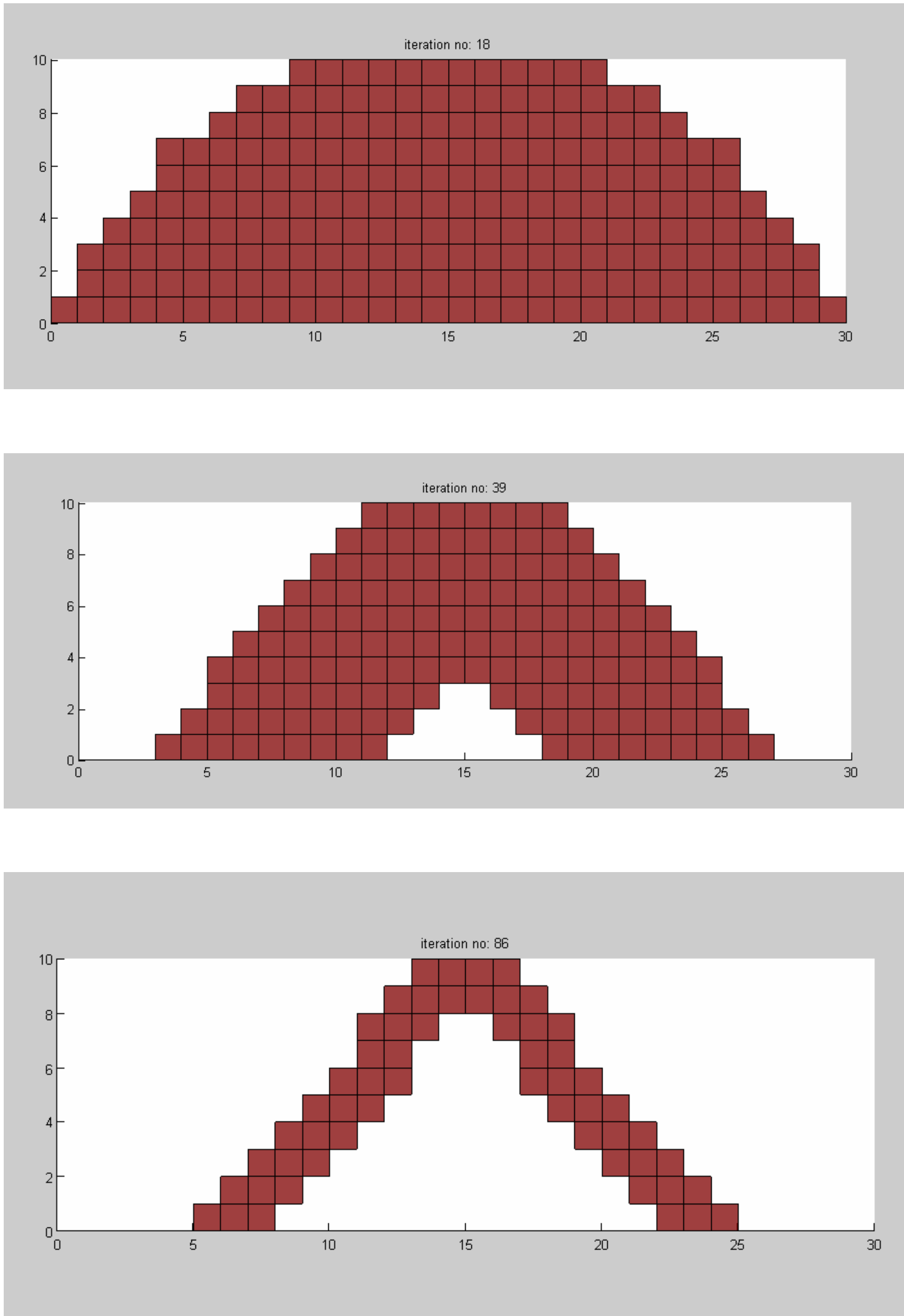

Figure 4.7 Obtained stress based topologies for 0.25, 0.50 and 0.80 M/M_0 values respectively

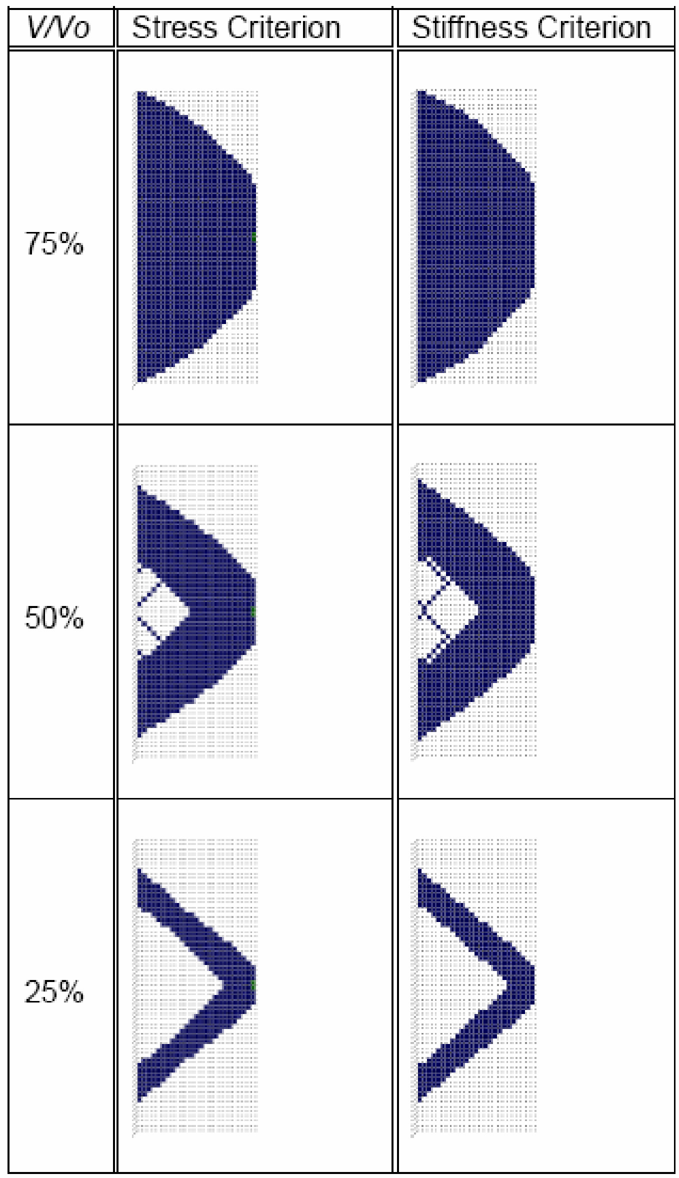

Figure 4.8. The stress and strain energy based optimizations taken from [ref 8].

The results may well be compared to those obtained from the literature. In the example from the literature given above, the initial design domain consists of 25x60 quadrilateral plane stress elements. There is a force at the middle of the top face which is directed downwards.

The optimization convergences, taking into account the stress based evolution and stiffness based evolution are compared on this figure. On the left side column, the ratios of volumes after iterations, to the initial volume of the design domain are given. There is a noticeable difference at the resulting topologies of 50% ratio of resulting topology volume over initial volume. This is an indicator that, not all the time should the stress based and stiffness based optimizations converge to the same topologies. The idea in using these two algorithms is not always to arrive at the same final topology but to find the optimum topologies depending on the main criterion taken into account which can be stress or displacement. Namely, when the search is for small deflections, the criteria should be stiffness and when the critical parameter is stress, should the stress based algorithm be employed.

Although the dimensions of the design domains in the examples solved in this study and of the example found from the literature are different, the solutions obtained are similar.

One other important point to note is that the elements used for this problem are quadrilateral plane stress elements. The elements used in this thesis are 3 dimensional 8 nodal brick elements. As mentioned before, these elements may be used to model plates, beams, bars and so on, but the disadvantage of using 3D meshes is that the solution time and the number of equations to be solved increase greatly compared to those simplification assumptions.

To show the versatility of these elements, some 3D solids are examined and optimized in the following examples. The comparisons of the solutions obtained to the ones in literature are given.

4.2 Evolutionary Structural Optimization of 3 D Continuum Solids

In this chapter the design of a 3-D space frame structure is accomplished from evolving from a block of 3-D solid elements. The results are obtained using the basic ESO method with stress and stiffness constraint.

4.2.1 Modelling Procedure

In this study, the designable domain was discretized with 4000 brick elements. Each element is of dimensions 1x1x1 while the material properties are given as E=200000 Mpa and v=0,3. The solid is a block of dimensions 20x20x10. The four bottom corners are fixed while the force is applied to the middle of the top floor. The figure 4.1 displays the design domain.

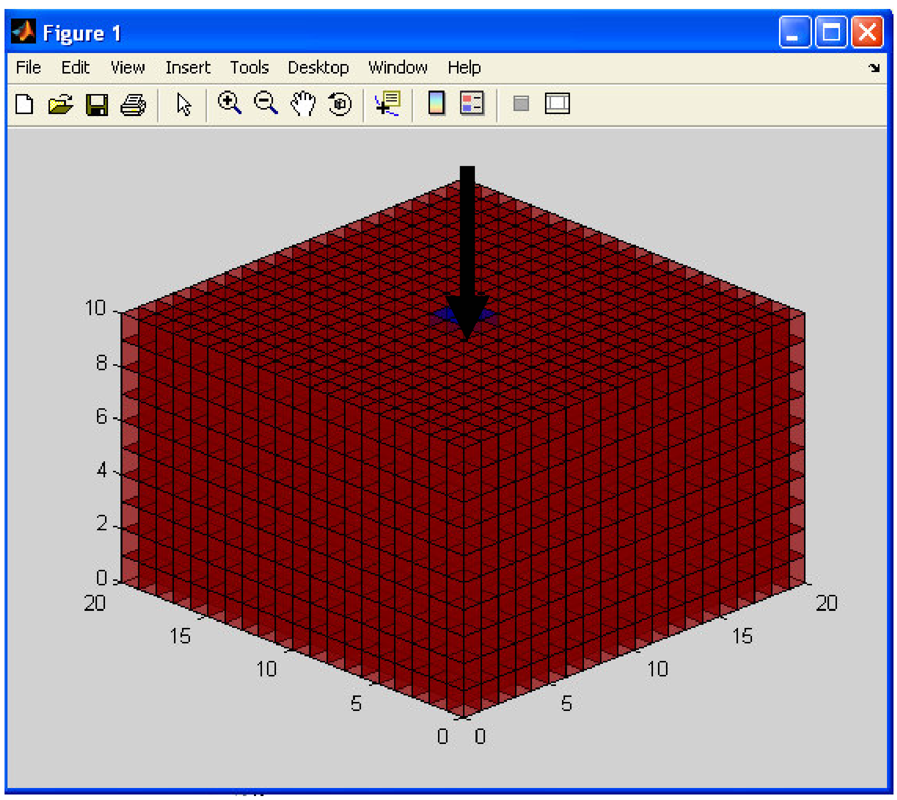

Figure 4.9. Design Domain of the 3-D Spatial Frame

4.2.2 Results

Although the mesh is course , the result obtained is interpretable. The solution is displayed in the following figures.

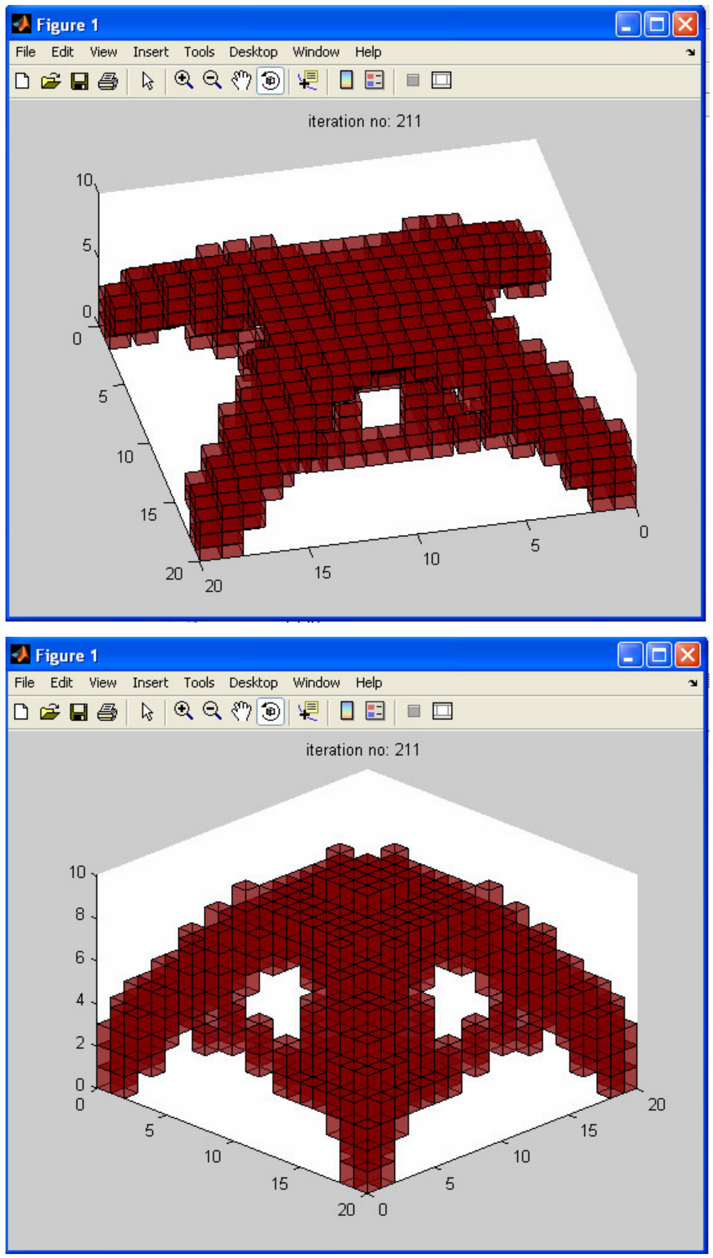

Figure 4.10. 3_D Display of Topology after 211 iterations

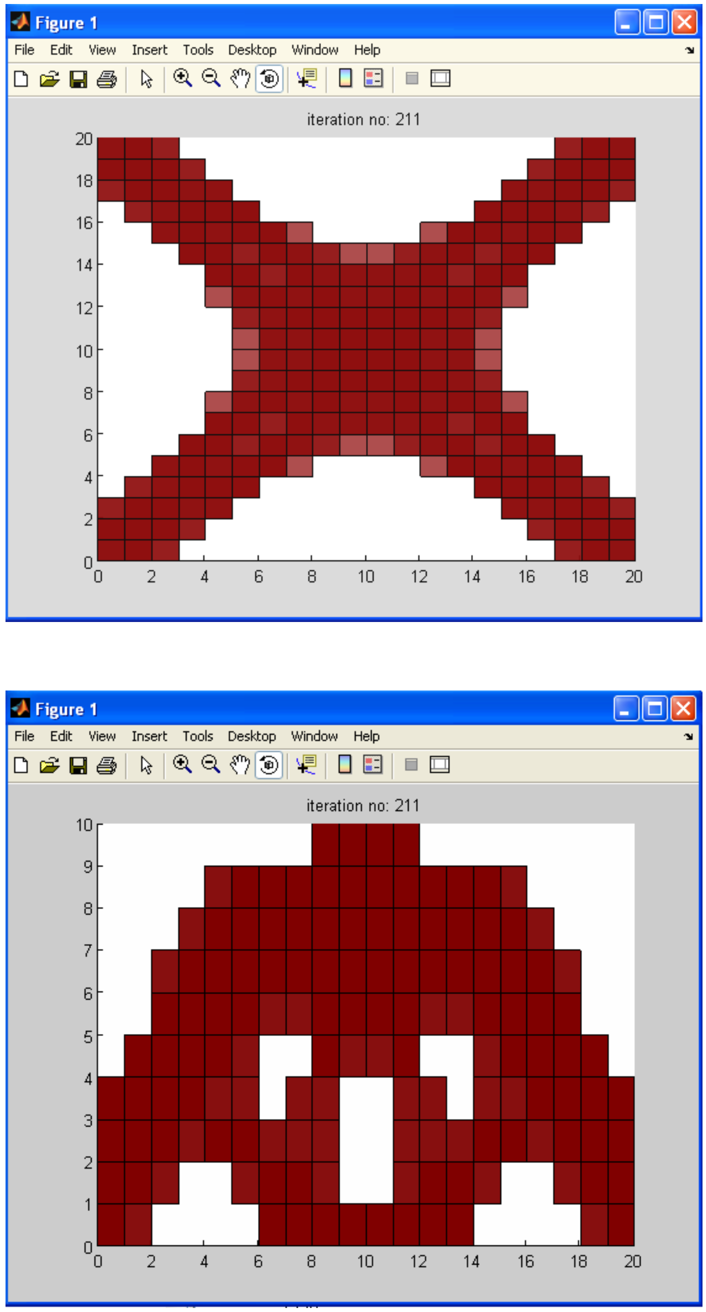

Figure 4.11. Side views of the topology after 211 iterations

Following examples are about the design of 3D Beams. The first example is solved to make it possible to compare the results with that found in literature. The example found in literature is taken from a product development conference and the design is accomplished by using MSC Nastran. The optimization is accomplished in the paper with the algorithm of compliance minimization. The study of this thesis solved the example with using both of the convergence criterions.

The second and third examples are other beam examples.

The design domain in the first example is a solid of dimensions 10x10x20 in x, y and z dimensions respectively. The example in literature uses a solid of dimensions 17x17x33.

Although the dimensions are not the same, the results obtained seem to be in well agreement. The aim in all cases is to minimize compliance until the mass is reduced to 20% of the initial mass.

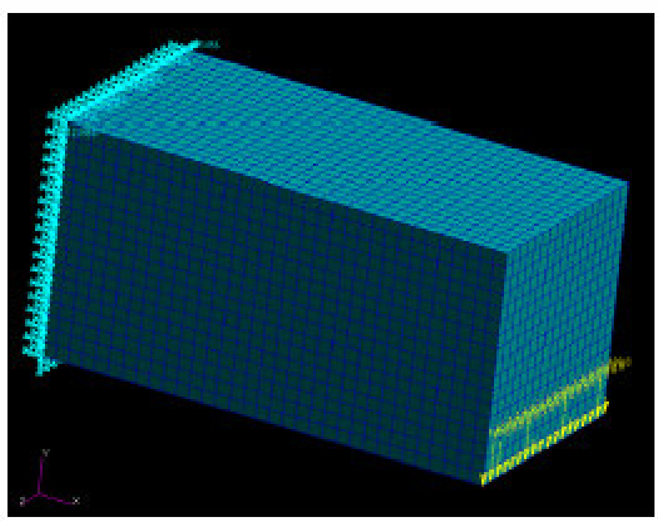

Figure 4.12. Design domain of a 3D beam taken from [ref 35]

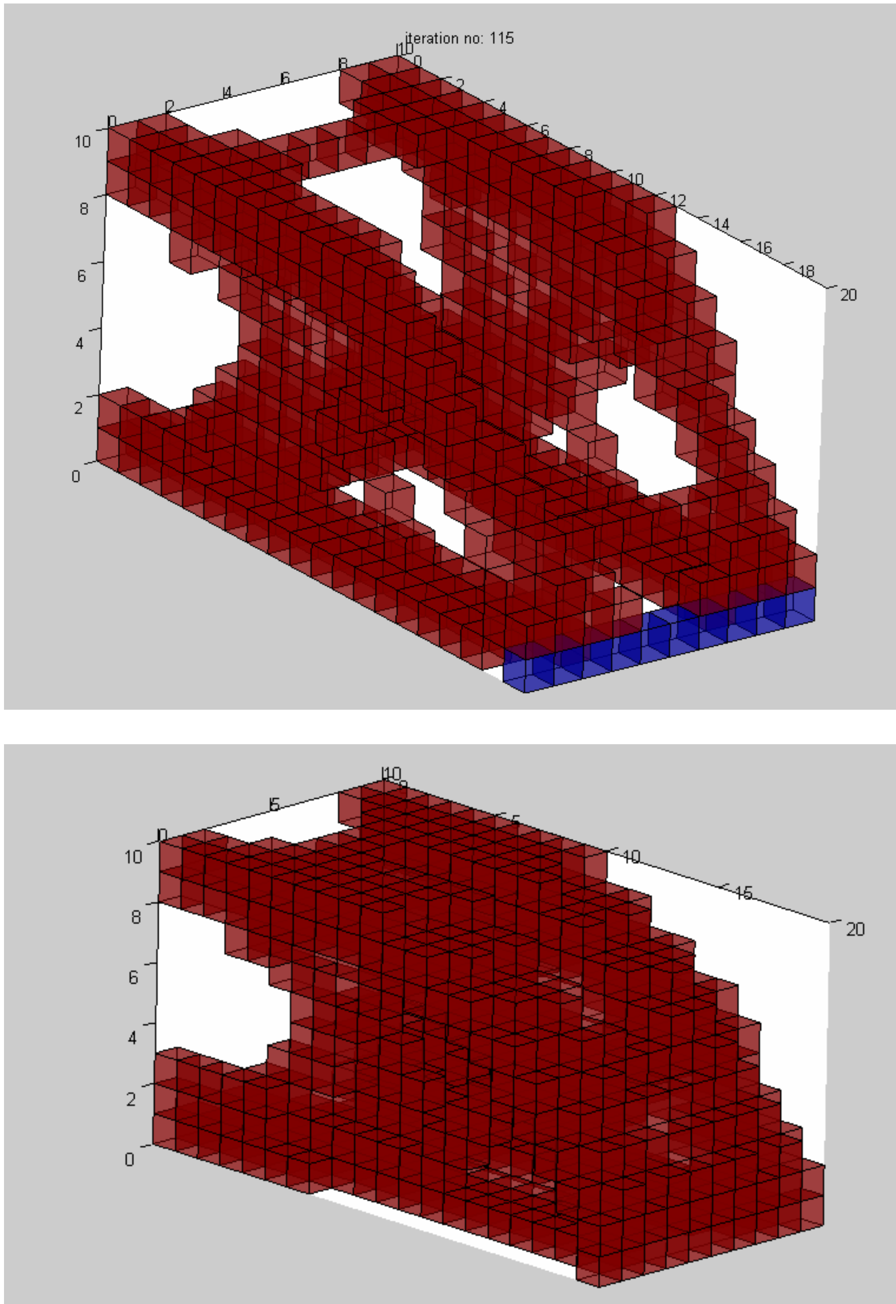

Figure 4.13. Stress based and Strain energy based convergences of the beam

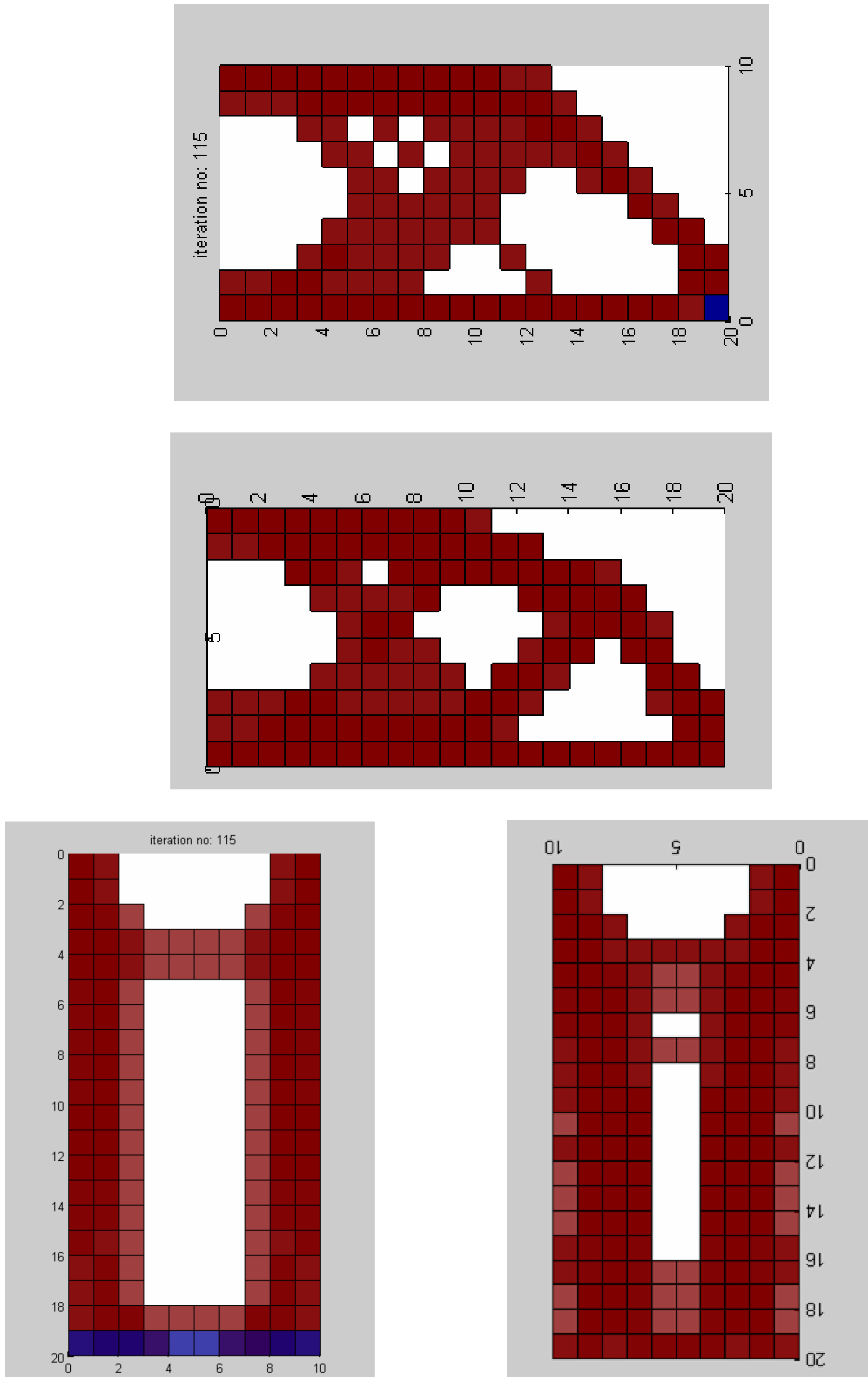

Figure 4.14. Side views of stress based and strain energy based convergences of the beam

There is an unneglicable difference between the stress based and strain energy based convergences. This difference may result from the evolution rates employed. The ratio of the maximum stress to the minimum stress is not equal to the ratio of the maximum strain energy to the minumum strain energy ratio in all iterations and when different number of elements are removed from the structure at each iteration, the convergences differ. As presented earlier in the 2D solid examples there were quiet differences in the removed elements at the given iterations. However because of the reason that there were a small amount of members, the optimum topologies were the same. In this example the total number of elements are 2000 therefore it is not a surprise to see a difference in the convergences.

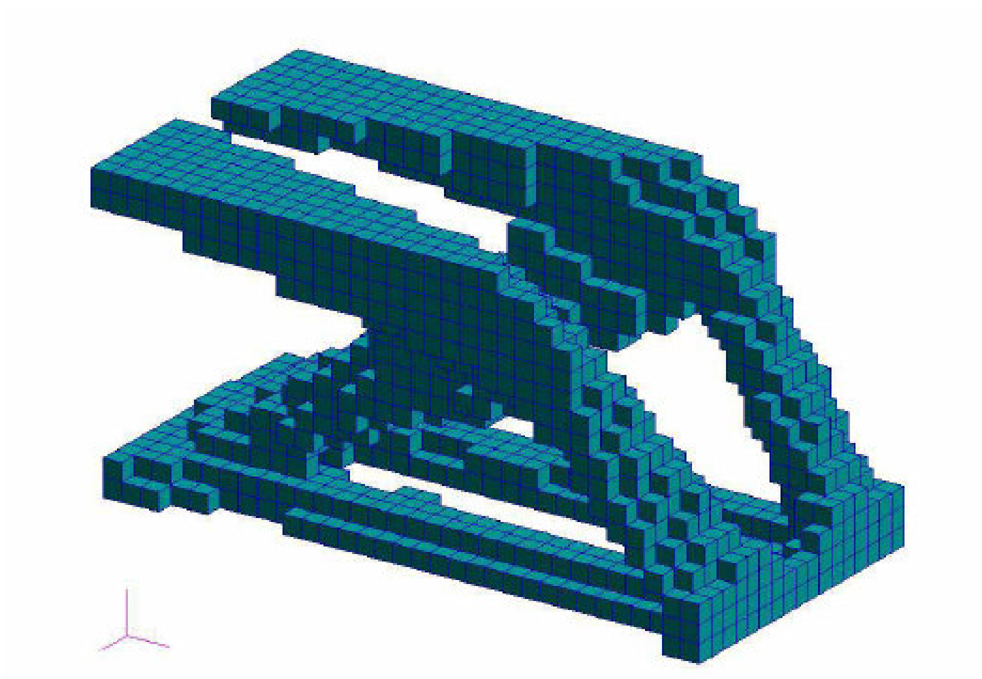

Figure 4.15. Topology optimization of a 3D beam taken from [ref 28]

The difference between the convergences of this study and the one from the literature is because of the number of elements used and the removal rate differences. The one in literature uses a finer mesh with 9537 elements and the removal criterion is not given. However the solution of this study still seems to be in agreement with the one from the literature.

In these two examples a beam of dimensions 6x12x48 is investigated. The nodes on the small faces of the beam in the first example are not allowed to move in any direction and the force is applied to the center of gravity of the beam. The following figures show the topology after 80% of the initial design domain is removed.

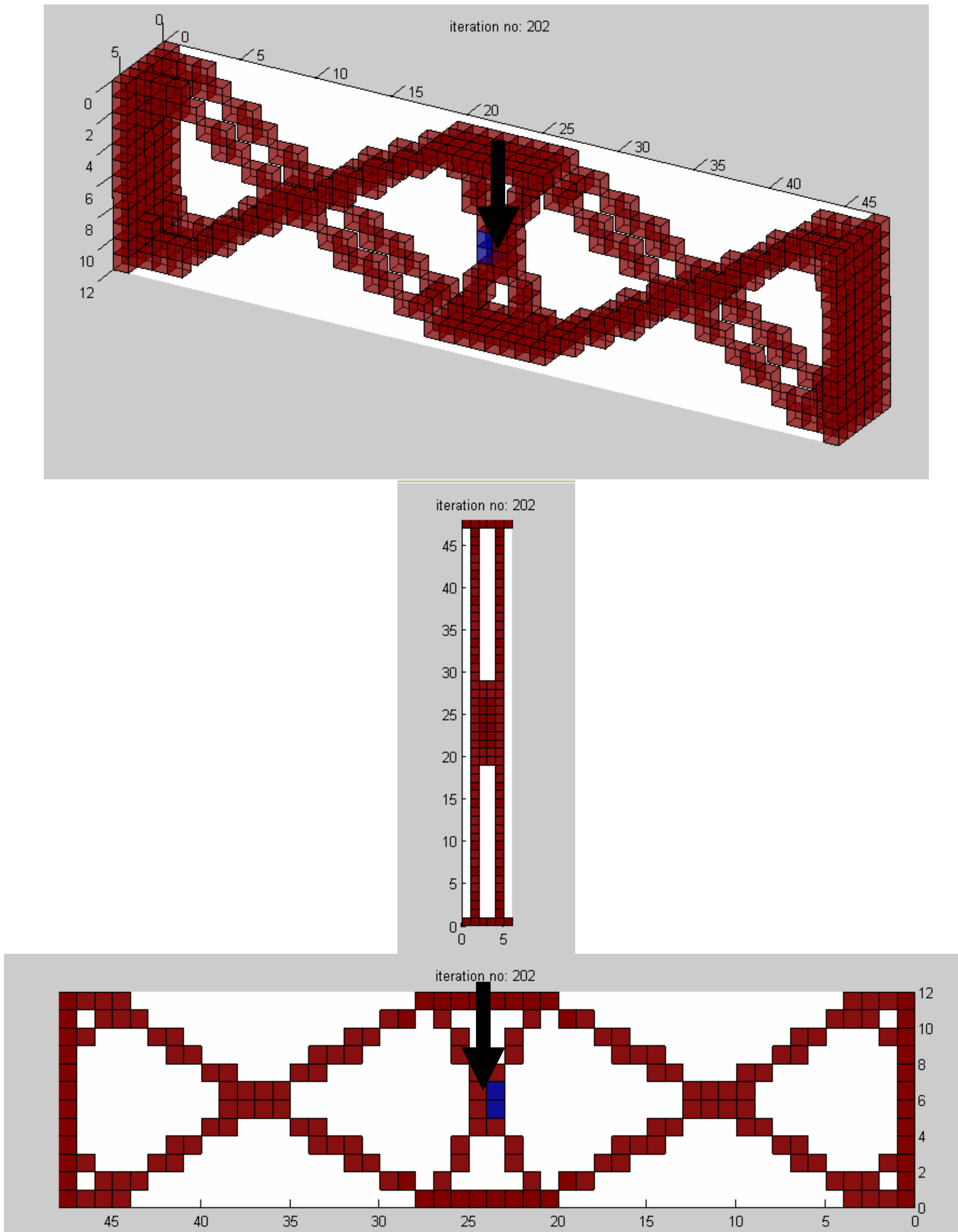

Figure 4.16. Topology optimization of a 3D beam

In the second example, the nodes on one of the small faces of the beam are not allowed to move in any direction as a boundary condition and the force is applied to the center of the other small face. The following figures show the topology after 80% of the initial design domain is removed.

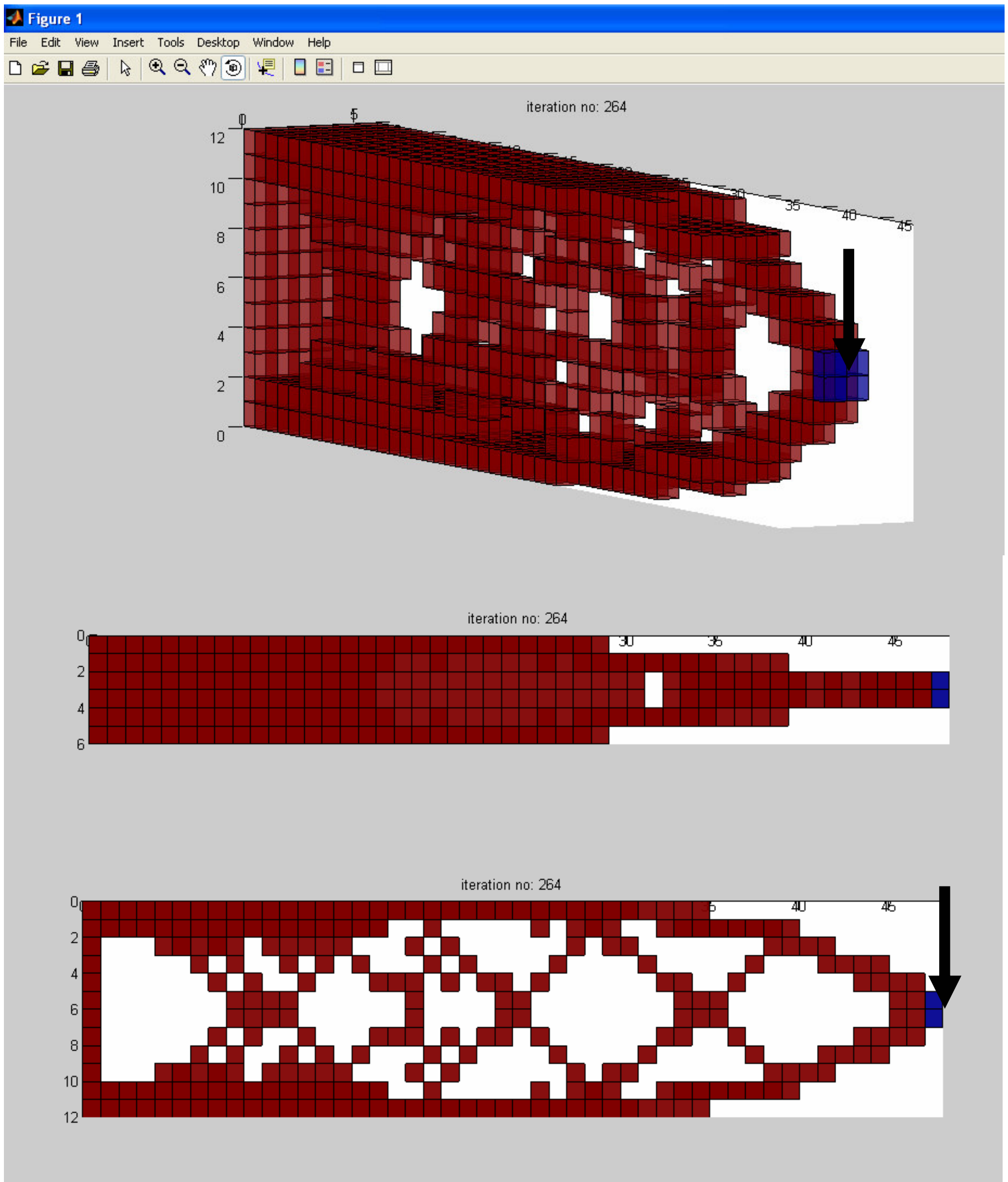

Figure 4.17. Topology optimization of another 3D beam

CHAPTER 5

SUMMARY AND CONCLUSION

In this chapter, the work done in the thesis study is summarized focusing on the algorithm developed. The major achievements of the study are concluded and some recommendations for the future work are given.

5.1. Summary

The aim of this study is to apply evolutionary structural optimization algorithm for the topology optimization of 3-D solids. To develop the program that evolves a solid to optimum topology, two main programs are written. The first one is a FEA code that solves the matrix equations to give displacements and stresses on the elements. The other main program is the optimization program that rejects the elements that are of very small contribution to the overall structural stiffness or that are subjected to relatively low von Mises stress.

The design domain to be evaluated in the code is prepared by another code that is written in MATLAB. This code generates the number of nodes, number of elements, element connectivities and node positions that are needed by the Fortran code.

In order to visualize the results, another MATLAB code is written that displayed the topology of the structure after each iteration. The results are displayed in figure windows which can be zoomed in and out and rotated to grasp the layout of the final structure.

5.2. Conclusions

The prepared program EVO lets the designer to define the design domain with the loads and possible boundary conditions. Once the program is run, the optimization iterations continue until the objective function is satisfied. The program may also terminate the iterations before the objective is reached if there is no more removable element. That is to say, when the rejection criteria is increased so much that the structure breaks, the program stops saying that there is a problem in one of the equations to be solved but still outputs the last possible optimization..

The case studies in this study are selected such that the comparison of the resulting topologies could be compared to those that exist in literature. There is enough number of examples in literature for plane stress problems. In this study some 2d plane stress problems are formulated with the 3D elements and the solutions obtained are compared to those with the original formulations topologies. The resulting topologies are the same. This shows that the formulation of finite element code and the optimization algorithm is well working for 3D elastic solid bodies. One example for a 3D design domain had been found in literature and the finalizing topology was similar. More 3D examples that are interpretable are also solved.

5.3. Recommendations for Future Work

It is possible to list some recommendations for the improvement of the models and software developed:

- An optimization algorithm that not only removes elements but also adds elements may be added to the program.
- Multicriteria optimization may be investigated with adding some subroutines to the program
- The display of the results is done by MATLAB program. The results may be formatted in such a way that a model builder builds the results for better and faster display.

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APPENDIX

TYPICAL PROGRAM INPUT AND OUTPUT FILES

Iteration No	Number of Elements Removed at the Iteration	Total Number of Elements Removed	Rejection Ratio RR
1	0	0	0.05
2	56	56	0.05
3	8	64	0.05
4	4	68	0.05
5	0	68	0.06
6	48	116	0.06
7	16	132	0.06
8	8	140	0.06
9	0	140	0.07
10	52	192	0.07
11	8	200	0.07
12	0	200	0.08
13	48	248	0.08
14	16	264	0.08
15	0	264	0.09
16	72	336	0.09
17	52	388	0.09
18	0	388	0.1
19	40	428	0.1
20	28	456	0.1
21	16	472	0.1
22	0	472	0.11
23	52	524	0.11
24	8	532	0.11
25	0	532	0.12
26	96	628	0.12
27	88	716	0.12
28	40	756	0.12
29	32	788	0.12
30	16	804	0.12
31	16	820	0.12
32	8	828	0.12
33	8	836	0.12

Table A.1. Element removal after each iteration

Node-	6	U=	0	V=	0	W=	0
Node-	7	U=	0	V=	0	W=	0
Node-	7	U=	0	V=	0	W=	0
Node-	8	U=	0	V=	0	W=	0
Node-	8	U=	0	V=	0	W=	0
Node-	9	U=	0	V=	0	W=	0
Node-	23	U=	0	V=	0	W=	0
Node-	24	U=	0	V=	0	W=	0
Node-	24	U=	0	V=	0	W=	0
Node-	25	U=	0	V=	0	W=	0
Node-	25	U=	0	V=	0	W=	0
Node-	26	U=	0	V=	0	W=	0
Node-	37	U=	-0.00005	V=	-0.000016	W=	-0.00001
Node-	38	U=	-0.000072	V=	-0.000025	W=	-0.00001
Node-	38	U=	-0.000072	V=	-0.000025	W=	-0.00001
Node-	38	U=	-0.000072	V=	-0.000025	W=	-0.00001
Node-	39	U=	-0.000091	V=	-0.000031	W=	-0.000012
Node-	39	U=	-0.000091	V=	-0.000031	W=	-0.000012
Node-	39	U=	-0.000091	V=	-0.000031	W=	-0.000012
Node-	39	U=	-0.000091	V=	-0.000031	W=	-0.000012
Node-	40	U=	-0.000114	V=	-0.000046	W=	-0.000016

Element-	6	Svm=	4.819
Element-	7	Svm=	6.189
Element-	8	Svm=	8.144
Element-	23	Svm=	8.144
Element-	24	Svm=	6.189
Element-	25	Svm=	4.819
Element-	37	Svm=	5.923
Element-	38	Svm=	6.916
Element-	39	Svm=	7.688
Element-	52	Svm=	7.688
Element-	53	Svm=	6.916
Element-	54	Svm=	5.923
Element-	68	Svm=	6.724
Element-	69	Svm=	7.291
Element-	70	Svm=	6.217

Table A.2. A typical output data of the program EVO

8	27.		1	1	1	0			
0.01	0.005								
1	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0
3	0	0	0	2	0	0	0	0	0
4	0	0	0	0	1	0	0	0	0
5	0	0	0	1	1	0	0	0	0
6	0	0	0	2	1	0	0	0	0
7	0	0	0	0	2	0	0	0	0
8	0	0	0	1	2	0	0	0	0
9	0	0	0	2	2	0	0	0	0
10	0	0	0	0	0	1	0	0	0
11	0	0	0	1	0	1	0	0	0
12	0	0	0	2	0	1	0	0	0
13	0	0	0	0	1	1	0	0	0
14	0	0	0	1	1	1	0	0	0
15	0	0	0	2	1	1	0	0	0
16	0	0	0	0	2	1	0	0	0
17	0	0	0	1	2	1	0	0	0
18	0	0	0	2	2	1	0	0	0
19	0	0	0	0	0	2	0	0	0
20	0	0	0	1	0	2	0	0	0
21	0	0	0	2	0	2	0	0	0
22	0	0	0	0	1	2	0	0	0
23	0	0	0	1	1	2	0	0	0
24	0	0	0	2	1	2	0	0	0
25	0	0	0	0	2	2	0	0	0
26	0	0	0	1	2	2	0	0	0
27	0	0	0	2	2	2	0	0	0
1	1	2	5	4	10	11	14	13	1
2	2	3	6	5	11	12	15	14	1
3	4	5	8	7	13	14	17	16	1
4	5	6	9	8	14	15	18	17	1
5	10	11	14	13	19	20	23	22	1
6	11	12	15	14	20	21	24	23	1
7	13	14	17	16	22	23	26	25	1
8	14	15	18	17	23	24	27	26	1

Table A.3. A typical input data for the program EVO

4.70E+04	1.60E+04	1.60E+04	-2.14E+04	3.21E+03	3.21E+03	-1.71E+04	-1.60E+04	1.60E+03	1.07E+04	-3.21E+03	8.01E+03
1.60E+04	4.70E+04	1.60E+04	-3.21E+03	1.07E+04	8.01E+03	-1.60E+04	-1.71E+04	1.60E+03	3.21E+03	-2.14E+04	3.21E+03
1.60E+04	1.60E+04	4.70E+04	-3.21E+03	8.01E+03	1.07E+04	-1.60E+03	-1.60E+03	-1.07E+03	8.01E+03	-3.21E+03	1.07E+04
-2.14E+04	-3.21E+03	-3.21E+03	4.70E+04	-1.60E+04	-1.60E+04	1.07E+04	3.21E+03	-8.01E+03	-1.71E+04	1.60E+04	-1.60E+03
3.21E+03	1.07E+04	8.01E+03	-1.60E+04	4.70E+04	1.60E+04	-3.21E+03	-2.14E+04	3.21E+03	1.60E+04	-1.71E+04	1.60E+03
3.21E+03	8.01E+03	1.07E+04	-1.60E+04	1.60E+04	4.70E+04	-8.01E+03	-3.21E+03	1.07E+04	1.60E+03	-1.60E+03	-1.07E+03
-1.71E+04	-1.60E+04	-1.60E+03	1.07E+04	-3.21E+03	-8.01E+03	4.70E+04	1.60E+04	-1.60E+04	-2.14E+04	3.21E+03	-3.21E+03
-1.60E+04	-1.71E+04	-1.60E+03	3.21E+03	-2.14E+04	-3.21E+03	1.60E+04	4.70E+04	-1.60E+04	-3.21E+03	1.07E+04	-8.01E+03
1.60E+03	1.60E+03	-1.07E+03	-8.01E+03	3.21E+03	1.07E+04	-1.60E+04	-1.60E+04	4.70E+04	3.21E+03	-8.01E+03	1.07E+04
1.07E+04	3.21E+03	8.01E+03	-1.71E+04	1.60E+04	1.60E+03	-2.14E+04	-3.21E+03	3.21E+03	4.70E+04	-1.60E+04	1.60E+04
-3.21E+03	-2.14E+04	-3.21E+03	1.60E+04	-1.71E+04	-1.60E+03	3.21E+03	1.07E+04	-8.01E+03	-1.60E+04	4.70E+04	-1.60E+04
8.01E+03	3.21E+03	1.07E+04	-1.60E+03	1.60E+03	-1.07E+03	-3.21E+03	-8.01E+03	1.07E+04	1.60E+04	-1.60E+04	4.70E+04
1.07E+04	8.01E+03	3.21E+03	-1.71E+04	1.60E+03	1.60E+04	-1.18E+04	-8.01E+03	8.01E+03	-1.07E+03	-1.60E+03	1.60E+03
8.01E+03	1.07E+04	3.21E+03	-1.60E+03	-1.07E+03	1.60E+03	-8.01E+03	-1.18E+04	8.01E+03	1.60E+03	-1.71E+04	1.60E+04
-3.21E+03	-3.21E+03	-2.14E+04	1.60E+04	-1.60E+03	-1.71E+04	8.01E+03	8.01E+03	-1.18E+04	-1.60E+03	1.60E+04	-1.71E+04
-1.71E+04	-1.60E+03	-1.60E+04	1.07E+04	-8.01E+03	-3.21E+03	-1.07E+03	1.60E+03	-1.60E+03	-1.18E+04	8.01E+03	-8.01E+03
1.60E+03	-1.07E+03	1.60E+03	-8.01E+03	1.07E+04	3.21E+03	-1.60E+03	-1.71E+04	1.60E+04	8.01E+03	-1.18E+04	8.01E+03
-1.60E+04	-1.60E+03	-1.71E+04	3.21E+03	-3.21E+03	-2.14E+04	1.60E+03	1.60E+04	-1.71E+04	-8.01E+03	8.01E+03	-1.18E+04
-1.07E+03	1.60E+03	1.60E+03	-1.18E+04	8.01E+03	8.01E+03	-1.71E+04	-1.60E+03	1.60E+04	1.07E+04	-8.01E+03	3.21E+03
-1.60E+03	-1.71E+04	-1.60E+04	8.01E+03	-1.18E+04	-8.01E+03	1.60E+03	-1.07E+03	-1.60E+03	-8.01E+03	1.07E+04	-3.21E+03
-1.60E+03	-1.60E+04	-1.71E+04	8.01E+03	-8.01E+03	-1.18E+04	1.60E+04	1.60E+03	-1.71E+04	-3.21E+03	3.21E+03	-2.14E+04
-1.18E+04	-8.01E+03	-8.01E+03	-1.07E+03	-1.60E+03	-1.60E+03	1.07E+04	8.01E+03	-3.21E+03	-1.71E+04	1.60E+03	-1.60E+04
-8.01E+03	-1.18E+04	-8.01E+03	1.60E+03	-1.71E+04	-1.60E+04	8.01E+03	1.07E+04	-3.21E+03	-1.60E+03	-1.07E+03	-1.60E+03
-8.01E+03	-8.01E+03	-1.18E+04	1.60E+03	-1.60E+04	-1.71E+04	3.21E+03	3.21E+03	-2.14E+04	-1.60E+04	1.60E+03	-1.71E+04

Table A.4. Columns 1 to 12 of the Element Stiffness matrix of the hexahedron element used in this study

1.07E+04	8.01E+03	-3.21E+03	-1.71E+04	1.60E+03	-1.60E+04	-1.07E+03	-1.60E+03	-1.60E+03	-1.18E+04	-8.01E+03	-8.01E+03
8.01E+03	1.07E+04	-3.21E+03	-1.60E+03	-1.07E+03	-1.60E+03	1.60E+03	-1.71E+04	-1.60E+04	-8.01E+03	-1.18E+04	-8.01E+03
3.21E+03	3.21E+03	-2.14E+04	-1.60E+04	1.60E+03	-1.71E+04	1.60E+03	-1.60E+04	-1.71E+04	-8.01E+03	-8.01E+03	-1.18E+04
-1.71E+04	-1.60E+03	1.60E+04	1.07E+04	-8.01E+03	3.21E+03	-1.18E+04	8.01E+03	8.01E+03	-1.07E+03	1.60E+03	1.60E+03
1.60E+03	-1.07E+03	-1.60E+03	-8.01E+03	1.07E+04	-3.21E+03	8.01E+03	-1.18E+04	-8.01E+03	-1.60E+03	-1.71E+04	-1.60E+04
1.60E+04	1.60E+03	-1.71E+04	-3.21E+03	3.21E+03	-2.14E+04	8.01E+03	-8.01E+03	-1.18E+04	-1.60E+03	-1.60E+04	-1.71E+04
-1.18E+04	-8.01E+03	8.01E+03	-1.07E+03	-1.60E+03	1.60E+03	-1.71E+04	1.60E+03	1.60E+04	1.07E+04	8.01E+03	3.21E+03
-8.01E+03	-1.18E+04	8.01E+03	1.60E+03	-1.71E+04	1.60E+04	-1.60E+03	-1.07E+03	1.60E+03	8.01E+03	1.07E+04	3.21E+03
8.01E+03	8.01E+03	-1.18E+04	-1.60E+03	1.60E+04	-1.71E+04	1.60E+04	-1.60E+03	-1.71E+04	-3.21E+03	-3.21E+03	-2.14E+04
-1.07E+03	1.60E+03	-1.60E+03	-1.18E+04	8.01E+03	-8.01E+03	1.07E+04	-8.01E+03	-3.21E+03	-1.71E+04	-1.60E+03	-1.60E+04
-1.60E+03	-1.71E+04	1.60E+04	8.01E+03	-1.18E+04	8.01E+03	-8.01E+03	1.07E+04	3.21E+03	1.60E+03	-1.07E+03	1.60E+03
1.60E+03	1.60E+04	-1.71E+04	-8.01E+03	8.01E+03	-1.18E+04	3.21E+03	-3.21E+03	-2.14E+04	-1.60E+04	-1.60E+03	-1.71E+04
4.70E+04	1.60E+04	-1.60E+04	-2.14E+04	3.21E+03	-3.21E+03	1.07E+04	-3.21E+03	-8.01E+03	-1.71E+04	-1.60E+04	-1.60E+03
1.60E+04	4.70E+04	-1.60E+04	-3.21E+03	1.07E+04	-8.01E+03	3.21E+03	-2.14E+04	-3.21E+03	-1.60E+04	-1.71E+04	-1.60E+03
-1.60E+04	-1.60E+04	4.70E+04	3.21E+03	-8.01E+03	1.07E+04	-8.01E+03	3.21E+03	1.07E+04	1.60E+03	1.60E+03	-1.07E+03
-2.14E+04	-3.21E+03	3.21E+03	4.70E+04	-1.60E+04	1.60E+04	-1.71E+04	1.60E+04	1.60E+03	1.07E+04	3.21E+03	8.01E+03
3.21E+03	1.07E+04	-8.01E+03	-1.60E+04	4.70E+04	-1.60E+04	1.60E+04	-1.71E+04	-1.60E+03	-3.21E+03	-2.14E+04	-3.21E+03
-3.21E+03	-8.01E+03	1.07E+04	1.60E+04	-1.60E+04	4.70E+04	-1.60E+03	1.60E+03	-1.07E+03	8.01E+03	3.21E+03	1.07E+04
1.07E+04	3.21E+03	-8.01E+03	-1.71E+04	1.60E+04	-1.60E+03	4.70E+04	-1.60E+04	-1.60E+04	-2.14E+04	-3.21E+03	-3.21E+03

-3.21E+03	-2.14E+04	3.21E+03	1.60E+04	-1.71E+04	1.60E+03	-1.60E+04	4.70E+04	1.60E+04	3.21E+03	1.07E+04	8.01E+03
-8.01E+03	-3.21E+03	1.07E+04	1.60E+03	-1.60E+03	-1.07E+03	-1.60E+04	1.60E+04	4.70E+04	3.21E+03	8.01E+03	1.07E+04
-1.71E+04	-1.60E+04	1.60E+03	1.07E+04	-3.21E+03	8.01E+03	-2.14E+04	3.21E+03	3.21E+03	4.70E+04	1.60E+04	1.60E+04
-1.60E+04	-1.71E+04	1.60E+03	3.21E+03	-2.14E+04	3.21E+03	-3.21E+03	1.07E+04	8.01E+03	1.60E+04	4.70E+04	1.60E+04
-1.60E+03	-1.60E+03	-1.07E+03	8.01E+03	-3.21E+03	1.07E+04	-3.21E+03	8.01E+03	1.07E+04	1.60E+04	1.60E+04	4.70E+04

Table A.5. Columns 13 to 24 of the Element Stiffness matrix of the hexahedron element used in this study